Constructing and Extending Description Logic Ontologies using Methods of Formal Concept Analysis

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Contents

Introduction

Description Logic (abbrv. DL) [Baa+17] belongs to the field of knowledge representation and reasoning. DL researchers have developed a large family of logic-based languages, so-called *description logics* (abbrv. DLs). These logics allow their users to explicitly represent knowledge as *ontologies*, which are finite sets of (human- and machine-readable) axioms, and provide them with automated inference services to derive implicit knowledge. The landscape of decidability and computational complexity of common reasoning tasks for various description logics has been explored in large parts: there is always a trade-off between expressibility and reasoning costs. It is therefore not surprising that DLs are nowadays applied in a large variety of domains [Baa+17]: agriculture, astronomy, biology, defense, education, energy management, geography, geoscience, medicine, oceanography, and oil and gas. Furthermore, the most notable success of DLs is that these constitute the logical underpinning of the *Web Ontology Language* (abbrv. OWL) [HKR10] in the *Semantic Web*.

Formal Concept Analysis (abbrv. FCA) [GW99] is subfield of lattice theory that allows to analyze data-sets that can be represented as formal contexts. Put simply, such a formal context binds a set of objects to a set of attributes by specifying which objects have which attributes. There are two major techniques that can be applied in various ways for purposes of conceptual clustering, data mining, machine learning, knowledge management, knowledge visualization, etc. On the one hand, it is possible to describe the hierarchical structure of such a data-set in form of a formal concept lattice [GW99]. On the other hand, the theory of implications (dependencies between attributes) valid in a given formal context can be axiomatized in a sound and complete manner by the so-called canonical base [GD86], which furthermore contains a minimal number of implications w.r.t. the properties of soundness and completeness.

In spite of the different notions used in FCA and in DLs, there has been a very fruitful interaction between these two research areas. My thesis continues this line of research and, more specifically, I will describe how methods from FCA can be used to support the automatic construction and extension of DL ontologies from data.

Related Work

So far, several approaches for axiomatizing concept inclusions (abbrv. CIs) in different description logics have been developed, and many of these utilize sophisticated techniques from Formal Concept Analysis [GO16; GW99]: on the one hand, there is the so-called *canonical base*, cf. Guigues and Duquenne in [GD86], that provides a concise representation of the implicative theory of a formal context in a sound and complete manner and, on the other hand, the interactive algorithm *Attribute Exploration* exists, which guides an expert through the process of axiomatizing the theory of implications that are valid in a domain of interest, cf. Ganter in [Gan84]. In particular, Attribute Exploration is an interactive variant of an algorithm for computing canonical bases [Gan84], and it works as follows: the input is a formal context that only partially describes the domain of interest (that is, there may be implications that are not valid, but for which this partial description does not provide a counterexample), and during the run of the exploration process a minimal number of questions is enumerated and posed to the expert (such a question is an implication for which no counterexample has been explored, and the expert can either confirm its validity or provide a suitable counterexample). On termination, a minimal sound and complete representation of the theory of implications that are valid in the considered domain has been generated.

Note that the notions of Formal Concept Analysis are strongly related to those in propositional logic. More specifically, a formal context is simply a set of (named) propositional models and the implication logic in FCA (for formal contexts with finite attribute set) is exactly the propositional Horn logic (without the unsatisfiable formula $\bot$).

Three recent doctoral theses [Ser07; Dis11; Bor14] already provide a broad overview on research in the intersection of FCA and of DL. In the following, we only briefly mention existing approaches.

The Canonical Base, Attribute Exploration, and their Variations in Formal Concept Analysis

1. Guigues and Duquenne [GD86] describe how for a given formal context a minimal implication base, the so-called *canonical base*, can be computed. The canonical base is thus sometimes also called Duquenne-Guigues-base.

2. Ganter [Gan84; Gan87] introduces the interactive algorithm *Attribute Exploration* which allows for constructing the canonical base of a formal context that is only indirectly accessible through an expert. During the algorithm's run a sequence of implications is generated and the expert can either confirm that the implication is valid or has to specify a counterexample.

3. Luxenburger [Lux93] defines the notion of *confidence* for implications in formal contexts and demonstrates how an implication base for the confident implications can be constructed. Put simply, an implication is confident (or has high confidence), if it does not have too many counterexamples in the formal context.

4. Stumme [Stu96] shows how an existing set of valid implications can be incorporated in the computation of the canonical base and in the algorithm *Attribute Exploration*. Distel [Dis11] proves that the result is minimal.

5. Krauße [Kra98; GK05] formulates a canonical base of so-called *cumulated clauses* for formal contexts. A cumulated clause generalizes the notion of an implication such that the conclusion is a disjunction of conjunctions of attributes (propositional variables).

6. Ganter [Gan99] equips his algorithm *Attribute Exploration* with means for using arbitrary valid propositional formulas as background knowledge. However, it still enumerates implications, which then not already follow from the background knowledge.

7. HOLZER [Hol01; Hol04a; Hol04b] generalizes the algorithm *Attribute Exploration* to a setting of incomplete knowledge, i.e., in the initial formal context it might be unknown whether an object has an attribute, and further the expert might not be able to completely specify counterexamples.
8. SERTKAYA [Ser07] provides another generalization of the algorithm *Attribute Exploration* that starts with a formal context that is only partially specified, but assumes that the expert has full knowledge, i.e., can answer with fully specified counterexamples.
9. BORCHMANN [Bor14] constitutes the algorithm *Attribute Exploration by Confidence*, which allows for exploring the theory of confident implications of a formal context with help of an expert.

Applications and Variations of the Canonical Base and of Attribute Exploration in Description Logic

1. ZICKWOLFF [Zic91] devises the algorithm *Rule Exploration* as an extension of *Attribute Exploration* to function-free first-order Horn logic (Datalog). The domain of interest, which is only accessible though an expert and from which only a part is initially known, is then not a formal context but a first-order logic interpretation.
2. RUDOLPH [Rud06] introduces the interactive algorithm *Relational Exploration*. The domain of interest is described by a description logic interpretation, and his algorithm enumerates valid $\mathcal{FLE}$ concept inclusions.
3. SERTKAYA [Ser07] proposes the algorithm *Ontology Completion* for completing the terminological part of a description logic ontology. Completeness is only achieved for concept inclusions consisting of conjunctions of a predefined set of concept descriptions. The underlying description logic is arbitrary.
4. DISTEL [Dis11] demonstrates how a minimal canonical base of $\mathcal{EL}^{\bot}_{\mathsf{gfp}}$ concept inclusions can be constructed for a description logic interpretation and also shows how such a base can be translated into $\mathcal{EL}^{\bot}$. He also describes how expert interaction can be integrated, yielding the two algorithms *Model Exploration* and *ABox Exploration*.
5. BORCHMANN [Bor14] shows how a base of confident $\mathcal{EL}^{\bot}_{\mathsf{gfp}}$ concept inclusions can be obtained from a description logic interpretation. His interactive algorithm *Model Exploration by Confidence* further allows for interaction with an expert.

Contributions

Parallel Computation of the Canonical Base

The canonical base of some formal context provides a concise representation of its implicative theory. In particular, it has minimal cardinality among all implication bases for a fixed formal context. So far, only two algorithms existed for computing the canonical base: the algorithm *NextClosure* by GANTER [Gan84], and an attribute-incremental algorithm by OBIEDKOV and DUQUENNE [OD07]. Unfortunately, both algorithms do not perform well on modern computer systems due to their enumerative character, i.e., the implications of the canonical base are constructed one by one. In order to resolve this bottleneck, I propose to generate the implications in subset inclusion order with respect to increasing cardinality of the premises instead of in lectic order. Experiments have shown that this allows for a highly parallel computation of the canonical base. More specifically, the computation time decreases linearly with the number of available CPU cores.

An initial chaotic draft of this new algorithm *NextClosures* was provided in a technical report [Kri15f], which has never been published. Its contents were thoroughly rewritten in a series of four publications. The algorithm specifically tailored to the case of a formal context can be found in [KB15].[1] An extension being able to incorporate an existing implication set as background knowledge that need not necessarily be valid in the input formal context was published in [KB17]. Furthermore, [Kri16b] provided a generalization to closure operators in lattices, and [Kri16c] described an interactive variant: *Parallel Attribute Exploration*.

My other contributions on computing bases of concept inclusions heavily depend on using canonical bases as an important intermediate result during the computation. Thus, having a highly parallel algorithm for this computation problem at hand greatly improves practicability and applicability.

Publications, Technical Reports, and Software

[Kri19a] Francesco Kriegel: **Concept Explorer FX**. Software for Formal Concept Analysis with Description Logic Extensions. 2010–2019
(cited on pages 8, 9, 15, 26, 31)

[Kri15f] Francesco Kriegel: **NextClosures – Parallel Exploration of Constrained Closure Operators**. LTCS-Report 15-01. Unpublished, since chaotic. Chair for Automata Theory, Institute for Theoretical Computer Science, Technische Universität Dresden, Dresden, Germany, 2015 (cited on pages viii, xviii, xix, 9)

[KB15] Francesco Kriegel and Daniel Borchmann: **NextClosures: Parallel Computation of the Canonical Base**. In: *Proceedings of the 12th International Conference on Concept Lattices and their Applications (CLA 2015), October 13–16, 2015, Clermont-Ferrand, France*. Ed. by Sadok Ben Yahia and Jan Konecny. Vol. 1466. CEUR Workshop Proceedings. Best Paper Award. CEUR-WS.org, 2015, 182–192
(cited on pages viii, xviii, xix, 19, 25)

[1] After presenting the results in [KB15] at the conference venue, Bazin mentioned that he had already published an algorithm for this task that is somehow similar. Checking his publication [BG13] showed that there are indeed some similarities but also obvious differences. In particular, Algorithm 2 in [BG13] uses a lectic order during computation with the goal to rule out some duplicate computations, since this lectic order induces a spanning tree of the set of intents and pseudo-intents to be computed. In contrast, Algorithm 1 in [KB15] is explicitly designed to perform large amounts of independent computation steps in parallel and further a thorough proof of its soundness and completeness is provided, i.e., it is shown that any computation order on parallel steps is admissible.

[Kri16b] Francesco Kriegel: **NextClosures with Constraints**. In: *Proceedings of the 13th International Conference on Concept Lattices and Their Applications (CLA 2016), July 18–22, 2016, Moscow, Russia*. Ed. by Marianne Huchard and Sergei Kuznetsov. Vol. 1624. CEUR Workshop Proceedings. CEUR-WS.org, 2016, 231–243 (cited on pages viii, xii, xviii, xix, 10, 15, 17, 19, 178, 194)

[Kri16c] Francesco Kriegel: **Parallel Attribute Exploration**. In: *Proceedings of the 22nd International Conference on Conceptual Structures (ICCS 2016), July 5–7, 2016, Annecy, France*. Ed. by Ollivier Haemmerlé, Gem Stapleton, and Catherine Faron-Zucker. Vol. 9717. Lecture Notes in Computer Science. Springer, 2016, 91–106 (cited on pages viii, xix, 9)

[KB17] Francesco Kriegel and Daniel Borchmann: **NextClosures: Parallel Computation of the Canonical Base with Background Knowledge**. In: *International Journal of General Systems* 46.5 (2017), 490–510 (cited on pages viii, xviii, xix, 8, 17, 19, 20, 21, 22, 23, 24, 25, 26)

Role-Depth-Bounded Axiomatization of Concept Inclusions

A first approach for axiomatizing DL interpretations in a sound and complete manner has been devised by Baader and Distel [BD08; Dis11]. In particular, they have shown that, for each finite interpretation, there is a (finite) TBox which entails exactly all those concept inclusions that are valid in the interpretation. The resulting TBox is formulated in $\mathcal{EL}^{\bot}_{\mathsf{gfp}}$, which is an extension of the description logic $\mathcal{EL}^{\bot}$ with greatest fixed-point semantics—this choice is necessary due to possible cycles occurring in the input interpretation. However, the usage of $\mathcal{EL}^{\bot}_{\mathsf{gfp}}$ might be seen as a weakness impairing practicability for two reasons. Although the computational complexity of reasoning in $\mathcal{EL}^{\bot}_{\mathsf{gfp}}$ is not harder than in $\mathcal{EL}^{\bot}$—in both logics the subsumption problem is **P**-complete—only few reasoner implementations exist and these are not as highly optimized as those for $\mathcal{EL}^{\bot}$. Furthermore, it might be the case that the *cyclic* concept descriptions expressible in $\mathcal{EL}^{\bot}_{\mathsf{gfp}}$ are harder to grasp for humans.

A natural adaptation, as proposed by Distel [Dis12], is to restrict attention to the concept inclusions involving only concept descriptions up to a predefined role depth, i.e., where the number of nestings of existential restrictions is bounded. The goal is then to find an $\mathcal{EL}^{\bot}$ TBox that is sound and complete for the concept inclusions satisfying the chosen bound on the role depth. On the one hand, performing reasoning with the result is cheaper and, on the other hand, readability is greatly improved—if the bound is not set too high, of course. A thorough study of this idea has been conducted by Borchmann, Distel, and the author in [BDK15; BDK16].

A further advancement is to expand on their results by replacing $\mathcal{EL}^{\bot}$ with a more expressive description logic. However, one should be cautious with choosing that logic, since higher expressivity could easily lead to *overfitting* of the result. As a thumb rule, any logic providing disjunction or full negation is not suitable in general. Nominals should not be allowed either, since these allow rewriting the input interpretation into a TBox without any terminological abstraction.

In [Kri15b] a first proposal towards the axiomatization of interpretations in the description logic $\mathcal{M}$ has been made, which in addition to the concept constructors of $\mathcal{EL}^{\bot}$ allows for primitive negations, qualified at-least restrictions, unqualified at-most restrictions, value restrictions, and existential self restrictions. The basic argumentation is similar, and the only major difficulty was the computation of model-based most specific concept descriptions in $\mathcal{M}$, which was not addressed in [Kri15b]. Some first steps have been made in an unpublished technical report [Kri15e] that corresponds to a rejected conference submission. A solution to the problem of computing such model-based most specific concept descriptions as well as an in-depth description of the whole approach including proofs has been published later [Kri17a].

Although reasoning in $\mathcal{EL}^{\bot}$ is tractable (**P**-complete) and using $\mathcal{EL}^{\bot}$ concept inclusion bases in practical applications is thus cheap, it might be the case that some terminological facts cannot be expressed in $\mathcal{EL}^{\bot}$. Contrary to that, computing $\mathcal{M}$ concept inclusion bases leads to a more fine-grained axiomatization of the input interpretation with the downside that reasoning with the result is more expensive or even intractable (**EXP**-complete). However, it is often the case that data complexity of assertional reasoning gets considerably cheaper in Horn fragments of DLs. So, I have investigated the problem of axiomatizing concept inclusions in the description logic Horn-$\mathcal{M}$ as well [Kri19b; Kri19c], which is the Horn fragment of $\mathcal{M}$ and has an expressivity between $\mathcal{EL}^{\bot}$ and $\mathcal{M}$. Indeed, instance checking is **coNP**-complete for $\mathcal{M}$, but **P**-complete for Horn-$\mathcal{M}$.[2] Hence, it makes sense to use a Horn-$\mathcal{M}$ TBox as the schema for *ontology-based data access* (abbrv. OBDA) applications.

Publications and Technical Reports

[BDK15] DANIEL BORCHMANN, FELIX DISTEL, and FRANCESCO KRIEGEL: **Axiomatization of General Concept Inclusions from Finite Interpretations**. LTCS-Report 15-13. Chair for Automata Theory, Institute for Theoretical Computer Science, Technische Universität Dresden, Dresden, Germany, 2015 (cited on pages ix, xviii, xix, 80)

[Kri15b] FRANCESCO KRIEGEL: **Extracting $\mathcal{ALEQR}(\mathsf{Self})$-Knowledge Bases from Graphs**. In: *Proceedings of the International Workshop on Social Network Analysis using Formal Concept Analysis (SNAFCA 2015) in conjunction with the 13th International Conference on Formal Concept Analysis (ICFCA 2015), June 23–26, 2015, Nerja, Spain*. Ed. by SERGEI O. KUZNETSOV, ROKIA MISSAOUI, and SERGEI A. OBIEDKOV. Vol. 1534. CEUR Workshop Proceedings. CEUR-WS.org, 2015 (cited on pages ix, x, xviii, xix)

[Kri15e] FRANCESCO KRIEGEL: **Model-Based Most Specific Concept Descriptions and Least Common Subsumers in $\mathcal{ALEQ}^{\geq}\mathcal{N}^{\leq}(\mathsf{Self})$**. LTCS-Report 15-02. Unpublished, since erroneous. Chair for Automata Theory, Institute for Theoretical Computer Science, Technische Universität Dresden, Dresden, Germany, 2015 (cited on pages x, xix)

[BDK16] DANIEL BORCHMANN, FELIX DISTEL, and FRANCESCO KRIEGEL: **Axiomatisation of General Concept Inclusions from Finite Interpretations**. In: *Journal of Applied Non-Classical Logics* 26.1 (2016), 1–46 (cited on pages ix, xviii, xix, 45, 69, 70, 71, 73, 168, 184, 196, 201, 202, 207, 208, 210, 211, 217, 220)

[2] Note that the mentioned computational complexities are only proven for the sublogic $\mathcal{M}^{-}$ of $\mathcal{M}$ without existential self-restrictions. The author conjectures that the same complexity results also hold true for $\mathcal{M}$ and Horn-$\mathcal{M}$.

[Kri17a] Francesco Kriegel: **Acquisition of Terminological Knowledge from Social Networks in Description Logic**. In: *Formal Concept Analysis of Social Networks*. Ed. by Rokia Missaoui, Sergei O. Kuznetsov, and Sergei Obiedkov. Lecture Notes in Social Networks. Springer, 2017, 97–142
(cited on pages x, xviii, xix, 33)

[Kri19b] Francesco Kriegel: **Joining Implications in Formal Contexts and Inductive Learning in a Horn Description Logic**. In: *Proceedings of the 15th International Conference on Formal Concept Analysis (ICFCA 2019), June 25–28, 2019, Frankfurt, Germany*. Ed. by Diana Cristea, Florence Le Ber, and Bariş Sertkaya. Vol. 11511. Lecture Notes in Computer Science. Springer, 2019, 110–129
(cited on pages x, xviii, xix, 2, 33)

[Kri19c] Francesco Kriegel: **Joining Implications in Formal Contexts and Inductive Learning in a Horn Description Logic (Extended Version)**. LTCS-Report 19-02. Chair of Automata Theory, Institute of Theoretical Computer Science, Technische Universität Dresden, Dresden, Germany, 2019
(cited on pages x, xviii, xix, 2, 33)

Axiomatization of Concept Inclusions from Closure Operators

A weakness of the existing approaches for generating concept inclusion bases is that these are *static*, i.e., once a TBox has been learned it is not possible to extend or modify it with new observations. For developing an *incremental* approach to learning concept inclusions we could assume that there is a sequence $(\mathcal{I}_n \mid n \in \mathbb{N})$ of interpretations such that $\mathcal{I}_n$ is available at time point n. The goal is now to construct a sequence $(\mathcal{T}_n \mid n \in \mathbb{N})$ such that, for each time point n, the TBox $\mathcal{T}_n$ is sound and complete for the concept inclusions being valid in all interpretations observed so far.

A first step towards this has been published in [Kri15c]. Therein, it is shown how relative concept inclusion bases can be obtained. More specifically, it is assumed that there is some TBox $\mathcal{S}$ as well as an interpretation $\mathcal{I}$ that is a model of $\mathcal{S}$. A relative base is then a TBox $\mathcal{T}$ such that the union $\mathcal{S} \cup \mathcal{T}$ is both sound and complete for $\mathcal{I}$. The mentioned publication then devises a way to construct such relative bases. However, the condition that $\mathcal{I}$ must be a model of $\mathcal{S}$ is rather restrictive and need not hold true in general. It might certainly be the case that a later observed interpretation in the sequence contains a counterexample against some previously valid concept inclusion. In such cases, it is necessary to also adjust the terminological knowledge that has been obtained so far. As one quickly verifies, we need some technique that allows us to compute (a finite base of) the *logical intersection* of the terminological knowledge entailed by the last TBox $\mathcal{T}_{n-1}$ and valid in the current interpretation $\mathcal{I}_n$ in order to construct the current TBox $\mathcal{T}_n$.

That same problem can easily be solved in Formal Concept Analysis. Note that each formal context $\mathbb{K} := (G, M, I)$ induces the closure operator $\phi_{\mathbb{K}} \colon X \mapsto X^{II}$ on the attribute set M. Now an implication $X \to Y$ is valid in $\mathbb{K}$ if, and only if, Y is a subset of X^{II}, i.e., if it is valid for this closure operator $\phi_{\mathbb{K}}$. Furthermore, any implication set $\mathcal{L}$ over M induces a closure operator $\phi_{\mathcal{L}} \colon X \mapsto X^{\mathcal{L}}$ on the underlying attribute set M, and an implication $X \to Y$ follows from $\mathcal{L}$ if, and only if, Y is a subset of the closure $X^{\mathcal{L}}$, that is, if $X \to Y$ is valid for $\phi_{\mathcal{L}}$. Given a formal context $\mathbb{K}$ and an implication set $\mathcal{L}$, we have that any implication is both valid in $\mathbb{K}$ and entailed by $\mathcal{L}$ if, and only if, it is valid in the closure operator that is the infimum of $\phi_{\mathbb{K}}$ and $\phi_{\mathcal{L}}$, as it has been explained in [Kri16b, Section 3.1].

While Distel [Dis11] has shown that each interpretation $\mathcal{I}$ induces a closure operator $\phi_{\mathcal{I}}\colon C \mapsto C^{\mathcal{I}\mathcal{I}}$ such a concept inclusion is valid in $\mathcal{I}$ if, and only if, it is valid for $\phi_{\mathcal{I}}$, we would also need a way to let a TBox $\mathcal{T}$ induce some closure operator $\phi_{\mathcal{T}}$ and suitably combine both in order to make incremental learning possible. A half-finished proposal towards this has been published in [Kri16a], where the notion of a *most specific consequence* has been introduced. For understanding this new notion, we go back to the closure operator $\phi_{\mathcal{L}}$ induced by some implication set $\mathcal{L}$. It is easy to see that, for each subset $X \subseteq M$, the implication $X \to X^{\mathcal{L}}$ follows from $\mathcal{L}$ and further that, for each superset $Z \supsetneq X^{\mathcal{L}}$, the implication $X \to Z$ is not entailed by $\mathcal{L}$. Hence, we may also refer to $X^{\mathcal{L}}$ as the *most specific consequence* of X with respect to $\mathcal{L}$. Translating this into the DL setting yields the following definition. If C is a concept description and $\mathcal{T}$ is some TBox, then we call a concept description D a *most specific consequence* of C with respect to $\mathcal{T}$ if $C \sqsubseteq D$ is entailed by $\mathcal{T}$ and if further D is more specific than each concept description E where $\mathcal{T}$ entails $C \sqsubseteq E$. Note that all most specific consequences of C w.r.t. $\mathcal{T}$ are equivalent and so it makes sense to speak of *the* most specific consequence and further denote it by $C^{\mathcal{T}}$. Thus, we obtain the necessary closure operator $\phi_{\mathcal{T}}$ as the mapping $C \mapsto C^{\mathcal{T}}$—provided the most specific consequence exists.

A more sophisticated study of the notion of a most specific consequence as well as for combining closure operators induced by an interpretation, a TBox, or an ABox has been conducted in a following publication [Kri18c; Kri19e]. It has been shown that most specific consequences always exist in extensions of $\mathcal{EL}$ with greatest fixed-point semantics and always exist for the role-depth-bounded case in $\mathcal{EL}$. Furthermore, the idea of axiomatizing concept inclusions from streams of interpretations has been cultivated further. It also described a more general setting of learning from ABoxes under open world assumption, domain closure assumption, and unique name assumption. While learning from interpretation streams makes use of the infimum of closure operators, one could also ask whether utilizing the supremum has some practical relevance. As it turns out, this is the case. When axiomatizing the supremum of $\phi_{\mathcal{I}}$ and $\phi_{\mathcal{T}}$, we only get concept inclusions that are satisfied by the subsets of $\Delta^{\mathcal{I}}$ which satisfy the concept inclusions in $\mathcal{T}$. That way, we get a method for error-tolerant axiomatization of an interpretation where the TBox is used to detect and ignore erroneous sets of objects within $\mathcal{I}$, and in consequence only concept inclusions are generated that are valid in the error-free part of $\mathcal{I}$.

Further applications described in my thesis are the following.

- Axiomatizing or approximating the logical intersection of TBoxes
- Foundation of an interactive, gentle ontology repair algorithm
- Keeping role depths small in a concept inclusion base by an iterated computation that increases the maximal role depth in each step

Publications and Technical Reports

[Kri15c] FRANCESCO KRIEGEL: **Incremental Learning of TBoxes from Interpretation Sequences with Methods of Formal Concept Analysis**. In: *Proceedings of the 28th International Workshop on Description Logics (DL 2015), June 7–10, 2015, Athens, Greece*. Ed. by DIEGO CALVANESE and BORIS KONEV. Vol. 1350. CEUR Workshop Proceedings. CEUR-WS.org, 2015, 452–464 (cited on pages xi, xviii, xix, 184, 196, 227)

[Kri16a] FRANCESCO KRIEGEL: **Axiomatization of General Concept Inclusions from Streams of Interpretations with Optional Error Tolerance**. In: *Proceedings of the 5th International Workshop "What can FCA do for Artificial Intelligence?" (FCA4AI 2016) co-located with the European Conference on Artificial Intelligence (ECAI 2016), August 29 – September 2, 2016, The Hague, The Netherlands*. Ed. by SERGEI KUZNETSOV, AMEDEO NAPOLI, and SEBASTIAN RUDOLPH. Vol. 1703. CEUR Workshop Proceedings. CEUR-WS.org, 2016, 9–16 (cited on pages xii, xviii, xix, 79)

[Kri16b] FRANCESCO KRIEGEL: **NextClosures with Constraints**. In: *Proceedings of the 13th International Conference on Concept Lattices and Their Applications (CLA 2016), July 18–22, 2016, Moscow, Russia*. Ed. by MARIANNE HUCHARD and SERGEI KUZNETSOV. Vol. 1624. CEUR Workshop Proceedings. CEUR-WS.org, 2016, 231–243 (cited on pages viii, xii, xviii, xix, 10, 15, 17, 19, 178, 194)

[Kri18c] FRANCESCO KRIEGEL: **Most Specific Consequences in the Description Logic $\mathcal{EL}$**. LTCS-Report 18-11. Chair of Automata Theory, Institute of Theoretical Computer Science, Technische Universität Dresden, Dresden, Germany, 2018 (cited on pages xii, xviii, xix, 227)

[Kri19e] FRANCESCO KRIEGEL: **Most Specific Consequences in the Description Logic $\mathcal{EL}$**. In: *Discrete Applied Mathematics* (2019). To appear. (cited on pages xii, xviii, xix, 184)

The Distributive, Graded Lattice of $\mathcal{EL}$ Concept Descriptions and its Neighborhood Relation

In a conference paper [Kri18e] and a following extended technical report [Kri18f], I have investigated the description logic $\mathcal{EL}$ from a perspective of *lattice theory*. It is easy to see that the subsumption relation $\sqsubseteq_{\mathcal{T}}$ with respect to some TBox $\mathcal{T}$ is a quasi-order. Furthermore, conjunctions correspond to infima and *least common subsumers* are suprema in this quasi-ordered set. These least common subsumers always exist if, e.g., the TBox is empty or if *greatest fixed-point semantics* is applied. Apart from that not much is known about the lattice of $\mathcal{EL}$ concept descriptions. The above mentioned publications start with considering the neighborhood relation $\prec_{\mathcal{T}}$ induced by the subsumption relation $\sqsubseteq_{\mathcal{T}}$, i.e., it holds true that $\prec_{\mathcal{T}} = \sqsubsetneq_{\mathcal{T}} \setminus (\sqsubsetneq_{\mathcal{T}} \circ \sqsubsetneq_{\mathcal{T}})$. Then, the question whether $\sqsubseteq_{\mathcal{T}}$ is *neighborhood generated*, i.e., if the reflexive-transitive closure of $\prec_{\mathcal{T}}$ equals $\sqsubseteq_{\mathcal{T}}$, is answered. As it turns out, $\sqsubseteq_{\mathcal{T}}$ is neighborhood generated in the following cases.

- The description logic is $\mathcal{EL}$ and the TBox is empty.
- The description logic is $\mathcal{EL}$ and the TBox is acyclic.
- The description logic is $\mathcal{EL}$ and the TBox is cycle-restricted.

Complementing these results, it was found that in the following cases the subsumption relation $\sqsubseteq_{\mathcal{T}}$ is not neighborhood generated in each of the following situations.

- The description logic is $\mathcal{EL}^{\bot}$, i.e., the bottom concept description $\bot$ is present.
- The description logic is $\mathcal{EL}_{\mathsf{gfp}}$, i.e., greatest fixed-point semantics is applied.
- The description logic is $\mathcal{EL}$, and the TBox is general.

For the cases where $\sqsubseteq_{\mathcal{T}}$ is neighborhood generated, procedures for deciding neighborhood between two concept descriptions as well as for enumerating all upper and all lower neighbors of a given concept descriptions have been devised and their computational complexities have been analyzed.

Continuing this investigation, it was further shown that the lattice of $\mathcal{EL}$ concept descriptions for the empty TBox is distributive, graded, and metric. More specifically, there exists a rank function that can be obtained using the neighborhood relation using a standard construction from *order theory*, and from this rank function a distance function can be constructed. While these results help understanding the structure of the lattice of $\mathcal{EL}$ concept descriptions, one should be cautious when trying to utilize the rank or distance function in practical applications. This is due to the fact that there is no elementary bound on the values of the rank function: for each number d, there is some $\mathcal{EL}$ concept description with role depth d and linear size such that its rank is asymptotically bounded above and below by

$$\underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2^{2^d}}}}}}}_{d\text{ times}}.$$

For instance, the rank of $\exists\, r^4.\,(A_1 \sqcap A_2 \sqcap \ldots \sqcap A_6)$ is greater than $\binom{2.33\cdot 10^{55614}}{1.16\cdot 10^{55614}}$. Consequently, algorithms that walk along the neighborhood relation would not terminate in acceptable time. However, there are at least two applications as follows.

- For analyzing the complexity of a problem deciding whether some concept description is most general or most specific with respect to some monotonic property, one can use the procedure that enumerates all upper neighbors or all lower neighbors, respectively, in the obvious way. For instance, if the TBox is empty and there is a problem in **P** (consisting of $\mathcal{EL}$ concept descriptions), then the subproblem containing all most specific concept descriptions is in **coNP**.
- For enumerating the closures of some given closure operator in $\mathcal{EL}$, one can use an algorithm in the style of *Close-By-One*: initially compute the closure of $\top$, and then inductively continue with a last obtained closure by enumerating its lower neighbors and computing their closures.

Publications and Technical Reports

[Kri18e] FRANCESCO KRIEGEL: **The Distributive, Graded Lattice of $\mathcal{EL}$ Concept Descriptions and its Neighborhood Relation**. In: *Proceedings of the 14th International Conference on Concept Lattices and Their Applications (CLA 2018), June 12–14, 2018, Olomouc, Czech Republic*. Ed. by DMITRY I. IGNATOV and LHOUARI NOURINE. Vol. 2123. CEUR Workshop Proceedings. CEUR-WS.org, 2018, 267–278 (cited on pages xiii, xviii, xix)

[Kri18f] FRANCESCO KRIEGEL: **The Distributive, Graded Lattice of $\mathcal{EL}$ Concept Descriptions and its Neighborhood Relation (Extended Version)**. LTCS-Report 18-10. Chair of Automata Theory, Institute of Theoretical Computer Science, Technische Universität Dresden, Dresden, Germany, 2018 (cited on pages xiii, xviii, xix)

Axiomatization of Concept Inclusions in Probabilistic Description Logic

Logics in their standard form only allow for representing and reasoning with *crisp* knowledge without any degree of *uncertainty*. Of course, this is a serious shortcoming for use cases where it is impossible to perfectly determine the truth of a statement or where there exist degrees of truth. For resolving this expressivity restriction, probabilistic variants of logics have been introduced. A thorough article on extending first-order logics with means for representing and reasoning with probabilistic knowledge was published by HALPERN [Hal90]. In particular, HALPERN explains why it is important to distinguish between two contrary types of probabilities: *statistical information* (type 1) and *degrees of belief* (type 2). The crucial difference between both types is that type-1 probabilities represent information about one particular world, the *real* world, and require a probability distribution on the objects, while type-2 probabilities represent information about a multi-world view such that there is a probability distribution on the set of possible worlds. In my thesis, I only consider probabilities of the second type.

Probabilistic multi-world interpretations can be seen as families of directed graphs in which the vertices and edges are labeled and for which there exists a probability measure on this graph family, i.e., these are *discrete probability distributions over description graphs*. Results of scientific experiments, e.g., in medicine, psychology, biology, finance, or economy, that are repeated several times can induce probabilistic interpretations in a natural way. Each repetition corresponds to a world, and the results of a particular repetition are encoded in the graph structure of that world.

There are two options for referring to probabilities in DL expressions: on the one hand, we can assign probabilities to concept inclusions, and on the other hand, we can probabilistically quantify concept descriptions within the axioms.

For the first option, an approach is presented in [Kri15g; Kri17d]. The notion of *probability* of an $\mathcal{EL}^{\bot}$ concept inclusion is defined as the probability of the set of worlds in which it is valid, and then a procedure is proposed for constructing a base for the concept inclusions having a probability exceeding a predefined threshold. Note that the two mentioned publications first consider the simpler version of this problem for the propositional logic used in *Formal Concept Analysis* and then extend the results to the description logic $\mathcal{EL}^{\bot}$. The resulting procedure is somewhat similar to *Luxenburger's base* [Lux93] of a given formal context, except that I do not consider the statistical probability (type 1) induced by a uniform probability distribution of the object set of the formal context. Furthermore, the $\mathcal{EL}^{\bot}$ case is basically comparable to the results of Borchmann [Bor14], except that I do not consider statistical probabilities (type 1) of concept inclusions, but rather degrees of belief (type 2) of concept inclusions.

The second approach does not utilize the notion of probability of a concept inclusion, but rather allows for using the probability restrictions provided by the DL Prob-$\mathcal{EL}^{\bot}$ [Gut+17]. In this DL, the concept descriptions themselves may refer to probabilities by containing probabilistically quantified subconcepts. These are of the form $\mathsf{d} \geq p.\, C$ and describe all objects for which the probability of being in the extension of C is at least p. A given probabilistic interpretation could be analyzed with the procedure proposed in my thesis, which produces a *sound and complete axiomatization* of it. In particular, the outcome would then be a *logical-statistical evaluation* of the input data. However, one should be cautious when interpreting the results, since the procedure, like any other existing statistical evaluation technique, cannot distinguish between *causality* and *correlation*. It should further be mentioned that for evaluating observations by means of the proposed technique no hypotheses are necessary.

Instead of a probabilistic interpretation, the input data may also come in form of a simple probabilistic ABox containing probabilistically quantified assertional axioms. Under open world assumption, unique name assumption, and domain closure assumption there is only a finite, bounded number of possible worlds in a model of such a probabilistic ABox. Using a linear optimization program it is possible to compute a unique model for which the entropy is maximal, the so-called *maximum entropy model*. This model can now be processed with the approaches for axiomatizing concept inclusions from probabilistic interpretations.

Publications and Technical Reports

[Kri15a] Francesco Kriegel: **Axiomatization of General Concept Inclusions in Probabilistic Description Logics**. In: *Proceedings of the 38th German Conference on Artificial Intelligence (KI 2015), September 21–25, 2015, Dresden, Germany*. Ed. by Steffen Hölldobler, Sebastian Rudolph, and Markus Krötzsch. Vol. 9324. Lecture Notes in Artificial Intelligence. Springer, 2015, 124–136 (cited on pages xix, 220)

[Kri15d] Francesco Kriegel: **Learning General Concept Inclusions in Probabilistic Description Logics**. LTCS-Report 15-14. Chair for Automata Theory, Institute for Theoretical Computer Science, Technische Universität Dresden, Dresden, Germany, 2015 (cited on page xix)

[Kri15g] Francesco Kriegel: **Probabilistic Implicational Bases in FCA and Probabilistic Bases of GCIs in $\mathcal{EL}^{\bot}$**. In: *Proceedings of the 12th International Conference on Concept Lattices and their Applications (CLA 2015), October 13–16, 2015, Clermont-Ferrand, France*. Ed. by Sadok Ben Yahia and Jan Konecny. Vol. 1466. CEUR Workshop Proceedings. CEUR-WS.org, 2015, 193–204 (cited on pages xv, xix)

[Kri17b] Francesco Kriegel: **First Notes on Maximum Entropy Entailment for Quantified Implications**. In: *Proceedings of the 14th International Conference on Formal Concept Analysis (ICFCA 2017), June 13–16, 2017, Rennes, France*. Ed. by Karell Bertet, Daniel Borchmann, Peggy Cellier, and Sébastien Ferré. Vol. 10308. Lecture Notes in Computer Science. Springer, 2017, 155–167 (cited on page xviii)

[Kri17c] Francesco Kriegel: **Implications over Probabilistic Attributes**. In: *Proceedings of the 14th International Conference on Formal Concept Analysis (ICFCA 2017), June 13–16, 2017, Rennes, France*. Ed. by Karell Bertet, Daniel Borchmann, Peggy Cellier, and Sébastien Ferré. Vol. 10308. Lecture Notes in Computer Science. Springer, 2017, 168–183 (cited on page xix)

[Kri17d] Francesco Kriegel: **Probabilistic Implication Bases in FCA and Probabilistic Bases of GCIs in $\mathcal{EL}^\bot$**. In: *International Journal of General Systems* 46.5 (2017), 511–546 (cited on pages xv, xix, 222)

[Kri18a] Francesco Kriegel: **Acquisition of Terminological Knowledge in Probabilistic Description Logic**. In: *Proceedings of the 41st German Conference on Artificial Intelligence (KI 2018), September 24–28, 2018, Berlin, Germany*. Ed. by Frank Trollmann and Anni-Yasmin Turhan. Vol. 11117. Lecture Notes in Computer Science. Springer, 2018, 46–53 (cited on pages xix, 220)

[Kri18b] Francesco Kriegel: **Learning Description Logic Axioms from Discrete Probability Distributions over Description Graphs (Extended Version)**. LTCS-Report 18-12. Chair of Automata Theory, Institute of Theoretical Computer Science, Technische Universität Dresden, Dresden, Germany, 2018 (cited on pages xviii, xix)

[Kri18d] Francesco Kriegel: **Terminological Knowledge Acquisition in Probabilistic Description Logic**. LTCS-Report 18-03. Chair of Automata Theory, Institute of Theoretical Computer Science, Technische Universität Dresden, Dresden, Germany, 2018 (cited on page xix)

[Kri19d] Francesco Kriegel: **Learning Description Logic Axioms from Discrete Probability Distributions over Description Graphs**. In: *Proceedings of the 16th European Conference on Logics in Artificial Intelligence (JELIA 2019), May 8–10, 2019, Rende, Italy*. Ed. by Francesco Calimeri, Nicola Leone, and Marco Manna. Vol. 11468. Lecture Notes in Computer Science. Springer, 2019, 399–417 (cited on pages xviii, xix, 220)

Meta-Information

Self-Citations

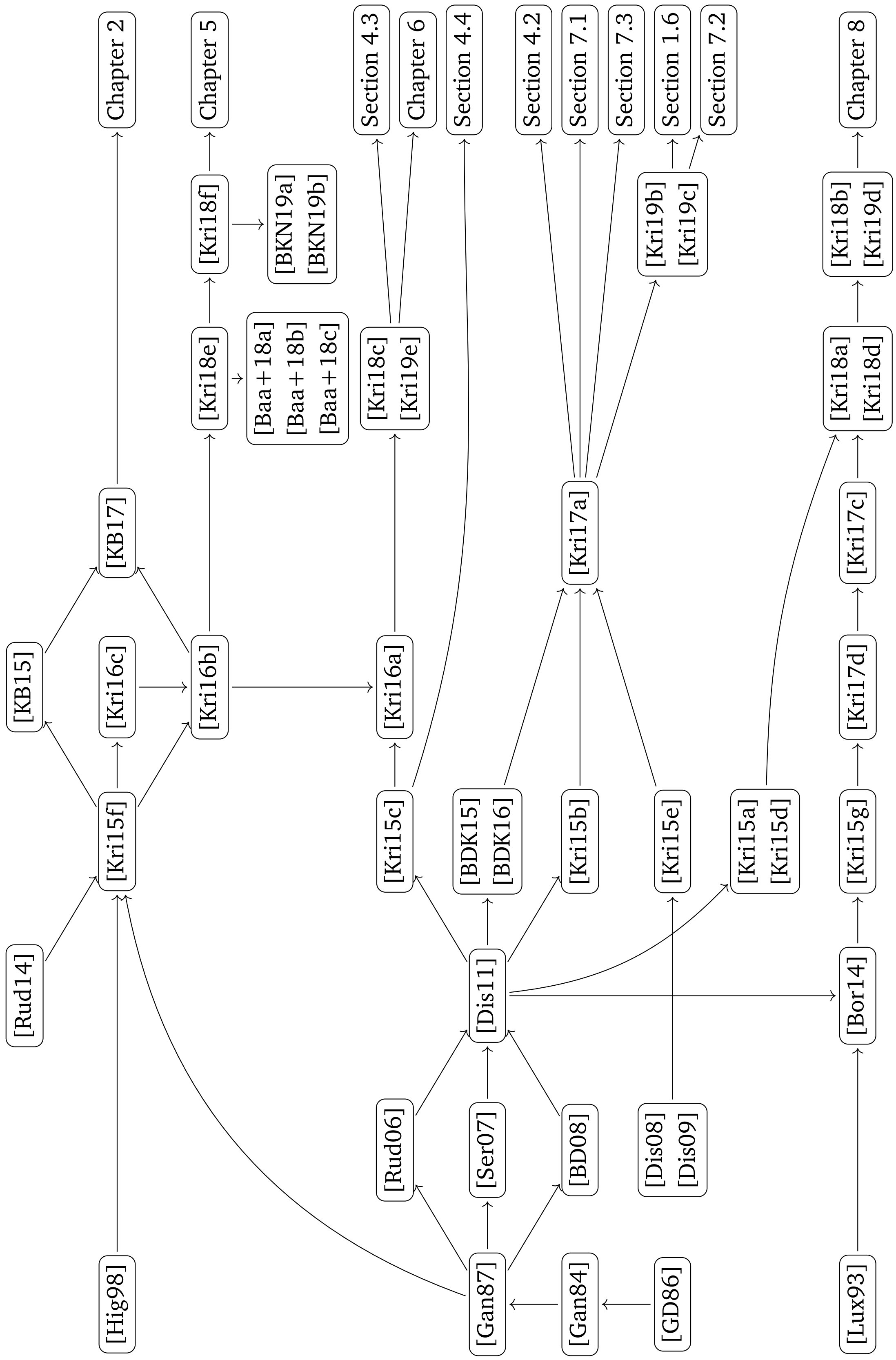
The following table lists all self-citations within this thesis.

	Directly Self-Cited References	**Indirectly Self-Cited References**
Section 1.6 and Section 7.2	[Kri19b; Kri19c]	—
Chapter 2	[KB17]	[KB15; Kri15f; Kri16b]
Section 4.1	—	[BDK15; BDK16]
Section 4.2	[Kri17a]	—
Section 4.3	[Kri18c; Kri19e]	[Kri16a]
Section 4.4	—	[Kri15c]
Chapter 5	[Kri18f]	[Kri18e]
Chapter 6	—	[Kri15c; Kri16a; Kri18c; Kri19e]
Sections 7.1 and 7.3	[Kri17a]	[Kri15b]
Chapter 8	[Kri17b; Kri18b; Kri19d]	—

In particular, this thesis contains text fragments from the above mentioned directly self-cited references. Since these referenced conference/workshop papers and journal articles have all been produced by the author of this thesis (with the exception [KB17], which is co-authored by Daniel Borchmann), such self-citations might not always be explicitly mentioned.

Graph of Influences and Evolution

During the research time prior to writing this thesis the author has published several conference/workshop papers and journal articles. The following graph displays the evolution of results that eventually lead to the chapters and sections within this thesis. Furthermore, influencing publications of other authors are mentioned and accordingly connected.


Chapter 2
Chapter 5
Section 4.3
Chapter 6
Section 4.4
Section 4.2
Section 7.1
Section 7.3
Section 1.6
Section 7.2
Chapter 8
[Kri18f]
[BKN19a]
[BKN19b]
[Kri19b]
[Kri19c]
[Kri18b]
[Kri19d]
[Kri18e]
[Baa+18a]
[Baa+18b]
[Baa+18c]
[Kri18c]
[Kri19e]
[Kri18a]
[Kri18d]
[KB17]
[Kri17a]
[Kri17c]
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[Kri15a]
[Kri15d]
[Kri15g]
[Rud14]
[Dis11]
[Bor14]
[Rud06]
[Ser07]
[BD08]
[Dis08]
[Dis09]
[Hig98]
[Gan87]
[Gan84]
[GD86]
[Lux93]

1 Formal Concept Analysis

We begin this section with a quote of the inventor RUDOLF WILLE of *Formal Concept Analysis* (abbrv. FCA) from his initial article [Wil82a].

> *"Lattice theory today reflects the general status of current mathematics: there is a rich production of theoretical concepts, results, and developments, many of which are reached by elaborate mental gymnastics; on the other hand, the connections of the theory to its surroundings are getting weaker and weaker, with the result that the theory and even many of its parts become more isolated. Restructuring lattice theory is an attempt to reinvigorate connections with our general culture by interpreting the theory as concretely as possible, and in this way to promote better communication between lattice theorists and potential users of lattice theory."* (RUDOLF WILLE)

This shows one of the major driving forces of the evolution of FCA: restructuring lattice theory and making it more accessible to potential users. The other major driving force for the evolution of FCA is thoroughly described in a later, more philosophical article [Wil05].

> *"The aim and meaning of Formal Concept Analysis as mathematical theory of concepts and concept hierarchies is to support the rational communication of humans by mathematically developing appropriate conceptual structures which can be logically activated."* (RUDOLF WILLE)

FCA provides a mathematical view on the philosophical notion of a *concept*, which is an object of thought. As such, the interpretation of a concept can differ between contexts of thinking. Moreover, a concept can be described in a certain context both extensionally and intentionally. While the extent of a concept is a collection of all objects belonging to that concept, the intent of a concept is a collection of all attributes that characterize it. In FCA, this gave rise to the notion of a *formal context* for describing a context of thinking. Put simply, it binds a set of objects to a set of attributes by specifying which object has which attributes. Such a formal context then fully describes the domain of interest by relating objects to their attributes. A *formal concept* is now a pair (A, B) consisting of a set A of objects and a set B of attributes such that A and B are both maximal w.r.t. the property that each object in A has all attributes in B. Put differently, whenever an arbitrary object has all attributes in B, then it is already contained in A, and if an arbitrary attribute is shared by all objects in A, then it is already contained in B. The computation of all formal concepts of a given formal context is a well-understood problem and many algorithms are available for computationally solving it. An application of FCA is thus *conceptual clustering* of data. One needs to suitably transform given data into a formal context, and then one enumerates all formal concepts. The clusters are then represented by the formal concepts. Furthermore, the formal concepts can be ordered, that is, a *"is subconcept of"* relation can be defined. It was shown that this ordering constitutes a complete lattice, which means that an arbitrary number of formal concepts always has a most general subconcept as well as a most specific superconcept. This leads to another important application of FCA: the *visualization* of data represented by a formal context. For this purpose, a *line diagram* of the lattice of formal concepts can be drawn in which each formal concept is displayed as a node and it is connected by an upwards directed edge to all its neighboring superconcepts. The hierarchy of the formal concepts is then easily read off from such a line diagram.

Another important notion in FCA is that of an (attribute) implication. Given a formal context and two attribute sets X and Y, it might be the case that each object having all attributes in X also has all attributes in Y. In such a case we say that the implication $X \to Y$ is *valid*. Exploring valid implications of a data set represented as a formal context can be an important utility for *data mining* applications. However, one should not try to naïvely enumerate all valid implications, since the resulting set would contain a huge amount of redundancy, meaning that some of the implications would already follow from others. Research on characterizing the theory of implications of a given formal context in a concise way has thus been conducted. On the one hand, there is the so-called *canonical base* [GD86] that provides a representation of the implicative theory of a formal context in a sound and complete manner with a minimal number of implications. The canonical base is sometimes also called Duquenne-Guigues-base. On the other hand, one might also think of a formal context containing only partial knowledge and for this purpose the interactive algorithm *Attribute Exploration* [Gan84; Gan87] has been devised by Ganter, which guides an expert through the process of axiomatizing the theory of implications that are valid in a domain of interest. In particular, Attribute Exploration is an interactive variant of an algorithm for computing canonical bases [Gan84; Gan87], and it works as follows. The input is a formal context that only partially describes the domain of interest, that is, there may be implications that are not valid, but for which this partial description does not provide a counterexample. During the run of the exploration process a minimal number of questions is enumerated and posed to the expert. Such a question is an implication for which no counterexample is present in the current formal context, and the expert can either confirm its validity or provide a suitable counterexample. If the implication is valid, then it is added to a list; otherwise the counterexample is inserted into the formal context. On termination, a minimal sound and complete representation of the theory of implications that are valid in the considered domain has been generated.

This Chapter 1 cites the standard notions and important results in the field of *Formal Concept Analysis* from [GW99] with the exception of Section 1.6 citing from [Kri19b; Kri19c], which introduces so-called *joining implications* and shows how these can be axiomatized from a given data-set.

1.1 Formal Contexts

A *formal context* $\mathbb{K} := (G, M, I)$ consists of a set G of *objects* (*Gegenstände* in German), a set M of *attributes* (*Merkmale* in German), and an *incidence relation* $I \subseteq G \times M$. For a pair $(g, m) \in I$, we say that *g has m*. The *derivation operators* of $\mathbb{K}$ are the mappings $\cdot^I\colon \wp(G) \to \wp(M)$ and $\cdot^I\colon \wp(M) \to \wp(G)$ such that for each object set $A \subseteq G$, the set A^I contains all attributes that are shared by all objects in A, and dually for each attribute set $B \subseteq M$, the set B^I contains all those objects that have all attributes from B. Formally, we define the derivation operators as follows.

$$A^I := \{ m \in M \mid (g,m) \in I \text{ for each } g \in A \} \quad \text{for object sets } A \subseteq G$$
$$B^I := \{ g \in G \mid (g,m) \in I \text{ for each } m \in B \} \quad \text{for attribute sets } B \subseteq M$$

For singleton sets, we may also use the abbreviations $g^I := \{g\}^I$ for all objects $g \in G$, as well as $m^I := \{m\}^I$ for all attributes $m \in M$.

It is well-known [GW99] that both derivation operators constitute a so-called *GALOIS connection* between the powersets $\wp(G)$ and $\wp(M)$, i.e., the following statements hold true for all subsets $A, C \subseteq G$ and $B, D \subseteq M$.

1. $A \subseteq B^I$ if, and only if, $B \subseteq A^I$ if, and only if, $A \times B \subseteq I$
2. $A \subseteq A^{II}$
3. $A^I = A^{III}$
4. $A \subseteq C$ implies $A^I \supseteq C^I$
5. $B \subseteq B^{II}$
6. $B^I = B^{III}$
7. $B \subseteq D$ implies $B^I \supseteq D^I$

For obvious reasons, formal contexts can be represented as binary tables the rows of which are labeled with the objects, the columns of which are labeled with the attributes, and the occurrence of a cross $\times$ in the cell at row g and column m indicates that the object g has the attribute m. Exemplary formal contexts are shown in Table 1.1.1 and later in Table 1.6.2.

1.2 Formal Concepts

A *formal concept* of a formal context $\mathbb{K} := (G, M, I)$ is a pair (A, B) consisting of a set $A \subseteq G$ of objects as well as a set $B \subseteq M$ of attributes such that $A^I = B$ and $B^I = A$. We then also refer to A as the *extent*, and to B as the *intent*, respectively, of (A, B). Another characterization of a formal concept is as follows: (A, B) is a formal concept of $\mathbb{K}$ if, and only if, $A \subseteq G$, $B \subseteq M$, and both A and B are maximal with respect to the property $A \times B \subseteq I$, i.e., $C \times B \not\subseteq I$ for each strict superset $C \supsetneq A$, and accordingly $A \times D \not\subseteq I$ for each strict superset $D \supsetneq B$. In the denotation of $\mathbb{K}$ as a cross table, those formal concepts are the maximal rectangles full of crosses (modulo reordering of rows and columns). Then, the set of all extents of $\mathbb{K}$ is symbolized as $\mathsf{Ext}(\mathbb{K})$, the set of all intents of $\mathbb{K}$ is denoted by $\mathsf{Int}(\mathbb{K})$, and the set of all formal concepts of $\mathbb{K}$ is denoted as $\mathfrak{B}(\mathbb{K})$. It is readily verified that the following equalities hold true.

$$\mathsf{Ext}(\mathbb{K}) = \{ A^{II} \mid A \subseteq G \} = \{ B^I \mid B \subseteq M \}$$
$$\mathsf{Int}(\mathbb{K}) = \{ B^{II} \mid B \subseteq M \} = \{ A^I \mid A \subseteq G \}$$
$$\mathfrak{B}(\mathbb{K}) = \{ (A^{II}, A^I) \mid A \subseteq G \} = \{ (B^I, B^{II}) \mid B \subseteq M \}$$

Formal concepts can be ordered: we say that (A, B) is a *subconcept* of (C, D), denoted by $(A, B) \leq (C, D)$, if $A \subseteq C$ holds true or, equivalently, if $B \supseteq D$ is satisfied. In [Wil82b; GW99]

$\mathbb{K}_1$	m_1	m_2	m_3	m_4	m_5	m_6	m_7
g_1	×	·	·	·	·	·	·
g_2	·	×	·	·	·	·	·
g_3	·	×	·	·	·	×	·
g_4	·	×	·	·	·	·	·
g_5	·	×	·	·	·	×	·
g_6	·	×	·	·	·	×	·
g_7	×	×	×	·	×	×	·
g_8	·	·	·	·	·	·	·
g_9	·	×	·	·	·	×	·
g_{10}	·	·	·	·	×	·	·
g_{11}	·	×	×	·	·	·	·
g_{12}	·	×	×	·	×	×	×
g_{13}	×	×	×	×	·	·	·

1.1.1 Table. An exemplary formal context $\mathbb{K}_1$

it was shown that this order always induces a complete lattice

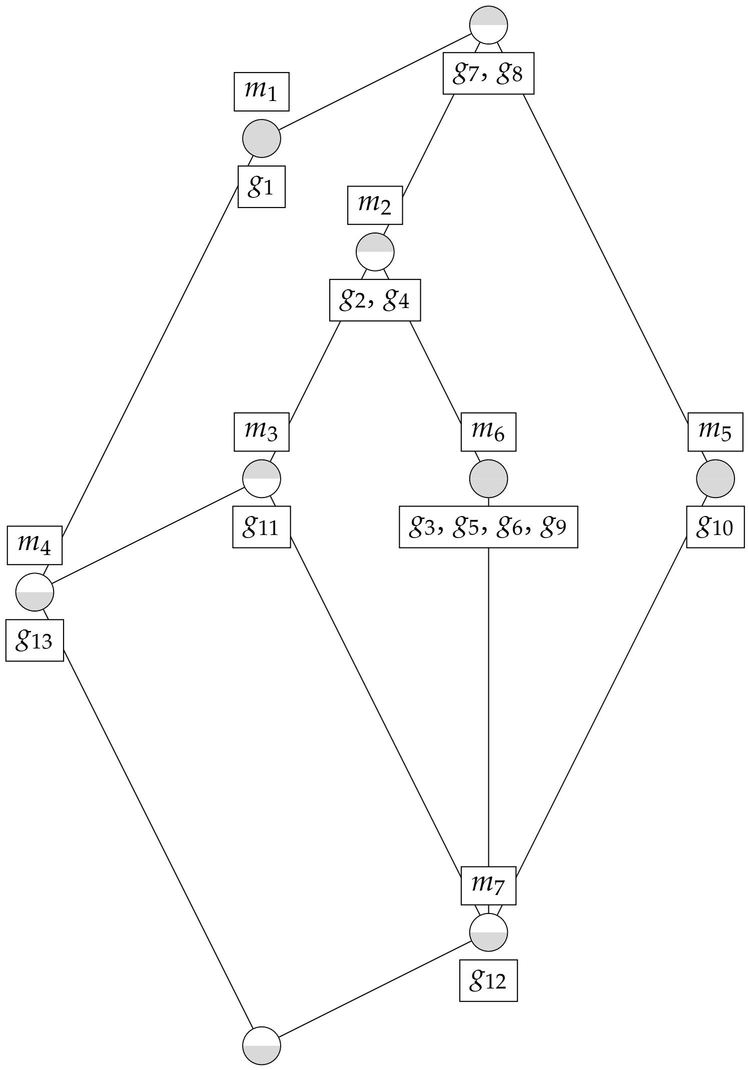
$$\underline{\mathfrak{B}}(\mathbb{K}) := (\mathfrak{B}(\mathbb{K}), \leq, \bigwedge, \bigvee, \top, \bot),$$

called the *concept lattice* of $\mathbb{K}$, in which the infimum and the supremum operation satisfy the equations

$$\bigwedge\{ (A_t, B_t) \mid t \in T \} = (\bigcap\{ A_t \mid t \in T \}, (\bigcup\{ B_t \mid t \in T \})^{II}),$$
$$\text{and} \quad \bigvee\{ (A_t, B_t) \mid t \in T \} = ((\bigcup\{ A_t \mid t \in T \})^{II}, \bigcap\{ B_t \mid t \in T \}),$$

and where $\top = (\emptyset^I, \emptyset^{II})$ is the greatest element and $\bot = (\emptyset^{II}, \emptyset^I)$ is the smallest element, respectively.

The number of formal concepts can be exponential in the size of the formal context. KUZNETSOV shows that determining this number is a #**P**-complete problem, cf. [Kuz01]. Furthermore, the problems of existence of a formal concept with restrictions on the size of the extent, intent, or both, respectively, are investigated in [Kuz01]—KUZNETSOV demonstrates that the existence of a formal concept (A, B) such that $|A| = k$, $|B| = k$, or $|A| + |B| = k$, respectively, are **NP**-complete problems; the similar problems with $\geq$ are all in **P**; and the problems with $\leq$ are also in **P**, except the problem where $|A| + |B| \leq k$ is **NP**-complete.



1.2.1 Figure. A line diagram of the concept lattice $\mathfrak{B}(\mathbb{K}_1)$ of the exemplary formal context in Table 1.1.1

Furthermore, the concept lattice of $\mathbb{K}$ can be nicely represented as a *line diagram* as follows: each formal concept is depicted as a vertex. Furthermore, there is an upward directed edge from each formal concept to its upper neighbors, i.e., to all those formal concepts which are greater with respect to $\leq$, but for which there is no other formal concept in between. The nodes are labeled as follows: an attribute $m \in M$ is an upper label of the *attribute concept* (m^I, m^{II}), and an object $g \in G$ is a lower label of the *object concept* (g^{II}, g^I). Then, the extent of the formal concept represented by a vertex consists of all objects which label vertices reachable by a downward directed path, and dually the intent is obtained by gathering all attribute labels of vertices reachable by an upward directed path. The concept lattice of the exemplary formal context from Table 1.1.1 is depicted in Figure 1.2.1, and Figure 1.6.3 shows the concept lattice of the formal context from Table 1.6.2.

1.3 Implications

An *implication* over M is an expression $X \to Y$ where $X, Y \subseteq M$. It is *valid* in $\mathbb{K}$, denoted as $\mathbb{K} \models X \to Y$, if $X^I \subseteq Y^I$, i.e., if each object of $\mathbb{K}$ that possesses all attributes in X also has all attributes in Y. An implication set $\mathcal{L}$ is *valid* in $\mathbb{K}$, denoted as $\mathbb{K} \models \mathcal{L}$, if all implications in $\mathcal{L}$ are valid in $\mathbb{K}$. Furthermore, the relation $\models$ is lifted to implication sets as follows: an implication set $\mathcal{L}$ *entails* an implication $X \to Y$, symbolized as $\mathcal{L} \models X \to Y$, if $X \to Y$ is valid in all formal contexts in which $\mathcal{L}$ is valid. More specifically, $\models$ is called the *semantic entailment relation*. We shall denote by $\mathsf{Imp}(M)$ the set of all implications over M.

A *model* of $X \to Y$ is an attribute set $Z \subseteq M$ such that $X \subseteq Z$ implies $Y \subseteq Z$, and we shall then write $Z \models X \to Y$. Of course, then an implication $X \to Y$ is valid in $\mathbb{K}$ if, and only if, for each object $g \in G$, the *object intent* g^I is a model of $X \to Y$. It is furthermore straightforward to verify that the following statements are equivalent.

1. $X \to Y$ is valid in $\mathbb{K}$.
2. Each object intent of $\mathbb{K}$ is a model of $X \to Y$.
3. Each intent of $\mathbb{K}$ is a model of $X \to Y$.
4. $Y \subseteq X^{II}$.

The equivalence between the first and the last statement indicates that X^{II} is the largest consequence of X in $\mathbb{K}$, i.e., $X \to X^{II}$ is valid in $\mathbb{K}$, and for each strict superset $W \supsetneq X^{II}$, the implication $X \to W$ is not valid in $\mathbb{K}$.

Consider an implication set $\mathcal{L} \subseteq \mathsf{Imp}(M)$. A *model* of $\mathcal{L}$ is an attribute set which is a model of each implication in $\mathcal{L}$. In particular, each model Z of $\mathcal{L}$ satisfies that $X \subseteq Z$ implies $Y \subseteq Z$ for each implication $X \to Y \in \mathcal{L}$, i.e., Z is a fixed point of the operator

$$Z \mapsto Z^{\mathcal{L}(1)} := Z \cup \bigcup \{\, Y \mid X \to Y \in \mathcal{L} \text{ and } X \subseteq Z \text{ for some } X \,\}.$$

We shall denote by $\mathsf{Mod}(\mathcal{L})$ the set of all models of $\mathcal{L}$. It is easy to verify that $\mathsf{Mod}(\mathcal{L})$ is closed under intersection. If we now consider some subset $Z \subseteq M$, then the intersection of all models of $\mathcal{L}$ that contain Z as a subset must clearly be the smallest model of $\mathcal{L}$ that contains Z. This smallest model is symbolized by $Z^{\mathcal{L}}$, and for a finite attribute set M it is obtained as the greatest fixed-point of successive exhaustive application of the operator $\cdot^{\mathcal{L}(1)}$, i.e.,

$$Z^{\mathcal{L}} = \bigcup \{\, Z^{\mathcal{L}(n)} \mid n \geq 1 \,\}$$

where $Z^{\mathcal{L}(n+1)} := (Z^{\mathcal{L}(1)})^{\mathcal{L}(n)}$ for all $n \geq 1$. Additionally, the following statements are equivalent.

1. $\mathcal{L}$ entails $X \to Y$.
2. Each model of $\mathcal{L}$ is a model of $X \to Y$.
3. $X \to Y$ is valid in all those formal contexts with attribute set M in which $\mathcal{L}$ is valid.
4. $Y \subseteq X^{\mathcal{L}}$.

We then infer that $X^{\mathcal{L}}$ is the largest consequence (the most specific consequence) of X with respect to the implication set $\mathcal{L}$, i.e., $\mathcal{L}$ entails $X \to X^{\mathcal{L}}$, and for all supersets $W \supsetneq X^{\mathcal{L}}$, the implication $X \to W$ does not follow from $\mathcal{L}$.

It was shown that entailment can also be decided *syntactically* by applying *deduction rules* to the implication set $\mathcal{L}$ without the requirement to consider all formal contexts in which $\mathcal{L}$ is valid, or all models of $\mathcal{L}$, respectively. Recall that an implication $X \to Y$ is *syntactically entailed* by an implication set $\mathcal{L}$, denoted as $\mathcal{L} \vdash X \to Y$, if $X \to Y$ can be constructed from $\mathcal{L}$ by the application of *inference axioms*, cf. [Mai83, Page 47], which are described as follows.

F1. *Reflexivity:* $\emptyset \vdash X \to X$

F2. *Augmentation:* $\{X \to Y\} \vdash X \cup Z \to Y$

F3. *Additivity:* $\{X \to Y, X \to Z\} \vdash X \to Y \cup Z$

F4. *Projectivity:* $\{X \to Y \cup Z\} \vdash X \to Y$

F5. *Transitivity:* $\{X \to Y, Y \to Z\} \vdash X \to Z$

F6. *Pseudo-Transitivity:* $\{X \to Y, Y \cup Z \to W\} \vdash X \cup Z \to W$

In the inference axioms above the symbols X, Y, Z, and W denote arbitrary subsets of the considered set M of attributes. Formally, we define $\mathcal{L} \vdash X \to Y$ if there is a finite sequence of implications $X_0 \to Y_0, \ldots, X_n \to Y_n$ such that the following conditions hold.

1. For each $i \in \{0, \ldots, n\}$, there is a subset $\mathcal{L}_i \subseteq \mathcal{L} \cup \{X_0 \to Y_0, \ldots, X_{i-1} \to Y_{i-1}\}$ such that $\mathcal{L}_i \vdash X_i \to Y_i$ matches one of the Axioms F1 – F6.
2. $X_n \to Y_n = X \to Y$.

Often, the Axioms F1, F2, and F6 are referred to as *Armstrong's axioms*. These three axioms constitute a *complete* and *independent* set of inference axioms for entailment, i.e., from it the other Axioms F3 – F5 can be derived, and none of them is derivable from the others.

The semantic entailment and the syntactic entailment coincide, i.e., an implication $X \to Y$ is semantically entailed by an implication set $\mathcal{L}$ if, and only if, $\mathcal{L}$ syntactically entails $X \to Y$, cf. [Mai83, Theorem 4.1 on Page 50; GW99, Proposition 21 on Page 81]. Consequently, we do not have to distinguish between both entailment relations $\models$ and $\vdash$ when it is up to decide whether an implication follows from a set of implications.

1.4 Implication Bases

Fix some formal context $\mathbb{K}$ as well as an implication set $\mathcal{S}$ such that $\mathbb{K} \models \mathcal{S}$. A *pseudo-intent* of $\mathbb{K}$ relative to $\mathcal{S}$ is an attribute set $P \subseteq M$ which is no intent of $\mathbb{K}$, but is a model of $\mathcal{S}$, and satisfies $Q^{II} \subseteq P$ for all pseudo-intents $Q \subsetneq P$. The set of all those pseudo-intents is symbolized by $\mathsf{PsInt}(\mathbb{K}, \mathcal{S})$. Then the implication set

$$\mathsf{Can}(\mathbb{K}, \mathcal{S}) := \{\, P \to P^{II} \mid P \in \mathsf{PsInt}(\mathbb{K}, \mathcal{S}) \,\}$$

constitutes an *implication base* of $\mathbb{K}$ relative to $\mathcal{S}$, i.e., for each implication $X \to Y$ over M, the following equivalence is satisfied.

$$\mathbb{K} \models X \to Y \text{ if, and only if, } \mathsf{Can}(\mathbb{K}, \mathcal{S}) \cup \mathcal{S} \models X \to Y$$

$\mathsf{Can}(\mathbb{K}, \mathcal{S})$ is called the *canonical base* of $\mathbb{K}$ relative to $\mathcal{S}$. It can be shown that it is a *minimal* implication base of $\mathbb{K}$ relative to $\mathcal{S}$, i.e., there is no implication base of $\mathbb{K}$ relative to $\mathcal{S}$ with smaller cardinality. Further information as well as proofs can be found in [GD86; Gan84; GW99, Section 2.3; Stu96, Section 2; Dis11, Theorem 3.8]. Sometimes we also say that an implication set is sound or complete for a formal context: in particular, $\mathcal{L}$ is *sound* for $\mathbb{K}$ if $\mathcal{L}$ is valid in $\mathbb{K}$ or, equivalently, if $\mathsf{Int}(\mathbb{K}) \subseteq \mathsf{Mod}(\mathcal{L})$ holds true, and $\mathcal{L}$ is *complete* for $\mathbb{K}$ if $\mathcal{L}$ entails each implication that is valid in $\mathbb{K}$ or, equivalently, if $\mathsf{Mod}(\mathcal{L}) \subseteq \mathsf{Int}(\mathbb{K})$ is satisfied.[1]

Example. For the given example of a formal context in Table 1.1.1, the canonical base with respect to the empty background knowledge $\emptyset$ contains the following implications.

$$\begin{aligned}
\{m_1, m_2\} &\to \{m_3\}, \\
\{m_1, m_2, m_3, m_4, m_5, m_6\} &\to \{m_7\}, \\
\{m_1, m_2, m_3, m_5, m_6, m_7\} &\to \{m_4\}, \\
\{m_1, m_5\} &\to \{m_2, m_3, m_6\}, \\
\{m_2, m_3, m_6\} &\to \{m_5\}, \\
\{m_2, m_5\} &\to \{m_3, m_6\}, \\
\{m_3\} &\to \{m_2\}, \\
\{m_4\} &\to \{m_1, m_2, m_3\}, \\
\{m_6\} &\to \{m_2\}, \\
\{m_7\} &\to \{m_2, m_3, m_5, m_6\}
\end{aligned}$$

$\triangle$

The most prominent algorithm for computing the canonical base is certainly *NextClosure* developed by Ganter [Gan84; Gan84]. Bazhanov and Obiedkov propose an optimized version of *NextClosure* in [BO14] which speeds up the computation of the lectically next closure, and furthermore they then perform some benchmarks to compare both versions. Additionally, they also utilize three different algorithms for computing closures with respect to implication sets, i.e., firstly the already presented and straightforward algorithm which computes the (least) fixed point of the operator $X \mapsto X^{\mathcal{L}(1)}$, see also [Mai83], secondly the *LinClosure* algorithm [BB79], which computes $X^{\mathcal{L}}$ in linear time, and thirdly *Wild's Closure* algorithm [Wil95], which is essentially an improved version of *LinClosure*. Please note that *LinClosure* is not always faster than computing the least fixed point of $X \mapsto X^{\mathcal{L}(1)}$, due to its initialization overhead. Furthermore, Obiedkov and Duquenne constitute an attribute-incremental algorithm for constructing the canonical base, cf. [OD07]. A parallel algorithm called *NextClosures* is also available [KB17], and an implementation is provided in *Concept Explorer FX* [Kri19a]; its advantage is that its

[1] Proof of the *only if* direction: Let $W \in \mathsf{Mod}(\mathcal{L})$. Then $\mathbb{K} \models W \to W^{II}$ holds true, which implies $\mathcal{L} \models W \to W^{II}$. We conclude that $W^{II} \subseteq W^{\mathcal{L}} = W$, i.e., $W \in \mathsf{Int}(\mathbb{K})$.

processing time scales almost inverse linear with respect to the number of available CPU cores. We shall describe the algorithm *NextClosures* in Section 2.1.

