Fundamentals of constitutive modelling for soils

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The constitutive modelling of soils is based on several basic principles enhanced with numerous advanced issues. Some basic principles can be understood even in 1D, like irreversibility of deformation or non-linearity of the stress-strain response. The standard features — soil stiffness, limit stress condition, critical state, dilatancy — are common to all modern constitutive models. Advanced features can take into account some additional effects. In spite of the progress in the field, the present state of the art of the constitutive modelling for soils is still far away from perfection.

1 Introduction

Models are simplifications of reality. They need to capture essential features of the modelling objects and neglect those of less importance. The distinction between important and negligible issues is problem- and purpose-dependent. A physical model of a house for an architecture exhibition will be different from a model house for a children playground, even if both houses are of the same scale.

Constitutive models should mathematically describe the material behaviour. An extraordinary abstraction level is required. Moreover, the model can focus on the micro-, meso- or macroscale, respectively. The material behaviour is not restricted to the stress-strain response only. In many applications the transport phenomena for liquid, gas or heat are of major interest. In other cases, chemical processes within the material need to be considered.

The constitutive modelling of soils can simulate single grains and their interactions using the framework of the discrete element method (DEM). Although impressive advances have been achieved in this field, the discrete modelling is not suitable for routine engineering applications yet. The latter remain in the domain of the continuum mechanics which profits from manifold and well-established theoretical principles.

Within constitutive models for soil as a continuum matter, single grains are smeared into an idealized material (Fig. 1). This material has, for the aspects of interest, the

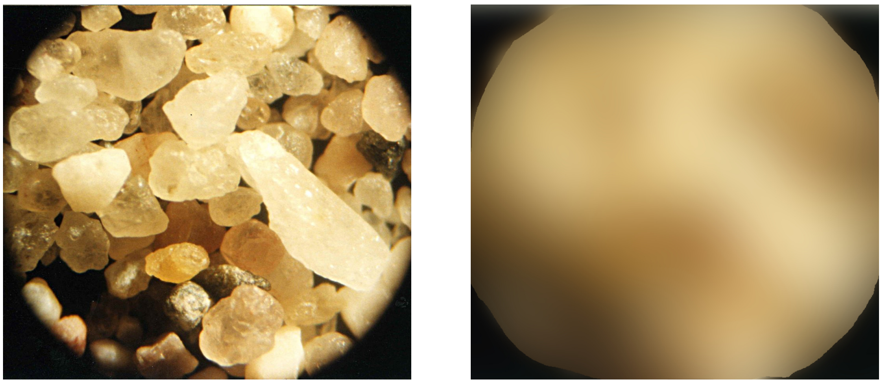

Figure 1: Single grains are not represented in constitutive models for continuum. The particulate nature of soil is smeared.

same response as a soil element. A suitable size of a representative elementary volume (REV) is being implicitly considered, although this size is not obvious. Which amount and fabric of soil grains are necessary for the REV in order to define stress and strain tensors from contact forces and mutual displacements of the grains? An objective answer is not possible since the REV is also inevitably influenced by the natural heterogeneity of soils as geological materials. What is the scale for such a heterogeneity? Where does the scatter of local quantities end and the natural heterogeneity start?

Common constitutive models for soils do not smear only the solid grains but also the liquid and gas phases between the grains. The principle of effective stresses serves as a link between those phases. Multiphase models can consider constitutive models for the phases separately. In this case, however, interaction relationships between the phases must be additionally specified. These interactions are, again, smeared over the REV and do not necessarily reflect the micromechanical response in a straightforward way.

Stress-strain relationships are typical products of constitutive models. Thus, stress-strain curves, like in Fig. 2, are related to the behaviour of a REV which, on the other hand, represents only a point in the continuum. In order to develop, calibrate and validate the constitutive models, the stress-strain curves of a REV must be accessible in experiments, at least for a few well controlled test conditions. The standard laboratory tests involve soil specimens which are definitely much larger than a corresponding REV of the tested material. Still, we interpret the specimen behaviour as identical with the one of the REV and base our constitutive models on this assumption.

