Incremental Learning of TBoxes from Interpretation Sequences with Methods of Formal Concept Analysis

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Abstract. Formal Concept Analysis and its methods for computing minimal implicational bases have been successfully applied to axiomatise minimal $\mathcal{EL}$-TBoxes from models, so called bases of GCIs. However, no technique for an adjustment of an existing $\mathcal{EL}$-TBox w.r.t. a new model is available, i.e., on a model change the complete TBox has to be recomputed. This document proposes a method for the computation of a minimal extension of a TBox w.r.t. a new model. The method is then utilised to formulate an incremental learning algorithm that requires a stream of interpretations, and an expert to guide the learning process, respectively, as input.

Keywords: description logics, formal concept analysis, base of GCIs, implicational base, TBox extension, incremental learning

1 Introduction

More and more data is generated and stored thanks to the ongoing technical development of computers in terms of processing speed and storage space. There is a vast number of databases, some of them freely available on the internet (e.g., `dbpedia.org` and `wikidata.org`), that are used in research and industry to store assertional knowledge, i.e., knowledge on certain objects and individuals. Examples are databases of online stores that besides contact data also store purchases and orders of their customers, or databases which contain results from experiments in biology, physics, psychology etc. Due to the large size of these databases it is difficult to quickly derive conclusions from the data, especially when only *terminological knowledge* is of interest, i.e., knowledge that does not reference certain objects or individuals but holds for all objects or individuals in the dataset.

So far there have been several successful approaches for the combination of description logics and formal concept analysis as follows. In [5, 6, 28] Baader, Ganter, and Sertkaya have developed a method for the completion of ontologies by means of the exploration algorithm for formal contexts. Rudolph has invented an exploration algorithm for $\mathcal{FLE}$-interpretations in [26, 27]. Furthermore, Baader and Distel presented in [3, 4, 12] a technique for the computation of bases of concept inclusions for finite interpretations in $\mathcal{EL}$ that has been extended with error-tolerance by Borchmann in [7, 8]. Unfortunately, none of these methods and algorithms provide the possibility of extension or adaption of an already existing ontology (or TBox). Hence, whenever a new dataset (in form of an interpretation or description graph) is observed then the whole base has to be recomputed completely which can be a costly operation, and moreover the changes are not explicitely shown to the user. In this document we propose an extension of the method of Baader and Distel in [3, 4, 12] that allows for the construction of a minimal extension of a TBox w.r.t. a model. The technique is then utilised to introduce an incremental learning algorithm that requires a stream of interpretations as input, and uses an expert to guide to exploration process.

In Sections 2 and 3 we introduce the neccessary notions from description logics, and formal concept analysis, respectively. Section 4 presents the results on bases of GCIs for interpretations relative to a TBox. Section 5 defines experts and adjustments that are neccessary to guide the incremental learning algorithm that is shown in Section 6. Finally, Section 7 compares the incremental learning approach with the existing single-step learning approach.

2 The Description Logic $\mathcal{EL}^{\bot}$

At first we introduce the light-weight description logic $\mathcal{EL}^{\bot}$. Let (N_C, N_R) be an arbitrary but fixed signature, i.e., N_C is a set of *concept names* and N_R is a set of *role names*. Every concept name $A \in N_C$, the *top concept* $\top$, and the *bottom concept* $\bot$ are $\mathcal{EL}^{\bot}$-concept descriptions. When C and D are $\mathcal{EL}^{\bot}$-concept descriptions and $r \in N_R$ is a role name then also the *conjunction* $C \sqcap D$ and the *existential restriction* $\exists\, r.\, C$ are $\mathcal{EL}^{\bot}$-concept descriptions. We denote the set of all $\mathcal{EL}^{\bot}$-concept descriptions over (N_C, N_R) by $\mathcal{EL}^{\bot}(N_C, N_R)$.

The semantics of $\mathcal{EL}^{\bot}$ are defined by means of interpretations. An *interpretation* $\mathcal{I}$ over (N_C, N_R) is a pair $(\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ consisting of a set $\Delta^{\mathcal{I}}$, called *domain*, and an *extension function* $\cdot^{\mathcal{I}}\colon N_C \cup N_R \to 2^{\Delta^{\mathcal{I}}} \cup 2^{\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}}$ that maps concept names $A \in N_C$ to subsets $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$ and role names $r \in N_R$ to binary relations $r^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$. Furthermore, the extension function is then canonically extended to all $\mathcal{EL}^{\bot}$-concept descriptions:

$$
\begin{aligned}
(C \sqcap D)^{\mathcal{I}} &:= C^{\mathcal{I}} \cap D^{\mathcal{I}} \\
(\exists\, r.\, C)^{\mathcal{I}} &:= \left\{ d \in \Delta^{\mathcal{I}} \,\middle|\, \exists e \in \Delta^{\mathcal{I}}\colon (d, e) \in r^{\mathcal{I}} \wedge e \in C^{\mathcal{I}} \right\}
\end{aligned}
$$

$\mathcal{EL}^{\bot}$ allows to express terminological knowledge with so called *concept inclusions*. A *general concept inclusion* (abbr. *GCI*) in $\mathcal{EL}^{\bot}$ over (N_C, N_R) is of the form $C \sqsubseteq D$ where C and D are $\mathcal{EL}^{\bot}$-concept descriptions over (N_C, N_R). An $\mathcal{EL}^{\bot}$-*TBox* $\mathcal{T}$ is a set of $\mathcal{EL}^{\bot}$-GCIs. An interpretation $\mathcal{I}$ is a *model* of an $\mathcal{EL}^{\bot}$-GCI $C \sqsubseteq D$, denoted as $\mathcal{I} \models C \sqsubseteq D$, if the set inclusion $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ holds; and $\mathcal{I}$ is a *model* of an $\mathcal{EL}^{\bot}$-TBox $\mathcal{T}$, symbolized as $\mathcal{I} \models \mathcal{T}$, if it is a model of all its $\mathcal{EL}^{\bot}$-concept inclusions. An $\mathcal{EL}^{\bot}$-GCI $C \sqsubseteq D$ *follows* from an $\mathcal{EL}^{\bot}$-TBox $\mathcal{T}$, denoted as $\mathcal{T} \models C \sqsubseteq D$, if every model of $\mathcal{T}$ is also a model of $C \sqsubseteq D$.