There are some important complexity problems related to the pseudo-intents and canonical bases. KUZNETSOV, and later together with OBIEDKOV, has proven in [Kuz04; KO06; KO08] that the number of pseudo-intents can be exponential in $|M|$ as well as in $|G| \cdot |M|$ or in $|I|$, and determining this number is #**P**-hard, furthermore that recognizing a pseudo-intent is in **coNP**, and that determining the number of non-pseudo-intents is #**P**-complete. SERTKAYA and DISTEL demonstrated in [Ser09a; Ser09b; Dis10; DS11] that the number of intents can be exponential in the number of pseudo-intents, i.e., the set of pseudo-intents cannot be enumerated in output-polynomial time by utilizing one of the existing algorithms, which all enumerate the closure system of both intents and pseudo-intents, and that the lectically first pseudo-intent can be computed in polynomial time, but recognizing the first n pseudo-intents is **coNP**-complete. Consequently, the pseudo-intents of a given formal context cannot be enumerated in the lectic order with polynomial delay, unless **P** = **NP**. Enumeration of pseudo-intents (in an arbitrary order) was also investigated, but concrete complexity results are outstanding. BABIN and KUZNETSOV showed in [BK10; BK13] that recognizing a pseudo-intent is **coNP**-complete, and furthermore that recognizing the lectically largest pseudo-intent is **coNP**-hard. Hence, computing pseudo-intents in the dual lectic order is also intractable, i.e., not possible with polynomial delay, unless **P** = **NP**. As a corollary BABIN and KUZNETSOV conclude that the maximal pseudo-intents cannot be enumerated with polynomial delay, unless **P** = **NP**. Further consequences which they found are, for example, that premises of minimal implication bases cannot be tractably recognized, since this problem is **coNP**-complete, and that there cannot be an algorithm that outputs a random pseudo-intent in polynomial time, unless **NP** = **coNP**.

Eventually, in case a given formal context is not complete in the sense that it does not contain enough objects to refute invalid implications, i.e., only contains some observed objects in the domain of interest, but one aims at exploring all valid implications over the given attribute set, a technique called *Attribute Exploration* can be utilized, which guides the user through the process of axiomatizing an implication base for the underlying domain such that the number of questions posed to the user is minimal. For a sophisticated introduction as well as for theoretical and technical details, the interested reader is rather referred to [Gan84; Stu96; Gan99; Gan84; Kri16c]. A parallel variant of *Attribute Exploration* also exists, cf. [Kri15f; Kri16c], which is implemented in *Concept Explorer FX* [Kri19a].

1.5 Closure Operators

We have seen in Section 1.3 that an implication $X \to Y$ is valid in a formal context $\mathbb{K}$ if, and only if, each intent of $\mathbb{K}$ is a model of $X \to Y$ or, equivalently, $Y \subseteq X^{II}$ is satisfied. Similarly, $X \to Y$ follows from some implication set $\mathcal{L}$ if, and only if, each model of $\mathcal{L}$ is a model of $X \to Y$ or, equivalently, $Y \subseteq X^{\mathcal{L}}$ holds true. Thus, validity of an implication in $\mathbb{K}$ can be characterized using the operator $\phi_{\mathbb{K}} \colon X \mapsto X^{II}$ while entailment of an implication from $\mathcal{L}$ can be characterized by means of the operator $\phi_{\mathcal{L}} \colon X \mapsto X^{\mathcal{L}}$. The commonality is that both are closure operators on M, a notion that is formally defined as follows. A *closure operator* on M is some mapping $\phi \colon \wp(M) \to \wp(M)$ with the following properties for all subsets $X, Y \subseteq M$.

1. $X \subseteq \phi(X)$ *(extensive)*
2. $X \subseteq Y$ implies $\phi(X) \subseteq \phi(Y)$ *(monotonic)*
3. $\phi(\phi(X)) = \phi(X)$ *(idempotent)*

Instead of $\phi(X)$ we often write X^ϕ. We shall call each X^ϕ a *closure* of ϕ, and the set of all closures is denoted by $\mathsf{Clo}(\phi)$. If we now define that an implication $X \to Y$ is *valid* for ϕ if each closure of ϕ is a model of $X \to Y$, which we denote by $\phi \models X \to Y$, then validity for $\phi_{\mathbb{K}}$ is the same as validity in $\mathbb{K}$ and, likewise validity for $\phi_{\mathcal{L}}$ is equivalent to entailment from $\mathcal{L}$.

We can now generalize the notion of a pseudo-intent by replacing the closure operator $\phi_{\mathbb{K}}$ with ϕ. In particular, assume that $\mathcal{S}$ is an implication set that is valid for ϕ; then a *pseudo-closure* for ϕ relative to $\mathcal{S}$ is a subset $P \subseteq M$ which is no closure of ϕ, but is a model of $\mathcal{S}$, and satisfies $Q^\phi \subseteq P$ for each pseudo-closure $Q \subsetneq P$. The set of all pseudo-closures is symbolized by $\mathsf{PsClo}(\phi, \mathcal{S})$. It is then straightforward to verify by generalizing the proofs in [GD86; Gan84; GW99, Section 2.3; Stu96, Section 2; Dis11, Theorem 3.8] that the implication set

$$\mathsf{Can}(\phi, \mathcal{S}) := \{\, P \to P^\phi \mid P \in \mathsf{PsClo}(\phi, \mathcal{S}) \,\},$$

called *canonical base* for ϕ relative to $\mathcal{S}$, is an *implication base* for ϕ relative to $\mathcal{S}$, i.e., $\phi \models X \to Y$ is equivalent to $\mathsf{Can}(\phi, \mathcal{S}) \cup \mathcal{S} \models X \to Y$ for each implication $X \to Y$, and further that $\mathsf{Can}(\phi, \mathcal{S})$ has *minimal cardinality* among all such implication bases.

A further generalization replaces the powerset lattice $(\wp(M), \subseteq, \bigcap, \bigcup, \emptyset, M)$ with an arbitrary complete lattice $\mathbf{M} := (M, \leq, \bigwedge, \bigvee, \bot, \top)$, i.e., $\leq$ is a partial order relation on M such that, for any subset $X \subseteq M$, the *infimum* $\bigwedge X$ as well as the *supremum* $\bigvee X$ exists, and further it holds true that $\bot$ is the smallest element and $\top$ is the greatest element. We do not refrain basic definitions from order theory and lattice theory here, and rather refer the interested reader to the following authors and references: Davey and Priestley [DP02], Ganter and Wille [GW99], Grätzer [Grä02], and Birkhoff [Bir40].

A *closure operator* in $\mathbf{M}$ is a mapping $\phi\colon M \to M$ that satisfies the following properties for all elements $x, y \in M$. Instead of $\phi(x)$ we shall write x^ϕ.

1. $x \leq x^\phi$ *(extensive)*
2. $x \leq y$ implies $x^\phi \leq y^\phi$ *(monotonic)*
3. $x^{\phi\phi} = x^\phi$ *(idempotent)*

The set of all closure operators in $\mathbf{M}$ is denoted by $\mathsf{ClOp}(\mathbf{M})$. The following notions can be easily generalized from the previous definitions: closure, implication, model (of an implication), validity (of an implication for a closure operator), entailment (of an implication from an implication set), the closure operator $\phi_{\mathcal{L}}\colon x \mapsto x^{\mathcal{L}}$ mapping x to the smallest model of $\mathcal{L}$ above x, pseudo-closure, implication base, canonical base. See also [Kri16b].

Further information on closure operators can be found in [Hig98; DP02; CM03], and we shall cite some important results in the sequel of this section. For any mapping $\phi\colon M \to M$, the following statements are equivalent.

1. ϕ is a closure operator on $\mathbf{M}$, i.e., $\phi \in \mathsf{ClOp}(\mathbf{M})$

2. $x \leq y^\phi$ if, and only if, $x^\phi \leq y^\phi$ for all $x, y \in M$
3. $x \vee y^{\phi\phi} \leq (x \vee y)^\phi$ for all $x, y \in M$
4. $x \leq x^\phi$ and $(x \vee y)^\phi = (x^\phi \vee y^\phi)^\phi$ for all $x, y \in M$

For every closure operator ϕ in $\mathbf{M}$, the following statements hold true.

1. $(x \wedge y)^\phi \leq x^\phi \wedge y^\phi$ for all $x, y \in M$
2. $(x^\phi \wedge y^\phi)^\phi = x^\phi \wedge y^\phi$ for all $x, y \in M$

A *closure system* in $\mathbf{M}$ is a $\bigwedge$-closed subset $P \subseteq M$, i.e., it holds true that $\bigwedge X \in P$ for each subset $X \subseteq P$. Note that the empty infimum $\bigwedge \emptyset$ in $\mathbf{M}$ equals $\top$, i.e., each closure system in $\mathbf{M}$ contains $\top$. A subset $P \subseteq M$ is a closure system in $\mathbf{M}$ if, and only if, $\{ p \in P \mid x \leq p \}$ has a smallest element for all $x \in M$. There exists a one-to-one-correspondence between closure operators and closure systems as follows. For every closure operator ϕ in $\mathbf{M}$, the set $\mathsf{Clo}(\phi)$ is a closure system in $\mathbf{M}$. For every closure system P in $\mathbf{M}$, the mapping

$$\phi_P \colon x \mapsto \bigwedge \{ p \mid p \in P \text{ and } x \leq p \}$$

is a closure operator in $\mathbf{M}$. Both operations are mutually inverse, i.e., $\phi_{\mathsf{Clo}(\phi)} = \phi$ for all closure operators ϕ, and $\mathsf{Clo}(\phi_P) = P$ for all closure systems P.

Indeed, closure operators can be ordered, cf. [Hig98; Rud14]. For closure operators ϕ and ψ in $\mathbf{M}$, we call ϕ *finer* than ψ and, dually, we call ψ *coarser* than ϕ, denoted as $\phi \trianglelefteq \psi$, if all ψ-closures are also ϕ-closed, that is, if $\mathsf{Clo}(\psi) \subseteq \mathsf{Clo}(\phi)$ holds true. It can be shown that the statements $\phi \trianglelefteq \psi$, $\phi \circ \psi = \psi$, and $\phi \leq \psi$ (pointwise order) are equivalent. As it turns out, the set of all closure operators in $\mathbf{M}$ ordered by $\trianglelefteq$ constitutes a complete lattice

$$\mathbf{ClOp}(\mathbf{M}) := (\mathsf{ClOp}(\mathbf{M}), \trianglelefteq, \triangle, \triangledown, \underline{\perp}, \overline{\top}).$$

In particular, every set Φ of closure operators in $\mathbf{M}$ has an infimum $\triangle\Phi$ as well as a supremum $\triangledown\Phi$ and these are given as follows.

$$\bigtriangleup \Phi \colon x \mapsto \bigwedge \{ x^\phi \mid \phi \in \Phi \}$$
$$\bigtriangledown \Phi \colon x \mapsto \bigwedge \{ y \mid x \leq y \text{ and } y = y^\phi \text{ for each } \phi \in \Phi \}$$

The finest closure operator is the identity mapping $\underline{\perp} \colon x \mapsto x$, and the coarsest closure operator is the constant mapping $\overline{\top} \colon x \mapsto \top$. It is easy to see that $\mathsf{Clo}(\triangle\Phi)$ is the smallest closure system that contains $\bigcup \{ \mathsf{Clo}(\phi) \mid \phi \in \Phi \}$, and that $\mathsf{Clo}(\triangledown\Phi)$ equals $\bigcap \{ \mathsf{Clo}(\phi) \mid \phi \in \Phi \}$.

1.6 Joining Implications

We are now interested in a restricted form of implications. In particular, we restrict the sets of attributes that may occur in the *premise* X and in the *conclusion* Y, respectively, of every implication $X \to Y$. Thus, let further M_p be a set of *premise attributes* and let M_c be a set of *conclusion attributes* such that $M_\mathsf{p} \cup M_\mathsf{c} \subseteq M$ holds true. Note that, in general, it is not required that the sets M_{p} and M_{c} form a partition of the attribute set. Both can overlap, one can be contained in the other, or their union can be a strict subset of the whole attribute set. For each $\mathsf{x} \in \{\mathsf{p}, \mathsf{c}\}$, we define the subcontext $\mathbb{K}_{\mathsf{x}} := (G, M_{\mathsf{x}}, I_{\mathsf{x}})$ where $I_{\mathsf{x}} := I \cap (G \times M_{\mathsf{x}})$. Furthermore, we may also write X^{x} instead of $X^{I_{\mathsf{x}}}$ for subsets $X \subseteq G$ or $X \subseteq M_{\mathsf{x}}$. Please note that then each pair $(\cdot^{\mathsf{x}}, \cdot^{\mathsf{x}})$ is a GALOIS connection between $(\wp(G), \subseteq)$ and $(\wp(M_{\mathsf{x}}), \subseteq)$, that is, similar statements like in Section 1.1 are valid.

1.6.1 Definition. A *joining implication* from M_{p} to M_{c}, or simply pc-*implication*, is an expression $X \to Y$ where $X \subseteq M_{\mathsf{p}}$ and $Y \subseteq M_{\mathsf{c}}$.[2] It is *valid* in $\mathbb{K}$, written $\mathbb{K} \models X \to Y$, if $X^{\mathsf{p}} \subseteq Y^{\mathsf{c}}$ holds true. △

$\mathbb{K}_{\text{illnesses}}$	Abrupt Onset	Fever	Aches	Chills	Fatigue	Sneezing	Cough	Stuffy Nose	Sore Throat	Headache	Cold	Flu
Bob	·	×	×	·	×	×	×	×	×	×	×	·
Alice	·	·	·	·	·	×	×	×	×	·	×	·
Tom	·	·	·	·	·	×	·	×	×	·	×	·
Julia	×	×	×	×	×	·	×	×	·	×	·	×
Keith	×	×	×	·	×	×	·	·	×	·	·	×
Wendy	×	×	×	×	×	×	×	·	·	×	·	×
	M_{p}										M_{c}	

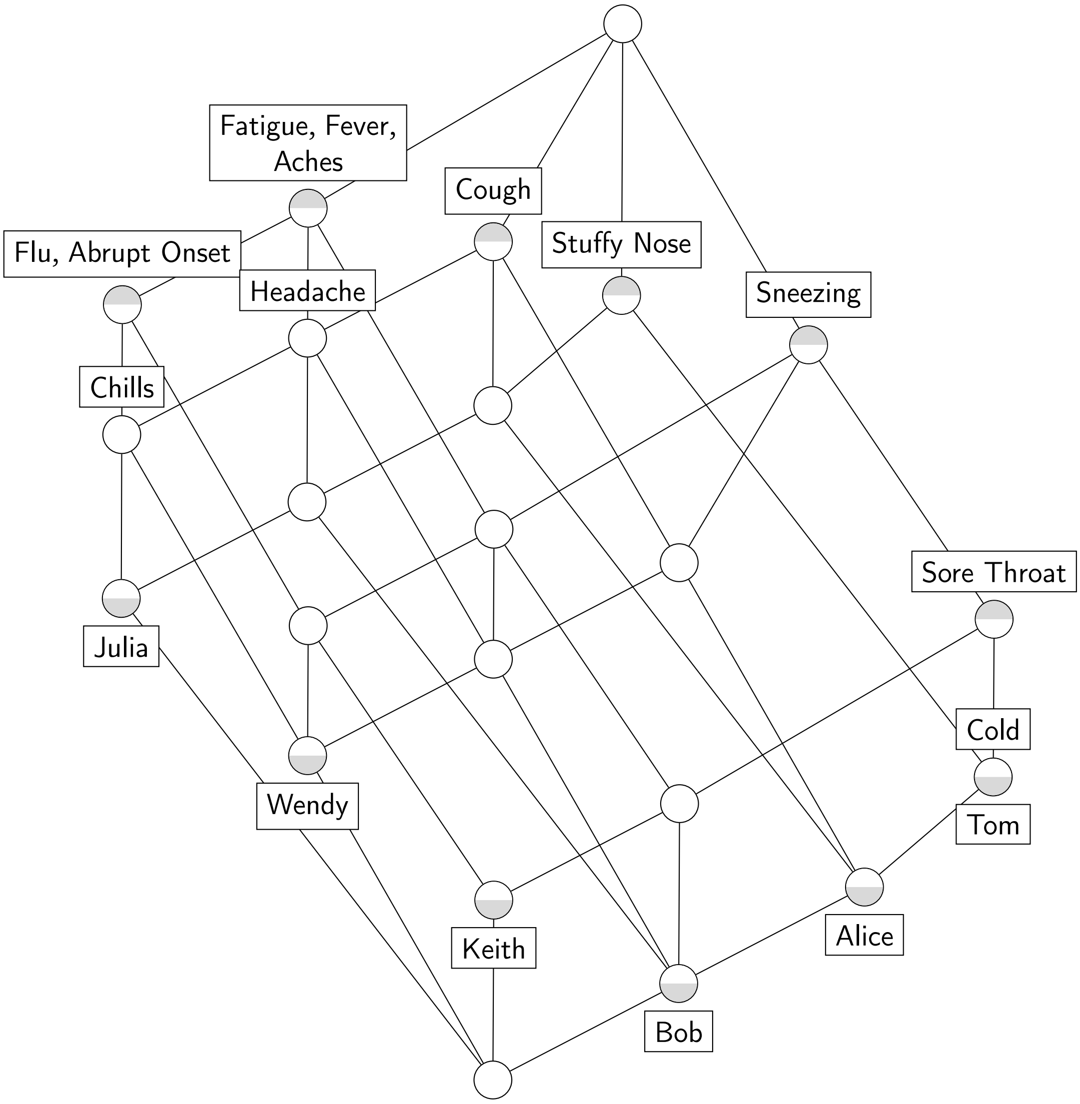
1.6.2 Table. The formal context $\mathbb{K}_{\text{illnesses}}$.

Example. Consider the formal context $\mathbb{K}_{\text{illnesses}}$ in Table 1.6.2. It considers six persons as objects and their symptoms and illnesses as attributes. Furthermore, we regard the symptoms as premise attributes and the illnesses as conclusion attributes. The concept lattice is displayed in Figure 1.6.3.

The expression {Cold, Cough} → {Chills} is no pc-implication, since the attribute Cold must not occur in a premise and, likewise, the attribute Chills must not occur in a conclusion. The expression {Sneezing, Cough, Stuffy Nose} → {Cold} is a well-formed joining implication and it is valid in $\mathbb{K}_{\text{illnesses}}$, since $\{\text{Sneezing, Cough, Stuffy Nose}\}^{\mathsf{p}} = \{\text{Bob, Alice}\}$ is a subset of $\{\text{Cold}\}^{\mathsf{c}} = \{\text{Bob, Alice, Tom}\}$. Furthermore, the expression {Abrupt Onset} → {Cold} is a

[2] SERGEI KUZNETSOV mentioned in his review on this thesis that joining implications have already been considered under different names in [Duq+03; Kuz05, Section 4]. However, the objectives of the respective authors were different than mine and, in particular, they have not demonstrated how minimal bases of joining implications can be characterized and computed.

well-formed joining implication as well, but it is not valid in $\mathbb{K}_{\mathsf{illnesses}}$, as $\{\mathsf{Abrupt\ Onset}\}^{\mathsf{p}} = \{\mathsf{Julia,\ Keith,\ Wendy}\}$ is not a subset of $\{\mathsf{Cold}\}^{\mathsf{c}} = \{\mathsf{Bob,\ Alice,\ Tom}\}$. △



1.6.3 Figure. The concept lattice of $\mathbb{K}_{\mathsf{illnesses}}$.

In the following, we shall characterize the set of all joining implications valid in $\mathbb{K}$. Of course, the pc-implication set

$$\mathsf{Imp}_{\mathsf{pc}}(\mathbb{K}) := \{\, X \to X^{\mathsf{pc}} \mid X \subseteq M_{\mathsf{p}} \,\}$$

contains only valid pc-implications and further entails any valid pc-implication, since $\mathbb{K} \models X \to Y$ is equivalent to $Y \subseteq X^{\mathsf{pc}}$ and so $\{X \to X^{\mathsf{pc}}\} \models X \to Y$. For the above joining implication set, we easily get that the operator $X \mapsto X^{\mathcal{L}(1)}$ from Page 8 for the case $\mathcal{L} = \mathsf{Imp}_{\mathsf{pc}}(\mathbb{K})$ equals

$$\phi_{\mathbb{K}}^{\mathsf{pc}} \colon X \mapsto X \cup (X \cap M_{\mathsf{p}})^{\mathsf{pc}}$$

for any $X \subseteq M$. This operator is idempotent, since the following holds true.

$$\begin{aligned}
&((X \cup (X \cap M_{\mathsf{p}})^{\mathsf{pc}}) \cap M_{\mathsf{p}})^{\mathsf{p}} \\
&= ((X \cap M_{\mathsf{p}}) \cup ((X \cap M_{\mathsf{p}})^{\mathsf{pc}} \cap M_{\mathsf{p}}))^{\mathsf{p}} \\
&= (X \cap M_{\mathsf{p}})^{\mathsf{p}} \cap ((X \cap M_{\mathsf{p}})^{\mathsf{pc}} \cap M_{\mathsf{p}})^{\mathsf{p}} \\
&= (X \cap M_{\mathsf{p}})^{I} \cap (\underbrace{(X \cap M_{\mathsf{p}})^{II} \cap M_{\mathsf{c}} \cap M_{\mathsf{p}}}_{\subseteq (X \cap M_{\mathsf{p}})^{II}})^{I} \\
&\supseteq (X \cap M_{\mathsf{p}})^{I} \cap (X \cap M_{\mathsf{p}})^{III} \\
&= (X \cap M_{\mathsf{p}})^{I} \\
&= (X \cap M_{\mathsf{p}})^{\mathsf{p}}
\end{aligned}$$

We conclude that $\phi_{\mathbb{K}}^{\mathsf{pc}}$ is the closure operator induced by the implication set $\mathsf{Imp}_{\mathsf{pc}}(\mathbb{K})$.

1.6.4 Definition. An implication set is *join-sound* or pc-*sound* if it is sound for $\phi_{\mathbb{K}}^{\mathsf{pc}}$, and it is *join-complete* or pc-*complete* if it is complete for $\phi_{\mathbb{K}}^{\mathsf{pc}}$. Fix some pc-sound implication set $\mathcal{S}$. A pc-implication set is called *joining implication base* or pc-*implication base* relative to $\mathcal{S}$ if it is pc-sound and its union with $\mathcal{S}$ is pc-complete. △

Obviously, the above $\mathsf{Imp}_{\mathsf{pc}}(\mathbb{K})$ is a joining implication base relative to $\emptyset$. Our aim for the sequel of this section is to find a *canonical* representation of the valid joining implications of some formal context, i.e., we shall provide a joining implication base that has *minimal* cardinality among all joining implication bases. For this purpose, we consider the canonical implication base of the above closure operator $\phi_{\mathbb{K}}^{\mathsf{pc}}$ and show how we can modify it to get a *canonical joining implication base*. We start with showing that we can rewrite any join-sound and join-complete implication set into a joining implication base in a certain normal form. For the remainder of this section, fix some arbitrary join-sound joining implication set $\mathcal{S}$ that is used as background knowledge.

1.6.5 Lemma. *Fix some join-sound implication set $\mathcal{L}$ over M. Further assume that $\mathcal{L} \cup \mathcal{S}$ is join-complete, and define the following set of joining implications.*

$$\mathcal{L}_{\mathsf{pc}} := \{\, X \cap M_{\mathsf{p}} \to (X \cap M_{\mathsf{p}})^{\mathsf{pc}} \mid X \to Y \in \mathcal{L} \,\}$$

Then, $\mathcal{L}_{\mathsf{pc}}$ is a joining implication base relative to $\mathcal{S}$.

Proof. Since $\mathcal{L}_{\mathsf{pc}} \subseteq \mathsf{Imp}_{\mathsf{pc}}(\mathbb{K})$ obviously holds true, we know that $\mathcal{L}_{\mathsf{pc}}$ is join-sound. For join-completeness we show that $\mathcal{L}_{\mathsf{pc}} \cup \mathcal{S} \models \mathsf{Imp}_{\mathsf{pc}}(\mathbb{K})$. Thus, consider some $Z \subseteq M_{\mathsf{p}}$. As $\mathcal{L} \cup \mathcal{S}$ is join-complete, it must hold true that $\mathcal{L} \cup \mathcal{S} \models Z \to Z^{\mathsf{pc}}$, that is, there are implications $X_1 \to Y_1$, …, $X_n \to Y_n$ in $\mathcal{L} \cup \mathcal{S}$ such that the following statements are satisfied.

$$\begin{aligned}
&X_1 \subseteq Z \\
&X_2 \subseteq Z \cup Y_1 \\
&X_3 \subseteq Z \cup Y_1 \cup Y_2 \\
&\quad\vdots
\end{aligned}$$

$$X_n \subseteq Z \cup Y_1 \cup Y_2 \cup \cdots \cup Y_{n-1}$$
$$Z^{\mathsf{pc}} \subseteq Z \cup Y_1 \cup Y_2 \cup \cdots \cup Y_{n-1} \cup Y_n$$

Let $L := \{\, k \mid k \in \{1,\ldots,n\} \text{ and } X_k \to Y_k \in \mathcal{L} \setminus \mathcal{S} \,\}$ and $S := \{1,\ldots,n\} \setminus L$. Since $\mathcal{L}$ is join-sound, we have $Y_k \subseteq X_k \cup (X_k \cap M_{\mathsf{p}})^{\mathsf{pc}}$ for each index $k \in L$. Define $X_{n+1} := Z^{\mathsf{pc}}$. An induction on $k \in \{1,\ldots,n+1\}$ shows the following.

$$\begin{aligned} X_k \subseteq Z &\cup \bigcup \{\, Y_i \mid i \in \{1,\ldots,k-1\} \cap S \,\} \\ &\cup \bigcup \{\, (X_i \cap M_{\mathsf{p}})^{\mathsf{pc}} \mid i \in \{1,\ldots,k-1\} \cap L \,\} \end{aligned}$$

Of course, $X_k \cap M_{\mathsf{p}} \subseteq X_k$ is satisfied for any index $k \in L$. We conclude that $\{\, X_k \to Y_k \mid k \in S \,\} \cup \{\, X_k \cap M_{\mathsf{p}} \to (X_k \cap M_{\mathsf{p}})^{\mathsf{pc}} \mid k \in L \,\}$ entails $Z \to Z^{\mathsf{pc}}$ and we are done. □

The transformation from Lemma 1.6.5 can now immediately be applied to the canonical implication base of the closure operator $\phi_{\mathbb{K}}^{\mathsf{pc}}$ to obtain a joining implication base, which we call *canonical*. This is due to fact that, by definition, $\mathsf{Can}(\phi_{\mathbb{K}}^{\mathsf{pc}})$ is both pc-sound and pc-complete.

1.6.6 Proposition. *The following is a joining implication base relative to* $\mathcal{S}$ *and is called* canonical joining implication base *or* canonical pc-implication base *of* $\mathbb{K}$ *relative to* $\mathcal{S}$.

$$\mathsf{Can}_{\mathsf{pc}}(\mathbb{K},\mathcal{S}) := \{\, P \cap M_{\mathsf{p}} \to (P \cap M_{\mathsf{p}})^{\mathsf{pc}} \mid P \in \mathsf{PsClo}(\phi_{\mathbb{K}}^{\mathsf{pc}},\mathcal{S}) \,\}$$

Proof. Note that $\phi_{\mathbb{K}}^{\mathsf{pc}}(P) = P \cup (P \cap M_{\mathsf{p}})^{\mathsf{pc}}$ holds true and, consequently, the canonical implication base for $\phi_{\mathbb{K}}^{\mathsf{pc}}$ relative to $\mathcal{S}$ evaluates to

$$\mathsf{Can}(\phi_{\mathbb{K}}^{\mathsf{pc}},\mathcal{S}) = \{\, P \to (P \cap M_{\mathsf{p}})^{\mathsf{pc}} \mid P \in \mathsf{PsClo}(\phi_{\mathbb{K}}^{\mathsf{pc}},\mathcal{S}) \,\}.$$

We already know that $\mathsf{Can}(\phi_{\mathbb{K}}^{\mathsf{pc}},\mathcal{S})$ is join-sound and its union with $\mathcal{S}$ is join-complete. Since $(\mathsf{Can}(\phi_{\mathbb{K}}^{\mathsf{pc}},\mathcal{S}))_{\mathsf{pc}} = \mathsf{Can}_{\mathsf{pc}}(\mathbb{K},\mathcal{S})$ holds true, an application of Lemma 1.6.5 shows that $\mathsf{Can}_{\mathsf{pc}}(\mathbb{K},\mathcal{S})$ is indeed a joining implication base relative to $\mathcal{S}$. □

Example. We continue with investigating our exemplary formal context $\mathbb{K}_{\mathsf{illnesses}}$. In order to compute the canonical joining implication base of it (relative to $\emptyset$), we first need to construct the canonical base of the closure operator $\phi_{\mathbb{K}_{\mathsf{illnesses}}}^{\mathsf{pc}}$. We get that $\mathsf{Can}(\phi_{\mathbb{K}_{\mathsf{illnesses}}}^{\mathsf{pc}},\emptyset)$ contains the following implications.[3]

$$\begin{aligned} \{\text{Headache, Sore Throat}\} &\to \{\text{Cold}\} \\ \{\text{Abrupt Onset}\} &\to \{\text{Flu}\} \\ \{\text{Sore Throat, Stuffy Nose}\} &\to \{\text{Cold}\} \\ \{\text{Flu, Sore Throat, Chills}\} &\to \{\text{Cold}\} \\ \{\text{Stuffy Nose, Sneezing}\} &\to \{\text{Cold}\} \\ \{\text{Chills}\} &\to \{\text{Flu}\} \end{aligned}$$

[3] The result has not been obtained by hand, but instead the implementation of the algorithm *NextClosures* [Kri16b] in *Concept Explorer FX* [Kri19a] has been utilized. Thus, no intermediate computation steps are provided.

$$\{\mathsf{Sore\ Throat},\ \mathsf{Cough}\} \to \{\mathsf{Cold}\}$$

Now applying the transformation from Lemma 1.6.5 yields the following set of joining implications, which is the canonical joining implication base. In particular, only the fourth implication is altered. We obtain that $\mathsf{Can}_{\mathsf{pc}}(\mathbb{K}_{\mathsf{illnesses}}, \emptyset)$ contains the following joining implications.

$$\begin{aligned}
\{\mathsf{Headache},\ \mathsf{Sore\ Throat}\} &\to \{\mathsf{Cold}\} \\
\{\mathsf{Abrupt\ Onset}\} &\to \{\mathsf{Flu}\} \\
\{\mathsf{Sore\ Throat},\ \mathsf{Stuffy\ Nose}\} &\to \{\mathsf{Cold}\} \\
\{\mathsf{Sore\ Throat},\ \mathsf{Chills}\} &\to \{\mathsf{Flu},\ \mathsf{Cold}\} \\
\{\mathsf{Stuffy\ Nose},\ \mathsf{Sneezing}\} &\to \{\mathsf{Cold}\} \\
\{\mathsf{Chills}\} &\to \{\mathsf{Flu}\} \\
\{\mathsf{Sore\ Throat},\ \mathsf{Cough}\} &\to \{\mathsf{Cold}\}
\end{aligned}$$

The canonical base $\mathsf{Can}(\mathbb{K}_{\mathsf{illnesses}}, \emptyset)$ of $\mathbb{K}_{\mathsf{illnesses}}$, which coincides with the canonical base of the induced closure operator $\phi_{\mathbb{K}_{\mathsf{illnesses}}}$, is as follows. Note that it is sound and complete for *all* implications valid in $\mathbb{K}_{\mathsf{illnesses}}$, i.e., no constraints on premises and conclusions are imposed.

$$\begin{aligned}
\{\mathsf{Fever}\} &\to \{\mathsf{Fatigue},\ \mathsf{Aches}\} \\
\{\mathsf{Sore\ Throat}\} &\to \{\mathsf{Sneezing}\} \\
\{\mathsf{Chills}\} &\to \left\{\begin{matrix}\mathsf{Headache},\ \mathsf{Flu},\ \mathsf{Fatigue},\ \mathsf{Cough}, \\ \mathsf{Fever},\ \mathsf{Aches},\ \mathsf{Abrupt\ Onset}\end{matrix}\right\} \\
\{\mathsf{Cold}\} &\to \{\mathsf{Sore\ Throat},\ \mathsf{Stuffy\ Nose},\ \mathsf{Sneezing}\} \\
\{\mathsf{Headache}\} &\to \{\mathsf{Fatigue},\ \mathsf{Cough},\ \mathsf{Fever},\ \mathsf{Aches}\} \\
\left\{\begin{matrix}\mathsf{Headache},\ \mathsf{Flu},\ \mathsf{Fatigue},\ \mathsf{Cough}, \\ \mathsf{Fever},\ \mathsf{Aches},\ \mathsf{Abrupt\ Onset}\end{matrix}\right\} &\to \{\mathsf{Chills}\} \\
\{\mathsf{Aches}\} &\to \{\mathsf{Fatigue},\ \mathsf{Fever}\} \\
\{\mathsf{Stuffy\ Nose},\ \mathsf{Sneezing}\} &\to \{\mathsf{Sore\ Throat},\ \mathsf{Cold}\} \\
\{\mathsf{Fatigue}\} &\to \{\mathsf{Fever},\ \mathsf{Aches}\} \\
\{\mathsf{Sore\ Throat},\ \mathsf{Sneezing},\ \mathsf{Cough}\} &\to \{\mathsf{Stuffy\ Nose},\ \mathsf{Cold}\} \\
\{\mathsf{Fatigue},\ \mathsf{Stuffy\ Nose},\ \mathsf{Fever},\ \mathsf{Aches}\} &\to \{\mathsf{Headache},\ \mathsf{Cough}\} \\
\{\mathsf{Fatigue},\ \mathsf{Cough},\ \mathsf{Fever},\ \mathsf{Aches}\} &\to \{\mathsf{Headache}\} \\
\{\mathsf{Abrupt\ Onset}\} &\to \{\mathsf{Flu},\ \mathsf{Fatigue},\ \mathsf{Fever},\ \mathsf{Aches}\} \\
\{\mathsf{Flu}\} &\to \{\mathsf{Fatigue},\ \mathsf{Fever},\ \mathsf{Aches},\ \mathsf{Abrupt\ Onset}\}
\end{aligned}$$

If we apply the transformation from Lemma 1.6.5 to $\mathsf{Can}(\mathbb{K}_{\mathsf{illnesses}}, \emptyset)$, then we obtain the following set of joining implications. Obviously, it is not complete, since it does not entail the valid joining implication $\{\mathsf{Headache},\ \mathsf{Sore\ Throat}\} \to \{\mathsf{Cold}\}$.

$$\{\mathsf{Chills}\} \to \{\mathsf{Flu}\}$$

$$\{\mathsf{Stuffy\ Nose,\ Sneezing}\} \to \{\mathsf{Cold}\}$$
$$\{\mathsf{Sore\ Throat,\ Sneezing,\ Cough}\} \to \{\mathsf{Cold}\}$$
$$\{\mathsf{Abrupt\ Onset}\} \to \{\mathsf{Flu}\} \qquad \triangle$$

We close this section with two further important properties of the canonical joining implication base. On the one hand, we shall show that it has minimal cardinality among all joining implication bases or, more generally, even among all join-sound, join-complete implication bases. On the other hand, we investigate the computational complexity of actually computing the canonical joining implication base.

1.6.7 Proposition. *The canonical joining implication base* $\mathsf{Can}_{\mathsf{pc}}(\mathbb{K}, \mathcal{S})$ *has minimal cardinality among all implication sets that are join-sound and have a union with* $\mathcal{S}$ *that is join-complete.*

Proof. Consider some implication set $\mathcal{L}$ such that $\mathcal{L} \cup \mathcal{S}$ is join-sound and join-complete. According to Lemma 1.6.5, we can assume that—without loss of generality—$\mathcal{L} \subseteq \mathsf{Imp}_{\mathsf{pc}}(\mathbb{K})$ holds true. In particular, note that $|\mathcal{L}_{\mathsf{pc}}| \leq |\mathcal{L}|$ is always true.

Join-soundness and join-completeness of $\mathcal{L} \cup \mathcal{S}$ yield that $\mathcal{L} \cup \mathcal{S}$ and $\mathsf{Can}(\phi_{\mathbb{K}}^{\mathsf{pc}}, \mathcal{S}) \cup \mathcal{S}$ are equivalent. It is well-known [Dis11, Theorem 3.8] that $\mathsf{Can}(\phi_{\mathbb{K}}^{\mathsf{pc}}, \mathcal{S})$ has minimal cardinality among all implication bases for $\phi_{\mathbb{K}}^{\mathsf{pc}}$ relative to $\mathcal{S}$, and so it follows that $|\mathcal{L}| \geq |\mathsf{Can}(\phi_{\mathbb{K}}^{\mathsf{pc}}, \mathcal{S})|$.

Clearly, the choice $\mathcal{L} := \mathsf{Can}_{\mathsf{pc}}(\mathbb{K}, \mathcal{S})$ implies $|\mathsf{Can}_{\mathsf{pc}}(\mathbb{K}, \mathcal{S})| \geq |\mathsf{Can}(\phi_{\mathbb{K}}^{\mathsf{pc}}, \mathcal{S})|$. It is further apparent that $|\mathsf{Can}_{\mathsf{pc}}(\mathbb{K}, \mathcal{S})| \leq |\mathsf{Can}(\phi_{\mathbb{K}}^{\mathsf{pc}}, \mathcal{S})|$ holds true and we infer that, in particular, $\mathsf{Can}_{\mathsf{pc}}(\mathbb{K}, \mathcal{S})$ and $\mathsf{Can}(\phi_{\mathbb{K}}^{\mathsf{pc}}, \mathcal{S})$ must contain the same number of implications. □

The next proposition shows that computing the canonical joining implication base is not more expensive than computing the canonical implication base where no constraints on the premises and conclusions must be satisfied. It uses the fact that canonical implication bases of closure operators can be computed using the algorithm *NextClosures*, cf. [Kri16b; KB17] or Chapter 2.

1.6.8 Proposition. *The canonical joining implication base can be computed in exponential time, and there exist formal contexts for which the canonical joining implication base cannot be encoded in polynomial space.*

Proof. The canonical implication base of the closure operator $\phi_{\mathbb{K}}^{\mathsf{pc}}$ relative to some background implication set $\mathcal{S}$ can be computed in exponential time by means of the algorithm *NextClosures*, cf. [KB17; Kri16b] and Section 2.1, which is easy to verify. The transformation of $\mathsf{Can}(\phi_{\mathbb{K}}^{\mathsf{pc}}, \mathcal{S})$ into $\mathsf{Can}_{\mathsf{pc}}(\mathbb{K}, \mathcal{S})$ as described in Lemma 1.6.5 can be done in polynomial time.

Kuznetsov and Obiedkov have shown in [KO08, Theorem 4.1] that the number of implications in the canonical implication base $\mathsf{Can}(\mathbb{K})$ of a formal context $\mathbb{K} := (G, M, I)$ can be exponential in $|G| \cdot |M|$. Clearly, if we let $\mathcal{S} := \emptyset$ and set both M_{p} and M_{c} to M, then $\mathsf{Can}(\mathbb{K})$ and $\mathsf{Can}_{\mathsf{pc}}(\mathbb{K}, \mathcal{S})$ coincide. □

We have seen in the running example that the canonical pc-implication base can be used to characterize implications between symptoms and diagnoses/illnesses. A further applications is, for instance, formal contexts encoding observations between attributes satisfied *yesterday* and *today*, i.e., we could construct the canonical base of pc-implications and then use it as a forecast stating which combinations of attributes being satisfied *today* would imply which combinations of attributes being satisfied *tomorrow*. In general, we could think of the premise attributes as *observable attributes* and the conclusion attributes as *goal/decision attributes*. By constructing the canonical pc-implication base from some formal context in which the goal/decision attributes have been manually assessed, we would obtain a set of rules with which we could analyze new data sets for which only the observable attributes are specified.

2 Parallel Computation of Canonical Bases

For each closure operator ϕ, the canonical base as introduced in Section 1.4 provides a sound and complete axiomatization of the implication theory of ϕ, and it further enjoys the property of being of minimal cardinality among all implication bases for ϕ. If background knowledge in form of an implication set $\mathcal{S}$ valid for ϕ is already present, then it can be incorporated in the construction of the canonical base, leading to a minimal extension of $\mathcal{S}$ that is then sound and complete for ϕ. This section is now concerned with developing an algorithm for computing such canonical bases. Later in this thesis, solutions for axiomatizing concept inclusions under different assumptions and in different description logics will be developed and, for each of these, we will reduce the respective axiomatization task to the task of axiomatizing implications of a suitable closure operator. Thus, such an algorithm is strongly required as a foundation for the techniques described later.

Of course, a suitable algorithm already exists: the algorithm *NextClosure* by GANTER [Gan84; Gan87]. However, it operates in a highly sequential fashion by enumerating the implications of the canonical base one by one (in the so-called lectic order). Nowadays computers possess many CPU cores, and so a procedure working in parallel by utilizing as many CPU cores as possible in order to lower computation time is desired. The algorithm *NextClosures* [KB15; Kri16b; KB17][1] solves this problem of constructing canonical bases in parallel, and we will describe it in the next Section 2.1. Afterwards, Section 2.2 shows benchmarks results and, in particular, demonstrates the speed–up gained by utilizing many CPU cores as well as shows that, on one CPU core, its performance is comparable to the existing *NextClosure* algorithm.

In [GW99, Proposition 26; Stu96, Corollary 4] it has been shown that, for each formal context $\mathbb{K}$ (and background knowledge $\mathcal{S}$), there is a closure operator ψ such that $\mathsf{Clo}(\psi) = \mathsf{Int}(\mathbb{K}) \cup \mathsf{PsInt}(\mathbb{K}, \mathcal{S})$ holds true. The very same also holds true for the (pseudo-)closures of a closure operator instead of the (pseudo-)intents of a formal context, and we shall describe in the following how this operator can be defined.

Fix some implication set $\mathcal{L}$. We have seen that, for each subset $Z \subseteq M$, the smallest model $Z^{\mathcal{L}}$ of $\mathcal{L}$ that is a superset of Z can be obtained by exhaustively saturating Z against the implications in $\mathcal{L}$. In particular, this saturation operator $Z \mapsto Z^{\mathcal{L}}$ has been defined as the repeated application of the one-step saturation operator $Z \mapsto Z^{\mathcal{L}(1)}$. We now define a similar operator $Z \mapsto Z^{\mathcal{L}^*(1)}$ but replace $X \subseteq Z$ with the stronger condition $X \subsetneq Z$, and further denote the

[1] After presenting the results in [KB15] at the conference venue, BAZIN mentioned that he had already published an algorithm for this task that is somehow similar. Checking his publication [BG13] showed that there are indeed some similarities but also obvious differences. In particular, Algorithm 2 in [BG13] uses a lectic order during computation with the goal to rule out some duplicate computations, since this lectic order induces a spanning tree of the set of intents and pseudo-intents to be computed. In contrast, Algorithm 1 in [KB15] is explicitly designed to perform large amounts of independent computation steps in parallel and further a thorough proof of its soundness and completeness is provided, i.e., it is shown that any computation order on parallel steps is admissible.

(greatest) fixed point of applying it starting from some subset $Z \subseteq M$ by $Z^{\mathcal{L}^*}$. More specifically, we define the monotonic operator

$$Z \mapsto Z^{\mathcal{L}^*(1)} := Z \cup \bigcup \{ Y \mid X \to Y \in \mathcal{L} \text{ and } X \subsetneq Z \text{ for some } X \}$$

and further set $Z^{\mathcal{L}^*} = \bigcup \{ Z^{\mathcal{L}^*(n)} \mid n \geq 1 \}$ where $Z^{\mathcal{L}^*(n+1)} := (Z^{\mathcal{L}^*(1)})^{\mathcal{L}^*(n)}$ for each $n \geq 1$. Obviously, the mapping $\phi_{\mathcal{L}^*} \colon Z \mapsto Z^{\mathcal{L}^*}$ is a closure operator. Now let ϕ be a closure operator and consider an implication set $\mathcal{S}$ that is valid for ϕ. It is not hard to verify that a subset $B \subseteq M$ is either a closure of ϕ or a pseudo-closure of ϕ relative to $\mathcal{S}$ if, and only if, B is a closure of the supremum $\phi_{\mathsf{Can}(\phi,\mathcal{S})^*} \triangledown \phi_{\mathcal{S}}$. For the case where the closure operator ϕ is induced by a formal context, this statement has already been proven by Stumme [Stu96]. In the following we shall use the abbreviation $\psi_{\phi,\mathcal{S}}$ for $\phi_{\mathsf{Can}(\phi,\mathcal{S})^*} \triangledown \phi_{\mathcal{S}}$.

The following text in Sections 2.1 and 2.2 is a citation of [KB17]. In particular, Section 2.1 is a modified copy of [KB17, Section 5] in which cross-references and citations have been updated, and the closure operator $\phi_{\mathbb{K}}$ induced by a formal context is replaced by an arbitrary closure operator ϕ where we only consider the case that the background knowledge $\mathcal{S}$ is valid for ϕ. This setting is suitable for the computation problems encountered in this thesis, more specifically in Sections 1.6, 6.6, 7.1, 7.2 and 8.1. Furthermore, Section 2.2 is an almost complete copy of [KB17, Section 6], again with some slight modifications in the notations.

2.1 The Algorithm *NextClosures*

The well-known *NextClosure* algorithm developed by Ganter [Gan84] can be used to enumerate the implications of the canonical base. The mathematical idea behind this algorithm is to compute all intents and pseudo-intents of a given formal context $\mathbb{K}$ in a certain linear order, namely the *lectic order*. As an advantage the next (pseudo-)intent is uniquely determined, but we potentially have to compute several candidate closures in order to find it. As we have seen above, those sets form a closure system, and the *NextClosure* algorithm uses the corresponding closure operator $\psi_{\phi_{\mathbb{K}},\emptyset}$, which equals $\phi_{\mathsf{Can}(\mathbb{K},\emptyset)^*}$, to enumerate the pseudo-intents (and the intents as a by-product) of $\mathbb{K}$ in the lectic order. Furthermore, this algorithm is inherently sequential, i.e., it is not possible to parallelize it.

In our approach we shall not make use of the lectic order. Indeed, our algorithm will enumerate all pseudo-closures of ϕ relative to $\mathcal{S}$ in the subset order, with no further restrictions. As a benefit we get a very easy and obvious way to parallelize this enumeration. Moreover, in multi-threaded implementations no communication between different threads is necessary. However, as it is the case with all other known algorithms for computing the canonical base, we also have to compute all closures of ϕ in addition to all pseudo-closures.

The main idea is very simple and works as follows. From the definition of pseudo-closures we see that in order to decide whether an attribute set $P \subseteq M$ is a pseudo-closure we only need all pseudo-closures $Q \subsetneq P$, i.e., it suffices to know all pseudo-closures with a smaller cardinality than P. This allows for the level-wise computation of the canonical base with respect to the subset order, i.e., we can enumerate the (pseudo-)closures with respect to increasing cardinality.

An algorithm that implements this idea works as follows. First we start by considering the empty set, as it is the only set of cardinality 0. Of course, it must either be a closure or a pseudo-closure, and the distinction can be made by checking whether $\emptyset = \emptyset^{\phi}$. Then assuming inductively that all pseudo-closures with cardinality $< k$ have been determined, we can correctly decide whether a subset $P \subseteq M$ with $|P| = k$ is a pseudo-closure or not.

To compute the set of all closures of ϕ and all pseudo-closures of ϕ relative to $\mathcal{S}$, the algorithm manages a set of *candidates* that contains the (pseudo-)closures on the current level. Then, whenever a pseudo-closure P has been found, the $\subseteq$-next closure is uniquely determined by its closure P^{ϕ}. If a closure B has been found, then the $\subseteq$-next (pseudo-)closures must be of the form $(B \cup \{m\})^{\psi_{\phi,\mathcal{S}}}$ for an attribute $m \notin B$. However, as we are not aware of the full implication base of ϕ relative to $\mathcal{S}$ yet, but only of an *approximation* $\mathcal{L}$ of it, the operators $\psi_{\phi,\mathcal{S}} = \phi_{\mathsf{Can}(\phi,\mathcal{S})^*} \triangledown \phi_{\mathcal{S}}$ and $\phi_{\mathcal{L}^*} \triangledown \phi_{\mathcal{S}}$ do not coincide on all subsets of M. We will show that they yield the same closure for attribute sets B with a cardinality $|B| \leq k$ if $\mathcal{L}$ contains all implications $P \to P^{\phi}$ where P is a pseudo-closure with a cardinality $|P| < k$. Consequently, the $\phi_{\mathcal{L}^*} \triangledown \phi_{\mathcal{S}}$-closure of a set $B \cup \{m\}$ may not be a closure or a pseudo-closure. Instead, they are added to the candidate list, and are processed when all pseudo-closures with smaller cardinality have been determined. We will formally prove that this technique is correct. Furthermore, the computation of all pseudo-closures and closures of cardinality k can be done in parallel, since they are independent of each other.

In summary, we can describe the inductive structure of the algorithm as follows. Let ϕ be a closure operator over a finite attribute set M, and let $\mathcal{S}$ be a set of background implications that are valid for ϕ. We use four variables: k denotes the current cardinality of candidates, $\mathsf{Candidates}$ is the set of candidates, $\mathsf{Closures}$ is a set of closures, and $\mathcal{L}$ is an implication set. Then the algorithm works as follows.

1. Set $k := 0$, $\mathsf{Candidates} := \{\emptyset\}$, $\mathsf{Closures} := \emptyset$, and $\mathcal{L} := \emptyset$.
2. In parallel: for each candidate set $C \in \mathsf{Candidates}$ with cardinality $|C| = k$, determine whether it is both a closure of $\phi_{\mathcal{L}^*}$ and a model of $\mathcal{S}$. If not, then add its closure[2] $C^{\mathcal{L}^* \triangledown \mathcal{S}}$ to the candidate set $\mathsf{Candidates}$, and go to Step 5.
3. If C is a closure of ϕ, then add C to $\mathsf{Closures}$. Otherwise, C must be a pseudo-closure of ϕ relative to $\mathcal{S}$, and thus we add the implication $C \to C^{\phi}$ to the set $\mathcal{L}$, and add the closure C^{ϕ} to the set $\mathsf{Closures}$.
4. For each observed closure C^{ϕ}, add all its upper neighbors $C^{\phi} \cup \{m\}$ where $m \notin C^{\phi}$ to the candidate set $\mathsf{Candidates}$.
5. Wait until all candidates of cardinality k have been processed. If $k < |M|$, then increase the candidate cardinality k by 1, and go to Step 2. Otherwise return $\mathsf{Closures}$ and $\mathcal{L}$.

In order to approximate the operator $\phi_{\mathcal{L}^*}$ we furthermore introduce the following notion: if $\mathcal{L}$ is a set of implications, then $\mathcal{L}{\restriction_k}$ denotes the subset of $\mathcal{L}$ that consists of all implications with premise cardinality not exceeding k.

2.1.1 Lemma. [KB17, Lemma 4] *For all attribute sets $X \subseteq M$, the following statements are equivalent.*

1. *X is either a closure of ϕ or a pseudo-closure of ϕ relative to $\mathcal{S}$.*

[2] We write $C^{\mathcal{L}^* \triangledown \mathcal{S}}$ instead of $C^{\phi_{\mathcal{L}^*} \triangledown \phi_{\mathcal{S}}}$.

2. *X is a closure of* $\phi_{\mathsf{Can}(\phi,\mathcal{S})^*} \nabla \phi_{\mathcal{S}}$.
3. *X is a closure of* $\phi_{(\mathsf{Can}(\phi,\mathcal{S})\restriction_{|X|-1})^*} \nabla \phi_{\mathcal{S}}$.

Proof. As mentioned in the beginning of Chapter 2, the equivalence of Statements 1 and 2 is a straightforward generalization of [GW99, Proposition 26; Stu96, Corollary 4]. Regarding the equivalence of Statements 2 and 3, the *if* direction follows from the fact that $P \subsetneq X$ implies $|P| \leq |X| - 1$, and the *only if* direction follows from $\mathsf{Can}(\phi, \mathcal{S})\restriction_{|X|-1} \subseteq \mathsf{Can}(\phi, \mathcal{S})$. □

As an immediate consequence of Lemma 2.1.1 we infer that in order to decide whether an attribute set X is a closure or a pseudo-closure it suffices to know all implications in the canonical base for which the premises have a lower cardinality than X. More specifically, if $\mathcal{L}$ contains all implications $P \to P^\phi$ where P is a pseudo-closure of ϕ relative to $\mathcal{S}$ with $|P| < k$, and otherwise only implications with premise cardinality k, then for all attribute sets $X \subseteq M$ with $|X| \leq k$, the following statements are equivalent.

1. X is either a closure of ϕ or a pseudo-closure of ϕ relative to $\mathcal{S}$.
2. X is a closure of $\phi_{\mathcal{L}^*}$ and a model of $\mathcal{S}$.

In a certain sense, this corollary allows us to approximate the set of all closures and pseudo-closures in the order of increasing cardinality, and thus also permits the approximation of the closure operator $\phi_{\mathsf{Can}(\phi,\mathcal{S})^*}$. In the following Lemma 2.1.2 we will characterize the structure of the set of all closures and pseudo-closures, and also give a method to compute upper neighbors. It is true that between comparable pseudo-closures there must always be a closure. In particular, the unique upper $\psi_{\phi,\mathcal{S}}$-closed neighbor of a pseudo-closure must be a closure.

2.1.2 Lemma. [KB17, Lemma 6] *The following statements hold true.*

1. *If $P \subseteq M$ is a pseudo-closure of ϕ relative to $\mathcal{S}$, then there is no closure or pseudo-closure strictly between P and P^ϕ.*
2. *If $B \subseteq M$ is a closure, then the next closures or pseudo-closures are of the form $(B \cup \{m\})^{\psi_{\phi,\mathcal{S}}}$ for attributes $m \notin B$.*
3. *If $X \subsetneq Y \subseteq M$ are neighboring $\psi_{\phi,\mathcal{S}}$-closures, then $Y = (X \cup \{m\})^{\psi_{\phi,\mathcal{S}}}$ for all attributes $m \in Y \setminus X$.*

Proof.

1. Assume that $P \in \mathsf{PsClo}(\phi, \mathcal{S})$. Then for every closure B between P and P^ϕ, i.e., $P \subseteq B \subseteq P^\phi$, we have that $B = B^\phi = P^\phi$. Thus, there cannot be a closure strictly between P and P^ϕ. Furthermore, if Q were a pseudo-closure such that $P \subsetneq Q \subseteq P^\phi$, then by definition of a pseudo-closure it follows that $P^\phi \subseteq Q$, which is an obvious contradiction. ↯
2. Let $B \subseteq M$ be a closure of ϕ. Consider a subset $X \supseteq B$ that is a closure or a pseudo-closure such that there is no other closure or pseudo-closure between them. Of course, then $B \subseteq B \cup \{m\} \subseteq X$ for all $m \in X \setminus B$. Consequently, $B = B^{\psi_{\phi,\mathcal{S}}} \subsetneq (B \cup \{m\})^{\psi_{\phi,\mathcal{S}}} \subseteq X^{\psi_{\phi,\mathcal{S}}} = X$. Then $(B \cup \{m\})^{\psi_{\phi,\mathcal{S}}}$ is a closure or a pseudo-closure between B and X which strictly contains B, and thus $(B \cup \{m\})^{\psi_{\phi,\mathcal{S}}} = X$.
3. Let $m \in Y \setminus X$. Then $X \cup \{m\} \subseteq Y$ implies that $X \subsetneq (X \cup \{m\})^{\psi_{\phi,\mathcal{S}}} \subseteq Y$, since Y is already closed. Consequently, $(X \cup \{m\})^{\psi_{\phi,\mathcal{S}}} = Y$. □

2.1.3 Algorithm. [KB17, Algorithm 1] *NextClosuresWithBackgroundKnowledge*

Input: a closure operator ϕ on M
Input: an implication set $\mathcal{S}$ over M such that $\phi \models \mathcal{S}$
Initialize: a candidate set $\mathsf{Candidates} := \{\emptyset\}$
Initialize: a closure set $\mathsf{Closures} := \emptyset$
Initialize: an implication set $\mathcal{L} := \emptyset$

```
for all k = 0,...,|M| do
  for all C ∈ Candidates with |C| = k do in parallel
    if C = C^{L*} and C = C^S then
      if C ≠ C^φ then
        L := L ∪ {C → C^φ}
      Closures := Closures ∪ {C^φ}
      Candidates := Candidates ∪ { C^φ ∪ {m} | m ∉ C^φ }
    else
      Candidates := Candidates ∪ {C^{L* ∇ S}}
  Wait for termination of all parallel processes.
```

Output: the set $\mathsf{Closures}$ of all closures of ϕ
Output: the canonical base $\mathcal{L}$ of ϕ relative to $\mathcal{S}$

We are now ready to formulate our algorithm *NextClosuresWithBackgroundKnowledge* in pseudo-code, see Algorithm 2.1.3. In the remainder of this section we shall show that this algorithm always terminates—of course, under the assumption that ϕ is computable and that the attribute set M is finite— and that it returns the canonical base of ϕ relative to $\mathcal{S}$ as well as the set of all closures of ϕ. Beforehand, let us introduce the following notation.

1. Algorithm 2.1.3 is *in state k* (where $-1 \leq k \leq |M|$) if it has processed all candidate sets with a cardinality $\leq k$, but none of cardinality $> k$.
2. $\mathsf{Candidates}_k$ denotes the set of candidates in state k.
3. $\mathcal{L}_k$ denotes the set of implications in state k.
4. $\mathsf{Closures}_k$ denotes the set of closures in state k.

2.1.4 Proposition. [KB17, Proposition 7] *Assume that Algorithm 2.1.3 has been started on inputs ϕ and $\mathcal{S}$ and is in state k. Then, the following statements are true.*

1. $\mathsf{Candidates}_k$ *contains all pseudo-closures with cardinality $k+1$, and contains all closures with cardinality $k+1$ that are not yet in* $\mathsf{Closures}_k$.
2. $\mathsf{Closures}_k$ *contains all closures with cardinality $\leq k$.*
3. $\mathcal{L}_k$ *contains all implications $P \to P^\phi$ where the premise P is a pseudo-closure with cardinality $\leq k$.*
4. *Between the states k and $k+1$ an attribute set with cardinality $k+1$ is a closure or a pseudo-closure if, and only if, it is both a closure of $\phi_{\mathcal{L}^*}$ and a model of $\mathcal{S}$.*

Proof. We prove the statements by induction on k. The base case handles the initial state $k = -1$. Of course, $\emptyset$ is always a closure or a pseudo-closure. Furthermore, it is contained in the candidate set Candidates. As there are no sets with cardinality ≤ -1, Closures_{-1} and $\mathcal{L}_{-1}$ trivially satisfy Statements 2 and 3, respectively. Finally, we have that $\mathcal{L}_{-1} = \emptyset$, and hence every attribute set is $\phi_{\mathcal{L}^*_{-1}}$-closed, in particular $\emptyset$.

We now assume that the induction hypothesis is true for k. For every implication set $\mathcal{L}$ between states k and $k+1$, i.e., $\mathcal{L}_k \subseteq \mathcal{L} \subseteq \mathcal{L}_{k+1}$, the induction hypothesis yields that $\mathcal{L}$ contains all implications $P \to P^{\phi}$ where P is a pseudo-closure with cardinality $\leq k$, and furthermore only implications the premises of which have cardinality $k+1$ (by definition of Algorithm 2.1.3). Additionally, we know that the candidate set Candidates_k contains all pseudo-closures P where $|P| = k+1$, and all closures B such that $|B| = k+1$ and $B \notin \text{Closures}_k$. Lemma 2.1.1 immediately yields the validity of Statements 2 and 3 for $k+1$, as those $\psi_{\phi,\mathcal{S}}$-closures are recognized correctly in Line 3. Then $\mathcal{L}_{k+1}$ contains all implications $P \to P^{\phi}$ where P is a pseudo-closure with $|P| \leq k+1$, and hence each implication set $\mathcal{L}$ with $\mathcal{L}_{k+1} \subseteq \mathcal{L} \subseteq \mathcal{L}_{k+2}$ contains all those implications, too, and furthermore only implications with a premise cardinality $k+2$. By another application of Lemma 2.1.1 we conclude that also Statement 4 is satisfied for $k+1$.

Finally, we show Statement 1 for $k+1$. Consider any $\psi_{\phi,\mathcal{S}}$-closed set X where $|X| = k+2$. Then Lemma 2.1.2 states that for all lower $\psi_{\phi,\mathcal{S}}$-neighbors Y and all $m \in X \setminus Y$ it is true that $(Y \cup \{m\})^{\psi_{\phi,\mathcal{S}}} = X$. We proceed with a case distinction.

If there is a lower $\psi_{\phi,\mathcal{S}}$-neighbor Y which is a pseudo-closure, then Lemma 2.1.2 yields that the (unique) next $\psi_{\phi,\mathcal{S}}$-neighbor is obtained as Y^{ϕ}, and the closure Y^{ϕ} is added to the set Closures in Line 6. Of course, it is true that $X = Y^{\phi}$.

Otherwise all lower $\psi_{\phi,\mathcal{S}}$-neighbors Y are closures, and in particular this is the case for X being a pseudo-closure, cf. Lemma 2.1.2. Then for all these Y we have $(Y \cup \{m\})^{\psi_{\phi,\mathcal{S}}} = X$ for all $m \in X \setminus Y$. Furthermore, all sets Z with $Y \cup \{m\} \subsetneq Z \subsetneq X$ are not $\psi_{\phi,\mathcal{S}}$-closed. Since $X \setminus Y$ is finite, the following sequence must also be finite:

$$C_0 := Y \cup \{m\} \quad \text{and} \quad C_{i+1} := C_i^{\mathcal{L}^* \nabla \mathcal{S}} \quad \text{where} \quad \mathcal{L}_{|C_i|-1} \subseteq \mathcal{L} \subseteq \mathcal{L}_{|C_i|}.$$

The sequence is well-defined, since implications from $\mathcal{L}_{|C_i|} \setminus \mathcal{L}_{|C_i|-1}$ have no influence on the closure of C_i. Furthermore, the sequence obviously ends with the set X, and contains no further $\psi_{\phi,\mathcal{S}}$-closed sets, and each of the sets $C_0, C_1, \ldots$ appears as a candidate during the run of the algorithm, cf. Lines 7 and 9. □

From the previous result we can infer that in the last state $|M|$ the set Closures contains all closures of ϕ, and that $\mathcal{L}$ is the canonical base of ϕ relative to $\mathcal{S}$. Both sets are returned from Algorithm 2.1.3, and hence we can conclude that *NextClosuresWithBackgroundKnowledge* is sound and complete. The following corollary summarizes our results obtained so far, and also shows termination.

2.1.5 Corollary. [KB17, Corollary 8] *If Algorithm 2.1.3 is started on a computable closure operator ϕ over a finite attribute set and an implication set $\mathcal{S}$ (where $\phi \models \mathcal{S}$) as inputs, then it terminates, and returns both the set of all closures of ϕ as well as the canonical base of ϕ relative to $\mathcal{S}$ as outputs.*

Proof. The second part of the statement is a direct consequence of Proposition 2.1.4. Finally, the computation time between states k and $k+1$ is finite, because there are only finitely many candidates of cardinality $k+1$, and the computation of closures of the operators $\phi_{\mathcal{L}^*}$, $\phi_{\mathcal{S}}$, and ϕ, can be done in finite time. As there are exactly $|M|$ states, the algorithm must terminate. □

One could ask whether there are closure operators that do not allow for a speed-up in the enumeration of all closures and pseudo-closures on parallel execution. This would happen for a closure operator for which the closures and pseudo-closures are linearly ordered, i.e., form a chain with respect to the subset inclusion order. However, the next Lemma 2.1.6 shows that this is impossible.

Note that a formal context $\mathbb{K} := (G, M, I)$ is *clarified* if $\{g\}^I = \{h\}^I$ implies $g = h$ for all objects $g, h \in G$, and dually $\{m\}^I = \{n\}^I$ implies $m = n$ for all attributes $m, n \in M$.

2.1.6 Lemma. [KB17, Lemma 9] *For each non-empty clarified formal context, the set of its intents and pseudo-intents is not linearly ordered with respect to the subset inclusion order.*

Proof. Assume that $\mathbb{K} := (G, M, I)$ with $G := \{g_1, \ldots, g_n\}$, $n > 0$, were a clarified formal context with (pseudo-)intents $P_1 \subsetneq P_2 \subsetneq \ldots \subsetneq P_\ell$. In particular, then all object intents form a chain $g_1^I \subsetneq g_2^I \subsetneq \ldots \subsetneq g_n^I$ where $n \leq \ell$. Since $\mathbb{K}$ is clarified, it follows $|g_{j+1}^I \setminus g_j^I| = 1$ for all j, and hence w.l.o.g. $M = \{m_1, \ldots, m_n\}$, and $g_i \; I \; m_j$ if, and only if, $i \geq j$. Hence, $\mathbb{K}$ is isomorphic to the ordinal scale $\mathbb{K}_n := (\{1, \ldots, n\}, \{1, \ldots, n\}, \leq)$. It is easy to verify that the pseudo-intents of $\mathbb{K}_n$ are either $\emptyset$, or of the form $\{m, n\}$ where $m < n-1$, a contradiction. ↯ □

In order to relate the above result to closure operators, we note that there is a strong correspondence between closure operators and formal contexts as follows. If ϕ is a closure operator on M, then $\mathbb{K}_\phi := (\mathsf{Clo}(\phi), M, \ni)$ is a formal context such that $\mathsf{Int}(\mathbb{K}_\phi) = \mathsf{Clo}(\phi)$ (and hence $\mathsf{PsInt}(\mathbb{K}_\phi, \mathcal{S}) = \mathsf{PsClo}(\phi, \mathcal{S})$ as well). For the converse direction, we have already seen that each formal context $\mathbb{K}$ with attribute set M induces the closure operator $\phi_{\mathbb{K}} \colon X \mapsto X^{II}$ on M and it holds true that $\mathsf{Clo}(\phi_{\mathbb{K}}) = \mathsf{Int}(\mathbb{K})$ (and thus also $\mathsf{PsClo}(\phi_{\mathbb{K}}, \mathcal{S}) = \mathsf{PsInt}(\mathbb{K}, \mathcal{S})$).

Consequently, there is no closure operator with a linearly ordered set of closures and pseudo-closures. Hence, a parallel enumeration of the closures and pseudo-closures will always result in a speed-up compared to a sequential enumeration.

In the case where no background knowledge is available, i.e., $\mathcal{S} = \emptyset$, we can easily see that Algorithm 2.1.3 may be simplified to Algorithm 2.1.7 which computes the canonical base of a closure operator ϕ, as it has been described for the case of a formal context in [KB15].

2.2 Benchmarks

The purpose of this section is to show that our parallel Algorithms 2.1.3 and 2.1.7 for computing the canonical base indeed yield a speed-up, both qualitatively and quantitatively, compared to the classical algorithm *NextClosure* [Gan84]. To this end, we shall present the running times of our algorithm *NextClosures* when applied to selected data sets and with a varying number of available CPU cores. We shall see that, up to a certain limit, the running time of our algorithm decreases proportional to the number of available CPU cores. Furthermore, we shall also show that this speed-up is not only qualitative, but indeed yields a real speed-up compared to the original sequential algorithm *NextClosure* for computing the canonical base.

2.1.7 Algorithm. [KB17, Algorithm 2] *NextClosures*

Input: a closure operator ϕ on M
Initialize: a candidate set $\mathsf{Candidates} := \{\emptyset\}$
Initialize: a closure set $\mathsf{Closures} := \emptyset$
Initialize: an implication set $\mathcal{L} := \emptyset$
1 `for all` $k = 0, \ldots, |M|$ `do`
2 `for all` $C \in \mathsf{Candidates}$ `with` $|C| = k$ `do in parallel`
3 `if` $C = C^{\mathcal{L}^*}$ `then`
4 `if` $C \neq C^{\phi}$ `then`
5 $\mathcal{L} := \mathcal{L} \cup \{C \to C^{\phi}\}$
6 $\mathsf{Closures} := \mathsf{Closures} \cup \{C^{\phi}\}$
7 $\mathsf{Candidates} := \mathsf{Candidates} \cup \{\, C^{\phi} \cup \{m\} \mid m \notin C^{\phi} \,\}$
8 `else`
9 $\mathsf{Candidates} := \mathsf{Candidates} \cup \{C^{\mathcal{L}^*}\}$
10 Wait for termination of all parallel processes.
Output: the set $\mathsf{Closures}$ of all closures of ϕ
Output: the canonical base $\mathcal{L}$ of ϕ

The presented algorithms *NextClosures* and *NextClosuresWithBackgroundKnowledge* have been integrated into *Concept Explorer FX* [Kri19a]. The implementations are straightforward adaptations of Algorithms 2.1.3 and 2.1.7 to the programming language Java 8, and heavily use the new Stream API and thread-safe concurrent collection classes (like `ConcurrentHashMap`). As we have described before, the processing of all candidates on the current cardinality level can be done in parallel, i.e., for each of them a separate thread is started that executes the necessary operations for Lines 3–9 in Algorithms 2.1.3 and 2.1.7, respectively. Furthermore, as the candidates on the same level cannot affect each other, no communication between the threads is needed. More specifically, we have seen that the decision whether a candidate is an closure or a pseudo-closure is independent of all other sets with the same or a higher cardinality.