Constitutive models are equations and, thus, they are composed of two different sets of quantities — constants and (state) variables. The material parameters are constants in constitutive equations and should not change their values throughout the modelled process. This requires that the material properties with respect to those parameters

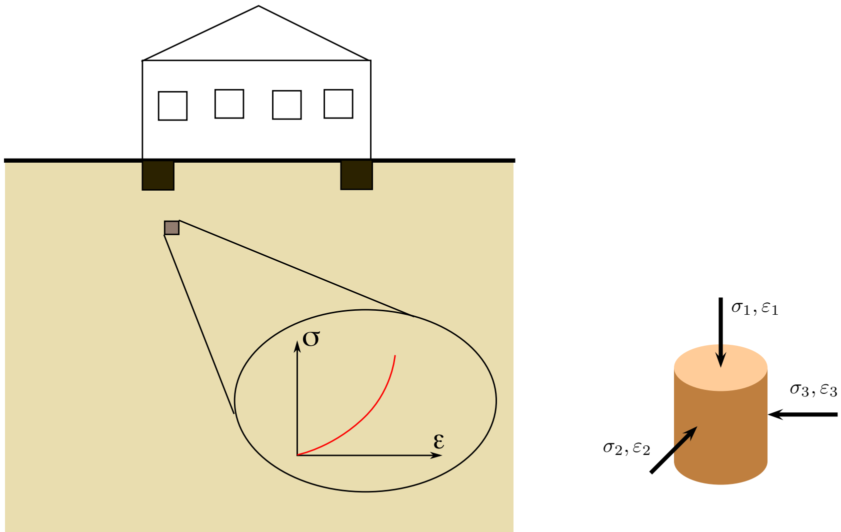

Figure 2: Soil element in situ as a REV and its representation in a laboratory test.

remain constant too. Although this requirement may seem self-evident, it is often violated in the practice of constitutive modelling.

The model variables characterize the actual soil state and, thus, can (and should) change during the modelled process. Equations for the evolution of state variables must be specified. A constitutive model for a stress-strain relationship is an example for an evolution equation for the stress as a state variable. If more (internal) variables are included in a constitutive model, evolution equations for all of them are required. Note that the strain tensor should not be considered as a state variable since soils do not possess a unique reference configuration in which strain corresponds to zero.

A determination (calibration) of the material parameters is crucial for a successful application of any constitutive model. The calibration can be seen as a fitting procedure attempting to achieve a good coincidence between the measured (observed) and calculated behaviour. Equally important is the determination of the initial values of state variables. This task can be much more difficult than the calibration of the model parameters (constants), especially for advanced constitutive models which use internal variables not accessible to measurements.

2 Basic features

Let us focus first on stress-strain relationships in one-dimensional representation. The equation

$$\sigma = E\varepsilon \tag{1}$$

describes a proportionality between the stress σ and the strain ε (Fig. 3 left). Stress should be the effective stress which controls the mechanical behaviour of soils. The constant in Eq. (1) is the material parameter E. It represents the stiffness of the material and in the model of elasticity it would be called Young's modulus.

2.1 Irreversibility

One of the most obvious soil properties is the irreversibility of deformation. With exception of extremely small deformations (e. g., during the passage of weak seismic waves through the soil), the soil skeleton does not recover its original configuration after a load reversal. Eq. (1) suggests a fully reversible behaviour with a unique relationship between stress and strain (Fig 3 left). However, for an irreversible response the stiffness must be different for loading and unloading, respectively (Fig 3 middle). Thus, a unique relationship between stress and strain does not exist (Fig 3 right).

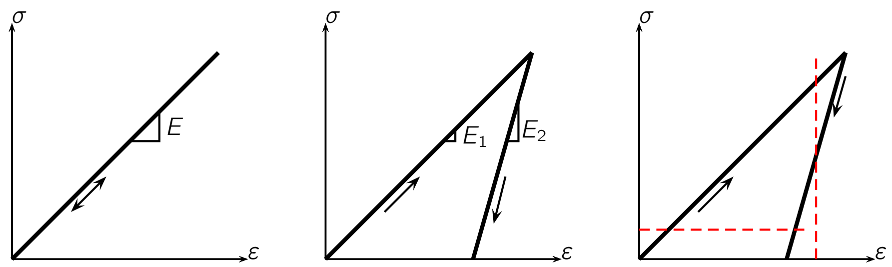

Figure 3: Reversible (elastic) and irreversible (inelastic) behaviour. Red dashed lines in the right diagram demonstrate two different strains for one stress and vice versa. Consequently, there is no unique relationship between stress and strain if the behaviour is irreversible.