3 Formal Concept Analysis

This section gives a brief overview on the basic definitions of formal concept analysis and the neccessary lemmata and theorems cited in this paper.

The basic structure of formal concept analysis is a *formal context* $\mathbb{K} = (G, M, I)$ that consists of a set G of *objects*, a set M of *attributes*, and an *incidence relation* $I \subseteq G \times M$. Instead of $(g, m) \in I$ we rather use the infix notation $g\ I\ m$, and say that g *has* m. For brevity we may sometimes drop the adjective "formal". From each formal context $\mathbb{K}$ two so-called *derivation operators* arise: For subsets $A \subseteq G$ we define A^I as a subset of M that contains all attributes which the objects in A have in common, i.e.,

$$A^I := \{m \in M \mid \forall g \in A \colon g\ I\ m\}.$$

Dually for $B \subseteq M$ we define B^I as the set of all objects that have all attributes in B, i.e., $B^I := \{g \in G \mid \forall m \in B \colon g\ I\ m\}$. An *intent* is a subset $B \subseteq M$ such that $B = B^{II}$ holds.

A *formal implication* over M is of the form $X \to Y$ where $X, Y \subseteq M$. It *holds* in the context (G, M, I) if all objects having all attributes from X also have all attributes from Y, i.e., iff $X^I \subseteq Y^I$ is satisfied. An *implication set* over M is a set of implications over M, and it *holds* in a context $\mathbb{K}$ if all its implications hold in $\mathbb{K}$. An implication $X \to Y$ *follows* from an implication set $\mathcal{L}$ if $X \to Y$ holds in all contexts in which $\mathcal{L}$ holds, or equivalently iff $Y \subseteq X^{\mathcal{L}}$ where $X^{\mathcal{L}}$ is defined as the least superset of X that satisfies the implication $A \subseteq X^{\mathcal{L}} \Rightarrow B \subseteq X^{\mathcal{L}}$ for all implications $A \to B \in \mathcal{L}$.

Stumme has extended the notion of implicational bases as defined by Duquenne and Guigues in [20] and by Ganter in [13] towards background knowledge in form of an implication set. We therefore skip the original definitions and theorems and just cite those from Stumme in [29]. If $\mathcal{S}$ is an implication set holding in a context $\mathbb{K}$, then an *implicational base of* $\mathbb{K}$ *relative to* $\mathcal{S}$ is defined as an implication set $\mathcal{L}$, such that $\mathcal{L}$ holds in $\mathbb{K}$, and furthermore each implication that holds in $\mathbb{K}$ follows from $\mathcal{S} \cup \mathcal{L}$.

A set $P \subseteq M$ is called a *pseudo-intent of* $\mathbb{K}$ *relative to* $\mathcal{S}$ if $P = P^{\mathcal{S}}$, $P \neq P^{II}$, and for each pseudo-intent $Q \subsetneq P$ of $\mathbb{K}$ relative to $\mathcal{S}$ it holds that $Q^{II} \subseteq P$. Then the following set is the *canonical implicational base of* $\mathbb{K}$ *relative to* $\mathcal{S}$:

$$\left\{P \to P^{II} \;\middle|\; P \text{ is a pseudo-intent of } \mathbb{K} \text{ relative to } \mathcal{S}\right\}.$$

4 Relative Bases of GCIs w.r.t. Background TBox

In this section we extend the definition of a base of GCIs for interpretations as introduced by Baader and Distel in [3, 4, 12] towards the possibility to handle background knowledge in form of a TBox. Therefore, we simply assume that there is already a set of GCIs that holds in an interpretation, and are just interested in a minimal extension of the TBox such that the union of the TBox and this *relative base* indeed entails all GCIs which hold in the interpretation.

In the following text we always assume that $\mathcal{I}$ is an interpretation, and $\mathcal{T}$ is a TBox that has $\mathcal{I}$ as a model, and both are defined over the signature (N_C, N_R).

Definition 1 (Relative Base w.r.t. Background TBox). *An* $\mathcal{EL}^{\perp}$-base for $\mathcal{I}$ relative to $\mathcal{T}$ *is defined as an* $\mathcal{EL}^{\perp}$*-TBox* $\mathcal{B}$ *that fulfills the following conditions:*

(sound) *All GCIs in $\mathcal{T} \cup \mathcal{B}$ hold in $\mathcal{I}$, i.e., $\mathcal{I} \models \mathcal{T} \cup \mathcal{B}$.*
(complete) *All GCIs that hold in $\mathcal{I}$ also follow from $\mathcal{T} \cup \mathcal{B}$, i.e., $\mathcal{I} \models C \sqsubseteq D$ implies $\mathcal{T} \cup \mathcal{B} \models C \sqsubseteq D$.*

Furthermore, we call $\mathcal{B}$ irredundant, *if none of its concept inclusions follows from the others, i.e., if $(\mathcal{T} \cup \mathcal{B}) \setminus \{C \sqsubseteq D\} \not\models C \sqsubseteq D$ holds for all $C \sqsubseteq D \in \mathcal{B}$; and* minimal, *if it has minimal cardinality among all $\mathcal{EL}^{\perp}$-bases for $\mathcal{I}$ relative to $\mathcal{T}$. Of course all minimal bases are irredundant but not vice versa.*

The previous definition is a straightforward generalization of bases of GCIs for interpretations since in case of an empty TBox $\mathcal{T} = \emptyset$ both definitions coincide.

