$\aleph_0$-categorical Structures: Endomorphisms and Interpretations

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Abstract

We extend the Ahlbrandt–Ziegler analysis of interpretability in $\aleph_0$-categorical structures by showing that existential interpretation is controlled by the monoid of self–embeddings and positive existential interpretation of structures without constant endomorphisms is controlled by the monoid of endomorphisms in the same way as general interpretability is controlled by the automorphism group.

1 Introduction

$\aleph_0$-categorical structures (often called ω-categorical structures) appear quite naturally in mathematics, and have extensively been studied by model theorists. They appear for example as countable universal structures for classes of finite structures with the amalgamation property. The best known example might be the countable random graph, which can be seen as a universal amalgam of the class of all finite graphs. The $\aleph_0$-categorical structures can also be characterised by a transitivity property of their automorphisms groups, which are so-called "oligomorphic permutation groups", and therefore they are also interesting for and have been studied by group theorists. More on $\aleph_0$-categorical structures can for example be found in [8], Sections 7.3 and 7.4, [9] and [6].

In fact, much of an $\aleph_0$-categorical structure is coded in its automorphism group. Ahlbrandt and Ziegler in [1] have shown that a countable $\aleph_0$-categorical structure is, up to bi-interpretability, determined by its automorphism group as a topological group. We extend this analysis and show that, with certain unavoidable restrictions, existential interpretability is controlled by the monoid of self-embeddings and positive existential interpretability by the endomorphism monoid.

It would be interesting to further extend the theory (as far as possible) to primitive positive interpretability on the one hand and polymorphism clones on the

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other hand; a characterisation of primitive positive interpretability in terms of the topological polymorphism clone would have interesting consequences for the study of the computational complexity of constraint satisfaction problems in theoretical computer science.

2 Endomorphisms

2.1 Preservation theorems for $\aleph_0$-categorical theories

In this paper, we only consider structures $\mathfrak{M}$ in a countable signature without function symbols (i.e. relational possibly with constants). We denote by $\mathrm{Aut}(\mathfrak{M})$ the automorphism group of $\mathfrak{M}$, by $\mathrm{Emb}(\mathfrak{M})$ the monoid of embeddings[1] of $\mathfrak{M}$ into $\mathfrak{M}$, and by $\mathrm{End}(\mathfrak{M})$ the monoid of all endomorphisms of $\mathfrak{M}$. Then $\mathrm{Aut}(\mathfrak{M}) \subseteq \mathrm{Emb}(\mathfrak{M}) \subseteq \mathrm{End}(\mathfrak{M}) \subseteq {}^M M$. All these monoids carry the topology of pointwise convergence, a basis of open neighbourhoods of which is given by the sets $U_{\bar{a},\bar{b}} = \{\sigma \mid \bar{a}^\sigma = \bar{b}\}$. Finally, $\mathrm{Sym}(M)$ denotes the symmetric group on M.

Remark 1 **(a)** $\mathrm{Emb}(\mathfrak{M})$ *and* $\mathrm{End}(\mathfrak{M})$ *are closed in* ${}^M M$*, because if a map is not a homomorphisms, not injective or not strong, then this is already witnessed by a finite tuple, hence a complete open neighbourhood does lack this property.* $\mathrm{Aut}(\mathfrak{M})$ *is closed in* $\mathrm{Sym}(M)$*, and more generally in the set of all surjections* $M \to M$*, but in general not in* ${}^M M$*.*

(b) $\mathrm{Aut}(\mathfrak{M}) = \mathrm{Emb}(\mathfrak{M}) \cap \mathrm{Sym}(M)$*, but in general there are more bijective endomorphisms than automorphisms as they need not to be strong. But then their inverse maps are not homomorphisms. It follows that* $\mathrm{Aut}(\mathfrak{M})$ *equals the set of invertible elements of* $\mathrm{End}(\mathfrak{M})$ *and therefore the largest subgroup of* $\mathrm{End}(\mathfrak{M})$*.*

Notations and conventions: For the sake of this paper, we call a structure $\aleph_0$-*categorical* if it is finite or a countable model of an $\aleph_0$-categorical theory with an at most countable language. We will freely use the characterisation of Engeler, Ryll–Nardzewski and Svenonius (see [8], Theorem 7.3.1), which in particular implies the *ultrahomogeneity* of an $\aleph_0$-categorical structure: any two tuples of same type are conjugate under the automorphism group.

We let endomorphisms act from the right side and write x^σ for $\sigma(x)$ and $x^{\sigma\tau}$ for $\tau(\sigma(x))$, and in particular $\sigma\tau$ for $\tau \circ \sigma$.

Formulae, definability etc, are meant without parameters, unless otherwise specified. For the present paper, it doesn't make a difference whether we understand "existential formula" and "positive formula" up to logical equivalence or not. It is a classical result that this works as well for "positive existential", i.e. positivity and existentiality can be realised simultaneously (see e.g. [7] Exercise 5.2.6).

Let Σ be a set of maps from M to M. Each $\sigma \in \Sigma$ induces a map $M^k \to M^k$ (acting component by component), also denoted by σ. The *orbit* of $\bar{m} \in M^k$ under Σ is the

[1] i.e. isomorphisms onto a substructure, or equivalently strong injective homomorphisms

set of images $\bar{m}^\Sigma = \{\bar{m}^\sigma \mid \sigma \in \Sigma\}$. In general, the orbits are not the classes of an equivalence relation. If $X \subseteq M^k$, then X is called *closed under* Σ if for all $x \in X$, the orbit x^Σ is contained in X.

Proposition 2 ((a),(b) in [5], Theorem 5) *Let M be an $\aleph_0$-categorical structure, and $X \subseteq M^k$.*

- **(a)** *X is existentially definable in $\mathfrak{M}$ if and only if X is closed under* $\mathrm{Emb}(\mathfrak{M})$.
- **(b)** *X is positive existentially definable in $\mathfrak{M}$ if and only if X is closed under* $\mathrm{End}(\mathfrak{M})$.
- **(c)** *X is positively[2] definable in $\mathfrak{M}$ if and only if X is closed under all surjective endomorphisms of $\mathfrak{M}$.*
- **(d)** *X is positive existentially definable in $\mathfrak{M}$ in the language with $\neq$ if and only if X is closed under all injective endomorphisms of $\mathfrak{M}$.*
- **(e)** *X is positively definable in $\mathfrak{M}$ in the language with $\neq$ if and only if X is closed under all bijective endomorphisms of $\mathfrak{M}$.*

Proof: It is well known by Ryll–Nardzewski etc. that X is definable if and only if X is invariant under $\mathrm{Aut}(\mathfrak{M})$, and because $\mathrm{Aut}(\mathfrak{M})$ is a group, this is equivalent to being closed under $\mathrm{Aut}(\mathfrak{M})$. Therefore, we may assume that X is definable by a formula ϕ.

