

FINITE TREES ARE RAMSEY UNDER TOPOLOGICAL EMBEDDINGS

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ABSTRACT. We show that the class of finite rooted binary plane trees is a Ramsey class (with respect to topological embeddings that map leaves to leaves). That is, for all such trees P, H and every natural number k there exists a tree T such that for every k -coloring of the (topological) copies of P in T there exists a (topological) copy H' of H in T such that all copies of P in H' have the same color. When the trees are represented by the so-called rooted triple relation, the result gives rise to a Ramsey class of relational structures with respect to induced substructures.

1. INTRODUCTION AND RESULT

All trees in this paper are finite, rooted, and *binary*, i.e., all vertices have outdegree two or zero. We also assume that the trees are *plane* in the sense that they are embedded without crossings into the half-plane such that all leaves lie on the boundary of the half-plane (and hence there is a linear order on the leaves of a tree). In the following, *tree* always means finite rooted binary plane tree. The set of all vertices of a tree T is denoted by $V(T)$, and the set of all leaves by $L(T)$.

Two trees H and T are said to be *isomorphic* if there exists a bijection f from $V(H)$ to $V(T)$ that preserves the tree structure and that preserves the linear order on the leaves given by the embedding of the tree. We say that H is a (*topological*) *subtree* of T if $L(H) \subseteq L(T)$ and if T can be obtained from H by adding isolated vertices, adding edges, and *subdividing* edges by replacing an edge with a path (so that all inner vertices of the path are new vertices). If H is a subtree of T that is isomorphic to G then we say that H is a (*topological*) *copy* of G in T .

We illustrate these concepts at the following example, drawn in Figure 1. The tree on the left with root g' contains a copy of the tree on the right: we can subdivide the edge from g' to b' by a new vertex e' , add an isolated vertex a' , and add an edge from e' to a' . The resulting graph is isomorphic to the graph on the left.

A *k-coloring* of a set S is a mapping χ from S into a set of cardinality k (the set of *colors*). We say that an element of S has *color* c (*under* χ) if it is mapped to c (by the mapping χ). In the statement of our result it will

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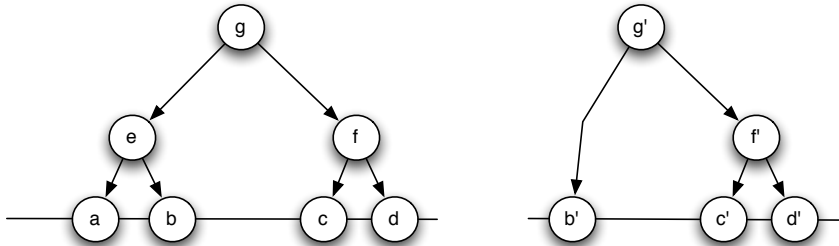


FIGURE 1. Text

be convenient to use the classical Ramsey theoretic notation $T \rightarrow (H)_k^P$ (for trees T, P, H and an integer $k \geq 2$) for the fact that for any k -coloring of the copies of P in T there is a copy H' of H in T such that all copies of P in H' have the same color.

We prove the following.

Theorem 1. *For all trees P, H and for all $k \geq 2$ there exists a tree T such that $T \rightarrow (H)_k^P$.*

Note. After this article has been written, we noticed that the main result is implied by stronger results in [Mil79] (Theorem 4.3), building on work in [Deu75]. The results in [Mil79] also imply results on infinite trees, cover many variations, and therefore involve more sophisticated terminology than here. Since we need the result in applications (to study the complexity of constraint satisfaction problems), and since we found both the statement and its proof easier accessible in the present paper, we decided to still post this article as a technical report, without claiming originality.

1.1. Ramsey classes. The result is of interest in structural Ramsey theory since it gives rise to a so-called *Ramsey class* of relational structures with respect to embeddings. Ramsey classes of relational structures are one of the central topics in Ramsey theory, and we give a brief introduction.