The term of a model-based most-specific concept description has been introduced by Baader and Distel in [3, 4, 12]. The next definition extends their notion to model-based most-specific concept descriptions relative to a TBox.

Definition 2 (Relative model-based most-specific concept description). *An $\mathcal{EL}^{\perp}$-concept description C over (N_C, N_R) is called* relative model-based most-specific concept description of $X \subseteq \Delta^{\mathcal{I}}$ w.r.t. $\mathcal{I}$ and $\mathcal{T}$ *if the following conditions are satisfied:*

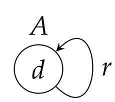
1. *$X \subseteq C^{\mathcal{I}}$.*
2. *If $X \subseteq D^{\mathcal{I}}$ holds for a concept description $D \in \mathcal{EL}^{\perp}(N_C, N_R)$ then $\mathcal{T} \models C \sqsubseteq D$.*

The definition implies that all relative model-based most-specific concept descriptions of a subset $X \subseteq \Delta^{\mathcal{I}}$ are equivalent w.r.t. the TBox $\mathcal{T}$. Hence, we use the symbol $X^{\mathcal{I}_{\mathcal{T}}}$ for *the* relative mmsc of X w.r.t. $\mathcal{I}$ and $\mathcal{T}$.

However, as an immediate consequence from the definition it follows that the model-based most-specific concept description of $X \subseteq \Delta^{\mathcal{I}}$ w.r.t. $\mathcal{I}$ is always a relative model-based most-specific concept description of X w.r.t. $\mathcal{I}$ and $\mathcal{T}$.

There are situations where the relative mmsc exists but not the standard mmsc. Consider the interpretation $\mathcal{I}$ described by the following graph:

$\mathcal{I}$:



Then d has no model-based most specific concept in $\mathcal{EL}^{\perp}$ but has a relative mmsc w.r.t. $\mathcal{T} := \{A \sqsubseteq \exists r.\, A\}$, in particular it holds $d^{\mathcal{I}_{\mathcal{T}}} = A$. Of course, d has a role-depth bounded mmsc, and a mmsc in $\mathcal{EL}^{\perp}_{\mathsf{gfp}}$ with greatest fixpoint semantics, respectively.

For the following statements on the construction of relative bases of GCIs we strongly need the fact that all model-based most-specific concept descriptions exist. If computability is neccessary, too, then we further have to enforce that there are only finitely many model-based most-specific concept descriptions (up to equivalence) and that the interpretation only contains finitely many individuals; of course the second requirement implies the first.

The model-based most-specific concept description of every individual x w.r.t. $\mathcal{I}$ clearly exists if the interpretation $\mathcal{I}$ is finite and acyclic. Relative model-based most-specific concept descriptions exist if we can find suitable synchronised simulations on a description graph constructed from the interpretation and the TBox. A detailed characterisation and appropriate proofs will be subject of a future paper.

In case we cannot ensure the existence of mmscs for all individuals of the interpretation we may also adopt role-depth bounds. Further details are given in [10]. Then we modify the definition of a relative base of GCIs to only involve GCIs whose subsumee and subsumer satisfy the role-depth bound. This is both applied to the GCIs in the base and the GCIs that must be entailed. As a consequence we are able to treat cyclic interpretations whose cycles are not already modeled in the background TBox.

As in the default case without a TBox, the definition of relative model-based most-specific concept descriptions yields a quasi-adjunction between the powerset lattice $(2^{\Delta^{\mathcal{I}}}, \subseteq)$ and the quasiordered set $(\mathcal{EL}^{\bot}(N_C, N_R), \sqsubseteq_{\mathcal{T}})$.

Lemma 1 (Properties of Relative mmscs). *For all subsets $X, Y \subseteq \Delta^{\mathcal{I}}$ and all concept descriptions $C, D \in \mathcal{EL}^{\bot}(N_C, N_R)$ the following statements hold:*

1. $X \subseteq C^{\mathcal{I}}$ *if and only if* $\mathcal{T} \models X^{\mathcal{I}_{\mathcal{T}}} \sqsubseteq C$
2. $X \subseteq Y$ *implies* $\mathcal{T} \models X^{\mathcal{I}_{\mathcal{T}}} \sqsubseteq Y^{\mathcal{I}_{\mathcal{T}}}$
3. $\mathcal{T} \models C \sqsubseteq D$ *implies* $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$
4. $X \subseteq X^{\mathcal{I}_{\mathcal{T}}\mathcal{I}}$
5. $\mathcal{T} \models C^{\mathcal{I}\mathcal{I}_{\mathcal{T}}} \sqsubseteq C$
6. $\mathcal{T} \models X^{\mathcal{I}_{\mathcal{T}}} \equiv X^{\mathcal{I}_{\mathcal{T}}\mathcal{I}\mathcal{I}_{\mathcal{T}}}$
7. $C^{\mathcal{I}} = C^{\mathcal{I}\mathcal{I}_{\mathcal{T}}\mathcal{I}}$

In order to obtain a first relative base of GCIs for $\mathcal{I}$ w.r.t. $\mathcal{T}$ we can prove that it suffices to have mmscs as the right-hand-sides of concept inclusions in a relative base. More specifically, it holds that the set

$$\left\{ C \sqsubseteq C^{\mathcal{I}\mathcal{I}_{\mathcal{T}}} \;\middle|\; C \in \mathcal{EL}^{\bot}(N_C, N_R) \right\}$$

is a relative of GCIs for $\mathcal{I}$ w.r.t. $\mathcal{T}$. This statement is a simple consequence of the fact that a GCI $C \sqsubseteq D$ only holds in $\mathcal{I}$ if it follows from $\mathcal{T} \cup \{ C \sqsubseteq C^{\mathcal{I}\mathcal{I}_{\mathcal{T}}} \}$.