(b) and (c): If $\mathfrak{M} \models \phi(\bar{a})$ and ϕ is positive, then $\sigma(\mathfrak{M}) \models \phi(\bar{a}^\sigma)$ for every homomorphism σ. If σ is surjective or if ϕ is in addition existential modulo T, then it follows that $\mathfrak{M} \models \phi(\bar{a}^\sigma)$.

For the other direction, we need the classical Łos–Tarski and Lyndon preservation theorems (see [8] Theorem 6.5.4 and Corollary 10.3.5). By these well-known theorems, if ϕ is not positive existential (not positive), then there are models $\mathfrak{M}_i \models T$, a (surjective) homomorphism $\sigma : \mathfrak{M}_1 \to \mathfrak{M}_2$ and $\bar{a}$ in $\mathfrak{M}_1$ with $\mathfrak{M}_1 \models \phi(\bar{a})$ and $\mathfrak{M}_2 \not\models \phi(\bar{a}^\sigma)$. Now choose a countable elementary substructure of $(\mathfrak{M}_1, \mathfrak{M}_2, \sigma, \bar{a})$. Up to isomorphism, it has the form $(\mathfrak{M}, \mathfrak{M}, \sigma', \bar{a})$, where σ' is a (surjective) endomorphism of $\mathfrak{M}$. Then we get $\mathfrak{M} \models \phi(\bar{a})$, but $\mathfrak{M} \not\models \phi(\bar{a}^{\sigma'})$.

(d) follows from (b) and (e) from (c) just by adding $\neq$ to the language. In the same way (a) follows from (d) by adding negations of all the basic relations to the language. □

Clearly, a set $X \subseteq M^k$ is closed under $\Sigma \subseteq {}^M M$ if and only if X is closed under the closure of Σ in ${}^M M$. Therefore, if Σ_1 and Σ_2 are dense in each other, then syntactical properties characterised by Σ_1, Σ_2 are equivalent. If $\Sigma_1 = \mathrm{Aut}(\mathfrak{M}) \subseteq \Sigma_2$, then the converse also holds, which we will prove for the example $\Sigma_2 = \mathrm{Emb}(\mathfrak{M})$:

Corollary 3 (Bodirsky, Pinsker) *An $\aleph_0$-categorical theory with countable model $\mathfrak{M}$ is model complete if and only if* $\mathrm{Aut}(\mathfrak{M})$ *is dense in* $\mathrm{Emb}(\mathfrak{M})$.

[2] $\top$ and $\bot$ are positive formulae.

PROOF: "$\Longleftarrow$" follows from the general remarks above.

"$\Longrightarrow$": According to Proposition 2 (a), self-embeddings preserve existential types, hence complete types in case the theory is model complete. This implies $\bar{a}^\sigma \equiv \bar{a}$ for all finite tuples $\bar{a}$ in $\mathfrak{M}$ and all $\sigma \in \mathrm{Emb}(\mathfrak{M})$. By the ultrahomogeneity of an $\aleph_0$-categorical model (tuples of same type are conjugate under the automorphism group), this is equivalent to $\mathrm{Aut}(\mathfrak{M})$ being dense in $\mathrm{Emb}(\mathfrak{M})$. $\square$

In the same style, every definable set is positively definable in $\mathfrak{M}$, if all surjective homomorphisms are automorphisms.

2.2 Topology

Let $\mathfrak{M}$ be an $\aleph_0$-categorical structure and T its theory. We consider the topological space $\mathrm{End}(\mathfrak{M})/\mathrm{Aut}(\mathfrak{M})$ of *right cosets* of $\mathrm{Aut}(\mathfrak{M})$ in $\mathrm{End}(\mathfrak{M})$, i.e. the quotient of $\mathrm{End}(\mathfrak{M})$ by the equivalence relation

$$\sigma \sim \sigma' \iff \text{there is } \alpha \in \mathrm{Aut}(\mathfrak{A}) \text{ with } \sigma' = \sigma\alpha,$$

equipped with the quotient topology, the finest topology which turns $\pi : \sigma \mapsto \sigma/_\sim$ into a continuous map. Inverse images of the open sets are open sets in $\mathrm{End}(\mathfrak{M})$ of the form $X \cdot \mathrm{Aut}(\mathfrak{M})$ for open $X \subseteq \mathrm{End}(\mathfrak{M})$.

Lemma 4 $\mathrm{End}(\mathfrak{M})/\mathrm{Aut}(\mathfrak{M})$ *and* $\mathrm{Emb}(\mathfrak{M})/\mathrm{Aut}(\mathfrak{M})$ *are compact.*

PROOF: As $\mathrm{Emb}(\mathfrak{M})$ is closed in $\mathrm{End}(\mathfrak{M})$ and a union of right cosets of $Aut(\mathfrak{M})$, it is sufficient to show the first claim. Consider an open covering $(U_i)_{i\in I}$ of $\mathrm{End}(\mathfrak{M})/\mathrm{Aut}(\mathfrak{M})$. We may assume that the inverse images $\tilde{U}_i := \pi^{-1}[U_i]$ in $\mathrm{End}(\mathfrak{M})$ are of the form $U_{\bar{c}_i,\bar{d}_i} \cdot \mathrm{Aut}(\mathfrak{M}) = \{\sigma \in \mathrm{End}(\mathfrak{M}) \mid \bar{c}_i^\sigma \equiv \bar{d}_i\}$. Thus the $\tilde{U}_i$ from an open covering of $\mathrm{End}(\mathfrak{M})$ by sets which are unions of right cosets. It is sufficient to show that $\mathrm{End}(\mathfrak{M})$ is covered by finitely many of the $\tilde{U}_i$. Fix an enumeration $(m_i)_{i\in\omega}$ of M. If p is an n-type of T, let $U_p := U_{(m_0,\dots,m_{n-1}),\bar{a}} \cdot \mathrm{Aut}(\mathfrak{M})$ where $\bar{a}$ is some/any realisation of p. Note that if $U_p \neq \emptyset$ and $\bar{a} \models p$, then $(m_0,\dots,m_{n-1}) \mapsto \bar{a}$ is a partial endomorphism. Finally, let us say that an open set O is "covered" if there is an $i \in I$ with $O \subseteq \tilde{U}_i$.

If U_p is covered for some $n \in \omega$ and each of the finitely many n-types p, then the coverings set form an open sub-covering of $\mathrm{End}(\mathfrak{M})$. Therefore, we may assume that for each n, there is an n-type $p_n(x_0,\dots,x_{n-1})$ such that U_{p_n} is not covered (and in particular, $U_{p_n} \neq \emptyset$). The types p_n form an infinite tree under inclusion, which is finitely branched because of the $\aleph_0$-categoricity. Hence, by König's Lemma, there is an infinite branch $(p_n)_{n\in\omega}$. If $(a_n)_{n\in\omega}$ realises $\bigcup_{n\in\omega} p_n$, then $\sigma : m_n \mapsto a_n$ defines an endomorphism of $\mathfrak{M}$.