A *relational signature* τ is a set of relation symbols R , each associated with an arity $ar(R) \geq 1$. A *relational structure* Γ with signature τ (short, a τ -structure) consists of a *domain* D_Γ and a relation $R^\Gamma \subseteq (D_\Gamma)^k$ for each relation symbol $R \in \tau$ of arity k .

Let Γ and Δ be two relational structures over the same signature τ . Then an *embedding* of Γ into Δ is an injective mapping f from D_Γ to D_Δ that satisfies that $(t_1, \dots, t_{ar(R)}) \in R^\Gamma$ if and only if $(f(t_1), \dots, f(t_{ar(R)})) \in R^\Delta$, for each relation symbol $R \in \tau$. A bijective embedding is called an *isomorphism*. All classes of structures in this paper will be closed under taking isomorphisms. If f is the identity mapping, then Γ is called an (*induced*) *substructure* of Δ .

A class of relational τ -structures \mathcal{C} is a *Ramsey class* (with respect to embeddings) if for all $P, H \in \mathcal{C}$ and every natural number k there exists a $T \in \mathcal{C}$ such that for every k -coloring of the substructures of T that are isomorphic to P there exists a substructure H' of T that is isomorphic to H such that all substructures of H' that are isomorphic to P have the same color; again, this is denoted by $T \rightarrow (H)_k^P$. Examples of Ramsey classes are

- the class of all finite structures over the empty signature (this is just the classical theorem of Ramsey);
- the class of all finite linear orders over the signature $\{<\}$ with a single binary relation symbol $<$ that defines the ordering (yet another form of the classical theorem of Ramsey);
- for any relational signature τ , the class of all ordered structures over τ (i.e., all finite structures over the signature $\tau \cup \{<\}$, where $<$ is a new binary relation symbol that defines the ordering) [NR89, AH78, NR83];
- the class of all finite posets that are equipped with a linear extension [NR84];
- the class of all finite metric spaces [Nes07].
- Canonically ordered Boolean Algebras [KPT05].

For more examples of Ramsey classes, see e.g. [Nes05, Nes95, KPT05]. It is known that all Ramsey classes \mathcal{C} of finite structures that are closed under taking induced substructures (those classes are sometimes also called *hereditary*) are *amalgamation classes* [Nes05], and hence there exists an (up to isomorphism unique) *homogeneous* countably infinite structure Γ such that \mathcal{C} is exactly the class of finite structures that embed into Γ (this follows from Fraïssé's theorem, see [Hod97]). Homogeneity is a very strong model-theoretic property, and it is therefore possible to use model-theoretic techniques and results to approach a classification of all Ramsey classes that are closed under taking induced substructures. This is the program that has basically been launched in [Nes05].

The result presented here gives rise to a new Ramsey class, and hence contributes to the classification program. We have to describe how to represent rooted binary plane trees as relational structures. Our relational structures will be ordered, i.e., the signature contains a binary relation symbol $<$ that is interpreted by a linear order. The tree structure is represented by a single ternary relation symbol as follows. For leaves a, b, c of a tree, we write $ab|c$ if the least common ancestor of a and b is below the least common ancestor of a and c in the tree; the relation $|$ is also called the *rooted triple relation*, following terminology in phylogenetic reconstruction [Ste92, NW96, HKW96]. It is known that a rooted binary tree is described up to isomorphism by the rooted triple relation (see e.g. [Ste92]).

We now associate to a rooted binary plane tree T the relational structure $\Gamma = (L(T); |, <)$ where a triple (a, b, c) of elements of Γ is in the rooted triple relation $|^\Gamma$ if T satisfies $ab|c$, and where a pair of elements (a, b) is in the

relation $<^\Gamma$ if the leaf a lies on the left of the leaf b with respect to the embedding of T into the half-plane. It is now clear that Theorem 1 implies that the class of relational structures \mathcal{T} over the signature $\{|\!, <\}$ obtained as described above from rooted binary plane trees is a Ramsey class.