Since change of the loading direction can take place in any admissible stress state, the constitutive models for soils must be formulated as incremental stress-strain relationships

$$\dot{\sigma} = E\dot{\varepsilon} \tag{2}$$

where the superimposed dots denote rates (time derivatives) of the quantities. A stress increment can be obtained by integration, e. g.,

$$\Delta\sigma = \int_{t_i}^{t_{i+1}} \dot{\sigma} dt = \dot{\sigma} \cdot \Delta t \; . \tag{3}$$

Obviously, E in Eq. (2) is not a material constant any more since it depends on the loading direction. This condition requires that $\dot{\sigma}(\dot{\varepsilon}) \neq \dot{\sigma}(-\dot{\varepsilon})$. The corresponding constitutive equation may be written as

$$\dot{\sigma} = E_1 \dot{\varepsilon} \quad \text{for } \dot{\varepsilon} > 0 \tag{4}$$

$$\dot{\sigma} = E_2 \dot{\varepsilon} \quad \text{for } \dot{\varepsilon} < 0 \tag{5}$$

It is possible to avoid the switch condition in the equations above if the absolute value $|\dot{\varepsilon}|$ is considered:

$$\dot{\sigma} = \frac{E_1 + E_2}{2}\,\dot{\varepsilon} + \frac{E_1 - E_2}{2}\,|\dot{\varepsilon}| = E_a\,\dot{\varepsilon} + E_b\,|\dot{\varepsilon}| \tag{6}$$

The latter approach is fundamental for the hypoplastic constitutive models. In elasto-plastic constitutive models, the reversible part of the deformation is usually considered elastic.

2.2 Nonlinearity

The incremental stiffness

$$E = \frac{\dot{\sigma}}{\dot{\varepsilon}} \tag{7}$$

does not depend only on loading direction but also on the stress state. It may increase with stress if we consider a compression loading (Fig. 4), or decrease if the stress approaches the limit state.

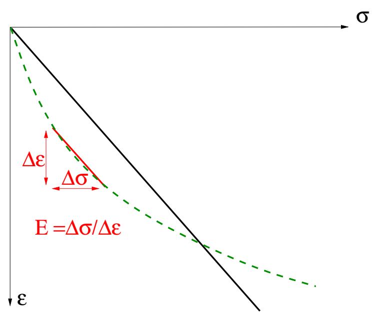

Figure 4: Nonlinear stress-strain behaviour due to stress-dependent stiffness.

In order to reproduce the nonlinear behaviour in (Fig. 4), the stiffness E can be written as a function of stress

$$\dot{\sigma} = E(\sigma)\dot{\varepsilon} \ . \tag{8}$$

The most simple is a linear dependence, i. e.,

$$E(\sigma) = C\sigma \ , \tag{9}$$

where C is a new soil parameter. The stiffness E becomes an auxiliary variable and is not a soil constant any more.

2.3 Rate-independence

The stress increment according to Eq. (3) depends on the time increment Δt. Often, however, the effects of real time on the soil behaviour are to be neglected and t should represent an integration parameter only. This requires that the constitutive equation is homogeneous of the first degree with respect to $\dot{\varepsilon}$, i. e.,

$$\dot{\sigma} = f(\sigma, k \cdot \dot{\varepsilon}) = k \cdot f(\sigma, \dot{\varepsilon}) \qquad \text{with } k > 0 \ . \tag{10}$$

The constitutive models (6) and (8) fulfil Eq. (10).

On the contrary, if time effects like creep, relaxation or dependence on the deformation velocity (rate-dependence, see Fig. 5) should be reproduced, the model must not obey Eq. (10).