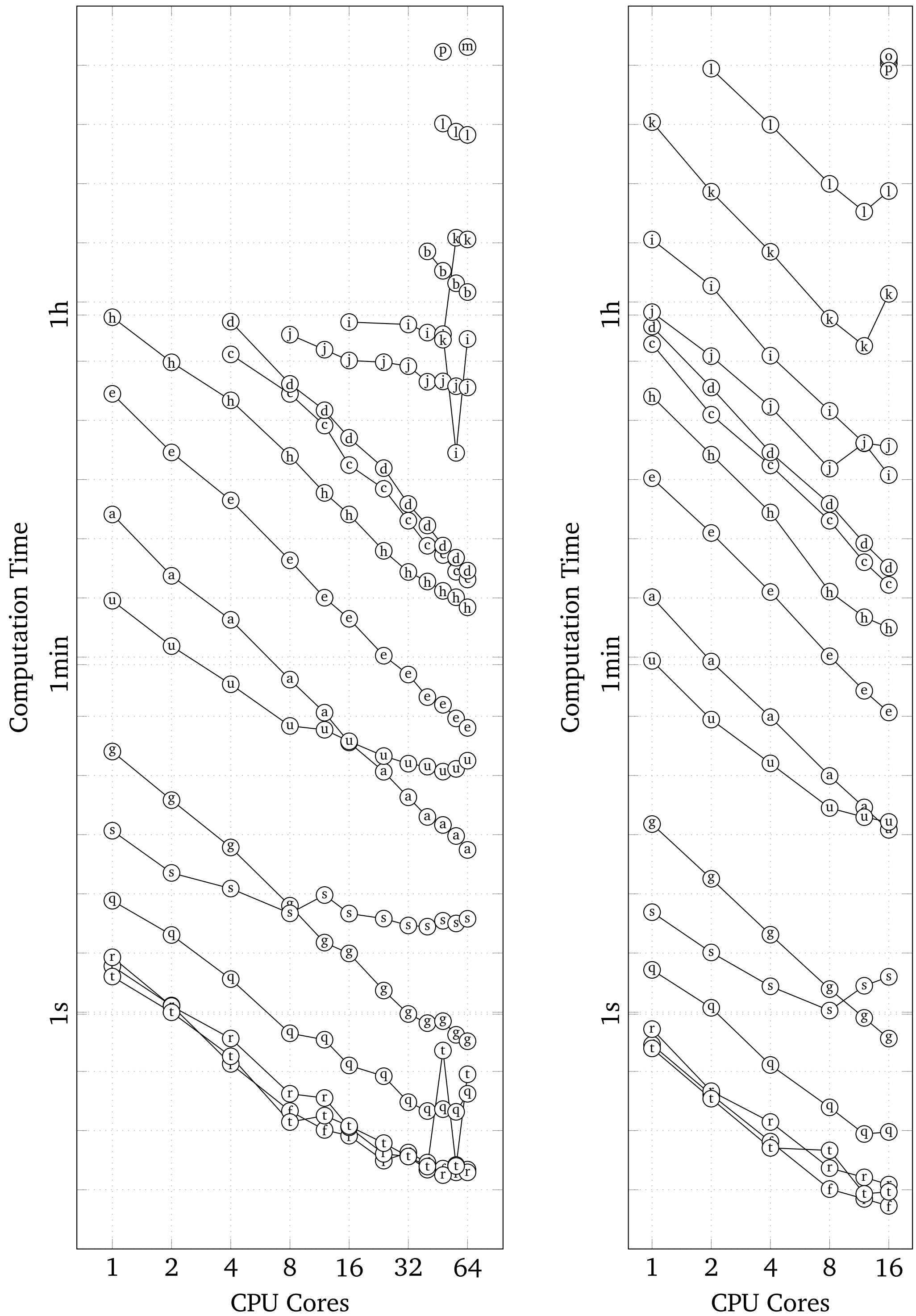
The formal contexts used for the benchmarks[3] are listed in Table 2.2.1, and are either obtained from the *FCA Data Repository* [FCADR] (ⓐ to ⓓ, and ⓕ to ⓟ), randomly created (ⓠ to ⓤ), or created from experimental results (ⓔ). For each of them we executed the implementation at least three times, and recorded the average computation times. The experiments were performed on the following two systems:

Taurus (1 Node of Bull HPC-Cluster)
CPU: 2x Intel Xeon E5-2690 with eight cores @ 2.9 GHz, RAM: 32 GB

Atlas (1 Node of Megware PC-Farm)
CPU: 4x AMD Opteron 6274 with sixteen cores @ 2.2 GHz, RAM: 64 GB

Please note that the experiments were only conducted for the implementation of the simpler Algorithm 2.1.7 without any background knowledge and where the closure operator is induced by a formal context. The execution of Algorithm 2.1.3 could possibly be slower, and the concrete slow-down depends on the size of the background knowledge $\mathcal{S}$, due to the additional costs for

[3] The test contexts used for the experiments can be obtained from the author via email.



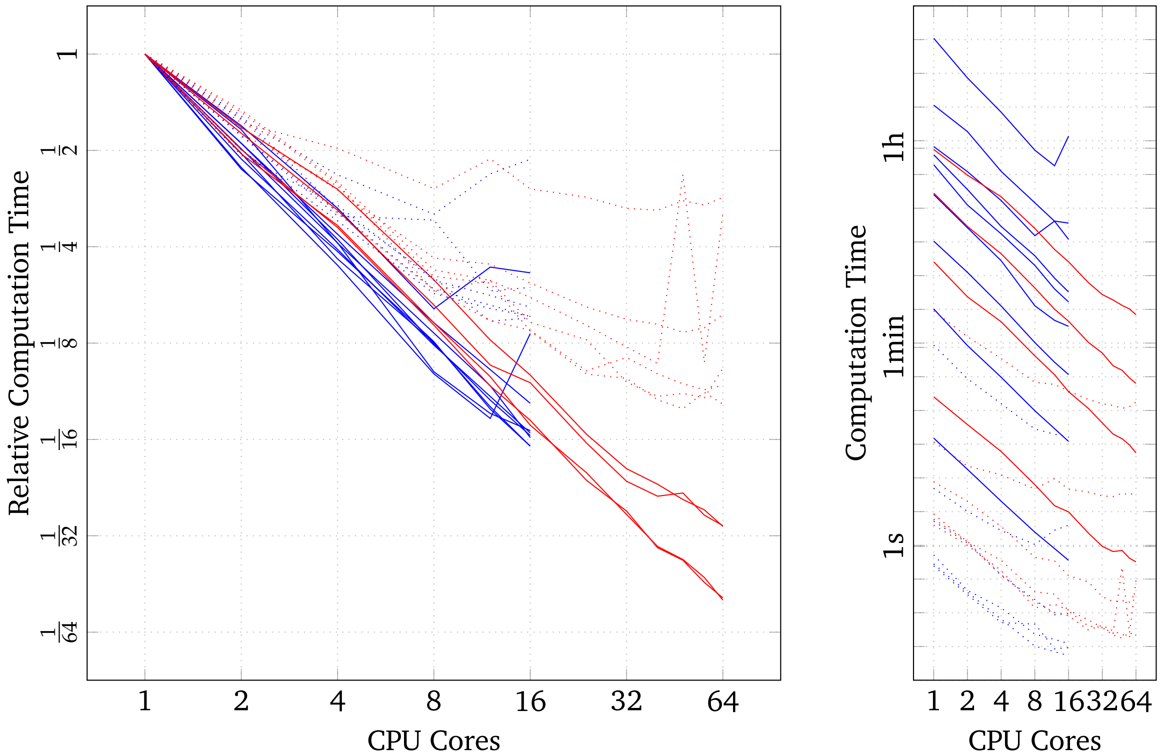
2.2.2 Figure. Benchmark Results (left: Atlas, right: Taurus)

	Formal Context	Objects	Attributes	Density
ⓐ	car.cxt	1728	25	28 %
ⓑ	mushroom.cxt	8124	119	19 %
ⓒ	tic-tac-toe.cxt	958	29	34 %
ⓓ	wine.cxt	178	68	20 %
ⓔ	algorithms.cxt	2688	54	22 %
ⓕ	o1000a10d10.cxt	1000	10	10 %
ⓖ	o1000a20d10.cxt	1000	20	10 %
ⓗ	o1000a36d17.cxt	1000	36	16 %
ⓘ	o1000a49d14.cxt	1000	49	14 %
ⓙ	o1000a50d10.cxt	1000	50	10 %
ⓚ	o1000a64d12.cxt	1000	64	12 %
ⓛ	o1000a81d11.cxt	1000	81	11 %
ⓜ	o1000a100d10-001.cxt	1000	100	11 %
ⓝ	o1000a100d10-002.cxt	1000	100	11 %
ⓞ	o1000a100d10.cxt	1000	100	11 %
ⓟ	o2000a81d11.cxt	2000	81	11 %
ⓠ	24.cxt	17	26	51 %
ⓡ	35.cxt	18	24	43 %
ⓢ	51.cxt	26	17	76 %
ⓣ	54.cxt	20	20	48 %
ⓤ	79.cxt	25	26	68 %

2.2.1 Table. Formal Contexts in Benchmarks

computing closures with respect to the induced closure operator $\phi_{\mathcal{S}}$, and for computing closures of the supremum of $\phi_{\mathcal{L}^*}$ and $\phi_{\mathcal{S}}$, respectively. However, the processing of all candidates with the same cardinality is still independent, i.e., the same scaling behavior is to be expected when more CPU cores are available and utilized, and of course if the input formal context is large enough.

The benchmark results are displayed in Figures 2.2.2 and 2.2.3. While in Figure 2.2.2 the individual results for the test contexts are tagged by their label as defined in Table 2.2.1, no individual labeling is done in Figure 2.2.3. However, solid lines represent large formal contexts with more than 20 attributes and more than 100 objects, and dotted lines denote smaller formal contexts. The charts have both axes logarithmically scaled, to emphasize the correlation between the execution times and the number of available CPU cores. We can see that the computation time is almost inverse linear proportional to the number of available CPU cores, provided that the context is large enough, meaning there are enough candidates on each cardinality level for the computation to be done in parallel. However, we note that there are some

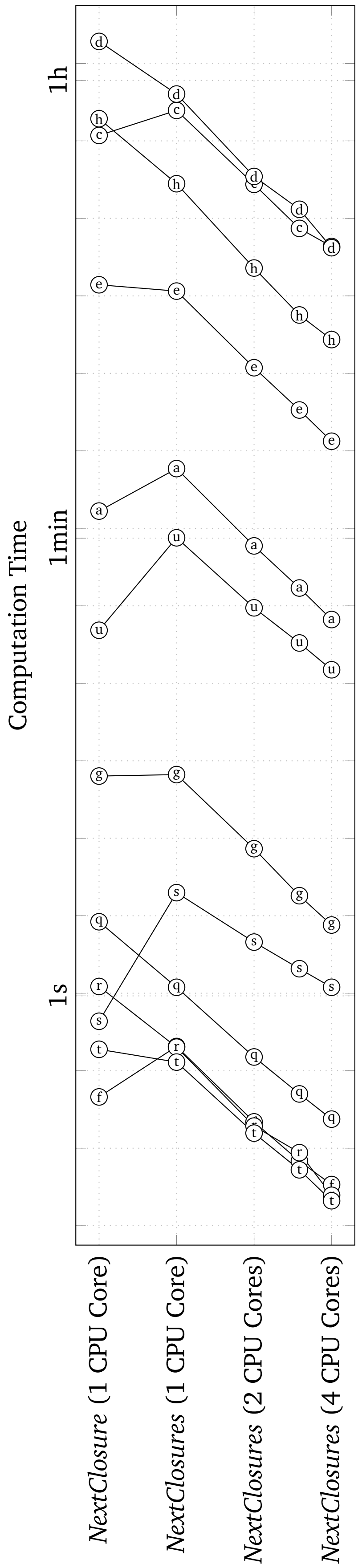


2.2.3 Figure. Benchmark Results (red: Atlas, blue: Taurus)

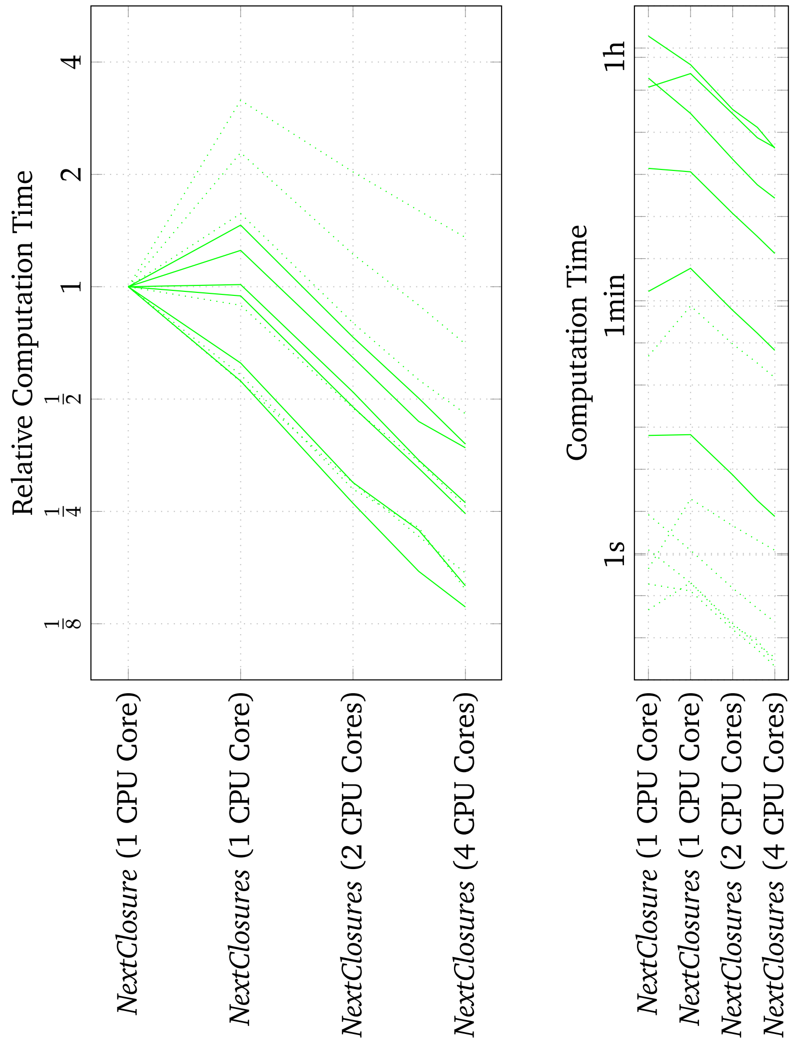
cases where the computation times increase when utilizing all available CPU cores. We are currently not aware of an explanation for this exception, but we conjecture that this is due to some technical details of the platforms or the operation systems, e.g., some background tasks that are executed during the benchmark, or overhead caused by thread maintenance. Note that we did not have full system access during the experiments, but could only execute tasks by scheduling them in a batch system. Additionally, for some of the test contexts only benchmarks for a large number of CPU cores could be performed, due to the time limitations on the test systems.

Furthermore, we have performed the same benchmark with small-sized contexts having at most 15 attributes. The computation times were far below one second. We have noticed that there is a certain number of available CPU cores for which there is no further increase in speed of the algorithm. This happens when the number of candidates is smaller than that of the available CPU cores.

Finally, we compared our two implementations of *NextClosure* and *NextClosures* when only one CPU core is utilized. The comparison was performed on a notebook with Intel Core i7-3720QM CPU with four cores @ 2.6 GHz and 8 GB RAM. The results are shown in Figures 2.2.4 and 2.2.5. We conclude that our proposed algorithm is on average as fast as *NextClosure* on the test contexts. The computation time ratio is between $\frac{1}{3}$ and 3, depending on the specific context. Low or no speed-ups are expected for formal contexts where *NextClosure* does not have to compute candidate closures in order to find the next, but where it can find the next intent or pseudo-intent immediately. Those formal contexts exist and some of them have been used in our benchmarks.



2.2.4 Figure. Performance Comparison



2.2.5 Figure. Performance Comparison

Please do not take the absolute computation times too seriously, as they can certainly be lowered by utilizing other more efficient data structures, or faster programming languages. For example, *NextClosures* was reimplemented in *Concept Explorer FX* [Kri19a], and the new version essentially operates on `java.util.BitSets`. Due to its smaller memory footprint and the faster execution of its methods (compared to `java.util.HashSet`), the absolute computation times were reduced by a factor of approximately 10.

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3 Description Logic

This chapter provides a brief introduction to *Description Logic* (abbrv. DL). It is out of scope to elaborate on the history and origins of DL—the interested reader is rather referred to a thorough summary in [Baa+17, Chapter 1] written by distinguished experts in this field of research. Put simply, the family of description logics is the condensed result of several attempts to defining languages for knowledge representation and reasoning and puts these on a strongly logical foundation.

Section 3.1 introduces two basic description logics $\mathcal{EL}$ and $\mathcal{M}$ and further explains fundamental notions that are also used in other description logics. Section 3.2 presents the Horn fragment Horn-$\mathcal{M}$, for which terminological axioms can be translated into Datalog and thus has a lower data complexity than $\mathcal{M}$ for both instance checking and query answering. Then, Section 3.3 defines the notion of a simulation for $\mathcal{EL}$, which characterizes the semantics of $\mathcal{EL}$ from another perspective. It also constitutes the foundation for the description logic $\mathcal{EL}_{\mathsf{si}}$ introduced in the following Section 3.4. $\mathcal{EL}_{\mathsf{si}}$ is an extension of $\mathcal{EL}$ with greatest fixed-point semantics and it is more expressive than $\mathcal{EL}$, since it allows for the construction of cyclic concept descriptions. In particular, results of many non-standard inference tasks can always be expressed in $\mathcal{EL}_{\mathsf{si}}$ but not in $\mathcal{EL}$. Section 3.5 continues with defining the notion of a simulation for the description logic $\mathcal{M}$ and provides separation results, i.e., it is demonstrated that some concept constructors cannot be expressed in $\mathcal{M}$. A probabilistic variant of $\mathcal{EL}$ is introduced in Section 3.6, which allows for logical reasoning with probabilistic knowledge. Eventually, Section 3.7 briefly mentions the Web Ontology Language.

The DLs $\mathcal{M}$ in Section 3.1 and Horn-$\mathcal{M}$ in Section 3.2 have been introduced by the author in [Kri17a] and in [Kri19b; Kri19c], respectively. Furthermore, note that new results are only contained in Sections 3.4.2 – 3.4.4 and 3.5, the other sections only cite existing basic notions and results from the literature.

3.1 The Description Logics $\mathcal{EL}$ and $\mathcal{M}$

In this section we shall introduce the syntax and semantics of two description logics: the light-weight DL $\mathcal{EL}$ [BBL05], which has tractable reasoning problems, and further the DL $\mathcal{M}$ [Kri17a], which is a large fragment of the DL $\mathcal{SROIQ}$ [HKS06] underlying OWL2. Note that the name $\mathcal{EL}$ is an abbreviation of *existential logic*, while $\mathcal{M}$ stands for *monotonicity*—a property that all concept constructors of $\mathcal{M}$ have as we shall see later.[1] Furthermore, we describe common reasoning problems in DLs as well as their computational complexity for the two exemplary description logics $\mathcal{EL}$ and $\mathcal{M}$.

[1] According to the default naming scheme for DLs, $\mathcal{M}$ equals $\mathcal{ALQ}^{\geq}\mathcal{N}^{\leq}(\mathsf{Self})$.

3.1.1 Syntax

Throughout the whole section assume that Σ is a *signature*, that is, a disjoint union of a set Σ_C of *concept names*, a set Σ_R of *role names*, and a set Σ_I of *individual names*. An $\mathcal{M}$ *concept description* C over Σ is a term that is constructed by means of the following inductive rule, where $A \in \Sigma_\mathsf{C}$ and $r \in \Sigma_\mathsf{R}$, and $n \in \mathbb{N}$.

$$\begin{aligned}
C ::= {} & \bot && \textit{(bottom concept description)} \\
\mid {} & \top && \textit{(top concept description)} \\
\mid {} & A && \textit{(concept name)} \\
\mid {} & \neg A && \textit{(negated concept name)} \\
\mid {} & C \sqcap C && \textit{(conjunction)} \\
\mid {} & \exists \geq n.\, r.\, C && \textit{(qualified at-least restriction)} \\
\mid {} & \exists \leq n.\, r && \textit{(unqualified at-most restriction)} \\
\mid {} & \exists\, r.\, \mathsf{Self} && \textit{(existential self restriction)} \\
\mid {} & \forall\, r.\, C && \textit{(value restriction)}
\end{aligned}$$

Furthermore, an $\mathcal{EL}^\bot$ *concept description* can be constructed with the inductive rule

$$C ::= \bot \mid \top \mid A \mid C \sqcap C \mid \exists\, r.\, C$$

where $\exists\, r.\, C$ is called *existential restriction*, and an $\mathcal{EL}$ *concept description* can be built with the rule $C ::= \top \mid A \mid C \sqcap C \mid \exists\, r.\, C$. For each description logic $\mathcal{DL}$, the set of all $\mathcal{DL}$ concept descriptions over Σ is denoted as $\mathcal{DL}(\Sigma)$. If C is a concept description, then the set $\mathsf{Sub}(C)$ contains all *subconcepts* of C, i.e., all substrings of C that are well-formed concept descriptions as well. Within this document, we stick to the default conventions and denote concept names by letters A or B, denote concept descriptions by letters C, D, E, etc., and denote role names by letters r, s, t, etc., each possibly with sub- or superscripts.

Example. The following is an example of an $\mathcal{EL}$ concept description.

$$\mathsf{Person} \sqcap \exists\, \mathsf{has}.\, \mathsf{UniversityDegree} \sqcap \exists\, \mathsf{published}.\, \mathsf{ScientificArticle} \tag{3.1.A}$$

It refers to the concept of all persons that have an university degree and have published a scientific article. △

A *concept inclusion* (abbrv. CI) is an expression $C \sqsubseteq D$ where both the *premise* C as well as the *conclusion* D are concept descriptions. A *concept equivalence* is an expression $C \equiv D$ such that C and D are concept descriptions, and furthermore a *concept definition* is a term $A \equiv C$ where A is a concept name and C is a concept description. A *terminological box* (abbrv. TBox) is a finite set of concept inclusions, concept equivalences, and concept definitions.

Example. An example of a terminological axiom is the following concept definition.

$$\mathsf{Researcher} \equiv \mathsf{Person} \sqcap \exists\, \mathsf{has}.\, \mathsf{UniversityDegree} \sqcap \exists\, \mathsf{published}.\, \mathsf{ScientificArticle} \tag{3.1.B}$$

It expresses that the class of researchers is exactly the class of persons that have obtained a university degree and further have published some scientific article. More specifically, the above axiom defines the concept name Researcher to coincide with the concept description in (3.1.A). A further example of a concept definition is as follows.

$$\begin{aligned}\mathsf{UniversityProfessor} \equiv \mathsf{Person} \sqcap \exists\, \mathsf{has}.\, \mathsf{DoctoralDegree} \sqcap \exists\, \mathsf{published}.\, \mathsf{ScientificArticle} \\ \sqcap \exists\, \mathsf{teaches}.\, \mathsf{UniversityLecture} \sqcap \exists\, \mathsf{published}.\, \mathsf{TextBook}\end{aligned} \tag{3.1.C}$$

An exemplary concept inclusion is

$$\mathsf{DoctoralDegree} \sqsubseteq \mathsf{UniversityDegree} \tag{3.1.D}$$

and it states that each doctoral degree is a university degree. △

A *concept assertion* is a term $a \mathrel{\varepsilon} C$ where $a \in \Sigma_{\mathsf{I}}$ is an individual name and C is a concept description, and a *role assertion* is a term $(a, b) \mathrel{\varepsilon} r$ where $a, b \in \Sigma_{\mathsf{I}}$ are individual names and $r \in \Sigma_{\mathsf{R}}$ is a role name. An *assertional box* (abbrv. ABox) is a finite set of concept assertions and role assertions; an ABox is called *simple* if all concept descriptions occurring in concept assertions are concept names.

While we have seen that terminological axioms describe the *schema* of the domain of interest, i.e., knowledge that holds true for all entities, assertional axioms can be used to describe the *data*, i.e., facts about certain entities.

Example. The following is an example of a concept assertion.

$$\mathsf{james} \mathrel{\varepsilon} \mathsf{UniversityProfessor} \tag{3.1.E}$$

It says that the individual name james belongs to the class of university professors in (3.1.C). △

An *ontology* $\mathcal{O}$ is a union of an assertional box and a terminological box, and elements that can occur in ontologies are also called *axioms*. For each axiom α, the set $\mathsf{Sub}(\alpha)$ of all *subconcepts* of α consists of all substrings of C being a well-formed concept description. We further set $\mathsf{Sub}(\mathcal{O}) := \bigcup\{\, \mathsf{Sub}(\alpha) \mid \alpha \in \mathcal{O} \,\}$ for each ontology $\mathcal{O}$.

A *role inclusion* (abbrv. RI) is an expression $r \sqsubseteq s$ where $r, s \in \Sigma_{\mathsf{R}}$ are role names. A *relational box* (abbrv. RBox) is a finite set of role inclusions. In case a description logic allows for the usage of these role inclusions, then its name contains the letter $\mathcal{H}$, i.e., we can use role inclusions in the description logics $\mathcal{ELH}$, $\mathcal{ELH}^{\bot}$, and $\mathcal{MH}$.

Informally, the *role depth* of a concept description is defined as the maximal number of nestings of role restrictions. More formally, we can recursively define the *role depth* $\mathsf{rd}(C)$ of a concept description C. We set 0 as the role depth of $\bot$, $\top$, A, and $\neg A$. Concept descriptions of the form $\exists \leq n.\, r$ and $\exists\, r.\, \mathsf{Self}$ have role depth 1. The role depth of a conjunction $C \sqcap D$ is defined as the maximum of $\mathsf{rd}(C)$ and $\mathsf{rd}(D)$. Eventually, the role depth of $\exists\, r.\, C$, $\exists \geq n.\, r.\, C$, and $\forall\, r.\, C$ is set to $1 + \mathsf{rd}(C)$. For a role depth bound $d \in \mathbb{N}$, we denote by $\mathcal{DL}_d(\Sigma)$ the set of all $\mathcal{DL}$ concept descriptions over Σ with a role depth not exceeding d.

We further introduce some syntactic sugar. Firstly, we allow using words of role names within existential restrictions: if $w \in \Sigma_{\mathsf{R}}^*$ and C is some concept description, then $\exists\, w.\, C$ is a well-formed concept description; it is defined by $\exists\, \varepsilon. C := C$ and $\exists\, rw. C := \exists\, r. \exists\, w. C$. Secondly, we allow conjunctions of any finite number of concept descriptions: if $\mathbf{C}$ is a finite set of concept descriptions, then $\bigsqcap \mathbf{C}$ is a well-formed concept description as well; it is defined by $\bigsqcap \emptyset := \top$ and $\bigsqcap\{C_1, \ldots, C_n\} := C_1 \sqcap \ldots \sqcap C_n$.

It is easy to see that each $\mathcal{EL}^\bot$ concept description essentially is a conjunction of atomic $\mathcal{EL}^\bot$ concept descriptions, where an *atomic* concept description is either $\bot$, or $\top$, or some concept name A, or an existential restriction $\exists\, r. C$. In particular, if we define $\mathsf{Conj}(C)$ as the set of all atomic top-level conjuncts in an $\mathcal{EL}^\bot$ concept description C, then C has the form $\bigsqcap \mathsf{Conj}(C)$ (modulo associativity and commutativity of $\sqcap$).

3.1.2 Semantics

An *interpretation* $\mathcal{I} := (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ over Σ consists of a non-empty set $\Delta^\mathcal{I}$ of *objects*, called the *domain*, and an *extension function* $\cdot^\mathcal{I}$ that maps concept names $A \in \Sigma_\mathsf{C}$ to subsets $A^\mathcal{I} \subseteq \Delta^\mathcal{I}$, maps role names $r \in \Sigma_\mathsf{R}$ to binary relations $r^\mathcal{I} \subseteq \Delta^\mathcal{I} \times \Delta^\mathcal{I}$, and maps individual names $a \in \Sigma_\mathsf{I}$ to elements $a^\mathcal{I} \in \Delta^\mathcal{I}$. Of course, we can also treat each relation $r^\mathcal{I}$ as a function of type $\Delta^\mathcal{I} \to \wp(\Delta^\mathcal{I})$ by defining $r^\mathcal{I}(\delta) := \{\, \epsilon \mid (\delta, \epsilon) \in r^\mathcal{I} \,\}$.

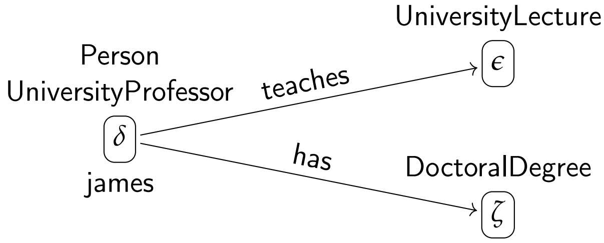
The *active signature* of an interpretation $\mathcal{I}$ is defined as the set $\Sigma^\mathcal{I}$ that contains all concept and role names from Σ with a non-empty extension in $\mathcal{I}$, that is, we define $\Sigma^\mathcal{I} := \{\, \sigma \mid \sigma \in \Sigma \text{ and } \sigma^\mathcal{I} \neq \emptyset \,\}$. Furthermore, we call an interpretation $\mathcal{I}$ *finite* if its domain $\Delta^\mathcal{I}$ and its active signature $\Sigma^\mathcal{I}$ are both finite.

Then, the extension function is canonically extended to all concept descriptions by the following recursive definitions.

$$
\begin{aligned}
\bot^\mathcal{I} &:= \emptyset \\
\top^\mathcal{I} &:= \Delta^\mathcal{I} \\
(\neg A)^\mathcal{I} &:= \Delta^\mathcal{I} \setminus A^\mathcal{I} \\
(C \sqcap D)^\mathcal{I} &:= C^\mathcal{I} \cap D^\mathcal{I} \\
(\exists\, r. C)^\mathcal{I} &:= \{\, \delta \mid r^\mathcal{I}(\delta) \cap C^\mathcal{I} \neq \emptyset \,\} \\
(\exists\, {\geq}\, n. r. C)^\mathcal{I} &:= \{\, \delta \mid |r^\mathcal{I}(\delta) \cap C^\mathcal{I}| \geq n \,\} \\
(\exists\, {\leq}\, n. r)^\mathcal{I} &:= \{\, \delta \mid |r^\mathcal{I}(\delta)| \leq n \,\} \\
(\exists\, r. \mathsf{Self})^\mathcal{I} &:= \{\, \delta \mid (\delta, \delta) \in r^\mathcal{I} \,\} \\
(\forall\, r. C)^\mathcal{I} &:= \{\, \delta \mid r^\mathcal{I}(\delta) \subseteq C^\mathcal{I} \,\}
\end{aligned}
$$

A concept inclusion $C \sqsubseteq D$ is *valid* in $\mathcal{I}$, written $\mathcal{I} \models C \sqsubseteq D$, if $C^\mathcal{I} \subseteq D^\mathcal{I}$. A concept equivalence $C \equiv D$ is *valid* in $\mathcal{I}$, denoted by $\mathcal{I} \models C \equiv D$, if $C^\mathcal{I} = D^\mathcal{I}$. A concept assertion $a \mathrel{\varepsilon} C$ is *valid* in $\mathcal{I}$, symbolized by $\mathcal{I} \models a \mathrel{\varepsilon} C$, if $a^\mathcal{I} \in C^\mathcal{I}$. A role assertion $(a, b) \mathrel{\varepsilon} r$ is *valid* in $\mathcal{I}$, written $\mathcal{I} \models (a, b) \mathrel{\varepsilon} r$, if $(a^\mathcal{I}, b^\mathcal{I}) \in r^\mathcal{I}$. We say that $r \sqsubseteq s$ is *valid* in $\mathcal{I}$, denoted as $\mathcal{I} \models r \sqsubseteq s$, if $r^\mathcal{I} \subseteq s^\mathcal{I}$ holds true. We also refer to $\mathcal{I}$ as a *model* of the axiom α if $\mathcal{I} \models \alpha$ holds true. Furthermore, $\mathcal{I}$ is a *model* of an ontology $\mathcal{O}$, symbolized as $\mathcal{I} \models \mathcal{O}$, if each axiom in $\mathcal{O}$ is valid in $\mathcal{I}$.

Example. The graph in Figure 3.1.1 shows an interpretation $\mathcal{I}_\mathsf{ex}$ over the signature that is



3.1.1 Figure. The exemplary interpretation $\mathcal{I}_{\mathsf{ex}}$

used in the axioms in (3.1.B)–(3.1.E). For instance, the object δ of $\mathcal{I}_{\mathsf{ex}}$ is in the extension of $\mathsf{Person} \sqcap \mathsf{UniversityProfessor} \sqcap \exists\, \mathsf{teaches}.\, \mathsf{UniversityLecture}$, but it is not in the extension of $\exists\, \mathsf{published}.\, \mathsf{Textbook}$. Furthermore, the interpretation $\mathcal{I}_{\mathsf{ex}}$ is a model of the axiom in (3.1.E), since the node assigned to the individual name james belongs to the set of nodes with label UniversityProfessor. However, it is not a model of the axiom in (3.1.C), since it does not contain a node with label DoctoralDegree that is connected to δ with an edge labeled by has. △

The entailment relation is lifted to ontologies as follows: an axiom α is *entailed* by an ontology $\mathcal{O}$, denoted as $\mathcal{O} \models \alpha$, if each model of $\mathcal{O}$ is a model of α too. An ontology $\mathcal{O}_1$ *entails* an ontology $\mathcal{O}_2$, symbolized as $\mathcal{O}_1 \models \mathcal{O}_2$, if $\mathcal{O}_1$ entails each axiom in $\mathcal{O}_2$ or, equivalently, if each model of $\mathcal{O}_1$ is also a model of $\mathcal{O}_2$. In case $\mathcal{O} \models C \sqsubseteq D$, we then also say that C is *subsumed* by D with respect to $\mathcal{O}$ or, alternatively, that C is *more specific* than D *modulo* $\mathcal{O}$. Two concept descriptions C and D are *equivalent* with respect to some ontology $\mathcal{O}$ if $\mathcal{O} \models C \equiv D$, and we may alternatively say that C and D are *equal modulo* $\mathcal{O}$. For an individual name a and a concept description C, we say that a is an *instance* of C with respect to some ontology $\mathcal{O}$ if $\mathcal{O} \models a \in C$.

Example. We now collect all aforementioned axioms in (3.1.B)–(3.1.E) in the ontology $\mathcal{O}_{\mathsf{ex}}$. It is easily verified that $\mathcal{O}_{\mathsf{ex}}$ entails the following two axioms.

$$\mathsf{UniversityProfessor} \sqsubseteq \mathsf{Researcher}$$
$$\mathsf{james} \in \mathsf{Researcher}$$
△

As a further abbreviation, let $\mathcal{O} \models C \sqsubsetneq D$ if both $\mathcal{O} \models C \sqsubseteq D$ and $\mathcal{O} \not\models C \sqsupseteq D$, and we then say that C is *strictly subsumed* by D with respect to $\mathcal{O}$ or, alternatively, that C is *strictly more specific* than D *modulo* $\mathcal{O}$. We say that two concept descriptions C and D are *incomparable* with respect to $\mathcal{O}$, written $\mathcal{O} \models C \parallel D$, if $\mathcal{O} \not\models C \sqsubseteq D$ as well as $\mathcal{O} \not\models C \sqsupseteq D$ holds true. In the sequel of this document we may also write $x \leq_{\mathcal{Z}} y$ instead of $\mathcal{Z} \models x \leq y$ where $\mathcal{Z}$ is either an interpretation or an ontology and $x \leq y$ is an axiom where $\leq$ is some suitable relation symbol, e.g., $\in$, $\sqsubseteq$, $\sqsubsetneq$, $\equiv$, or $\parallel$.

It is readily verified that the concept descriptions $C \sqcap D$ and $D \sqcap C$ are always equivalent, i.e., we can treat the conjunction $\sqcap$ as a *commutative* operation. Furthermore, we have that $C \sqcap (D \sqcap E)$ and $(C \sqcap D) \sqcap E$ are equivalent, which means that the conjunction $\sqcap$ is also an *associative* operation. We will sometimes say that two concept descriptions C and D are *equal modulo* AC to express that we can syntactically rewrite C into D using associativity and commutativity of $\sqcap$. Usually, we do not need to distinguish between concept descriptions that are equal modulo AC, and we can thus treat $(\mathcal{DL}(\Sigma), \sqcap, \top)$ as a commutative monoid in which each element is idempotent.

To justify the choice of the abbreviation $\mathcal{M}$ for $\mathcal{ALQ}^{\geq}\mathcal{N}^{\leq}(\mathsf{Self})$, we remark that each of the constructors having a concept description filler is *monotonic*, since the following entailment holds true.

$$\{C \sqsubseteq D\} \models \{C \sqcap E \sqsubseteq D \sqcap E,\ \forall r.C \sqsubseteq \forall r.D,\ \exists \geq n.r.C \sqsubseteq \exists \geq n.r.D\}$$

Of course, we can express *existential restrictions* in $\mathcal{M}$, since $\exists r.C$ is obviously equivalent to the quantified at-least restriction $\exists \geq 1.r.C$ modulo $\emptyset$.

Reduced Forms of $\mathcal{EL}^{\bot}$ Concept Descriptions

It is not hard to find $\mathcal{EL}^{\bot}$ concept descriptions which are equivalent w.r.t. $\emptyset$, i.e., have the same extension in *all* interpretations, but are not equal. For instance, consider $C := A \sqcap \bot$ and $D := \exists r.\bot$; both concepts must always have an empty extension, and hence both are equivalent to $\bot$. It is therefore helpful for technical details to have a unique *reduced form* for $\mathcal{EL}^{\bot}$ concept descriptions. According to [BM10; Küs01] an $\mathcal{EL}^{\bot}$ concept description C can be transformed into a *reduced form* that is equivalent to C by exhaustive application of the *reduction rule* $D \sqcap E \mapsto D$ whenever $D \sqsubseteq_{\emptyset} E$ to the subconcepts of C (modulo commutativity and associativity of $\sqcap$), and this reduced form is unique modulo AC. We shall denote by $\mathsf{reduce}(C)$ the reduced form of C. From the definition of syntax and semantics of $\mathcal{EL}^{\bot}$ it is immediately clear that each reduced $\mathcal{EL}^{\bot}$ concept description C is either the bottom concept description $\bot$ or an $\mathcal{EL}$ concept description.

Example. The concept description $A \sqcap \exists r.A \sqcap \exists s.(B \sqcap C) \sqcap \exists r.(A \sqcap C) \sqcap A$ has the reduced form $A \sqcap \exists s.(B \sqcap C) \sqcap \exists r.(A \sqcap C)$. △

The Lattice of $\mathcal{EL}^{\bot}$ Concept Descriptions

It is readily verified that, for each TBox $\mathcal{T}$, the *subsumption relation* $\sqsubseteq_{\mathcal{T}}$ constitutes a quasi-order—a reflexive, transitive binary relation—on the set $\mathcal{EL}^{\bot}(\Sigma)$ of all $\mathcal{EL}^{\bot}$ concept descriptions over the signature Σ. Hence, the *quotient* of $\mathcal{EL}^{\bot}(\Sigma)$ with respect to the induced *equivalence relation* $\equiv_{\mathcal{T}}$ is a partially ordered set (abbrv. poset). In particular, this quotient consists of *equivalence classes*, that are defined as

$$[C]_{\mathcal{T}} := \{\, D \mid D \in \mathcal{EL}^{\bot}(\Sigma) \text{ and } C \equiv_{\mathcal{T}} D \,\}$$

for each $\mathcal{EL}^{\bot}$ concept description C. If $\mathbf{C} \subseteq \mathcal{EL}^{\bot}(\Sigma)$ is a set of concept descriptions, then we may also simply write $\mathbf{C}/\mathcal{T}$ for the quotient if $\mathbf{C}$ w.r.t. $\equiv_{\mathcal{T}}$. Furthermore, we shall denote by $\mathsf{Min}_{\mathcal{T}}(\mathbf{C})$ the set of concept descriptions from $\mathbf{C}$ that are most specific w.r.t. $\mathcal{T}$; analogously, $\mathsf{Max}_{\mathcal{T}}(\mathbf{C})$ contains those concept descriptions from $\mathbf{C}$ which are most general w.r.t. $\mathcal{T}$. Formally, we define the following.

$$\mathsf{Min}_{\mathcal{T}}(\mathbf{C}) := \{\, C \mid C \in \mathbf{C} \text{ and there does not exist any } D \in \mathbf{C} \text{ such that } D \sqsubsetneq_{\mathcal{T}} C \,\}$$
$$\mathsf{Max}_{\mathcal{T}}(\mathbf{C}) := \{\, C \mid C \in \mathbf{C} \text{ and there does not exist any } D \in \mathbf{C} \text{ such that } C \sqsubsetneq_{\mathcal{T}} D \,\}$$

In the following we will not distinguish between the equivalence classes and their representatives, and we consider only the case $\mathcal{T} = \emptyset$. Of course, $\bot$ is the smallest element, $\top$ is the greatest element, and the quotient set $\mathcal{EL}^\bot(\Sigma)/\emptyset$ is also a lattice that we shall symbolize by $\boldsymbol{\mathcal{EL}}^\bot(\Sigma)$. It is easy to verify that the conjunction $\sqcap$ corresponds to the finitary *infimum* operation. In a description logic allowing for disjunction $\sqcup$, it dually holds true that the disjunction $\sqcup$ corresponds to the finitary *supremum* operation. Unfortunately, this does not apply to our considered description logic $\mathcal{EL}^\bot$. As an obvious solution, we can simply define the lattice-theoretic notion of a *supremum* specifically tailored to the case of $\mathcal{EL}^\bot$ concept descriptions as follows. The *supremum* or *least common subsumer* (abbrv. LCS) of two $\mathcal{EL}^\bot$ concept descriptions C and D is a concept description E such that

1. $C \sqsubseteq_\emptyset E$ as well as $D \sqsubseteq_\emptyset E$, and
2. for each $\mathcal{EL}^\bot$ concept description F, if $C \sqsubseteq_\emptyset F$ and $D \sqsubseteq_\emptyset F$, then $E \sqsubseteq_\emptyset F$.

Since all least common subsumers of C and D are unique up to equivalence, we may denote a representative of the corresponding equivalence class by $C \vee D$. It is well known that least common subsumers always exist in $\mathcal{EL}^\bot$; in particular, the least common subsumer $C \vee D$ can be computed, modulo equivalence, by means of the following recursive formula.

$$\begin{aligned} C \vee D = {} & \bigsqcap(\Sigma_\mathsf{C} \cap \mathsf{Conj}(C) \cap \mathsf{Conj}(D)) \\ & \sqcap \bigsqcap\{\, \exists\, r.\,(E \vee F) \mid r \in \Sigma_\mathsf{R},\ \exists\, r.\, E \in \mathsf{Conj}(C),\ \text{and}\ \exists\, r.\, F \in \mathsf{Conj}(D)\,\} \end{aligned}$$

Note that this formula follows from Proposition 3.4.3.

Example. For the concept descriptions $A \sqcap B \sqcap \exists\, r.\,(A \sqcap B) \sqcap \exists\, s.\, C$ and $B \sqcap C \sqcap \exists\, r.\, A \sqcap \exists\, r.\,(B \sqcap C)$, the least common subsumer evaluates to $B \sqcap \exists\, r.\, A \sqcap \exists\, r.\, B$. $\triangle$

Of course, the definition of a least common subsumer can be extended to an arbitrary number of arguments in the obvious way, and we shall then denote the least common subsumer of a set $\mathbf{C}$ of concept descriptions by $\bigvee \mathbf{C}$. We say that two concept descriptions $C, D \in \mathcal{EL}^\bot(\Sigma)$ are *orthogonal* or *disjoint* w.r.t. $\emptyset$, written $\emptyset \models C \perp D$ or $C \perp_\emptyset D$, if it holds true that $C \vee D \equiv_\emptyset \top$.

It is easy to see that the equivalence $\equiv$ is compatible with both $\sqcap$ and $\vee$. In the sequel of this document, we shall denote this bounded lattice by $\boldsymbol{\mathcal{EL}}^\bot(\Sigma) := (\mathcal{EL}^\bot(\Sigma), \sqsubseteq_\emptyset)/{\equiv_\emptyset}$, and accordingly $\boldsymbol{\mathcal{EL}}^\bot_d(\Sigma) := (\mathcal{EL}^\bot_d(\Sigma), \sqsubseteq_\emptyset)/{\equiv_\emptyset}$ symbolizes the corresponding bounded lattice of (equivalence classes of) $\mathcal{EL}^\bot$ concept descriptions. Note that $\boldsymbol{\mathcal{EL}}^\bot_d(\Sigma)$ is complete if the underlying signature Σ is finite.

3.1.3 Reasoning and Computational Complexity

So far, we have seen that DLs are suitable for knowledge representation and have introduced syntax and semantics of two exemplary description logics that are used within this thesis. Furthermore, we have shown that it is possible to derive other axioms from ontologies, i.e., to infer implicit knowledge from the explicit knowledge contained in an ontology—a task called *reasoning*. Apparently, this is the other major goal that DLs have been invented for. Deriving implicit knowledge from an ontology is, however, not the only type of inference. Another common question is whether an ontology does not contain contradictory knowledge, i.e., if it has some model, or whether a concept description can be satisfied with respect to some ontology. There are the following common reasoning tasks, cf. [BHS08, Section 3.2.2].

Ontology Consistency. Let $\mathcal{O}$ be an ontology. Is $\mathcal{O}$ *consistent*, i.e., does it have some model?

Concept Satisfiability. Let $\mathcal{O}$ be an ontology and consider a concept description C. Is there a model of $\mathcal{O}$ in which C has a non-empty extension?

Concept Subsumption. Let $\mathcal{O}$ be an ontology and consider a concept inclusion $C \sqsubseteq D$. Is C *subsumed* by D w.r.t. $\mathcal{O}$, i.e., does $\mathcal{O}$ entail $C \sqsubseteq D$?

Instance Checking. Let $\mathcal{O}$ be an ontology and consider a concept assertion $a \mathrel{\varepsilon} C$. Is a an *instance* of C w.r.t. $\mathcal{O}$, i.e., does $\mathcal{O}$ entail $a \mathrel{\varepsilon} C$?

Of course, the above reasoning tasks are *decision problems*, i.e., the answer can either be *"Yes"* or *"No"*, and their decidability and computational complexity has nowadays been explored for most DLs. As a thumb rule, one can say that there is always a trade-off between expressibility and complexity, that is, the more knowledge representation features a DL provides the more expensive, or even undecidable, reasoning becomes. For solving a decidable reasoning problem, one is interested in a suitable *decision procedure*, i.e., a sequence of instructions to be followed for solving the problem which is

- *sound*, i.e., all *"Yes"* answers are correct, and
- *complete*, i.e., all *"No"* answers are correct, and
- *terminating*, i.e., an answer is given after a finite amount of time.

It is readily verified that, in a sufficiently expressive DL, the above reasoning problems are mutually reducible to each other. For instance, $\mathcal{O}$ is consistent if, and only if, $\top$ is satisfiable w.r.t. $\mathcal{O}$. Furthermore, C is subsumed by D w.r.t. $\mathcal{O}$ if, and only if, $\mathcal{O} \cup \{a \mathrel{\varepsilon} C\}$ entails $a \mathrel{\varepsilon} D$ (where a is a fresh individual name not occurring in $\mathcal{O}$). As a consequence, it often suffices to find a decision procedure for one of the problems. For instance, the well-known *tableaux algorithm* [BHS08, Subsection 3.4; Baa+17, Chapter 4; HKS06] decides ontology consistency, and thus also all other above reasoning problems. The tableaux algorithm takes as input an ontology and then tries to construct a model, i.e., a witness for consistency.

For determining the *computational complexity* of reasoning problems, we need to be able to quantify how "large" a concept description, an axiom, or an ontology is when it is used as input for a computing device. As usual, we say that the *size* of some object is the length of an *efficient* string encoding for that object. For our purposes, of course, we can assume that the alphabet used for encoding contains

- all names from the signature Σ,
- the symbols used in the concept constructors, e.g., $\top$, $\sqcap$, $\exists$, etc.,
- 0 and 1 for binary encodings of numbers, e.g., in qualified greater-than restrictions $\exists \geq n.\, r.\, C$, and

- the symbols from the axiom constructors, e.g., $\sqsubseteq$, $\in$, etc.

For instance, the size $||\exists \geq n.r.C||$ is then defined by $1 + \log_2(n) + 1 + ||C||$. It should now be obvious how to recursively define the size of the other forms of concept descriptions. Furthermore, encoding a subsumption axiom $C \sqsubseteq D$ yields a string starting with the encoding of C followed by the symbol $\sqsubseteq$ and then followed by the encoding of D, i.e., the size $||C \sqsubseteq D||$ is defined as $||C|| + 1 + ||D||$. For an instance axiom $a \in C$, we can set $||a \in C|| := 1 + 1 + ||C||$. Sizes of other types of axioms are obtained similarly. Eventually, an ontology $\mathcal{O}$ can be encoded by simply concatenating the encodings of all axioms in $\mathcal{O}$ (possibly separated by some delimiter), i.e., the size of an ontology $\mathcal{O}$ is given as the sum of the sizes of all axioms in $\mathcal{O}$ (plus the number of delimiters).

Now there are two approaches to measuring the computational complexity of the above reasoning tasks. Beforehand, note that we often call the assertional part of an ontology the *data* and the terminological part of an ontology the *schema*. If a question of the form $\mathcal{O} \models \alpha$? is to be decided, then we also call the axiom α the *query*.

Combined Complexity. This is the default. Necessary time and space for solving the reasoning problem is measured as a function in the size of the whole input. For instance, if $a \in_{\mathcal{O}} E$ is to be decided, then time and space requirements are measured as a function of $||a \in E|| + ||\mathcal{O}||$.

Data Complexity. Determining data complexity is more meaningful for practical purposes, as in most cases the size of the stored data easily outgrows the size of the schema and query. In particular, time and space needed for solving the reasoning problem is measured as a function in the size of the ABox only. If, e.g., $a \in_{\mathcal{O}} E$ is to be decided where $\mathcal{O}$ is the union of an ABox $\mathcal{A}$ and some TBox $\mathcal{T}$, then necessary time and space is only measured as a function of $||\mathcal{A}||$.

We shall now elaborate on the computational complexity of the description logics $\mathcal{EL}$, $\mathcal{EL}^{\bot}$, and $\mathcal{M}$.

It is easy to see that in the DL $\mathcal{EL}$ we cannot express unsatisfiability. Thus, deciding consistency of an ontology as well as deciding satisfiability of a concept description w.r.t. some ontology is trivial, since the answers to all instances of these problems are always *"Yes"*. In particular, the canonical model $\mathcal{I}_{C,\mathcal{T}}$ [LW10, Definition 11], which is also cited in Section 4.3.1, is always a witness model for the satisfiability of an $\mathcal{EL}$ concept description C w.r.t. some $\mathcal{EL}$ TBox $\mathcal{T}$. Henceforth, it is more interesting to investigate the problems of concept subsumption and of instance checking. In [BBL05] it is shown that concept subsumption is in **P**. Furthermore, it is also **P**-hard for the following reason. The *path system accessibility* problem is a well-known **P**-complete problem [GJ79]. A path system is a tuple (N, E, S) consisting of a set N of nodes, an accessibility relation $E \subseteq N \times N \times N$, and a set $S \subseteq N$ of source nodes. All source nodes in S are accessible, and a node n is accessible if $(n, p, q) \in E$ and both p and q are accessible. The **P**-complete accessibility problem consists now in deciding whether a target node $t \in N$ is accessible in some path system (N, E, S). This problem can easily be **LOGSpace**-reduced to the subsumption problem: t is accessible in (N, E, S) if, and only if, the $\mathcal{EL}$ TBox $\{ A_p \sqcap A_q \sqsubseteq A_n \mid (n, p, q) \in E \}$ entails the $\mathcal{EL}$ concept inclusion $\bigsqcap\{ A_n \mid n \in S \} \sqsubseteq A_t$.

The upper bound for the data complexity of instance checking in $\mathcal{EL}$ follows immediately from the results in [BBL05]: it is in **P** as well. Hardness for **P** in data complexity can again be shown by a reduction of path system accessibility, see [Baa19, Proposition 7.12] (since the used conjunctive query is an instance query) or see [Cal+13, Theorem 4.3].

Since $\mathcal{EL}$ is a sublogic of $\mathcal{EL}^{\bot}$, both **P**-hardness results also apply to $\mathcal{EL}^{\bot}$. Containment in **P** of concept subsumption for combined complexity and of instance checking for data complexity is again proven in [BBL05]. We conclude that reasoning in the description logics $\mathcal{EL}$ and $\mathcal{EL}^{\bot}$ is always *tractable*.

We now determine the combined complexity of concept subsumption and the data complexity of instance checking for $\mathcal{M}$. In order to be able to use existing complexity results for other description logics, we consider the sublogic $\mathcal{M}^{-}$ instead in which we disallow existential self-restrictions $\exists\, r.\,\mathsf{Self}$. We conjecture that the results then also hold true for $\mathcal{M}$. Some further comments on this are given in Section 3.2.2. Obviously, the hardness results transfer immediately from $\mathcal{M}^{-}$ to $\mathcal{M}$.

In [BBL05] it was shown that concept subsumption for $\mathcal{FL}_0$ is **EXP**-complete. Since $\mathcal{FL}_0$ is a sublogic of $\mathcal{M}^{-}$, we conclude that concept subsumption for $\mathcal{M}^{-}$ is **EXP**-hard. Furthermore, since $\mathcal{M}^{-}$ is a sublogic of $\mathcal{SHIQ}$ in which concept subsumption is **EXP**-complete [Tob01], we conclude that concept subsumption in $\mathcal{M}^{-}$ is **EXP**-complete.

For the following reasons, instance checking for $\mathcal{M}^{-}$ is **co NP**-complete in data complexity. Instance checking in $\mathcal{SHIQ}$ is in **co NP** (data complexity) [HMS05] and $\mathcal{M}^{-}$ is a sublogic of $\mathcal{SHIQ}$. Furthermore, $\mathcal{EL}^{kf}$ is a sublogic of $\mathcal{M}^{-}$ in which instance checking is **co NP**-hard (data complexity) [KL07].

Since the description logic $\mathcal{SHIQ}$ used for obtaining the upper complexity bounds allows for using role inclusions, we further conclude that the subsumption problem for $\mathcal{M}^{-}\mathcal{H}$ is **EXP**-complete as well and instance checking for $\mathcal{M}^{-}\mathcal{H}$ is **co NP**-complete for data complexity.

However, the results only refer to worst-case complexities, that is, we need for instance not expect that a decision procedure for deciding concept subsumption in $\mathcal{M}^{-}$ always needs exponential time in the size of the input. In particular, the popular tableaux algorithms [BHS08, Subsection 3.4; Baa+17, Chapter 4; HKS06] behave very well in practice for most ontologies.

In the prominent DL $\mathcal{ALC}$, which is obtained from $\mathcal{EL}$ by allowing negation of arbitrary concept descriptions, all of the above mentioned inference problems are decidable in exponential time and, more specifically, these are **EXP**-complete [Baa+17, Chapter 5]. However, it is not always the case that reasoning is decidable for each DL. There are examples of very expressive DLs in which undecidable problems can be encoded as a reasoning problem, e.g., the extension of $\mathcal{ALC}$ with Boolean operators on roles ($\cap$, $\cup$, $\neg$) and composition of roles ($\circ$), called the DL $\mathcal{ALC}(\cup, \neg, \circ)$, is undecidable [Sch89]. Other undecidable extensions of $\mathcal{ALC}$ include *role value maps* or *concrete domains* [Baa+17, Section 5.3].

3.2 The Description Logic Horn-$\mathcal{M}$

A *Horn description logic* [Her+18; HMS07; KRH13] can be obtained from some existing DL by, roughly speaking, disallowing any disjunctions. While Hornness decreases expressivity, it often also significantly lowers the computational complexity of some common reasoning tasks, e.g., *instance checking* or *query answering*. Reasoning procedures can then work deterministically, i.e., *reasoning by case* is not required [HMS05]. These are, thus, of importance in practical applications where computation times and costs must not be too high.

Hornness is not a new notion: Horn clauses in first order logic are disjunctions of an arbitrary number of negated atomic formulas and at most one non-negated atomic formula. It is easy to see that such Horn clauses have an implicative character, since $\neg\phi_1 \vee \ldots \vee \neg\phi_n \vee \psi$ is equivalent to $\phi_1 \wedge \ldots \wedge \phi_n \to \psi$ and likewise $\neg\phi_1 \vee \ldots \vee \neg\phi_n$ is equivalent to $\phi_1 \wedge \ldots \wedge \phi_n \to \bot$ and similarly ψ is equivalent to $\top \to \psi$. A *logic program* is a set of Horn clauses, and a *Datalog program* is a function-free logic program [Dan+01]. All commonly known Horn description logics can be translated into Datalog—more specifically, each Horn-$\mathcal{DL}$ TBox $\mathcal{T}$ can be translated into some Datalog program $\mathcal{D}$ such that, for each simple ABox $\mathcal{A}$, the ontology $\mathcal{T} \cup \mathcal{A}$ is satisfiable if, and only if, the Datalog program $\mathcal{D} \cup \mathcal{A}$ is satisfiable. For deeper insights please consider [Her+18; HMS07; KRH13].

Note that $\mathcal{EL}$ and $\mathcal{EL}^{\bot}$ are already Horn DLs, i.e., both coincide with their Horn fragments.

3.2.1 Syntax and Semantics

We shall now introduce the description logic Horn-$\mathcal{M}$, which is the Horn fragment of the description logic $\mathcal{M}$ from Section 3.1.1. Syntax and semantics are the same as for $\mathcal{M}$, and restrictions are imposed on concept inclusions only. Generally speaking, premises must always be $\mathcal{EL}^* := \mathcal{EL}^{\bot}(\mathsf{Self})$ concept descriptions while conclusions may be arbitrary $\mathcal{M}^{\leq 1} := \mathcal{ALQ}^{\geq}\mathcal{N}^{\leq 1}(\mathsf{Self})$ concept descriptions, that is, $\mathcal{M}$ concept descriptions except that in unqualified smaller-than restrictions $\exists \leq n.\, r$ only the case $n = 1$ is allowed. More specifically, a Horn-$\mathcal{M}$ *concept inclusion* is an expression $C \sqsubseteq D$ where the *concept descriptions* C and D are built by means of the following inductive rules.

$$C := \bot \mid \top \mid A \mid C \sqcap C \mid \exists\, r.\, C \mid \exists\, r.\, \mathsf{Self} \qquad (\mathcal{EL}^*)$$

$$D := \bot \mid \top \mid A \mid \neg A \mid D \sqcap D \mid \exists \geq n.\, r.\, D \mid \exists \leq 1.\, r \mid \forall\, r.\, D \mid \exists\, r.\, \mathsf{Self} \qquad (\mathcal{M}^{\leq 1})$$

We call a concept description $\exists \leq 1.\, r$ a *local functionality restriction*. Note that the above syntactic characterization follows easily from the results in [Her+18; HMS07; KRH13].

As it has already been pointed out in [HMS07], the following properties can be expressed in a sufficiently strong Horn DL, e.g., in Horn-$\mathcal{M}$.

Inclusion of Simple Concepts. $A \sqsubseteq B$ states that each individual being A is also B.

Concept Disjointness. $A \sqcap B \sqsubseteq \bot$ states that there are no individuals that are both A and B.

Domain Restrictions. $\exists\, r.\, \top \sqsubseteq A$ states that each individual having an r-successor must be an A.

Range Restrictions. $\top \sqsubseteq \forall\, r.\, A$ states that each individual being an r-successor must be an A.

Functionality Restrictions. $\top \sqsubseteq \exists \leq 1.\, r$ states that each individual has at most one r-successor.

Participation Constraints. $A \sqsubseteq \exists\, r.\, B$ states that each individual that is an A has an r-successor that is a B.

3.2.2 Computational Complexity

We have already seen in Section 3.1.3 that concept subsumption for $\mathcal{M}^-$ is an **EXP**-complete problem for combined complexity and further that instance checking for $\mathcal{M}^-$ is a **coNP**-complete problem for data complexity. We shall now catch up on this and determine the complexity of these two reasoning problems for Horn-$\mathcal{M}$. Again we only consider Horn-$\mathcal{M}^-$ for the instance checking problem to be able to reuse existing results.

In [BBL05, Theorem 11] it was shown that concept subsumption is **EXP**-complete for the description logic $\mathcal{EL}^{\leq 1}$, i.e., $\mathcal{EL}$ extended with local functionality restrictions $\exists \leq 1. r$. The proof shows hardness even if the local functionality restrictions only occur in the right-hand sides of concept inclusions, i.e., concept subsumption is **EXP**-hard for Horn-$\mathcal{EL}^{\leq 1}$. We can thus conclude that, since Horn-$\mathcal{EL}^{\leq 1}$ is a sublogic of Horn-$\mathcal{M}^-$, concept subsumption is also **EXP**-hard for Horn-$\mathcal{M}^-$. Furthermore, since Horn-$\mathcal{M}^-$ is a sublogic of $\mathcal{M}^-$, we conclude that concept subsumption in Horn-$\mathcal{M}^-$ is **EXP**-complete (combined complexity) as well.

As instance checking in Horn-$\mathcal{SHIQ}$ is in **P** (data complexity) [HMS05] and Horn-$\mathcal{M}^-$ is a sublogic of Horn-$\mathcal{SHIQ}$, we conclude that the similar problem in Horn-$\mathcal{M}^-$ is in **P** (data complexity) as well. Furthermore, $\mathcal{EL}$ is a sublogic of Horn-$\mathcal{M}^-$ and instance checking in $\mathcal{EL}$ is **P**-hard (data complexity), cf. Section 3.1.3. Consequently, instance checking in Horn-$\mathcal{M}^-$ is **P**-hard as well.

We see that terminological reasoning in Horn-$\mathcal{M}^-$ is not cheaper than in $\mathcal{M}^-$, but that assertional reasoning with ontologies containing both a schema (TBox) and data (ABox) is considerably cheaper in Horn-$\mathcal{M}^-$ than in $\mathcal{M}^-$ if we only take into account the size of the ABox (data complexity), unless $\mathbf{P} = \mathbf{NP}$. It is obvious that the hardness results transfer from Horn-$\mathcal{M}^-$ to Horn-$\mathcal{M}$. Unfortunately, the author cannot provide sharp upper bounds. A comment regarding the data complexity of instance checking in $\mathcal{M}$ and Horn-$\mathcal{M}$ is as follows. If one takes a closer look on the proofs in [HMS07], one could get the impression that it might suffice to include the case $\pi_y(\exists R. \mathsf{Self}, X) := R(X, X)$ for the translation of concept descriptions into first-order logic. The author conjectures that this extended translation allows for obtaining the same complexity results.

Henceforth, it makes sense to use a Horn-$\mathcal{M}$ TBox—or, more cautiously, some Horn-$\mathcal{M}^-$ TBox—as the schema for *ontology-based data access* (abbrv. OBDA) applications. This motivates the development of a procedure that can learn Horn-$\mathcal{M}$ concept inclusions from observations in form of an interpretation, which will be presented later in Chapter 7.

3.3 Simulations for $\mathcal{EL}$

The semantics of $\mathcal{EL}$ and of its fixed-point extensions, some of which are described in Section 3.4, can be characterized by means of so-called simulations. A short overview is given below.

A *pointed interpretation* is a pair $(\mathcal{I}, \delta)$ consisting of an interpretation $\mathcal{I}$ and an element $\delta \in \Delta^{\mathcal{I}}$. Now let $(\mathcal{I}, \delta)$ and $(\mathcal{J}, \epsilon)$ be two pointed interpretations, and assume that $\Gamma \subseteq \Sigma$. A Γ-*simulation* from $(\mathcal{I}, \delta)$ to $(\mathcal{J}, \epsilon)$ is a relation $\mathfrak{S} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{J}}$ that satisfies $(\delta, \epsilon) \in \mathfrak{S}$ as well as the following conditions for all pairs $(\zeta, \eta) \in \mathfrak{S}$.

1. For all concept names $A \in \Gamma_{\mathsf{C}}$, if $\zeta \in A^{\mathcal{I}}$, then $\eta \in A^{\mathcal{J}}$.

2. For all role names $r \in \Gamma_\mathsf{R}$, if there is an element $\theta \in \Delta^\mathcal{I}$ such that $(\zeta, \theta) \in r^\mathcal{I}$, then there is an element $\iota \in \Delta^\mathcal{J}$ such that $(\eta, \iota) \in r^\mathcal{J}$ and $(\theta, \iota) \in \mathfrak{S}$.

We then also write $\mathfrak{S}\colon (\mathcal{I}, \delta) \precsim_\Gamma (\mathcal{J}, \epsilon)$, and to express the mere existence of a Γ-simulation from $(\mathcal{I}, \delta)$ to $(\mathcal{J}, \epsilon)$ we may write $(\mathcal{I}, \delta) \precsim_\Gamma (\mathcal{J}, \epsilon)$. Furthermore, if $\Gamma = \Sigma$, then we speak of *simulations* instead of Γ-simulations, and we leave out the subscript Γ, i.e., we use the symbol $\precsim$ instead of $\precsim_\Gamma$. Two pointed simulations $(\mathcal{I}, \delta)$ and $(\mathcal{J}, \epsilon)$ are *equi-similar* if there is a simulation in each direction, and we shall then write $(\mathcal{I}, \delta) \approx (\mathcal{J}, \epsilon)$.

Assume that $(\mathcal{I}, \delta) \precsim (\mathcal{J}, \epsilon)$. It is easily verified by structural induction that $\delta \in C^\mathcal{I}$ implies $\epsilon \in C^\mathcal{J}$ for all $\mathcal{EL}^\bot$ concept description C, and we also say that $\mathcal{EL}^\bot$ concept descriptions are *preserved under simulations*. In [LW10, Theorem 5] it has been shown that simulations can be used to characterize the description logic $\mathcal{EL}$. More specifically, Lutz and Wolter [LW10, Theorem 5] state that the set of (FOL translations of) $\mathcal{EL}$ concept descriptions is a maximal set of FOL formulas that is preserved under simulations and has finite minimal models, where we say that a set $\mathcal{L}$ of FOL formulas *has finite minimal models* if, for each $\phi(x) \in \mathcal{L}$, there is some finite pointed interpretation $(\mathcal{I}, \delta)$ such that, for each $\psi(x) \in \mathcal{L}$, it holds true that $\mathcal{I}, \{x \mapsto \delta\} \models \psi(x)$ if, and only if, $\forall x.\, (\phi(x) \to \psi(x))$ is a tautology. It is easy to see that these finite minimal models are the canonical models $\mathcal{I}_C$ defined below.

Simulations can be used to characterize subsumption between $\mathcal{EL}$ concept descriptions w.r.t. the empty TBox. More specifically, Baader, Küsters, and Molitor [BKM98] have shown that $C \sqsubseteq_\emptyset D$ holds true if, and only if, there is a simulation from $(\mathcal{I}_D, D)$ to $(\mathcal{I}_C, C)$. The interpretation $\mathcal{I}_C$ is the *canonical model* of C and it is defined as follows.[2]

$$\Delta^{\mathcal{I}_C} := \{C\} \cup \{\, D \mid \exists\, r.\, D \in \mathsf{Sub}(C) \text{ for some } r \in \Sigma_\mathsf{R} \,\}$$
$$\cdot^{\mathcal{I}_C}\colon \begin{cases} A \mapsto \{\, D \mid A \in \mathsf{Conj}(D) \,\} & \text{for each } A \in \Sigma_\mathsf{C} \\ r \mapsto \{\, (D, E) \mid \exists\, r.\, E \in \mathsf{Conj}(D) \,\} & \text{for each } r \in \Sigma_\mathsf{R} \end{cases}$$

As an immediate corollary, we get that an $\mathcal{EL}$ concept description C is more specific than another $\mathcal{EL}$ concept description D modulo $\emptyset$ if, and only if,

- $A \in \mathsf{Conj}(D)$ implies $A \in \mathsf{Conj}(C)$ for each concept name A, and
- for each existential restriction $\exists\, r.\, F \in \mathsf{Conj}(D)$, there is an existential restriction $\exists\, r.\, E \in \mathsf{Conj}(C)$ such that $E \sqsubseteq_\emptyset F$.

This yields a recursive procedure for deciding subsumption modulo $\emptyset$.