Now choose i such that $\sigma \in \tilde{U}_i$, and let n be big enough such that $\bar{c}_i$ is contained in $\bar{m} := (m_0,\dots,m_{n-1})$. Then $U_{p_n} = U_{\bar{m},\bar{m}^\sigma} \subseteq \tilde{U}_i$: contradiction. This shows quasi-compactness.

If $\sigma \not\sim \sigma'$, then there is a tuple $\bar{a}$ with $\bar{a}^\sigma \not\equiv \bar{a}^{\sigma'}$. Thus the open neighbourhoods $U_{\bar{a},\bar{a}^\sigma} \cdot \mathrm{Aut}(\mathfrak{M})$ and $U_{\bar{a},\bar{a}^{\sigma'}} \cdot \mathrm{Aut}(\mathfrak{M})$ separate σ and σ'. $\square$

Remark 5 $\mathrm{Aut}(\mathfrak{M})$ *is not "normal" in* $\mathrm{Emb}(\mathfrak{M})$*, i.e. the left coset* $\sigma \cdot \mathrm{Aut}(\mathfrak{M})$ *is in general different from the right coset* $\mathrm{Aut}(\mathfrak{M}) \cdot \sigma$*.*

Example 1 Let $\mathfrak{M}$ be an equivalence relation with two classes, both countably infinite; α is an automorphism that exchanges both classes, and σ is an embedding that is the identity on one class and non surjective on the other class. Then $\sigma^{-1}\alpha\sigma$ can't be extended to an automorphism of $\mathfrak{M}$, i.e. $\alpha\sigma$ is not of the form $\sigma\alpha'$ for some automorphism α'.

3 Interpretations

The classical theory of interpretations of $\aleph_0$-categorical theories as developed by Ahlbrandt and Ziegler in [1] is briefly as follows. (An account of the theory and more about interpretations can be found in Section 1 of [9] and in Section 5 of [8]). In [1], $\aleph_0$-categorical structures are considered as a category with interpretations as morphisms, and "Aut" is made into a functor into the category of topological groups with continuous group homomorphisms, where $\mathrm{Aut}(i)$ for an interpretation i of $\mathfrak{B}$ in $\mathfrak{A}$ is the natural map $\mathrm{Aut}(\mathfrak{A}) \to \mathrm{Aut}(\mathfrak{B})$ induced by i.

Theorem 1.2 in [1] *A continuous group homomorphism* $f : \mathrm{Aut}(\mathfrak{A}) \to \mathrm{Aut}(\mathfrak{B})$ *is of the form* $\mathrm{Aut}(i)$ *for an interpretation* i *of* $\mathfrak{B}$ *in* $\mathfrak{A}$ *if and only if* B *is covered by finitely many orbits under the image of* f*.*

Two interpretations i_1, i_2 of $\mathfrak{B}$ in $\mathfrak{A}$ are called homotopic if $\{(\bar{x}, \bar{y}) \mid i_1(\bar{x}) = i_2(\bar{y})\}$ is definable in $\mathfrak{A}$. Two structures $\mathfrak{A}, \mathfrak{B}$ are bi-interpretable if there are mutual interpretations i and j such that $i \circ j$ and $j \circ i$ are homotopic to the identity interpretations $\mathrm{id}_{\mathfrak{A}}, \mathrm{id}_{\mathfrak{B}}$ respectively.

Theorem 1.3 and Corollary 1.4 in [1] *Two interpretations* i_1, i_2 *of* $\mathfrak{B}$ *in* $\mathfrak{A}$ *are homotopic if and only if* $\mathrm{Aut}(i_1) = \mathrm{Aut}(i_2)$*. The structures are bi-interpretable if and only if there automorphism groups are isomorphic as topological groups.*

Remark 6 *In Theorem 1.2 of [1], one could as well have considered a continuous monoid homomorphism* $\mathrm{Aut}(\mathfrak{A}) \to \mathrm{End}(\mathfrak{B})$ *instead of a continuous group homomorphism* $\mathrm{Aut}(\mathfrak{A}) \to \mathrm{Aut}(\mathfrak{B})$*. This is because a monoid homomorphism defined on a group is a group homomorphism, and thus the group* $\mathrm{Aut}(\mathfrak{A})$ *has to be mapped into the largest group contained in* $\mathrm{End}(\mathfrak{B})$ *which is* $\mathrm{Aut}(\mathfrak{B})$*.*

Our aim is to extend the classical results to endomorphisms on the one hand and to syntactically restricted interpretations on the other hand.

3.1 The existential case

Let us call *basic sets of a structure* the universe, the diagonal, the interpretations of the relational symbols in the language and the graphs of the interpretations of the functions symbols in the language. An interpretation of a structure $\mathfrak{N}$ in a structure $\mathfrak{M}$ is existential (positive existential) if all inverse images of basic sets of $\mathfrak{N}$ are existentially (positive existentially) definable in $\mathfrak{M}$.

Theorem 7 *Let $\mathfrak{A}$ be an $\aleph_0$-categorical structure with at least two elements. Then $\mathfrak{B}$ is existentially interpretable in $\mathfrak{A}$ if and only if there is a continuous monoid homomorphism $f : \mathrm{Emb}(\mathfrak{A}) \to \mathrm{End}(\mathfrak{B})$ such that B is covered by finitely many orbits under the image of f, or, equivalently, such that B is covered by finitely many orbits under $f[\mathrm{Aut}(\mathfrak{A})]$.*

Proof: If B is covered by finitely many orbits under $f[\mathrm{Aut}(\mathfrak{A})]$, then it is also covered by finitely many orbits under $f[\mathrm{Emb}(\mathfrak{A})]$. We are going to show "$\Longleftarrow$" with the weaker and "$\Longrightarrow$" with the stronger of the two covering conditions.

"$\Longleftarrow$": Choose $\bar{b} = (b_1, \dots, b_k)$ with $b_i \in B$ such that B is covered by the orbits of the b_i under $f[\mathrm{Emb}(\mathfrak{A})]$.

Claim: There is a finite tuple $\bar{a}$ in A with the following property: If $\bar{a}^\sigma = \bar{a}^\tau$ for $\sigma, \tau \in \mathrm{Emb}(\mathfrak{A})$, then $\bar{b}^{f(\sigma)} = \bar{b}^{f(\tau)}$.