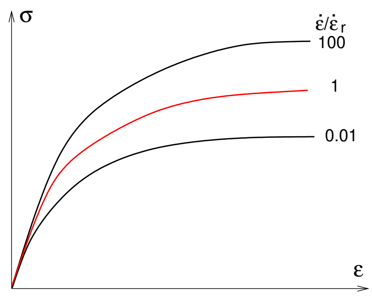

Figure 5: Rate-dependence of the stress-strain behaviour. $\dot{\varepsilon}_r$ denotes a reference strain rate.

3 Standard features

Only a few basic features of constitutive models can be outlined in an one-dimensional representation. A generalisation for at least two dimensions is necessary to capture some further standard features of the soil models. Finally, all constitutive models need a tensorial formulation in 3D, in order to be able to implement the models into general-purpose finite elements codes. The model formulations must be independent on the reference frame and should not predict any deformation for a rigid body rotation.

In more than one spatial dimension, the understanding of stress (and strain) paths is crucial for constitutive modelling. The evolution of the stress tensor can be captured in various coordinates. The stress invariants p (mean stress) and q (stress deviator) are the most common ones. Nevertheless, some effects, like the rotation of principal stress axes, cannot be observed in the $p - q$ representation.

3.1 Stiffness

The incremental modulus defined in Eq. (8) obscures the spatial character of the stiffness. Its magnitude should depend on the direction of deformation at a particular soil state, i. e.,

$$E_{ij}(\sigma_i) = \frac{\dot{\sigma}_i}{\dot{\varepsilon}_j} \tag{11}$$

(here, the state variable is the principal effective stress σ_i). This feature can be well represented by response envelopes shown in Fig. 6.

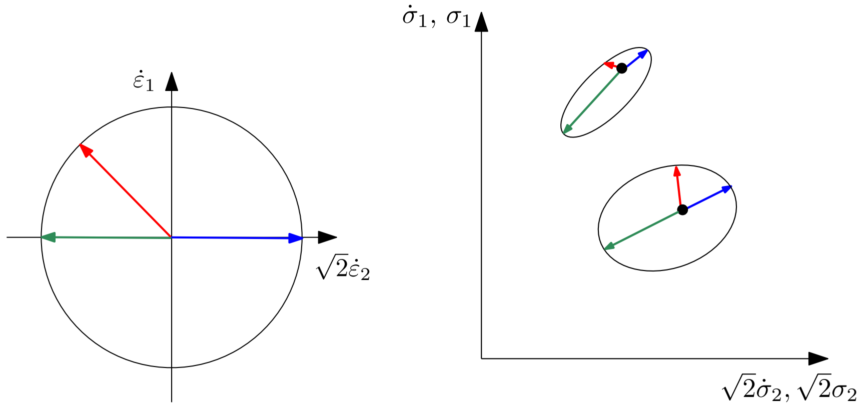

Figure 6: The concept of response envelopes. Each direction of the strain rate $|\dot{\varepsilon}| = 1$ (left) is mapped to the corresponding stress rate $\dot{\sigma}$ (right) which represents a state- and direction-dependent stiffness.

Consider a cylindrical specimen at radially symmetric (triaxial) conditions loaded in three different strain rates of the same magnitude but of different directions (Fig. 6 left): the blue arrow represents isotropic compression, the green one isotropic extension and the red one the undrained (constant volume) compression. The corresponding strain rates, as predicted by a constitutive model, are shown in Fig. 6 right. The particular stress states are marked by the big black dots. The stress ratio for isotropic extension (unloading) is higher than for isotropic loading. For the stress state with a higher ratio of principal stresses, i. e., closer to the limit stress condition, the stiffness in shear (undrained compression) is much lower than for the stress state close to the isotropic one.

If the constitutive model is rate-independent, the magnitude of the strain rate can be considered as one, $|\dot{\varepsilon}| = \sqrt{\dot{\varepsilon}_1 + 2\dot{\varepsilon}_2} = 1$, and thus the magnitude of the stress rate $|\dot{\sigma}| = \sqrt{\dot{\sigma}_1 + 2\dot{\sigma}_2}$ corresponds to the state- and direction-dependent stiffness $\dot{\sigma}/\dot{\varepsilon}$. Connecting stress rates calculated for all strain rates at one particular state, so-called stress response envelopes (Fig. 6 right) are obtained [Gud79].