Furthermore, we can also utilize simulations to characterize the extension of an $\mathcal{EL}$ concept description for some interpretation. More specifically, [BDK16, Proposition A.6] shows that $\delta \in C^\mathcal{I}$ holds true if, and only if, there exists a simulation from $(\mathcal{I}_C, C)$ to $(\mathcal{I}, \delta)$. It follows that $C^\mathcal{I} = \{\, \delta \mid (\mathcal{I}_C, C) \precsim (\mathcal{I}, \delta) \,\}$.

It is easy to see that $\precsim$ is a partial order relation. Furthermore, infima w.r.t. $\precsim$ always exist and can be characterized by products. The *product* of interpretations $\mathcal{I}$ and $\mathcal{J}$ over the same signature Σ is defined as the interpretation $\mathcal{I} \times \mathcal{J}$ consisting of the following components.

$$\Delta^{\mathcal{I} \times \mathcal{J}} := \Delta^\mathcal{I} \times \Delta^\mathcal{J}$$

[2] We have $\mathcal{I}_C = \mathcal{I}_{C,\emptyset}$ for the canonical models $\mathcal{I}_{C,\mathcal{T}}$ w.r.t. TBoxes that are cited later in Section 4.3.1.

$$\cdot^{\mathcal{I}\times\mathcal{J}}\colon \begin{cases} A \mapsto \{\, (\delta,\zeta) \mid \delta \in A^{\mathcal{I}} \text{ and } \zeta \in A^{\mathcal{J}} \,\} & \text{for each } A \in \Sigma_{\mathsf{C}} \\ r \mapsto \{\, ((\delta,\zeta),(\epsilon,\eta)) \mid (\delta,\epsilon) \in r^{\mathcal{I}} \text{ and } (\zeta,\eta) \in r^{\mathcal{J}} \,\} & \text{for each } r \in \Sigma_{\mathsf{R}} \end{cases}$$

Given two pointed interpretations $(\mathcal{I},\delta)$ and $(\mathcal{J},\epsilon)$, their *product* $(\mathcal{I},\delta) \times (\mathcal{J},\epsilon)$ is defined as the pointed interpretation $(\mathcal{I} \times \mathcal{J},(\delta,\epsilon))$. Now, LUTZ, PIRO, and WOLTER [LPW10, Observation 3] have found that the product operation $\times$ is the infimum operation in the set of (equivalence classes of) pointed interpretations ordered by $\precsim$. It is immediate to extend the notion of a product to an arbitrary number of (pointed) interpretations used as factors, and we shall denote the product of a set $\mathfrak{I}$ of (pointed) interpretations as $\bigtimes \mathfrak{I}$.

3.4 The Description Logic $\mathcal{EL}_{\mathsf{si}}$

In Section 3.3 we have seen that $C^{\mathcal{I}} = \{\, \delta \mid (\mathcal{I}_C,C) \precsim (\mathcal{I},\delta) \,\}$ holds true for each $\mathcal{EL}$ concept description C and each interpretation $\mathcal{I}$. For defining a new concept constructor, the idea of LUTZ, PIRO, and WOLTER in [LPW10] is to replace the acyclic pointed interpretation $(\mathcal{I}_C,C)$ by an arbitrary, possibly cyclic, pointed interpretation that is finite. This extension yields the more expressive description logic $\mathcal{EL}_{\mathsf{si}}$. Since it allows for cyclic concept descriptions, results of non-standard reasoning tasks can often be expressed in $\mathcal{EL}_{\mathsf{si}}$ while the same is not true for $\mathcal{EL}$ that only has acyclic concept descriptions. In this section, we start with formally introducing syntax and semantics. In Section 3.4.2, we show that there is a polynomial translation between $\mathcal{EL}_{\mathsf{si}}$ and $\mathcal{EL}_{\mathsf{gfp}}$, which means that $\mathcal{EL}_{\mathsf{si}}$ is an extension of $\mathcal{EL}$ with greatest fixed-point semantics. Then, Section 3.4.3 explains how simulations can be utilized for reasoning in $\mathcal{EL}_{\mathsf{si}}$ and it further cites results on the computational complexity of common reasoning problems. Finally, Section 3.4.4 develops an approach for computing reduced forms of $\mathcal{EL}_{\mathsf{si}}$ concept descriptions.

3.4.1 Syntax and Semantics

The description logic $\mathcal{EL}_{\mathsf{si}}$ extends $\mathcal{EL}$ by the concept constructor $\exists^{\mathsf{sim}}(\mathcal{I},\delta)$ where $(\mathcal{I},\delta)$ is a pointed interpretation such that $\mathcal{I}$ is finite. The semantics of the additional concept constructor is defined as follows: for each interpretation $\mathcal{J}$ and any object $\epsilon \in \Delta^{\mathcal{J}}$, it holds true that $\epsilon \in (\exists^{\mathsf{sim}}(\mathcal{I},\delta))^{\mathcal{J}}$ if $(\mathcal{I},\delta) \precsim (\mathcal{J},\epsilon)$. As shown in [LPW10, Lemma 7], every $\mathcal{EL}_{\mathsf{si}}$ concept description is equivalent to a concept description of the form $\exists^{\mathsf{sim}}(\mathcal{I},\delta)$, and furthermore, such an equivalent concept description can be constructed in linear time. Adding the bottom concept description $\bot$ yields the description logic $\mathcal{EL}_{\mathsf{si}}^{\bot}$.

For a given $\mathcal{EL}_{\mathsf{si}}$ concept description $\exists^{\mathsf{sim}}(\mathcal{I},\delta)$, we define the set of its *top-level conjuncts* in the following way.

$$\mathsf{Conj}(\exists^{\mathsf{sim}}(\mathcal{I},\delta)) := \{\, A \mid A \in \Sigma_{\mathsf{C}} \text{ and } \delta \in A^{\mathcal{I}} \,\} \cup \{\, \exists\, r.\exists^{\mathsf{sim}}(\mathcal{I},\epsilon) \mid r \in \Sigma_{\mathsf{R}} \text{ and } (\delta,\epsilon) \in r^{\mathcal{I}} \,\}$$

It then holds true that $\exists^{\mathsf{sim}}(\mathcal{I},\delta)$ is equivalent to $\bigsqcap \mathsf{Conj}(\exists^{\mathsf{sim}}(\mathcal{I},\delta))$ modulo $\emptyset$. Furthermore, the set $\mathsf{Sub}(\exists^{\mathsf{sim}}(\mathcal{I},\delta))$ of *subconcepts* is the smallest set that contains $\{\exists^{\mathsf{sim}}(\mathcal{I},\epsilon)\} \cup \mathsf{Conj}(\exists^{\mathsf{sim}}(\mathcal{I},\epsilon))$ for each object ϵ that is reachable from δ on some (possibly empty) path.

The *size* of an $\mathcal{EL}_{\mathsf{si}}$ concept description $\exists^{\mathsf{sim}}(\mathcal{I},\delta)$ is defined as $||\exists^{\mathsf{sim}}(\mathcal{I},\delta)|| := 1 + ||\mathcal{I}||$

where the size of $\mathcal{I}$ is defined as

$$||\mathcal{I}|| := |\Delta^{\mathcal{I}}| + \sum(\, |\sigma^{\mathcal{I}}| \mid \sigma \in \Sigma\,).$$

It then follows that $|\mathsf{Conj}(\exists^{\mathsf{sim}}(\mathcal{I},\delta))| \leq ||\mathcal{I}|| \leq ||\exists^{\mathsf{sim}}(\mathcal{I},\delta)||$ and $|\mathsf{Sub}(\exists^{\mathsf{sim}}(\mathcal{I},\delta))| \leq ||\mathcal{I}||^2 \leq ||\exists^{\mathsf{sim}}(\mathcal{I},\delta)||^2$.

3.4.2 A Comparison with $\mathcal{EL}_{\mathsf{gfp}}$

We show in this section that the description logic $\mathcal{EL}_{\mathsf{si}}$ is equi-expressive to $\mathcal{EL}_{\mathsf{gfp}}$ [Baa03b; Dis11], which is an existing extension of $\mathcal{EL}$ with greatest fixed-point semantics. An $\mathcal{EL}_{\mathsf{gfp}}$ concept description C over a signature Σ is a pair $(A_C, \mathcal{T}_C)$ consisting of a concept name A_C and a possibly cyclic $\mathcal{EL}$ TBox $\mathcal{T}_C$ of concept definitions such that

- there is a set Γ_C of concept names where $\Sigma \cap \Gamma_C = \emptyset$ and $A_C \in \Gamma_C$, and
- $\mathcal{T}_C$ is formulated over the extended signature $\Sigma \cup \Gamma_C$ and uniquely defines each concept name in Γ_C, i.e., there is exactly one concept definition $A \equiv D_A$ for each $A \in \Gamma_C$.

Without loss of generality we assume that the sets Γ_C are mutually disjoint for all $\mathcal{EL}_{\mathsf{gfp}}$ concept descriptions.

Let $C = (A_C, \mathcal{T}_C)$ be an $\mathcal{EL}_{\mathsf{gfp}}$ concept description and consider an interpretation over Σ. The extension of C for $\mathcal{I}$ is defined by means of the *gfp-model* of $\mathcal{I}$ w.r.t. $\mathcal{T}_C$, which we shall introduce in the following. An interpretation $\mathcal{J}$ over $\Sigma \cup \Gamma_C$ *extends* $\mathcal{I}$ if $\mathcal{J}$ coincides with $\mathcal{I}$ on Σ. The extensions of $\mathcal{I}$ can be ordered: we let $\mathcal{J}_1 \leq \mathcal{J}_2$ if $A^{\mathcal{J}_1} \subseteq A^{\mathcal{J}_2}$ holds true for all concept names $A \in \Gamma_C$. We now define a mapping f on extensions of $\mathcal{I}$: if $\mathcal{J}$ is an extension of $\mathcal{I}$, then $A^{f(\mathcal{J})} := D_A^{\mathcal{J}}$ for all concept names $A \in \Gamma_C$ and $f(\mathcal{J})$ coincides with $\mathcal{J}$ on all names in Σ. Obviously, the mapping f is order-preserving and thus, according to TARSKI's fixed-point theorem [Tar55], f has a greatest fixed point $\mathcal{I}^*$, which is called *gfp-model* of $\mathcal{I}$ w.r.t. $\mathcal{T}_C$. Then, the extension of C for $\mathcal{I}$ is given by $C^{\mathcal{I}} := A_C^{\mathcal{I}^*}$. Furthermore, the fixed points of f are exactly the models of $\mathcal{T}_C$, since $\mathcal{J} = f(\mathcal{J})$ implies $A^{\mathcal{J}} = A^{f(\mathcal{J})} = D_A^{\mathcal{J}}$ is satisfied for each $A \in \Gamma_C$, and vice versa $A^{\mathcal{J}} = D_A^{\mathcal{J}}$ for each $A \in \Gamma_C$ implies that $A^{f(\mathcal{J})} = D_A^{\mathcal{J}} = A^{\mathcal{J}}$, i.e., $f(\mathcal{J}) = \mathcal{J}$.

It is easy to see that every $\mathcal{EL}$ concept description C is equivalent to the $\mathcal{EL}_{\mathsf{gfp}}$ concept description $(A, \{A \equiv C\})$ where A is a concept name not occurring in C.

3.4.1 Proposition. *The description logics $\mathcal{EL}_{\mathsf{gfp}}$ and $\mathcal{EL}_{\mathsf{si}}$ are polynomially equivalent.*

Proof. Let $\exists^{\mathsf{sim}}(\mathcal{I},\delta)$ be some $\mathcal{EL}_{\mathsf{si}}$ concept description and define the following $\mathcal{EL}_{\mathsf{gfp}}$ concept description.

$$\begin{aligned} &C := (A_\delta, \mathcal{T}) \\ \text{where}\quad &\mathcal{T} := \{\, A_\epsilon \equiv \textstyle\bigsqcap\{\, A \mid \epsilon \in A^{\mathcal{I}} \,\} \sqcap \textstyle\bigsqcap\{\, \exists\, r.\, A_\zeta \mid (\epsilon,\zeta) \in r^{\mathcal{I}} \,\} \mid \epsilon \in \Delta^{\mathcal{I}} \,\} \end{aligned}$$

We show that both concept descriptions are equivalent. Consider an interpretation $\mathcal{J}$. If $\alpha \in (\exists^{\mathsf{sim}}(\mathcal{I},\delta))^{\mathcal{J}}$, then there is a simulation $\mathfrak{S}$ from $(\mathcal{I},\delta)$ to $(\mathcal{J},\alpha)$. Now [Baa02, Proposition 18; Baa03b, Proposition 6] immediately yields that $\alpha \in A_\delta^{\mathcal{J}^*}$ where $\mathcal{J}^*$ is the gfp-model of $\mathcal{J}$ w.r.t. $\mathcal{T}$, i.e., we have $\alpha \in C^{\mathcal{J}}$.

Vice versa, let $(A_C, \mathcal{T}_C)$ be some $\mathcal{EL}_{\mathsf{gfp}}$ concept description. As explained in [Baa02, Section 3.1], we can assume without loss of generality that $\mathcal{T}_C$ is normalized, i.e., the fillers of existential restrictions occurring in right-hand sides of concept definitions in $\mathcal{T}_C$ are always concept names that are defined in $\mathcal{T}_C$. It is now straightforward to transform $\mathcal{T}_C$ into an interpretation $\mathcal{I}_C$ such that each defined concept name $A_{C'}$ corresponds to the object $\delta_{C'}$. If we now assume that $\alpha \in (A_C, \mathcal{T}_C)^{\mathcal{J}}$ holds true for some interpretation $\mathcal{J}$, then [Baa02, Proposition 18; Baa03b, Proposition 6] implies that there is a simulation from $(\mathcal{I}_C, \delta_C)$ to $(\mathcal{J}, \alpha)$, which means that $\alpha \in (\exists^{\mathsf{sim}}(\mathcal{I}_C, \delta_C))^{\mathcal{J}}$. Thus, $(A_C, \mathcal{T}_C)$ is equivalent to the $\mathcal{EL}_{\mathsf{si}}$ concept description $\exists^{\mathsf{sim}}(\mathcal{I}_C, \delta_C)$.

Eventually, it is easy to see that both transformations can be computed in polynomial time. Note that, according to [Baa02], the normalization of $\mathcal{T}_C$ can be obtained in polynomial time. □

3.4.3 Computational Complexity and Reasoning with Simulations

As shown in [LPW10, Theorem 12], all common reasoning problems in $\mathcal{EL}_{\mathsf{si}}$ can be decided in polynomial time. Since $\mathcal{EL}$ is a sublogic, we also have **P**-hardness for the reasoning problems. Note that the results from Section 4.3.1 can also be utilized for deciding the subsumption problem in $\mathcal{EL}_{\mathsf{si}}$. We are going to show in Propositions 4.3.27 and 4.3.30 that the concept satisfiability problem as well as the concept subsumption problem in $\mathcal{EL}_{\mathsf{si}}^{\bot}$ are both **P**-complete too.

In the sequel of this section, we shall show how simulations can be used for characterizing subsumption in $\mathcal{EL}_{\mathsf{si}}$ without any TBox, and we also provide some important results, like the finite model property.

3.4.2 Proposition. $\exists^{\mathsf{sim}}(\mathcal{I}, \delta) \sqsubseteq_{\emptyset} \exists^{\mathsf{sim}}(\mathcal{J}, \epsilon)$ *if, and only if, there is a simulation from* $(\mathcal{J}, \epsilon)$ *to* $(\mathcal{I}, \delta)$.

Proof. Assume that $\exists^{\mathsf{sim}}(\mathcal{I}, \delta)$ is more specific than $\exists^{\mathsf{sim}}(\mathcal{J}, \epsilon)$ modulo $\emptyset$. Since the reflexive relation on $\Delta^{\mathcal{I}}$ is a simulation from $(\mathcal{I}, \delta)$ to $(\mathcal{I}, \delta)$, we have that $\delta \in (\exists^{\mathsf{sim}}(\mathcal{I}, \delta))^{\mathcal{I}}$ holds true. It follows that $\delta \in (\exists^{\mathsf{sim}}(\mathcal{J}, \epsilon))^{\mathcal{I}}$ and so there exists some simulation from $(\mathcal{J}, \epsilon)$ to $(\mathcal{I}, \delta)$.

For the converse direction, let $\mathfrak{S}$ be a simulation from $(\mathcal{J}, \epsilon)$ to $(\mathcal{I}, \delta)$. If for some interpretation $\mathcal{K}$ it holds true that $\zeta \in (\exists^{\mathsf{sim}}(\mathcal{I}, \delta))^{\mathcal{K}}$, then there is a simulation from $(\mathcal{I}, \delta)$ to $(\mathcal{K}, \zeta)$. Composing it with $\mathfrak{S}$ yields a simulation from $(\mathcal{J}, \epsilon)$ to $(\mathcal{K}, \zeta)$, implying that $\zeta \in (\exists^{\mathsf{sim}}(\mathcal{J}, \epsilon))^{\mathcal{K}}$. □

As a direct consequence of the definition and the above proposition we obtain that the following statements are all equivalent.

1. $\epsilon \in (\exists^{\mathsf{sim}}(\mathcal{I}, \delta))^{\mathcal{J}}$
2. $(\mathcal{J}, \epsilon) \precsim (\mathcal{I}, \delta)$
3. $\exists^{\mathsf{sim}}(\mathcal{J}, \epsilon) \sqsupseteq_{\emptyset} \exists^{\mathsf{sim}}(\mathcal{I}, \delta)$

Another immediate consequence of the above proposition is that we can easily compute least common subsumers as follows.

3.4.3 Proposition. $\exists^{\mathsf{sim}}((\mathcal{I}, \delta) \times (\mathcal{J}, \epsilon))$ *is the least common subsumer of* $\exists^{\mathsf{sim}}(\mathcal{I}, \delta)$ *and* $\exists^{\mathsf{sim}}(\mathcal{J}, \epsilon)$.

Proof. Note that in [LPW10, Observation 3] is has been found that the product operation $\times$ is the infimum operation in the set of (equivalence classes of) pointed interpretations ordered by $\precsim$. Together with Proposition 3.4.2 this immediately implies the claim. □

It is straightforward to generalize the above proposition to the case of finitely many $\mathcal{EL}_{\mathsf{si}}$ concept descriptions instead of only two. Furthermore, in $\mathcal{EL}_{\mathsf{si}}^{\bot}$ the least common subsumer $\bigvee \mathbf{C}$ of some set $\mathbf{C}$ of $\mathcal{EL}_{\mathsf{si}}^{\bot}$ concept descriptions is equivalent to $\bot$ modulo $\emptyset$ if, and only if, there exists some $C \in \mathbf{C}$ such that $C \equiv_{\emptyset} \bot$; otherwise the least common subsumer $\bigvee \mathbf{C}$ can be computed as above. As an immediate consequence we infer that $\boldsymbol{\mathcal{EL}}_{\mathsf{si}}(\Sigma) := (\mathcal{EL}_{\mathsf{si}}(\Sigma), \sqsubseteq_{\emptyset}) / {\equiv_{\emptyset}}$ as well as $\boldsymbol{\mathcal{EL}}_{\mathsf{si}}^{\bot}(\Sigma) := (\mathcal{EL}_{\mathsf{si}}^{\bot}(\Sigma), \sqsubseteq_{\emptyset}) / {\equiv_{\emptyset}}$ are lattices in which the finitary infimum operation is the conjunction operation $\bigsqcap$ and where the finitary supremum operation is the least common subsumer operation $\bigvee$.

3.4.4 Proposition. *For each $\mathcal{EL}_{\mathsf{si}}$ concept description $\exists^{\mathsf{sim}}(\mathcal{I}, \delta)$, the following equivalence holds true.*

$$\exists^{\mathsf{sim}}(\mathcal{I}, \delta) \equiv_{\emptyset} \bigsqcap\{\, A \mid \delta \in A^{\mathcal{I}} \,\} \sqcap \bigsqcap\{\, \exists\, r. \exists^{\mathsf{sim}}(\mathcal{I}, \epsilon) \mid (\delta, \epsilon) \in r^{\mathcal{I}} \,\}$$

Proof. The subsumption $\sqsubseteq_{\emptyset}$ is obvious. For the converse direction we first transform the right-hand side into some concept description of the form $\exists^{\mathsf{sim}}(\mathcal{J}, \zeta)$. We add ζ to the domain $\Delta^{\mathcal{J}}$ as well as to the extension $A^{\mathcal{J}}$ for each concept name A where $\delta \in A^{\mathcal{I}}$. For each (ϵ, r) with $(\delta, \epsilon) \in r^{\mathcal{I}}$, we create a copy $(\mathcal{I}_{(\epsilon,r)}, \eta_{(\epsilon,r)})$ of $(\mathcal{I}, \epsilon)$ such that the domains $\Delta^{\mathcal{I}_{(\epsilon,r)}}$ are pairwise disjoint. We then add all these interpretations $\mathcal{I}_{(\epsilon,r)}$ to $\mathcal{J}$, and we further add the pair $(\zeta, \eta_{(\epsilon,r)})$ to $r^{\mathcal{J}}$ for each role name r where $(\delta, \epsilon) \in r^{\mathcal{I}}$. It is then easy to construct a simulation from $(\mathcal{I}, \delta)$ to $(\mathcal{J}, \zeta)$, i.e., the converse subsumption $\sqsupseteq_{\emptyset}$ is satisfied as well. □

Lutz, Piro, and Wolter [LPW10, Definition 28] define the *nth characteristic concept description* $\mathsf{X}^n(\mathcal{I}, \delta)$ of a pointed interpretation $(\mathcal{I}, \delta)$ that has a finite active signature recursively as follows.

$$\mathsf{X}^0(\mathcal{I}, \delta) := \bigsqcap\{\, A \mid A \in \Sigma_{\mathsf{C}} \text{ and } \delta \in A^{\mathcal{I}} \,\}$$
$$\mathsf{X}^{n+1}(\mathcal{I}, \delta) := \mathsf{X}^0(\mathcal{I}, \delta) \sqcap \bigsqcap\{\, \exists\, r. \mathsf{X}^n(\mathcal{I}, \epsilon) \mid r \in \Sigma_{\mathsf{R}} \text{ and } (\delta, \epsilon) \in r^{\mathcal{I}} \,\}$$

We also call $\mathsf{X}^n(\mathcal{I}, \delta)$ the *nth approximation* of $\exists^{\mathsf{sim}}(\mathcal{I}, \delta)$. In general, we shall denote the nth approximation of an $\mathcal{EL}_{\mathsf{si}}^{\bot}$ concept description C as $C{\restriction_n}$ where we additionally need to define that $\bot{\restriction_n} := \bot$ for each $n \in \mathbb{N}$. Clearly, if C is an $\mathcal{EL}^{\bot}$ concept description with role depth d, then $C \equiv_{\emptyset} C{\restriction_n}$ holds true for each $n \geq d$. Alternatively, we may call an nth approximation $C{\restriction_n}$ also a *restriction* of C to a role depth of n.

The approximations can also be obtained from the tree unravellings of $\mathcal{I}$. Fix some object $\delta \in \Delta^{\mathcal{I}}$. A *finite path* p in $\mathcal{I}$ starting at δ is a sequence

$$\delta =: \delta_0 \xrightarrow{r_1} \delta_1 \xrightarrow{r_2} \delta_2 \ldots \xrightarrow{r_n} \delta_n$$

such that $(\delta_{i-1}, \delta_i) \in r_i^{\mathcal{I}}$ for each index $i \in \{1, \ldots, n\}$. We further define the *last* object of p as $\mathsf{last}(p) := \delta_n$ and the *length* of p is $\mathsf{length}(p) := n$. Then, the *tree unraveling* $\mathsf{tree}(\mathcal{I}, \delta)$ of $\mathcal{I}$ at δ

is defined as the interpretation with the following components.

$$\Delta^{\mathsf{tree}(\mathcal{I},\delta)} := \{\, p \mid p \text{ is a finite path in } \mathcal{I} \text{ starting at } \delta \,\}$$

$$\cdot^{\mathsf{tree}(\mathcal{I},\delta)} : \begin{cases} A \mapsto \{\, p \mid \mathsf{last}(p) \in A^{\mathcal{I}} \,\} & \text{for any } A \in \Sigma_{\mathsf{C}} \\ r \mapsto \{\, (p,q) \mid p \xrightarrow{r} \mathsf{last}(q) = q \,\} & \text{for any } r \in \Sigma_{\mathsf{R}} \end{cases}$$

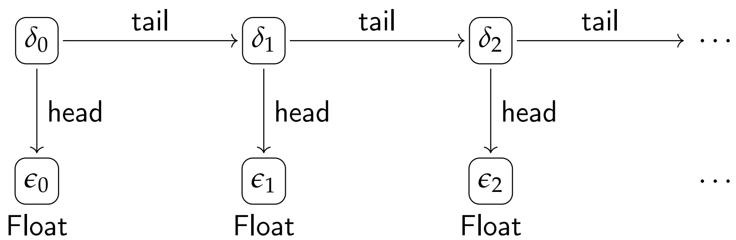
The relation $\{\, (\epsilon, p) \mid \epsilon = \mathsf{last}(p) \,\}$ is a simulation from $(\mathcal{I}, \delta)$ to its tree unraveling $(\mathsf{tree}(\mathcal{I}, \delta), \delta)$, and the inverse relation is a simulation in the converse direction, showing that the pointed interpretations $(\mathcal{I}, \delta)$ and $(\mathsf{tree}(\mathcal{I}, \delta), \delta)$ are equi-similar.

Furthermore, the *tree unraveling* $\mathsf{tree}(\mathcal{I}, \delta)\restriction_d$ of $\mathcal{I}$ at δ *restricted to depth* d is the induced subinterpretation of $\mathsf{tree}(\mathcal{I}, \delta)$ such that the domain only contains the finite paths in $\mathcal{I}$ starting at δ with lengths not exceeding d. It is easy to see that the concept descriptions $\mathsf{X}^d(\mathcal{I}, \delta)$, and $\mathsf{X}^d(\mathsf{tree}(\mathcal{I}, \delta)\restriction_d, \delta)$, and $\exists^{\mathsf{sim}}(\mathsf{tree}(\mathcal{I}, \delta)\restriction_d, \delta)$ are equivalent modulo $\emptyset$.

Example. Consider the interpretation $\mathcal{I}$ depicted below.

The approximations of the concept description $\exists^{\mathsf{sim}}(\mathcal{I}, \delta)$ are then the concept descriptions $\mathsf{X}^n(\mathcal{I}, \delta) = \exists\, r^n.\, \top$. △

Example. As another example, consider the interpretation $\mathcal{I}_{\mathsf{List}}$ shown below.



The finite approximations of $\exists^{\mathsf{sim}}(\mathcal{I}_{\mathsf{List}}, \delta_0)$ are the following concept descriptions.

$$\begin{aligned}
\mathsf{X}^0(\mathcal{I}_{\mathsf{List}}, \delta_0) &= \top \\
\mathsf{X}^1(\mathcal{I}_{\mathsf{List}}, \delta_0) &= \exists\, \mathsf{head}.\, \mathsf{Float} \sqcap \exists\, \mathsf{tail}.\, \top \\
\mathsf{X}^2(\mathcal{I}_{\mathsf{List}}, \delta_0) &= \exists\, \mathsf{head}.\, \mathsf{Float} \sqcap \exists\, \mathsf{tail}.\, (\exists\, \mathsf{head}.\, \mathsf{Float} \sqcap \exists\, \mathsf{tail}.\, \top) \\
\mathsf{X}^3(\mathcal{I}_{\mathsf{List}}, \delta_0) &= \exists\, \mathsf{head}.\, \mathsf{Float} \sqcap \exists\, \mathsf{tail}.\, (\exists\, \mathsf{head}.\, \mathsf{Float} \sqcap \exists\, \mathsf{tail}.\, (\exists\, \mathsf{head}.\, \mathsf{Float} \sqcap \exists\, \mathsf{tail}.\, \top)) \\
&\;\;\vdots \\
\mathsf{X}^{n+1}(\mathcal{I}_{\mathsf{List}}, \delta_k) &= \exists\, \mathsf{head}.\, \mathsf{Float} \sqcap \exists\, \mathsf{tail}.\, \mathsf{X}^n(\mathcal{I}_{\mathsf{List}}, \delta_{k+1})
\end{aligned}$$

△

In [LPW10] it has been shown that $\mathcal{EL}_{\mathsf{si}}$ has the *finite model property*. In particular, this implies that the following statements hold true.

1. If a concept description C is satisfiable w.r.t. some TBox $\mathcal{T}$, then there exists a finite model of $\mathcal{T}$ in which C has a non-empty extension.
2. If a concept inclusion $C \sqsubseteq D$ is not entailed by some TBox $\mathcal{T}$, then there exists a finite model of $\mathcal{T}$ that contains some counterexample against $C \sqsubseteq D$.

We will see later that both statements are justified by the existence of so-called canonical models $\mathcal{I}_{C,\mathcal{T}}$.

3.4.5 Lemma. *Fix some $\mathcal{EL}_{\mathsf{si}}$ concept description C as well as a finite interpretation $\mathcal{I}$. Then C has a* finite closure ordinal *for $\mathcal{I}$, i.e., there exists some number $m \in \mathbb{N}$ such that $C^{\mathcal{I}} = (C\restriction_m)^{\mathcal{I}}$ holds true.*

Proof. It is obvious that $k \leq \ell$ implies $C\restriction_k \sqsupseteq_\emptyset C\restriction_\ell$. In particular, then $(C\restriction_k)^{\mathcal{I}} \supseteq (C\restriction_\ell)^{\mathcal{I}}$ holds true. Since the domain $\Delta^{\mathcal{I}}$ is finite, there must exist some number $m \in \mathbb{N}$ such that $(C\restriction_m)^{\mathcal{I}} = (C\restriction_n)^{\mathcal{I}}$ for all $n \geq m$. We continue with proving that $(C\restriction_m)^{\mathcal{I}} = C^{\mathcal{I}}$ is satisfied. The direction $\supseteq$ is trivial.

Assume that C has the form $\exists^{\mathsf{sim}}(\mathcal{J},\alpha)$. Furthermore, consider some object $\delta \in \Delta^{\mathcal{I}}$ where $\delta \in (C\restriction_m)^{\mathcal{I}} = (\mathsf{X}^m(\mathcal{J},\alpha))^{\mathcal{I}}$. Note that $\bigcap\{\, (\mathsf{X}^n(\mathcal{J},\alpha))^{\mathcal{I}} \mid n \in \mathbb{N} \,\} = (\mathsf{X}^m(\mathcal{J},\alpha))^{\mathcal{I}}$ holds true. With analogous arguments as above we infer that, for each object $\beta \in \Delta^{\mathcal{J}}$, there exists some (finite) number $m_\beta \in \mathbb{N}$ satisfying $\bigcap\{\, (\mathsf{X}^n(\mathcal{J},\beta))^{\mathcal{I}} \mid n \in \mathbb{N} \,\} = (\mathsf{X}^{m_\beta}(\mathcal{J},\beta))^{\mathcal{I}}$.

We now construct a simulation $\mathfrak{S} \colon (\mathcal{J},\alpha) \precsim (\mathcal{I},\delta)$ as follows, and its existence then justifies that $\delta \in (\exists^{\mathsf{sim}}(\mathcal{J},\alpha))^{\mathcal{I}} = C^{\mathcal{I}}$.

$$\mathfrak{S} \coloneqq \{\, (\beta,\epsilon) \mid \epsilon \in \bigcap\{\, (\mathsf{X}^n(\mathcal{J},\beta))^{\mathcal{I}} \mid n \in \mathbb{N} \,\} \,\}$$

Of course, $(\alpha,\delta) \in \mathfrak{S}$ is satisfied. Now fix some pair $(\beta,\epsilon) \in \mathfrak{S}$.

- Let $\beta \in A^{\mathcal{J}}$. Then $\epsilon \in (\mathsf{X}^0(\mathcal{J},\beta))^{\mathcal{I}} \subseteq A^{\mathcal{I}}$.
- Assume that $(\beta,\gamma) \in r^{\mathcal{J}}$. We then have $\epsilon \in (\mathsf{X}^{n+1}(\mathcal{J},\beta))^{\mathcal{I}} \subseteq (\exists\, r.\, \mathsf{X}^n(\mathcal{J},\gamma))^{\mathcal{I}}$ for each $n \in \mathbb{N}$, that is, for each number $n \in \mathbb{N}$, there exists some object ζ_n such that $(\epsilon,\zeta_n) \in r^{\mathcal{I}}$ and $\zeta_n \in (\mathsf{X}^n(\mathcal{J},\gamma))^{\mathcal{I}}$. Now set $\zeta \coloneqq \zeta_{m_\gamma}$. It follows that $(\epsilon,\zeta) \in r^{\mathcal{I}}$ and $\zeta \in (\mathsf{X}^n(\mathcal{J},\gamma))^{\mathcal{I}}$ for each number $n \in \mathbb{N}$. We conclude that $(\gamma,\zeta) \in \mathfrak{S}$. □

3.4.6 Lemma. *Let $\mathcal{T} \cup \{C \sqsubseteq D\}$ be some $\mathcal{EL}_{\mathsf{si}}$ TBox. Then the following statements are equivalent.*

1. *$\mathcal{T}$ entails $C \sqsubseteq D$*
2. *$\mathcal{T}$ entails $C \sqsubseteq D\restriction_n$ for each $n \in \mathbb{N}$*

Proof. Since $D \sqsubseteq_\emptyset D\restriction_n$ holds true for each $n \in \mathbb{N}$, it is obvious that the first implies the second statement. Now assume that $\mathcal{T}$ does not entail $C \sqsubseteq D$. Since $\mathcal{EL}_{\mathsf{si}}$ has the finite model property, there exists a finite model $\mathcal{I}$ of $\mathcal{T}$ containing a counterexample against $C \sqsubseteq D$. Lemma 3.4.5 shows that D has a finite closure ordinal m for $\mathcal{I}$. Thus, we conclude that $C \sqsubseteq D\restriction_m$ is not valid in $\mathcal{I}$, which shows that $\mathcal{T}$ does not entail $C \sqsubseteq D\restriction_m$. □

3.4.7 Lemma. *Let $\exists^{\mathsf{sim}}(\mathcal{I},\delta)$ and $\exists^{\mathsf{sim}}(\mathcal{J},\epsilon)$ be $\mathcal{EL}_{\mathsf{si}}$ concept descriptions and fix some number $d \in \mathbb{N}$. The following statements are equivalent.*

1. $\exists^{\mathsf{sim}}(\mathcal{I},\delta) \sqsubseteq_\emptyset \mathsf{X}^d(\mathcal{J},\epsilon)$
2. $\mathsf{X}^d(\mathcal{I},\delta) \sqsubseteq_\emptyset \mathsf{X}^d(\mathcal{J},\epsilon)$

Proof. Since $\exists^{\mathsf{sim}}(\mathcal{I},\delta)$ is more specific than the approximation $\mathsf{X}^d(\mathcal{I},\delta)$, Statement 2 implies Statement 1.

For the converse direction, assume that $\exists^{\mathsf{sim}}(\mathcal{I},\delta) \sqsubseteq_\emptyset \mathsf{X}^d(\mathcal{J},\epsilon)$ is satisfied. Now note that $\mathsf{X}^d(\mathcal{J},\epsilon)$ is equivalent to $\exists^{\mathsf{sim}}(\mathsf{tree}(\mathcal{J},\epsilon)\restriction_d,\epsilon)$. According to Proposition 3.4.2 there exists some simulation $\mathfrak{S}$ from $(\mathsf{tree}(\mathcal{J},\epsilon)\restriction_d,\epsilon)$ to $(\mathcal{I},\delta)$. Since $(\mathcal{I},\delta)$ and its tree unraveling $(\mathsf{tree}(\mathcal{I},\delta),\delta)$ are equi-similar, we infer that also a simulation $\mathfrak{S}$ from the bounded tree unraveling $(\mathsf{tree}(\mathcal{J},\epsilon)\restriction_d,\epsilon)$ to the tree unraveling $(\mathsf{tree}(\mathcal{I},\delta),\delta)$ exists. It is apparent that we do not need the paths in $\mathcal{I}$ issuing at δ with a length exceeding d, i.e., $\mathfrak{S}$ must be a simulation from $(\mathsf{tree}(\mathcal{J},\epsilon)\restriction_d,\epsilon)$ to $(\mathsf{tree}(\mathcal{I},\delta)\restriction_d,\delta)$ as well. We conclude that

$$\mathsf{X}^d(\mathcal{I},\delta) \equiv_\emptyset \exists^{\mathsf{sim}}(\mathsf{tree}(\mathcal{I},\delta)\restriction_d,\delta) \sqsubseteq_\emptyset \exists^{\mathsf{sim}}(\mathsf{tree}(\mathcal{J},\epsilon)\restriction_d,\epsilon) \equiv_\emptyset \mathsf{X}^d(\mathcal{J},\epsilon). \qquad \square$$

3.4.8 Proposition. *Let $\exists^{\mathsf{sim}}(\mathcal{I},\delta)$ and $\exists^{\mathsf{sim}}(\mathcal{J},\epsilon)$ be $\mathcal{EL}_{\mathsf{si}}$ concept descriptions. The following statements are equivalent.*

1. $\exists^{\mathsf{sim}}(\mathcal{I},\delta) \sqsubseteq_\emptyset \exists^{\mathsf{sim}}(\mathcal{J},\epsilon)$
2. $\exists^{\mathsf{sim}}(\mathcal{I},\delta) \sqsubseteq_\emptyset \mathsf{X}^d(\mathcal{J},\epsilon)$ *for each number* $d \in \mathbb{N}$
3. $\mathsf{X}^d(\mathcal{I},\delta) \sqsubseteq_\emptyset \mathsf{X}^d(\mathcal{J},\epsilon)$ *for each number* $d \in \mathbb{N}$

Proof. The equivalence of Statements 1 and 2 follows from Lemma 3.4.6, and further Lemma 3.4.7 implies that Statements 1 and 2 are equivalent. $\square$

3.4.9 Definition. Let $\mathbf{C}$ be a set of $\mathcal{EL}_{\mathsf{si}}$ concept descriptions. We say that an $\mathcal{EL}_{\mathsf{si}}$ concept description D is the *infimum* or *greatest common subsumee* of $\mathbf{C}$ if it satisfies the following conditions.

1. $D \sqsubseteq_\emptyset C$ for each $C \in \mathbf{C}$.
2. If E is some $\mathcal{EL}_{\mathsf{si}}^\bot$ concept description such that $E \sqsubseteq_\emptyset C$ for each $C \in \mathbf{C}$, then $E \sqsubseteq_\emptyset D$.

Clearly, all infima of $\mathbf{C}$ are equivalent modulo $\emptyset$, and so we denote *the* infimum as $\bigwedge \mathbf{C}$. If now $(\, C_n \mid n \in \mathbb{N}\,)$ is some decreasing sequence of $\mathcal{EL}_{\mathsf{si}}^\bot$ concept descriptions, then its *limit* $\mathsf{lim}_{n\to\infty} C_n$ is defined as the infimum of $\{\, C_n \mid n \in \mathbb{N}\,\}$. Clearly, if such a sequence is ultimately constant modulo $\emptyset$, i.e., if there is some index k such that $C_k \equiv_\emptyset C_{k+1} \equiv_\emptyset C_{k+2} \equiv_\emptyset \ldots$ holds true, then the limit must be C_k. $\triangle$

It is easy to see that, for each finite set $\mathbf{C}$, the infimum $\bigwedge \mathbf{C}$ is equivalent to the conjunction $\bigsqcap \mathbf{C}$ modulo $\emptyset$.

3.4.10 Proposition. *Each $\mathcal{EL}_{\mathsf{si}}$ concept description $\exists^{\mathsf{sim}}(\mathcal{I},\delta)$ is the infimum of all its approximations $\mathsf{X}^d(\mathcal{I},\delta)$, that is, the following holds true.*

$$\bigwedge\{\, \mathsf{X}^d(\mathcal{I},\delta) \mid d \in \mathbb{N}\,\} \equiv_\emptyset \exists^{\mathsf{sim}}(\mathcal{I},\delta)$$

Proof. Since $\exists^{\mathsf{sim}}(\mathcal{I},\delta) \sqsubseteq_{\emptyset} \exists^{\mathsf{sim}}(\mathcal{I},\delta)$ is satisfied, Proposition 3.4.8 implies that $\exists^{\mathsf{sim}}(\mathcal{I},\delta) \sqsubseteq_{\emptyset} \mathsf{X}^d(\mathcal{I},\delta)$ holds true for each number $d \in \mathbb{N}$. Now consider some $\mathcal{EL}_{\mathsf{si}}$ concept description $\exists^{\mathsf{sim}}(\mathcal{J},\epsilon)$ such that $\exists^{\mathsf{sim}}(\mathcal{J},\epsilon) \sqsubseteq_{\emptyset} \mathsf{X}^d(\mathcal{I},\delta)$ for each $d \in \mathbb{N}$. Then Proposition 3.4.8 yields that $\exists^{\mathsf{sim}}(\mathcal{J},\epsilon) \sqsubseteq_{\emptyset} \exists^{\mathsf{sim}}(\mathcal{I},\delta)$. Eventually, we conclude that $\exists^{\mathsf{sim}}(\mathcal{I},\delta)$ is indeed the limit of the sequence of its approximations $\mathsf{X}^d(\mathcal{I},\delta)$. □

3.4.4 Reduced Forms of Interpretations

We have seen before that each $\mathcal{EL}$ concept description can be reduced by applications of the rule $C \sqcap D \mapsto C$ whenever $C \sqsubseteq_{\emptyset} D$ to its subconcepts. In the sequel of this section we are going to develop tools for reducing $\mathcal{EL}_{\mathsf{si}}$ concept descriptions. Since each $\mathcal{EL}_{\mathsf{si}}$ concept description can be transformed in linear time into the form $\exists^{\mathsf{sim}}(\mathcal{I},\delta)$, it suffices to find means for transforming such a pointed interpretation $(\mathcal{I},\delta)$ into an equi-similar, but "smaller" pointed interpretation. We shall split such a reduction process into three steps. At first, the domain is reduced by identifying equi-similar objects. Then, we utilize an existing approach [ET13; EPT15] for minimizing the number of role edges in the interpretation's graph. Eventually, we delete anything outside the connected component of δ.

Domain-Reduced Interpretations

3.4.11 Definition. Let $\mathcal{I}$ be a finite interpretation. We define the equivalence relation $\sim_{\mathcal{I}}$ on $\Delta^{\mathcal{I}}$ by $\delta \sim_{\mathcal{I}} \epsilon$ if $(\mathcal{I},\delta) \rightleftarrows (\mathcal{I},\epsilon)$. Then, the *domain reduction* of $\mathcal{I}$ is defined as the interpretation $\mathcal{I}_*$ that has the following components.

$$\Delta^{\mathcal{I}_*} := \Delta^{\mathcal{I}}/\sim_{\mathcal{I}}$$

$$\cdot^{\mathcal{I}_*} : \begin{cases} A \mapsto \{\, [\delta] \mid \exists^{\mathsf{sim}}(\mathcal{I},\delta) \sqsubseteq_{\emptyset} A \,\} & \text{for any } A \in \Sigma_{\mathsf{C}} \\ r \mapsto \{\, ([\delta],[\epsilon]) \mid \exists^{\mathsf{sim}}(\mathcal{I},\delta) \sqsubseteq_{\emptyset} \exists\, r.\exists^{\mathsf{sim}}(\mathcal{I},\epsilon) \,\} & \text{for any } r \in \Sigma_{\mathsf{R}} \end{cases}$$

Furthermore, $\mathcal{I}$ is *domain-reduced* if $\delta \sim_{\mathcal{I}} \epsilon$ implies $\delta = \epsilon$ for any objects $\delta,\epsilon \in \Delta^{\mathcal{I}}$. △

The next proposition shows that we do not have to distinguish between an interpretation $\mathcal{I}$ and its domain reduction $\mathcal{I}_*$.

3.4.12 Proposition. *Let $\mathcal{I}$ be a finite interpretation. Then, the following statements hold true.*

1. $(\mathcal{I},\delta) \rightleftarrows (\mathcal{I}_*,[\delta])$ *for each object* $\delta \in \Delta^{\mathcal{I}}$
2. $\mathcal{I} \models C \sqsubseteq D$ *if, and only if,* $\mathcal{I}_* \models C \sqsubseteq D$ *for any concept descriptions* C *and* D

Proof. Firstly, we shall show that the relation

$$\mathfrak{S} := \{\, (\epsilon,[\epsilon]) \mid \epsilon \in \Delta^{\mathcal{I}} \,\}$$

is a simulation from $(\mathcal{I},\delta)$ to $(\mathcal{I}_*,[\delta])$. Of course, it holds true that $(\delta,[\delta]) \in \mathfrak{S}$. Now consider some pair $(\epsilon,[\epsilon]) \in \mathfrak{S}$.

- If $\epsilon \in A^{\mathcal{I}}$ for a concept name $A \in \Sigma_{\mathsf{C}}$, then we infer that $\exists^{\mathsf{sim}}(\mathcal{I},\epsilon) \sqsubseteq_\emptyset A$ and, thus, that $[\epsilon] \in A^{\mathcal{I}_*}$.
- Furthermore, assume that $(\epsilon,\zeta) \in r^{\mathcal{I}}$ for some role name $r \in \Sigma_{\mathsf{R}}$. Since $\zeta \in (\exists^{\mathsf{sim}}(\mathcal{I},\zeta))^{\mathcal{I}}$ is satisfied, we infer that $\epsilon \in (\exists\, r.\, \exists^{\mathsf{sim}}(\mathcal{I},\zeta))^{\mathcal{I}}$ or, equivalently, that $\exists^{\mathsf{sim}}(\mathcal{I},\epsilon) \sqsubseteq_\emptyset \exists\, r.\, \exists^{\mathsf{sim}}(\mathcal{I},\zeta)$. We conclude that $([\epsilon],[\zeta]) \in r^{\mathcal{I}_*}$. By its very definition, $\mathfrak{S}$ contains $(\zeta,[\zeta])$.

Secondly, we prove that, vice versa, the relation

$$\mathfrak{T} := \{\, ([\epsilon],\zeta) \mid \exists^{\mathsf{sim}}(\mathcal{I},\zeta) \sqsubseteq_\emptyset \exists^{\mathsf{sim}}(\mathcal{I},\epsilon) \,\}$$

is a simulation in the converse direction. It is obvious that $([\delta],\delta) \in \mathfrak{T}$ is satisfied. Let $([\epsilon],\zeta) \in \mathfrak{T}$, i.e., $\exists^{\mathsf{sim}}(\mathcal{I},\zeta) \sqsubseteq_\emptyset \exists^{\mathsf{sim}}(\mathcal{I},\epsilon)$ holds true.

- If $[\epsilon] \in A^{\mathcal{I}_*}$, then $\exists^{\mathsf{sim}}(\mathcal{I},\epsilon) \sqsubseteq_\emptyset A$. We infer that $\exists^{\mathsf{sim}}(\mathcal{I},\zeta) \sqsubseteq_\emptyset A$, and so it follows that $\zeta \in A^{\mathcal{I}}$.
- If $([\epsilon],[\eta]) \in r^{\mathcal{I}_*}$, then it follows that $\exists^{\mathsf{sim}}(\mathcal{I},\epsilon) \sqsubseteq_\emptyset \exists\, r.\, \exists^{\mathsf{sim}}(\mathcal{I},\eta)$. We further infer that $\exists^{\mathsf{sim}}(\mathcal{I},\zeta) \sqsubseteq_\emptyset \exists\, r.\, \exists^{\mathsf{sim}}(\mathcal{I},\eta)$, that is, $\zeta \in (\exists\, r.\, \exists^{\mathsf{sim}}(\mathcal{I},\eta))^{\mathcal{I}}$. Consequently, there is some object $\theta \in (\exists^{\mathsf{sim}}(\mathcal{I},\eta))^{\mathcal{I}}$ such that $(\zeta,\theta) \in r^{\mathcal{I}}$. It follows that $\exists^{\mathsf{sim}}(\mathcal{I},\theta) \sqsubseteq_\emptyset \exists^{\mathsf{sim}}(\mathcal{I},\eta)$, which means that $([\eta],\theta) \in \mathfrak{T}$.

Thus, Statement 1 holds true. Eventually, it is apparent that Statement 2 is a consequence of Statement 1. □

Successor-Reduced Interpretations

The following notion has been defined by Ecke and Turhan [ET13, Definition 7] and Ecke, Peñaloza, and Turhan [EPT15, Definition 5.4] under the name *normal form*. However, we shall choose another.

3.4.13 Definition. (Reformulation of [EPT15, Definition 5.4]) Let $\mathcal{I}$ be an interpretation. Then, $\mathcal{I}$ is *successor-reduced* if there does not exist an object $\delta \in \Delta^{\mathcal{I}}$ that has distinct successors $\epsilon,\zeta \in \Delta^{\mathcal{I}}$, i.e., $\epsilon \neq \zeta$, which satisfy $\{(\delta,\epsilon),(\delta,\zeta)\} \subseteq r^{\mathcal{I}}$ for some role name Σ_{R} and $(\mathcal{I},\epsilon) \precsim (\mathcal{I},\zeta)$. △

Clearly, each interpretation $\mathcal{I}$ can be transformed into an equi-similar successor-reduced interpretation by exhaustively applying the following rule. Furthermore, we call any interpretation that results from exhaustive application of this rule also a *successor reduction* of $\mathcal{I}$.

Successor-Reduce-Rule Choose an object $\delta \in \Delta^{\mathcal{I}}$ as well as a role name $r \in \Sigma_{\mathsf{R}}$. If there are distinct r-successors $\epsilon,\zeta \in \Delta^{\mathcal{I}}$, i.e., $\epsilon \neq \zeta$ as well as $\{(\delta,\epsilon),(\delta,\zeta)\} \subseteq r^{\mathcal{I}}$, that satisfy $(\mathcal{I},\epsilon) \precsim (\mathcal{I},\zeta)$, then define the interpretation $\mathcal{J} := \mathcal{I}$, but where $r^{\mathcal{J}} := r^{\mathcal{I}} \setminus \{(\delta,\epsilon)\}$, and return $\mathcal{J}$; otherwise the rule is not applicable for the specific choice of δ and r.

3.4.14 Lemma. (Reformulation of [ET13, Lemma 8]) *Let $(\mathcal{I},\delta)$ and $(\mathcal{J},\epsilon)$ be pointed interpretations such that $\mathcal{I}'$ is a successor-reduced form of $\mathcal{I}$ and $\mathcal{J}'$ is a successor-reduced form of $\mathcal{J}$. Then, the following statements hold true.*

1. $(\mathcal{I},\delta) \precsim (\mathcal{J},\epsilon)$ *implies* $(\mathcal{I}',\delta) \precsim (\mathcal{J}',\epsilon)$
2. *If* $(\mathcal{I}',\delta) \simeq (\mathcal{J}',\epsilon)$ *and* $(\delta,\zeta) \in r^{\mathcal{I}'}$, *then there exists a unique* $\eta \in \Delta^{\mathcal{J}'}$ *such that* $(\mathcal{I}',\zeta) \simeq (\mathcal{J}',\eta)$ *and* $(\epsilon,\eta) \in r^{\mathcal{J}'}$. □

3.4.15 Lemma. *Let* $(\mathcal{I}',[\delta])$ *be the successor reduction of the domain reduction of* $(\mathcal{I},\delta)$. *Then, the following relation* $\mathfrak{S}$ *is a simulation from* $(\mathcal{I}',[\delta])$ *to* $(\mathcal{I},\delta)$.

$$\mathfrak{S} := \{ ([\epsilon],\epsilon) \mid [\epsilon] \in \Delta^{\mathcal{I}'} \}$$

Proof. By definition we have that $([\delta],\delta)$ is in $\mathfrak{S}$. Now fix some arbitrary pair $([\epsilon],\epsilon)$ in $\mathfrak{S}$.

- If $[\epsilon] \in A^{\mathcal{I}'}$, then $\exists^{\mathsf{sim}}(\mathcal{I},\epsilon) \sqsubseteq_\emptyset A$ holds true by definition of the domain reduction. We conclude that $\epsilon \in A^{\mathcal{I}}$.
- Let $([\epsilon],[\zeta]) \in r^{\mathcal{I}'}$, that is, we have $\exists^{\mathsf{sim}}(\mathcal{I},\epsilon) \sqsubseteq_\emptyset \exists\, r.\, \exists^{\mathsf{sim}}(\mathcal{I},\zeta)$. We conclude that ϵ is an element of the extension $(\exists\, r.\, \exists^{\mathsf{sim}}(\mathcal{I},\zeta))^{\mathcal{I}}$, and so there is some object η such that $(\epsilon,\eta) \in r^{\mathcal{I}}$ and $\eta \in (\exists^{\mathsf{sim}}(\mathcal{I},\zeta))^{\mathcal{I}}$. Of course, the latter statement is equivalent to $(\mathcal{I},\zeta) \precsim (\mathcal{I},\eta)$. Proposition 3.4.12 further implies that $(\mathcal{I}_*,[\zeta]) \precsim (\mathcal{I}_*,[\eta])$.

  Now $(\epsilon,\eta) \in r^{\mathcal{I}}$ implies that $\exists^{\mathsf{sim}}(\mathcal{I},\epsilon) \sqsubseteq_\emptyset \exists\, r.\, \exists^{\mathsf{sim}}(\mathcal{I},\eta)$ is satisfied, which means that $([\epsilon],[\eta]) \in r^{\mathcal{I}_*}$. Since $\mathcal{I}'$ is the successor reduction of $\mathcal{I}_*$, we conclude that $[\zeta] = [\eta]$. □

Connected Interpretations

We call a pointed interpretation $(\mathcal{I},\delta)$ *connected* if each object $\epsilon \in \Delta^{\mathcal{I}}$ is reachable from δ. It is obvious that, for each pointed interpretation $(\mathcal{I},\delta)$, there exists a pointed interpretation $(\mathcal{J},\delta)$ which is connected and equi-similar to $(\mathcal{I},\delta)$. Such a $(\mathcal{J},\delta)$ can simply be obtained by removing all objects not reachable from δ, and is called *connected component* of $(\mathcal{I},\delta)$.

Reduced Interpretations

Combining all three notions leads us to the following. A pointed interpretation $(\mathcal{I},\delta)$ is *reduced* if it is domain-reduced, successor-reduced, and connected.

3.4.16 Lemma. *Each pointed interpretation* $(\mathcal{I},\delta)$ *can be transformed into a reduced pointed interpretation by means of the following procedure.*

1. *Firstly, construct the domain reduction* $(\mathcal{I}_d,\delta_d)$ *of* $(\mathcal{I},\delta)$.
2. *Secondly, construct the successor reduction* $(\mathcal{I}_s,\delta_s)$ *of* $(\mathcal{I}_d,\delta_d)$.
3. *Thirdly, construct the connected component* $(\mathcal{I}_c,\delta_c)$ *of* $(\mathcal{I}_s,\delta_s)$.
4. *Eventually, return* $(\mathcal{I}_c,\delta_c)$.

Proof. Let $(\mathcal{J},\epsilon)$ be the result of applying the above procedure to some pointed interpretation $(\mathcal{I},\delta)$. It is easy to see that $(\mathcal{J},\epsilon)$ is successor-reduced and connected. It remains to show that $(\mathcal{J},\epsilon)$ is domain-reduced. For this purpose, fix two objects $\eta,\theta \in \Delta^{\mathcal{J}}$ where $\eta \neq \theta$. In particular, both η and θ are contained in the domain of $\mathcal{I}_d$ and it follows that $(\mathcal{I}_d,\eta) \not\precsim (\mathcal{I}_d,\theta)$, since $\mathcal{I}_d$ is domain-reduced. We further know that $(\mathcal{I}_d,\eta) \simeq (\mathcal{I}_s,\eta) \simeq (\mathcal{I}_c,\eta) = (\mathcal{J},\eta)$ and similarly for θ. Thus, it must as well hold true that $(\mathcal{J},\eta) \not\precsim (\mathcal{J},\theta)$. □

Note that a *homomorphism* from $(\mathcal{I},\delta)$ to $(\mathcal{J},\epsilon)$ is a left-total, functional[3] simulation from $(\mathcal{I},\delta)$ to $(\mathcal{J},\epsilon)$. In particular, such a homomorphism satisfies the following conditions.

1. $\phi(\delta) = \epsilon$
2. $\zeta \in A^{\mathcal{I}}$ implies $\phi(\zeta) \in A^{\mathcal{J}}$ for any concept name $A \in \Sigma_{\mathsf{C}}$.
3. $(\zeta,\eta) \in r^{\mathcal{I}}$ implies $(\phi(\zeta),\phi(\eta)) \in r^{\mathcal{J}}$ for each role name $r \in \Sigma_{\mathsf{R}}$.

An *isomorphism* between $(\mathcal{I},\delta)$ and $(\mathcal{J},\epsilon)$ is a bijective homomorphism from $(\mathcal{I},\delta)$ to $(\mathcal{J},\epsilon)$ such that its inverse is a homomorphism in the converse direction.

3.4.17 Proposition. *Let $(\mathcal{I},\delta)$ and $(\mathcal{J},\epsilon)$ be reduced pointed interpretations. If $(\mathcal{I},\delta)$ and $(\mathcal{J},\epsilon)$ are equi-similar, then there exists an isomorphism between $(\mathcal{I},\delta)$ and $(\mathcal{J},\epsilon)$.*

Proof. We define a mapping $\phi\colon \Delta^{\mathcal{I}} \to \Delta^{\mathcal{J}}$ inductively. Initially, set $\phi(\delta) := \epsilon$. If $\phi(\delta_1) = \epsilon_1$ has been defined and $(\delta_1,\delta_2) \in r^{\mathcal{I}}$, then according to Lemma 3.4.14 there exists a unique $\epsilon_2 \in \Delta^{\mathcal{J}}$ such that $(\mathcal{I},\delta_2) \rightleftarrows (\mathcal{J},\epsilon_2)$ and $(\epsilon_1,\epsilon_2) \in r^{\mathcal{J}}$. Now set $\phi(\delta_2) := \epsilon_2$.

ϕ is well-defined for the following reasons. Assume that $\phi(\delta_1) = \epsilon_1$ has been defined, $(\delta_1,\delta_2) \in r^{\mathcal{I}} \cap s^{\mathcal{I}}$, and that there are objects $\epsilon_2,\epsilon_2' \in \Delta^{\mathcal{J}}$ such that $(\epsilon_1,\epsilon_2) \in r^{\mathcal{J}}$, $(\epsilon_1,\epsilon_2') \in s^{\mathcal{J}}$, $(\mathcal{I},\delta_2) \rightleftarrows (\mathcal{J},\epsilon_2)$, and $(\mathcal{I},\delta_2) \rightleftarrows (\mathcal{J},\epsilon_2')$. It then follows that $(\mathcal{J},\epsilon_2) \rightleftarrows (\mathcal{J},\epsilon_2')$, and since $\mathcal{J}$ is domain-reduced we conclude that $\epsilon_2 = \epsilon_2'$. This shows that $\phi(\delta_2) = \epsilon_2$ is well-defined.

We proceed with demonstrating that ϕ is a homomorphism. By construction, $(\mathcal{I},\delta_1) \rightleftarrows (\mathcal{J},\phi(\delta_1))$ holds true for all objects $\delta_1 \in \Delta^{\mathcal{I}}$. Of course, $\delta_1 \in A^{\mathcal{I}}$ immediately implies $\phi(\delta_1) \in A^{\mathcal{J}}$. Further assume that $(\delta_1,\delta_2) \in r^{\mathcal{I}}$. By definition, it then follows that $(\phi(\delta_1),\phi(\delta_2)) \in r^{\mathcal{J}}$.

Let $\phi(\delta_1) = \phi(\delta_2)$. It follows that $(\mathcal{I},\delta_1) \rightleftarrows (\mathcal{J},\phi(\delta_1)) = (\mathcal{J},\phi(\delta_2)) \rightleftarrows (\mathcal{I},\delta_2)$, which yields that $\delta_1 = \delta_2$, since $\mathcal{I}$ is domain-reduced. We conclude that ϕ is injective.

Now we show that ϕ is surjective. Since $(\mathcal{J},\epsilon)$ is connected, we can reach every object in $\Delta^{\mathcal{J}}$ by a path issuing from ϵ. Let $\epsilon \xrightarrow{r_1} \epsilon_1 \xrightarrow{r_2} \epsilon_2 \xrightarrow{r_3} \ldots \xrightarrow{r_n} \epsilon_n$ be an arbitrary path in $\mathcal{J}$. Then, [ET13, Lemma 8] yields the existence of a unique path $\delta \xrightarrow{r_1} \delta_1 \xrightarrow{r_2} \delta_2 \xrightarrow{r_3} \ldots \xrightarrow{r_n} \delta_n$ such that $(\mathcal{I},\delta_i) \rightleftarrows (\mathcal{J},\epsilon_i)$ for all indexes $i \in \{1,\ldots,n\}$. By definition of ϕ, we then have the path $\phi(\delta) \xrightarrow{r_1} \phi(\delta_1) \xrightarrow{r_2} \phi(\delta_2) \xrightarrow{r_3} \ldots \xrightarrow{r_n} \phi(\delta_n)$ in $\mathcal{J}$, and further it holds true that $(\mathcal{I},\delta_i) \rightleftarrows (\mathcal{J},\phi(\delta_i))$ for all $i \in \{1,\ldots,n\}$. We infer that $(\mathcal{J},\epsilon_i) \rightleftarrows (\mathcal{J},\phi(\delta_i))$, and thus $\epsilon_i = \phi(\delta_i)$ for all i.

It remains to show that the inverse of ϕ is a homomorphism as well. By construction, we have that $(\mathcal{I},\phi^{-1}(\epsilon_1)) \rightleftarrows (\mathcal{J},\epsilon_1)$ is satisfied for all objects $\epsilon_1 \in \Delta^{\mathcal{J}}$. Hence, $\epsilon_1 \in A^{\mathcal{J}}$ implies $\phi^{-1}(\epsilon_1) \in A^{\mathcal{I}}$. Furthermore, if $(\epsilon_1,\epsilon_2) \in r^{\mathcal{J}}$, then there is a unique $\delta_2 \in \Delta^{\mathcal{I}}$ such that $(\phi^{-1}(\epsilon_1),\delta_2) \in r^{\mathcal{I}}$ and $(\mathcal{I},\delta_2) \rightleftarrows (\mathcal{J},\epsilon_2)$. Since $\mathcal{I}$ is successor-reduced, we conclude that $\phi^{-1}(\epsilon_2) = \delta_2$. □

Since it holds true that any pointed interpretation $(\mathcal{I},\delta)$ is equi-similar to each of its reduced forms, we infer by means of Proposition 3.4.17 that all reduced forms of $(\mathcal{I},\delta)$ must be isomorphic. Consequently, we do not need to distinguish between them, and may symbolize *one arbitrary* reduced form as $\mathsf{reduce}(\mathcal{I},\delta)$. Furthermore, the next proposition shows that this reduced form can be computed in polynomial time.

[3] A relation $R \subseteq X \times Y$ is *left-total* and *functional* if, for each element $x \in X$, there is exactly one element $y \in Y$ with $(x,y) \in R$. We then also write $R\colon X \to Y$ to indicate that R is left-total and functional, and further we write $R(x) = y$ instead of $(x,y) \in R$.

3.4.18 Proposition. *The reduced form* $\mathsf{reduce}(\mathcal{I},\delta)$ *of some pointed interpretation* $(\mathcal{I},\delta)$ *can be computed in time polynomial in* $||\mathcal{I}||$.

Proof. We start with considering the task of computing the domain reduction $(\mathcal{I}_d,\delta_d)$ of $(\mathcal{I},\delta)$. Its domain is obtained by factorizing the domain of $\mathcal{I}$ with respect to the equivalence relation $\sim_{\mathcal{I}}$. Since checking the existence of a simulation can be done in polynomial time, and only polynomially many such checks whether $\delta \sim_{\mathcal{I}} \epsilon$ for all $\delta,\epsilon \in \Delta^{\mathcal{I}}$ are necessary, we infer that the domain $\Delta^{\mathcal{I}_d}$ can be obtained in polynomial time. Since reasoning in $\mathcal{EL}_{\mathsf{si}}$ has polynomial time complexity [LPW10, Theorem 12], it follows that all extensions can be computed in polynomial time as well. Since $(\mathcal{I},\delta)$ is finite, there are at most $||\mathcal{I}||$ non-empty extensions in $\mathcal{I}_d$, i.e., which need to be computed.

As next step we investigate the required time for constructing the successor reduction $(\mathcal{I}_s,\delta_s)$ from the domain reduction $(\mathcal{I}_d,\delta_d)$. Clearly, the number of choices $(\delta,\epsilon,\zeta,r)$ for applying the Successor-Reduce-Rule is polynomial in the size $||\mathcal{I}||$. For each such choice, we need to check whether $\epsilon \neq \zeta$ and $\{(\delta,\epsilon),(\delta,\zeta)\} \subseteq r^{\mathcal{I}}$, which can be done in polynomial time. (The concrete time complexity depends on the data structure that is used to store and represent the interpretation $\mathcal{I}$.) Furthermore, it is well known that deciding the existence of a simulation can be made in polynomial time as well. Summing up, we conclude that computing $(\mathcal{I}_s,\delta_s)$ needs time polynomial in $||\mathcal{I}||$.

Eventually, for computing the connected reduction $(\mathcal{I}_c,\delta_c)$ of $(\mathcal{I}_s,\delta_s)$, we have to determine which objects are reachable from δ_s. Since the Floyd-Warshall algorithm solves this task in polynomial time—more specifically, it can compute the transitive closure E^+ of a finite directed graph (V,E) in time $\mathcal{O}(|V|^3)$ and in space $\mathcal{O}(|V|^2)$—it is immediate to conclude that $(\mathcal{I}_c,\delta_c)$ can be constructed in polynomial time as well. □

Of course, we can utilize the above approach for reducing interpretations for defining a reduced form of $\mathcal{EL}_{\mathsf{si}}$ concept descriptions: we call $\exists^{\mathsf{sim}}(\mathsf{reduce}(\mathcal{I},\delta))$ the *reduced form* of $\exists^{\mathsf{sim}}(\mathcal{I},\delta)$. Both concept descriptions are then equivalent modulo $\emptyset$. Furthermore, the reduced form can be computed in polynomial time and is unique modulo renaming of objects. Obviously, Proposition 3.4.17 implies that there cannot exist a reduced pointed interpretation that is equi-similar to $(\mathcal{I},\delta)$ but has a smaller domain than $\mathsf{reduce}(\mathcal{I},\delta)$.

3.5 Simulations for $\mathcal{M}$

We have already seen that the description logic $\mathcal{EL}$ can be characterized by means of simulations and the existence of finite minimal models. More specifically, [LPW10, Theorem 5] shows that the set of (FOL translations of) $\mathcal{EL}$ concept descriptions is a maximal set of FOL formulas that is preserved under simulations and has finite minimal models. The goal of this section now is to also find a suitable characterization for the more expressive description logic $\mathcal{M}$. In particular, we define a notion of simulation specifically tailored to $\mathcal{M}$ and then show that the set of (FOL translations) of $\mathcal{M}$ concept descriptions is a maximal set of FOL formulas that is preserved under $\mathcal{M}$ simulations. Note that the technique of proving preservation under the new notion of simulation is strongly influenced by Kurtonina and de Rijke [Kd99].

As pointed out earlier, the extension $r^{\mathcal{I}}$ of a role name r can also be treated as a function $r^{\mathcal{I}}\colon \Delta^{\mathcal{I}} \to \wp(\Delta^{\mathcal{I}})$ such that $r^{\mathcal{I}}(x) \ni y$ if, and only if, $(x,y) \in r^{\mathcal{I}}$. In case $y \in r^{\mathcal{I}}(x)$ we call y an *r-successor* of x in $\mathcal{I}$. Furthermore, for each subset $X \subseteq \Delta^{\mathcal{I}}$ we denote by $r^{\mathcal{I}}(X)$ the set of all r-successors of elements in X, i.e., we set $r^{\mathcal{I}}(X) := \bigcup\{\, r^{\mathcal{I}}(x) \mid x \in X \,\}$. The following *quantified successor sets* are now defined.

$$\mathsf{Succ}_{\mathcal{I}}(X, \exists \geq n.\, r) := \{\, Y \mid Y \subseteq r^{\mathcal{I}}(X) \text{ and } \forall\, x \in X\colon |r^{\mathcal{I}}(x) \cap Y| \geq n \,\}$$
$$\mathsf{Succ}_{\mathcal{I}}(X, \forall\, r) := \{\, Y \mid Y \subseteq r^{\mathcal{I}}(X) \text{ and } \forall\, x \in X\colon r^{\mathcal{I}}(x) \subseteq Y \,\}.$$

It is readily verified that $\mathsf{Succ}_{\mathcal{I}}(X, \forall\, r) = \{r^{\mathcal{I}}(X)\}$ always holds true.

3.5.1 Definition. Let $\mathcal{I}$ and $\mathcal{J}$ be interpretations over the same signature Σ. Then an $\mathcal{M}$ *simulation* from $\mathcal{I}$ to $\mathcal{J}$ is a relation $\mathfrak{S} \subseteq \wp(\Delta^{\mathcal{I}}) \times \wp(\Delta^{\mathcal{J}})$ such that, for all pairs $(X, Y) \in \mathfrak{S}$, the following conditions are satisfied.

S1. $X \subseteq A^{\mathcal{I}}$ implies $Y \subseteq A^{\mathcal{J}}$ for each concept name $A \in \Sigma_{\mathsf{C}}$.

S2. $X \cap A^{\mathcal{I}} = \emptyset$ implies $Y \cap A^{\mathcal{J}} = \emptyset$ for each concept name $A \in \Sigma_{\mathsf{C}}$.

S3. For all positive natural numbers $n \in \mathbb{N}_+$ and for all role names $r \in \Sigma_{\mathsf{R}}$, if $X' \in \mathsf{Succ}_{\mathcal{I}}(X, \exists \geq n.\, r)$, then there is a subset $Y' \subseteq \Delta^{\mathcal{J}}$ such that $(X', Y') \in \mathfrak{S}$ and $Y' \in \mathsf{Succ}_{\mathcal{J}}(Y, \exists \geq n.\, r)$.