In the following text we want to make a strong connection to formal concept analysis in a similar way as Baader and Distel did in [3, 4, 12]. We therefore define a set $\mathcal{M}_{\mathcal{I},\mathcal{T}}$ of $\mathcal{EL}^{\bot}$-concept descriptions such that all relative model-based most-specific concept descriptions can be expressed as a conjunction over a subset of $\mathcal{M}_{\mathcal{I},\mathcal{T}}$. We use similar techniques like lower approximations and induced contexts but in an extended way to be explicitly able to handle background knowledge in a TBox.

For an $\mathcal{EL}$-concept description in its normal form $C \equiv \bigsqcap_{A \in U} A \sqcap \bigsqcap_{(r,D) \in \Pi} \exists r. D$ we define its *lower approximation* w.r.t. $\mathcal{I}$ and $\mathcal{T}$ as the $\mathcal{EL}$-concept description

$$\lfloor C \rfloor_{\mathcal{I},\mathcal{T}} \coloneqq \bigsqcap_{A \in U} A \sqcap \bigsqcap_{(r,D) \in \Pi} \exists r. D^{\mathcal{I}\mathcal{I}_{\mathcal{T}}}.$$

As a consequence of the definition we get that $\mathcal{T}$ entails the concept inclusion $\lfloor C \rfloor_{\mathcal{I},\mathcal{T}} \sqsubseteq C$. The explicit proof uses Lemma 1 5 and the fact that both conjunction and existential restrictions are monotonic. This also justifies the name of a lower approximation of C.

Furthermore, it is readily verified that all lower approximations can be expressed in terms of the set $\mathcal{M}_{\mathcal{I},\mathcal{T}}$ which is defined as follows:

$$\mathcal{M}_{\mathcal{I},\mathcal{T}} \coloneqq \{\bot\} \cup N_C \cup \left\{ \exists r. X^{\mathcal{I}_{\mathcal{T}}} \;\middle|\; r \in N_r, \emptyset \neq X \subseteq \Delta^{\mathcal{I}} \right\}$$

In order to prove that also each model-based most-specific concept description is expressible in terms of $\mathcal{M}_{\mathcal{I},\mathcal{T}}$ it suffices to show that every model-based most-specific concept description is equivalent to its lower approximation. We already know from Lemma 1 5 that $\mathcal{T}$ entails the concept inclusion $C^{\mathcal{I}\mathcal{I}_\mathcal{T}} \sqsubseteq C$. Furthermore, for all concept descriptions $C, D \in \mathcal{EL}^\perp(N_C, N_R)$ and all role names $r \in N_R$ it may be easily shown by means of Lemma 1 7 that the following two statements hold:

1. $(C \sqcap D)^\mathcal{I} = (C \sqcap D^{\mathcal{I}\mathcal{I}_\mathcal{T}})^\mathcal{I}$.
2. $(\exists\, r.\, C)^\mathcal{I} = (\exists\, r.\, C^{\mathcal{I}\mathcal{I}_\mathcal{T}})^\mathcal{I}$.

As a consequence it follows that both the mmsc $C^{\mathcal{I}\mathcal{I}_\mathcal{T}}$ and the lower approximation $\lfloor C \rfloor_{\mathcal{I},\mathcal{T}}$ have the same extensions w.r.t. $\mathcal{I}$, and then Lemma 1 1 yields that $\mathcal{T}$ entails $C^{\mathcal{I}\mathcal{I}_\mathcal{T}} \sqsubseteq \lfloor C \rfloor_{\mathcal{I},\mathcal{T}}$. In summary, it follows that $\mathcal{T}$ entails the concept inclusions

$$C^{\mathcal{I}\mathcal{I}_\mathcal{T}} \equiv C^{\mathcal{I}\mathcal{I}_\mathcal{T}\mathcal{I}\mathcal{I}_\mathcal{T}} \sqsubseteq \lfloor C^{\mathcal{I}\mathcal{I}_\mathcal{T}} \rfloor_{\mathcal{I},\mathcal{T}} \sqsubseteq C^{\mathcal{I}\mathcal{I}_\mathcal{T}},$$

and hence each relative model-based most-specific concept description is equivalent to its lower approximation w.r.t. $\mathcal{T}$, and thus is expressible in terms of $\mathcal{M}_{\mathcal{I},\mathcal{T}}$.

Now we are ready to use methods of formal concept analysis to construct a minimal base of GCIs for $\mathcal{I}$ w.r.t. $\mathcal{T}$. We therefore first introduce the *induced context* w.r.t. $\mathcal{I}$ and $\mathcal{T}$ which is defined as $\mathbb{K}_{\mathcal{I},\mathcal{T}} := \left(\Delta^\mathcal{I}, \mathcal{M}_{\mathcal{I},\mathcal{T}}, I\right)$ where $(d, C) \in I$ if and only if $d \in C^\mathcal{I}$ holds. Additionally, the *background knowledge* is defined as the implication set

$$\mathcal{S}_{\mathcal{I},\mathcal{T}} := \{\{C\} \to \{D\} \mid C, D \in \mathcal{M}_{\mathcal{I},\mathcal{T}} \text{ and } \mathcal{T} \models C \sqsubseteq D\}\,.$$

One the one hand we need this background implications to skip the computation of trivial GCIs $C \sqsubseteq D$ where the implication $\{C\} \to \{D\}$ is not neccessarily trivial in $\mathbb{K}_{\mathcal{I},\mathcal{T}}$, and on the other hand it is needed to avoid the generation of GCIs that are already entailed by $\mathcal{T}$.