3.2 Limit stress condition

The effective stresses are bounded in the stress space. However, it is not possible to consider a unique limit stress condition for a soil. The magnitude of the limit stress depends on the amount of deformation and the soil state (Fig. 7 left). The state-dependence of the limit stress state results in a non-linear stress envelope (Fig. 7 middle). Moreover, various proposals can be found for the shape of the limit stress surface in the deviatoric plane (Fig. 7 right).

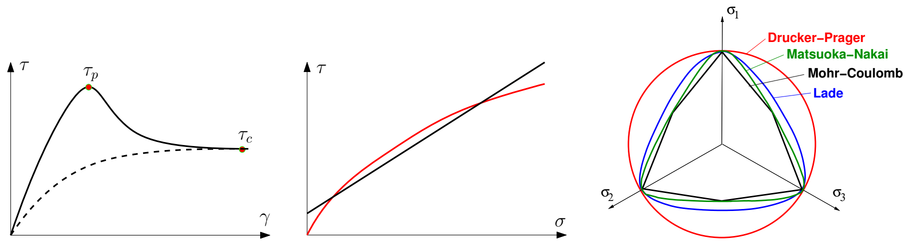

Figure 7: Various aspects of the limit stress condition: state- and strain-dependence of shear strength (left), non-linearity with respect to normal (mean) stress (middle) and cross-section in the deviatoric plane (right).

The limit stress state is characterised by vanishing stiffness, i. e., $\dot{\sigma}_i = 0$. Thus, for a particular soil state, it should be possible to calculate the limit stress from the constitutive equation analytically.

3.3 Void ratio and critical states

Void ratio e plays a crucial role for the state-dependent description of the soil behaviour. It has been established as a state variable practically in all advanced constitutive models. The evolution equation

$$\dot{e} = (1 + e)\dot{\varepsilon}_v \tag{12}$$

relates the change of e to the change of volumetric strain ε_v and, thus, implies incompressibility of soil grains.

The critical state as a steady state during constant volume deformation is a fundamental concept of the modern soil mechanics [Mui90]. It is common to assume a unique relationship between mean stress and void ratio in the critical state, although some experimental results question it [MFV98, FR03]. The critical state as an attractor during shear deformation is necessary for a robust performance of any constitutive model for soils.

3.4 Dilatancy

Dilatancy D expresses the maximum rate of the volume increase during shearing, which can be formulated, e. g., in triaxial (axially symmetric) conditions as

$$D = \frac{\dot{\varepsilon}_v}{\dot{\varepsilon}_q} = \frac{\dot{\varepsilon}_1 + 2\dot{\varepsilon}_2}{\frac{2}{3}(\dot{\varepsilon}_1 - \dot{\varepsilon}_2)} \quad . \tag{13}$$

By increasing relative soil density, the maximum shear strength and also dilatancy rise. Thus, a unique relationship between dilatancy D (for plastic strain rates in case of elasto-plastic models) and the maximum ratio of principal stresses $R = \sigma_1/\sigma_2$ is included in many constitutive equations (Fig. 8 left).

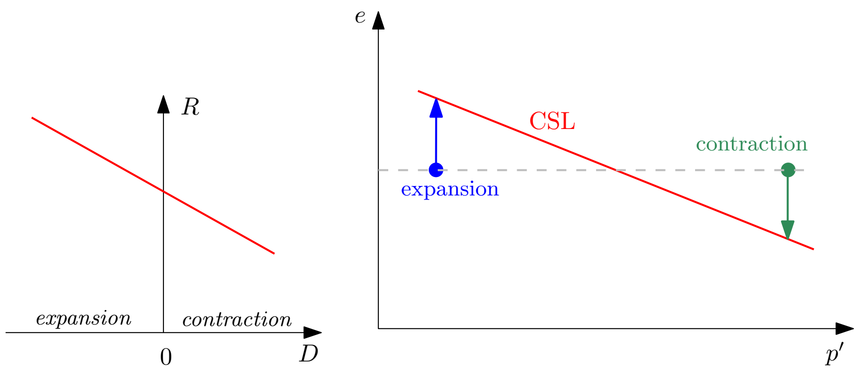

Figure 8: Dilatancy D as a function of the maximum stress ratio R (left), pressure-dependent volumetric response (right).