S4. For all natural numbers $n \in \mathbb{N}$ and for all role names $r \in \Sigma_{\mathsf{R}}$, if it holds true that $|r^{\mathcal{I}}(x)| \leq n$ for all $x \in X$, then it also holds true that $|r^{\mathcal{J}}(y)| \leq n$ for all $y \in Y$.

S5. For all role names $r \in \Sigma_{\mathsf{R}}$, if $X' \in \mathsf{Succ}_{\mathcal{I}}(X, \forall\, r)$, then there is a subset $Y' \subseteq \Delta^{\mathcal{J}}$ such that $(X', Y') \in \mathfrak{S}$ and $Y' \in \mathsf{Succ}_{\mathcal{J}}(Y, \forall\, r)$.
Alternative formulation: For all role names $r \in \Sigma_{\mathsf{R}}$, it holds true that $(r^{\mathcal{I}}(X), r^{\mathcal{J}}(Y)) \in \mathfrak{S}$.

S6. For all role names $r \in \Sigma_{\mathsf{R}}$, if $(x, x) \in r^{\mathcal{I}}$ for all $x \in X$, then $(y, y) \in r^{\mathcal{J}}$ for all $y \in Y$.

We shall denote by $\mathfrak{S}\colon \mathcal{I} \precsim \mathcal{J}$ the fact that $\mathfrak{S}$ is an $\mathcal{M}$ simulation from $\mathcal{I}$ to $\mathcal{J}$. Furthermore, we say that $\mathfrak{S}$ is an $\mathcal{M}$ simulation from $(\mathcal{I}, X)$ to $(\mathcal{J}, Y)$, symbolized by $\mathfrak{S}\colon (\mathcal{I}, X) \precsim (\mathcal{J}, Y)$, if $\mathfrak{S}$ is an $\mathcal{M}$ simulation from $\mathcal{I}$ to $\mathcal{J}$ such that $(X, Y) \in \mathfrak{S}$. In case only the existence of a simulation shall be expressed, then we may use the notations $\mathcal{I} \precsim \mathcal{J}$ as well as $(\mathcal{I}, X) \precsim (\mathcal{J}, Y)$ with their obvious meanings. △

3.5.2 Theorem. *Let Σ be a signature, and consider a FOL formula $\phi(x)$ over Σ with one free variable x. Then the following statements are equivalent.*

1. *ϕ is equivalent to (the FOL translation of) an $\mathcal{M}$ concept description, i.e., there is an $\mathcal{M}$ concept description C over Σ such that $\forall\, x.\, (\phi(x) \leftrightarrow C^{\#}(x))$ is a tautology.*[4]
2. *ϕ is preserved under $\mathcal{M}$ simulations, i.e., if $\mathfrak{S}\colon (\mathcal{I}, X) \precsim (\mathcal{J}, Y)$ is an $\mathcal{M}$ simulation and $X \subseteq \phi^{\mathcal{I}}$, then $Y \subseteq \phi^{\mathcal{J}}$.*[5]

[4]We denote by $C^{\#}$ the translation of an $\mathcal{M}$ concept description C into a FOL formula.
[5]We define $\phi^{\mathcal{I}} := \{\, \delta \in \Delta^{\mathcal{I}} \mid \mathcal{I}, \{x \mapsto \delta\} \models \phi \,\}$ for each FOL formula $\phi(x)$ with one free variable x and each interpretation $\mathcal{I}$.

Proof. It is a finger exercise to prove by structural induction on C that Statement 1 implies Statement 2. We now turn our attention to the more difficult part of demonstrating that Statement 2 holds true only if Statement 1 holds true. Hence, assume that $\phi(x)$ is a FOL formula over Σ which has one free variable x and is preserved under $\mathcal{M}$ simulations. Define

$$\mathsf{Con}_{\mathcal{M}}(\phi) := \{ C^\# \mid C \in \mathcal{M}(\Sigma) \text{ and } \phi \models C^\# \}$$

as the set of all FOL translations of $\mathcal{M}$ concept descriptions which are entailed by ϕ. Our objective is to show that $\mathsf{Con}_{\mathcal{M}}(\phi) \models \phi$, since then we can conclude by compactness of FOL that there is a finite subset $\mathcal{F}_\phi \subseteq \mathsf{Con}_{\mathcal{M}}(\phi)$ with $\mathcal{F}_\phi \models \phi$, and thus ϕ is equivalent to the $\mathcal{M}$ concept description

$$C_\phi := \bigsqcap \{ C \mid C^\# \in \mathcal{F}_\phi \}.$$

Let $\mathcal{I}$ be an interpretation and $\delta \in \Delta^{\mathcal{I}}$ such that $\mathcal{I}, \{x \mapsto \delta\} \models \mathsf{Con}_{\mathcal{M}}(\phi)$. We are going to show that $\mathcal{I}, \{x \mapsto \delta\} \models \phi$. For this purpose define

$$\mathbf{M}_\delta := \{ C \mid C \in \mathcal{M}(\Sigma) \text{ and } \delta \notin C^{\mathcal{I}} \}.$$

For each $C \in \mathbf{M}_\delta$, since it holds true that $C^\#$ cannot be a consequence of ϕ, i.e., $C^\# \notin \mathsf{Con}_{\mathcal{M}}(\phi)$, we conclude that the set $\{\phi, \neg C^\#\}$ is consistent. Consequently, for each $C \in \mathbf{M}_\delta$, there is a pointed interpretation $(\mathcal{J}_C, \delta_C)$ such that $\delta_C \in \phi^{\mathcal{J}_C} \cap (\neg C)^{\mathcal{J}_C}$. Now let

$$\mathcal{J} := \biguplus \{ \mathcal{J}_C \mid C \in \mathbf{M}_\delta \}$$

be the disjoint union of all these interpretations, and let κ be an infinite cardinal greater than $|\mathbb{N} \times \Delta^{\mathcal{I}}|$. By [CK90] there is a κ-saturated elementary extension $\mathcal{I}^*$ of $\mathcal{I}$, i.e., in particular then for all $\mathcal{M}$ concept descriptions D over Σ and for all $\delta \in \Delta^{\mathcal{I}}$, the statements $\delta \in D^{\mathcal{I}}$ and $\delta \in D^{\mathcal{I}^*}$ are equivalent. Let $\mathfrak{S} \subseteq \wp(\Delta^{\mathcal{J}}) \times \wp(\Delta^{\mathcal{I}^*})$ be defined as follows.

$$\mathfrak{S} := \{ (X, Y) \mid X \subseteq D^{\mathcal{J}} \text{ implies } Y \subseteq D^{\mathcal{I}^*} \text{ for each } D \in \mathcal{M}(\Sigma) \}$$

We demonstrate that $\mathfrak{S}$ is an $\mathcal{M}$ simulation. So consider an arbitrary pair $(X, Y) \in \mathfrak{S}$.

S1. Since each concept name $A \in \Sigma_{\mathsf{C}}$ is an $\mathcal{M}$ concept description, $X \subseteq A^{\mathcal{J}} \Rightarrow Y \subseteq A^{\mathcal{I}^*}$ holds by the very definition of $\mathfrak{S}$.

S2. For each concept name $A \in \Sigma_{\mathsf{C}}$, its primitive negation $\neg A$ is an $\mathcal{M}$ concept description, and so the definition of $\mathfrak{S}$ immediately yields that $X \subseteq (\neg A)^{\mathcal{J}}$ implies $Y \subseteq (\neg A)^{\mathcal{I}^*}$.

S3. Let $X' \in \mathsf{Succ}_{\mathcal{J}}(X, \exists{\geq}n.r)$ for an $n \in \mathbb{N}_+$ and an $r \in \Sigma_{\mathsf{R}}$, i.e., $|r^{\mathcal{J}}(x) \cap X'| \geq n$ for all $x \in X$, and $X' \subseteq r^{\mathcal{I}}(X)$. Define $\mathcal{D} := \{ D \mid D \in \mathcal{M}(\Sigma) \text{ and } X' \subseteq D^{\mathcal{J}} \}$, and let Φ be the following set of FOL formulas the free variables of which are in $\mathsf{Var}(\Phi) := \{ x_i^y \mid i \in \{1, \ldots, n\} \text{ and } y \in Y \}$.

$$\begin{aligned}\Phi := &\{ x_i^y \neq x_j^y \mid i, j \in \{1, \ldots, n\},\ i < j, \text{ and } y \in Y \} \\ &\cup \{ D^\#(x_i^y) \mid D \in \mathcal{D},\ i \in \{1, \ldots, n\}, \text{ and } y \in Y \}\end{aligned}$$

$$\cup \{\, r(y, x_i^y) \mid i \in \{1, \ldots, n\} \text{ and } y \in Y \,\}$$

Then Φ is finitely satisfiable in $\mathcal{I}^*$ for the following reasons. First, each finite subset $\Psi \subseteq \Phi$ is contained in a finite subset $\Psi_{Z,\mathcal{F}}$ of Φ which has the form

$$\begin{aligned}\Psi_{Z,\mathcal{F}} \coloneqq \{\, & x_i^y \neq x_j^y \mid i, j \in \{1, \ldots, n\},\ i < j, \text{ and } y \in Z \,\} \\ \cup \{\, & D^{\#}(x_i^y) \mid D \in \mathcal{F},\ i \in \{1, \ldots, n\}, \text{ and } y \in Z \,\} \\ \cup \{\, & r(y, x_i^y) \mid i \in \{1, \ldots, n\} \text{ and } y \in Z \,\}\end{aligned}$$

for some finite subset $Z \subseteq Y$ and some finite subset $\mathcal{F} \subseteq \mathcal{D}$. Second, each such finite subset $\Psi_{Z,\mathcal{F}} \subseteq \Phi$ is satisfiable. $X' \subseteq (\bigsqcap \mathcal{F})^{\mathcal{J}}$ implies $X \subseteq (\exists \geq n. r. \bigsqcap \mathcal{F})^{\mathcal{J}}$, and so $(X, Y) \in \mathfrak{S}$ yields $Y \subseteq (\exists \geq n. r. \bigsqcap \mathcal{F})^{\mathcal{I}^*}$. In particular, $(\bigsqcap \mathcal{F})^{\mathcal{I}^*} \cap r^{\mathcal{I}^*}(Y) \in \mathsf{Succ}_{\mathcal{I}^*}(Y, \exists \geq n. r)$, and as $Z \subseteq Y$ holds true we also know that $(\bigsqcap \mathcal{F})^{\mathcal{I}^*} \cap r^{\mathcal{I}^*}(Z) \in \mathsf{Succ}_{\mathcal{I}^*}(Z, \exists \geq n. r)$, i.e., there is a set $\{\, d_1^y, \ldots, d_n^y \mid y \in Z \,\} \subseteq \Delta^{\mathcal{I}^*}$ such that, for each $y \in Z$, the set $\{d_1^y, \ldots, d_n^y\}$ only contains pairwise distinct elements, $\{\, (y, d_i^y) \mid i \in \{1, \ldots, n\} \,\} \subseteq r^{\mathcal{I}^*}$, and $\{\, d_i^y \mid i \in \{1, \ldots, n\} \,\} \subseteq (\bigsqcap \mathcal{F})^{\mathcal{I}^*}$. As a corollary we have that

$$\mathcal{I}^*, \{\, x_i^y \mapsto d_i^y \mid i \in \{1, \ldots, n\} \text{ and } y \in Z \,\} \models \Psi_{Z,\mathcal{F}},$$

i.e., $\Psi_{Z,\mathcal{F}}$ is satisfiable in $\mathcal{I}^*$.

Since $\mathcal{I}^*$ is κ-saturated, and $|\mathsf{Var}(\Phi)| < \kappa$ as well as $|Y| < \kappa$, it follows that Φ is satisfiable in $\mathcal{I}^*$. Let hence $\mathcal{Z} \colon \mathsf{Var}(\Phi) \to \Delta^{\mathcal{I}^*}$ be a variable assignment such that $\mathcal{I}^*, \mathcal{Z} \models \Phi$, i.e., for all $y \in Y$, for all $i, j, k \in \{1, \ldots, n\}$ with $j < k$, and for all $D \in \mathcal{D}$, it holds true that

$$\begin{aligned} & \mathcal{Z}(x_i^y) \in \Delta^{\mathcal{I}^*}, \\ & \mathcal{Z}(x_j^y) \neq \mathcal{Z}(x_k^y), \\ & \mathcal{Z}(x_i^y) \in D^{\mathcal{I}^*}, \\ \text{and} \quad & (y, \mathcal{Z}(x_i^y)) \in r^{\mathcal{I}^*}. \end{aligned}$$

If we set $Y' \coloneqq \{\, \mathcal{Z}(x) \mid x \in \mathsf{Var}(\Phi) \,\}$, then $(X', Y') \in \mathfrak{S}$ and $Y' \in \mathsf{Succ}_{\mathcal{I}^*}(Y, \exists \geq n. r)$.

S4. For each natural number $n \in \mathbb{N}$ and for each role name $r \in \Sigma_{\mathsf{R}}$, the term $\exists \leq n. r$ is an $\mathcal{M}$ concept description. Consequently, $X \subseteq (\exists \leq n. r)^{\mathcal{J}}$ implies $Y \subseteq (\exists \leq n. r)^{\mathcal{I}^*}$.

S5. Let $r \in \Sigma_{\mathsf{R}}$ be a role name. If $C \in \mathcal{M}(\Sigma)$ with $r^{\mathcal{J}}(X) \subseteq C^{\mathcal{J}}$, then $X \subseteq (\forall r. C)^{\mathcal{J}}$, and so $Y \subseteq (\forall r. C)^{\mathcal{I}^*}$, i.e., $r^{\mathcal{I}^*}(Y) \subseteq C^{\mathcal{I}^*}$. Of course, this immediately shows that $(r^{\mathcal{J}}(X), r^{\mathcal{I}^*}(Y)) \in \mathfrak{S}$.

S6. Of course, $\exists r. \mathsf{Self}$ is an $\mathcal{M}$ concept description for every role name $r \in \Sigma_{\mathsf{R}}$. Hence, $X \subseteq (\exists r. \mathsf{Self})^{\mathcal{J}}$ only if $Y \subseteq (\exists r. \mathsf{Self})^{\mathcal{I}^*}$.

In summary, we have so far proven that $\mathfrak{S}$ is an $\mathcal{M}$ simulation from $\mathcal{J}$ to $\mathcal{I}^*$. We can now conclude that $(\{\, \delta_C \mid C \in \mathbf{M}_\delta \,\}, \{\delta\}) \in \mathfrak{S}$ is satisfied for the following reasons. If D is an $\mathcal{M}$ concept description over Σ such that $\{\, \delta_C \mid C \in \mathbf{M}_\delta \,\} \subseteq D^{\mathcal{J}}$, then it follows that $D \notin \mathbf{M}_\delta$. To see this, assume to the contrary that $D \in \mathbf{M}_\delta$, which implies $\delta_D \notin D^{\mathcal{J}_D}$ and $\delta_D \in D^{\mathcal{J}}$. It is readily verified that the reflexive relation on $\wp(\Delta^{\mathcal{J}_D})$ is an $\mathcal{M}$ simulation from $(\mathcal{J}, \{\delta_D\})$ to $(\mathcal{J}_D, \{\delta_D\})$. Consequently, $\delta_D \notin D^{\mathcal{J}_D}$ implies $\delta_D \notin D^{\mathcal{J}}$. $\lightning$ By definition of $\mathbf{M}_\delta$, we have that $D \notin \mathbf{M}_\delta$ implies $\delta \in D^{\mathcal{I}}$, and so we conclude that $\delta \in D^{\mathcal{I}^*}$ holds true as $\mathcal{I}^*$ is an elementary extension of $\mathcal{I}$.

Fix some $C \in \mathbf{M}_\delta$, then $\delta_C \in \phi^{\mathcal{J}_C}$ is satisfied. Since ϕ is preserved under $\mathcal{M}$ simulations and the reflexive relation on $\wp(\Delta^{\mathcal{J}_C})$ is an $\mathcal{M}$ simulation from $(\mathcal{J}_C, \{\delta_C\})$ to $(\mathcal{J}, \{\delta_C\})$, we infer that $\delta_C \in \phi^{\mathcal{J}_C}$ implies $\delta_C \in \phi^{\mathcal{J}}$. As a consequence, we obtain $\{\, \delta_C \mid C \in \mathbf{M}_\delta \,\} \subseteq \phi^{\mathcal{J}}$. From the result in the last paragraph we infer that $\delta \in \phi^{\mathcal{I}^*}$. As $\mathcal{I}^*$ is an elementary extension of $\mathcal{I}$ and $\delta \in \Delta^{\mathcal{I}}$, we eventually conclude that $\delta \in \phi^{\mathcal{I}}$ holds true, i.e., $\mathcal{I}, \{x \mapsto \delta\} \models \phi$. □

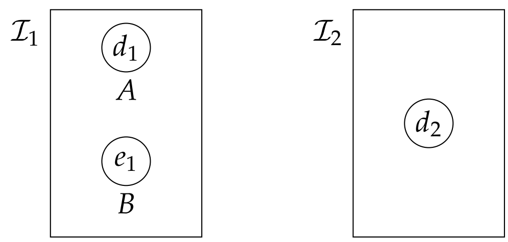
We have seen that the set of $\mathcal{M}$ concept descriptions can be exactly characterized by preservation under $\mathcal{M}$ simulations. This fact is now used to separate $\mathcal{M}$ from other description logics, i.e., we show that some extensions of $\mathcal{M}$ with further concept constructors are more expressive than $\mathcal{M}$. We do this by specifying a pair of interpretations between which there exists an $\mathcal{M}$ simulation $\mathfrak{S}$ and then show that there is one concept description C in the extension $\mathcal{MY}$ of $\mathcal{M}$ that is not preserved under the particular $\mathcal{M}$ simulation $\mathfrak{S}$. Of course, this then implies that the considered $\mathcal{MY}$ concept description C cannot be equivalent to some $\mathcal{M}$ concept description, i.e., the extension $\mathcal{MY}$ is more expressive than $\mathcal{M}$.

We first consider the concept constructor *disjunction*, i.e., for concept descriptions C and D also $C \sqcup D$ is a concept description, and it has the semantics

$$(C \sqcup D)^{\mathcal{I}} := C^{\mathcal{I}} \cup D^{\mathcal{I}}.$$

3.5.3 Proposition. *The description logic $\mathcal{MU}$, i.e., $\mathcal{M}$ extended with disjunctions, is more expressive than $\mathcal{M}$.*

Proof. We show that $A \sqcup B$ is not expressible in $\mathcal{M}$. Therefore, consider the two interpretations $\mathcal{I}_1$ and $\mathcal{I}_2$ depicted below. It is easy to verify that $\{(\{d_1, e_1\}, \{d_2\}), (\emptyset, \emptyset)\}$ is an $\mathcal{M}$ simulation from $\mathcal{I}_1$ to $\mathcal{I}_2$.



However, $\{d_1, e_1\} \subseteq (A \cup B)^{\mathcal{I}_1}$ and $\{d_2\} \not\subseteq (A \cup B)^{\mathcal{I}_2}$. □

As next extension, we consider the concept constructor *full negation*, i.e., if C is a concept description, then also $\neg C$ is a concept description as well, and we have

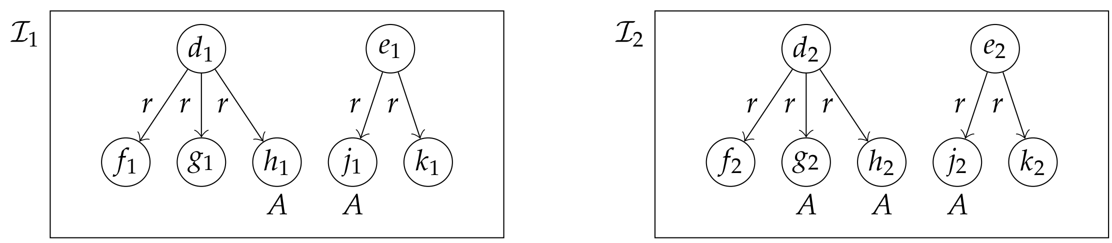
$$(\neg C)^{\mathcal{I}} := \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}.$$

3.5.4 Proposition. *The description logic $\mathcal{MC}$, i.e., $\mathcal{M}$ extended with full negations, is more expressive than $\mathcal{M}$.*

Proof. We show that $\neg\exists \geq 2.\, r.\, A$ is not expressible in $\mathcal{M}$. Therefore, consider the two interpretations $\mathcal{I}_1$ and $\mathcal{I}_2$ depicted below. It is easy to verify that

$$\begin{aligned}
\{(\{d_1,e_1\},\{d_2,e_2\}),\\
(\{f_1,j_1\},\{f_2,j_2\}),\\
(\{f_1,g_1,j_1\},\{f_2,j_2\}),\\
(\{f_1,h_1,j_1\},\{f_2,j_2\}),\\
(\{f_1,g_1,h_1,j_1\},\{f_2,j_2\}),\\
(\{f_1,j_1,k_1\},\{f_2,j_2\}),\\
(\{f_1,k_1\},\{f_2,k_2\}),\\
(\{f_1,g_1,k_1\},\{f_2,k_2\}),\\
(\{f_1,h_1,k_1\},\{h_2,k_2\}),\\
(\{f_1,g_1,h_1,k_1\},\{h_2,k_2\}),\\
(\{g_1,j_1\},\{f_2,j_2\}),\\
(\{g_1,h_1,j_1\},\{f_2,j_2\}),\\
(\{g_1,j_1,k_1\},\{f_2,j_2\}),\\
(\{g_1,k_1\},\{f_2,k_2\}),\\
(\{g_1,h_1,k_1\},\{h_2,k_2\}),\\
(\{h_1,j_1\},\{h_2,j_2\}),\\
(\{h_1,j_1,k_1\},\{h_2,k_2\}),\\
(\{h_1,k_1\},\{h_2,k_2\}),\\
(\{f_1,g_1,j_1,k_1\},\{f_2,g_2,j_2,k_2\}),\\
(\{f_1,h_1,j_1,k_1\},\{f_2,h_2,j_2,k_2\}),\\
(\{g_1,h_1,j_1,k_1\},\{g_2,h_2,j_2,k_2\}),\\
(\{f_1,g_1,h_1,j_1,k_1\},\{f_2,g_2,h_2,j_2,k_2\}),\\
(\emptyset,\emptyset)\}
\end{aligned}$$

is an $\mathcal{M}$ simulation from $\mathcal{I}_1$ to $\mathcal{I}_2$.



However, $\{d_1,e_1\} \subseteq (\neg\exists \geq 2.\, r.\, A)^{\mathcal{I}_1}$ and $\{d_2,e_2\} \nsubseteq (\neg\exists \geq 2.\, r.\, A)^{\mathcal{I}_2}$. □

For each number $n \in \mathbb{N}$, each role name $r \in \Sigma_\mathsf{R}$, and each concept description C, also the *quantified at-most restriction* $\exists \leq n.\, r.\, C$ is a concept description. Its semantics are defined by

$$(\exists \leq n.\, r.\, C)^{\mathcal{I}} := \{\, \delta \mid |r^{\mathcal{I}}(\delta) \cap C^{\mathcal{I}}| \leq n \,\}.$$

3.5.5 Proposition. *The description logic $\mathcal{MQ}$, i.e., $\mathcal{M}$ extended with quantified at-most restrictions, is more expressive than $\mathcal{M}$.*

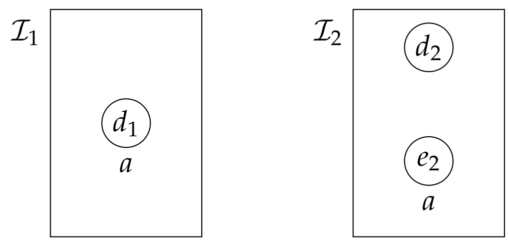
Proof. The previous example also demonstrates that the concept description $\exists \leq 1.\, r.\, A$ is not expressible in $\mathcal{M}$, since we have that $\{d_1, e_1\} \subseteq (\exists \leq 1.\, r.\, A)^{\mathcal{I}_1}$ and $\{d_2, e_2\} \not\subseteq (\exists \leq 1.\, r.\, A)^{\mathcal{I}_2}$. □

Eventually, we consider *nominals*. If $a \in \Sigma_{\mathsf{I}}$ is an individual name, then $\{a\}$ is a concept description, and we interpret it by

$$\{a\}^{\mathcal{I}} := \{a^{\mathcal{I}}\}.$$

3.5.6 Proposition. *The description logic $\mathcal{MO}$, i.e., $\mathcal{M}$ extended with nominals, is more expressive than $\mathcal{M}$.*

Proof. We show that $\{a\}$ is not expressible in $\mathcal{M}$. Therefore, consider the two interpretations $\mathcal{I}_1$ and $\mathcal{I}_2$ depicted below. It is easy to verify that $\{(\{d_1\}, \{d_2\})\}$ is an $\mathcal{M}$ simulation from $\mathcal{I}_1$ to $\mathcal{I}_2$.



However, $\{d_1\} \subseteq \{a\}^{\mathcal{I}_1}$ and $\{d_2\} \not\subseteq \{a\}^{\mathcal{I}_2}$. □

3.6 The Probabilistic Description Logic $\mathsf{Prob}^{>}\mathcal{EL}^{\bot}$

Logics in their standard form only allow for representing and reasoning with *crisp* knowledge without any degree of *uncertainty*. Of course, this is a serious shortcoming for use cases where it is impossible to perfectly determine the truth of a statement or where there exist degrees of truth. For resolving this expressivity restriction, probabilistic variants of logics have been introduced. A thorough article on extending first-order logics with means for representing and reasoning with probabilistic knowledge was published by HALPERN [Hal90]. In particular, HALPERN explains why it is important to distinguish between two contrary types of probabilities: *statistical information* (type 1) and *degrees of belief* (type 2). The crucial difference between both types is that type-1 probabilities represent information about one particular world, the *real* world, and assume that there is a probability distribution on the objects, while type-2 probabilities represent information about a multi-world view such that there is a probability distribution on the set of possible worlds. Following his arguments and citing two of his examples, the first following statement can only be expressed in type-1 probabilistic logics and the second one is only expressible in type-2 probabilistic logics.

1. *"The probability that a randomly chosen bird will fly is greater than 0.9."*
2. *"The probability that Tweety (a particular bird) flies is greater than 0.9."*

BACCHUS has published a further early work on probabilistic logics [Bac88]. In particular, he defined the probabilistic first-order logic **Lp**, which allows to express various kinds of probabilistic/statistical knowledge: relative, interval, functional, conditional, independence. It is of type 1, since its semantics is based on probability measures over the domain of discourse (the objects). However, it also supports the deduction of degrees of belief (type 2) from given knowledge by means of an inference mechanism that is called *belief formation* and is based on an *inductive assumption of randomization*.

In [Hei94], HEINSOHN introduced the probabilistic description logic $\mathcal{ALCP}$ as an extension of $\mathcal{ALC}$. An $\mathcal{ALCP}$ ontology is a union of some acyclic $\mathcal{ALC}$ TBox and a finite set of so-called *p-conditionings*, which are expressions of the form $C \overset{[p,q]}{\to} D$ where C and D are Boolean combinations of concept names and where p and q are real numbers from the unit interval $[0,1]$. $\mathcal{ALCP}$ allows for expressing type-1 probabilities only, since a p-conditioning $C \overset{[p,q]}{\to} D$ is defined to be valid in an interpretation $\mathcal{I}$ if it holds true that $p \leq \frac{|C^{\mathcal{I}} \cap D^{\mathcal{I}}|}{|C^{\mathcal{I}}|} \leq q$, that is, a uniform distribution on the domain of $\mathcal{I}$ is assumed and it is measured which percentage of the objects satisfying the premise C also satisfies the conclusion D. In particular, this means that only finite models are considered, which is a major restriction. HEINSOHN shows how important reasoning problems (consistency and determining minimal p-conditionings) can be translated into problems of linear algebra. Please note that there is a strong correspondence with the notion of *confidence* of a concept inclusion as utilized by BORCHMANN in [Bor14].

Another probabilistic extension of $\mathcal{ALC}$ was devised by JAEGER [Jae94]: the description logic $\mathcal{PALC}$. Probabilities can be assigned to both terminological information and assertional information, rendering it a mixture of means for expressing type-1 and type-2 probabilities. A $\mathcal{PALC}$ ontology is a union of an acyclic $\mathcal{ALC}$ TBox, a finite set of *probabilistic terminological axioms* of the form $P(C \mid D) = p$, and a finite set of *probabilistic assertions* of the form $P(a \in C) = p$. The model-theoretic semantics are defined by extending the usual notion of a model with probability measures: one measure μ dedicated to the probabilistic terminological axioms, and one measure ν_a dedicated to the probabilistic assertions for each individual a. Furthermore, these probability measures are defined on some finite subalgebra of the LINDENBAUM-TARSKI algebra of $\mathcal{ALC}$ concept descriptions that is generated by the concept descriptions occurring in the ontology, and it is further required that each ABox measure ν_a has minimal *cross entropy* to the TBox measure μ.

LUKASIEWICZ introduced in [Luk08] the description logics P-*DL-Lite*, P-$\mathcal{SHIF}(\mathbf{D})$, and P-$\mathcal{SHOIN}(\mathbf{D})$ that are probabilistic extensions of *DL-Lite* and of the DLs underlying OWL Lite and OWL DL, respectively. We shall now briefly explain P-$\mathcal{SHOIN}(\mathbf{D})$, the others are analogous. It allows for expressing *conditional constraints* of the form $(\phi|\psi)[l,u]$ where ϕ and ψ are elements from some fixed, finite set $\mathcal{C}$ of $\mathcal{SHOIN}(\mathbf{D})$ concept descriptions, so-called *basic classification concepts*, and where l and u are real numbers from the unit interval $[0,1]$. Similar to $\mathcal{PALC}$, P-$\mathcal{SHOIN}(\mathbf{D})$ ontologies are unions of some $\mathcal{SHOIN}(\mathbf{D})$ ontology, a finite set of conditional constraints (*PTBox*) as probabilistic terminological knowledge, and a finite set of conditional constraints (*PABox*) as probabilistic assertional knowledge for each probabilistic individual. The semantics are then defined using interpretations that are additionally equipped with a discrete probability measure on the LINDENBAUM-TARSKI algebra generated by $\mathcal{C}$. Note that, in contrast to $\mathcal{PALC}$, there is only one probability measure available in each interpretation. While the terminological knowledge is, just like for $\mathcal{PALC}$, the default knowledge from which we only differ for a particular individual if the corresponding knowledge requires us to do so, the inference process is different, i.e., cross entropy is not utilized in any way. In order to allow for drawing inferences from a P-$\mathcal{SHOIN}(\mathbf{D})$ ontology, *lexicographic entailment* is defined for deciding whether a conditional constraint follows from the terminological part or for a certain individual. A thorough complexity analysis shows that the decision problems in these three logics are **NP**-complete, **EXP**-complete, and **NEXP**-complete, respectively.

GUTIÉRREZ-BASULTO, JUNG, LUTZ, and SCHRÖDER consider in [Gut+17] the probabilistic description logics Prob-$\mathcal{ALC}$ and Prob-$\mathcal{EL}$ where probabilities are always interpreted as degrees of belief (type 2). Among other language constructs, a new concept constructor is introduced that allows to probabilistically quantify a concept description. The semantics are based on multi-world interpretations where a discrete probability measure on the set of worlds is defined. Consistency and entailment is then defined just as usual, but using such probabilistic interpretations. A thorough investigation of computational complexity for various probabilistic extensions of DLs is provided: for instance, the common reasoning problems in Prob-$\mathcal{EL}$ and in Prob-$\mathcal{ALC}$ are **EXP**-complete, that is, not more expensive than the same problems in $\mathcal{ALC}$.

One should never mix up probabilistic and fuzzy variants of (description) logics. Although at first sight one could get the impression that both are suitable for any use cases where imprecise knowledge is to be represented and reasoned with, this is definitely not the case. A very simple argument against this is that in fuzzy logics we can easily evaluate conjunctions by means of the underlying fixed triangular norm (abbrv. t-norm), while it is not (always) possible to deduce the probability of a conjunction given the probabilities of the conjuncts. For instance, consider statements α and β. If both have fuzzy truth degree $\frac{1}{2}$ and the t-norm is GÖDEL's minimum, then $\alpha \wedge \beta$ has the fuzzy truth degree $\frac{1}{2}$ as well. In contrast, if both have probabilistic truth degree $\frac{1}{2}$, then the probability of $\alpha \wedge \beta$ might be any value in the interval $[0, \frac{1}{2}]$, but without additional information we cannot bound it further or even determine it exactly.

The probabilistic description logic $\mathsf{Prob}^{>}\mathcal{EL}^{\bot}$ constitutes an extension of the tractable description logic $\mathcal{EL}^{\bot}$ [Baa+17] that allows for expressing and reasoning with probabilities. More specifically, it is a sublogic of Prob-$\mathcal{EL}$ introduced by GUTIÉRREZ-BASULTO, JUNG, LUTZ, and SCHRÖDER [Gut+17] in which only the relation symbols $>$ and $\geq$ are available for the probability restrictions, and in which the bottom concept description $\bot$ is present.[6] In the sequel of this section, we shall introduce the syntax and semantics of $\mathsf{Prob}^{>}\mathcal{EL}^{\bot}$. Furthermore, we will show that a common inference problem in $\mathsf{Prob}^{>}\mathcal{EL}^{\bot}$ is **EXP**-complete and, thus, more expensive than in $\mathcal{EL}^{\bot}$ where the same problem is **P**-complete.

3.6.1 Syntax and Semantics

The description logic $\mathsf{Prob}^{>}\mathcal{EL}^{\bot}$ extends $\mathcal{EL}^{\bot}$ by concept descriptions of the form $\mathsf{d} \gtrdot p.\,C$, so-called *probability restrictions*, where $\gtrdot \in \{\geq, >\}$, and $p \in [0,1] \cap \mathbb{Q}$, and where C is a

[6] We merely introduce $\bot$ as syntactic sugar; of course, it is semantically equivalent to the unsatisfiable probabilistic restriction $\mathsf{d} > 1.\,\top$.

$\mathsf{Prob}^{>}\mathcal{EL}^{\bot}$ concept description.[7]

Example. An example of a $\mathsf{Prob}^{>}\mathcal{EL}^{\bot}$ concept description is the following.

$$\begin{aligned} &\mathsf{Cat} \sqcap \mathsf{d} \geq \tfrac{1}{2}.\, \exists\, \mathsf{hasPhysicalCondition}.\, \mathsf{Alive} \\ &\quad \sqcap \mathsf{d} \geq \tfrac{1}{2}.\, \exists\, \mathsf{hasPhysicalCondition}.\, \mathsf{Dead} \end{aligned} \tag{3.6.A}$$

It describes cats that are both alive and dead with a respective probability of at least 50 %. In particular, we could consider the above concept description as a formalization of the famous thought experiment *Schrödinger's Cat*. △

The *probability depth* $\mathsf{pd}(C)$ of a $\mathsf{Prob}^{>}\mathcal{EL}^{\bot}$ concept description C is defined as the maximal nesting depth of probability restrictions within C, and we formally define it as follows: $\mathsf{pd}(A) := 0$ for each $A \in \Sigma_{\mathsf{C}} \cup \{\bot, \top\}$, $\mathsf{pd}(C \sqcap D) := \mathsf{pd}(C) \vee \mathsf{pd}(D)$, $\mathsf{pd}(\exists\, r.\, C) := \mathsf{pd}(C)$, and $\mathsf{pd}(\mathsf{d} \gtrdot p.\, C) := 1 + \mathsf{pd}(C)$. Then, $\mathsf{Prob}_n^{>}\mathcal{EL}^{\bot}(\Sigma)$ denotes the set of all $\mathsf{Prob}^{>}\mathcal{EL}^{\bot}$ concept descriptions over Σ the probability depth of which does not exceed n.

Much like $\mathcal{EL}$, our considered logic $\mathsf{Prob}^{>}\mathcal{EL}^{\bot}$ possesses a model-theoretic semantics. However, in order to suitably give meaning to the probability restrictions, we need to extend the notion of an interpretation to a so-called probabilistic interpretation. Such a *probabilistic interpretation* over Σ is a tuple $\mathcal{I} := (\Delta^{\mathcal{I}}, \Omega^{\mathcal{I}}, \cdot^{\mathcal{I}}, \mathbb{P}^{\mathcal{I}})$ that consists of a non-empty set $\Delta^{\mathcal{I}}$ of *objects*, called the *domain*, a non-empty, countable set $\Omega^{\mathcal{I}}$ of *worlds*, a discrete probability measure $\mathbb{P}^{\mathcal{I}}$ on $\Omega^{\mathcal{I}}$, and an *extension function* $\cdot^{\mathcal{I}}$ such that, for each world $\omega \in \Omega^{\mathcal{I}}$, any concept name $A \in \Sigma_{\mathsf{C}}$ is mapped to a subset $A^{\mathcal{I}(\omega)} \subseteq \Delta^{\mathcal{I}}$ and each role name $r \in \Sigma_{\mathsf{R}}$ is mapped to a binary relation $r^{\mathcal{I}(\omega)} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$. We remark that the discrete probability measure is a mapping $\mathbb{P}^{\mathcal{I}}\colon \wp(\Omega^{\mathcal{I}}) \to [0,1]$ which satisfies $\mathbb{P}^{\mathcal{I}}(\emptyset) = 0$ and $\mathbb{P}^{\mathcal{I}}(\Omega^{\mathcal{I}}) = 1$, and which is *σ-additive*, that is, for all countable families $(\, U_n \mid n \in \mathbb{N}\,)$ of pairwise disjoint sets $U_n \subseteq \Omega^{\mathcal{I}}$ it holds true that $\mathbb{P}^{\mathcal{I}}(\bigcup\{\, U_n \mid n \in \mathbb{N}\,\}) = \sum(\, \mathbb{P}^{\mathcal{I}}(U_n) \mid n \in \mathbb{N}\,)$.

We shall follow the assumption in [Gut+17, Section 2.6] and consider only probabilistic interpretations without any infinitely improbable worlds, i.e., which do not contain any world $\omega \in \Omega^{\mathcal{I}}$ with $\mathbb{P}^{\mathcal{I}}\{\omega\} = 0$. Furthermore, a probabilistic interpretation $\mathcal{I}$ is *finite* if $\Delta^{\mathcal{I}}$ is finite, $\Omega^{\mathcal{I}}$ is finite, the *active signature*

$$\Sigma^{\mathcal{I}} := \{\, \sigma \mid \sigma \in \Sigma \text{ and } \sigma^{\mathcal{I}(\omega)} \neq \emptyset \text{ for some } \omega \in \Omega^{\mathcal{I}}\,\}$$

is finite, and if $\mathbb{P}^{\mathcal{I}}$ has only rational values.

It is easy to see that, for any probabilistic interpretation $\mathcal{I}$, each world $\omega \in \Omega^{\mathcal{I}}$ can be represented as a labeled, directed graph: the node set is the domain $\Delta^{\mathcal{I}}$, the edge set is $\bigcup\{\, r^{\mathcal{I}(\omega)} \mid r \in \Sigma_{\mathsf{R}}\,\}$, any node δ is labeled with all those concept names A that satisfy $\delta \in A^{\mathcal{I}(\omega)}$, and any edge (δ, ϵ) has a role name r as a label if $(\delta, \epsilon) \in r^{\mathcal{I}(\omega)}$ holds true. That way, we can regard probabilistic interpretations also as discrete probability distributions over description graphs.

Let $\mathcal{I}$ be a probabilistic interpretation. Then, the *extension* $C^{\mathcal{I}(\omega)}$ of a $\mathsf{Prob}^{>}\mathcal{EL}^{\bot}$ concept description C in a world ω of $\mathcal{I}$ is recursively defined as follows.

$$\bot^{\mathcal{I}(\omega)} := \emptyset$$

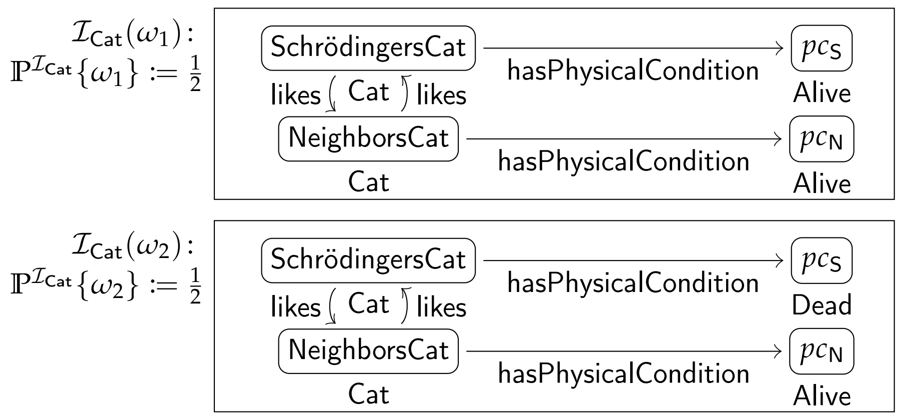
[7] Note that we do not use the denotation $P_{\gtrdot p}C$ as in [Gut+17].

$$
\begin{aligned}
\top^{\mathcal{I}(\omega)} &:= \Delta^{\mathcal{I}} \\
(C \sqcap D)^{\mathcal{I}(\omega)} &:= C^{\mathcal{I}(\omega)} \cap D^{\mathcal{I}(\omega)} \\
(\exists\, r.\, C)^{\mathcal{I}(\omega)} &:= \{\, \delta \mid \delta \in \Delta^{\mathcal{I}} \text{ and } r^{\mathcal{I}(\omega)}(\delta) \cap C^{\mathcal{I}(\omega)} \neq \emptyset \,\} \\
(\mathsf{d} \gtrdot p.\, C)^{\mathcal{I}(\omega)} &:= \{\, \delta \mid \delta \in \Delta^{\mathcal{I}} \text{ and } \mathbb{P}^{\mathcal{I}}\{\delta \in C^{\mathcal{I}}\} \gtrdot p \,\}
\end{aligned}
$$

In the last of the above definitions we use the abbreviation

$$\{\delta \in C^{\mathcal{I}}\} := \{\, \omega \mid \omega \in \Omega^{\mathcal{I}} \text{ and } \delta \in C^{\mathcal{I}(\omega)} \,\}.$$

Please note that, in accordance with [Gut+17], there is nothing wrong with the above definition of extensions; in particular, it is true that the extension $(\mathsf{d} \gtrdot p.\, C)^{\mathcal{I}(\omega)}$ of a probabilistic restriction $\mathsf{d} \gtrdot p.\, C$ is indeed independent of the concrete world ω, i.e., it holds true that $(\mathsf{d} \gtrdot p.\, C)^{\mathcal{I}(\omega)} = (\mathsf{d} \gtrdot p.\, C)^{\mathcal{I}(\psi)}$ whenever ω and ψ are arbitrary worlds in $\Omega^{\mathcal{I}}$. This is due to the intended meaning of $\mathsf{d} \gtrdot p.\, C$: it describes the class of objects for which the probability of being a C is $\gtrdot p$. As a probabilistic interpretation $\mathcal{I}$ provides a multi-world view where probabilities can be assigned to sets of worlds, the probability of an object $\delta \in \Delta^{\mathcal{I}}$ being a C is defined as the probability of the set of all those worlds in which δ is some C, just like we have defined it above.



3.6.1 Figure. An exemplary probabilistic interpretation

Example. A toy example of a probabilistic interpretation is $\mathcal{I}_{\mathsf{Cat}}$ shown in Figure 3.6.1. As one quickly verifies, only the object SchrödingersCat belongs to the extension of the concept description in (3.6.A). △

A concept inclusion $C \sqsubseteq D$ is *valid* in $\mathcal{I}$ if, for each world $\omega \in \Omega^{\mathcal{I}}$, it holds true that $C^{\mathcal{I}(\omega)} \subseteq D^{\mathcal{I}(\omega)}$, and we shall then write $\mathcal{I} \models C \sqsubseteq D$ as usual.

3.6.2 Computational Complexity

The next proposition shows that the subsumption problem in $\mathsf{Prob}^{>}\mathcal{EL}^{\bot}$ is **EXP**-complete and, consequently, more expensive than deciding subsumption w.r.t. a TBox in its non-probabilistic sibling $\mathcal{EL}^{\bot}$—a problem which is well-known to be **P**-complete. We conclude that reasoning in $\mathsf{Prob}^{>}\mathcal{EL}^{\bot}$ is worst-case intractable, while reasoning in $\mathcal{EL}^{\bot}$ is always tractable.

3.6.2 Proposition. *In* $\mathsf{Prob}^{>}\mathcal{EL}^{\bot}$*, the subsumption problem is* **EXP***-complete.*

Proof. Containment in **EXP** follows from [Gut+17, Theorem 3] and the fact that $\mathsf{Prob}^{>}\mathcal{EL}^{\bot}$ is a sublogic of $\mathsf{Prob}\text{-}\mathcal{ALC}$. **EXP**-hardness is a consequence of [Gut+17, Theorem 13 and Sections A.1, A.2, and A.3], where **EXP**-hardness of the logics $\mathsf{Prob}\text{-}\mathcal{EL}^{\sim p}$ for $\sim \in \{\geq, >\}$, that is, of sublogics of $\mathsf{Prob}^{>}\mathcal{EL}^{\bot}$, is demonstrated. □

3.7 Web Ontology Language

The advent of increasing popularity of DL was the invention of the *Web Ontology Language* (abbrv. OWL) as a DL-based language used in the *Semantic Web*. The *World Wide Web Consortium* (abbrv. W3C) has published its second version, called OWL2 [Cue+08], in 2009, after its predecessor OWL1 had attracted significant interest in several areas. In particular, then-recent advances from DL research were incorporated into the second version. The full version of OWL2 is based on the DL $\mathcal{SROIQ}$ [HKS06], for which reasoning is **N2EXP**-complete [Kaz08]. Furthermore, profiles of OWL2 are defined that only provide a limited expressivity of full OWL2 and have, thus, cheaper reasoning complexity and allow for simpler implementations of appropriate reasoners. For instance, the profile OWL2EL is based on the $\mathcal{EL}$ family of DLs, more specifically, on the DL $\mathcal{EL}^{++}$, and has **P**-complete reasoning problems [BLB08].

The ontology editor *Protégé* [Mus15] has become a popular, widely used software for working with ontologies. In particular, it supports the full specification of OWL2. Its workbench supports creating ontologies, maintaining ontologies, and reasoning with ontologies. Due to its plug-in architecture, new features can be easily added. *Protégé* uses the *OWL API* [HB11] at its core, which is an application program interface written in Java supporting all language constructs of OWL2.

4 Non-Standard Inferences

As explained in Section 3.1.3, standard reasoning tasks in Description Logic are usually concerned with subsumption of concept descriptions, satisfiability of ontologies, instance checking, or query answering. However, there are also other reasoning tasks, which are often called *non-standard*, since they do not address questions immediately arising from the semantics, but are rather used in procedures for solving specific tasks, e.g., for knowledge acquisition or construction of ontologies.

This chapter shall introduce several non-standard inferences that are needed for later purposes, and it shall describe computation means as well as algebraic properties of these. We start with defining *model-based most specific concept descriptions* in Section 4.1 for the description logic $\mathcal{EL}$. These are defined for subsets X of the domain of a given interpretation as the most specific concept description such that the extension contains X. The following Section 4.2 shows how model-based most specific concept descriptions can be computed for the more expressive description logic $\mathcal{M}$. Then in Section 4.3, we introduce the notion of a *most specific consequence* as the most specific concept description that subsumes a given concept description C with respect to a TBox $\mathcal{T}$, and consider $\mathcal{EL}$ as the target description logic. Both for model-based most specific concept descriptions and for most specific consequences it might happen that these do not always exist in a description logic with standard semantics. More specifically, if the given interpretation $\mathcal{I}$ or the given TBox $\mathcal{T}$ is cyclic in a certain sense, then these cycles cannot be expressed within an acyclic concept description. As a solution to guaranteeing the existence we can, on the one hand, restrict the role depth, i.e., we only consider concept descriptions up to a certain degree of nesting role restrictions and can thus only approximate the cycles, or we can, on the other hand, utilize a description logic with greatest fixed-point semantics as we have introduced in Section 3.4.

Finally, in Section 4.4 we introduce model-based most specific concept descriptions *relative to* a given TBox and we characterize these for the case of $\mathcal{EL}$ by means of the results from the preceding Sections 4.1 and 4.3.

4.1 Model-Based Most Specific $\mathcal{EL}$ Concept Descriptions

This section presents the existing definition as well as existing results on the notion of a model-based most specific concept description in a slightly different way so that it integrates better into this thesis. Furthermore, a few new results are provided. These results are needed for later purposes, in particular for the upcoming Section 4.3 on so-called most specific consequences. In the following, we partially cite from [Dis11; BDK16].

In Chapter 1 we have seen that in *Formal Concept Analysis* the pair of the derivation operators $\cdot^I\colon \wp(G) \to \wp(M)$ and $\cdot^I\colon \wp(M) \to \wp(G)$ of a formal context $\mathbb{K} := (G, M, I)$ constitutes a GALOIS connection. In *Description Logic* however, for an interpretation $\mathcal{I} := (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ we only have an *extension mapping* $\cdot^{\mathcal{I}} \colon \mathcal{EL}^{\bot}_{\mathsf{si}}(\Sigma) \to \wp(\Delta^{\mathcal{I}})$, i.e., the adjoint mapping of type $\wp(\Delta^{\mathcal{I}}) \to \mathcal{EL}^{\bot}_{\mathsf{si}}(\Sigma)$ is missing. Further details on GALOIS connections can also be found in [DP02, Definition 7.23; DP02, Lemma 7.26].

By definition, the extension mapping $\cdot^{\mathcal{I}} \colon \mathcal{EL}^{\bot}_{\mathsf{si}}(\Sigma) \to \wp(\Delta^{\mathcal{I}})$ preserves finitary joins, i.e., we have that $(\bigsqcap\{ C_t \mid t \in T \})^{\mathcal{I}} = \bigcap\{ C_t^{\mathcal{I}} \mid t \in T \}$ for all finite families $\{ C_t \mid t \in T \}$ of $\mathcal{EL}^{\bot}_{\mathsf{si}}$ concept descriptions over Σ. When imposing a role-depth bound d on the concept descriptions, then we know that there are only finitely many concept descriptions in case of a finite signature, and thus the extension mapping $\cdot^{\mathcal{I}} \colon \mathcal{EL}^{\bot}_{d}(\Sigma) \to \wp(\Delta^{\mathcal{I}})$ preserves arbitrary joins—then [DP02, 7.34] yields that there is another mapping $\wp(\Delta^{\mathcal{I}}) \to \mathcal{EL}^{\bot}_{d}(\Sigma)$, which together with $\cdot^{\mathcal{I}}$ constitutes a GALOIS connection, and in terms of lattice theory this mapping is called the *upper adjoint* of the extension mapping $\cdot^{\mathcal{I}}$. Furthermore, [DP02, 7.33] then states that this other mapping can be found as $X \mapsto \mathsf{Min}\{ C \in \mathcal{EL}^{\bot}_{d}(\Sigma) \mid X \subseteq C^{\mathcal{I}} \}$,[1] i.e., the mapping which assigns to each subset $X \subseteq \Delta^{\mathcal{I}}$ the most specific concept description having an extension that contains X. Such a concept description is then called the model-based most specific concept description of X for $\mathcal{I}$. While for the role-depth-bounded case it is immediately clear that $\mathsf{Min}\{ C \in \mathcal{EL}^{\bot}_{d}(\Sigma) \mid X \subseteq C^{\mathcal{I}} \}$ always exists, the existence in the unbounded case of $\mathcal{EL}^{\bot}_{\mathsf{si}}$ in not obvious, since we cannot simply construct the conjunction of the infinite set $\{ C \in \mathcal{EL}^{\bot}_{\mathsf{si}}(\Sigma) \mid X \subseteq C^{\mathcal{I}} \}$. However, we will see that such unbounded most specific concept descriptions exist in $\mathcal{EL}^{\bot}_{\mathsf{si}}$ too.

4.1.1 Definition. [Dis11, Definition 4.1; BDK16, Definition 4.1] Fix some description logic $\mathcal{DL}$. Let $\mathcal{I}$ be an interpretation over the signature Σ, and let $X \subseteq \Delta^{\mathcal{I}}$. A $\mathcal{DL}$ concept description C is called *model-based most specific concept description* (abbrv. MMSC) of X in $\mathcal{I}$ if it satisfies the following conditions.

1. $X \subseteq C^{\mathcal{I}}$
2. C is more specific than D modulo $\emptyset$ for each $\mathcal{DL}$ concept descriptions D over Σ such that $X \subseteq D^{\mathcal{I}}$.

We shall denote the set of all MMSCs in $\mathcal{I}$ by $\mathsf{MMSC}(\mathcal{I}, \mathcal{DL})$ or simply by $\mathsf{MMSC}(\mathcal{I})$ if $\mathcal{DL}$ is clear from the context. △

Firstly, all model-based most specific concept descriptions of X in $\mathcal{I}$ are equivalent, and a representative of the equivalence class is hence denoted as $X^{\mathcal{I}_{\mathcal{DL}}}$ or just as $X^{\mathcal{I}}$ if the description logic $\mathcal{DL}$ is clear from the context. Secondly, it need not be the case that $X^{\mathcal{I}}$ exists for each description logic $\mathcal{DL}$.

For the choice $\mathcal{DL} = \mathcal{EL}^{\bot}_{d}$, we can easily convince ourselves that $X^{\mathcal{I}}$ always exists – provided that the underlying signature is finite. This is due to the fact that for a finite signature, only finitely many $\mathcal{EL}^{\bot}$ concept descriptions with a role depth of at most d exist. Consequently, in order to construct $X^{\mathcal{I}}$ we may just build the (finite) conjunction of all those concept descriptions with a role depth not exceeding d and for which the extension is a superset of X. Of course,

[1] For a subset $X \subseteq P$ of a quasi-ordered set $(P, \leq)$, we use the expression $\mathsf{Min}(X)$ to denote the set of all those elements in X which are minimal with respect to $\leq$, i.e., $x \in \mathsf{Min}(X)$ if, and only if, $x \in X$ and there is no other element $y \in X$ such that $y \leq x$ and $y \not\equiv x$.

this does not yield a practical means for the construction of model-based most specific concept descriptions, but we will investigate an appropriate computation method later.

For the choice $\mathcal{DL} = \mathcal{EL}^{\bot}$, the model-based most specific concept description need not always exist. A counterexample is provided by some cyclic interpretation. More specifically, consider the interpretation $\mathcal{I}$ where the domain contains only one element δ and where δ is an r-successor of itself. It is then obvious that δ is in the extension of $\exists\, r^n.\top$ for each number $n \in \mathbb{N}$, and further there is no most specific concept description among these.

Eventually, model-based most specific concept descriptions always exist for the case $\mathcal{DL} = \mathcal{EL}^{\bot}_{\mathsf{si}}$ if the considered interpretation is finite. In the following, it is demonstrated how MMSCs can be computed in the description logic $\mathcal{EL}^{\bot}_{\mathsf{si}}$.

4.1.2 Lemma. [Dis11, Lemma 4.1; BDK16, Lemmas 4.3 and 4.4] *Consider a description logic $\mathcal{DL}$ as well as some interpretation such that MMSCs for $\mathcal{I}$ always exist in $\mathcal{DL}$. Then, the extension mapping $\cdot^{\mathcal{I}}\colon \mathcal{DL}(\Sigma) \to \wp(\Delta^{\mathcal{I}})$ and the MMSC mapping $\cdot^{\mathcal{I}}\colon \wp(\Delta^{\mathcal{I}}) \to \mathcal{DL}(\Sigma)$ constitute a* Galois *connection between $(\wp(\Delta^{\mathcal{I}}), \subseteq)$ and the quotient of $(\mathcal{DL}(\Sigma), \sqsupseteq_{\emptyset})$ w.r.t. $\equiv_{\emptyset}$. In particular, the following statements hold true for all subsets $X, Y \subseteq \Delta^{\mathcal{I}}$ and for all $\mathcal{DL}$ concept descriptions C, D over Σ.*

1. $X \subseteq C^{\mathcal{I}}$ *if, and only if,* $X^{\mathcal{I}} \sqsubseteq_{\emptyset} C$
2. $X \subseteq X^{\mathcal{I}\mathcal{I}}$
3. $X^{\mathcal{I}} \equiv_{\emptyset} X^{\mathcal{I}\mathcal{I}\mathcal{I}}$
4. $X \subseteq Y$ *implies* $X^{\mathcal{I}} \sqsubseteq_{\emptyset} Y^{\mathcal{I}}$
5. $C \sqsupseteq_{\emptyset} C^{\mathcal{I}\mathcal{I}}$
6. $C^{\mathcal{I}} = C^{\mathcal{I}\mathcal{I}\mathcal{I}}$
7. $C \sqsubseteq_{\emptyset} D$ *implies* $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$

Proof. It suffices to prove the first statement, since the others are then obtained as consequences, cf. [DP02, Definition 7.23 and Lemma 7.26]. Hence, assume that $X \subseteq C^{\mathcal{I}}$. Then by Statement 2 of Definition 4.1.1 we conclude that $X^{\mathcal{I}} \sqsubseteq_{\emptyset} C$. Vice versa, if $X^{\mathcal{I}}$ is subsumed by C with respect to the empty TBox $\emptyset$, then in particular it follows that $X^{\mathcal{I}\mathcal{I}} \subseteq C^{\mathcal{I}}$. An application of Statement 1 of Definition 4.1.1 then yields $X \subseteq X^{\mathcal{I}\mathcal{I}} \subseteq C^{\mathcal{I}}$. □

4.1.3 Lemma. [Dis11, Lemma 4.5] *For each finite interpretation $\mathcal{I}$ and each object δ in the domain of $\mathcal{I}$, the $\mathcal{EL}_{\mathsf{si}}$ concept description $\exists^{\mathsf{sim}}(\mathcal{I}, \delta)$ is the model-based most specific concept description of $\{\delta\}$ in $\mathcal{I}$.*

Proof. It is trivial that the identity is a simulation from $(\mathcal{I}, \delta)$ to $(\mathcal{I}, \delta)$, i.e., it holds true that $\delta \in (\exists^{\mathsf{sim}}(\mathcal{I}, \delta))^{\mathcal{I}}$. Furthermore, consider some $\mathcal{EL}_{\mathsf{si}}$ concept description $\exists^{\mathsf{sim}}(\mathcal{J}, \epsilon)$ such that $\delta \in (\exists^{\mathsf{sim}}(\mathcal{J}, \epsilon))^{\mathcal{I}}$, which means that there exists some simulation from $(\mathcal{J}, \epsilon)$ to $(\mathcal{I}, \delta)$. By an application of Proposition 3.4.2 we conclude that the concept description $\exists^{\mathsf{sim}}(\mathcal{I}, \delta)$ is more specific than $\exists^{\mathsf{sim}}(\mathcal{J}, \epsilon)$ modulo $\emptyset$. □

4.1.4 Proposition. [Dis11, Lemma 4.6] *For each finite interpretation $\mathcal{I}$ and each non-empty subset X of the domain of $\mathcal{I}$, the least common subsumer $\bigvee\{\, \exists^{\mathsf{sim}}(\mathcal{I}, \delta) \mid \delta \in X \,\}$ is the model-based most specific concept description of X in $\mathcal{I}$.*

Proof. We show more generally that the least common subsumer $X_1^{\mathcal{I}} \vee \ldots \vee X_n^{\mathcal{I}}$ is equivalent to the model-based most specific concept description $(X_1 \cup \ldots \cup X_n)^{\mathcal{I}}$ for subsets $X_1, \ldots, X_n \subseteq \Delta^{\mathcal{I}}$. The above claim then follows for the case when all subsets X_i are singletons.

We know that $X_i \subseteq X_i^{\mathcal{I}\mathcal{I}}$ is satisfied for each index $i \in \{1, \ldots, n\}$. Since each $X_i^{\mathcal{I}}$ is more specific than $X_1^{\mathcal{I}} \vee \ldots \vee X_n^{\mathcal{I}}$, we conclude that $X_1 \cup \ldots \cup X_n$ is a subset of $(X_1^{\mathcal{I}} \vee \ldots \vee X_n^{\mathcal{I}})^{\mathcal{I}}$.

Now fix some $\mathcal{EL}_{\mathsf{si}}$ concept description C such that $X_1 \cup \ldots \cup X_n \subseteq C^{\mathcal{I}}$ holds true. In particular, we have $X_i \subseteq C^{\mathcal{I}}$ for each index $i \in \{1, \ldots, n\}$, which implies that $X_i^{\mathcal{I}} \sqsubseteq_{\emptyset} C$. It is now straightforward to conclude that the least common subsumer $X_1^{\mathcal{I}} \vee \ldots \vee X_n^{\mathcal{I}}$ must be more specific than C modulo $\emptyset$ as well, and we are done. □

So far, we have seen that model-based most specific concept descriptions can easily be characterized in the description logic $\mathcal{EL}_{\mathsf{si}}$. In particular, MMSCs of non-singleton sets X can be obtained as least common subsumers of the MMSCs of the singletons $\{\delta\}$ for $\delta \in X$. We shall now show that we can also use a *powerset construction* on interpretations for computing MMSCs of non-singletons. This construction has the benefit that it produces smaller representations of non-singleton MMSCs $X^{\mathcal{I}}$ if the size of X is linear in the size of the domain $\Delta^{\mathcal{I}}$ and, furthermore, *all* MMSCs $X^{\mathcal{I}}$ for non-empty subsets $X \subseteq \Delta^{\mathcal{I}}$ can be directly read off from it.

Fix some interpretation $\mathcal{I}$. For an object δ in the domain of $\mathcal{I}$, we define the set $\mathsf{succ}(\delta, r) := \{\, \epsilon \mid (\delta, \epsilon) \in r^{\mathcal{I}} \,\}$ containing all r-successors of δ in $\mathcal{I}$ and, dually, we define the set $\mathsf{pred}(\delta, r) := \{\, \epsilon \mid (\epsilon, \delta) \in r^{\mathcal{I}} \,\}$ containing all r-predecessors of δ in $\mathcal{I}$. If X is a subset of the domain of $\mathcal{I}$, then we further let $\mathsf{succ}(X, r) := \bigcup\{\, \mathsf{succ}(x, r) \mid x \in X \,\}$ and $\mathsf{pred}(X, r) := \bigcup\{\, \mathsf{pred}(x, r) \mid x \in X \,\}$.[2] With these notions, we now define $\mathsf{Succ}(X, r) := \{\, Y \mid Y \subseteq \mathsf{succ}(X, r) \text{ and } X \subseteq \mathsf{pred}(Y, r) \,\}$, that is, $\mathsf{Succ}(X, r)$ contains all subsets of $\Delta^{\mathcal{I}}$ consisting only of objects that are an r-successor of some object in X and which contain at least one r-successor of each object in X.

Furthermore, we define the interpretation $\wp(\mathcal{I})$, which is called the *powering* of $\mathcal{I}$, as follows.

$$\Delta^{\wp(\mathcal{I})} := \wp(\Delta^{\mathcal{I}}) \setminus \{\emptyset\}$$

$$\cdot^{\wp(\mathcal{I})} \colon \begin{cases} A \mapsto \{\, X \mid X \subseteq A^{\mathcal{I}} \,\} & \text{for any } A \in \Sigma_{\mathsf{C}} \\ r \mapsto \{\, (X, Y) \mid Y \in \mathsf{Min}(\mathsf{Succ}(X, r)) \,\} & \text{for any } r \in \Sigma_{\mathsf{R}} \end{cases}$$

4.1.5 Lemma. *Fix some finite interpretation $\mathcal{I}$ as well as some non-empty subset $X \subseteq \Delta^{\mathcal{I}}$. Further let $\exists^{\mathsf{sim}}(\mathcal{J}, \epsilon)$ be an $\mathcal{EL}_{\mathsf{si}}$ concept description. Then, $X \subseteq (\exists^{\mathsf{sim}}(\mathcal{J}, \epsilon))^{\mathcal{I}}$ is equivalent to $(\mathcal{J}, \epsilon) \precsim (\wp(\mathcal{I}), X)$.*

Proof. Let $X \subseteq (\exists^{\mathsf{sim}}(\mathcal{J}, \epsilon))^{\mathcal{I}}$. We immediately conclude that there exists some simulation $\mathfrak{S}_\delta$ from $(\mathcal{J}, \epsilon)$ to $(\mathcal{I}, \delta)$ for each $\delta \in X$. We now define the following relation and show that it is a simulation from $(\mathcal{J}, \epsilon)$ to $(\wp(\mathcal{I}), X)$.

$$\mathfrak{S} := \{\, (\eta, Y) \mid (\eta, \zeta) \in \bigcup\{\, \mathfrak{S}_\delta \mid \delta \in X \,\} \text{ for each } \zeta \in Y \,\}$$

Since $(\epsilon, \delta) \in \mathfrak{S}_\delta$ is satisfied for each $\delta \in X$, it follows that $(\epsilon, X) \in \mathfrak{S}$. Now fix some pair $(\eta, Y) \in \mathfrak{S}$.

- Let $\eta \in A^{\mathcal{J}}$. For each $\zeta \in Y$, it holds true that $(\eta, \zeta) \in \mathfrak{S}_\delta$ for some $\delta \in X$, which implies $\zeta \in A^{\mathcal{I}}$. Thus, we conclude that $Y \subseteq A^{\mathcal{I}}$, and this means that $Y \in A^{\wp(\mathcal{I})}$.

[2]Note that $\mathsf{succ}(\delta, r) = r^{\mathcal{I}}(\delta)$ and $\mathsf{succ}(X, r) = r^{\mathcal{I}}(X)$ holds true for the notions introduced earlier.

- Consider some pair $(\eta, \iota) \in r^{\mathcal{J}}$. For each $\zeta \in Y$, it holds true that $(\eta, \zeta) \in \mathfrak{S}_\delta$ for some $\delta \in X$, which implies that there exists some θ such that $(\zeta, \theta) \in r^{\mathcal{I}}$ and $(\iota, \theta) \in \mathfrak{S}_\delta$. Now define Z as the set containing all these θ; it holds true that $Z \in \mathsf{Succ}(Y, r)$. Of course, there exists some Z' such that $Z' \subseteq Z$ and $Z' \in \mathsf{Min}(\mathsf{Succ}(Y, r))$, i.e., $(Y, Z') \in r^{\wp(\mathcal{I})}$ is satisfied. The above also shows that (ι, Z) as well as (ι, Z') are both elements of $\mathfrak{T}$.

For the converse direction, assume that $\mathfrak{T}$ is a simulation from $(\mathcal{J}, \epsilon)$ to $(\wp(\mathcal{I}), X)$. We need to find simulations from $(\mathcal{J}, \epsilon)$ to $(\mathcal{I}, \delta)$ for each object $\delta \in X$. For this purpose, we define the following relation.

$$\mathfrak{U} := \{ (\eta, \zeta) \mid (\eta, Y) \in \mathfrak{T} \text{ and } \zeta \in Y \}$$

Obviously, we have that $(\epsilon, \delta) \in \mathfrak{U}$ for each $\delta \in X$. Now consider some pair $(\eta, \zeta) \in \mathfrak{U}$, i.e., there is some Y such that $(\eta, Y) \in \mathfrak{T}$ and $\zeta \in Y$.

- If $\eta \in A^{\mathcal{J}}$, then $Y \in A^{\wp(\mathcal{I})}$ follows, since $\mathfrak{T}$ is a simulation. We conclude that $Y \subseteq A^{\mathcal{I}}$, which immediately implies $\zeta \in A^{\mathcal{I}}$.
- If $(\eta, \iota) \in r^{\mathcal{J}}$, then there is some Z such that $(\iota, Z) \in \mathfrak{T}$ and $(Y, Z) \in r^{\wp(\mathcal{I})}$, i.e., $Z \in \mathsf{Min}(\mathsf{Succ}(Y, r))$. In particular, Z contains some θ such that $(\zeta, \theta) \in r^{\mathcal{I}}$. Of course, we also have that $(\iota, \theta) \in \mathfrak{U}$. □

4.1.6 Proposition. *For each finite interpretation $\mathcal{I}$ and each non-empty subset X of the domain of $\mathcal{I}$, the $\mathcal{EL}_{\mathsf{si}}$ concept description $\exists^{\mathsf{sim}}(\wp(\mathcal{I}), X)$ is the model-based most specific concept description of X in $\mathcal{I}$ for the description logic $\mathcal{EL}_{\mathsf{si}}$.*

Proof. Lemma 4.1.5 immediately implies that X is a subset of $(\exists^{\mathsf{sim}}(\wp(\mathcal{I}), X))^{\mathcal{I}}$. Furthermore, consider some $\mathcal{EL}_{\mathsf{si}}$ concept description $\exists^{\mathsf{sim}}(\mathcal{J}, \epsilon)$ such that $X \subseteq (\exists^{\mathsf{sim}}(\mathcal{J}, \epsilon))^{\mathcal{I}}$ is satisfied. Lemma 4.1.5 shows that then there exists a simulation from $(\mathcal{J}, \epsilon)$ to $(\wp(\mathcal{I}), X)$, which yields that $\exists^{\mathsf{sim}}(\wp(\mathcal{I}), X)$ is more specific than $\exists^{\mathsf{sim}}(\mathcal{J}, \epsilon)$ modulo $\emptyset$. □

Consider an interpretation $\mathcal{I}$ with n objects in the domain. Our objective is to compute the MMSC of the whole domain $\Delta^{\mathcal{I}}$. According to Propositions 3.4.3 and 4.1.4 we would now construct the n-fold product $\mathcal{I}^n := \mathcal{I} \times \cdots \times \mathcal{I}$ and can then obtain the MMSC as $\exists^{\mathsf{sim}}(\mathcal{I}^n, (\delta_1, \ldots, \delta_n))$ where $\Delta^{\mathcal{I}} = \{\delta_1, \ldots, \delta_n\}$. However, the domain of $\mathcal{I}^n$ is quite large, since it contains n^n objects (and all of them might be reachable from $(\delta_1, \ldots, \delta_n)$, e.g., if $r^{\mathcal{I}} = \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$). In contrast, the domain of the powering $\wp(\mathcal{I})$ contains only $2^n - 1$ objects and Proposition 4.1.6 shows that we can obtain the desired MMSC as $\exists^{\mathsf{sim}}(\wp(\mathcal{I}), \Delta^{\mathcal{I}})$ too.