As an immediate consequence of the definition it follows that $(\bigsqcap U)^\mathcal{I} = U^I$ holds for all subsets $U \subseteq \mathcal{M}_{\mathcal{I},\mathcal{T}}$, and hence we infer that for all subsets $U, V \subseteq \mathcal{M}_{\mathcal{I},\mathcal{T}}$ the GCI $\bigsqcap U \sqsubseteq \bigsqcap V$ holds in $\mathcal{I}$ if and only if the implication $U \to V$ holds in the induced context $\mathbb{K}_{\mathcal{I},\mathcal{T}}$. Furthermore, it is true that conjunctions of intents of $\mathbb{K}_{\mathcal{I},\mathcal{T}}$ are exactly the mmscs w.r.t. $\mathcal{I}$ and $\mathcal{T}$, i.e., $\mathcal{T} \models \bigsqcap U^{II} \equiv (\bigsqcap U)^{\mathcal{I}\mathcal{I}_\mathcal{T}}$ holds for all subsets $U \subseteq \mathcal{M}_{\mathcal{I},\mathcal{T}}$. Eventually, the previous statements allow for the transformation of a minimal implicational base of the induced context $\mathbb{K}_{\mathcal{I},\mathcal{T}}$ w.r.t. the background knowledge $\mathcal{S}_{\mathcal{I},\mathcal{T}}$ into a minimal base of GCIs for $\mathcal{I}$ relative to the background TBox $\mathcal{T}$.

Theorem 1 (Minimal Relative Base of GCIs). *Assume that all model-based most-specific concept descriptions of $\mathcal{I}$ relative to $\mathcal{T}$ exist. Let $\mathcal{L}$ be a minimal implicational base of the induced context $\mathbb{K}_{\mathcal{I},\mathcal{T}}$ w.r.t. the background knowledge $\mathcal{S}_{\mathcal{I},\mathcal{T}}$. Then $\left\{\bigsqcap U \sqsubseteq \bigsqcap U^{II} \,\middle|\, U \to U^{II} \in \mathcal{L}\right\}$ is a minimal base of GCIs for $\mathcal{I}$ relative to $\mathcal{T}$.*

Eventually, the following set is the (minimal) *canonical base for $\mathcal{I}$ relative to $\mathcal{T}$*:

$$\mathcal{B}_{\mathcal{I},\mathcal{T}} := \left\{\bigsqcap P \sqsubseteq \bigsqcap P^{II} \,\middle|\, P \text{ is a pseudo-intent of } \mathbb{K}_{\mathcal{I},\mathcal{T}} \text{ relative to } \mathcal{S}_{\mathcal{I},\mathcal{T}}\right\}.$$

All of the results presented in this section are generalisations of those from Baader and Distel in [3, 4, 12], and for the special case of an empty background TBox $\mathcal{T} = \emptyset$ the definitions and propositions coincide. In particular, the last Theorem 1 constructs the same base of GCIs as [12, Theorem 5.12, Corollary 5.13] for $\mathcal{T} = \emptyset$.

5 Experts in the Domain of Interest

We have seen how to extend an existing TBox $\mathcal{T}$ with concept inclusions holding in an interpretation $\mathcal{I}$ that is a model of $\mathcal{T}$. However, there might be situations where we want to adjust a TBox $\mathcal{T}$ with information from an interpretation $\mathcal{I}$ that is not a model of $\mathcal{T}$. In order to use the results from the previous section on relative bases it is neccessary to adjust the interpretation or the TBox such that as much knowledge as possible is preserved and the adjusted interpretation models the adjusted TBox. However, an automatic approach can hardly decide whether counterexamples in the interpretation are valid in the domain of interest, or whether concept inclusions hold in the domain of interest. We therefore need some external information to decide whether a concept inclusion should be considered true or false in the domain of interest.

Beforehand, we define the notion of adjustments as follows.

Definition 3 (Adjustment). *Let $\mathcal{I}$ be an interpretation that does not model the GCI $C \sqsubseteq D$.*

1. *An interpretation $\mathcal{J}$ is called an* adjustment *of $\mathcal{I}$ w.r.t. $C \sqsubseteq D$ if it satisfies the following conditions:*
   (a) $\mathcal{J} \models C \sqsubseteq D$.
   (b) $\Delta^{\mathcal{I}} \setminus X \subseteq \Delta^{\mathcal{J}}$.
   (c) $A^{\mathcal{I}} \setminus X \subseteq A^{\mathcal{J}}$ *holds for all concept names* $A \in N_C$.
   (d) $r^{\mathcal{I}} \setminus (\Delta^{\mathcal{I}} \times X \cup X \times \Delta^{\mathcal{I}}) \subseteq r^{\mathcal{J}}$ *holds for all role names* $r \in N_R$.
   The set $X := C^{\mathcal{I}} \setminus D^{\mathcal{I}}$ denotes the set of all counterexamples in $\mathcal{I}$ against $C \sqsubseteq D$.
   We call an adjustment $\mathcal{J}$ minimal *if the value $\sum_{A \in N_C} \left|A^{\mathcal{I}} \triangle A^{\mathcal{J}}\right| + \sum_{r \in N_R} \left|r^{\mathcal{I}} \triangle r^{\mathcal{J}}\right|$ is minimal among all adjustments of $\mathcal{I}$ w.r.t. $C \sqsubseteq D$.*
2. *A general concept inclusion $E \sqsubseteq F$ is called an* adjustment *of $C \sqsubseteq D$ w.r.t. $\mathcal{I}$ if it satisfies the following conditions:*
   (a) $\mathcal{I} \models E \sqsubseteq F$.
   (b) $E \sqsubseteq C$.
   (c) $D \sqsubseteq F$.
   An adjustment $E \sqsubseteq F$ is called minimal *if there is no adjustment $X \sqsubseteq Y$ such that $E \sqsubseteq X$ and $Y \sqsubseteq F$ holds.*

As next step we introduce the definition of an expert that is used to guide the incremental exploration process, i.e., it ensures that the new interpretation is always adjusted such that it models the adjusted TBox.