1.2. Infinite Permutation Groups. As we have mentioned in the previous subsection, any Ramsey class that is closed under taking substructures (and our Ramsey class \mathcal{T} is obviously closed under taking substructures) is an amalgamation class, and therefore there exists a unique countable homogeneous structure $\Lambda = (\Omega; |\!, <)$ such that \mathcal{T} is exactly the class of all finite structures that embed into Λ . The structure $(\Omega; |)$ (i.e., the reduct of Λ that only contains the rooted triple relation without an ordering on the domain) is well-known to model-theorists and in the theory of infinite permutation groups, and also has many explicit constructions; see e.g. [Cam90, AN98]. Its automorphism group is oligomorphic, 2-transitive, and 3-set-transitive, but not 3-transitive. The rooted triple relation $|$ is a C -relation in the terminology of [AN98].

2. PROOF OF THE MAIN RESULT

We start with the easy special case where we color only the leaves of a tree (and thus the copies of trees of order 1); this will serve us as an induction basis in the proof of the main result. We denote by $T(c)$ the rooted binary tree with 2^c leaves of height c , i.e., any leaf is at distance c to the root. In particular, $T(0)$ denotes the one-vertex tree.

Proposition 2. *For all trees H and all $k \geq 2$ there exists a tree T such that $T \rightarrow (H)_k^{T(0)}$.*

Proof. Note that the k -colorings of the copies of $T(0)$ in a tree T are just colorings of the leaves of T ; we therefore just speak of k -colorings of T .

We apply the following operation to construct trees. Let G and H be trees. Then $G[H]$ denotes the tree obtained from G by replacing each leaf of G by a tree isomorphic to H (for each leaf v of G the children of the root of the copy of H in $G[H]$ are the children of v). It is clear that the resulting tree has an embedding into the half-plane so that $G[H]$ is again a rooted binary plane tree. We can iterate the construction: let $H^{(1)}$ be H , and define $H^{(i+1)}$ for $i \geq 1$ to be $H[H^{(i)}]$.

Clearly, $H[H] \rightarrow (P)_2^{T(0)}$, because for all 2-colorings of $H[H]$ either one of the ‘lower’ copies of H in $H[H]$ (i.e., one of the copies of H in $H[H]$ that replaced a leaf in H) is monochromatic, and we are done, or otherwise all these copies of H contain both colors. Let a_1, \dots, a_n be the leaves of H , and let c_i be a leaf in the copy of H that replaced a_i in $H[H]$ and that has color 0. Then the subtree of $H[H]$ with leaf set $\{c_1, \dots, c_n\}$ is a 0-chromatic copy of H .

Therefore

$$H[H^{(2)}] \rightarrow (H)_3^{T(0)}$$

because either one of the lower copies of $H^{(2)}$ in $H[H^{(2)}]$ is 2-chromatic, in which case we have already shown that this copy contains a monochromatic copy of H , or all copies contain all three colors. But then, by an analogous argument as above, we have a monochromatic copy of H in $H^{(3)}$. By iterating this argument it follows that $H^{(k)} \rightarrow (H)_k^{T(0)}$. \square

In the proof of Theorem 1, the following notation will be convenient. The set of all copies of P in H is denoted by $\binom{H}{P}$. If all copies of P in H have the same color, we say that H is χ -*monochromatic* (or simply *monochromatic* if the coloring is clear from the context). If the color is k , we also say that H is k -*chromatic*.

If T is a tree with more than one vertex, then the root of T has exactly two children; we denote the subtree T rooted at the left child by T_{\swarrow} , and the subtree of T rooted at the right child by T_{\searrow} (and we speak of the *left subtree of T* and the *right subtree of T* , respectively). Finally, suppose that H_1 and H_2 are disjoint subtrees of T . Then $\langle H_1, H_2 \rangle$ denotes the (uniquely defined) subtree of T with leaves $L(H_1) \cup L(H_2)$.