Proof of the Claim: We call a tuple $\bar{a}$ *good for* σ if $\bar{a}^\sigma = \bar{a}^\tau$ implies $\bar{b}^{f(\sigma)} = \bar{b}^{f(\tau)}$ for all τ. Fix $\sigma_0 \in \mathrm{Emb}(\mathfrak{A})$. Because f is continuous, $f^{-1}[U_{\bar{b},\bar{b}^{f(\sigma_0)}}]$ is an open set containing σ_0 and thus contains a basic open neighbourhood $U_{\bar{c},\bar{c}^{\sigma_0}}$ of σ_0. Then $\bar{c}$ is good for σ_0 because if $\bar{c}^{\sigma_0} = \bar{c}^\tau$, then $\tau \in U_{\bar{c},\bar{c}^{\sigma_0}}$, hence $f(\tau) \in U_{\bar{b},\bar{b}^{f(\sigma_0)}}$ and thus $\bar{b}^{f(\sigma_0)} = \bar{b}^{f(\tau)}$.

Note that $\bar{c}$ clearly is good for each other $\sigma \in U_{\bar{c},\bar{c}^{\sigma_0}}$, and also for all $\sigma_0\alpha$ with $\alpha \in \mathrm{Aut}(\mathfrak{A})$, i.e. for the whole neighbourhood $U_{\bar{c},\bar{c}^{\sigma_0}} \cdot \mathrm{Aut}(\mathfrak{A})$. For suppose $\bar{c}^{\sigma_0\alpha} = \bar{c}^\tau$, then $\bar{c}^{\sigma_0} = \bar{c}^{\tau\alpha^{-1}}$, hence $\bar{b}^{f(\sigma_0)} = \bar{b}^{f(\tau\alpha^{-1})} = \bar{b}^{f(\tau)f(\alpha)^{-1}}$ because f is a monoid homomorphism and thus maps automorphisms onto automorphisms. Finally $\bar{b}^{f(\sigma_0\alpha)} = \bar{b}^{f(\tau)}$ follows.

Now we have found a $\bar{c}_i$ for each σ_i which is good for the neighbourhood $U_i := U_{\bar{c}_i,\bar{c}_i^{\sigma_i}} \cdot \mathrm{Aut}(\mathfrak{A})$. By the compactness of $\mathrm{Emb}(\mathfrak{A})/\mathrm{Aut}(\mathfrak{A})$ shown in Lemma 4, finitely many of these neighbourhoods, say $U_1, \dots, U_l$, cover $\mathrm{Emb}(\mathfrak{A})$. Then $\bar{a} := \bar{c}_1{}^\frown \cdots {}^\frown\bar{c}_l$ is a tuple which is good for all $\mathrm{Emb}(\mathfrak{A})$. ◊

Let $a'_1, \dots, a'_k$ be arbitrary pairwise distinct elements of A (or, if $|A| < k$, of some sufficiently large power of A). We may assume that the a'_i appear in the tuple $\bar{a}$ (otherwise extend $\bar{a}$ by the a'_i). Now we can continue as in Ahlbrandt–Ziegler:

Definition:
Let $U := \{(a'_i, \bar{a})^\sigma \mid i = 1, \dots, k,\ \sigma \in \mathrm{Emb}(\mathfrak{A})\}$ and define $\underline{f} : U \to B$ by $(a'_i, \bar{a})^\sigma \mapsto b_i^{f(\sigma)}$.

Note that by definition, U is closed under $\mathrm{Emb}(\mathfrak{A})$, hence existentially definable after Proposition 2.

Claim: $\underline{f}$ is well defined and surjective.

If $(a'_i, \bar{a})^\sigma = (a'_j, \bar{a})^\tau$, then ${a'_i}^\sigma = {a'_j}^\tau$, and, as a'_i is contained in the tuple $\bar{a}$, also ${a'_i}^\sigma = {a'_i}^\tau$. Because τ is an embedding, hence injective, we get $i = j$. Now $\bar{a}^\sigma = \bar{a}^\tau$

implies $b_i^{f(\sigma)} = b_i^{f(\tau)}$ by the construction of $\bar{a}$, proofing $\underline{f}$ to be well defined. The surjectivity is clear by the choice of the $\bar{b}_i$. $\Diamond$

Claim: $\underline{f}$ is an existential interpretation.

Let $X \subseteq B^l$ and consider $\underline{f}^{-1}[X] \subseteq A^{(m+1)l}$. An element $\bar{y}$ therein has the form

$$\bar{y} = \big((a'_{i_1}, \bar{a})^{\sigma_1}, \dots, (a'_{i_l}, \bar{a})^{\sigma_l}\big) \text{ with } \big(b_{i_1}^{f(\sigma_1)}, \dots, b_{i_l}^{f(\sigma_l)}\big) \in X.$$

Let $\sigma \in \mathrm{Emb}(\mathfrak{A})$. Then

$$\begin{aligned} \bar{y}^\sigma &= \big((a'_{i_1}, \bar{a})^{\sigma_1\sigma}, \dots, (a'_{i_l}, \bar{a})^{\sigma_l\sigma}\big) \in \underline{f}^{-1}[X] \\ &\iff \big(b_{i_1}^{f(\sigma_1\sigma)}, \dots, b_{i_l}^{f(\sigma_l\sigma)}\big) = \big(b_{i_1}^{f(\sigma_1)}, \dots, b_{i_l}^{f(\sigma_l)}\big)^{f(\sigma)} \in X. \end{aligned}$$

If X is a basic set of the structure $\mathfrak{B}$, then X is closed under $\mathrm{End}(\mathfrak{B})$, thus the second condition is satisfied, whence $\underline{f}^{-1}[X]$ is existentially definable by Proposition 2. $\Diamond$

"$\Longrightarrow$": Let $\mathfrak{B}$ be existentially interpreted in $\mathfrak{A}$ by a surjection $i : A^l \supseteq U \twoheadrightarrow B$. Then $U = i^{-1}[B]$ and $E := i^{-1}[=_B]$ are existentially definable in $\mathfrak{A}$, hence closed under $\mathrm{Emb}(\mathfrak{A})$ by Proposition 2. It follows that every $\sigma \in \mathrm{Emb}(\mathfrak{A})$ induces a map $\sigma^* : B \to B, uE \mapsto u^\sigma E$. Because the inverse image $i^{-1}[R]$ of every basic set R of $\mathfrak{B}$ is also existentially definable and thus closed under $\mathrm{Emb}(\mathfrak{A})$, the map σ^* is even a homomorphism. This defines a mapping $\mathrm{Emb}(i) : \mathrm{Emb}(\mathfrak{A}) \to \mathrm{End}(\mathfrak{B}), \sigma \mapsto \sigma^*$, which clearly is a monoid homomorphism. The homomorphism is continuous: if $\bar{a}, \bar{a}'$ are inverse images of $\bar{b}, \bar{b}'$, then the open set $U_{\bar{a},\bar{a}'}$ lies in the inverse image of $U_{\bar{b},\bar{b}'}$. By the general theory developed in [1], the interpretation i induces a continuous group homomorphism $\mathrm{Aut}(i) : \mathrm{Aut}(\mathfrak{A}) \to \mathrm{Aut}(\mathfrak{B})$ in the same way as above, that is $\mathrm{Aut}(i)$ is the map induced by $\mathrm{Emb}(i)$. Then by Theorem 1.2 in [1], B is covered by finitely many orbits under the image of $\mathrm{Aut}(\mathfrak{A})$ under $\mathrm{Aut}(i)$. $\square$