Definition 4 (Expert). *An* expert *is a mapping from pairs of interpretations $\mathcal{I}$ and GCIs $C \sqsubseteq D$ where $\mathcal{I} \not\models C \sqsubseteq D$ to adjustments. We say that the expert* accepts $C \sqsubseteq D$ *if it adjusts the interpretation, and that it* declines $C \sqsubseteq D$ *if it adjusts the GCI.*

Furthermore, the following requirements must be satisfied:

1. *Acceptance must be independent of $\mathcal{I}$, i.e., if χ accepts $C \sqsubseteq D$ w.r.t. $\mathcal{I}$ then χ must also accept $C \sqsubseteq D$ w.r.t. any other interpretation $\mathcal{J}$.*
2. *Adjusted interpretations must model previously accepted GCIs and must not model previously declined GCIs.*
3. *Adjustments of declined GCIs must be accepted.*

An expert is called minimal *if it always returns minimal adjustments.*

5.1 Examples for Experts

Of course, we may use a *human expert* who is aware of the full knowledge holding in the domain of interest. However, the problem of the construction of *automatically acting experts* is left for future research. We will only present some first ideas.

An expert may be defined by means of the confidence measure that has been introduced by Borchmann in [7, 8]. For a GCI $C \sqsubseteq D$ and an interpretation $\mathcal{I}$ it is defined by

$$\mathsf{conf}_{\mathcal{I}}(C \sqsubseteq D) := \frac{\left|(C \sqcap D)^{\mathcal{I}}\right|}{\left|C^{\mathcal{I}}\right|} \in [0,1].$$

Note that $\mathsf{conf}_{\mathcal{I}}(C \sqsubseteq D) = 1$ iff $\mathcal{I} \models C \sqsubseteq D$. This confidence can give a hint whether an expert should accept or decline the GCI. Assume that $c \in [0,1)$ is a *confidence threshold*. In case $1 > \mathsf{conf}_{\mathcal{I}}(C \sqsubseteq D) \geq c$, i.e., if there are some but not too many counterexamples against $C \sqsubseteq D$ in $\mathcal{I}$, the expert accepts the GCI and has to adjust $\mathcal{I}$, and otherwise declines the GCI and returns an adjustment of it.

Another approach is as follows. Let $\mathcal{I} = \mathcal{I}_t \uplus \mathcal{I}_u$ be a disjoint union of the *trusted subinterpretation* $\mathcal{I}_t$ (which is assumed to be error-free) and the *untrusted subinterpretation* $\mathcal{I}_u$. Then the expert accepts $C \sqsubseteq D$ if it holds in $\mathcal{I}_t$, and declines otherwise.

Of course, the automatic construction of adjustments is not addressed with both approaches as they only provide methods for the decision whether the expert should accept or decline. The next section presents possibilities for adjustment construction.

5.2 Construction of Adjustments

Adjusting the general concept inclusion Consider a general concept inclusion $C \sqsubseteq D$ that does not hold in the interpretation $\mathcal{I}$. The expert now wants to decline the GCI by adjusting it. According to the definition of adjustments of GCIs it is both possible to shrink the premise C and to enlarge the conclusion D to construct a GCI that holds in $\mathcal{I}$ but is more general than $C \sqsubseteq D$. Of course, it is always simply possible to return the adjustment $\bot \sqsubseteq \top$ but this may not be a good practise since then no knowledge that is enclosed in $C \sqsubseteq D$ and holds in $\mathcal{I}$ would be preserved.

In order to adjust the GCI more carefully the expert has the following options:

1. Add a conjunct to C, or choose an existential restriction $\exists r.E$ in C and modify E such that the resulting concept description is more specific than E, e.g., by adding a conjunct, or by adjusting an existential restriction.
2. Choose a conjunct in D and remove it, or choose an existential restriction $\exists r.E$ in D and modify E such that the resulting concept description is more general than E, e.g., by removing a conjunct, or adjusting an existential restriction.

In order to obtain a minimal adjustment the expert should only apply as few changes as possible.

Adjusting the interpretation To generate an adjustment of $\mathcal{I}$ w.r.t. $C \sqsubseteq D$ the expert may either remove or modify the counterexamples in $\mathcal{I}$, or introduce new individuals, to enforce the GCI. The simplest solution is to just remove all counterexamples against $C \sqsubseteq D$ from $\mathcal{I}$, and this may always be done by automatic experts. Of course, the removal of a counterexample from the interpretation is an impractical solution since in most cases there will only be few errors in the dataset. A more intelligent solution involves the modification of the concept and role name extensions occuring in the premise or conclusion of the GCI at hand.

Let $x \in C^{\mathcal{I}} \setminus D^{\mathcal{I}}$ be a counterexample. Then the expert may proceed as follows:

1. Remove x from the interpretation.
2. For a *modification of the premise* it suffices to choose one conjunct E of C and modify its extension such that $x \notin E^{\mathcal{I}}$ holds. The expert may choose either a concept name A in C and remove the element x from the extension $A^{\mathcal{I}}$, or an existential restriction $\exists\, r.\, E$ in C and modify the interpretation such that x is not in the extension $(\exists\, r.\, E)^{\mathcal{I}}$ anymore. This may either be done by removing all r-successors of x that are elements of $E^{\mathcal{I}}$, or by modifying all r-successors such that they are not elements of the extension $E^{\mathcal{I}}$ anymore.
3. For a *modification of the conclusion* the expert has to modify all conjuncts E of D with $x \notin E^{\mathcal{I}}$. If $E = A$ is a concept name in D then the expert simply has to add x to the extension of A. If $E = \exists\, r.\, F$ is an existential restriction in D then the expert has the following choices:
   (a) Choose an existing r-successor y of x and modify y such that $y \in E^{\mathcal{I}}$ holds. In case of E containing an existential restriction as a subconcept a modification or introduction of further successors may be neccessary.
   (b) Introduce a new r-successor y of x such that $y \in E^{\mathcal{I}}$ holds. If E contains an existential restriction as a subconcept then this action requires the introduction of further new elements in the interpretation.