Proof of Theorem 1. We prove the theorem by induction on the size of P . For $P = T(0)$ the statement holds by Proposition 2. We now assume that the statement holds for all proper subtrees of P ; we want to prove it for P .

We start with the case $k = 2$, and proceed by induction on the size of H . Observe that trivially $P \rightarrow (P)_2^P$. So assume that the theorem holds for proper subtrees H' of H , i. e., we assume that there exists a tree $T_{H'}$ such that $T_{H'} \rightarrow (H')_2^{P}$. In particular, we assume that the theorem holds for H_{\swarrow} and H_{\searrow} , the left and right subtree of H . In the inductive step, we first prove the following claim.

Claim (Asymmetric step). *There exists a tree F such that for any 2-coloring $\chi : \binom{F}{P} \rightarrow \{0, 1\}$ of the copies of P in F*

- *there is a 0-chromatic copy of H_{\searrow} in F_{\searrow} or of H_{\swarrow} in F_{\swarrow} , or*
- *there exists a 1-chromatic copy of H in F .*

Proof of the asymmetric step. For a sufficiently large n (whose choice will be discussed at the end of the proof), let F be isomorphic to $T(n)$. Suppose that there is no 0-chromatic copy of H_{\swarrow} and no 0-chromatic copy of H_{\searrow} in F under the coloring χ . We show that there exists a 1-chromatic copy of H in F . Let $\psi : \binom{F_{\swarrow}}{P_{\swarrow}} \rightarrow 2^{\binom{F_{\searrow}}{P_{\searrow}}}$ be the mapping that assigns to each copy of P_{\swarrow} in F_{\swarrow} the coloring of $\binom{F_{\searrow}}{P_{\searrow}}$ induced in the following way: if P_1 is a copy of P_{\swarrow} in F_{\swarrow} , then $\psi(P_1)$ is the mapping that maps a copy P_2 of P_{\searrow} in F_{\searrow} to $\chi(\langle P_1, P_2 \rangle)$. Hence, we color the set $\binom{F_{\swarrow}}{P_{\swarrow}}$ where the colors are themselves 2-colorings of $\binom{F_{\searrow}}{P_{\searrow}}$.

By inductive hypothesis, and because F_{\swarrow} is large enough, there exists a ψ -monochromatic copy F_1 of $T(m)$ in F_{\swarrow} , where m is sufficiently large. Let ϕ be the color of the copies of P_{\swarrow} in F_1 ; recall that ϕ is a 2-coloring of $\binom{F_{\searrow}}{P_{\searrow}}$. Since F_{\searrow} is large enough, there is a copy F_2 of $T(m)$ that is ϕ -monochromatic, say all copies of P_{\searrow} in F_2 are *blue*. Note that if P_1 is a copy of P_{\swarrow} in F_1 , and P_2 is a copy of P_{\searrow} in F_2 , then $\langle P_1, P_2 \rangle$ is a copy of P in F that is colored *blue*.

Assume first that *blue* = 1. By inductive assumption, $F_1 \rightarrow (H_{\swarrow})_2^P$ and $F_2 \rightarrow (H_{\searrow})_2^P$. The color of the copies of P in the monochromatic copy H_1 of H_{\swarrow} in F_1 and in the monochromatic copy H_2 of H_{\searrow} in F_2 must be 1, or otherwise the first disjunct of the conclusion of the statement is fulfilled. Because $\langle H_1, H_2 \rangle$ is a copy of H in F , all copies of P in $\langle H_1, H_2 \rangle$ are colored by 1, and we are done in this case. So we can assume that *blue* = 0.