Remark 8 *In "$\Longleftarrow$", it follows in particular that $\mathfrak{B}$ is $\aleph_0$-categorical, too.*

If this is known before, and if $\mathrm{Aut}(\mathfrak{B})$ *is contained in the image of f —in particular if f is surjective— then B is automatically covered by finitely many orbits under the image of f.*

If the image of f is contained in $\mathrm{Emb}(\mathfrak{B})$*, then Proposition 2 can be applied to $\mathfrak{B}$, and the same argument as above shows that not only the inverse images of the basic sets, but of all existentially defined sets of $\mathfrak{B}$ are existentially definable in $\mathfrak{A}$.*

3.2 The positive existential case

The proof of Theorem 7 works as well if one replaces "Emb" by "End" and "existential" by "positive existential", except for the well definedness of $\underline{f}$. The remark after Lemma 14 will show that there is no general solution to this problem. Therefore, we have to restrict our attention to a well behaved class of structures.

Definition 9 *An $\aleph_0$-categorical structure is called* contractible *if it has a constant endomorphism.*

Lemma 10 *An $\aleph_0$-categorical structure $\mathfrak{A}$ is contractible if and only if for each two tuples $\bar{c}_0, \bar{c}_1$ out of A of same length there is an endomorphism $\sigma \in \mathrm{End}(\mathfrak{A})$ such that $\bar{c}_0^\sigma = \bar{c}_1^\sigma$.*

PROOF: Clearly, a contractible structure satisfies the condition. Assume now that the condition is satisfied. Given a tuple $\bar{a} = (a_1, \dots, a_k)$, choose an endomorphism σ with $(a_1, \dots, a_k)^\sigma = (a_1, \dots, a_1)^\sigma$. Then σ is constant $= c$ on $\bar{a}$. By multiplying with automorphisms we can assume that c is an element of a fixed representation system $\{c_1, \dots, c_l\}$ of the 1-types and moreover that it only depends on the type of $\bar{a}$. If we do this for a long tuple composed from representations of all k-types, we see that we can choose the value for each k even independently from the type of $\bar{a}$. But then one of the finitely many values in question $c_1, \dots, c_l$ must work for every finite tuple, say c. Now $\mathrm{End}(\mathfrak{A})$ is closed ${}^A A$, therefore the constant map c is an endomorphism. □

Theorem 11 *Let $\mathfrak{A}$ be an $\aleph_0$-categorical, non-contractible structure. Then $\mathfrak{B}$ is positive existentially interpretable in $\mathfrak{A}$ if and only if there is a continuous monoid homomorphism $f : \mathrm{End}(\mathfrak{A}) \to \mathrm{End}(\mathfrak{B})$ such that B is covered by finitely many orbits under the image of f, or, equivalently, such that B is covered by finitely many orbits under $f[\mathrm{Aut}(\mathfrak{A})]$.*

PROOF: Take the proof of Theorem 7, replace "Emb" by "End" and "existential" by "positive existential", and change the definition of U as follows: Choose tuples $\bar{c}_0, \bar{c}_1$ of length l such that $\bar{c}_0^\sigma \neq \bar{c}_1^\sigma$ for all endomorphisms σ as given by Lemma 10. Then let a'_i be the ml-tuple $(\bar{c}_0, \dots, \bar{c}_0, \bar{c}_1, \bar{c}_0, \dots, \bar{c}_0)$ where $\bar{c}_1$ is at the ith position. We may assume $m \geqslant 3$.

Now if $(a'_i, \bar{a})^\sigma = (a'_j, \bar{a})^\tau$ for $i \neq j$, then by comparing the appropriate coordinates we get $\bar{c}_1^\sigma = \bar{c}_0^\tau = \bar{c}_0^\sigma$: contradiction. Thus again $\underline{f}$ is well defined and everything goes through as in the proof of Theorem 7. □

In fact, for the direction "$\Longrightarrow$" we do not need $\mathfrak{A}$ to be non-contractible. Therefore:

Proposition 12 *If $\mathfrak{A}$ is an $\aleph_0$-categorical structure and if i is a positive existential interpretation of $\mathfrak{B}$ in $\mathfrak{A}$, then there is a continuous monoid homomorphism $\mathrm{End}(i) : \mathrm{End}(\mathfrak{A}) \to \mathrm{End}(\mathfrak{B})$ such that B is covered by finitely many orbits under $f[\mathrm{Aut}(\mathfrak{A})]$.*

Corollary 13 *"End" is a functor from the category of $\aleph_0$-categorical structures together with positive existential interpretations as morphisms into the category of topological monoids with continuous monoid homomorphisms.*

PROOF: Check that the composition of positive existential interpretations is again positive existential (replacing a quantifier-free sub-formula of a positive existential formula by a positive existential formula yields again a positive existential formula). The rest follows from Proposition 12. □

Finally we remark that Theorem 11 can't be extended to arbitrary $\aleph_0$-categorical structures:

Lemma 14 *If $\mathfrak{A}$ is contractible and if $\mathfrak{B}$ is positive existentially interpretable in $\mathfrak{A}$, then $\mathfrak{B}$ is contractible, too.*

PROOF: Let σ be a constant endomorphism of $\mathfrak{A}$ and let $\mathfrak{B}$ be positive existentially interpreted in $\mathfrak{A}$ by the interpretation i. Then $\sigma^* = \mathrm{End}(i)(\sigma)$ is a constant endomorphism of $\mathfrak{B}$. □

Note that there are non-contractible finite structures $\mathfrak{B}$, which by Lemma 14 are not positive existentially interpretable in a contractible structure as for example $(\mathbb{N}, =)$, but the conditions of Theorem 11 are trivially satisfied: the trivial monoid homomorphism $\mathrm{End}(\mathbb{N}, =) \to \mathrm{End}(\mathfrak{B})$ is continuous and B, being finite, is covered by finitely many orbits of the image $\{\mathrm{id}\}$.

3.3 Bi-interpretability

For non-contractible structures, the theory of bi-interpretability of [1] can be extended to positive existential interpretations. In the general case, or for existential interpretations, only partial results hold.