6 An Incremental Learning Algorithm

By means of experts it is possible to adjust an interpretation $\mathcal{I}$ and a TBox $\mathcal{T}$ such that $\mathcal{I} \models \mathcal{T}$. This enables us to use the techniques for the computation of relative bases as described in Section 4. Based on these results and definitions, we now want to formulate an *incremental learning algorithm* which takes a possibly empty initial TBox $\mathcal{T}_0$ and a sequence $(\mathcal{I}_n)_{n\geq 1}$ of interpretations as input, and iteratively adjusts the TBox in order to compute a TBox of GCIs holding in the domain of interest. This is modeled as a sequence $(\mathcal{T}_n)_{n\geq 0}$ of TBoxes where each TBox $\mathcal{T}_n$ is defined as the base of the adjustment of $\mathcal{I}_n$ relative to the adjustment of $\mathcal{T}_{n-1}$. Of course, we also have to presuppose an expert χ that has full knowledge on the domain of interest and provides the neccessary adjustments during the algorithm's run. The algorithm is briefly described as follows and also given in pseudo-code in Algorithm 1.

(Start) Assume that the TBox $\mathcal{T}_{n-1}$ has been constructed and a new interpretation $\mathcal{I}_n$ is available. In case $\mathcal{I}_n \models \mathcal{T}_{n-1}$ we may skip the next step, and otherwise we first have to adjust both the TBox and interpretation in the next step.

(Adjustment) For each GCI $C \sqsubseteq D \in \mathcal{T}_{n-1}$ ask the expert χ whether it accepts it. If yes then set $\mathcal{I}_n$ to the returned adjustment $\chi(C \sqsubseteq D, \mathcal{I}_n)$. If it otherwise declines it then replace the GCI $C \sqsubseteq D$ with the returned adjustment $\chi(C \sqsubseteq D, \mathcal{I}_n)$ in $\mathcal{T}_n$. After all GCIs have been processed then we have that $\mathcal{I}_n \models \mathcal{T}_n$ holds.

(Computation of the Relative Base) As a next step we compute the base $\mathcal{B}_n$ of $\mathcal{I}_n$ relative to $\mathcal{T}_{n-1}$ and set $\mathcal{T}_n := \mathcal{T}_{n-1} \cup \mathcal{B}_n$. Set $n := n + 1$ and goto (Start).

It may occur that a previously answered question is posed to the expert again during the algorithm's run. Of course, we may apply caching techniques, i.e., store a set of accepted GCIs and a set of declined GCIs but this will raise the problem how an adjustment of the interpretation (for acceptance), or of the GCI (for decline), respectively, can be constructed, when it is not returned from the expert itself. Some simple solutions are given in the previous section, e.g., one may just remove all counterexamples for an accepted GCI from the interpretation, or replace the GCI with the adjusted one that has been previously returned by the expert. For this purpose the algorithm may build a set $\mathcal{T}_\chi$ of accepted GCIs to avoid a second question for the same concept inclusion, and a set $\mathcal{F}_\chi$ of pairs of declined GCIs and their adjustments.

Algorithm 1 Incremental Learning Algorithm

`Require:` a domain expert χ, a TBox $\mathcal{T}$ (initial knowledge)
Let $\mathcal{T}_\chi := \emptyset$ and $\mathcal{F}_\chi := \emptyset$.
while a new interpretation $\mathcal{I}$ has been observed **do**
    while $\mathcal{I} \not\models \mathcal{T}$ **do**
        for all $C \sqsubseteq D \in \mathcal{T}$ **do**
            if $\mathcal{I} \not\models C \sqsubseteq D$ **then**
                if $C \sqsubseteq D \in \mathcal{T}_\chi$ **then**
                    Remove all counterexamples against $C \sqsubseteq D$ from $\mathcal{I}$.
                else if $(C \sqsubseteq D, E \sqsubseteq F) \in \mathcal{F}_\chi$ **then**
                    Remove $C \sqsubseteq D$ from $\mathcal{T}$.
                else if χ accepts $C \sqsubseteq D$ **then**
                    Set $\mathcal{I} := \chi(C \sqsubseteq D, \mathcal{I})$.
                    Set $\mathcal{T}_\chi := \mathcal{T}_\chi \cup \{C \sqsubseteq D\}$.
                else
                    Let $E \sqsubseteq F := \chi(C \sqsubseteq D, \mathcal{I})$ be the adjustment of the GCI.
                    Set $\mathcal{T} := (\mathcal{T} \setminus \{C \sqsubseteq D\}) \cup \{E \sqsubseteq F\}$.
                    Set $\mathcal{F}_\chi := \mathcal{F}_\chi \cup \{(C \sqsubseteq D, E \sqsubseteq F)\}$.
                    Set $\mathcal{T}_\chi := \mathcal{T}_\chi \cup \{E \sqsubseteq F\}$.
    Let $\mathcal{T} := \mathcal{T} \cup \mathcal{B}$ where $\mathcal{B}$ is a base of $\mathcal{I}$ relative to $\mathcal{T}$.
return $\mathcal{T}$.