However, the soil state changes during deformation and is, in fact, pressure-dependent. This means that a "dense" soil at a low mean pressure can behave as a loose soil at a high mean pressure (Fig. 8 right). In (water-saturated) fine grained soils, analogous effects can be observed for water content related to consistency limits. Consequently, the relationship in Fig. 8 (left) is pressure-dependent [Bol86, LD00].

The distance between the soil state and the critical state line CSL at one particular mean pressure p' is often denoted as state parameter $\psi = e - e_c$ [BJ85]. An analogous role plays a so-called pressure-dependent relative density in hypoplastic models for sand [Gud96]. For fine grained soils, the horizontal distance between the soil state and the critical state line, similar to the Hvorslev's equivalent pressure (e.g., [Mui90]), can be used.

3.5 Constant volume deformation

The undrained conditions are being produced in laboratory testing of fully saturated soils when the drainage is completely prohibited. Under assumption of incompressibility of water and soil grains, constant volume of the specimen, i. e. $\dot{\varepsilon}_v = 0$, is preserved during the test.

The undrained response is an important benchmark for the constitutive models. The shape of the stress path is linked to the evolution of the pore water pressure and, thus, to the dilatancy effect. The maximum stress difference $q = \sigma_1 - \sigma_2$ is essential, e. g., for analyses of liquefaction or short term slope stability. The predicted undrained shear strength should be state-dependent and should reflect the soil loading history like overconsolidation.

4 Advanced features

Recent constitutive models can take into account a number of additional features of the soil behaviour. Obviously, by adding further ingredients, the complexity of the models increases. The increased complexity results not only in more equations but also in more material parameters which may be mutually dependent. In many cases, additional state variables are introduced which, in turn, need their evolution equations.

The following list of effects, which can be implemented in advanced constitutive models (stress-strain relationships), brings only a few typical examples and does not represent a comprehensive state of the art.

- *Stress and deformation history*

  Memory of soil preserves its stress and deformation history in a manifold way. A typical scalar memory variable is the overconsolidation ratio OCR. In elastoplastic models, the latter is usually related to the size of the yield surface. The OCR can be also linked to the equivalent pressure or another similar quantity.

  Recent deformation history related to the so-called small-strain stiffness [ARS90] is often taken into account by the kinematic nature of yield surfaces [ATMW89]. Another option may be the so-called intergranular strain concept [NH97].

- *Anisotropy*

  Properties of anisotropic materials depend on the orientation with reference to the coordinate system. An essential induced anisotropy evolves with non-isotropic stress tensor since in most models stiffness depends on stress. A fabric-related anisotropy (e. g., the distribution of the grain contact normals

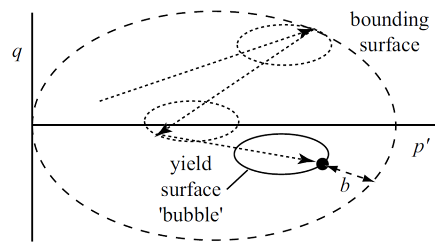

Figure 9: Kinematic yield surface of the ”Bubble” model [Mui04].

in space) is reflecting the deformation history. It may be modelled by a structure tensor linked to the kinematic hardening of the yield surface(s) [WNKL03, TD08].

- *Cementation*

  In natural soils, brittle bonds at grain contacts can evolve with time due to various physical and chemical processes. Such a cementation may result in an increased apparent preconsolidation pressure. Elastoplastic constitutive models consider this effect by increasing the yield stress and thus expanding the (quasi) elastic stress range, followed by a fast structure degradation (collapse) [LN95]. The limit stress condition and further soil features can be affected by cementation as well [LT14].

- *Chemical and weathering effects*

  The modelling methodology for the degradation of bonding (cementation) can be also applied to weathering and chemical degradation [NCT03, Bus12, CdP16], sometimes in coupling with effects of partial saturation [PAV07]. Purely chemical processes can impact the mechanical properties of soils as well [HHH16].