Definition 15 *Following [1], we call two interpretations i_1 and i_2 of $\mathfrak{B}$ in $\mathfrak{A}$* End-homotopic *if* $\mathrm{End}(i_1) = \mathrm{End}(i_2)$. *Two $\aleph_0$-categorical structures $\mathfrak{A}$ and $\mathfrak{B}$ are* positive existentially bi-interpretable *if there are mutual positive existential interpretations i and j such that $i \circ j$ and $j \circ i$ are* End*-homotopic to the identical interpretations* $\mathrm{id}_{\mathfrak{A}}, \mathrm{id}_{\mathfrak{B}}$ *respectively.*

Lemma 16 *Let i_1, i_2 be two interpretations of $\mathfrak{B}$ in $\mathfrak{A}$. Then* $\mathrm{End}(i_1) = \mathrm{End}(i_2)$ *holds if and only if the set $I_{i_1,i_2} := \{(\bar{x}, \bar{y}) \mid i_1(\bar{x}) = i_2(\bar{y})\}$ is positive existentially definable in $\mathfrak{A}$.*

PROOF: $\mathrm{End}(i_1), End(i_2)$ associate with an endomorphism $\sigma \in \mathrm{End}(\mathfrak{A})$ the maps induced by $\bar{x} \mapsto \bar{x}^\sigma, \bar{y} \mapsto \bar{y}^\sigma$, respectively. Both are the same if and only if $(\bar{x}, \bar{y}) \in I_{i_1,i_2}$ implies $(\bar{x}, \bar{y})^\sigma = (\bar{x}^\sigma, \bar{y}^\sigma) \in I_{i_1,i_2}$. But according to Proposition 2 this is exactly the case if I_{i_1,i_2} is positive existentially definable. □

Proposition 17 *Let $\mathfrak{A}$ and $\mathfrak{B}$ be $\aleph_0$-categorical structures. If they are positive existentially bi-interpretable, then* $\mathrm{End}(\mathfrak{A})$ *and* $\mathrm{End}(\mathfrak{B})$ *are isomorphic as topological monoids. The converse holds for non-contractible structures.*

PROOF: "$\Longrightarrow$": Let i and j be mutual interpretations witnessing the positive existential bi-interpretability. Then $j \circ i$ is End-homotopic to the identical interpretation, hence $\mathrm{End}(j) \circ \mathrm{End}(i) = \mathrm{End}(j \circ i) = \mathrm{End}(\mathrm{id}) = \mathrm{id}$. Symmetrically, $\mathrm{End}(i) \circ \mathrm{End}(j) = \mathrm{id}$, hence $\mathrm{End}(i) = \mathrm{End}(j)^{-1}$ has to be a bi-continuous isomorphism.

"$\Longleftarrow$": If $f : \mathrm{End}(\mathfrak{A}) \to \mathrm{End}(\mathfrak{B})$ is an isomorphism, then by Theorem 11, f and f^{-1} yield interpretations $\underline{f}$ and $\underline{f^{-1}}$. The composition $j := \underline{f^{-1}} \circ \underline{f} : \mathfrak{A} \to \mathfrak{A}$ then induces the map $\mathrm{End}(\mathfrak{A}) \to \mathrm{End}(\mathfrak{B}) \to \mathrm{End}(\mathfrak{A}), \sigma \mapsto f^{-1}(\sigma) \mapsto f(f^{-1}(\sigma)) = \sigma$, hence $\mathrm{End}(j) = \mathrm{id} = \mathrm{End}(\mathrm{id})$. By symmetry, also $\mathrm{End}(\underline{f} \circ \underline{f}^{-1}) = \mathrm{End}(\mathrm{id})$. □

The converse of Proposition 17 does not hold for arbitrary $\aleph_0$-categorical structures, as Lemma 14 together with the following lemma shows.

Lemma 18 *The isomorphism type of* $\mathrm{End}(\mathfrak{A})$ *does not determine whether* $\mathfrak{A}$ *is contractible.*

PROOF: Let $\mathfrak{A}$ be contractible $\mathcal{L}$-structure. We may assume the language $\mathcal{L}$ to be relational. Let $\mathfrak{B}$ be an $\mathcal{L} \cup \{c, P\}$-structure that results from joining a new element c to $\mathfrak{A}$ and a predicate P for the set A. Then $\mathfrak{B}$ is not contractible, but clearly $\mathrm{End}(\mathfrak{A})$ and $\mathrm{End}(\mathfrak{B})$ are isomorphic. □

On the other hand, each contractible structure contains an absorbing endomorphism σ, i.e. $\tau\sigma = \sigma$ for every τ (and if there are constant endomorphisms, then they are exactly the absorbing elements). So non-contractibility can sometimes be seen from the endomorphism monoid.

Whether there are similar interpretability results for contractible structures is unclear.

We will see in Section 4 that "Emb" is not a functor as "Aut" and "End" are. Therefore, the characterisation of existential bi-interpretability via the embedding monoids only holds in one direction. With definitions analogously to Definition 15 and the same proofs as for Lemma 16 and Proposition 17, we get:

Proposition 19 *Let* i_1, i_2 *be two interpretations of* $\mathfrak{B}$ *in* $\mathfrak{A}$. *Then* $\mathrm{Emb}(i_1) = \mathrm{Emb}(i_2)$ *holds if and only if the set* $I_{i_1,i_2} := \{(\bar{x}, \bar{y}) \mid i_1(\bar{x}) = i_2(\bar{y})\}$ *is existentially definable in* $\mathfrak{A}$.

If $\mathrm{Emb}(\mathfrak{A})$ *and* $\mathrm{Emb}(\mathfrak{B})$ *are isomorphic as topological monoids, then* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are existentially bi-interpretable.*

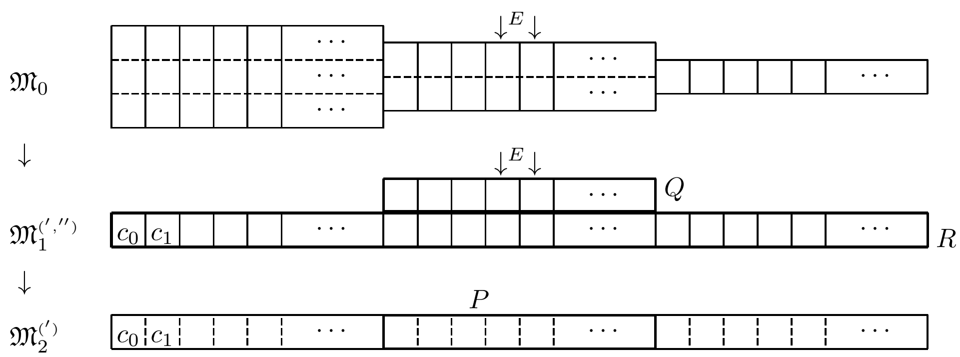
The converse of the second part does not hold, as Example 5 below shows.