As next step, we now consider the computation of MMSCs for the role-depth-bounded case. Existing results immediately show that these bounded MMSCs can be obtained as the approximations of the unbounded MMSCs.

4.1.7 Lemma. [BDK16, Theorem 4.17 and Corollary 4.22] *Fix some finite interpretation $\mathcal{I}$, some non-empty subset $X \subseteq \Delta^{\mathcal{I}}$, and a role depth bound $d \in \mathbb{N}$. If C is the model-based most specific concept description of X in $\mathcal{I}$ for the description logic $\mathcal{EL}_{\mathsf{si}}$, then the dth approximation $C\restriction_d$ is the model-based most specific concept description of X in $\mathcal{I}$ for the description logic $\mathcal{EL}_d$.*

Proof. We already know that X is a subset of the extension $C^{\mathcal{I}}$. Since C is more specific than its dth approximation $C\restriction_d$ modulo $\emptyset$, we infer that $X \subseteq (C\restriction_d)^{\mathcal{I}}$ holds true as well.

Now let D be some $\mathcal{EL}_d$ concept description such that $X \subseteq D^{\mathcal{I}}$ is satisfied. Since D is an $\mathcal{EL}_{\mathsf{si}}$ concept description as well, we infer that $C \sqsubseteq_\emptyset D$. An application of Lemma 3.4.7 eventually shows that $C\restriction_d \sqsubseteq_\emptyset D$. □

Until now we have only been concerned with computing MMSCs for non-empty sets. It is easy to see that then these MMSCs coincide in $\mathcal{EL}_{\mathsf{si}}$ and $\mathcal{EL}^\bot_{\mathsf{si}}$, as well as in $\mathcal{EL}_d$ and $\mathcal{EL}^\bot_d$, since the MMSCs of non-empty sets can never be unsatisfiable. The following lemma shows how the MMSC of $\emptyset$ can be characterized.

4.1.8 Lemma. [Dis11, Lemma 4.4] *Let $\mathcal{I}$ be a finite interpretation. In both description logics $\mathcal{EL}^\bot_{\mathsf{si}}$ and $\mathcal{EL}^\bot_d$ the model-based most specific concept description of $\emptyset$ in $\mathcal{I}$ exists and is the bottom concept description $\bot$. If the signature Σ is finite, then $\emptyset^{\mathcal{I}}$ exists in $\mathcal{EL}_{\mathsf{si}}$ and is the concept description $\exists^{\mathsf{sim}}(\mathcal{I}_\bot,\delta)$ where $\mathcal{I}_\bot$ has the following components*

$$\Delta^{\mathcal{I}_\bot} := \{\delta\}$$
$$\cdot^{\mathcal{I}_\bot}\colon \begin{cases} A \mapsto \{\delta\} & \text{for any } A \in \Sigma_{\mathsf{C}} \\ r \mapsto \{(\delta,\delta)\} & \text{for any } r \in \Sigma_{\mathsf{R}} \end{cases}$$

and $\emptyset^{\mathcal{I}}$ exists in $\mathcal{EL}_d$ as well and is the concept description $\mathsf{X}^d(\mathcal{I}_\bot,\delta)$. □

The next theorem collects all results that we have obtained so far.

4.1.9 Theorem. *Let $\mathcal{DL}$ be one of the description logics $\mathcal{EL}_{\mathsf{si}}$, $\mathcal{EL}^\bot_{\mathsf{si}}$, $\mathcal{EL}_d$, or $\mathcal{EL}^\bot_d$. If $\mathcal{DL}$ does not provide the bottom concept description $\bot$, then assume that the signature Σ is finite. Further fix some finite interpretation $\mathcal{I}$. For each subset $X \subseteq \Delta^{\mathcal{I}}$, the model-based most specific concept description of X in $\mathcal{I}$ for the description logic $\mathcal{DL}$ exists, and these have the following representations.*

- *In $\mathcal{EL}_{\mathsf{si}}$ we have $\emptyset^{\mathcal{I}} \equiv_\emptyset \exists^{\mathsf{sim}}(\mathcal{I}_\bot,\delta)$, and in $\mathcal{EL}_d$ we have $\emptyset^{\mathcal{I}} \equiv_\emptyset \mathsf{X}^d(\mathcal{I}_\bot,\delta)$. Both in $\mathcal{EL}^\bot_{\mathsf{si}}$ and in $\mathcal{EL}^\bot_d$ it holds true that $\emptyset^{\mathcal{I}} \equiv_\emptyset \bot$.*
- *For each $X \neq \emptyset$, we have $X^{\mathcal{I}} \equiv_\emptyset \bigvee\{\,\exists^{\mathsf{sim}}(\mathcal{I},\delta) \mid \delta \in X\,\} \equiv_\emptyset \exists^{\mathsf{sim}}(\wp(\mathcal{I}),X)$ in $\mathcal{EL}_{\mathsf{si}}$ as well as in $\mathcal{EL}^\bot_{\mathsf{si}}$, and further it holds true that $X^{\mathcal{I}} \equiv_\emptyset \bigvee\{\,\mathsf{X}^d(\mathcal{I},\delta) \mid \delta \in X\,\} \equiv_\emptyset \mathsf{X}^d(\wp(\mathcal{I}),X)$ in $\mathcal{EL}_d$ as well as in $\mathcal{EL}^\bot_d$.*

Furthermore, the mapping

$$\phi_{\mathcal{I}}\colon \mathcal{DL}(\Sigma) \to \mathcal{DL}(\Sigma)$$
$$C \mapsto C^{\mathcal{I}\mathcal{I}}$$

is a closure operator in $\boldsymbol{\mathcal{DL}}(\Sigma)$ and a $\mathcal{DL}$ concept inclusion is valid in $\mathcal{I}$ if, and only if, it is valid for $\phi_{\mathcal{I}}$.

Proof. The existence of all MMSCs as well as the equivalences follow from Lemmas 4.1.3, 4.1.7, and 4.1.8 and Propositions 4.1.4 and 4.1.6. The fact that $\phi_{\mathcal{I}}$ is a closure operator is a direct consequence of Lemma 4.1.2.

extensive. This is Statement 7 of Lemma 4.1.2.

monotonic. Monotonicity follows from Statements 3 and 6 of Lemma 4.1.2.

idempotent. Statement 5 of Lemma 4.1.2 shows that $C^{\mathcal{II}} \sqsupseteq_\emptyset C^{\mathcal{IIII}}$. The converse subsumption follows from Statements 3 and 5 of Lemma 4.1.2.

Eventually, fix some $\mathcal{EL}_{\mathsf{si}}$ concept inclusion $C \sqsubseteq D$. The following equivalences are implied by the definitions of validity of a concept inclusion in an interpretation and in a closure operator, respectively, as well as by Statement 1 of Lemma 4.1.2.

$$C \sqsubseteq_\mathcal{I} D \text{ if, and only if, } C^\mathcal{I} \subseteq D^\mathcal{I} \text{ if, and only if, } C^{\mathcal{II}} \sqsubseteq_\emptyset D \text{ if, and only if, } C \sqsubseteq_{\phi_\mathcal{I}} D \qquad \square$$

4.1.10 Proposition. *Let $\mathcal{I}$ be some finite interpretation.*

1. *The MMSC $\emptyset^\mathcal{I}$ does not exist in $\mathcal{EL}$, but it exists in $\mathcal{EL}^\bot$.*
2. *The MMSC $X^\mathcal{I}$ exists in $\mathcal{EL}$ and in $\mathcal{EL}^\bot$ for each non-empty subset $X \subseteq \Delta^\mathcal{I}$ if, and only if, the interpretation $\mathcal{I}$ is acyclic.*
3. *The problem whether all MMSC exist in $\mathcal{EL}$ or in $\mathcal{EL}^\bot$ can be decided in polynomial time.*
4. *The problem whether the MMSC $X^\mathcal{I}$ exists in $\mathcal{EL}$ or in $\mathcal{EL}^\bot$ can be decided in polynomial time.*
5. *The representation $\exists^{\mathsf{sim}}(\mathcal{I},\delta)$ of $\{\delta\}^\mathcal{I}$ for $\mathcal{EL}_{\mathsf{si}}$ and $\mathcal{EL}^\bot_{\mathsf{si}}$ has a size that is linear in $||\mathcal{I}||$. If $X \neq \emptyset$, then representations of $X^\mathcal{I}$ for $\mathcal{EL}_{\mathsf{si}}$ and $\mathcal{EL}^\bot_{\mathsf{si}}$ can have an exponential size; more specifically it holds true that the concept description $\exists^{\mathsf{sim}}(\bigtimes\{\, (\mathcal{I},\delta) \mid \delta \in X \,\})$ can have a size that is linear in $||\mathcal{I}||^{|X|}$.*
6. *The representation $\mathsf{X}^d(\mathcal{I},\delta)$ of $\{\delta\}^\mathcal{I}$ for $\mathcal{EL}_d$ and $\mathcal{EL}^\bot_d$ has a size that is linear in $||\mathcal{I}||$ and exponential in d. If $X \neq \emptyset$, then representations of $X^\mathcal{I}$ for $\mathcal{EL}_d$ and $\mathcal{EL}^\bot_d$ can have an exponential size both in $||\mathcal{I}||$ and in d; more specifically it holds true that the concept description $\mathsf{X}^d(\bigtimes\{\, (\mathcal{I},\delta) \mid \delta \in X \,\})$ can have a size that is linear in $||\mathcal{I}||^{|X|}$ and exponential in d.*

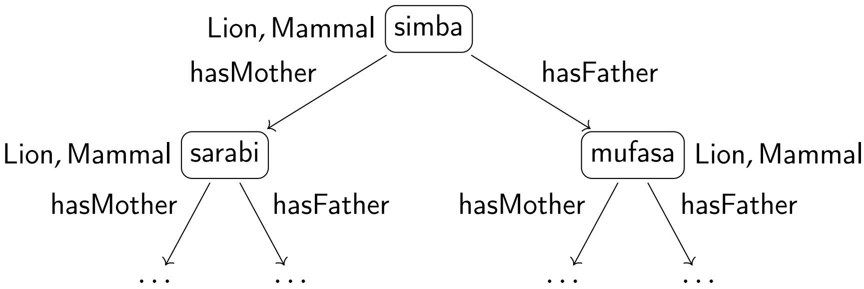
Proof. Clearly, we have that $\emptyset$ is a subset of the extension $C^\mathcal{I}$ for each $\mathcal{EL}$ concept description C. Furthermore, the concept description $C \sqcap \exists\, r.\, C$ is strictly more specific than C modulo the empty TBox, which implies that there cannot exist a most specific $\mathcal{EL}$ concept description for $\emptyset$ in any interpretation.

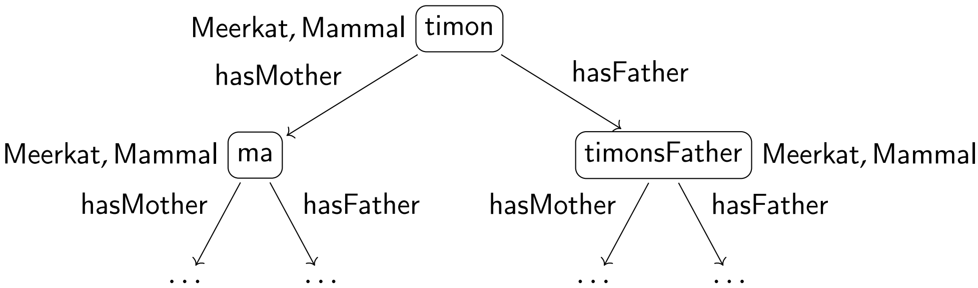
It has already been mentioned in [Dis11, Section 4.1.2] that cycles within the given interpretation $\mathcal{I}$ can prevent the existence of model-based most specific concept descriptions in the description logics $\mathcal{EL}$ and $\mathcal{EL}^\bot$. Obviously, one can utilize the FLOYD-WARSHALL algorithm for checking whether $\mathcal{I}$ is cyclic, and this check can be done in polynomial time.

For checking the existence of the MMSC of a singleton $\{\delta\}$ in $\mathcal{I}$ for $\mathcal{EL}$, it is sufficient to determine whether a cycle in $\mathcal{I}$ is reachable from δ. For a subset $X \subseteq \Delta^\mathcal{I}$ where $|X| \geq 2$, we have that $X^\mathcal{I}$ is equivalent to the least common subsumer $\bigvee\{\, \{\delta\}^\mathcal{I} \mid \delta \in X \,\}$. Obviously, if there is at least one object $\delta \in X$ for which $\{\delta\}^\mathcal{I}$ exists in $\mathcal{EL}$, then the least common subsumer must be an $\mathcal{EL}$ concept description as well, since its role depth cannot exceed the role depth of $\{\delta\}^\mathcal{I}$.

By the very definition, the size of $\exists^{\mathsf{sim}}(\mathcal{I},\delta)$ is linear in $||\mathcal{I}||$. For the general case where X is an arbitrary subset of the domain $\Delta^\mathcal{I}$, the product $\bigtimes\{\, (\mathcal{I},\delta) \mid \delta \in X \,\}$ can clearly have a size that is exponential in $|X|$. By unraveling the MMSCs in $\mathcal{EL}^\bot_{\mathsf{si}}$ up to depth d we obtain the MMSCs in $\mathcal{EL}^\bot_d$, and so the upper bounds on their sizes are obviously satisfied. $\square$

Example. Fix the following interpretation $\mathcal{I}_{\mathsf{LionKing}}$ that contains the objects simba and timon as well as corresponding ancestors.

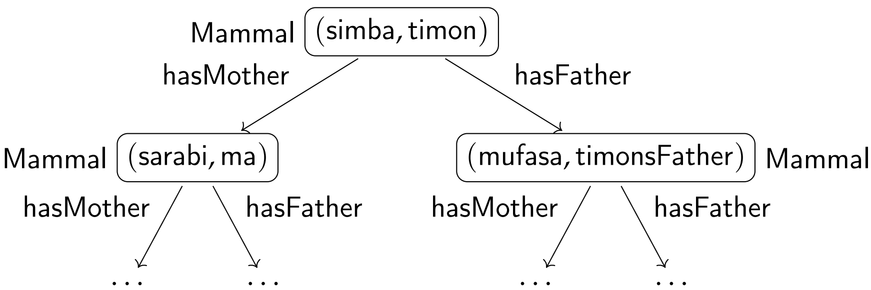




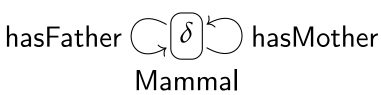
We now want to construct the model-based most specific concept description of $\{\mathsf{simba}, \mathsf{timon}\}$ in $\mathcal{I}_{\mathsf{LionKing}}$, which is equivalent to

$$\exists^{\mathsf{sim}}((\mathcal{I}_{\mathsf{LionKing}}, \mathsf{simba}) \times (\mathcal{I}_{\mathsf{LionKing}}, \mathsf{timon})).$$

The product $\mathcal{I}_{\mathsf{LionKing}} \times \mathcal{I}_{\mathsf{LionKing}}$ is as follows where we only construct objects that are reachable from $(\mathsf{simba}, \mathsf{timon})$.



It is not hard to verify that $(\mathcal{I}_{\mathsf{LionKing}} \times \mathcal{I}_{\mathsf{LionKing}}, (\mathsf{simba}, \mathsf{timon}))$ is equi-similar to the following pointed interpretation $(\mathcal{I}, \delta)$.



We conclude that $\{\mathsf{simba}, \mathsf{timon}\}^{\mathcal{I}_{\mathsf{LionKing}}}$ is equivalent to $\exists^{\mathsf{sim}}(\mathcal{I}, \delta)$. △

4.2 Model-Based Most Specific $\mathcal{M}$ Concept Descriptions

We have defined the notion of a model-based most specific concept description in Section 4.1. In the sequel of this section, we are going to develop a method for the computation of MMSCs in $\mathcal{M}$ with respect to an upper bound on the role depth. Throughout the whole chapter, we use the abbreviation $X^{\mathcal{I}_d}$ for $X^{\mathcal{I}_{\mathcal{M}_d}}$.

Note that we have defined the following quantified successor sets in Section 3.5.

$$\mathsf{Succ}_{\mathcal{I}}(X, \exists \geq n.\, r) := \{\, Y \mid Y \subseteq r^{\mathcal{I}}(X) \text{ and } \forall\, x \in X \colon |r^{\mathcal{I}}(x) \cap Y| \geq n \,\}$$
$$\mathsf{Succ}_{\mathcal{I}}(X, \forall\, r) := \{\, Y \mid Y \subseteq r^{\mathcal{I}}(X) \text{ and } \forall\, x \in X \colon r^{\mathcal{I}}(x) \subseteq Y \,\}.$$

We shall see in the following that it suffices to only consider the minimal sets in $\mathsf{Succ}_{\mathcal{I}}(X, \exists \geq n.\, r)$ and in $\mathsf{Succ}_{\mathcal{I}}(X, \forall\, r)$ for constructing MMSCs. Of course, it holds true that $\mathsf{Min}(\mathsf{Succ}_{\mathcal{I}}(X, \forall\, r)) = \{r^{\mathcal{I}}(X)\}$. We further define $n(X, r) := \max\{\, |r^{\mathcal{I}}(\delta)| \mid \delta \in X \,\}$, i.e., $n(X, r)$ is the smallest number n such that $X \subseteq (\exists \leq n.\, r)^{\mathcal{I}}$ is satisfied.

The first step now is to define a so-called canonical MMSC by induction on the role depth. Afterwards, we prove that it always equals the MMSC.

4.2.1 Definition. Let $\mathcal{I}$ be a finite interpretation over a finite signature Σ, $X \subseteq \Delta^{\mathcal{I}}$ with $X \neq \emptyset$ be a subset of the domain, and $d \in \mathbb{N}$ be a role-depth bound. Then, the *canonical MMSC* of X for $\mathcal{I}$ in the description logic $\mathcal{M}_d$ is the concept description $\mathsf{mmsc}(X, \mathcal{I}, d)$ which is defined by induction on the role depth as follows.

$$\begin{aligned}
\mathsf{mmsc}(X, \mathcal{I}, 0) &:= \textstyle\bigsqcap\{\, A \mid X \subseteq A^{\mathcal{I}} \,\} \sqcap \bigsqcap\{\, \neg A \mid X \cap A^{\mathcal{I}} = \emptyset \,\} \\
\mathsf{mmsc}(X, \mathcal{I}, d) &:= \mathsf{mmsc}(X, \mathcal{I}, 0) \\
&\qquad \sqcap \textstyle\bigsqcap\{\, \forall\, r.\, \mathsf{mmsc}(Y, \mathcal{I}, d-1) \mid Y \in \mathsf{Min}(\mathsf{Succ}_{\mathcal{I}}(X, \forall\, r)) \,\} \\
&\qquad \sqcap \textstyle\bigsqcap\{\, \exists \geq n.\, r.\, \mathsf{mmsc}(Y, \mathcal{I}, d-1) \mid Y \in \mathsf{Min}(\mathsf{Succ}_{\mathcal{I}}(X, \exists \geq n.\, r) \text{ and } n \leq |\Delta^{\mathcal{I}}| \,\} \\
&\qquad \sqcap \textstyle\bigsqcap\{\, \exists \leq n(X, r).\, r \mid r \in \Sigma_{\mathsf{R}} \,\} \\
&\qquad \sqcap \textstyle\bigsqcap\{\, \exists\, r.\, \mathsf{Self} \mid \{\, (x, x) \mid x \in X \,\} \subseteq r^{\mathcal{I}} \,\}
\end{aligned}$$

Furthermore, we define $\mathsf{mmsc}(\emptyset, \mathcal{I}, d) := \bot$ for all $d \in \mathbb{N}$. △

4.2.2 Theorem. *Let $\mathcal{I}$ be a finite interpretation over the signature Σ, consider a subset $X \subseteq \Delta^{\mathcal{I}}$, and let $d \in \mathbb{N}$ be a role depth bound. Then,* $\mathsf{mmsc}(X, \mathcal{I}, d)$ *is the model-based most specific concept description of X for $\mathcal{I}$ in the description logic $\mathcal{M}_d$, i.e., it holds true that $X^{\mathcal{I}_d} \equiv_{\emptyset} \mathsf{mmsc}(X, \mathcal{I}, d)$.*

Proof. The case $X = \emptyset$ is obvious. Hence, consider a non-empty subset $X \subseteq \Delta^{\mathcal{I}}$. It is easy to see, that for a finite interpretation $\mathcal{I}$, it always holds true that $\mathsf{Min}(\mathsf{Succ}_{\mathcal{I}}(X, \exists \geq n.\, r)) = \emptyset$ for all numbers $n > |\Delta^{\mathcal{I}}|$ and all role names $r \in \Sigma_{\mathsf{R}}$. Consequently $\mathsf{mmsc}(X, \mathcal{I}, d)$ consists of finitely many conjunctions, and is thus a well-defined $\mathcal{M}$ concept description.

We now show the three properties of Definition 4.1.1 by induction on the role-depth bound d. We start with the induction base where $d = 0$. It is readily verified that $\mathsf{mmsc}(X, \mathcal{I}, 0)$ has role depth 0 that that $X \subseteq \mathsf{mmsc}(X, \mathcal{I}, 0)^{\mathcal{I}}$ is satisfied. Assume that D is an $\mathcal{M}$ concept description over Σ with a role depth of 0, i.e., D is a conjunction of concept names and primitive negations only, and let $X \subseteq D^{\mathcal{I}}$. Then, for each concept name $A \in \Sigma_{\mathsf{C}}$ occurring in D, it certainly holds that $X \subseteq A^{\mathcal{I}}$, and hence A must be a top-level conjunct in $\mathsf{mmsc}(X, \mathcal{I}, 0)$, too. Analogously, for a primitive negation $\neg A$ in D, we know that $X \subseteq (\neg A)^{\mathcal{I}}$ must be satisfied, and so also $\neg A$ is contained in the top-level conjunction of $\mathsf{mmsc}(X, \mathcal{I}, 0)$. We just showed that each conjunct in D also occurs in $\mathsf{mmsc}(X, \mathcal{I}, 0)$, and hence $\mathsf{mmsc}(X, \mathcal{I}, 0) \sqsubseteq_{\emptyset} D$ follows.

For the induction step now let $d > 0$. By induction hypothesis, we know that the role depth of each $\mathsf{mmsc}(Y, \mathcal{I}, d-1)$ where $Y \in \mathsf{Min}(\mathsf{Succ}_{\mathcal{I}}(X, \mho\, r))$ does not exceed $d-1$, and so we conclude that the role depth of $\mathsf{mmsc}(X, \mathcal{I}, d)$ is at most d.

Consider a top-level conjunct $\mho\, r.\, \mathsf{mmsc}(Y, \mathcal{I}, d-1)$ occurring in $\mathsf{mmsc}(X, \mathcal{I}, d)$, i.e., $Y \in \mathsf{Min}(\mathsf{Succ}_{\mathcal{I}}(X, \mho\, r))$ holds true. By induction hypothesis, Y is a subset of $\mathsf{mmsc}(Y, \mathcal{I}, d-1)^{\mathcal{I}}$. We continue with a case distinction on the quantifier $\mho$.

($\mho = \exists \geq n$) By definition of the successor sets, it holds true that all elements in Y are r-successors of some element in X, since $Y \subseteq r^{\mathcal{I}}(X)$. Furthermore, Y satisfies the condition that, for each element $x \in X$, the cardinality of the intersection $r^{\mathcal{I}}(x) \cap Y$ is at least n, i.e., each element $x \in X$ has n or more r-successors in Y. Consequently, it holds true that $X \subseteq (\exists \geq n.\, r.\, \mathsf{mmsc}(Y, \mathcal{I}, d-1))^{\mathcal{I}}$.

($\mho = \forall$) In this case, we have that $Y = r^{\mathcal{I}}(X)$. Consider an arbitrary $x \in X$. If $y \in \Delta^{\mathcal{I}}$ and $(x, y) \in r^{\mathcal{I}}$, then $y \in Y$, and so $x \in (\forall\, r.\, \mathsf{mmsc}(Y, \mathcal{I}, d-1))^{\mathcal{I}}$ follows easily.

It is obvious that X is also a subset of the extension of the other top-level conjuncts in $\mathsf{mmsc}(X, \mathcal{I}, d)$.

Eventually, fix some $\mathcal{M}$ concept description D such that $\mathsf{rd}(D) \leq d$ and $X \subseteq D^{\mathcal{I}}$. Further let E be a conjunct on the top level of D. Of course, it then holds true that $X \subseteq E^{\mathcal{I}}$. We proceed with a case distinction on E, and prove that there is always a top-level conjunct in $\mathsf{mmsc}(X, \mathcal{I}, d)$ which is more specific than E modulo $\emptyset$. As a consequence we then obtain that $\mathsf{mmsc}(X, \mathcal{I}, d) \sqsubseteq_{\emptyset} D$ is satisfied.

($E = \forall\, r.\, F$) Since $X \subseteq (\forall\, r.\, F)^{\mathcal{I}}$, we infer that each r-successor of each element in X is in the extension $F^{\mathcal{I}}$, i.e.,

$$\forall x \in X\ \forall y \in \Delta^{\mathcal{I}} \colon (x, y) \in r^{\mathcal{I}} \text{ implies } y \in F^{\mathcal{I}}.$$

As the set $r^{\mathcal{I}}(X)$ contains all r-successors of any element in X and no additional elements, we conclude that $r^{\mathcal{I}}(X) \subseteq F^{\mathcal{I}}$. Applying Statement 1 of Lemma 4.1.2 yields $(r^{\mathcal{I}}(X))^{\mathcal{I}_{d-1}} \sqsubseteq_{\emptyset} F$. An application of the induction hypothesis implies that $(r^{\mathcal{I}}(X))^{\mathcal{I}_{d-1}} \equiv_{\emptyset} \mathsf{mmsc}(r^{\mathcal{I}}(X), \mathcal{I}, d-1)$. Eventually, it follows that

$$\forall\, r.\, \mathsf{mmsc}(r^{\mathcal{I}}(X), \mathcal{I}, d-1) \sqsubseteq_{\emptyset} \forall\, r.\, F.$$

($E = \exists \geq n.\, r.\, F$) By assumption, we have that $X \subseteq (\exists \geq n.\, r.\, F)^{\mathcal{I}}$, i.e., every element $x \in X$ has n or more r-successors which are in the extension of F. Thus, $|r^{\mathcal{I}}(x) \cap F^{\mathcal{I}}| \geq n$ for all $x \in X$, and consequently there is a set $Y \in \mathsf{Min}(\mathsf{Succ}_{\mathcal{I}}(X, \exists \geq n.\, r))$ such that $Y \subseteq F^{\mathcal{I}}$. By applying Statement 1 of Lemma 4.1.2 we conclude that $Y^{\mathcal{I}_{d-1}} \sqsubseteq_{\emptyset} F$, and since the induction hypothesis yields that $Y^{\mathcal{I}_{d-1}} \equiv_\emptyset \mathsf{mmsc}(Y, \mathcal{I}, d-1)$, it eventually follows that $\exists \geq n. r. \mathsf{mmsc}(Y, \mathcal{I}, d-1) \sqsubseteq_\emptyset \exists \geq n. r. F$.

($E = \exists \leq n. r$) The set inclusion $X \subseteq (\exists \leq n. r)^\mathcal{I}$ yields that, for every element $x \in X$, the number of r-successors of x does not exceed n. It is readily verified that then $n(X, r) \leq n$, and thus $\exists \leq n(X, r). r \sqsubseteq_\emptyset \exists \leq n. r$. Of course, $\exists \leq n(X, r). r$ is contained as a top-level conjunct in $\mathsf{mmsc}(X, \mathcal{I}, d)$.

($E = \exists r. \mathsf{Self}$) From $X \subseteq (\exists r. \mathsf{Self})^\mathcal{I}$ it follows that each element $x \in X$ is an r-successor of itself, i.e., $\{ (x, x) \mid x \in X \} \subseteq r^\mathcal{I}$. By definition, $\mathsf{mmsc}(X, \mathcal{I}, d)$ then also contains $\exists r. \mathsf{Self}$ as a top-level conjunct. □

4.3 Most Specific Consequences in $\mathcal{EL}$

The notion of a *most specific consequence* was introduced by the author in [Kri16a]. However, no conditions for their existence have been known, and it has been unclear how and whether these could be computed—problems that will be solved in Section 4.3.1. For describing the origin of that notion, we first take a short detour to *Formal Concept Analysis*, cf. Chapter 1.

Consider a set $\mathcal{L}$ of implications over an attribute set M and let $X \subseteq M$ be a subset. On Page 7 we have seen that there is always a *most specific consequence* of X with respect to $\mathcal{L}$, i.e., there is some superset $Y \supseteq X$ such that $\mathcal{L}$ entails $X \to Y$ and, if $\mathcal{L}$ entails $X \to Z$, then $Z \subseteq Y$. This most specific consequence Y equals $X^\mathcal{L}$, which is the result of exhaustively saturating X with the implications in $\mathcal{L}$ and, alternatively, it can be characterized as the smallest model of $\mathcal{L}$ that contains X.

As it turns out, there is no such notion in the field of *Description Logic*. Anyways, it is readily verified that sets of implications correspond to TBoxes, and consequently we can simply define the following.

4.3.1 Definition. Let $\mathcal{DL}_1$ and $\mathcal{DL}_2$ denote description logics, and fix some $\mathcal{DL}_1$ TBox $\mathcal{T}$ as well as a $\mathcal{DL}_1$ concept description C. Then, a $\mathcal{DL}_2$ concept description D is called a *most specific consequence* or *most specific subsumer* (abbrv. MSS) in $\mathcal{DL}_2$ of C with respect to $\mathcal{T}$ if it satisfies the following conditions.

1. The concept inclusion $C \sqsubseteq D$ follows from $\mathcal{T}$, i.e., $C \sqsubseteq_\mathcal{T} D$.
2. D is most specific with respect to the property of subsuming C w.r.t. $\mathcal{T}$, that is, for each $\mathcal{DL}_2$ concept description E, if $C \sqsubseteq_\mathcal{T} E$, then $D \sqsubseteq_\emptyset E$. △

Note that simply choosing $D := C$ does not do the job in general. For instance, consider the TBox $\mathcal{T} := \{A \sqsubseteq B\}$. Clearly, $A \sqcap B$ is more specific than A w.r.t. $\emptyset$ and both A and $A \sqcap B$ subsume A w.r.t. $\mathcal{T}$, i.e., A is not the *most specific* consequence of A w.r.t. $\mathcal{T}$.

Within this document, we only consider the description logics $\mathcal{EL}$ and $\mathcal{EL}^\bot$ or its extensions with greatest fixed-point semantics, e.g., $\mathcal{EL}_\mathsf{si}$, and $\mathcal{EL}^\bot_\mathsf{si}$, as possible choices for $\mathcal{DL}_1$, and for $\mathcal{DL}_2$ we investigate the cases $\mathcal{EL}$, $\mathcal{EL}^\bot$, $\mathcal{EL}_\mathsf{si}$, $\mathcal{EL}^\bot_\mathsf{si}$, $\mathcal{EL}_d$, and $\mathcal{EL}^\bot_d$ for some $d \in \mathbb{N}$.

As one quickly verifies, *all* most specific consequences of C with respect to $\mathcal{T}$ are unique up to equivalence, and hence we shall denote *the* most specific consequence of C with respect to

$\mathcal{T}$ by $C^{\mathcal{T}}$—provided that it exists. Another immediate consequence of Definition 4.3.1 is that C and its most specific consequence $C^{\mathcal{T}}$ are equivalent with respect to $\mathcal{T}$, since, on the one hand, $C \sqsubseteq_{\mathcal{T}} C^{\mathcal{T}}$ is satisfied, and, on the other hand, C clearly is a consequence of itself w.r.t. $\mathcal{T}$, that is, $C^{\mathcal{T}} \sqsubseteq_{\emptyset} C$. Please note that writing $C^{\mathcal{T}}$ can cause an abuse of notation, since the target DL $\mathcal{DL}_2$ is not specified; however, in this document this will not cause any issues.

Note that Distel [Dis11, Chapter 7] has investigated a dual notion, namely that of a *minimal possible consequence*, which he utilized to constitute an algorithm for the exploration of ontologies, called *ABox Exploration*. To emphasize this duality, it is also reasonable to use the name of a *minimal certain consequence* for a most specific consequence.[3]

As an example, consider the TBox $\mathcal{T} := \{\top \sqsubseteq \exists\, r.\,\top\}$. It can be readily verified that, for each $n \in \mathbb{N}$, the concept description $\exists\, r^n.\,\top$ is a *consequence* (i.e., a subsumer) of $\top$ with respect to $\mathcal{T}$. However, $\exists\, r^{n+1}.\,\top$ is more specific than $\exists\, r^n.\,\top$, and thus a most specific consequence of $\top$ w.r.t. $\mathcal{T}$ does not exist in the description logic $\mathcal{EL}^{\bot}$ with *descriptive semantics* (the standard semantics as introduced in Section 3.1.2). There are two solutions to tackle this problem of existence of most specific consequences. The first one is to use an extension of $\mathcal{EL}^{\bot}$ with *greatest fixed-point semantics*, as these have been introduced in Section 3.4. We have already seen that concept descriptions for standard semantics are finite trees, while concept descriptions for greatest fixed-point semantics can be seen as arbitrary, possibly cyclic, finite graphs. Such extensions have been extensively studied [Baa03a; Baa03b; Dis11; LPW10; LPW10], and in particular it has been shown that these extensions can handle terminological cycles (as present in the given TBox $\mathcal{T}$ above) also within concept descriptions. We are going to prove in the upcoming Section 4.3.1 that most specific consequences always exist in variants of $\mathcal{EL}^{\bot}$ that are equipped with greatest fixed-point semantics. Another solution for ensuring the existence of most specific consequences is to *restrict the role depth* of the concept descriptions under consideration, as this has been done by Borchmann, Distel, and Kriegel [BDK15] to ensure the existence of model-based most specific concept descriptions in $\mathcal{EL}^{\bot}$ with descriptive semantics. This approach shall be considered in Section 4.3.1 as well. Returning to our above example, we can readily verify that, for each role-depth bound $d \in \mathbb{N}$, the $\mathcal{EL}^{\bot}_{d}$ concept description $\exists\, r^d.\,\top$ is the most specific consequence of $\top$ w.r.t. $\mathcal{T}$ in $\mathcal{EL}^{\bot}_{d}$ (for the standard semantics).

4.3.1 Existence and Computation of Most Specific Consequences

Within this section, we shall investigate whether most specific consequences exist in $\mathcal{EL}$ and some of its variants. In particular, we also consider the extension $\mathcal{EL}^{\bot}$ with the bottom concept description, which can be used to express unsatisfiability, and we consider the variant $\mathcal{EL}^{\bot}_{\mathsf{si}}$ that is equipped with greatest fixed-point semantics. As we will demonstrate, most specific consequences always exist in $\mathcal{EL}^{\bot}_{\mathsf{si}}$, most specific consequences always exist in $\mathcal{EL}^{\bot}$ for so-called cycle-restricted TBoxes, and further most specific consequences always exist in $\mathcal{EL}^{\bot}_{d}$ for any $d \in \mathbb{N}$. Additionally, we shall provide means for the computation of most specific consequences, and analyze the complexity of computing these.

[3]Being minimal w.r.t. the subsumption order means being most specific.

The Unrestricted Case

We start our investigations with the unrestricted case, that is, we do not impose any bound on the role depths. More specifically, we will show that in $\mathcal{EL}_{\mathsf{si}}$ most specific consequences always exist and can be computed in polynomial time. Furthermore, it holds true that most specific consequences need not exist in $\mathcal{EL}$, but we can decide in polynomial time whether these exist in $\mathcal{EL}$. The only reason that prevents the existence of $C^{\mathcal{T}}$ in $\mathcal{EL}$ is that $\mathcal{T}$ induces a cycle for some subconcept of C or, more generally, for some concept description that subsumes some subconcept of C w.r.t. $\mathcal{T}$. By such a cycle we mean a concept description D together with a non-empty word $r_1 r_2 \ldots r_n$ of role names such that

$$D \sqsubseteq_{\mathcal{T}} \exists\, r_1 r_2 \ldots r_n. D.$$

It turns out that $C^{\mathcal{T}}$ can be constructed from the canonical model $\mathcal{I}_{C,\mathcal{T}}$ (see Page 83), and that $C^{\mathcal{T}}$ is equivalent to the model-based most specific concept description of $\{C\}$ in $\mathcal{I}_{C,\mathcal{T}}$, which is an $\mathcal{EL}_{\mathsf{si}}$ concept description in general due to the possible presence of cycles in the canonical model. Of course, if $\mathcal{I}_{C,\mathcal{T}}$ does not contain cycles, then $C^{\mathcal{T}}$ is equivalent to an $\mathcal{EL}$ concept description. Thus, in order to check existence of $C^{\mathcal{T}}$ in $\mathcal{EL}$, it suffices to construct the canonical model, which can be done in polynomial time, then compute its reachability relation, i.e., the transitive closure of its set of edges, and finally test if there is some vertex reachable from itself on a path of length at least 1. The task of computing the reachability relation can be solved with the FLOYD-WARSHALL algorithm, which is well-known to run in polynomial time. Furthermore, we shall prove that, for cycle-restricted TBoxes $\mathcal{T}$, all canonical models $\mathcal{I}_{C,\mathcal{T}}$ for arbitrary concept descriptions C are acyclic, which means that most specific consequences with respect to cycle-restricted TBoxes always exist in $\mathcal{EL}$.

The Unrestricted Case for $\mathcal{EL}_{\mathsf{st}}$

The description logic $\mathcal{EL}_{\mathsf{st}}$ [LPW10] extends $\mathcal{EL}$ by the concept constructor $\exists^{\mathsf{sim}}\, \Gamma. (\mathcal{T}, C)$, where $\Gamma \subseteq \Sigma$ is a finite signature, $\mathcal{T}$ is a TBox, and C is a concept description. More specifically, $\mathcal{EL}_{\mathsf{st}}$ concept descriptions, $\mathcal{EL}_{\mathsf{st}}$ concept inclusions, and $\mathcal{EL}_{\mathsf{st}}$ TBoxes are defined by simultaneous induction as follows.

1. Every $\mathcal{EL}$ concept description, every $\mathcal{EL}$ concept inclusion, and every $\mathcal{EL}$ TBox is an $\mathcal{EL}_{\mathsf{st}}$ concept description, $\mathcal{EL}_{\mathsf{st}}$ concept inclusion, and $\mathcal{EL}_{\mathsf{st}}$ TBox, respectively;
2. if $\mathcal{T}$ is an $\mathcal{EL}_{\mathsf{st}}$ TBox, C an $\mathcal{EL}_{\mathsf{st}}$ concept description, and $\Gamma \subseteq \Sigma$ a finite signature, then $\exists^{\mathsf{sim}}\, \Gamma. (\mathcal{T}, C)$ is an $\mathcal{EL}_{\mathsf{st}}$ concept description;
3. if C and D are $\mathcal{EL}_{\mathsf{st}}$ concept descriptions, then $C \sqsubseteq D$ is an $\mathcal{EL}_{\mathsf{st}}$ concept inclusion;
4. an $\mathcal{EL}_{\mathsf{st}}$ TBox is a finite set of $\mathcal{EL}_{\mathsf{st}}$ concept inclusions.

The semantics of the additional concept constructor is defined as follows: let $\mathcal{I}$ be an interpretation, then $\delta \in (\exists^{\mathsf{sim}}\, \Gamma. (\mathcal{T}, C))^{\mathcal{I}}$ if there exists a pointed interpretation $(\mathcal{J}, \epsilon)$ such that $\mathcal{J}$ is a model of $\mathcal{T}$, $\epsilon \in C^{\mathcal{J}}$, and $(\mathcal{J}, \epsilon) \rightleftarrows_{\Sigma \setminus \Gamma} (\mathcal{I}, \delta)$. In case $\Gamma = \emptyset$ we may abbreviate $\exists^{\mathsf{sim}}\, \Gamma. (\mathcal{T}, C)$ as $\exists^{\mathsf{sim}}(\mathcal{T}, C)$. Adding the bottom concept description $\bot$ yields the description logic $\mathcal{EL}^{\bot}_{\mathsf{st}}$. In

[LPW10, Theorem 10] it is demonstrated that $\mathcal{EL}_{\mathsf{st}}$ and $\mathcal{EL}_{\mathsf{si}}$ are polynomially equivalent, i.e., both are equi-expressive and there are translations between both DLs that are computable in polynomial time.

The next proposition demonstrates that most specific consequences of $\mathcal{EL}_{\mathsf{st}}$ concept descriptions with respect to $\mathcal{EL}_{\mathsf{st}}$ TBoxes always exist in $\mathcal{EL}_{\mathsf{st}}$.

4.3.2 Proposition. *For each $\mathcal{EL}_{\mathsf{st}}$ TBox $\mathcal{T}$ and each $\mathcal{EL}_{\mathsf{st}}$ concept description C, the most specific consequence $C^{\mathcal{T}}$ exists in $\mathcal{EL}_{\mathsf{st}}$. More specifically, it holds true that*

$$C^{\mathcal{T}} \equiv_{\emptyset} \exists^{\mathsf{sim}}(\mathcal{T}, C).$$

Proof. Firstly, we show that $\exists^{\mathsf{sim}}(\mathcal{T}, C)$ is a consequence of C with respect to $\mathcal{T}$. Let $\mathcal{I}$ be a model of $\mathcal{T}$ such that $\delta \in C^{\mathcal{I}}$. It is trivial that $(\mathcal{I}, \delta) \precsim (\mathcal{I}, \delta)$, and hence we immediately conclude that $\delta \in (\exists^{\mathsf{sim}}(\mathcal{T}, C))^{\mathcal{I}}$.

Secondly, we prove that $\exists^{\mathsf{sim}}(\mathcal{T}, C)$ is indeed most specific. For this purpose, consider an $\mathcal{EL}_{\mathsf{st}}$ concept description E such that $C \sqsubseteq_{\mathcal{T}} E$, and let $\mathcal{I}$ be an arbitrary interpretation such that $\delta \in (\exists^{\mathsf{sim}}(\mathcal{T}, C))^{\mathcal{I}}$. Of course, then there is a pointed interpretation $(\mathcal{J}, \epsilon)$ such that $(\mathcal{J}, \epsilon) \precsim (\mathcal{I}, \delta)$, $\epsilon \in C^{\mathcal{J}}$, and $\mathcal{J} \models \mathcal{T}$. We proceed with a case distinction on E. If E is an $\mathcal{EL}$ concept description, then we immediately conclude that $\epsilon \in E^{\mathcal{J}}$, and so $\delta \in E^{\mathcal{I}}$. Otherwise, let $E = \exists^{\mathsf{sim}} \Gamma. (\mathcal{U}, D)$ be an $\mathcal{EL}_{\mathsf{st}}$ concept description.[4] It then follows that $\epsilon \in (\exists^{\mathsf{sim}} \Gamma. (\mathcal{U}, D))^{\mathcal{J}}$, and so there is another pointed interpretation $(\mathcal{K}, \zeta)$ with $\mathcal{K} \models \mathcal{U}$, $\zeta \in D^{\mathcal{K}}$, and $(\mathcal{K}, \zeta) \precsim_{\Sigma \setminus \Gamma} (\mathcal{J}, \epsilon)$. We may conclude that $(\mathcal{K}, \zeta) \precsim_{\Sigma \setminus \Gamma} (\mathcal{I}, \delta)$, and consequently $\delta \in (\exists^{\mathsf{sim}} \Gamma. (\mathcal{U}, D))^{\mathcal{I}}$. □

Since $\mathcal{EL}$ is a sublogic of $\mathcal{EL}_{\mathsf{st}}$, we can immediately draw the following conclusion.

4.3.3 Corollary. *For each $\mathcal{EL}$ TBox $\mathcal{T}$ and each $\mathcal{EL}$ concept description C, the most specific consequence $C^{\mathcal{T}}$ exists in $\mathcal{EL}_{\mathsf{st}}$.* □

Furthermore, as $\mathcal{EL}_{\mathsf{st}}$ is polynomially equivalent to $\mathcal{EL}_{\mathsf{si}}$, we can also conclude that most specific consequences always exist in $\mathcal{EL}_{\mathsf{si}}$.

The Unrestricted Case for $\mathcal{EL}$

Furthermore, we can interconnect the notions of most specific consequences and of model-based most specific concept descriptions. In particular, according to the following proposition it holds true that the most specific consequence $C^{\mathcal{T}}$ is equivalent to the model-based most specific concept description $\{C\}^{\mathcal{I}_{C,\mathcal{T}}}$. This important result will later be used to analyze the complexity of computing $C^{\mathcal{T}}$ as well as for deciding existence of $C^{\mathcal{T}}$ in $\mathcal{EL}$.

The Canonical Model of an $\mathcal{EL}$ Concept Description w.r.t. some $\mathcal{EL}$ TBox

The following definition of a canonical model and the following properties are cited from Lutz and Wolter [LW10].

[4] Since each $\mathcal{EL}$ concept description E is equivalent to the $\mathcal{EL}_{\mathsf{st}}$ concept description $\exists^{\mathsf{sim}}(\emptyset, E)$, the first case also follows from this second case.

[LW10, Definition 11]. Let $\mathcal{T}$ be an $\mathcal{EL}$ TBox, and C be an $\mathcal{EL}$ concept description. The *canonical model* $\mathcal{I}_{C,\mathcal{T}}$ of $\mathcal{T}$ and C consists of the following components.

$$\Delta^{\mathcal{I}_{C,\mathcal{T}}} := \{C\} \cup \{\, D \mid \exists r \in \Sigma_{\mathsf{R}} \colon \exists r.\, D \in \mathsf{Sub}(\mathcal{T}) \cup \mathsf{Sub}(C) \,\}$$
$$\cdot^{\mathcal{I}_{C,\mathcal{T}}} \colon \begin{cases} A \mapsto \{\, D \mid D \sqsubseteq_{\mathcal{T}} A \,\} & \text{for any } A \in \Sigma_{\mathsf{C}} \\ r \mapsto \left\{ (D,E) \;\middle|\; \begin{array}{l} D \sqsubseteq_{\mathcal{T}} \exists r.\, E \text{ and } \exists r.\, E \in \mathsf{Sub}(\mathcal{T}), \\ \text{or } \exists r.\, E \in \mathsf{Conj}(D) \end{array} \right\} & \text{for any } r \in \Sigma_{\mathsf{R}} \end{cases}$$
△

Note that $\mathcal{I}_C = \mathcal{I}_{C,\emptyset}$ holds true for any $C \in \mathcal{EL}(\Sigma)$.

[LW10, Lemma 12]. *Let $\mathcal{T}$ be an $\mathcal{EL}$ TBox, and C be an $\mathcal{EL}$ concept description. Then, the following statements hold true.*

1. $D \in D^{\mathcal{I}_{C,\mathcal{T}}}$ *for all* $D \in \Delta^{\mathcal{I}_{C,\mathcal{T}}}$
2. $\mathcal{I}_{C,\mathcal{T}} \models \mathcal{T}$
3. $(\mathcal{I}_{C,\mathcal{T}}, E) \precsim (\mathcal{I}_{D,\mathcal{T}}, E)$ *for all* $D \in \mathcal{EL}(\Sigma)$ *and all* $E \in \Delta^{\mathcal{I}_{C,\mathcal{T}}} \cap \Delta^{\mathcal{I}_{D,\mathcal{T}}}$ □

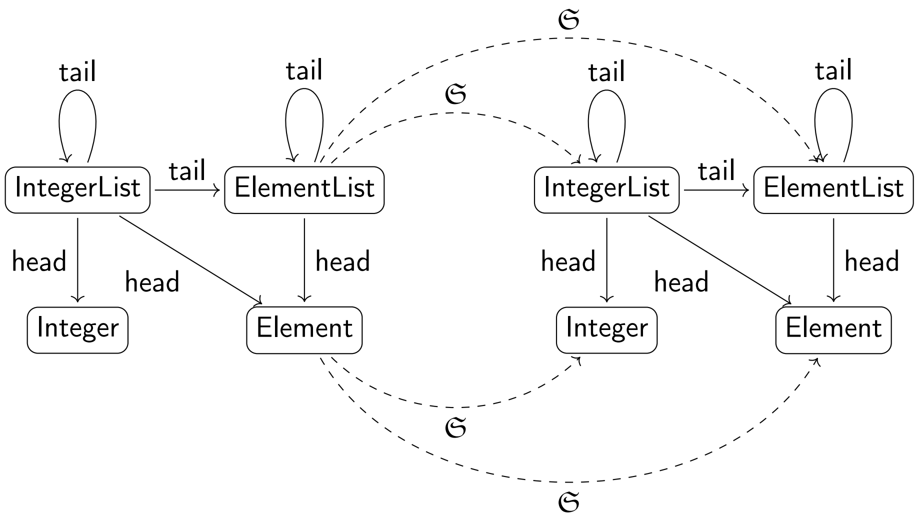
[LW10, Lemma 13]. *Let $\mathcal{T}$ be an $\mathcal{EL}$ TBox, and C be an $\mathcal{EL}$ concept description.*

1. *For all models $\mathcal{I}$ of $\mathcal{T}$ and all objects $\delta \in \Delta^{\mathcal{I}}$, the following statements are equivalent.*
   a) $\delta \in C^{\mathcal{I}}$
   b) $(\mathcal{I}_{C,\mathcal{T}}, C) \precsim (\mathcal{I}, \delta)$
2. *For all $\mathcal{EL}$ concept descriptions D, the following statements are equivalent.*
   a) $\mathcal{T} \models C \sqsubseteq D$
   b) $C \in D^{\mathcal{I}_{C,\mathcal{T}}}$
   c) $(\mathcal{I}_{D,\mathcal{T}}, D) \precsim (\mathcal{I}_{C,\mathcal{T}}, C)$ □

Example. As an example, consider the concept descriptions $\mathsf{IntegerList}$ and $\mathsf{ElementList}$ as well as the following terminological box $\mathcal{T}$ over the signature Σ.

$$\Sigma_{\mathsf{C}} := \{\mathsf{ElementList}, \mathsf{IntegerList}, \mathsf{Element}, \mathsf{Integer}\}$$
$$\Sigma_{\mathsf{R}} := \{\mathsf{head}, \mathsf{tail}\}$$
$$\mathcal{T} := \left\{ \begin{array}{r} \mathsf{ElementList} \equiv \exists\, \mathsf{head}.\, \mathsf{Element} \sqcap \exists\, \mathsf{tail}.\, \mathsf{ElementList}, \\ \mathsf{IntegerList} \equiv \exists\, \mathsf{head}.\, \mathsf{Integer} \sqcap \exists\, \mathsf{tail}.\, \mathsf{IntegerList}, \\ \mathsf{Integer} \sqsubseteq \mathsf{Element} \end{array} \right\}$$

It is easy to see that $\mathsf{IntegerList} \sqsubseteq_{\mathcal{T}} \mathsf{ElementList}$ holds true. Figure 4.3.4 depicts an according simulation $\mathfrak{S}$ from the canonical model $\mathcal{I}_{\mathsf{ElementList},\mathcal{T}}$ to the canonical model $\mathcal{I}_{\mathsf{IntegerList},\mathcal{T}}$ that contains $(\mathsf{ElementList}, \mathsf{IntegerList})$. Note that, in order to ease readability, we have not included node labels; to be complete, we list these as follows. Each node C is labeled with C itself, and further $\mathsf{IntegerList}$ has label $\mathsf{ElementList}$ and $\mathsf{Integer}$ has label $\mathsf{Element}$. △



4.3.4 Figure. A simulation $\mathfrak{S}$ from $\mathcal{I}_{\mathsf{ElementList},\mathcal{T}}$ to $\mathcal{I}_{\mathsf{IntegerList},\mathcal{T}}$

Most Specific Consequences of $\mathcal{EL}$ Concept Descriptions w.r.t. $\mathcal{EL}$ TBoxes

4.3.5 Proposition. *Let $\mathcal{T}$ be an $\mathcal{EL}$ TBox, and C be an $\mathcal{EL}$ concept description. Then the most specific consequence of C with respect to $\mathcal{T}$ is equivalent to the model-based most specific concept description of $\{C\}$ with respect to the canonical model of $\mathcal{T}$ and C, i.e.,*

$$C^{\mathcal{T}} \equiv_{\emptyset} \{C\}^{\mathcal{I}_{C,\mathcal{T}}}.$$

Proof. Note that the model-based most specific concept description of $\{C\}$ with respect to $\mathcal{I}_{C,\mathcal{T}}$ is described by the $\mathcal{EL}_{\mathsf{si}}$ concept description $\exists^{\mathsf{sim}}(\mathcal{I}_{C,\mathcal{T}}, C)$. Hence, it suffices to show that the concept descriptions $\exists^{\mathsf{sim}}(\mathcal{T}, C)$ and $\exists^{\mathsf{sim}}(\mathcal{I}_{C,\mathcal{T}}, C)$ are equivalent. For this purpose consider an arbitrary interpretation $\mathcal{I}$ and an element $\delta \in \Delta^{\mathcal{I}}$. By definition of the semantics, $\delta \in (\exists^{\mathsf{sim}}(\mathcal{T}, C))^{\mathcal{I}}$ if, and only if, there is a pointed interpretation $(\mathcal{J}, \epsilon)$ such that $(\mathcal{J}, \epsilon) \precsim (\mathcal{I}, \delta)$, $\mathcal{J} \models \mathcal{T}$, and $\epsilon \in C^{\mathcal{J}}$. Furthermore, $\delta \in (\exists^{\mathsf{sim}}(\mathcal{I}_{C,\mathcal{T}}, C))^{\mathcal{I}}$ if, and only if, $(\mathcal{I}_{C,\mathcal{T}}, C) \precsim (\mathcal{I}, \delta)$, i.e., if there is a simulation from $(\mathcal{I}_{C,\mathcal{T}}, C)$ to $(\mathcal{I}, \delta)$.

Firstly, assume that $\delta \in (\exists^{\mathsf{sim}}(\mathcal{T}, C))^{\mathcal{I}}$. Then [LW10, Lemma 13] yields that there is a simulation from $(\mathcal{I}_{C,\mathcal{T}}, C)$ to $(\mathcal{I}, \delta)$, since simulations are closed under composition, and thus $\delta \in (\exists^{\mathsf{sim}}(\mathcal{I}_{C,\mathcal{T}}, C))^{\mathcal{I}}$.

Vice versa, let $\delta \in (\exists^{\mathsf{sim}}(\mathcal{I}_{C,\mathcal{T}}, C))^{\mathcal{I}}$, i.e., $(\mathcal{I}_{C,\mathcal{T}}, C) \precsim (\mathcal{I}, \delta)$. By [LW10, Lemma 12] we have that $\mathcal{I}_{C,\mathcal{T}}$ is a model of $\mathcal{T}$, and that $C \in C^{\mathcal{I}_{C,\mathcal{T}}}$. Consequently, $\delta \in (\exists^{\mathsf{sim}}(\mathcal{T}, C))^{\mathcal{I}}$. □

As already mentioned, the existence of cycles induced by the TBox $\mathcal{T}$ can require that also a description of the most specific consequence $C^{\mathcal{T}}$ must contain a cycle, which can be expressed in $\mathcal{EL}_{\mathsf{si}}$, but not in $\mathcal{EL}$. This observation yields a sufficient condition for the existence of $C^{\mathcal{T}}$ in $\mathcal{EL}$, namely that most specific consequences always exist w.r.t. cycle-restricted TBoxes—a notion that is cited below.

[BBM12b; BBM12a, Definition 2]. *An $\mathcal{EL}$ TBox $\mathcal{T}$ is* cycle-restricted *if there does not exist an*

$\mathcal{EL}$ concept description C and a non-empty role word $w \in \Sigma_{\mathsf{R}}^{+}$ such that $C \sqsubseteq_{\mathcal{T}} \exists\, w.\, C$. □

Example. The TBox $\mathcal{T}$ defined below is not cycle-restricted, since it entails the cyclic concept inclusion $A \sqsubseteq \exists\, r^3.\, A$.

$$\mathcal{T} := \left\{ \begin{aligned} A &\sqsubseteq \exists\, rs.\, (B \sqcap \exists\, r.\, B), \\ \exists\, s.\, \exists\, r.\, \top &\sqsubseteq B, \\ \exists\, r.\, B &\sqsubseteq \exists\, rrr.\, A \end{aligned} \right\}$$

△

In the following, we shall consider the directed graphs $(\Delta^{\mathcal{I}}, \bigcup\{\, r^{\mathcal{I}} \mid r \in \Sigma_{\mathsf{R}} \,\})$ for interpretations $\mathcal{I}$ over Σ. That way, we can utilize graph-theoretic notions when speaking about interpretations.

4.3.6 Proposition. *For each $\mathcal{EL}$ TBox $\mathcal{T}$, the following statements are equivalent.*

1. *$\mathcal{T}$ is cycle-restricted.*
2. *For each $\mathcal{EL}$ concept description C, the canonical model $\mathcal{I}_{C,\mathcal{T}}$ is acyclic.*
3. *For each $\mathcal{EL}$ concept description C, the most specific consequence $C^{\mathcal{T}}$ exists in $\mathcal{EL}$.*

Proof. We start with demonstrating that Statement 1 implies Statement 2. Consider a TBox $\mathcal{T}$ and a concept description C, and assume that the canonical model $\mathcal{I}_{C,\mathcal{T}}$ is not acyclic. Then, $\mathcal{I}_{C,\mathcal{T}}$ contains some cycle

$$D \xrightarrow{r_1} D_1 \xrightarrow{r_2} D_2 \xrightarrow{r_3} \ldots \xrightarrow{r_n} D.$$

It immediately follows that $D \sqsubseteq_{\mathcal{T}} \exists\, w.\, D$ where $w := r_1 r_2 r_3 \ldots r_n \in \Sigma_{\mathsf{R}}^{+}$, which yields that $\mathcal{T}$ is not cycle-restricted.

We proceed with proving that Statement 2 implies Statement 1. Let $w \in \Sigma_{\mathsf{R}}^{+}$ and consider some concept description D such that $D \sqsubseteq_{\mathcal{T}} \exists\, w.\, D$. Firstly, assume that the word w has length 1, i.e., $w = r$ for some role name $r \in \Sigma_{\mathsf{R}}$. By the very definition of a canonical model, it is then apparent that $(D, D) \in r^{\mathcal{I}_{C,\mathcal{T}}}$ is a loop in the canonical model $\mathcal{I}_{C,\mathcal{T}}$, that is, $\mathcal{I}_{C,\mathcal{T}}$ is not acyclic. Secondly, assume that w has a length of at least 2, i.e., there are role names $r, s \in \Sigma_{\mathsf{R}}$ and a role word $v \in \Sigma_{\mathsf{R}}^{*}$ such that $w = rvs$. Our assumption implies that $D \sqsubseteq_{\mathcal{T}} \exists\, r.\, \exists\, v.\, \exists\, s.\, D$, and so [LW10, Lemma 13] shows that $D \in (\exists\, r.\, \exists\, v.\, \exists\, s.\, D)^{\mathcal{I}_{C,\mathcal{T}}}$, i.e., there is a path $D \xrightarrow{r} E \xrightarrow{v} F \xrightarrow{s} G$ in $\mathcal{I}_{C,\mathcal{T}}$ such that $G \in D^{\mathcal{I}_{C,\mathcal{T}}}$. In particular, we infer that $F \sqsubseteq_{\mathcal{T}} \exists\, s.\, G$ and $G \sqsubseteq_{\mathcal{T}} D$, and thus $F \sqsubseteq_{\mathcal{T}} \exists\, s.\, D$. Consequently, we have found a path $D \xrightarrow{r} E \xrightarrow{v} F \xrightarrow{s} D$ in the canonical model $\mathcal{I}_{C,\mathcal{T}}$, which is hence not acyclic.

Of course, an $\mathcal{EL}_{\mathsf{si}}$ concept description $\exists^{\mathsf{sim}}(\mathcal{I}, \delta)$ is equivalent to an $\mathcal{EL}$ concept description if, and only if, the connected component of $\mathcal{I}$ that contains δ is acyclic. Since we have shown in Proposition 4.3.5 that $C^{\mathcal{T}} \equiv_{\emptyset} \exists^{\mathsf{sim}}(\mathcal{I}_{C,\mathcal{T}}, C)$ holds true, we immediately conclude that Statement 2 implies Statement 3. For the converse direction, assume that Statement 3 is satisfied and consider some $\mathcal{EL}$ concept description C. First note that the canonical model $\mathcal{I}_{C,\mathcal{T}}$ is a disjoint union of connected components. More specifically, there are $D_1, \ldots, D_n \in \Delta^{\mathcal{I}_{C,\mathcal{T}}}$ such that $\mathcal{I}_{C,\mathcal{T}}$ is the disjoint union of the connected components of $\mathcal{I}_{D,\mathcal{T}}$ containing D where D ranges over $\{C, D_1, \ldots, D_n\}$. By assumption we now have that, for each concept description

$D \in \{C, D_1, \ldots, D_n\}$, the most specific consequence $D^{\mathcal{T}}$ exists, i.e., in $\mathcal{I}_{D,\mathcal{T}}$ the connected component containing D is acyclic. It follows that all connected components of $\mathcal{I}_{C,\mathcal{T}}$ must be acyclic, i.e., the whole canonical model $\mathcal{I}_{C,\mathcal{T}}$ is acyclic, which yields Statement 2. □

4.3.7 Corollary. *The problem whether all most specific consequences with respect to some $\mathcal{EL}$ TBox $\mathcal{T}$ exist in $\mathcal{EL}$ can be decided in deterministic polynomial time.*

Proof. The problem whether an $\mathcal{EL}$ TBox is cycle-restricted can be decided in deterministic polynomial time, cf. [BBM12b, Lemma 21]. Thus, the statement follows from Proposition 4.3.6. □

However, the condition that $\mathcal{T}$ is cycle-restricted is not necessary for the existence of $C^{\mathcal{T}}$ in $\mathcal{EL}$. To see this, consider the TBox $\mathcal{T} := \{A \sqsubseteq \exists\, r.\, A\}$. It is apparent that $\mathcal{T}$ is not cycle-restricted, although the most specific consequence of B w.r.t. $\mathcal{T}$ exists in $\mathcal{EL}$, and is (equivalent to) B. We see that $\mathcal{T}$ induces a cycle which does not affect the concept description B or, more specifically, B does not contain any subconcept that entails A, and so the cycle in $\mathcal{T}$ does not induce a cycle in a description of $B^{\mathcal{T}}$. This idea is utilized in the proof of the upcoming proposition, which shows that existence of $C^{\mathcal{T}}$ in $\mathcal{EL}$ can always be decided in polynomial time.

4.3.8 Proposition. *The problem whether the most specific consequence $C^{\mathcal{T}}$ of an $\mathcal{EL}$ concept description C with respect to an $\mathcal{EL}$ TBox $\mathcal{T}$ exists in $\mathcal{EL}$ can be decided in deterministic polynomial time.*

Proof. We have shown in Proposition 4.3.5 that the most specific consequence $C^{\mathcal{T}}$ is equivalent to the model-based most specific concept description $\{C\}^{\mathcal{I}_{C,\mathcal{T}}}$, which is equivalent to $\exists^{\mathsf{sim}}(\mathcal{I}_{C,\mathcal{T}}, C)$. Furthermore, an $\mathcal{EL}_{\mathsf{si}}$ concept description $\exists^{\mathsf{sim}}(\mathcal{I}, \delta)$ is equivalent to an $\mathcal{EL}$ concept description if, and only if, the connected component of $\mathcal{I}$ that contains δ is acyclic. According to Lutz and Wolter [LW10], the canonical model $\mathcal{I}_{C,\mathcal{T}}$ can be constructed in time polynomial in the size of C and $\mathcal{T}$. By means of the Floyd-Warshall algorithm, the transitive closure E^{+} for a given directed graph (V, E) can be computed in deterministic time $\mathcal{O}(|V|^3)$ and in deterministic space $\mathcal{O}(|V|^2)$. It follows that reachability in the canonical model $\mathcal{I}_{C,\mathcal{T}}$ can be decided in time polynomial in the size of C and $\mathcal{T}$. We now only need to check whether in $\mathcal{I}_{C,\mathcal{T}}$ there is some object in the connected component containing C that is reachable from itself on a path of length at least 1. Clearly, such an object exists if, and only if, $C^{\mathcal{T}}$ does not exist in $\mathcal{EL}$. □

Eventually, we analyze the complexity of computing $C^{\mathcal{T}}$. This is an easy task, since we have already shown that $C^{\mathcal{T}}$ can be computed as a model-based concept description for the canonical model $\mathcal{I}_{C,\mathcal{T}}$, and since canonical models can be constructed in polynomial time.

4.3.9 Proposition. *The most specific consequence $C^{\mathcal{T}}$ of an $\mathcal{EL}$ concept description C with respect to an $\mathcal{EL}$ TBox $\mathcal{T}$ can be computed in deterministic polynomial time, and its size is polynomial in $||C|| + ||\mathcal{T}||$.*

Proof. The statements are obtained as immediate corollaries from Proposition 4.3.5 and the fact that the canonical model $\mathcal{I}_{C,\mathcal{T}}$ can be computed in polynomial time, cf. Lutz and Wolter [LW10]. □

The Canonical Model of an $\mathcal{EL}_{\mathsf{si}}$ Concept Description w.r.t. some $\mathcal{EL}_{\mathsf{si}}$ TBox

In the following, we shall extend the results from the previous section to the description logic $\mathcal{EL}_{\mathsf{si}}$. Thus, fix some $\mathcal{EL}_{\mathsf{si}}$ concept description C as well as some $\mathcal{EL}_{\mathsf{si}}$ terminological box $\mathcal{T}$. The *canonical model* $\mathcal{I}_{C,\mathcal{T}}$ of C and $\mathcal{T}$ consists of the following components. Note that the definition is essentially the same as [LW10, Definition 11] except that all occurring concept descriptions are now formulated in $\mathcal{EL}_{\mathsf{si}}$ instead of $\mathcal{EL}$.

$$\Delta^{\mathcal{I}_{C,\mathcal{T}}} := \{C\} \cup \{\, D \mid \exists\, r \in \Sigma_{\mathsf{R}} \colon \exists\, r.\, D \in \mathsf{Sub}(\mathcal{T}) \cup \mathsf{Sub}(C) \,\}$$

$$\cdot^{\mathcal{I}_{C,\mathcal{T}}} \colon \begin{cases} A \mapsto \{\, D \mid D \sqsubseteq_{\mathcal{T}} A \,\} & \text{for any } A \in \Sigma_{\mathsf{C}} \\ r \mapsto \left\{ (D,E) \;\middle|\; \begin{array}{l} D \sqsubseteq_{\mathcal{T}} \exists\, r.\, E \text{ and } \exists\, r.\, E \in \mathsf{Sub}(\mathcal{T}), \\ \text{or } \exists\, r.\, E \in \mathsf{Conj}(D) \end{array} \right\} & \text{for any } r \in \Sigma_{\mathsf{R}} \end{cases}$$

Some of the following statements and corresponding proofs might be similar to [LW10, Lemmas 12 and 13]. However, note that we use $\mathcal{EL}_{\mathsf{si}}$ instead of $\mathcal{EL}$ and our goal is obtaining a computation method for most specific consequences. As a consequence, we only formulate and prove statements that are directly required for demonstrating that the $\mathcal{EL}_{\mathsf{si}}$ concept description $\exists^{\mathsf{sim}}(\mathcal{I}_{C,\mathcal{T}}, C)$ is the most specific consequence of C w.r.t. $\mathcal{T}$. Afterwards, analogs of [LW10, Lemmas 12 and 13] will be provided.

4.3.10 Lemma. *The following statements hold true.*

1. $\exists^{\mathsf{sim}}(\mathcal{I}_{C,\mathcal{T}}, \exists^{\mathsf{sim}}(\mathcal{J}, \epsilon)) \sqsubseteq_{\emptyset} \exists^{\mathsf{sim}}(\mathcal{J}, \epsilon)$ *for each* $\exists^{\mathsf{sim}}(\mathcal{J}, \epsilon) \in \Delta^{\mathcal{I}_{C,\mathcal{T}}}$.
2. *If* $\mathcal{K}$ *is a model of* $\mathcal{T}$ *and* $a \in X^{\mathcal{K}}$ *for some* $X \in \Delta^{\mathcal{I}_{C,\mathcal{T}}}$, *then there is a simulation from* $(\mathcal{I}_{C,\mathcal{T}}, X)$ *to* $(\mathcal{K}, a)$.
3. *If there is a simulation from* $(\mathcal{J}, \epsilon)$ *to* $(\mathcal{I}_{C,\mathcal{T}}, X)$, *then it follows that* $X \sqsubseteq_{\mathcal{T}} \exists^{\mathsf{sim}}(\mathcal{J}, \epsilon)$.
4. $X \sqsubseteq_{\mathcal{T}} \exists^{\mathsf{sim}}(\mathcal{I}_{C,\mathcal{T}}, X)$ *for each* $X \in \Delta^{\mathcal{I}_{C,\mathcal{T}}}$.
5. $\exists^{\mathsf{sim}}(\mathcal{I}_{C,\mathcal{T}}, X) \sqsubseteq_{\mathcal{T}} Y$ *implies* $\exists^{\mathsf{sim}}(\mathcal{I}_{C,\mathcal{T}}, X) \sqsubseteq_{\emptyset} Y$ *for each* $Y \in \Sigma_{\mathsf{C}} \cup \{\, \exists\, r.\, Z \mid \exists\, r.\, Z \in \mathsf{Sub}(\mathcal{T}) \,\}$.