With slight restrictions on the expert and the interpretations used as input data during the algorithm's run we may prove soundness (w.r.t. the domain of interest) and completeness (w.r.t. the processed input interpretations) of the final TBox that is returned after no new interpretation has been observed.

Proposition 1 (Soundness and Completeness). *Assume that $\mathcal{I}$ is the domain of interest, and $\mathcal{T}_0$ is the initial TBox where $\mathcal{I} \models \mathcal{T}_0$. Furthermore, let χ be an expert that has full knowledge of $\mathcal{I}$, i.e., does not decline any GCI holding in $\mathcal{I}$; and let $\mathcal{I}_1, \ldots, \mathcal{I}_n$ be a sequence of sound interpretations, i.e., each $\mathcal{I}_k$ only models GCIs holding in $\mathcal{I}$. Then the final TBox $\mathcal{T}_n$ is sound for $\mathcal{I}$, i.e., only contains GCIs holding in $\mathcal{I}$, and is complete for the adjustment of each $\mathcal{I}_k$, i.e., all GCIs holding in the adjustment of any $\mathcal{I}_k$ are entailed by $\mathcal{T}_n$.*

Proof. We prove the claim by induction on k. Since we have that $\mathcal{T}_k$ holds in $\mathcal{I}$ the expert does not adjust $\mathcal{T}_k$ but constructs an adjustment $\mathcal{I}'_{k+1}$ of $\mathcal{I}_{k+1}$ that is a model of $\mathcal{T}_k$. Then the next TBox is obtained as $\mathcal{T}_{k+1} := \mathcal{T}_k \cup \mathcal{B}_{k+1}$ where $\mathcal{B}_{k+1}$ is a base of

$\mathcal{I}'_{k+1}$ relative to $\mathcal{T}_k$. By assumption, $\mathcal{I}$ is a model of $\mathcal{B}_{k+1}$, i.e., also of $\mathcal{T}_{k+1}$, and by construction $\mathcal{T}_{k+1}$ is complete for $\mathcal{I}'_{k+1}$. Finally, $\mathcal{T}_\ell \subseteq \mathcal{T}_k$ holds for all $\ell \leq k$, and thus we conclude that $\mathcal{T}_{k+1}$ must be complete for all adjusted interpretations $\mathcal{I}'_1, \ldots, \mathcal{I}'_{k+1}$. $\square$

7 Comparison with the existing single-step-learning approaches

In comparison with a single-step approach that explores the canonical base of the disjoint union $\biguplus \mathcal{I}_k$ there are several drawbacks. The first problem is to store all the observed interpretations. Secondly, upon input of a new interpretation there is no output of the refuted old GCIs, and newly obtained GCIs, respectively. Thirdly, a major disadvantage is the computational complexity. In order to iteratively compute the relative bases the model-based most-specific concept descriptions of each interpretation $\mathcal{I}_k$ are neccessary, and their number can be exponential in the size of the domain of $\mathcal{I}_k$. Hence for n input interpretations up to $m := (2^{|\Delta^{\mathcal{I}_1}|} - 1) + \ldots + (2^{|\Delta^{\mathcal{I}_n}|} - 1)$ mmscs have to be constructed. In order to compute the canonical base of the disjoint sum $\biguplus \mathcal{I}_k$ we have to compute up to $m' := 2^{|\Delta^{\mathcal{I}_1}| + \ldots + |\Delta^{\mathcal{I}_n}|} - 1$ mmscs. Obviously, m is much smaller than m' for a sufficently large number n of input interpretations.

Furthermore, the upper bound for the size of the induced context $\mathbb{K}_{\mathcal{I}_k}$ is $|\Delta^{\mathcal{I}_k}| \cdot (1 + |N_C| + |N_R| \cdot (2^{|\Delta^{\mathcal{I}_k}|} - 1))$, and the number of implications may be exponential in the size of the context, i.e., double-exponential in the size of the interpretation $\mathcal{I}_k$, hence in the iterative approach we get an upper bound of

$$2^{|\Delta^{\mathcal{I}_1}| \cdot (1 + |N_C| + |N_R| \cdot (2^{|\Delta^{\mathcal{I}_1}|} - 1))} + \ldots + 2^{|\Delta^{\mathcal{I}_n}| \cdot (1 + |N_C| + |N_R| \cdot (2^{|\Delta^{\mathcal{I}_n}|} - 1))}$$

GCIs, and in the single-step approach the upper bound is given as

$$2^{(|\Delta^{\mathcal{I}_1}| + \ldots + |\Delta^{\mathcal{I}_n}|) \cdot (1 + |N_C| + |N_R| \cdot (2^{|\Delta^{\mathcal{I}_1}| + \ldots + |\Delta^{\mathcal{I}_n}|} - 1))}.$$

It is easy to see that $\sum_k 2^{2^{|\Delta^{\mathcal{I}_k}|}}$ is much smaller than $2^{2^{\sum_k |\Delta^{\mathcal{I}_k}|}}$.

8 Conclusion

We have presented a method for the construction of a minimal extension of a TBox w.r.t. a model, and utilised it to formulate an algorithm that learns $\mathcal{EL}^\perp$-concept inclusions from interpretation streams with external support by means of an expert in the domain of interest. The approach can be applied to a wide range of input data as there are various cryptomorphic structures for interpretations, like description graphs, binary power context families, RDF-graphs (with some effort for a transformation) etc. It may be extended towards more expressive description logics, e.g., $\mathcal{FLE}$, or $\mathcal{ALE}$, or to include the construction of RBoxes in DLs allowing for role hierarchies or complex role inclusions. Another direction for future research is the construction of bases for temporal terminological knowledge.

This document has not considered the problem of the computation of relative model-based most-specific concept descriptions. However, it is possible to use synchronised simulations for their construction, and a detailed characterisation of the existence and construction will be subject of a future paper.

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