4 Examples

We have seen that "End" can be considered as a functor of the category of $\aleph_0$-categorical structures with positive existential interpretations into the category of topological monoids with continuous monoid homomorphisms. This is not possible for "Emb" and existential interpretations, for at least two reasons: $\aleph_0$-categorical structures with existential interpretations do not form a category, and the natural way to define Emb on morphisms leads to non-embeddings. We start with an example for the second problem:

Example 2 The image of a monoid homomorphism $f : \mathrm{Emb}(\mathfrak{A}) \to \mathrm{End}(\mathfrak{B})$ is not in general contained in $\mathrm{Emb}(\mathfrak{B})$.

Let $\mathfrak{M}_1$ be the following structure: M_1 is a countably infinite set, E an equivalence relation on M_1 with infinitely many one-element classes, infinitely many two-element classes and no others. The language just contains a symbol for E. In $\mathfrak{M}_1$, the structure $\mathfrak{M}_2$ of an infinite, co-infinite predicate P is existentially definable as M_1/E with P being the image of the two-element classes. Now there are embeddings of $\mathfrak{M}_1$ mapping one-element classes into two-element classes. Their image in $\mathfrak{M}_2$ are endomorphisms that are not strong. □



Squares correspond to elements of the structures; dotted lines do not correspond to structure named in the signature.

Figure 1: Examples 2, 3, 4 and 5.

This phenomenon is in connection with the following: $\mathrm{Aut}(\mathfrak{M})$ can be characterised in the abstract monoid $\mathrm{End}(\mathfrak{M})$ as the subgroup of invertible elements $\mathrm{End}(\mathfrak{M})^*$. Therefore a homomorphism between endomorphism monoids restricts to a homomorphism between the automorphism group. $\mathrm{Emb}(\mathfrak{M})$, on the other hand, can only be defined in the "permutation monoid"[3] $\mathrm{End}(\mathfrak{M})$; no characterisation in the abstract monoid is possible as Example 4 shows.

The following holds in general, with $E = \mathrm{End}(\mathfrak{M})$:

$$\{\sigma \in E \mid \exists \tau \in E\ \sigma\tau \in E^*\} \subseteq \mathrm{Emb}(\mathfrak{M}) \subseteq \{\sigma \in E \mid \forall \tau_1, \tau_2 \in E\ (\tau_1\sigma = \tau_2\sigma \Rightarrow \tau_1 = \tau_2)\}$$

Example 3 Take $\mathfrak{M}_1, \mathfrak{M}_2$ as in Example 2 and expand $\mathfrak{M}_1$ to the structure $\mathfrak{M}_1''$ by adding a predicate Q that picks exactly one element out of each two-element class, and a predicate R for its complement. It is easy to see that the interpretation of $\mathfrak{M}_2$ in $\mathfrak{M}_1''$ induces an isomorphism $\mathrm{End}(\mathfrak{M}_2) \to \mathrm{End}(\mathfrak{M}_1'')$. The image of the injective endomorphisms of $\mathfrak{M}_2$ are exactly the injective endomorphisms of $\mathfrak{M}_1''$, but the image of $\mathrm{Emb}(\mathfrak{M}_2)$ is only a proper subset of $\mathrm{Emb}(\mathfrak{M}_1'')$. Thus "Emb" does not allow an abstract characterisation. (Note that $\mathfrak{M}_2$ is contractible, but $\mathfrak{M}_1$ is not, so they are not positive existentially bi-interpretable.) □

[3] I.e. the monoid with its action on the set M; in analogy to "permutation group", though the elements of the monoid are not in generally acting as permutations.

Example 4 The composition of existential interpretations need not to be existential:

Let $\mathfrak{M}_0$ be the structure on a countably infinite set M_0 with an equivalence relation with infinitely many three-element, two-element and one-element classes and no others. It interprets existentially the structure $\mathfrak{M}_1$ in Example 2 by collapsing each three-element class to one element. The composition with the interpretation in Example 2 however is not existential: it yields the interpretation of $\mathfrak{M}_2$ in $\mathfrak{M}_0$, which on M_0/E define a predicate for the images of the two-element classes. This predicate is not existential. $\square$

We conclude with an example of two structures $\mathfrak{M}'_1, \mathfrak{M}'_2$ that are mutually positive existentially interpretable, existentially but not positive existentially bi-interpretable, and with non isomorphic embedding monoids.

Example 5 $\mathfrak{M}'_1$ is the structure $\mathfrak{M}_1$ from Example 2 with an additional predicate Q, that takes exactly one element out of each two-element equivalence class, and with two additional non-equivalent constants c_0, c_1,living in one-element classes. $\mathfrak{M}'_2$ is the structure $\mathfrak{M}_2$ of an infinite, co-infinite predicate P from Example 2 together with two distinct constants c_0, c_1 not in P.

We interpret $\mathfrak{M}'_2$ positive existentially in $\mathfrak{M}'_1$ as M_1/E with $\exists y(Qy \wedge Exy)$ providing the predicate P and by keeping the two constants. We interpret $\mathfrak{M}'_1$ positive existentially in $\mathfrak{M}'_2$ as follows: the universe is $(M_2 \times \{c_0\}) \cup (P \times \{c_1\})$; the equivalence relation E is "same first coordinate", the predicate Q is $P \times \{c_1\}$ and the two constants are (c_0, c_0) and (c_1, c_0).

Both structures are bi-interpretable as they have the same automorphism group $S_\omega \times S_\omega$. But the endomorphism monoids are not isomorphic: In $\mathrm{End}(\mathfrak{M}'_1)$, there is the endomorphism σ that collapses all two-element classes and is the identity on Q and the one-element classes. This endomorphism satisfies $\sigma^2 = \sigma$ and commutes with all automorphisms, hence is definable in the structure. There are three such elements in $\mathrm{End}(\mathfrak{M}'_2)$: identity on $P \cup \{c_0, c_1\}$ and either identity or constant $= c_0$ or c_1 on the rest, but six in $\mathrm{End}(\mathfrak{M}'_1)$: the corresponding maps and their compositions with σ.