We now iterate this argument h -times as follows, where h is the height of H (the maximal distance from the root of H to one of its leaves). In the i -th step, we define disjoint subtrees $F_1^i, \dots, F_{2^i}^i$ of F . Initially, in the first step, let $F_1^1 := F_1$ and $F_2^1 := F_2$. In the $i + 1$ -st step, for $i \geq 1$ and $j \leq 2^i$, let ψ_j^i be the mapping that assigns to each copy of P_{\swarrow} in $(F_j^i)_{\swarrow}$ the coloring of $\binom{(F_j^i)_{\searrow}}{P_{\searrow}}$ induced by χ as before.

By inductive hypothesis, and because $(F_j^i)_{\swarrow}$ is large enough, there exists a ψ_j^i -monochromatic copy F_j^{i+1} of $T(n_i)$ in $(F_j^i)_{\swarrow}$, where n_i is sufficiently large. Let ϕ_j^i be the color of the copies of P_{\swarrow} in F_j^{i+1} . Since $(F_j^i)_{\searrow}$ is large enough, there is a copy $F_{j+2^i}^{i+1}$ of $T(n_i)$ that is ϕ_j^i -monochromatic. We can argue as before to conclude that if P_1 is a copy of P_{\swarrow} in F_j^{i+1} and P_2 is a copy of P_{\searrow} in $F_{j+2^i}^{i+1}$, then $\langle P_1, P_2 \rangle$ is a copy of P in F that has color 0. Finally, we select one leaf in each of the trees $F_1^h, \dots, F_{2^h}^h$. These vertices show that there is a copy of $T(h)$ and hence also a copy of H in F where every copy of P is colored by 0.

We can certainly find appropriate (large) values for n, m , and n_i , for $i \geq 1$, since we can choose $n_h = 1$, and for $i < h$ we can choose n_i large enough depending on the size of n_{i+1} , so that we can finally also choose an appropriate value for m and for n . \square

To conclude the inductive proof of Theorem 1, let T be a copy of $T(d)$ where d is large enough (again we discuss the choice of d at the end of the proof). We will show that for any $\chi : \binom{T}{P} \rightarrow \{0, 1\}$ there exists a monochromatic copy of H . Let $\psi : \binom{T_{\swarrow}}{P_{\swarrow}} \rightarrow 2^{\binom{T_{\searrow}}{P_{\searrow}}}$ be the function that assigns to a copy P_1 of P_{\swarrow} in T the function that maps a copy P_2 of P_{\searrow} in T_{\searrow} to $\chi(\langle P_1, P_2 \rangle)$. By our inductive hypothesis, and since T_{\swarrow} is large enough, we find a ψ -monochromatic copy T_1 of the tree F given by the asymmetric step. This gives us a 2-coloring ϕ of $\binom{T_{\searrow}}{P_{\searrow}}$. Since T_{\searrow} is large enough, we find

a ϕ -monochromatic copy T_2 of the tree F from the asymmetric step; let us assume that all copies of P in T_2 are colored by 1. Note that if P_1 is a copy of P_{\swarrow} in T_1 and P_2 is a copy of P_{\searrow} in T_2 , then $\langle P_1, P_2 \rangle$ is a subtree of T that is colored by 1 under χ .

We apply the asymmetric step to T_1 and T_2 and the restriction of χ to T_1 and T_2 , respectively. If there is a 1-chromatic copy of H in T_1 or in T_2 , we are done. So we may assume that the colorings of $\binom{T_1}{P}$ and $\binom{T_2}{P}$ are such that there is a 0-chromatic copy H_1 of H_{\swarrow} in T_1 and a 0-chromatic copy H_2 of H_{\searrow} in T_2 . Then $\langle H_1, H_2 \rangle$ is a copy of H in T and 1-chromatic with respect to χ .