- *Thermal effects*

  Soil behaviour is also sensitive to temperature. Constitutive models usually distinguish the effects of high temperatures (e.g., in clay barriers for the radioactive waste) [MK12, HPTC13] and freezing phenomena [NW19] separately. With respect to energy geostructures, the constitutive modelling of temperature oscillations may be of special interest, too [DL15].

- *Grain crushing*

  Usually, the soil parameters are constant for one particular soil which is characterized, among others, by its grain size distribution curve. Consequently, the grains of such a soil are considered to be permanent. However, especially coarse grains undergo degradation during soil deformation. This degradation starts with an abrasion of asperities at the grain surface at lower stresses and continues with grain breakage at higher stresses. The modelling of grain crushing and the resulting change of the grain size distribution may be linked to the consumed energy [Ein07]. The modification of the grading can be related to the classical elastoplastic concepts [KMR10].

- *Partial saturation*

  If the soil is not fully saturated, it must be considered as a three-phase material. The definition of the effective stress becomes less obvious. An additional stress variable, mostly the suction as a difference between air and water pressure, is needed in order to model the observed phenomena [GGSV03]. A short overview of the modelling concepts can be found, e. g., in [GSS06, SGFS08, NZC20].

5 Evaluation and validation

Even if there exists a perfect constitutive model for the soil behaviour, it is of no value until its parameters (constants) are known. Thus, the calibration of the material parameters is crucial for a successful application of constitutive models.

However, the constitutive models for soils are by far not perfect. They represent a compromise with respect to numerous effects which can be observed in experiments. There is a number of publications comparing the performance of advanced constitutive models, e. g., [RM10, WFT19]. They confirm the necessity for further (sometimes substantial) improvements. A unified approach for the software routines of constitutive models [GAG$^+$07] is helpful for such comparisons.

It must be also taken into account that practically all soil mechanics tests treat the soil specimens as idealized elements and attribute only unique values of the measured quantities to the whole specimen. E. g., vertical stresses and strains in a triaxial specimen are calculated from the measurements of a single force and displacement at the specimen boundary, assuming a homogeneous deformation. Thus, the scatter of the soil state (and, eventually, of the soil parameters) over the high number of REVs within the specimen is not taken into account.

The development, evaluation and validation of the constitutive models is, thus, affected by many uncertainties. A perfect coincidence between the measured and calculated curves is not necessarily admirable. Exaggerated requirements on the agreement between experimental and numerical results in element tests are not meaningful. General trends are usually much more important. During the model calibration and evaluation, a decision linked to the later application must be often made, see, e. g., Fig. 10.

6 Final remarks

The constitutive modelling of soils is a challenging discipline. Its fundamentals require a firm knowledge of soil testing and behaviour, paired with advanced mathemat-

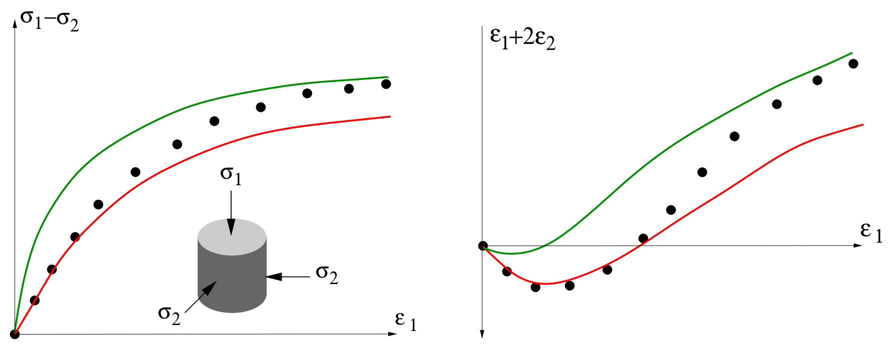

Figure 10: Which model (or parameter set) is better?

ics and a high level of abstraction. Although numerous advanced constitutive models are available for soils, their performance is not fully satisfactory under general conditions. A further research is needed.

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