Proof. 1. It is easy to see that the relation

$$\{\, (\zeta, \exists^{\mathsf{sim}}(\mathcal{J}, \zeta)) \mid \zeta \in \Delta^{\mathcal{J}} \,\}$$

is a simulation from $(\mathcal{J}, \epsilon)$ to $(\mathcal{I}_{C,\mathcal{T}}, \exists^{\mathsf{sim}}(\mathcal{J}, \epsilon))$.

2. Define the relation $\mathfrak{S} := \{\, (Y, b) \mid b \in Y^{\mathcal{K}} \,\}$. By assumption it holds true that (X, a) is in $\mathfrak{S}$. Now consider an arbitrary pair $(Y, b) \in \mathfrak{S}$.
   - If $Y \in A^{\mathcal{I}_{C,\mathcal{T}}}$, i.e., $Y \sqsubseteq_{\mathcal{T}} A$, then we have $Y^{\mathcal{K}} \subseteq A^{\mathcal{K}}$. It follows that $b \in A^{\mathcal{K}}$.
   - If $(Y, Z) \in r^{\mathcal{I}_{C,\mathcal{T}}}$, i.e., $Y \sqsubseteq_{\mathcal{T}} \exists\, r.\, Z$, then we have $Y^{\mathcal{K}} \subseteq (\exists\, r.\, Z)^{\mathcal{K}}$. It follows that $b \in (\exists\, r.\, Z)^{\mathcal{K}}$, i.e., there is some object $c \in \Delta^{\mathcal{K}}$ such that $(b, c) \in r^{\mathcal{K}}$ and $c \in Z^{\mathcal{K}}$, which means that $(Z, c) \in \mathfrak{S}$.
3. Let $\mathcal{K}$ be some model of $\mathcal{T}$ such that $a \in X^{\mathcal{K}}$. From Statement 2 we then get some simulation from $(\mathcal{I}_{C,\mathcal{T}}, X)$ to $(\mathcal{K}, a)$. Composing it with the existing simulation from

$(\mathcal{J},\epsilon)$ to $(\mathcal{I}_{C,\mathcal{T}},X)$ yields a simulation from $(\mathcal{J},\epsilon)$ to $(\mathcal{K},a)$. Of course, this shows $a \in (\exists^{\mathsf{sim}}(\mathcal{J},\epsilon))^{\mathcal{K}}$.

4. Since the reflexive relation is a simulation from $(\mathcal{I}_{C,\mathcal{T}},X)$ to $(\mathcal{I}_{C,\mathcal{T}},X)$, Statement 3 implies that $X \sqsubseteq_{\mathcal{T}} \exists^{\mathsf{sim}}(\mathcal{I}_{C,\mathcal{T}},X)$.
5. Consider some $Y \in \Sigma_{\mathsf{C}} \cup \{\, \exists\, r.\, Z \mid \exists\, r.\, Z \in \mathsf{Sub}(\mathcal{T}) \,\}$ such that $\exists^{\mathsf{sim}}(\mathcal{I}_{C,\mathcal{T}},X) \sqsubseteq_{\mathcal{T}} Y$. Using Statement 4 we conclude that $X \sqsubseteq_{\mathcal{T}} Y$ holds true as well.
   - If $Y = A$ is a concept name, then according to the definition of $\mathcal{I}_{C,\mathcal{T}}$ it follows that $X \in A^{\mathcal{I}_{C,\mathcal{T}}}$, i.e., $\exists^{\mathsf{sim}}(\mathcal{I}_{C,\mathcal{T}},X) \sqsubseteq_{\emptyset} A$.
   - If $Y = \exists\, r.\, Z$ is an existential restriction which occurs as a subconcept in $\mathcal{T}$, then we have that $(X,Z) \in r^{\mathcal{I}_{C,\mathcal{T}}}$ and $Z \in \Delta^{\mathcal{I}_{C,\mathcal{T}}}$. This implies that $\exists^{\mathsf{sim}}(\mathcal{I}_{C,\mathcal{T}},X) \sqsubseteq_{\emptyset} \exists\, r.\, \exists^{\mathsf{sim}}(\mathcal{I}_{C,\mathcal{T}},Z)$. Since further $\exists^{\mathsf{sim}}(\mathcal{I}_{C,\mathcal{T}},Z) \sqsubseteq_{\emptyset} Z$ is implied by Statement 1, we obtain $\exists^{\mathsf{sim}}(\mathcal{I}_{C,\mathcal{T}},X) \sqsubseteq_{\emptyset} \exists\, r.\, Z$. □

4.3.11 Proposition. *$\mathcal{I}_{C,\mathcal{T}}$ is a model of $\mathcal{T}$.*

Proof. Let $X \in \Delta^{\mathcal{I}_{C,\mathcal{T}}}$ and consider some concept inclusion $E \sqsubseteq F$ in $\mathcal{T}$ such that $X \in E^{\mathcal{I}_{C,\mathcal{T}}}$. We shall prove that $X \in F^{\mathcal{I}_{C,\mathcal{T}}}$ holds true as well. $X \in E^{\mathcal{I}_{C,\mathcal{T}}}$ implies $\exists^{\mathsf{sim}}(\mathcal{I}_{C,\mathcal{T}},X) \sqsubseteq_{\emptyset} E$ and so it follows that $\exists^{\mathsf{sim}}(\mathcal{I}_{C,\mathcal{T}},X) \sqsubseteq_{\mathcal{T}} F$, i.e., $\exists^{\mathsf{sim}}(\mathcal{I}_{C,\mathcal{T}},X) \sqsubseteq_{\mathcal{T}} G$ for each $G \in \mathsf{Conj}(F)$. We can now apply Statement 5 of Lemma 4.3.10 to infer that $\exists^{\mathsf{sim}}(\mathcal{I}_{C,\mathcal{T}},X) \sqsubseteq_{\emptyset} G$ for each $G \in \mathsf{Conj}(F)$. Of course, this shows that $\exists^{\mathsf{sim}}(\mathcal{I}_{C,\mathcal{T}},X) \sqsubseteq_{\emptyset} F$, i.e., $X \in F^{\mathcal{I}_{C,\mathcal{T}}}$. □

4.3.12 Proposition. *$\exists^{\mathsf{sim}}(\mathcal{I}_{C,\mathcal{T}},C)$ is the most specific consequence of C w.r.t. $\mathcal{T}$.*

Proof. Statement 4 of Lemma 4.3.10 shows that $\exists^{\mathsf{sim}}(\mathcal{I}_{C,\mathcal{T}},C)$ is a consequence of C w.r.t. $\mathcal{T}$. It remains to show that $\exists^{\mathsf{sim}}(\mathcal{I}_{C,\mathcal{T}},C)$ is most specific. Thus, consider some $\mathcal{EL}_{\mathsf{si}}$ concept description $\exists^{\mathsf{sim}}(\mathcal{I},\delta)$ where $C \sqsubseteq_{\mathcal{T}} \exists^{\mathsf{sim}}(\mathcal{I},\delta)$. We need to prove that $\exists^{\mathsf{sim}}(\mathcal{I}_{C,\mathcal{T}},C) \sqsubseteq_{\emptyset} \exists^{\mathsf{sim}}(\mathcal{I},\delta)$, i.e., that there is some simulation from $(\mathcal{I},\delta)$ to $(\mathcal{I}_{C,\mathcal{T}},C)$.

Since $\mathcal{I}_{C,\mathcal{T}} \models \mathcal{T}$, we have $C^{\mathcal{I}_{C,\mathcal{T}}} \subseteq (\exists^{\mathsf{sim}}(\mathcal{I},\delta))^{\mathcal{I}_{C,\mathcal{T}}}$. Furthermore, Statement 1 of Lemma 4.3.10 implies that $C \in C^{\mathcal{I}_{C,\mathcal{T}}}$ holds true, and so we infer that $C \in (\exists^{\mathsf{sim}}(\mathcal{I},\delta))^{\mathcal{I}_{C,\mathcal{T}}}$. This in turn means that $\exists^{\mathsf{sim}}(\mathcal{I}_{C,\mathcal{T}},C) \sqsubseteq_{\emptyset} \exists^{\mathsf{sim}}(\mathcal{I},\delta)$ as claimed. □

4.3.13 Lemma. *For each $D \in \Delta^{\mathcal{I}_{C,\mathcal{T}}}$, the pointed interpretations $(\mathcal{I}_{C,\mathcal{T}},D)$ and $(\mathcal{I}_{D,\mathcal{T}},D)$ are equi-similar.*

Proof. We know from Proposition 4.3.11 that $\mathcal{I}_{D,\mathcal{T}}$ is a model of $\mathcal{T}$. Furthermore, Statement 1 of Lemma 4.3.10 shows that $\exists^{\mathsf{sim}}(\mathcal{I}_{D,\mathcal{T}},D)$ is more specific than D modulo $\emptyset$, which yields that $D \in D^{\mathcal{I}_{D,\mathcal{T}}}$ by an application of the laws of Galois connections, cf. Lemmas 4.1.2 and 4.1.3. According to Statement 2 of Lemma 4.3.10, there must exist some simulation from $(\mathcal{I}_{C,\mathcal{T}},D)$ to $(\mathcal{I}_{D,\mathcal{T}},D)$.

Of course, we have that D is an object of $\mathcal{I}_{D,\mathcal{T}}$ as well and $\mathcal{I}_{C,\mathcal{T}}$ is a model of $\mathcal{T}$. Statement 1 of Lemma 4.3.10 implies that $\exists^{\mathsf{sim}}(\mathcal{I}_{C,\mathcal{T}},D) \sqsubseteq_{\emptyset} D$ holds true, and we conclude that $D \in D^{\mathcal{I}_{C,\mathcal{T}}}$. Now Statement 2 of Lemma 4.3.10 ensures the existence of some simulation from $(\mathcal{I}_{D,\mathcal{T}},D)$ to $(\mathcal{I}_{C,\mathcal{T}},D)$. □

4.3.14 Corollary. *Fix some $\mathcal{EL}_{\mathsf{si}}$ concept description C as well as an $\mathcal{EL}_{\mathsf{si}}$ TBox $\mathcal{T}$. For each object D in the domain of $\mathcal{I}_{C,\mathcal{T}}$ and for each $\mathcal{EL}_{\mathsf{si}}$ concept description E, the following statements are equivalent.*

1. $D \in E^{\mathcal{I}_{C,\mathcal{T}}}$
2. $\exists^{\mathsf{sim}}(\mathcal{I}_{C,\mathcal{T}}, D) \sqsubseteq_{\emptyset} E$
3. $D \sqsubseteq_{\mathcal{T}} E$

Proof. Statements 1 and 2 are equivalent according to Lemmas 4.1.2 and 4.1.3. Furthermore, Lemma 4.3.13 shows that $(\mathcal{I}_{C,\mathcal{T}}, D)$ and $(\mathcal{I}_{D,\mathcal{T}}, D)$ are always equi-similar, i.e., the concept descriptions $\exists^{\mathsf{sim}}(\mathcal{I}_{C,\mathcal{T}}, D)$ and $\exists^{\mathsf{sim}}(\mathcal{I}_{D,\mathcal{T}}, D)$ are equivalent modulo $\emptyset$. We also know from Proposition 4.3.12 that $\exists^{\mathsf{sim}}(\mathcal{I}_{D,\mathcal{T}}, D)$ is the most specific consequence of D w.r.t. $\mathcal{T}$. On the one hand, this immediately shows that Statement 3 implies Statement 2. On the other hand, this in particular means that $D \sqsubseteq_{\mathcal{T}} \exists^{\mathsf{sim}}(\mathcal{I}_{D,\mathcal{T}}, D)$ is satisfied. It is now easy to verify that Statement 2 implies Statement 3 as well. □

Eventually, we note that similar statements as given in [LW10, Lemmas 12 and 13] hold true for the generalization of the canonical model $\mathcal{I}_{C,\mathcal{T}}$ to the description logic $\mathcal{EL}_{\mathsf{si}}$.

4.3.15 Corollary. (Generalization of [LW10, Lemma 12]) *Fix some $\mathcal{EL}_{\mathsf{si}}$ concept description C and some $\mathcal{EL}_{\mathsf{si}}$ TBox $\mathcal{T}$. Then the following statements hold true.*

1. $D \in D^{\mathcal{I}_{C,\mathcal{T}}}$ *for all* $D \in \Delta^{\mathcal{I}_{C,\mathcal{T}}}$
2. $\mathcal{I}_{C,\mathcal{T}} \models \mathcal{T}$
3. $(\mathcal{I}_{C,\mathcal{T}}, E) \rightleftarrows (\mathcal{I}_{D,\mathcal{T}}, E)$ *for all* $D \in \mathcal{EL}_{\mathsf{si}}(\Sigma)$ *and all* $E \in \Delta^{\mathcal{I}_{C,\mathcal{T}}} \cap \Delta^{\mathcal{I}_{D,\mathcal{T}}}$

Proof. In Statement 1 of Lemma 4.3.10 we have shown that $\exists^{\mathsf{sim}}(\mathcal{I}_{C,\mathcal{T}}, D) \sqsubseteq_{\emptyset} D$ is satisfied for each $D \in \Delta^{\mathcal{I}_{C,\mathcal{T}}}$. By means of Lemmas 4.1.2 and 4.1.3 we infer that $D \in D^{\mathcal{I}_{C,\mathcal{T}}}$ must be true, which shows Statement 1.

Statement 2 has already been shown in Proposition 4.3.11.

Regarding Statement 3 let now $E \in \Delta^{\mathcal{I}_{C,\mathcal{T}}} \cap \Delta^{\mathcal{I}_{D,\mathcal{T}}}$. According to Lemma 4.3.13 it follows that, on the one hand, $(\mathcal{I}_{C,\mathcal{T}}, E)$ and $(\mathcal{I}_{E,\mathcal{T}}, E)$ are equi-similar and, on the other hand, $(\mathcal{I}_{D,\mathcal{T}}, E)$ and $(\mathcal{I}_{E,\mathcal{T}}, E)$ are equi-similar. Since equi-similarity is transitive, we obtain that the pointed interpretations $(\mathcal{I}_{C,\mathcal{T}}, E)$ and $(\mathcal{I}_{D,\mathcal{T}}, E)$ are equi-similar as well. □

4.3.16 Corollary. (Generalization of [LW10, Lemma 13]) *Fix some $\mathcal{EL}_{\mathsf{si}}$ concept description C and some $\mathcal{EL}_{\mathsf{si}}$ TBox $\mathcal{T}$.*

1. *For each model $\mathcal{I}$ of $\mathcal{T}$ and each object $\delta \in \Delta^{\mathcal{I}}$, the following statements are equivalent.*
   a) $\delta \in C^{\mathcal{I}}$
   b) $(\mathcal{I}_{C,\mathcal{T}}, C) \rightleftarrows (\mathcal{I}, \delta)$
2. *For each $\mathcal{EL}_{\mathsf{si}}$ concept description D, the following statements are equivalent.*
   a) $C \sqsubseteq_{\mathcal{T}} D$
   b) $C \in D^{\mathcal{I}_{C,\mathcal{T}}}$

c) $(\mathcal{I}_{D,\mathcal{T}}, D) \precsim (\mathcal{I}_{C,\mathcal{T}}, C)$

Proof. Assume that C has the form $\exists^{\mathsf{sim}}(\mathcal{J}, \epsilon)$. Then $\delta \in C^{\mathcal{I}}$ is equivalent to $(\mathcal{J}, \epsilon) \precsim (\mathcal{I}, \delta)$. Since $(\mathcal{J}, \epsilon) \precsim (\mathcal{I}_{C,\mathcal{T}}, C)$ obviously holds true, we conclude that Statement 1b implies Statement 1a.

Now regarding the converse direction. We know that $(\mathcal{J}, \epsilon) \precsim (\mathcal{I}, \delta)$ means that $\exists^{\mathsf{sim}}(\mathcal{I}, \delta)$ is more specific than C modulo $\emptyset$. Proposition 4.3.35 implies that $(\exists^{\mathsf{sim}}(\mathcal{I}, \delta))^{\mathcal{T}}$ is more specific than $C^{\mathcal{T}}$ modulo $\emptyset$. Note that $C^{\mathcal{T}}$ is equivalent to $\exists^{\mathsf{sim}}(\mathcal{I}_{C,\mathcal{T}}, C)$ modulo $\emptyset$. It remains to show that $\exists^{\mathsf{sim}}(\mathcal{I}, \delta)$ and $(\exists^{\mathsf{sim}}(\mathcal{I}, \delta))^{\mathcal{T}}$ are equivalent as well, which then implies that there exists a simulation from $(\mathcal{I}_{C,\mathcal{T}}, C)$ to $(\mathcal{I}, \delta)$. The subsumption $(\exists^{\mathsf{sim}}(\mathcal{I}, \delta))^{\mathcal{T}} \sqsubseteq_{\emptyset} \exists^{\mathsf{sim}}(\mathcal{I}, \delta)$ is obvious. Since $\mathcal{I}$ is a model of $\mathcal{T}$, we infer by an application of Proposition 4.3.50 that $(\exists^{\mathsf{sim}}(\mathcal{I}, \delta))^{\mathcal{I}\mathcal{I}} \sqsubseteq_{\emptyset} (\exists^{\mathsf{sim}}(\mathcal{I}, \delta))^{\mathcal{T}}$. Furthermore, Lemmas 4.1.2 and 4.1.3 shows that $(\exists^{\mathsf{sim}}(\mathcal{I}, \delta))^{\mathcal{I}\mathcal{I}}$ and $\exists^{\mathsf{sim}}(\mathcal{I}, \delta)$ are equivalent, and we are done.

Statement 2 follows from Corollary 4.3.14, since $C \in D^{\mathcal{I}_{C,\mathcal{T}}}$ is equivalent to $\exists^{\mathsf{sim}}(\mathcal{I}_{C,\mathcal{T}}, C) \sqsubseteq_{\emptyset} D$ by Lemmas 4.1.2 and 4.1.3, and since further $(\mathcal{I}_{D,\mathcal{T}}, D) \precsim (\mathcal{I}_{C,\mathcal{T}}, C)$ is equivalent to $C^{\mathcal{T}} \sqsubseteq_{\emptyset} D^{\mathcal{T}}$ where the latter statement is equivalent to $C \sqsubseteq_{\mathcal{T}} D$ by Proposition 4.3.39. □

Rule-Based Approaches
for Constructing Canonical Models of $\mathcal{EL}_{\mathsf{si}}$ Concept Descriptions w.r.t. $\mathcal{EL}_{\mathsf{si}}$ TBoxes

We have shown in Proposition 4.3.12 that most specific consequences in the description logic $\mathcal{EL}_{\mathsf{si}}$ can be described by the canonical models from Page 101. However, a reasoner that decides concept subsumption w.r.t. a TBox is necessary for constructing such canonical models. For resolving this issue, we will now devise a rule-based procedure for constructing canonical models that only needs a reasoner for deciding concept subsumption without a TBox (i.e., only existence of a simulation needs to be checked).

Fix some $\mathcal{EL}_{\mathsf{si}}$ concept description C as well as an $\mathcal{EL}_{\mathsf{si}}$ TBox $\mathcal{T}$. Let $\mathcal{A}_{C,\mathcal{T}}$ be the ABox that is obtained from the initial ABox $\{C \sqsubseteq C\}$ by exhaustively applying the following rules.

$\sqcap$-*rule.* If $D \sqsubseteq E \in \mathcal{A}$ and $\{\, D \sqsubseteq F \mid F \in \mathsf{Conj}(E) \,\} \not\subseteq \mathcal{A}$,
then $\mathcal{A} \to \mathcal{A} \cup \{\, D \sqsubseteq F \mid F \in \mathsf{Conj}(E) \,\}$.

$\exists$-*rule.* If $D \sqsubseteq \exists\, r.\, E \in \mathcal{A}$, but $(D, E) \sqsubseteq r \notin \mathcal{D}$,
then $\mathcal{A} \to \mathcal{A} \cup \{(D, E) \sqsubseteq r,\ E \sqsubseteq E\}$.

$\sqsubseteq$-*rule.* If $D \in \mathsf{Ind}(\mathcal{A})$, $E \in \Sigma_{\mathsf{C}} \cup \exists\mathsf{Sub}(\mathcal{T})$, and $D \sqsubseteq_{\mathcal{T}} E$, but $D \sqsubseteq E \notin \mathcal{A}$,
then $\mathcal{A} \to \mathcal{A} \cup \{D \sqsubseteq E\}$.

The rule precedence is $\sqcap$-rule $>$ $\exists$-rule $>$ $\sqsubseteq$-rule. It is easy to see that the resulting ABox $\mathcal{A}_{C,\mathcal{T}}$ is unique. Furthermore, we shall denote by $\mathcal{J}_{C,\mathcal{T}}$ the interpretation that is induced by $\mathcal{A}_{C,\mathcal{T}}$ using the standard construction: $\Delta^{\mathcal{J}_{C,\mathcal{T}}} := \mathsf{Ind}(\mathcal{A}_{C,\mathcal{T}})$, and $A^{\mathcal{J}_{C,\mathcal{T}}} := \{\, D \mid D \sqsubseteq A \in \mathcal{A}_{C,\mathcal{T}} \,\}$ for each $A \in \Sigma_{\mathsf{C}}$, and $r^{\mathcal{J}_{C,\mathcal{T}}} := \{\, (D, E) \mid (D, E) \sqsubseteq r \in \mathcal{A}_{C,\mathcal{T}} \,\}$ for each $r \in \Sigma_{\mathsf{R}}$. Note that we only use the ABox notation for the intermediate construction, in the end we will consider just the interpretation $\mathcal{J}_{C,\mathcal{T}}$.

4.3.17 Lemma. *Let $C \sqsubseteq_{\mathcal{T}} \exists\, r.\, D$, then there exists a subconcept $\exists\, r.\, E \in \mathsf{Conj}(C) \cup \mathsf{Sub}(\mathcal{T})$ such that $C \sqsubseteq_{\mathcal{T}} \exists\, r.\, E$ and $E \sqsubseteq_{\mathcal{T}} D$.*

Proof. Corollary 4.3.14 shows that $C \sqsubseteq_{\mathcal{T}} \exists\, r.\, D$ implies $C \in (\exists\, r.\, D)^{\mathcal{I}_{C,\mathcal{T}}}$. We infer that there is some $E \in \Delta^{\mathcal{I}_{C,\mathcal{T}}}$ such that $(C, E) \in r^{\mathcal{I}_{C,\mathcal{T}}}$ and $E \in D^{\mathcal{I}_{C,\mathcal{T}}}$. The latter statement immediately yields that $E \sqsubseteq_{\mathcal{T}} D$ holds true, cf. Corollary 4.3.14.

Furthermore, $(C, E) \in r^{\mathcal{I}_{C,\mathcal{T}}}$ implies that $C \sqsubseteq_{\mathcal{T}} \exists\, r.\, E$ and $\exists\, r.\, E \in \mathsf{Sub}(\mathcal{T})$, or $\exists\, r.\, E \in \mathsf{Conj}(C)$. In both cases, we can be sure that $\exists\, r.\, E \in \mathsf{Conj}(C) \cup \mathsf{Sub}(\mathcal{T})$ and $C \sqsubseteq_{\mathcal{T}} \exists\, r.\, E$ are satisfied. □

4.3.18 Proposition. *The pointed interpretations $(\mathcal{I}_{C,\mathcal{T}}, C)$ and $(\mathcal{J}_{C,\mathcal{T}}, C)$ are equi-similar.*

Proof. We start with demonstrating that the following relation $\mathfrak{S}$ is a simulation from $(\mathcal{I}_{C,\mathcal{T}}, C)$ to $(\mathcal{J}_{C,\mathcal{T}}, C)$.

$$\mathfrak{S} := \{\, (D, E) \mid D \sqsupseteq_{\mathcal{T}} E \,\}$$

Of course, the pair (C, C) is contained in $\mathfrak{S}$. Now fix an arbitrary pair (D, E) in $\mathfrak{S}$.

- If $D \in A^{\mathcal{I}_{C,\mathcal{T}}}$, then $D \sqsubseteq_{\mathcal{T}} A$ holds true. It follows that $E \sqsubseteq_{\mathcal{T}} A$. Since the $\sqsubseteq$-rule has been applied exhaustively, we conclude that $E \in A^{\mathcal{J}_{C,\mathcal{T}}}$ must be true.
- If $(D, F) \in r^{\mathcal{I}_{C,\mathcal{T}}}$, then $\exists\, r.\, F \in \mathsf{Conj}(D) \cup \mathsf{Sub}(\mathcal{T})$ and $D \sqsubseteq_{\mathcal{T}} \exists\, r.\, F$. It follows that $E \sqsubseteq_{\mathcal{T}} \exists\, r.\, F$. If $\exists\, r.\, F \in \mathsf{Sub}(\mathcal{T})$, then the axiom $E \sqsubseteq \exists\, r.\, F$ must have been created by an application of the $\sqsubseteq$-rule. This shows that $(E, F) \in r^{\mathcal{J}_{C,\mathcal{T}}}$. It is further obvious that $(F, F) \in \mathfrak{S}$.

  In the remaining case where $\exists\, r.\, F$ is a conjunct in D, we only know that $E \sqsubseteq_{\mathcal{T}} \exists\, r.\, F$. Lemma 4.3.17 shows that there exists some subconcept $\exists\, r.\, G \in \mathsf{Conj}(E) \cup \mathsf{Sub}(\mathcal{T})$ such that $E \sqsubseteq_{\mathcal{T}} \exists\, r.\, G$ and $G \sqsubseteq_{\mathcal{T}} F$. Since both the $\sqcap$-rule and the $\sqsubseteq$-rule have been applied exhaustively, we conclude that $G \in \Delta^{\mathcal{J}_{C,\mathcal{T}}}$ and $(E, G) \in r^{\mathcal{J}_{C,\mathcal{T}}}$ are satisfied. Furthermore, we get $(F, G) \in \mathfrak{S}$ for free.

Furthermore, we shall show that the relation $\mathfrak{T}$ defined as follows is a simulation from $(\mathcal{J}_{C,\mathcal{T}}, C)$ to $(\mathcal{I}_{C,\mathcal{T}}, C)$. Beforehand note that $\Delta^{\mathcal{J}_{C,\mathcal{T}}} \subseteq \Delta^{\mathcal{I}_{C,\mathcal{T}}}$ must hold true.

$$\mathfrak{T} := \{\, (D, D) \mid D \in \Delta^{\mathcal{J}_{C,\mathcal{T}}} \,\}$$

It is trivial that (C, C) is in $\mathfrak{T}$. Consider an arbitrary pair $(D, D) \in \mathfrak{T}$.

- Assume $D \in A^{\mathcal{J}_{C,\mathcal{T}}}$. According to the rule-based construction of $\mathcal{A}_{C,\mathcal{T}}$ (and of $\mathcal{J}_{C,\mathcal{T}}$), this can only be the case if either A is a conjunct in D or A is a consequence of D w.r.t. $\mathcal{T}$. In either case $D \sqsubseteq_{\mathcal{T}} A$ must be satisfied, and it now immediately follows that $D \in A^{\mathcal{I}_{C,\mathcal{T}}}$.
- Fix $(D, E) \in r^{\mathcal{J}_{C,\mathcal{T}}}$. According to the construction of $\mathcal{J}_{C,\mathcal{T}}$, the axiom $(D, E) \sqsubseteq r$ needs to be in $\mathcal{A}_{C,\mathcal{T}}$ and this axiom can have only been created by the $\exists$-rule from the axiom $D \sqsubseteq \exists\, r.\, E$ in $\mathcal{A}_{C,\mathcal{T}}$. Furthermore, this last axiom $D \sqsubseteq \exists\, r.\, E$ must be created either by the $\sqcap$-rule or by the $\sqsubseteq$-rule, which means that either $\exists\, r.\, E \in \mathsf{Conj}(D)$ holds true or $\exists\, r.\, E \in \mathsf{Sub}(\mathcal{T})$ and $D \sqsubseteq_{\mathcal{T}} \exists\, r.\, E$ is satisfied. For both cases we infer that $(D, E) \in r^{\mathcal{I}_{C,\mathcal{T}}}$ holds true. Of course, we also have that $(E, E) \in \mathfrak{T}$, and we are done. □

4.3.19 Lemma. *Let $\exists^{\mathsf{sim}}(\mathcal{I}, \delta) \sqsubseteq \exists^{\mathsf{sim}}(\mathcal{J}, \epsilon)$ be a non-tautological $\mathcal{EL}_{\mathsf{si}}$ concept inclusion which is entailed by an $\mathcal{EL}_{\mathsf{si}}$ TBox $\mathcal{T}$. Then, there exists some concept inclusion $E \sqsubseteq F$ in $\mathcal{T}$ as well as some object $\zeta \in \Delta^{\mathcal{I}}$ such that $\exists^{\mathsf{sim}}(\mathcal{I}, \zeta) \sqsubseteq_{\emptyset} E$ and $\exists^{\mathsf{sim}}(\mathcal{I}, \zeta) \not\sqsubseteq_{\emptyset} F$.*

Proof. Since $\exists^{\mathsf{sim}}(\mathcal{I},\delta)$ is not more specific than $\exists^{\mathsf{sim}}(\mathcal{J},\epsilon)$ modulo $\emptyset$, there is no simulation from $(\mathcal{J},\epsilon)$ to $(\mathcal{I},\delta)$. It follows that $\mathcal{I}$ cannot be a model of $\exists^{\mathsf{sim}}(\mathcal{I},\delta) \sqsubseteq \exists^{\mathsf{sim}}(\mathcal{J},\epsilon)$, and thus $\mathcal{I} \not\models \mathcal{T}$. Consequently, there must exist some concept inclusion $E \sqsubseteq F \in \mathcal{T}$ that is not valid in $\mathcal{I}$, i.e., the domain of $\mathcal{I}$ must contain an object ζ satisfying $\zeta \in E^{\mathcal{I}} \setminus F^{\mathcal{I}}$. Utilizing the laws of Galois connections implies that $\exists^{\mathsf{sim}}(\mathcal{I},\zeta) \sqsubseteq_{\emptyset} E$ and $\exists^{\mathsf{sim}}(\mathcal{I},\zeta) \not\sqsubseteq_{\emptyset} F$ hold true. □

As usual, we denote by $\mathcal{I}_{\mathcal{A}}$ the interpretation that is induced by some ABox $\mathcal{A}$. For instance, $\mathcal{I}_{\mathcal{A}}$ has been defined in [LW10] as follows. The domain is $\Delta^{\mathcal{I}_{\mathcal{A}}} := \mathsf{Ind}(\mathcal{A})$, and $A^{\mathcal{I}_{\mathcal{A}}} := \{\, a \mid a \sqsubseteq A \in \mathcal{A} \,\}$ for each concept name $A \in \Sigma_{\mathsf{C}}$, and $r^{\mathcal{I}_{\mathcal{A}}} := \{\, (a,b) \mid (a,b) \sqsubseteq r \in \mathcal{A} \,\}$ for each role name $r \in \Sigma_{\mathsf{R}}$. If $\mathcal{A}$ is an $\mathcal{EL}$ ABox, then [LW10, Statement 2 of Lemma 27] claims that $\mathcal{I}_{\mathcal{A}}$ is a model of $\mathcal{A}$ and, furthermore, [LW10, Statement 3 of Lemma 27] claims that $a \sqsubseteq_{\mathcal{A}} C$ is equivalent to $a \in C^{\mathcal{I}_{\mathcal{A}}}$ for each $\mathcal{EL}$ concept description C. However, the last two statements only hold true for $\sqcap$-$\exists$-*complete* ABoxes $\mathcal{A}$, i.e., such ABoxes $\mathcal{A}$ satisfying the following two conditions.

1. For each assertion $a \sqsubseteq C$ in $\mathcal{A}$ and each concept name $A \in \mathsf{Conj}(C)$, it holds true that the assertion $a \sqsubseteq A$ is in $\mathcal{A}$ as well.
2. For each assertion $a \sqsubseteq C$ in $\mathcal{A}$ and each existential restriction $\exists\, r.\, D \in \mathsf{Conj}(C)$, it holds true that there is some individual name b such that the assertion $(a,b) \sqsubseteq r$ is in $\mathcal{A}$ and $b \sqsubseteq D$ is entailed by $\mathcal{A}$.

Obviously, any *simple* ABox not containing assertions $a \sqsubseteq C$ for a complex concept description C is $\sqcap$-$\exists$-complete.

The next lemma shows that Statements 2 and 3 of Lemma 27 in [LW10] can be adapted to the more general case of $\sqcap$-$\exists$-complete $\mathcal{EL}_{\mathsf{si}}$ ABoxes and $\mathcal{EL}_{\mathsf{si}}$ concept descriptions.

4.3.20 Lemma. (Generalization of [LW10, Statements 2 and 3 of Lemma 27]) *The following statements hold true for each $\sqcap$-$\exists$-complete ABox $\mathcal{A}$.*

1. *$\mathcal{I}_{\mathcal{A}}$ is a model of $\mathcal{A}$.*
2. *$a \sqsubseteq_{\mathcal{A}} C$ is equivalent to $a \in C^{\mathcal{I}_{\mathcal{A}}}$ for each individual name $a \in \mathsf{Ind}(\mathcal{A})$ and for each $\mathcal{EL}_{\mathsf{si}}$ concept description C.*

Proof. By construction, $\mathcal{I}_{\mathcal{A}}$ satisfies all role assertions as well as all concept assertions involving only a concept name. Now fix some concept assertion $a \sqsubseteq C$ in $\mathcal{A}$ where C is not a concept name, and assume that C has the form $\exists^{\mathsf{sim}}(\mathcal{J},\epsilon)$. We show that the relation

$$\mathfrak{S} := \{\, (\zeta,b) \mid b \sqsubseteq_{\mathcal{A}} \exists^{\mathsf{sim}}(\mathcal{J},\zeta) \,\}$$

is a simulation from $(\mathcal{J},\epsilon)$ to $(\mathcal{I}_{\mathcal{A}},a)$. The assumption that $\mathcal{A}$ contains $a \sqsubseteq \exists^{\mathsf{sim}}(\mathcal{J},\epsilon)$ immediately implies that $(\epsilon,a) \in \mathfrak{S}$. Let (ζ,b) be an arbitrary pair in $\mathfrak{S}$.

- If $\zeta \in A^{\mathcal{J}}$, then $\exists^{\mathsf{sim}}(\mathcal{J},\zeta) \sqsubseteq_{\emptyset} A$ follows, and we obtain that $\mathcal{A}$ entails $b \sqsubseteq A$. This can only be true if $b \sqsubseteq A$ is contained in $\mathcal{A}$, and so we conclude that $b \in A^{\mathcal{I}_{\mathcal{A}}}$ is satisfied.
- If $(\zeta,\eta) \in r^{\mathcal{J}}$, then $\exists^{\mathsf{sim}}(\mathcal{J},\zeta) \sqsubseteq_{\emptyset} \exists\, r.\, \exists^{\mathsf{sim}}(\mathcal{J},\eta)$ must hold true, which implies that $b \sqsubseteq_{\mathcal{A}} \exists\, r.\, \exists^{\mathsf{sim}}(\mathcal{J},\eta)$. We infer that either there is some individual name c such that $\mathcal{A}$ contains the role assertion $(b,c) \sqsubseteq r$ and entails the concept assertion $c \sqsubseteq \exists^{\mathsf{sim}}(\mathcal{J},\eta)$, or

$\mathcal{A}$ contains some concept assertion $b \mathrel{\varepsilon} \exists^{\mathsf{sim}}(\mathcal{K},\alpha)$ where $\exists^{\mathsf{sim}}(\mathcal{K},\alpha) \sqsubseteq_\emptyset \exists\, r.\, \exists^{\mathsf{sim}}(\mathcal{J},\eta)$. Due to the characterization of $\sqsubseteq_\emptyset$ by simulations we conclude for the second case that there must exist an object β where $(\alpha,\beta) \in r^{\mathcal{K}}$ and $\exists^{\mathsf{sim}}(\mathcal{K},\beta) \sqsubseteq_\emptyset \exists^{\mathsf{sim}}(\mathcal{J},\eta)$. Then, $\exists\, r.\, \exists^{\mathsf{sim}}(\mathcal{K},\beta)$ is a top-level conjunct of $\exists^{\mathsf{sim}}(\mathcal{K},\alpha)$, and $\sqcap$-$\exists$-completeness of $\mathcal{A}$ yields that there is some individual name c such that $(b,c) \mathrel{\varepsilon} r$ is in $\mathcal{A}$ and $c \mathrel{\varepsilon}_{\mathcal{A}} \exists^{\mathsf{sim}}(\mathcal{K},\beta)$, i.e., $c \mathrel{\varepsilon}_{\mathcal{A}} \exists^{\mathsf{sim}}(\mathcal{J},\eta)$. Summing up, it follows that $(b,c) \in r^{\mathcal{I}_{\mathcal{A}}}$ and $(\eta,c) \in \mathfrak{S}$ in both cases.

We conclude that $a \in C^{\mathcal{I}_{\mathcal{A}}}$ is satisfied. We have shown that $\mathcal{I}_{\mathcal{A}}$ models all assertions in $\mathcal{A}$, i.e., Statement 1 is true.

For the only if direction of Statement 2 we can use the same proof as above involving the construction of the simulation $\mathfrak{S}$. Vice versa, let $a \in C^{\mathcal{I}_{\mathcal{A}}}$. It follows that $\exists^{\mathsf{sim}}(\mathcal{I}_{\mathcal{A}},a)$ is more specific than C. We proceed with proving that $\mathcal{A}$ entails $a \mathrel{\varepsilon} \exists^{\mathsf{sim}}(\mathcal{I}_{\mathcal{A}},a)$. Let $\mathcal{J}$ be a model of $\mathcal{A}$. It is easy to verify that $\{\, (b,b^{\mathcal{J}}) \mid b \in \mathsf{Ind}(\mathcal{A}) \,\}$ is a simulation from $(\mathcal{I}_{\mathcal{A}},a)$ to $(\mathcal{J},a^{\mathcal{J}})$, which implies $a^{\mathcal{J}} \in (\exists^{\mathsf{sim}}(\mathcal{I}_{\mathcal{A}},a))^{\mathcal{J}}$. □

4.3.21 Lemma. *Fix some $\sqcap$-$\exists$-complete $\mathcal{EL}_{\mathsf{si}}$ ABox $\mathcal{A}$ as well as an $\mathcal{EL}_{\mathsf{si}}$ TBox $\mathcal{T}$, and let a be an individual name occurring in $\mathcal{A}$ and let C be some $\mathcal{EL}_{\mathsf{si}}$ concept description. If $a \not\mathrel{\varepsilon}_{\mathcal{A}} C$ but $a \mathrel{\varepsilon}_{\mathcal{A}\cup\mathcal{T}} C$, then there exists some individual name b occurring in $\mathcal{A}$ as well as a concept inclusion $D \sqsubseteq E$ in $\mathcal{T}$ such that $b \mathrel{\varepsilon}_{\mathcal{A}} D$ and $b \not\mathrel{\varepsilon}_{\mathcal{A}} E$.*

Proof. Since $a \not\mathrel{\varepsilon}_{\mathcal{A}} C$, it follows that $a \notin C^{\mathcal{I}_{\mathcal{A}}}$. Together with $a \mathrel{\varepsilon}_{\mathcal{A}\cup\mathcal{T}} C$ this implies that $\mathcal{I}_{\mathcal{A}}$ cannot be a model of $\mathcal{A} \cup \mathcal{T}$. As $\mathcal{I}_{\mathcal{A}} \models \mathcal{A}$, we conclude that $\mathcal{I}_{\mathcal{A}}$ is not a model of $\mathcal{T}$. Thus, there exists some individual name $b \in \mathsf{Ind}(\mathcal{A})$ and a concept inclusion $D \sqsubseteq E \in \mathcal{T}$ such that $b \in D^{\mathcal{I}_{\mathcal{A}}}$ and $b \notin E^{\mathcal{I}_{\mathcal{A}}}$. Of course, the last two statements are equivalent to $b \mathrel{\varepsilon}_{\mathcal{A}} D$ and $b \not\mathrel{\varepsilon}_{\mathcal{A}} E$, respectively. □

We are now going to consider a variation where we replace the $\sqsubseteq$-rule by the following rule and show that we can obtain equivalent results.

$\sqsubseteq'$-rule. If $D \in \mathsf{Ind}(\mathcal{A})$, $E \sqsubseteq F \in \mathcal{T}$, and $D \mathrel{\varepsilon}_{\mathcal{A}} E$, but $D \not\mathrel{\varepsilon}_{\mathcal{A}} F$,
then $\mathcal{A} \to \mathcal{A} \cup \{D \mathrel{\varepsilon} F\}$.

We shall denote by $\mathcal{A}'_{C,\mathcal{T}}$ the unique result of exhaustively applying the rules to the initial ABox $\{C \mathrel{\varepsilon} C\}$, and the induced interpretation is then symbolized by $\mathcal{J}'_{C,\mathcal{T}}$. Note that the resulting ABox $\mathcal{A}'_{C,\mathcal{T}}$ is always $\sqcap$-$\exists$-complete.

4.3.22 Lemma. *$D \mathrel{\varepsilon}_{\mathcal{A}'_{C,\mathcal{T}}\cup\mathcal{T}} E$ implies $D \mathrel{\varepsilon}_{\mathcal{A}'_{C,\mathcal{T}}} E$ for each individual name D occurring in $\mathcal{A}'_{C,\mathcal{T}}$ and for each $\mathcal{EL}_{\mathsf{si}}$ concept description E.*

Proof. Since the $\sqsubseteq'$-rule has been exhaustively applied to $\mathcal{A}'_{C,\mathcal{T}}$, we know that $D \mathrel{\varepsilon}_{\mathcal{A}'_{C,\mathcal{T}}} X$ implies $D \mathrel{\varepsilon}_{\mathcal{A}'_{C,\mathcal{T}}} Y$ for each individual name D occurring in $\mathcal{A}'_{C,\mathcal{T}}$ and for each concept inclusion $X \sqsubseteq Y$ in $\mathcal{T}$. Now assume that the claim does not hold true, i.e., there is some $D \in \mathsf{Ind}(\mathcal{A}'_{C,\mathcal{T}})$ and some $E \in \mathcal{EL}_{\mathsf{si}}(\Sigma)$ such that $D \mathrel{\varepsilon}_{\mathcal{A}'_{C,\mathcal{T}}\cup\mathcal{T}} E$ and $D \not\mathrel{\varepsilon}_{\mathcal{A}'_{C,\mathcal{T}}} E$. Then Lemma 4.3.21 would imply that the $\sqsubseteq'$-rule is still applicable to some individual name in $\mathcal{A}'_{C,\mathcal{T}}$, which is a contradiction. ↯ □

4.3.23 Lemma. *$D \sqsubseteq_{\mathcal{T}} E$ implies $D \mathrel{\varepsilon}_{\mathcal{A}'_{C,\mathcal{T}}} E$ for each individual name D occurring in $\mathcal{A}'_{C,\mathcal{T}}$ and for each $\mathcal{EL}_{\mathsf{si}}$ concept description E.*

Proof. Let $D \sqsubseteq_{\mathcal{T}} E$. As $D \in_{\mathcal{A}'_{C,\mathcal{T}}} D$ must obviously hold true, we infer that $D \in_{\mathcal{A}'_{C,\mathcal{T}} \cup \mathcal{T}} E$. Applying Lemma 4.3.22 yields the claim. □

4.3.24 Lemma. *Assume that $(\mathcal{J}'_{C,\mathcal{T}}, D) \precsim (\mathcal{I}_{C,\mathcal{T}}, D)$. Then $D \in_{\mathcal{A}'_{C,\mathcal{T}}} E$ implies $D \sqsubseteq_{\mathcal{T}} E$ for each individual name D occurring in $\mathcal{A}'_{C,\mathcal{T}}$ and for each $\mathcal{EL}_{\mathsf{si}}$ concept description E.*

Proof. Let $D \in_{\mathcal{A}'_{C,\mathcal{T}}} E$, i.e., $D \in E^{\mathcal{J}'_{C,\mathcal{T}}}$. Then $(\mathcal{J}'_{C,\mathcal{T}}, D) \precsim (\mathcal{I}_{C,\mathcal{T}}, D)$ implies $D \in E^{\mathcal{I}_{C,\mathcal{T}}}$, i.e., $\{D\}^{\mathcal{I}_{C,\mathcal{T}}} \sqsubseteq_{\emptyset} E$. Since $\{D\}^{\mathcal{I}_{C,\mathcal{T}}} \equiv_{\emptyset} D^{\mathcal{T}}$ is satisfied, we conclude that $D \sqsubseteq_{\mathcal{T}} E$. □

4.3.25 Proposition. *The pointed interpretations $(\mathcal{I}_{C,\mathcal{T}}, C)$ and $(\mathcal{J}'_{C,\mathcal{T}}, C)$ are equi-similar.*

Proof. We start with proving that the relation $\mathfrak{S} := \{\, (D,D) \mid D \in \mathsf{Ind}(\mathcal{A}'_{C,\mathcal{T}}) \,\}$ is a simulation from $\mathcal{J}'_{C,\mathcal{T}}$ to $\mathcal{I}_{C,\mathcal{T}}$. We do this by induction along the sequence of rule applications. More specifically, assume that

$$\{C \in C\} =: \mathcal{A}_0 \to \mathcal{A}_1 \to \mathcal{A}_2 \to \ldots \to \mathcal{A}_n := \mathcal{A}'_{C,\mathcal{T}}$$

is the sequence of intermediate ABoxes $\mathcal{A}_k$ obtained by applying the $\sqcap$-rule, the $\exists$-rule, and the $\sqsubseteq'$-rule exhaustively. Without loss of generality, assume that the $\sqsubseteq'$-rule is only applied if the other two rules are not applicable, i.e., the $\sqsubseteq'$-rule is only applied to $\sqcap$-$\exists$-complete ABoxes. Furthermore, let $\mathcal{J}_k$ be the induced interpretation of $\mathcal{A}_k$ for each suitable index k. We shall now show that, for each index k, the relation

$$\mathfrak{S}_k := \{\, (D,D) \mid D \in \mathsf{Ind}(\mathcal{A}_k) \,\}$$

is a simulation from $\mathcal{J}_k$ to $\mathcal{I}_{C,\mathcal{T}}$. Note that each $\mathfrak{S}_k$ is well-defined, since $\Delta^{\mathcal{J}_k}$ is a subset of $\Delta^{\mathcal{I}_{C,\mathcal{T}}}$.

The induction base for $k = 0$ is obvious. For the induction step we proceed with a case distinction on the rule that has been applied for the step $\mathcal{A}_k \to \mathcal{A}_{k+1}$.

$\sqcap$-rule. By assumption there is some axiom $D \in E$ in $\mathcal{A}_k$ such that

$$\mathcal{A}_{k+1} = \mathcal{A}_k \cup \{\, D \in F \mid F \in \mathsf{Conj}(E) \,\}.$$

The only differences in the induced interpretations $\mathcal{J}_k$ and $\mathcal{J}_{k+1}$ arise from the new assertions $D \in A$ for a concept name $A \in \mathsf{Conj}(E)$, since it holds true that $A^{\mathcal{J}_{k+1}} = A^{\mathcal{J}_k} \cup \{D\}$ for each such $A \in \mathsf{Conj}(E)$ and all other extensions of names do not differ between $\mathcal{J}_k$ and $\mathcal{J}_{k+1}$. The case $D = E$ is easy, as it then immediately follows that $D \sqsubseteq_{\emptyset} A$, which shows that $D \in A^{\mathcal{I}_{C,\mathcal{T}}}$ for each $A \in \mathsf{Conj}(E)$.

Otherwise, the axiom $D \in E$ must have been created by an application of the $\sqsubseteq'$-rule in a previous step, i.e., there is some index $\ell < k$ such that $\mathcal{A}_{\ell+1} = \mathcal{A}_\ell \cup \{D \in E\}$ and there is some F such that $F \sqsubseteq E$ is a concept inclusion in $\mathcal{T}$ where $D \in_{\mathcal{A}_\ell} F$.

Since $\mathcal{A}_\ell$ is $\sqcap$-$\exists$-complete, $D \in_{\mathcal{A}_\ell} F$ implies $D \in F^{\mathcal{J}_\ell}$. By induction hypothesis, $\mathfrak{S}_\ell$ is a simulation from $\mathcal{J}_\ell$ to $\mathcal{I}_{C,\mathcal{T}}$, which implies $D \in F^{\mathcal{I}_{C,\mathcal{T}}}$. Corollary 4.3.14 yields $D \sqsubseteq_{\mathcal{T}} F$. As $F \sqsubseteq E$ is in $\mathcal{T}$, it follows that $D \sqsubseteq_{\mathcal{T}} E$ and so we conclude that $D \sqsubseteq_{\mathcal{T}} A$ is satisfied for each concept name $A \in \mathsf{Conj}(E)$. Consequently, $D \in A^{\mathcal{I}_{C,\mathcal{T}}}$ for each $A \in \mathsf{Conj}(E)$.

∃-rule. Assume that there is some assertion $D \sqsubseteq \exists\, r.\, E$ in $\mathcal{A}_k$ such that

$$\mathcal{A}_{k+1} = \mathcal{A}_k \cup \{(D,E) \sqsubseteq r,\ E \sqsubseteq E\}.$$

If E is not already an individual name occurring in $\mathcal{A}_k$, then we have that

- $\Delta^{\mathcal{J}_{k+1}} = \Delta^{\mathcal{J}_k} \cup \{E\}$,
- $A^{\mathcal{J}_{k+1}} = A^{\mathcal{J}_k} \cup \{E\}$ if $E = A$,
- $A^{\mathcal{J}_{k+1}} = A^{\mathcal{J}_k}$ otherwise, and
- extensions of other concept names do not differ between $\mathcal{J}_{k+1}$ and $\mathcal{J}_k$;

otherwise $\Delta^{\mathcal{J}_{k+1}} = \Delta^{\mathcal{J}_k}$ holds true and the extensions of each concept name do not differ between $\mathcal{J}_{k+1}$ and $\mathcal{J}_k$. Furthermore, we have $r^{\mathcal{J}_{k+1}} = r^{\mathcal{J}_k} \cup \{(D,E)\}$, and extensions of other role names do not differ between $\mathcal{J}_{k+1}$ and $\mathcal{J}_k$. First note that E must be in the domain of $\mathcal{I}_{C,\mathcal{T}}$, since $\exists\, r.\, E$ is in $\mathsf{Sub}(C) \cup \mathsf{Sub}(\mathcal{T})$. Now if $E = A$, then $E \in A^{\mathcal{I}_{C,\mathcal{T}}}$ must surely hold true.

If $\exists\, r.\, E \in \mathsf{Conj}(D)$, then we can immediately conclude that $(D,E) \in r^{\mathcal{I}_{C,\mathcal{T}}}$. Otherwise, the assertion $D \sqsubseteq \exists\, r.\, E$ must have been created by an application of the $\sqsubseteq'$-rule (possibly followed by the $\sqcap$-rule) in some previous step, that is, we have some index $\ell < k$ and some concept inclusion $F \sqsubseteq G$ in $\mathcal{T}$ where $D \sqsubseteq_{\mathcal{A}_\ell} F$, and $\mathcal{A}_{\ell+1} = \mathcal{A}_\ell \cup \{D \sqsubseteq G\}$, and $\exists\, r.\, E \in \mathsf{Conj}(G)$.

Since $\mathcal{A}_\ell$ is $\sqcap$-∃-complete, $D \sqsubseteq_{\mathcal{A}_\ell} F$ implies $D \in F^{\mathcal{J}_\ell}$. According to the induction hypothesis the relation $\mathfrak{S}_\ell$ is a simulation from $\mathcal{J}_\ell$ to $\mathcal{I}_{C,\mathcal{T}}$, and we hence conclude that $D \in F^{\mathcal{I}_{C,\mathcal{T}}}$. An application of Corollary 4.3.14 yields $D \sqsubseteq_{\mathcal{T}} F$. We further infer that $D \sqsubseteq_{\mathcal{T}} \exists\, r.\, E$, and thus it follows that $(D,E) \in r^{\mathcal{I}_{C,\mathcal{T}}}$.

$\sqsubseteq'$-rule. Eventually, consider the case where there exists some individual name D occurring in $\mathcal{A}_k$ as well as a concept inclusion $E \sqsubseteq F$ in $\mathcal{T}$ such that $D \sqsubseteq_{\mathcal{A}_k} E$ and

$$\mathcal{A}_{k+1} = \mathcal{A}_k \cup \{D \sqsubseteq F\}.$$

If F is not a concept name, then the interpretations $\mathcal{J}_{k+1}$ and $\mathcal{J}_k$ are equal and we are done. Otherwise for $F = A$ the only difference between the interpretations $\mathcal{J}_{k+1}$ and $\mathcal{J}_k$ is possibly that $A^{\mathcal{J}_{k+1}} = A^{\mathcal{J}_k} \cup \{D\}$. As above, $D \sqsubseteq_{\mathcal{A}_k} E$ implies $D \in E^{\mathcal{J}_k}$. It follows that $D \in E^{\mathcal{I}_{C,\mathcal{T}}}$ and thus $D \sqsubseteq_{\mathcal{T}} E$. We further conclude that $D \sqsubseteq_{\mathcal{T}} A$, which shows that $D \in A^{\mathcal{I}_{C,\mathcal{T}}}$ is always satisfied.

In the sequel of this proof we show that $(\mathcal{I}_{C,\mathcal{T}}, C) \precsim (\mathcal{J}'_{C,\mathcal{T}}, C)$. For this purpose, define the following relation.

$$\mathfrak{T} := \{\, (D,E) \mid D \sqsupseteq_{\mathcal{T}} E \,\}$$

It is trivial that $(C,C) \in \mathfrak{T}$ holds true. Now fix some $(D,E) \in \mathfrak{T}$.

- Let $D \in A^{\mathcal{I}_{C,\mathcal{T}}}$, i.e., $D \sqsubseteq_{\mathcal{T}} A$ holds true. We infer that $E \sqsubseteq_{\mathcal{T}} A$, and now Lemma 4.3.23 implies that $E \sqsubseteq_{\mathcal{A}'_{C,\mathcal{T}}} A$ is satisfied, which can only be true if $E \in A^{\mathcal{J}'_{C,\mathcal{T}}}$.

- Assume that $(D, F) \in r^{\mathcal{I}_{C,\mathcal{T}}}$, i.e., $D \sqsubseteq_{\mathcal{T}} \exists r. F$ holds true. It follows that $E \sqsubseteq_{\mathcal{T}} \exists r. F$. Lemma 4.3.23 yields that $E \sqsubseteq_{\mathcal{A}'_{C,\mathcal{T}}} \exists r. F$, i.e., $E \in (\exists r. F)^{\mathcal{J}'_{C,\mathcal{T}}}$ is satisfied and so there is some G where $(E, G) \in r^{\mathcal{J}'_{C,\mathcal{T}}}$ and $G \in F^{\mathcal{J}'_{C,\mathcal{T}}}$, i.e., $G \sqsubseteq_{\mathcal{A}'_{C,\mathcal{T}}} F$. According to Lemma 4.3.24 the existence of the above simulation $\mathfrak{S}$ eventually implies $(F, G) \in \mathfrak{T}$, and we are done. □

Most Specific Consequences of $\mathcal{EL}_{\mathsf{si}}$ Concept Descriptions w.r.t. $\mathcal{EL}_{\mathsf{si}}$ TBoxes

We already know that most specific consequences always exist in the description logic $\mathcal{EL}_{\mathsf{si}}$. The following theorem now collects our previous results on three computation means for most specific consequences in $\mathcal{EL}_{\mathsf{si}}$. In particular, since the three pointed interpretations $(\mathcal{I}_{C,\mathcal{T}}, C)$, $(\mathcal{J}_{C,\mathcal{T}}, C)$, and $(\mathcal{J}'_{C,\mathcal{T}}, C)$ are equi-similar, and $\exists^{\mathsf{sim}}(\mathcal{I}_{C,\mathcal{T}}, C)$ is the most specific consequence of C w.r.t. $\mathcal{T}$, we get two further ways to compute most specific consequences.

4.3.26 Theorem. *Fix some $\mathcal{EL}_{\mathsf{si}}$ concept description C and an $\mathcal{EL}_{\mathsf{si}}$ TBox $\mathcal{T}$. Each of the concept descriptions $\exists^{\mathsf{sim}}(\mathcal{I}_{C,\mathcal{T}}, C)$, $\exists^{\mathsf{sim}}(\mathcal{J}_{C,\mathcal{T}}, C)$, and $\exists^{\mathsf{sim}}(\mathcal{J}'_{C,\mathcal{T}}, C)$ is the most specific consequence of C w.r.t. $\mathcal{T}$. In particular, the following equivalences hold true.*

$$C^{\mathcal{T}} \equiv_{\emptyset} \exists^{\mathsf{sim}}(\mathcal{I}_{C,\mathcal{T}}, C) \equiv_{\emptyset} \exists^{\mathsf{sim}}(\mathcal{J}_{C,\mathcal{T}}, C) \equiv_{\emptyset} \exists^{\mathsf{sim}}(\mathcal{J}'_{C,\mathcal{T}}, C)$$

Furthermore, each of the three representations can be computed in polynomial time w.r.t. $||C||$ and $||\mathcal{T}||$.

Proof. The first part of the claim follows from Propositions 4.3.12, 4.3.18, and 4.3.25. For the second part we shall show that each of the canonical models $\mathcal{I}_{C,\mathcal{T}}$, $\mathcal{J}_{C,\mathcal{T}}$, and $\mathcal{J}'_{C,\mathcal{T}}$ can be constructed in polynomial time.

We have already seen that $|\mathsf{Sub}(\exists^{\mathsf{sim}}(\mathcal{I}, \delta))| \leq ||\mathcal{I}||^2 \leq ||\exists^{\mathsf{sim}}(\mathcal{I}, \delta)||^2$ holds true for each $\mathcal{EL}_{\mathsf{si}}$ concept description $\exists^{\mathsf{sim}}(\mathcal{I}, \delta)$. It follows that $|\Delta^{\mathcal{I}_{C,\mathcal{T}}}|$ is polynomial in $||C||$ and $||\mathcal{T}||$. Since the subsumption relation $\sqsubseteq_{\mathcal{T}}$ can be decided in polynomial time, cf. [LPW10, Theorem 12], we conclude that the whole canonical model $\mathcal{I}_{C,\mathcal{T}}$ can be constructed in polynomial time w.r.t. $||C||$ and $||\mathcal{T}||$.

Furthermore, we have $\Delta^{\mathcal{J}_{C,\mathcal{T}}} \subseteq \Delta^{\mathcal{I}_{C,\mathcal{T}}}$ and there are only polynomially many rule applications each of which need polynomial time during the construction of $\mathcal{J}_{C,\mathcal{T}}$. Consequently, $\mathcal{J}_{C,\mathcal{T}}$ can be constructed in polynomial time as well. Similar arguments apply to the second rule-based variant $\mathcal{J}'_{C,\mathcal{T}}$. □

The rule-based canonical model $\mathcal{J}'_{C,\mathcal{T}}$ can be constructed without any calls to some reasoner. Henceforth, we can utilize it to constitute a new polynomial-time procedure for deciding the subsumption relation $\sqsubseteq_{\mathcal{T}}$ in the description logic $\mathcal{EL}_{\mathsf{si}}$. This is due to the fact that $C \sqsubseteq_{\mathcal{T}} D$ is equivalent to $\exists^{\mathsf{sim}}(\mathcal{J}'_{C,\mathcal{T}}, C) \sqsubseteq_{\emptyset} D$ and so we only need to normalize D to some $\mathcal{EL}_{\mathsf{si}}$ concept description of the form $\exists^{\mathsf{sim}}(\mathcal{I}, \delta)$ and afterwards check whether there is a simulation from $(\mathcal{I}, \delta)$ to $(\mathcal{J}'_{C,\mathcal{T}}, C)$. Since normalizing D, constructing $\mathcal{J}'_{C,\mathcal{T}}$, and checking existence of a simulation can all be done in polynomial time, we conclude that this yields a polynomial-time decision procedure.

The Bottom Concept Description

As next step, we investigate the problems of existence and computation of most specific consequences as well as their complexities when we further incorporate the bottom concept description $\bot$ in our considered description logics $\mathcal{EL}$ and $\mathcal{EL}_{\mathsf{si}}$. Since there has not been published any notion of canonical models for $\mathcal{EL}^{\bot}$ and $\mathcal{EL}^{\bot}_{\mathsf{si}}$, and extending the existing results from $\mathcal{EL}$ and $\mathcal{EL}_{\mathsf{si}}$, respectively, would take plenty of space herein, we are taking the lazy way and rather reduce the mentioned problems to the solved cases in Section 4.3.1.

For the upcoming complexity analysis, we need the following result.

4.3.27 Proposition. *The concept satisfiability problem for $\mathcal{EL}^{\bot}_{\mathsf{si}}$ is* **P***-complete.*

Proof. We first show that it is possible to decide in polynomial time whether an $\mathcal{EL}^{\bot}_{\mathsf{si}}$ concept description is satisfiable w.r.t. an $\mathcal{EL}^{\bot}_{\mathsf{si}}$ TBox. Thus, fix some $\mathcal{EL}^{\bot}_{\mathsf{si}}$ TBox $\mathcal{T}$ as well as an $\mathcal{EL}^{\bot}_{\mathsf{si}}$ concept description C. Obviously, C is not satisfiable w.r.t. $\mathcal{T}$ if $C = \bot$. For the remaining case let C be an $\mathcal{EL}_{\mathsf{si}}$ concept description and further let $\mathcal{U}$ be the sub-TBox of $\mathcal{T}$ not containing CIs of the form $E \sqsubseteq \bot$. Since CIs of the form $\bot \sqsubseteq F$ are tautologies, we can w.l.o.g. assume that such CIs do not occur in $\mathcal{T}$, that is, $\mathcal{U}$ is an $\mathcal{EL}_{\mathsf{si}}$ TBox. In the following, we prove that C is not satisfiable w.r.t. $\mathcal{T}$ if, and only if, there is an object $D \in \Delta^{\mathcal{I}_{C,\mathcal{U}}}$ as well as a concept inclusion $E \sqsubseteq \bot \in \mathcal{T} \setminus \mathcal{U}$ such that $D \in E^{\mathcal{I}_{C,\mathcal{U}}}$. Note that we can safely assume that the canonical model $\mathcal{I}_{C,\mathcal{U}}$ equals its connected component containing C, i.e., any object in the domain is reachable from C by some path of role names.

- Let C be not satisfiable w.r.t. $\mathcal{T}$, i.e., either $\mathcal{T}$ is inconsistent or $C^{\mathcal{J}} = \emptyset$ holds true for each model $\mathcal{J}$ of $\mathcal{T}$. Since $\mathcal{T} \supseteq \mathcal{U}$ and $C^{\mathcal{U}} \equiv_{\mathcal{U}} C$, it follows that $C^{\mathcal{U}} \equiv_{\mathcal{T}} C$. Consequently, $(C^{\mathcal{U}})^{\mathcal{J}} = \emptyset$ holds true for each model $\mathcal{J}$ of $\mathcal{T}$. We know that $C^{\mathcal{U}} \equiv_{\emptyset} \exists^{\mathsf{sim}}(\mathcal{I}_{C,\mathcal{U}}, C)$, and $\mathcal{I}_{C,\mathcal{U}} \models \mathcal{U}$, and $(C^{\mathcal{U}})^{\mathcal{I}_{C,\mathcal{U}}} \neq \emptyset$ (since the reflexive relation is a simulation from $(\mathcal{I}_{C,\mathcal{U}}, C)$ to itself). We conclude that $\mathcal{I}_{C,\mathcal{U}}$ cannot be a model of $\mathcal{T}$, and so there exists a CI $E \sqsubseteq \bot \in \mathcal{T} \setminus \mathcal{U}$ such that $E^{\mathcal{I}_{C,\mathcal{U}}} \neq \emptyset$. Thus, there is some object $D \in \Delta^{\mathcal{I}_{C,\mathcal{U}}}$ with $D \in E^{\mathcal{I}_{C,\mathcal{U}}}$.
- Assume that $D \in E^{\mathcal{I}_{C,\mathcal{U}}}$ is satisfied for some $D \in \Delta^{\mathcal{I}_{C,\mathcal{U}}}$ and some $E \sqsubseteq \bot \in \mathcal{T} \setminus \mathcal{U}$. It then immediately follows that $\exists^{\mathsf{sim}}(\mathcal{I}_{C,\mathcal{U}}, D) \sqsubseteq_{\emptyset} E \sqsubseteq_{\mathcal{T}} \bot$. Now let D be reachable from C via the path $w \in \Sigma_{\mathsf{R}}^{*}$. We then have $\exists^{\mathsf{sim}}(\mathcal{I}_{C,\mathcal{U}}, C) \sqsubseteq_{\emptyset} \exists\, w.\, \exists^{\mathsf{sim}}(\mathcal{I}_{C,\mathcal{U}}, D) \sqsubseteq_{\mathcal{T}} \exists\, w.\, \bot \equiv_{\emptyset} \bot$, i.e., $C^{\mathcal{U}} \equiv_{\mathcal{T}} \bot$. We have seen above that $C^{\mathcal{U}} \equiv_{\mathcal{T}} C$ is satisfied, and so we conclude that $C \equiv_{\mathcal{T}} \bot$, i.e., C is not satisfiable w.r.t. $\mathcal{T}$.

Eventually, we can decide satisfiability of C w.r.t. $\mathcal{T}$ in polynomial time, since the canonical model $\mathcal{I}_{C,\mathcal{U}}$ can be constructed in polynomial time.

Since the concept subsumption problem for $\mathcal{EL}^{\bot}$ is **P**-hard, and an $\mathcal{EL}^{\bot}$ concept description C is not satisfiable w.r.t. some $\mathcal{EL}^{\bot}$ TBox $\mathcal{T}$ if, and only if, $\mathcal{T}$ entails $C \sqsubseteq \bot$, the concept satisfiability problem is already **P**-hard for the sublogic $\mathcal{EL}^{\bot}$ of $\mathcal{EL}^{\bot}_{\mathsf{si}}$. □

We continue with showing an unsurprising result, namely that, for any $\mathcal{EL}^{\bot}_{\mathsf{si}}$ concept description C which is not satisfiable with respect to some $\mathcal{EL}^{\bot}_{\mathsf{si}}$ TBox $\mathcal{T}$, the most specific consequence $C^{\mathcal{T}}$ always exists in $\mathcal{EL}^{\bot}_{\mathsf{si}}$ and is (equivalent to) the bottom concept description $\bot$. For the remaining cases, we argue that it suffices to consider only the satisfiable part $\mathcal{T}_{\mathsf{sat}}$ of $\mathcal{T}$, i.e., the subset of $\mathcal{T}$ that contains only those concept inclusions the premises of which are satisfiable with respect to $\mathcal{T}$. More specifically, the most specific consequence $C^{\mathcal{T}}$ is then equivalent to $C^{\mathcal{T}_{\mathsf{sat}}}$ if C is satisfiable w.r.t. $\mathcal{T}$.

4.3.28 Lemma. *Fix some $\mathcal{EL}^{\bot}_{\mathsf{si}}$ TBox $\mathcal{T}$ and an $\mathcal{EL}^{\bot}_{\mathsf{si}}$ concept description C. Then, C is unsatisfiable with respect to $\mathcal{T}$ if, and only if, $\bot$ is the most specific consequence of C with respect to $\mathcal{T}$.*

Proof. If C is not $\mathcal{T}$-satisfiable, then $C \sqsubseteq_{\mathcal{T}} \bot$, i.e., $\bot$ is a consequence of C w.r.t. $\mathcal{T}$. Obviously, there does not exist any more specific consequence, and so $\bot$ is the most specific consequence. Vice versa, let $\bot$ be the most specific consequence of C. It then immediately follows that $C \sqsubseteq_{\mathcal{T}} \bot$, which is equivalent to $\mathcal{T}$-unsatisfiability of C. □

Fix some $\mathcal{EL}^{\bot}_{\mathsf{si}}$ TBox $\mathcal{T}$ and define the following TBox $\mathcal{T}_{\mathsf{sat}}$, which we call the *satisfiable part* of $\mathcal{T}$.

$$\mathcal{T}_{\mathsf{sat}} := \{\, C \sqsubseteq D \mid C \sqsubseteq D \in \mathcal{T} \text{ and } C \text{ is satisfiable w.r.t. } \mathcal{T} \,\}$$

It then follows that, for each concept inclusion $C \sqsubseteq D \in \mathcal{T}_{\mathsf{sat}}$, both concept descriptions C and D are satisfiable with respect to $\mathcal{T}$ and are, thus, also satisfiable w.r.t. $\emptyset$. In particular, we infer that $\mathcal{T}_{\mathsf{sat}}$ must be an $\mathcal{EL}_{\mathsf{si}}$ TBox. We continue with demonstrating that, for each $\mathcal{EL}^{\bot}_{\mathsf{si}}$ concept description which is satisfiable w.r.t. $\mathcal{T}$, its most specific consequences w.r.t. $\mathcal{T}$ and w.r.t. $\mathcal{T}_{\mathsf{sat}}$ are equivalent. That way, we infer that most specific consequences of $\mathcal{EL}^{\bot}_{\mathsf{si}}$ concept descriptions with respect to $\mathcal{EL}^{\bot}_{\mathsf{si}}$ TBoxes always exist in $\mathcal{EL}^{\bot}_{\mathsf{si}}$, and that these can be constructed from the canonical model $\mathcal{I}_{C,\mathcal{T}_{\mathsf{sat}}}$ if C is $\mathcal{T}$-satisfiable.