We have proved that there exists a tree T such that $T \rightarrow (H)_2^P$, and now prove the theorem for any finite number of colors k . Let l be $\lceil \log_2 k \rceil$. Define T^0 to be H , and let T^i for $1 \leq i \leq l$ be such that $T^i \rightarrow (T^{i-1})_2^P$; we already know that such a tree T^i exists. We claim that $T^l \rightarrow (H)_k^P$. Let $\chi : \binom{T^l}{P} \rightarrow \{0, \dots, k-1\}$, and consider for $1 \leq i \leq l$ the colorings $\psi_i : \binom{T^i}{P} \rightarrow \{0, 1\}$ where ψ_i colors a copy P' of P by 1 if the i -th bit in the binary representation of $\chi(P')$ is 1, and it colors P' by 0 otherwise. For $1 \leq i \leq \lceil \log_2 k \rceil$, let S_{i-1} be the ψ_i -monochromatic copy of T^{i-1} in S_i , and let b_i be the color of the copies of P in S_{i-1} . Note that S_0 is isomorphic to H . All copies of P in S_0 have color b_i with respect to ψ_i , for all $0 \leq i \leq \lceil \log_2 k \rceil$, and hence they have the same color with respect to χ . \square

Remark. With minor modifications of the proof a similar result can be shown for the class of trees with respect to embeddings where leaves are not necessarily mapped to leaves.

REFERENCES

- [AH78] Fred G. Abramson and Leo Harrington. Models without indiscernibles. *J. Symb. Log.*, 43(3):572–600, 1978.
- [AN98] S.A. Adeleke and Peter M. Neumann. Relations related to betweenness: their structure and automorphisms. *AMS Memoir*, 131(623), 1998.
- [Cam90] Peter J. Cameron. *Oligomorphic Permutation Groups*. Cambridge Univ. Press, Cambridge, 1990.
- [Deu75] W. Deuber. A generalization of ramsey’s theorem for regular trees. *Journal of Combinatorial Theory, ser. B*, 18:18–23, 1975.
- [HKW96] Monika Henzinger, Valerie King, and Tandy Warnow. Constructing a tree from homeomorphic subtrees, with applications to computational evolutionary biology. In *Proceedings of the 7th Symposium on Discrete Algorithms (SODA’96)*, pages 333–340, 1996.
- [Hod97] Wilfrid Hodges. *A shorter model theory*. Cambridge University Press, Cambridge, 1997.
- [KPT05] Kechris, Pestov, and Todorcevic. Fraissé limits, ramsey theory, and topological dynamics of automorphism groups. *Geometric and Functional Analysis*, 15(1):106–189, 2005.
- [Mil79] Keith R. Milliken. A ramsey theorem for trees. *J. Comb. Theory, Ser. A*, 26(3):215–237, 1979.
- [Nes95] Jaroslav Nešetřil. Ramsey theory. *Handbook of Combinatorics*, page 13311403, 1995.

- [Nes05] Jaroslav Nešetřil. Ramsey classes and homogeneous structures. *Combinatorics, Probability & Computing*, 14(1-2):171–189, 2005.
- [Nes07] Jaroslav Nešetřil. Metric spaces are ramsey. *Eur. J. Comb.*, 28(1):457–468, 2007.
- [NR83] Jaroslav Nešetřil and Vojtech Rödl. Ramsey classes of set systems. *J. Comb. Theory, Ser. A*, 34(2):183–201, 1983.
- [NR84] Jaroslav Nešetřil and Vojtech Rödl. Combinatorial partitions of finite posets and lattices. *Algebra Universalis*, 19:106–119, 1984.
- [NR89] Jaroslav Nešetřil and Vojtech Rödl. The partite construction and Ramsey set systems. *Discrete Mathematics*, 75(1-3):327–334, 1989.
- [NW96] Meei Pyng Ng and Nicholas C. Wormald. Reconstruction of rooted trees from subtrees. *Discrete Applied Mathematics*, 69(1-2):19–31, 1996.
- [Ste92] Michael Steel. The complexity of reconstructing trees from qualitative charaters and subtrees. *Journal of Classification*, 9:91–116, 1992.

We apologize for the improper spelling of the name of Jaroslav Nešetřil in three of the references; the latex system does not allow for the symbol š in the automatically generated short-cuts for the citations.

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