It is easy to verify that the bi-interpretation above is in fact an existential bi-interpretation. But $\mathrm{Emb}(\mathfrak{M}'_1) \ncong \mathrm{Emb}(\mathfrak{M}'_2)$, as can be seen with the following argument: Because of the bi-interpretability, both structures have isomorphic automorphism groups, of isomorphism type $S_\omega \times S_\omega$. If the two embedding monoids were isomorphic, an isomorphism had to respect this decomposition as it is unique in this group. Now in $\mathfrak{M}'_2$, each embedding σ is a (commuting) product of the two embeddings $\sigma\restriction_P \cup \mathrm{id}_{M_2 \setminus P}$ and $\mathrm{id}_P \cup \sigma\restriction_{M_2 \setminus P}$, and each of the two commutes with one of the factors S_ω. In $\mathfrak{M}_1$ however, there are embeddings which move one-element equivalence classes into two-element classes. Such an endomorphism cannot be decomposed in that way. $\square$

5 Concluding remarks

We have shown characterisations of existential and positive existential interpretability in $\aleph_0$-categorical structures:

- A structure $\mathfrak{B}$ has an existential interpretation in an $\aleph_0$-categorical structure $\mathfrak{A}$ if and only if there is a continuous monoid homomorphism f from the monoid of self-embeddings of $\mathfrak{A}$ to the endomorphism monoid of $\mathfrak{B}$ such that the domain of $\mathfrak{B}$ is covered by finitely many orbits under the image of f.
- A structure $\mathfrak{B}$ has a positive existential interpretation in an $\aleph_0$-categorical structure $\mathfrak{A}$ without constant endomorphisms if and only if there is a continuous monoid homomorphism f from the endomorphism monoid of $\mathfrak{A}$ to the endomorphism monoid of $\mathfrak{B}$ such that the domain of $\mathfrak{B}$ is covered by finitely many orbits under the image of f.

It is open whether the second result also holds for $\aleph_0$-categorical structures $\mathfrak{A}, \mathfrak{B}$ with constant endomorphisms.

It would be very interesting to find an analogous characterisation of *primitive positive interpretability.* A formula is called *primitive positive* if it is of the form

$$\exists x_1 \dots \exists x_n (\psi_1 \wedge \cdots \wedge \psi_m)$$

where $\psi_1, \dots, \psi_m$ are atomic formulas. Primitive positive interpretations play an important role for the study of the computational complexity of *constraint satisfaction problems.* For a structure $\mathfrak{A}$ with finite relational signature τ, the *constraint satisfaction problem for* $\mathfrak{A}$*, CSP*$(\mathfrak{A})$*,* is the computational problem to decide whether a given primitive positive τ-sentence is true in $\mathfrak{A}$. Such problems are abundant in many areas of computer science.

It is well-known that if (every relation of) a structure $\mathfrak{B}$ is primitive positively definable in a structure $\mathfrak{A}$, then CSP$(\mathfrak{B})$ has a polynomial-time reduction to CSP$(\mathfrak{A})$. Indeed, an important technique to show that CSP$(\mathfrak{A})$ is NP-hard is to find another structure $\mathfrak{B}$ such that CSP$(\mathfrak{B})$ is already known to be NP-hard, and to give a primitive positive definition of $\mathfrak{B}$ in $\mathfrak{A}$.

Primitive positive definability in an $\aleph_0$-categorical structure $\mathfrak{A}$ is captured by the *polymorphisms* of $\mathfrak{A}$. A polymorphism of $\mathfrak{M}$ is a homomorphism of some power $\mathfrak{M}^n$ (with the product structure) to $\mathfrak{M}$. A subset $X \subseteq M^k$ is called *closed under polymorphisms* if for all n, every polymorphism $\sigma : \mathfrak{M}^n \to \mathfrak{M}$ and all $\bar{a}_1, \dots, \bar{a}_n \in X$ we have $(\bar{a}_1, \dots, \bar{a}_n)^\sigma \in X$. The following has been shown in [4]:

Theorem 20 *Let M be an $\aleph_0$-categorical structure and $X \subseteq M^k$. Then X is positive primitive modulo the theory of $\mathfrak{M}$ if and only if X is closed under polymorphisms.*

The classification of the computational complexity of CSP$(\mathfrak{A})$ for all highly set-transitive structures $\mathfrak{A}$ obtained in [3] makes essential use of this theorem.

An even more powerful tool to classify the computational complexity of CSP($\mathfrak{A}$) is *primitive positive interpretability.* It has been shown in [2] that if a structure $\mathfrak{B}$ has a primitive positive interpretation in $\mathfrak{A}$, then there is a polynomial-time reduction from CSP($\mathfrak{B}$) to CSP($\mathfrak{A}$). Hence, it would be interesting to have algebraic characterisations of primitive positive interpretability in $\aleph_0$-categorical structures.

Note that the set of all polymorphisms of a structure $\mathfrak{A}$ can be seen as an algebra whose operations are precisely the polymorphisms of $\mathfrak{A}$; we will refer to this algebra as the *polymorphism clone* of $\mathfrak{A}$. In fact, the set of all polymorphisms forms an object called *clone* in universal algebra. The following characterisation of primitive positive interpretability has also been given in [2].

Theorem 21 *Let $\mathfrak{A}$ be finite or $\aleph_0$-categorical. Then a structure $\mathfrak{B}$ has a primitive positive interpretation in $\mathfrak{A}$ if and only if there is an algebra $\mathbb{B}$ in the pseudo-variety generated by the polymorphism clone of $\mathfrak{A}$ such that all operations of $\mathbb{B}$ are polymorphisms of $\mathfrak{B}$.*

It follows that for $\aleph_0$-categorical structures $\mathfrak{A}$ the computational complexity of CSP($\mathfrak{A}$) is determined by the pseudo-variety generated by the polymorphism clone of $\mathfrak{A}$. We would like to give an alternative characterisation of primitive positive interpretability in terms of the *topological polymorphism clone* of $\mathfrak{A}$, in analogy to the theorems shown in this paper. In fact, we conjecture that the computational complexity of CSP($\mathfrak{A}$) is indeed determined by the topological polymorphism clone of $\mathfrak{A}$.

References

[1] Gisela Ahlbrandt and Martin Ziegler. Quasi-finitely axiomatizable totally categorical theories. *Ann. Pure Appl. Logic*, 30(1):63–82, 1986. Stability in model theory (Trento, 1984).

[2] Manuel Bodirsky. Constraint satisfaction problems with infinite templates. In *Complexity of Constraints*, pages 196–228, 2008.

[3] Manuel Bodirsky and Jan Kára. The complexity of temporal constraint satisfaction problems. In *STOC*, pages 29–38, 2008.

[4] Manuel Bodirsky and Jaroslav Nesetril. Constraint satisfaction with countable homogeneous templates. *J. Log. Comput.*, 16(3):359–373, 2006.

[5] Manuel Bodirsky and Michael Pinsker. All reducts of the random graph are model-complete, 2009. arxiv: 0903.2553.

[6] Peter J. Cameron. *Oligomorphic permutation groups*, volume 152 of *London Mathematical Society Lecture Note Series.* Cambridge University Press, Cambridge, 1990.

[7] C. C. Chang and H. J. Keisler. *Model theory*, volume 73 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, third edition, 1990.

[8] Wilfrid Hodges. *Model theory*, volume 42 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1993.

[9] Richard Kaye and Dugald Macpherson, editors. *Automorphisms of first-order structures*. Oxford Science Publications. The Clarendon Press Oxford University Press, New York, 1994.