Piro [Pir12, Proposition 5.1.13] has shown that $\mathcal{EL}$ is *invariant under direct products*, that is, $C^{\mathcal{I}\times\mathcal{J}} = C^{\mathcal{I}} \times C^{\mathcal{J}}$ holds true for each $\mathcal{EL}$ concept description C. This result immediately extends to $\mathcal{EL}^{\bot}$, since $\bot^{\mathcal{I}\times\mathcal{J}} = \bot^{\mathcal{I}} \times \bot^{\mathcal{J}}$. Furthermore, since the product operation $\times$ is the infimum operation in the set of (equivalence classes of) pointed interpretations ordered by $\precsim$, we can immediately conclude that also $\mathcal{EL}^{\bot}_{\mathsf{si}}$ is invariant under products, that is,

$$(\exists^{\mathsf{sim}}(\mathcal{I},\delta))^{\mathcal{J}\times\mathcal{K}} = (\exists^{\mathsf{sim}}(\mathcal{I},\delta))^{\mathcal{J}} \times (\exists^{\mathsf{sim}}(\mathcal{I},\delta))^{\mathcal{K}}$$

holds true for all finite pointed interpretations $(\mathcal{I},\delta)$ and for all interpretations $\mathcal{J}$ and $\mathcal{K}$. Consequently, each $\mathcal{EL}^{\bot}_{\mathsf{si}}$ concept inclusion $C \sqsubseteq D$ that is valid in both $\mathcal{I}$ and $\mathcal{J}$ is also valid in the product $\mathcal{I} \times \mathcal{J}$.

4.3.29 Lemma. *Let $\mathcal{T}$ be an $\mathcal{EL}^{\bot}_{\mathsf{si}}$ TBox, and consider $\mathcal{EL}^{\bot}_{\mathsf{si}}$ concept descriptions C and D such that C is satisfiable with respect to $\mathcal{T}$. Then, the concept inclusion $C \sqsubseteq D$ is entailed by $\mathcal{T}$ if, and only if, it is entailed by $\mathcal{T}_{\mathsf{sat}}$.*

Proof. Since $\mathcal{T}_{\mathsf{sat}} \subseteq \mathcal{T}$, the *if* direction is trivial. We shall show the contraposition of the *only if* direction; consider a model $\mathcal{I}_{\mathsf{sat}}$ of $\mathcal{T}_{\mathsf{sat}}$ that contains a counterexample against $C \sqsubseteq D$, that is, $\mathcal{I}_{\mathsf{sat}}$ is such that $C^{\mathcal{I}_{\mathsf{sat}}} \setminus D^{\mathcal{I}_{\mathsf{sat}}} \neq \emptyset$. Since C is satisfiable with respect to $\mathcal{T}$, there exists some model $\mathcal{J}_C$ of $\mathcal{T}$ such that $C^{\mathcal{J}_C} \neq \emptyset$. By definition, each premise E of a concept inclusion $E \sqsubseteq F$ in $\mathcal{T} \setminus \mathcal{T}_{\mathsf{sat}}$ is not satisfiable with respect to $\mathcal{T}$ and, thus, we have that $E^{\mathcal{J}_C} = \emptyset$.

Of course, the direct product $\mathcal{I}_{\mathsf{sat}} \times \mathcal{J}_C$ is a model of $\mathcal{T}_{\mathsf{sat}}$. It also follows that $\mathcal{I}_{\mathsf{sat}} \times \mathcal{J}_C$ is a model of $\mathcal{T}$, since $E^{\mathcal{I}_{\mathsf{sat}} \times \mathcal{J}_C} = E^{\mathcal{I}_{\mathsf{sat}}} \times E^{\mathcal{J}_C} = \emptyset$ holds true for each premise E of a concept inclusion $E \sqsubseteq F \in \mathcal{T} \setminus \mathcal{T}_{\mathsf{sat}}$. Additionally, $C^{\mathcal{I}_{\mathsf{sat}}} \setminus D^{\mathcal{I}_{\mathsf{sat}}} \neq \emptyset$ in conjunction with $C^{\mathcal{J}_C} \neq \emptyset$ yields that

$$C^{\mathcal{I}_{\mathsf{sat}} \times \mathcal{J}_C} \setminus D^{\mathcal{I}_{\mathsf{sat}} \times \mathcal{J}_C} = (C^{\mathcal{I}_{\mathsf{sat}}} \times C^{\mathcal{J}_C}) \setminus (D^{\mathcal{I}_{\mathsf{sat}}} \times D^{\mathcal{J}_C}) \neq \emptyset,$$

that is, $\mathcal{I}_{\mathsf{sat}} \times \mathcal{J}_C$ contains a counterexample against $C \sqsubseteq D$ too. Eventually, we conclude that $C \not\sqsubseteq_{\mathcal{T}} D$. □

4.3.30 Proposition. *The concept subsumption problem for $\mathcal{EL}^{\bot}_{\mathsf{si}}$ is* **P***-complete.*

Proof. Fix some $\mathcal{EL}^{\bot}_{\mathsf{si}}$ TBox $\mathcal{T} \cup \{C \sqsubseteq D\}$. By means of Proposition 4.3.27, we can first check in polynomial time if C is satisfiable w.r.t. $\mathcal{T}$. If not, then it immediately holds true that $\mathcal{T}$ entails $C \sqsubseteq D$. Otherwise, we determine whether D is satisfiable w.r.t. $\mathcal{T}$. If not, then $\mathcal{T}$ cannot entail $C \sqsubseteq D$. For the remaining case where both C and D are satisfiable, we know that both are $\mathcal{EL}_{\mathsf{si}}$ concept descriptions. Lemma 4.3.29 shows that $\mathcal{T} \models C \sqsubseteq D$ is equivalent to $\mathcal{T}_{\mathsf{sat}} \models C \sqsubseteq D$. The satisfiable part $\mathcal{T}_{\mathsf{sat}}$ can be obtained in polynomial time, cf. Proposition 4.3.27, and is an $\mathcal{EL}_{\mathsf{si}}$ TBox. Since the concept subsumption problem for $\mathcal{EL}_{\mathsf{si}}$ is in **P**, we conclude that we can decide if $\mathcal{T}_{\mathsf{sat}} \models C \sqsubseteq D$ in polynomial time, and we are done. Eventually, concept subsumption is already **P**-hard for the sublogic $\mathcal{EL}^{\bot}$, and so the problem is **P**-complete for $\mathcal{EL}^{\bot}_{\mathsf{si}}$. □

Now we are ready to show that, for each concept description C that is satisfiable w.r.t. $\mathcal{T}$, its most specific consequence $C^{\mathcal{T}}$ exists and can furthermore be constructed from the satisfiable part $\mathcal{T}_{\mathsf{sat}}$, which is an $\mathcal{EL}_{\mathsf{si}}$ TBox. Thus, for the construction of $C^{\mathcal{T}}$ we can utilize our previous results on most specific consequences in $\mathcal{EL}_{\mathsf{si}}$ from Section 4.3.1. Beforehand, we need the following proposition.

4.3.31 Proposition. *If C is satisfiable with respect to $\mathcal{T}$, then the most specific consequence $C^{\mathcal{T}}$ exists in $\mathcal{EL}_{\mathsf{si}}$ and is equivalent to $C^{\mathcal{T}_{\mathsf{sat}}}$.*

Proof. Since C is satisfiable w.r.t. $\mathcal{T}$, it is satisfiable w.r.t. $\emptyset$, and it follows that C does not contain $\bot$ as a subconcept, that is, C is an $\mathcal{EL}_{\mathsf{si}}$ concept description. Furthermore, Lemma 4.3.28 yields that the most specific concept description $C^{\mathcal{T}}$—if it exists—is satisfiable w.r.t. $\emptyset$. In order to prove that $C^{\mathcal{T}_{\mathsf{sat}}}$ is the most specific consequence of C with respect to $\mathcal{T}$, we need to show the following two statements.

- $C \sqsubseteq_{\mathcal{T}} C^{\mathcal{T}_{\mathsf{sat}}}$
- $C \sqsubseteq_{\mathcal{T}} D$ implies $C^{\mathcal{T}_{\mathsf{sat}}} \sqsubseteq_{\emptyset} D$ or, equivalently, $C \sqsubseteq_{\mathcal{T}_{\mathsf{sat}}} D$ for any $\mathcal{EL}_{\mathsf{si}}$ concept description D.

As $\mathcal{T}_{\mathsf{sat}}$ is a subset of $\mathcal{T}$, the first statement is clearly true. The second statement has been proven in Lemma 4.3.29. □

The following statements are immediate consequences of combining the results from Section 4.3.1 with Lemma 4.3.28 and Proposition 4.3.31, and provide answers concerning the complexity of deciding existence of $C^{\mathcal{T}}$ in $\mathcal{EL}^{\bot}$ as well as of computing $C^{\mathcal{T}}$. Note that deciding subsumption w.r.t. a TBox in $\mathcal{EL}^{\bot}$ and $\mathcal{EL}^{\bot}_{\mathsf{si}}$ has polynomial time complexity, and so the satisfiable part $\mathcal{T}_{\mathsf{sat}}$ can be computed in polynomial time as well.

4.3.32 Corollary. *1. For each $\mathcal{EL}^{\bot}_{\mathsf{si}}$ TBox $\mathcal{T}$ and each $\mathcal{EL}^{\bot}_{\mathsf{si}}$ concept description C, the most specific consequence $C^{\mathcal{T}}$ exists in $\mathcal{EL}^{\bot}_{\mathsf{si}}$ and can be computed in polynomial time.*

2. *For each $\mathcal{EL}^{\bot}$ TBox $\mathcal{T}$ and each $\mathcal{EL}^{\bot}$ concept description C, the most specific consequence $C^{\mathcal{T}}$ exists in $\mathcal{EL}^{\bot}_{\mathsf{si}}$ and can be computed in polynomial time.*
3. *The problem whether all most specific consequences of $\mathcal{EL}^{\bot}$ concept descriptions with respect to some $\mathcal{EL}^{\bot}$ TBox $\mathcal{T}$ exist in $\mathcal{EL}^{\bot}$ can be decided in polynomial time.*
4. *The problem whether the most specific consequence $C^{\mathcal{T}}$ of an $\mathcal{EL}^{\bot}$ concept description C with respect to an $\mathcal{EL}^{\bot}$ TBox $\mathcal{T}$ exists in $\mathcal{EL}^{\bot}$ can be decided in polynomial time.* □

The Role-Depth-Bounded Case

We close this section with an investigation of the role-depth-bounded case, that is, for any role-depth bound $d \in \mathbb{N}$, we consider the problem whether most specific consequences of $\mathcal{EL}^{\bot}$ concept descriptions with respect to $\mathcal{EL}^{\bot}$ TBoxes exist in $\mathcal{EL}^{\bot}_{d}$ and, if so, how these can be computed. Obviously, existence is always guaranteed, simply because there are only finitely many appropriate candidates and this set of candidates is closed under conjunction.

To avoid confusion with the unrestricted case, we shall denote by $C^{\mathcal{T}_d}$ the most specific consequence of C w.r.t. $\mathcal{T}$ in $\mathcal{EL}^{\bot}_{d}$.

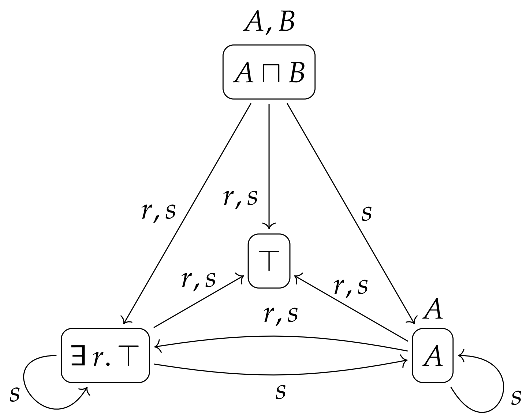
4.3.33 Proposition. *Fix some $\mathcal{EL}^{\bot}$ TBox $\mathcal{T}$, an $\mathcal{EL}^{\bot}$ concept description C, and a role depth bound $d \in \mathbb{N}$. Then, the most specific consequence of C w.r.t. $\mathcal{T}$ exists in $\mathcal{EL}^{\bot}_{d}$. More specifically, if C is not satisfiable w.r.t. $\mathcal{T}$, then $C^{\mathcal{T}_d} \equiv_{\emptyset} \bot$ holds true, and otherwise the following equivalences are satisfied.*

$$C^{\mathcal{T}_d} \equiv_{\emptyset} C^{\mathcal{T}}\restriction_d \equiv_{\emptyset} \mathsf{X}^d(\mathcal{I}_{C,\mathcal{T}_{\mathsf{sat}}}, C)$$

Proof. The proof for the case where C is not satisfiable w.r.t. $\mathcal{T}$ is analogous to the proof of Lemma 4.3.28. Otherwise, let C be satisfiable w.r.t. $\mathcal{T}$. Proposition 4.3.31 shows that then $C^{\mathcal{T}}$ is equivalent to $C^{\mathcal{T}_{\mathsf{sat}}}$. Furthermore, Proposition 4.3.12 states that $C^{\mathcal{T}_{\mathsf{sat}}}$ is equivalent to $\exists^{\mathsf{sim}}(\mathcal{I}_{C,\mathcal{T}_{\mathsf{sat}}}, C)$. We conclude that $C^{\mathcal{T}}\restriction_d \equiv_{\emptyset} \mathsf{X}^d(\mathcal{I}_{C,\mathcal{T}_{\mathsf{sat}}}, C)$. It remains to show that $C^{\mathcal{T}_d}$ and $C^{\mathcal{T}}\restriction_d$ are equivalent. Since $C \sqsubseteq_{\mathcal{T}} C^{\mathcal{T}}$ as well as $C^{\mathcal{T}} \sqsubseteq_{\emptyset} C^{\mathcal{T}}\restriction_d$ is satisfied, we infer that $C^{\mathcal{T}_d} \sqsubseteq_{\emptyset} C^{\mathcal{T}}\restriction_d$. Regarding the converse subsumption: we have $C \sqsubseteq_{\mathcal{T}} C^{\mathcal{T}_d}$, which implies $C^{\mathcal{T}} \sqsubseteq_{\emptyset} C^{\mathcal{T}_d}$. Then, Lemma 3.4.7 yields $C^{\mathcal{T}}\restriction_d \sqsubseteq_{\emptyset} C^{\mathcal{T}_d}$. □

Example. For illustrating the computation of most specific consequences, we consider the exemplary TBox $\mathcal{T} := \{A \sqsubseteq \exists\, rr.\top,\ \exists\, r.\top \sqsubseteq \exists\, s.A\}$ and the concept description $C := A \sqcap B$. The canonical model $\mathcal{I}_{C,\mathcal{T}}$ is shown in Figure 4.3.34.

Now the most specific consequence $C^{\mathcal{T}}$ can be read off from the canonical model $\mathcal{I}_{C,\mathcal{T}}$ as the model-based most specific concept description of $\{C\}$. As there is a cycle reachable from C, the most specific consequence does not exist in $\mathcal{EL}$, but only in $\mathcal{EL}_{\mathsf{si}}$ or in $\mathcal{EL}_{d}$ for each role depth bound $d \in \mathbb{N}$. Of course, $C^{\mathcal{T}}$ is equivalent to the $\mathcal{EL}_{\mathsf{si}}$ concept description $\exists^{\mathsf{sim}}(\mathcal{I}_{C,\mathcal{T}}, C)$, and we further list the first three approximations in the following, which are the most specific



4.3.34 Figure. A canonical model

consequences in $\mathcal{EL}_0$, in $\mathcal{EL}_1$, and in $\mathcal{EL}_2$, respectively.

$$C^{\mathcal{T}_0} \equiv_\emptyset A \sqcap B$$
$$C^{\mathcal{T}_1} \equiv_\emptyset A \sqcap B \sqcap \exists r. \top \sqcap \exists s. A$$
$$C^{\mathcal{T}_2} \equiv_\emptyset A \sqcap B \sqcap \exists r. (\exists r. \top \sqcap \exists s. A) \sqcap \exists s. (A \sqcap \exists r. \top \sqcap \exists s. A) \qquad \triangle$$

4.3.2 Algebraic Properties of Most Specific Consequences

This section's aim is to explore algebraic properties of most specific consequences. In particular, we shall connect Sections 1.5 and 4.3.1. For instance, the mappings $C \mapsto C^{\mathcal{T}}$ and $C \mapsto C^{\mathcal{T}_d}$ for each $d \in \mathbb{N}$ constitute closure operators in the respective lattices of concept descriptions, which immediately implies a series of mathematical laws and properties. Recursion formulas that are satisfied by most specific consequences are also provided within this section. We split our exploration in two cases: firstly, we consider the unrestricted case, and secondly, we investigate the role-depth-bounded case.

The Unrestricted Case

As announced, we shall start with the unrestricted case. The next proposition formulates that, for any TBox $\mathcal{T}$, the function which maps concept descriptions to their most specific consequence with respect to $\mathcal{T}$ constitutes a closure operator. Then, the following corollary shows some statements that immediately follow from the fact that the most specific consequence mapping is a closure operator.

4.3.35 Proposition. *For any $\mathcal{EL}^\bot_{\mathsf{si}}$ TBox $\mathcal{T}$, the mapping $\phi_{\mathcal{T}} \colon C \mapsto C^{\mathcal{T}}$ is a closure operator in the dual of $\boldsymbol{\mathcal{EL}}^\bot_{\mathsf{si}}(\Sigma)$, i.e., for all $\mathcal{EL}^\bot_{\mathsf{si}}$ concept descriptions C and D, the following conditions hold true.*

1. $C^{\mathcal{T}} \sqsubseteq_\emptyset C$ (extensive)
2. $C \sqsubseteq_\emptyset D$ *implies* $C^{\mathcal{T}} \sqsubseteq_\emptyset D^{\mathcal{T}}$ (monotonic)
3. $C^{\mathcal{T}} \equiv_\emptyset C^{\mathcal{T}\mathcal{T}}$ (idempotent)

Proof. Since C is a consequence of itself with respect to $\mathcal{T}$, it follows by Definition 4.3.1 that $C^{\mathcal{T}} \sqsubseteq_{\emptyset} C$.

Of course, $C^{\mathcal{T}} \sqsubseteq_{\mathcal{T}} C^{\mathcal{T}}$ is trivially valid, and so it follows that $C^{\mathcal{T}\mathcal{T}} \sqsubseteq_{\emptyset} C^{\mathcal{T}}$. Furthermore, it holds true that $C \sqsubseteq_{\mathcal{T}} C^{\mathcal{T}} \sqsubseteq_{\mathcal{T}} C^{\mathcal{T}\mathcal{T}}$, that is, $C^{\mathcal{T}\mathcal{T}}$ is a consequence of C with respect to $\mathcal{T}$. Since $C^{\mathcal{T}}$ is most specific, we conclude that $C^{\mathcal{T}} \sqsubseteq_{\emptyset} C^{\mathcal{T}\mathcal{T}}$.

Eventually, assume that $C \sqsubseteq_{\emptyset} D$. Since $D \sqsubseteq_{\mathcal{T}} D^{\mathcal{T}}$, it follows that $C \sqsubseteq_{\mathcal{T}} D^{\mathcal{T}}$, i.e., $D^{\mathcal{T}}$ is a consequence of C w.r.t. $\mathcal{T}$. Since $C^{\mathcal{T}}$ is most specific, we infer that $C^{\mathcal{T}} \sqsubseteq_{\emptyset} D^{\mathcal{T}}$. □

4.3.36 Corollary. *Let $\mathcal{T}$ be an $\mathcal{EL}^{\bot}_{\mathsf{si}}$ TBox, and assume that C as well as D are $\mathcal{EL}^{\bot}_{\mathsf{si}}$ concept descriptions. Then, the following statements hold true.*

1. $(C \sqcap D)^{\mathcal{T}} \sqsubseteq_{\emptyset} C^{\mathcal{T}} \sqcap D^{\mathcal{T}}$
2. $(C \sqcap D)^{\mathcal{T}} \equiv_{\emptyset} (C^{\mathcal{T}} \sqcap D^{\mathcal{T}})^{\mathcal{T}}$
3. $C^{\mathcal{T}} \vee D^{\mathcal{T}} \sqsubseteq_{\emptyset} (C \vee D)^{\mathcal{T}}$
4. $C^{\mathcal{T}} \vee D^{\mathcal{T}} \equiv_{\emptyset} (C^{\mathcal{T}} \vee D^{\mathcal{T}})^{\mathcal{T}}$

Proof. The statements are obtained as corollaries of Proposition 4.3.35 and Section 1.5. □

The notion of a least common subsumer can also be relativized with respect to a TBox. More specifically, we say that some concept description E is the *least common subsumer* of C and D w.r.t. $\mathcal{T}$ if it satisfies the following two conditions.

1. $C \sqsubseteq_{\mathcal{T}} E$ and $D \sqsubseteq_{\mathcal{T}} E$
2. $C \sqsubseteq_{\mathcal{T}} F$ together with $C \sqsubseteq_{\mathcal{T}} F$ implies $E \sqsubseteq_{\mathcal{T}} F$ for each concept description F.

Of course, all least common subsumers are equivalent modulo $\mathcal{T}$, and so we shall denote *the* least common subsumer by $C \vee_{\mathcal{T}} D$. According to [ZT13a, Lemma 12], we further have that $C \vee_{\mathcal{T}} D \equiv_{\mathcal{T}} \exists^{\mathsf{sim}}((\mathcal{I}_{C,\mathcal{T}}, C) \times (\mathcal{I}_{D,\mathcal{T}}, D))$ if C, D, and $\mathcal{T}$ are formulated in $\mathcal{EL}$. It is easy to verify that the same equivalence is also valid in $\mathcal{EL}_{\mathsf{si}}$.

4.3.37 Lemma. *It holds true that $C^{\mathcal{T}} \vee D^{\mathcal{T}} \equiv_{\emptyset} (C \vee_{\mathcal{T}} D)^{\mathcal{T}}$.*

Proof. It holds true that $C^{\mathcal{T}} \equiv_{\emptyset} \exists^{\mathsf{sim}}(\mathcal{I}_{C,\mathcal{T}}, C)$ and $D^{\mathcal{T}} \equiv_{\emptyset} \exists^{\mathsf{sim}}(\mathcal{I}_{D,\mathcal{T}}, D)$; it follows that $C^{\mathcal{T}} \vee D^{\mathcal{T}} \equiv_{\emptyset} \exists^{\mathsf{sim}}((\mathcal{I}_{C,\mathcal{T}}, C) \times (\mathcal{I}_{D,\mathcal{T}}, D))$. By means of the generalization of [ZT13a, Lemma 12] to $\mathcal{EL}_{\mathsf{si}}$, we conclude that $C^{\mathcal{T}} \vee D^{\mathcal{T}} \equiv_{\mathcal{T}} C \vee_{\mathcal{T}} D$, which implies that $(C^{\mathcal{T}} \vee D^{\mathcal{T}})^{\mathcal{T}} \equiv_{\emptyset} (C \vee_{\mathcal{T}} D)^{\mathcal{T}}$. Since $(C^{\mathcal{T}} \vee D^{\mathcal{T}})^{\mathcal{T}}$ is equivalent to $C^{\mathcal{T}} \vee D^{\mathcal{T}}$ with respect to $\emptyset$, we eventually infer that $C^{\mathcal{T}} \vee D^{\mathcal{T}} \equiv_{\emptyset} (C \vee_{\mathcal{T}} D)^{\mathcal{T}}$ holds true as claimed. □

It is straightforward to generalize the previous result to an arbitrary number of concept descriptions as follows. For each set $\mathbf{C}$ of $\mathcal{EL}^{\bot}$ concept descriptions, it holds true that $(\bigvee_{\mathcal{T}} \mathbf{C})^{\mathcal{T}} \equiv_{\emptyset} \bigvee\{ C^{\mathcal{T}} \mid C \in \mathbf{C} \}$.

Each TBox $\mathcal{T}$ can be normalized by means of the closure operator $\phi_{\mathcal{T}}$ in the sense that there is a TBox which is equivalent to $\mathcal{T}$ and only contains concept inclusions of the form $C \sqsubseteq C^{\mathcal{T}}$. On the one hand, this holds true for the infinite TBox that contains all these concept inclusions for all concept descriptions C and, on the other hand, it suffices to only take those concept descriptions C that occur as a premise in $\mathcal{T}$. A more sophisticated characterization is provided in the following lemma.

4.3.38 Lemma. *Let $\mathcal{T}$ be an $\mathcal{EL}^{\bot}_{\mathsf{si}}$ TBox. We define $\mathsf{Prem}(\mathcal{T})$ as the set of all premises of concept inclusions in $\mathcal{T}$, i.e., we set $\mathsf{Prem}(\mathcal{T}) := \{\, C \mid C \sqsubseteq D \in \mathcal{T} \,\}$. Then, the following sets of concept inclusions are both equivalent to $\mathcal{T}$.*[5]

$$\mathcal{T}^{\circ} := \{\, C \sqsubseteq C^{\mathcal{T}} \mid C \in \mathcal{EL}^{\bot}_{\mathsf{si}}(\Sigma) \,\}$$
$$\mathcal{T}^{*} := \{\, C \sqsubseteq C^{\mathcal{T}} \mid C \in \mathsf{Prem}(\mathcal{T}) \,\}$$

Proof. Since $C \sqsubseteq_{\mathcal{T}} C^{\mathcal{T}}$ for all $\mathcal{EL}^{\bot}_{\mathsf{si}}$ concept descriptions C, it immediately follows that $\mathcal{T}$ entails both $\mathcal{T}^{\circ}$ and $\mathcal{T}^{*}$. Furthermore, since $\mathcal{T}^{\circ} \supseteq \mathcal{T}^{*}$ and hence $\mathcal{T}^{\circ} \models \mathcal{T}^{*}$, it suffices to show that $\mathcal{T}^{*} \models \mathcal{T}$. Consider a concept inclusion $C \sqsubseteq D \in \mathcal{T}$, then $C \sqsubseteq C^{\mathcal{T}} \in \mathcal{T}^{*}$. Since D is a consequence of C with respect to $\mathcal{T}$, we infer that $C^{\mathcal{T}} \sqsubseteq_{\emptyset} D$, and as a consequence it then follows that $C \sqsubseteq_{\mathcal{T}^{*}} D$. Since $C \sqsubseteq D$ is an arbitrary concept inclusion from $\mathcal{T}$, we have just proven that $\mathcal{T}^{*} \models \mathcal{T}$. □

As a further important result, we shall show that entailment with respect to some TBox $\mathcal{T}$ and validity for the associated closure operator $\phi_{\mathcal{T}}$ are equivalent for any concept inclusion. It also holds true that subsumption w.r.t. a TBox is equivalent to subsumption of the corresponding most specific consequences w.r.t. $\emptyset$. In particular, subsumption reasoning in $\mathcal{EL}^{\bot}$ with respect to cycle-restricted TBoxes can, thus, be reduced to the simpler task of subsumption reasoning in $\mathcal{EL}^{\bot}$ with respect to the empty TBox where in the reduction the most specific consequence of the premise needs to be computed.

4.3.39 Proposition. *For each $\mathcal{EL}^{\bot}_{\mathsf{si}}$ TBox $\mathcal{T} \cup \{C \sqsubseteq D\}$, the following statements are equivalent.*

1. $C \sqsubseteq_{\mathcal{T}} D$
2. $C \sqsubseteq_{\mathcal{T}^{\circ}} D$
3. $C \sqsubseteq_{\mathcal{T}^{*}} D$
4. $C^{\mathcal{T}} \sqsubseteq_{\emptyset} D$
5. $C^{\mathcal{T}} \sqsubseteq_{\emptyset} D^{\mathcal{T}}$
6. $E^{\mathcal{T}} \sqsubseteq_{\emptyset} C$ *implies* $E^{\mathcal{T}} \sqsubseteq_{\emptyset} D$ *for each* $\mathcal{EL}^{\bot}_{\mathsf{si}}$ *concept description* E.

Proof. The equivalence of Statements 1 to 3 follows from Lemma 4.3.38.

Statements 1 and 4 are equivalent by the following observations. If $C \sqsubseteq_{\mathcal{T}} D$, then D is a consequence of C with respect to $\mathcal{T}$, and consequently $C^{\mathcal{T}} \sqsubseteq_{\emptyset} D$ by Definition 4.3.1. Vice versa, if $C^{\mathcal{T}} \sqsubseteq_{\emptyset} D$, then since $C \sqsubseteq_{\mathcal{T}} C^{\mathcal{T}}$, we infer that $C \sqsubseteq_{\mathcal{T}} D$.

Proposition 4.3.35 and Section 1.5 yield the equivalence of Statements 4 and 5.

Eventually, we demonstrate that Statement 6 is equivalent to the other statements. If $C \sqsubseteq_{\mathcal{T}} D$ and the empty TBox $\emptyset$ entails $E^{\mathcal{T}} \sqsubseteq C$, then it follows that $\mathcal{T}$ entails $E \sqsubseteq C$, and hence $E \sqsubseteq_{\mathcal{T}} D$. Consequently, $E^{\mathcal{T}} \sqsubseteq_{\emptyset} D$ as claimed.

Vice versa, assume that $E^{\mathcal{T}} \sqsubseteq_{\emptyset} C$ implies $E^{\mathcal{T}} \sqsubseteq_{\emptyset} D$ for all $\mathcal{EL}^{\bot}_{\mathsf{si}}$ concept descriptions E. Of course, then $E \sqsubseteq_{\mathcal{T}} C$ only if $E \sqsubseteq_{\mathcal{T}} D$. Since it trivially holds true that $C \sqsubseteq_{\mathcal{T}} C$, we immediately conclude that $C \sqsubseteq_{\mathcal{T}} D$. □

[5] Note that since $\mathcal{T}$ is a TBox and hence finite, also $\mathcal{T}^{*}$ is a finite set of concept inclusions, i.e., a TBox.

The following two corollaries collect previous results and further connect these to notions from the theory of closure operators.

4.3.40 Corollary. *Let $\mathcal{T}$ be an $\mathcal{EL}^{\bot}_{\mathsf{si}}$ TBox and assume that $C \sqsubseteq D$ is an $\mathcal{EL}^{\bot}_{\mathsf{si}}$ concept inclusion. Then, the following statements are equivalent.*

1. $\mathcal{T} \models C \sqsubseteq D$
2. $\phi_{\mathcal{T}} \models C \sqsubseteq D$
3. $\mathcal{I} \models \mathcal{T}$ *implies* $\mathcal{I} \models C \sqsubseteq D$ *for any interpretation* $\mathcal{I}$.
4. $\mathcal{I} \models \mathcal{T}$ *implies* $\phi_{\mathcal{I}} \models C \sqsubseteq D$ *for each interpretation* $\mathcal{I}$.
5. $\Delta\{\, \phi_{\mathcal{I}} \mid \mathcal{I} \models \mathcal{T} \,\} \models C \sqsubseteq D$
6. $\emptyset \models \bigvee\{\, C^{\mathcal{I}\mathcal{I}} \mid \mathcal{I} \models \mathcal{T} \,\} \sqsubseteq D$
7. $\emptyset \models C^{\mathcal{T}} \sqsubseteq D$

Proof. The equivalence of Statements 1, 2, and 7 has just been shown in Proposition 4.3.39. By the very definition of the semantics, also Statements 1 and 3 are equivalent. Since according to Lemma 4.1.2 $X \subseteq C^{\mathcal{I}}$ is equivalent to $X^{\mathcal{I}} \sqsubseteq_{\emptyset} C$, we conclude that $C \sqsubseteq_{\mathcal{I}} D$ is equivalent to $C^{\mathcal{I}\mathcal{I}} \sqsubseteq_{\emptyset} D$, which is equivalent to the validity of $C \sqsubseteq D$ for the closure operator $\phi_{\mathcal{I}}$. Consequently, Statements 3 and 4 are equivalent too. Eventually, Section 1.5 provides the equivalence of Statements 4 to 6. □

4.3.41 Corollary. *Consider an $\mathcal{EL}^{\bot}_{\mathsf{si}}$ TBox $\mathcal{T}$ as well as an $\mathcal{EL}^{\bot}_{\mathsf{si}}$ concept description C. If C is satisfiable w.r.t. $\mathcal{T}$, then the following equivalences hold true.*

$$C^{\mathcal{T}} \equiv_{\emptyset} C^{\mathcal{T}_{\mathsf{sat}}} \equiv_{\emptyset} \{C\}^{\mathcal{I}_{C,\mathcal{T}_{\mathsf{sat}}}} \equiv_{\emptyset} C^{\mathcal{I}_{C,\mathcal{T}_{\mathsf{sat}}}\mathcal{I}_{C,\mathcal{T}_{\mathsf{sat}}}} \equiv_{\emptyset} \bigvee\{\, C^{\mathcal{I}\mathcal{I}} \mid \mathcal{I} \models \mathcal{T}_{\mathsf{sat}} \,\} \equiv_{\emptyset} \bigvee\{\, C^{\mathcal{I}\mathcal{I}} \mid \mathcal{I} \models \mathcal{T} \,\}$$

Otherwise, if C is not satisfiable w.r.t. $\mathcal{T}$, then $C^{\mathcal{T}} \equiv_{\emptyset} \bot \equiv_{\emptyset} \bigvee\{\, C^{\mathcal{I}\mathcal{I}} \mid \mathcal{I} \models \mathcal{T} \,\}$.

Proof. Let C be $\mathcal{T}$-satisfiable. The first equivalence is proven in Proposition 4.3.31 and the second equivalence has been shown in Proposition 4.3.5. The equivalence of $C^{\mathcal{T}_{\mathsf{sat}}}$ and $\bigvee\{\, C^{\mathcal{I}\mathcal{I}} \mid \mathcal{I} \models \mathcal{T}_{\mathsf{sat}} \,\}$ as well as of $C^{\mathcal{T}}$ and $\bigvee\{\, C^{\mathcal{I}\mathcal{I}} \mid \mathcal{I} \models \mathcal{T} \,\}$ with respect to the empty TBox follows from the equivalence of Statements 6 and 7 in Corollary 4.3.40. Since the canonical model $\mathcal{I}_{C,\mathcal{T}_{\mathsf{sat}}}$ is a model of $\mathcal{T}_{\mathsf{sat}}$, we can infer that $\bigvee\{\, C^{\mathcal{I}\mathcal{I}} \mid \mathcal{I} \models \mathcal{T}_{\mathsf{sat}} \,\} \sqsupseteq_{\emptyset} C^{\mathcal{I}_{C,\mathcal{T}_{\mathsf{sat}}}\mathcal{I}_{C,\mathcal{T}_{\mathsf{sat}}}}$. Furthermore, $C \in C^{\mathcal{I}_{C,\mathcal{T}_{\mathsf{sat}}}}$ implies that $\{C\}^{\mathcal{I}_{C,\mathcal{T}_{\mathsf{sat}}}} \sqsubseteq_{\emptyset} C^{\mathcal{I}_{C,\mathcal{T}_{\mathsf{sat}}}\mathcal{I}_{C,\mathcal{T}_{\mathsf{sat}}}}$. The case where C is unsatisfiable is obvious. □

Eventually, we formulate a recursive characterization of most specific consequences. It is readily verified that it follows from Propositions 3.4.4 and 4.3.5.

4.3.42 Corollary. *Let $\mathcal{T}$ be an $\mathcal{EL}^{\bot}_{\mathsf{si}}$ TBox, and C be an $\mathcal{EL}^{\bot}_{\mathsf{si}}$ concept description. If C is satisfiable with respect to $\mathcal{T}$, then the following recursion formula for the most specific consequence of C with respect to $\mathcal{T}$ in $\mathcal{EL}^{\bot}_{\mathsf{si}}$ holds true. Otherwise, we have $C^{\mathcal{T}} \equiv_{\emptyset} \bot$.*

$$\begin{aligned} C^{\mathcal{T}} \equiv_{\emptyset} & \textstyle\bigsqcap\{\, A \mid C \sqsubseteq_{\mathcal{T}} A \,\} \\ & \sqcap \textstyle\bigsqcap\{\, \exists\, r.\, D^{\mathcal{T}} \mid C \sqsubseteq_{\mathcal{T}} \exists\, r.\, D \text{ and } \exists\, r.\, D \in \mathsf{Sub}(\mathcal{T}_{\mathsf{sat}}), \text{ or } \exists\, r.\, D \in \mathsf{Conj}(C) \,\} \end{aligned}$$

□

The Role-Depth Bounded Case

In this section, we shall continue with our investigations on algebraic properties of most specific consequences for the role-depth-bounded case. It is no surprise that we find similar results as in the unrestricted case. We skip a proof if it can be easily obtained from the proof of the corresponding analog.

4.3.43 Proposition. (Analog of Proposition 4.3.35) *Let $\mathcal{T}$ be an $\mathcal{EL}^\bot$ TBox and consider some role depth bound $d \in \mathbb{N}$. Then, the mapping $\phi_{\mathcal{T},d} \colon C \mapsto C^{\mathcal{T}_d}$ is a closure operator in the dual of $\mathcal{EL}^\bot_d(\Sigma)$, i.e., for all $\mathcal{EL}^\bot_d$ concept descriptions C and D, the following conditions are satisfied.*

1. $C^{\mathcal{T}_d} \sqsubseteq_\emptyset C$ (extensive)
2. $C \sqsubseteq_\emptyset D$ *implies* $C^{\mathcal{T}_d} \sqsubseteq_\emptyset D^{\mathcal{T}_d}$ (monotonic)
3. $C^{\mathcal{T}_d} \equiv_\emptyset C^{\mathcal{T}_d\mathcal{T}_d}$ (idempotent) □

Please note that, if C has a role depth exceeding d, then it may not follow that $C^{\mathcal{T}_d} \sqsubseteq_\emptyset C$. It is readily verified that, for any $\mathcal{EL}^\bot$ concept description C, the most specific consequence $C^{\emptyset_d}$ and the dth approximation $C\restriction_d$ coincide—thus, we have that $(\exists\, r.\, A)^{\emptyset_0} \equiv_\emptyset \top \not\sqsubseteq_\emptyset \exists\, r.\, A$. However, monotonicity is ensured even if C and D are arbitrary $\mathcal{EL}^\bot$ concept descriptions. Let $C \sqsubseteq_\emptyset D$, then Proposition 4.3.35 implies $C^{\mathcal{T}} \sqsubseteq_\emptyset D^{\mathcal{T}}$. An application of Proposition 3.4.8 yields that $C^{\mathcal{T}}\restriction_d \sqsubseteq_\emptyset D^{\mathcal{T}}\restriction_d$, and finally Proposition 4.3.33 shows $C^{\mathcal{T}_d} \sqsubseteq_\emptyset D^{\mathcal{T}_d}$. Furthermore, idempotency is satisfied for all $\mathcal{EL}^\bot$ concept descriptions C. In summary, we obtain that the extension of $\phi_{\mathcal{T},d}$ to the domain $\mathcal{EL}^\bot(\Sigma)$, or to $\mathcal{EL}^\bot_{\mathsf{si}}(\Sigma)$, is a monotonic, idempotent mapping, but it is not extensive.

4.3.44 Corollary. (Analog of Corollary 4.3.36) *Let $\mathcal{T}$ be an $\mathcal{EL}^\bot$ TBox, fix some role depth bound $d \in \mathbb{N}$, and assume that C as well as D are $\mathcal{EL}^\bot_d$ concept descriptions. Then, the following statements hold true.*

1. $(C \sqcap D)^{\mathcal{T}_d} \sqsubseteq_\emptyset C^{\mathcal{T}_d} \sqcap D^{\mathcal{T}_d}$
2. $(C \sqcap D)^{\mathcal{T}_d} \equiv_\emptyset (C^{\mathcal{T}_d} \sqcap D^{\mathcal{T}_d})^{\mathcal{T}_d}$
3. $C^{\mathcal{T}_d} \vee D^{\mathcal{T}_d} \sqsubseteq_\emptyset (C \vee D)^{\mathcal{T}_d}$
4. $C^{\mathcal{T}_d} \vee D^{\mathcal{T}_d} \equiv_\emptyset (C^{\mathcal{T}_d} \vee D^{\mathcal{T}_d})^{\mathcal{T}_d}$ □

4.3.45 Proposition. (Analog of Lemma 4.3.38 and Proposition 4.3.39) *Let $\mathcal{T} \cup \{C \sqsubseteq D\}$ be an $\mathcal{EL}^\bot$ TBox such that D has a role depth of at most d. Then the following statements are equivalent.*

1. $C \sqsubseteq_\mathcal{T} D$
2. $C^{\mathcal{T}_d} \sqsubseteq_\emptyset D$
3. $C \sqsubseteq_{\mathcal{T}_d^\circ} D$ *where* $\mathcal{T}_d^\circ := \{\, E \sqsubseteq E^{\mathcal{T}_d} \mid E \in \mathcal{EL}^\bot_d(\Sigma) \,\}$

If all conclusions of concept inclusions in $\mathcal{T}$ have role depths not exceeding d, then furthermore the following statement is equivalent to Statements 1 to 3.

4. $C \sqsubseteq_{\mathcal{T}_d^*} D$ *where* $\mathcal{T}_d^* := \{\, E \sqsubseteq E^{\mathcal{T}_d} \mid E \in \mathsf{Prem}(\mathcal{T}) \,\}$

If the concept description C has a role depth not exceeding d, then the following statement is equivalent to Statements 1 to 3, too.

5. *$E^{\mathcal{T}_d} \sqsubseteq_\emptyset C$ implies $E^{\mathcal{T}_d} \sqsubseteq_\emptyset D$ for each $\mathcal{EL}_d^\bot$ concept description E.* □

4.3.46 Corollary. (Analog of Corollary 4.3.40) *Let $\mathcal{T} \cup \{C \sqsubseteq D\}$ be an $\mathcal{EL}^\bot$ TBox such that the role depths of C and of D do not exceed d. Then, the following statements are equivalent.*

1. $\mathcal{T} \models C \sqsubseteq D$
2. $\phi_{\mathcal{T},d} \models C \sqsubseteq D$.
3. *$\mathcal{I} \models \mathcal{T}$ implies $\mathcal{I} \models C \sqsubseteq D$ for every interpretation $\mathcal{I}$.*
4. *$\mathcal{I} \models \mathcal{T}$ implies $\phi_{\mathcal{I},d} \models C \sqsubseteq D$ for any interpretation $\mathcal{I}$.*
5. $\Delta\{\, \phi_{\mathcal{I},d} \mid \mathcal{I} \models \mathcal{T} \,\} \models C \sqsubseteq D$
6. $\emptyset \models \bigvee\{\, C^{\mathcal{I}\mathcal{I}_d} \mid \mathcal{I} \models \mathcal{T} \,\} \sqsubseteq D$
7. $\emptyset \models C^{\mathcal{T}_d} \sqsubseteq D$. □

4.3.47 Corollary. (Analog of Corollary 4.3.41) *Consider an $\mathcal{EL}^\bot$ TBox $\mathcal{T}$, an $\mathcal{EL}^\bot$ concept description C, and some role depth bound $d \in \mathbb{N}$. If C is satisfiable w.r.t. $\mathcal{T}$, then the following equivalences hold true.*

$$\begin{aligned} C^{\mathcal{T}_d} &\equiv_\emptyset C^{(\mathcal{T}_{\mathsf{sat}})_d} \equiv_\emptyset \{C\}^{(\mathcal{I}_{C,\mathcal{T}_{\mathsf{sat}}})_d} \equiv_\emptyset C^{\mathcal{I}_{C,\mathcal{T}_{\mathsf{sat}}}(\mathcal{I}_{C,\mathcal{T}_{\mathsf{sat}}})_d} \\ &\equiv_\emptyset \bigvee\{\, C^{\mathcal{I}\mathcal{I}_d} \mid \mathcal{I} \models \mathcal{T}_{\mathsf{sat}} \,\} \equiv_\emptyset \bigvee\{\, C^{\mathcal{I}\mathcal{I}_d} \mid \mathcal{I} \models \mathcal{T} \,\} \end{aligned}$$

Otherwise, if C is not satisfiable w.r.t. $\mathcal{T}$, then $C^{\mathcal{T}_d} \equiv_\emptyset \bot \equiv_\emptyset \bigvee\{\, C^{\mathcal{I}\mathcal{I}} \mid \mathcal{I} \models \mathcal{T} \,\}$. □

4.3.48 Corollary. (Analog of Corollary 4.3.42) *Let $\mathcal{T}$ be an $\mathcal{EL}^\bot$ TBox, and C be an $\mathcal{EL}^\bot$ concept description. If C is satisfiable with respect to $\mathcal{T}$, then the following recursion formulas for the most specific consequence of C with respect to $\mathcal{T}$ in $\mathcal{EL}_d^\bot$ hold true. Otherwise, we have $C^{\mathcal{T}_d} \equiv_\emptyset \bot$ for any $d \in \mathbb{N}$.*

$$\begin{aligned} C^{\mathcal{T}_0} &\equiv_\emptyset \bigsqcap\{\, A \mid C \sqsubseteq_\mathcal{T} A \,\} \\ C^{\mathcal{T}_{d+1}} &\equiv_\emptyset \bigsqcap\{\, A \mid C \sqsubseteq_\mathcal{T} A \,\} \\ &\quad \sqcap \bigsqcap\{\, \exists\, r.\, D^{\mathcal{T}_d} \mid C \sqsubseteq_\mathcal{T} \exists\, r.\, D \text{ and } \exists\, r.\, D \in \mathsf{Sub}(\mathcal{T}_{\mathsf{sat}}),\ \text{or } \exists\, r.\, D \in \mathsf{Conj}(C) \,\} \end{aligned}$$ □

4.3.3 A Characterization of Entailment

We have seen in Section 4.3.2 that each TBox $\mathcal{T}$ induces a closure operator $\phi_\mathcal{T}$ in a way such that any concept inclusion is entailed by $\mathcal{T}$ if, and only if, it is valid for $\phi_\mathcal{T}$. In the following lemma, we shall use these closure operators to provide a characterization of entailment between two TBoxes. A similar result can, of course, be found for the role-depth-bounded case too using our results from Section 4.3.2.

4.3.49 Proposition. *Let $\mathcal{T}_1 \cup \mathcal{T}_2$ be an $\mathcal{EL}_{\mathsf{si}}^\bot$ TBox. Then, the following statements are equivalent.*

1. $\mathcal{T}_1 \models \mathcal{T}_2$.
2. *$C \sqsubseteq_{\mathcal{T}_2} D$ implies $C \sqsubseteq_{\mathcal{T}_1} D$ for all $\mathcal{EL}_{\mathsf{si}}^\bot$ concept inclusions $C \sqsubseteq D$.*

3. $C^{\mathcal{T}_1} \sqsubseteq_\emptyset C^{\mathcal{T}_2}$ *for all* $\mathcal{EL}^\bot_{\mathsf{si}}$ *concept descriptions* C.
4. $C^{\mathcal{T}_1\mathcal{T}_2} \equiv_\emptyset C^{\mathcal{T}_1}$ *for all* $\mathcal{EL}^\bot_{\mathsf{si}}$ *concept descriptions* C.
5. *Each most specific consequence of* $\mathcal{T}_1$ *is a most specific consequence of* $\mathcal{T}_2$*, modulo equivalence with respect to the empty TBox* $\emptyset$.
6. $\phi_{\mathcal{T}_1} \trianglerighteq \phi_{\mathcal{T}_2}$

Proof. We start with demonstrating the equivalence of Statements 1 and 3. Assume that $\mathcal{T}_1 \models \mathcal{T}_2$ and consider an arbitrary $\mathcal{EL}^\bot_{\mathsf{si}}$ concept description C. By Definition 4.3.1 it holds true that $C \sqsubseteq_{\mathcal{T}_2} C^{\mathcal{T}_2}$, and consequently $C \sqsubseteq_{\mathcal{T}_1} C^{\mathcal{T}_2}$. An application of Proposition 4.3.39 then yields $C^{\mathcal{T}_1} \sqsubseteq_\emptyset C^{\mathcal{T}_2}$. Conversely, let $C^{\mathcal{T}_1} \sqsubseteq_\emptyset C^{\mathcal{T}_2}$ for all $\mathcal{EL}^\bot_{\mathsf{si}}$ concept descriptions C. Of course, Proposition 4.3.39 implies $C \sqsubseteq_{\mathcal{T}_1} C^{\mathcal{T}_2}$ for all $C \in \mathcal{EL}^\bot_{\mathsf{si}}(\Sigma)$. Now consider a concept inclusion $C \sqsubseteq D \in \mathcal{T}_2$. It is immediately clear that then D is a consequence of C with respect to $\mathcal{T}_2$, and hence $C^{\mathcal{T}_2} \sqsubseteq_\emptyset D$. It follows that $C^{\mathcal{T}_1} \sqsubseteq_\emptyset D$, and thus $C \sqsubseteq_{\mathcal{T}_1} D$. Since $C \sqsubseteq D$ is an arbitrary concept inclusion from $\mathcal{T}_2$, we have just demonstrated that $\mathcal{T}_1 \models \mathcal{T}_2$.

Furthermore, it is readily verified that Statements 1 and 2 are equivalent. Eventually, Section 1.5 implies the equivalence of Statements 3 to 6. □

4.3.4 A Characterization of Soundness and Completeness

As we have already mentioned, there are several works on the axiomatization of concept inclusions from interpretations. In particular, these approaches can be used to compute so-called concept inclusion bases for interpretations, and a TBox $\mathcal{T}$ is such a *concept inclusion base* for some interpretation $\mathcal{I}$ if $\mathcal{T}$ is *sound* for $\mathcal{I}$, that is, $\mathcal{I} \models \mathcal{T}$, and is *complete* for $\mathcal{I}$, that is, $C \sqsubseteq_\mathcal{I} D$ implies $C \sqsubseteq_\mathcal{T} D$ for every concept inclusion $C \sqsubseteq D$. The aim of this section is to characterize these two notions of soundness and completeness using the notions of most specific consequences and of model-based most specific concept inclusions as well as their induced closure operators.

4.3.50 Proposition. *Let* $\mathcal{I}$ *be an interpretation, and assume that* $\mathcal{T}$ *is an* $\mathcal{EL}^\bot_{\mathsf{si}}$ *TBox. Then, the following statements are equivalent.*

1. $\mathcal{T}$ *is* sound *for* $\mathcal{I}$.
2. $\mathcal{I} \models \mathcal{T}$
3. $C \sqsubseteq_\mathcal{T} D$ *implies* $C \sqsubseteq_\mathcal{I} D$ *for all* $\mathcal{EL}^\bot_{\mathsf{si}}$ *concept inclusions* $C \sqsubseteq D$.
4. $C^{\mathcal{I}\mathcal{I}} \sqsubseteq_\emptyset C^\mathcal{T}$ *for all* $\mathcal{EL}^\bot_{\mathsf{si}}$ *concept descriptions* C.
5. *Each model-based most specific concept description of* $\mathcal{I}$ *is a most specific consequence of* $\mathcal{T}$*, modulo equivalence with respect to the empty TBox* $\emptyset$.
6. $\phi_\mathcal{I} \trianglerighteq \phi_\mathcal{T}$

Proof. The equivalence of Statements 1 to 3 is either true by definition or trivial.

By Proposition 4.3.39 we have that $C \sqsubseteq_\mathcal{T} D$ is equivalent to $C^\mathcal{T} \sqsubseteq_\emptyset D$. Furthermore, we know that $C \sqsubseteq_\mathcal{I} D$ if, and only if, $C^{\mathcal{I}\mathcal{I}} \sqsubseteq_\emptyset D$.

We are now going to show that Statement 4 implies Statement 3. Therefore assume that $\mathcal{T}$ entails $C \sqsubseteq D$, i.e., the concept inclusion $C^\mathcal{T} \sqsubseteq D$ is valid in all interpretations. Of course, then $C^{\mathcal{I}\mathcal{I}} \sqsubseteq_{\emptyset} C^{\mathcal{T}}$ yields that also the concept inclusion $C^{\mathcal{I}\mathcal{I}} \sqsubseteq D$ is valid in all interpretations, and consequently $C \sqsubseteq D$ is valid in $\mathcal{I}$. Vice versa, if $C \sqsubseteq_{\mathcal{T}} D$ implies $C \sqsubseteq_{\mathcal{I}} D$, then $C^{\mathcal{T}} \sqsubseteq_{\emptyset} D$ implies $C^{\mathcal{I}\mathcal{I}} \sqsubseteq_{\emptyset} D$. It readily verified that then $C^{\mathcal{I}\mathcal{I}} \sqsubseteq_{\emptyset} C^{\mathcal{T}}$.

Eventually, the equivalence of Statements 4 and 5 is an immediate consequence of Section 1.5. □

4.3.51 Proposition. *Let $\mathcal{I}$ be an interpretation, and assume that $\mathcal{T}$ is an $\mathcal{EL}^{\bot}_{\mathsf{si}}$ TBox. Then, the following statements are equivalent.*

1. $\mathcal{T}$ *is* complete *for* $\mathcal{I}$.
2. $C \sqsubseteq_{\mathcal{I}} D$ *implies* $C \sqsubseteq_{\mathcal{T}} D$ *for all* $\mathcal{EL}^{\bot}_{\mathsf{si}}$ *concept inclusions* $C \sqsubseteq D$.
3. $C^{\mathcal{T}} \sqsubseteq_{\emptyset} C^{\mathcal{I}\mathcal{I}}$ *for all* $\mathcal{EL}^{\bot}_{\mathsf{si}}$ *concept descriptions* C.
4. *Each most specific consequence of* $\mathcal{T}$ *is a model-based most specific concept description of* $\mathcal{I}$*, modulo equivalence with respect to the empty TBox* $\emptyset$.
5. $\phi_{\mathcal{T}} \trianglerighteq \phi_{\mathcal{I}}$

Proof. Statements 1 and 2 are equivalent just by definition, and the equivalence of Statements 3 and 4 follows from Section 1.5. It remains to prove, e.g., that Statements 2 and 3 are equivalent. Hence, assume that $C^{\mathcal{I}\mathcal{I}} \sqsubseteq_{\emptyset} D$ implies $C^{\mathcal{T}} \sqsubseteq_{\emptyset} D$ for all $\mathcal{EL}^{\bot}_{\mathsf{si}}$ concept inclusions $C \sqsubseteq D$. Of course, it easily follows that $C^{\mathcal{T}} \sqsubseteq_{\emptyset} C^{\mathcal{I}\mathcal{I}}$. For the converse direction, let $C^{\mathcal{I}\mathcal{I}} \sqsubseteq_{\emptyset} D$. Then $C^{\mathcal{T}} \sqsubseteq_{\emptyset} C^{\mathcal{I}\mathcal{I}}$ implies $C^{\mathcal{T}} \sqsubseteq_{\emptyset} D$. □

Summing up, Propositions 4.3.50 and 4.3.51 yield the following corollary.

4.3.52 Corollary. *Let $\mathcal{I}$ be an interpretation, and assume that $\mathcal{T}$ is an $\mathcal{EL}^{\bot}_{\mathsf{si}}$ TBox. Then, the following statements are equivalent.*

1. $\mathcal{T}$ *is a base of concept inclusions for* $\mathcal{I}$.
2. $\mathcal{T}$ *is sound as well as complete for* $\mathcal{I}$.
3. $C \sqsubseteq_{\mathcal{T}} D$ *if, and only if,* $C \sqsubseteq_{\mathcal{I}} D$ *for all* $\mathcal{EL}^{\bot}_{\mathsf{si}}$ *concept inclusions* $C \sqsubseteq D$.
4. $C^{\mathcal{T}} \equiv_{\emptyset} C^{\mathcal{I}\mathcal{I}}$ *for all* $\mathcal{EL}^{\bot}_{\mathsf{si}}$ *concept descriptions* C.
5. *The most specific consequences of* $\mathcal{T}$ *are exactly the model-based most specific concept descriptions of* $\mathcal{I}$*, modulo equivalence with respect to the empty TBox* $\emptyset$.
6. $\phi_{\mathcal{T}} = \phi_{\mathcal{I}}$

4.4 Relative Model-Based Most Specific $\mathcal{EL}$ Concept Descriptions

In the previous Section 4.1 we have introduced model-based most specific concept descriptions. We consider now a more general setting where being most specific is considered with respect to some non-empty TBox $\mathcal{T}$. It is easy to verify that each MMSC w.r.t. $\emptyset$ is also an MMSC w.r.t. $\mathcal{T}$, but it might happen that a MMSC w.r.t. $\mathcal{T}$ exists in $\mathcal{EL}$ even if the given interpretation is cyclic. This is the case if the cycle in the interpretation is already expressed in the TBox. For instance, consider the interpretation $\mathcal{I}$ that has only one object δ, and δ is an r-successor of itself. The MMSC $\{\delta\}^{\mathcal{I}}$ can not be expressed in the description logic $\mathcal{EL}$. However, if we now further define the TBox $\mathcal{T}$ to consist of the single concept inclusion $\top \sqsubseteq \exists\, r.\top$, then we can easily see that the MMSC of $\{\delta\}$ for $\mathcal{I}$ relative to $\mathcal{T}$ is $\top$, i.e., it exists in $\mathcal{EL}$.

4.4.1 Definition. Let $\mathcal{I}$ be an interpretation, let $\mathcal{T}$ be an $\mathcal{EL}^{\bot}$ TBox such that $\mathcal{I} \models \mathcal{T}$, and consider some subset $X \subseteq \Delta^{\mathcal{I}}$. Then, a *model-based most specific concept description* of X with respect to $\mathcal{I}$ *relative* to $\mathcal{T}$ is a concept description C that satisfies the following conditions.

1. $X \subseteq C^{\mathcal{I}}$
2. For any concept description D, it holds true that $X \subseteq D^{\mathcal{I}}$ implies $C \sqsubseteq_{\mathcal{T}} D$. △

Since all model-based most specific concept descriptions of X with respect to $\mathcal{I}$ relative to $\mathcal{T}$ are equivalent, we shall denote one of these equivalent concept descriptions by $X^{\mathcal{I}_{\mathcal{T}}}$. We can readily verify that $\emptyset^{\mathcal{I}_{\mathcal{T}}} = \bot$ is always true and, furthermore, that $X^{\mathcal{I}_{\mathcal{T}}} \sqsupset_{\not\equiv_\emptyset} \bot$ holds true for every non-empty subset $X \subseteq \Delta^{\mathcal{I}}$.

4.4.2 Proposition. *Let $\mathcal{I}$ be a finite interpretation and let $\mathcal{T}$ be an $\mathcal{EL}^{\bot}$ TBox such that $\mathcal{I} \models \mathcal{T}$. Then, for each concept description C and for every subset $X \subseteq \Delta^{\mathcal{I}}$, the following statement holds true.*

$$X^{\mathcal{I}_{\mathcal{T}}} \equiv_\emptyset C \text{ if, and only if, } X^{\mathcal{I}} \equiv_\emptyset C^{\mathcal{T}}$$

Proof. We begin with proving the *only if* direction; assume that $X^{\mathcal{I}_{\mathcal{T}}} \equiv_\emptyset C$ is satisfied. It then follows that $X \subseteq C^{\mathcal{I}}$, and so $X^{\mathcal{I}} \sqsubseteq_\emptyset C$. Applying the closure operator $\phi_{\mathcal{T}}$ yields that $X^{\mathcal{I}\mathcal{T}} \sqsubseteq_\emptyset C^{\mathcal{T}}$. Since $\mathcal{I} \models \mathcal{T}$, we have that $\phi_{\mathcal{I}} \trianglelefteq \phi_{\mathcal{T}}$, which means that $Y^{\mathcal{I}\mathcal{T}} \equiv_\emptyset Y^{\mathcal{I}}$ for any subset $Y \subseteq \Delta^{\mathcal{I}}$. We conclude that $X^{\mathcal{I}} \sqsubseteq_\emptyset C^{\mathcal{T}}$. Furthermore, from $X \subseteq X^{\mathcal{I}\mathcal{I}}$ it follows that $X^{\mathcal{I}_{\mathcal{T}}} \sqsubseteq_{\mathcal{T}} X^{\mathcal{I}}$, which means that $C \sqsubseteq_{\mathcal{T}} X^{\mathcal{I}}$, and so we conclude that $C^{\mathcal{T}} \sqsubseteq_\emptyset X^{\mathcal{I}}$ holds true as well.

Regarding a proof for the *if* direction, let $X^{\mathcal{I}} \equiv_\emptyset C^{\mathcal{T}}$. Since $\phi_{\mathcal{T}}$ is extensive, we infer that $X^{\mathcal{I}} \sqsubseteq_\emptyset C$ and, thus, that $X \subseteq C^{\mathcal{I}}$. Now consider some concept description D with $X \subseteq D^{\mathcal{I}}$. Then, we have that $X^{\mathcal{I}} \sqsubseteq_\emptyset D$, i.e., it holds true that $C^{\mathcal{T}} \sqsubseteq_\emptyset D$, which immediately yields that $C \sqsubseteq_{\mathcal{T}} D$. In summary, we conclude that $X^{\mathcal{I}_{\mathcal{T}}} \equiv_\emptyset C$. □

As a straightforward consequence of Propositions 4.3.31 and 4.4.2 we get the following.

4.4.3 Corollary. *Fix some finite interpretation $\mathcal{I}$ as well as some $\mathcal{EL}^{\bot}$ TBox $\mathcal{T}$ such that $\mathcal{I} \models \mathcal{T}$. For each subset $X \subseteq \Delta^{\mathcal{I}}$ it then holds true that $X^{\mathcal{I}_{\mathcal{T}}}$ is equivalent to $X^{\mathcal{I}_{\mathcal{T}_{\mathsf{sat}}}}$ with respect to $\emptyset$.* □

Combining our previous results from Propositions 3.4.3, 4.1.4, 4.3.5, and 4.4.2 and Corollary 4.4.3 immediately shows that the below corollary is true.

4.4.4 Corollary. *Let $\mathcal{I}$ be an interpretation and let $\mathcal{T}$ be an $\mathcal{EL}^{\bot}$ TBox such that $\mathcal{I} \models \mathcal{T}$. Then, for each concept description C and for every non-empty subset $X \subseteq \Delta^{\mathcal{I}}$, the following statement holds true.*

$$X^{\mathcal{I}_{\mathcal{T}}} \equiv_\emptyset C \text{ if, and only if, } \bigtimes\{\, (\mathcal{I}, \delta) \mid \delta \in X \,\} \rightleftarrows (\mathcal{I}_{C,\mathcal{T}_{\mathsf{sat}}}, C)$$ □

The non-relative MMSC of some subset X for $\mathcal{I}$ can be obtained as the least common subsumer of the singleton MMSCs, cf. Proposition 4.1.4. A similar statement is also true for the relative MMSCs, i.e., it suffices to develop techniques for computing relative MMSCs of single objects, since it is clear how least common subsumers can be computed. Note that the LCS $C \vee_{\mathcal{T}} D$ is equivalent to $\exists^{\mathsf{sim}}((\mathcal{I}_{C,\mathcal{T}}, C) \times (\mathcal{I}_{D,\mathcal{T}}, D))$ modulo $\mathcal{T}$.

4.4.5 Proposition. *Fix some finite interpretation $\mathcal{I}$ that is a model of an $\mathcal{EL}$ TBox $\mathcal{T}$. Then, for each subset $X \subseteq \Delta^{\mathcal{I}}$, the following equivalence holds true.*

$$X^{\mathcal{I}_{\mathcal{T}}} \equiv_{\emptyset} \bigvee_{\mathcal{T}} \{\, \{\delta\}^{\mathcal{I}_{\mathcal{T}}} \mid \delta \in X \,\}$$

Proof. We know that that $X^{\mathcal{I}}$ is equivalent to $\bigvee\{\, \{\delta\}^{\mathcal{I}} \mid \delta \in X \,\}$ with respect to $\emptyset$. From Proposition 4.4.2 it immediately follows that $Y^{\mathcal{I}_{\mathcal{T}}\mathcal{T}} \equiv_{\emptyset} Y^{\mathcal{I}}$ is satisfied for each subset $Y \subseteq \Delta^{\mathcal{I}}$, which shows that $X^{\mathcal{I}}$ must be equivalent to $\bigvee\{\, \{\delta\}^{\mathcal{I}_{\mathcal{T}}\mathcal{T}} \mid \delta \in X \,\}$ w.r.t. $\emptyset$. Eventually, an application of Lemma 4.3.37 yields that $X^{\mathcal{I}} \equiv_{\emptyset} (\bigvee_{\mathcal{T}}\{\, \{\delta\}^{\mathcal{I}_{\mathcal{T}}} \mid \delta \in X \,\})^{\mathcal{T}}$ holds true and then Proposition 4.4.2 implies the claim. □

Deciding the Existence in $\mathcal{EL}$

It is easy to see that the MMSC $X^{\mathcal{I}}$ is always a relative MMSC too. However, there are cases where we want to check whether $X^{\mathcal{I}_{\mathcal{T}}}$ exists in $\mathcal{EL}$ if $X^{\mathcal{I}}$ is not expressible in $\mathcal{EL}$. One possibility to obtain a decision procedure for checking the existence in $\mathcal{EL}$ is by suitably adapting the results in [ZT13b; ZT13a]. In particular, Zarrieß and Turhan have shown how existence of least common subsumers and of most specific concept descriptions (for ABoxes) relative to a given TBox can be decided. It is straightforward to modify their results and proofs to verify that $\{\delta\}^{\mathcal{J}_{\mathcal{T}}}$ exists in $\mathcal{EL}$ and is equivalent to $\mathsf{X}^{\ell}(\mathcal{J}, \delta)$ if, and only if, $(\mathcal{J}, \delta) \precsim (\mathcal{I}_{\mathsf{X}^{\ell}(\mathcal{J},\delta),\mathcal{T}_{\mathsf{sat}}}, \mathsf{X}^{\ell}(\mathcal{J}, \delta))$ where $\ell := |\Delta^{\mathcal{J}}|^2 + \bigvee\{\, \mathsf{rd}(C) \mid C \sqsubseteq D \in \mathcal{T}_{\mathsf{sat}} \,\}$. In particular, a proof for this statement utilizes a technique for shortening simulation paths that is similar to what Zarrieß and Turhan did in the technical report [ZT13b]. However, the characteristic concept description $\mathsf{X}^{\ell}(\mathcal{J}, \delta)$ has an exponential size, and so it follows that checking the existence of a simulation from $(\mathcal{J}, \delta)$ to $(\mathcal{I}_{\mathsf{X}^{\ell}(\mathcal{J},\delta),\mathcal{T}_{\mathsf{sat}}}, \mathsf{X}^{\ell}(\mathcal{J}, \delta))$ needs exponential time.

Another option for deciding the existence in $\mathcal{EL}$ is by defining a reduction system on pointed interpretations. There are two reduction rules: the first one deletes an object from the extension of a concept name, and the second one deletes a pair of objects from the extension of a role name, but in both cases only if this piece of information can be restored from the TBox. More specifically, we can define the following rules.

C-Rule Let $(\mathcal{I}, \delta) \to_{\mathsf{C}} (\mathcal{J}, \delta)$ if there exists some object $\epsilon \in \Delta^{\mathcal{I}}$ as well as a concept name $A \in \Sigma_{\mathsf{C}}$ such that $\epsilon \in A^{\mathcal{I}}$ and $(\mathcal{I}, \delta) \precsim (\mathcal{I}_{\exists^{\mathsf{sim}}(\mathcal{J},\delta),\mathcal{T}_{\mathsf{sat}}}, \exists^{\mathsf{sim}}(\mathcal{J}, \delta))$ is satisfied for the interpretation $\mathcal{J} := \mathcal{I}$, but where $A^{\mathcal{J}} := A^{\mathcal{I}} \setminus \{\epsilon\}$.

R-Rule Define $(\mathcal{I}, \delta) \to_{\mathsf{R}} (\mathcal{J}, \delta)$ if there are objects $\epsilon, \zeta \in \Delta^{\mathcal{I}}$ as well as a role name $r \in \Sigma_{\mathsf{R}}$ such that $(\epsilon, \zeta) \in r^{\mathcal{I}}$ and $(\mathcal{I}, \delta) \precsim (\mathcal{I}_{\exists^{\mathsf{sim}}(\mathcal{J},\delta),\mathcal{T}_{\mathsf{sat}}}, \exists^{\mathsf{sim}}(\mathcal{J}, \delta))$ holds true for the interpretation $\mathcal{J} := \mathcal{I}$, but where $r^{\mathcal{J}} := r^{\mathcal{I}} \setminus \{(\epsilon, \zeta)\}$.

Soundness of these rules is an immediate consequence of Proposition 4.4.2. Furthermore, the reduction system is obviously terminating. Moreover, each rule application needs only polynomial time w.r.t. $\mathcal{I}$, and the number of rule applications is polynomial in $\mathcal{I}$ as well. However, it remains an open problem whether this reduction system is also complete, i.e., if it can construct an acyclic pointed interpretation from an input $(\mathcal{I}, \delta)$ whenever the relative MMSC $\{\delta\}^{\mathcal{I}\mathcal{T}}$ exists in $\mathcal{EL}$.

If we assume that $\{\delta\}^{\mathcal{I}\mathcal{T}} \equiv_\emptyset C$ for some $\mathcal{EL}$ concept description C, then Proposition 4.4.2 implies that $(\mathcal{I}, \delta)$ and $(\mathcal{I}_{C,\mathcal{T}}, C)$ are equi-similar. Of course, we could use the above reduction rules to obtain $(\mathcal{I}_C, C)$ from $(\mathcal{I}_{C,\mathcal{T}}, C)$. However, this is not immediately helpful, as our reduction system starts with the pointed interpretation $(\mathcal{I}, \delta)$.

In Section 3.4.4 we have seen that each pointed interpretation is equi-similar to its reduction. Since $(\mathcal{I}, \delta)$ and $(\mathcal{I}_{C,\mathcal{T}}, C)$ are equi-similar, Proposition 3.4.17 implies that $(\mathcal{I}, \delta)$ and $(\mathcal{I}_{C,\mathcal{T}}, C)$ can be reduced to isomorphic pointed interpretations, that is, we can w.l.o.g. assume that both have the same reduction. We could now define a further reduction rule as follows and assign to it the highest application precedence, i.e., it is immediately applied as first rule. After the first rule application we would thus reduce the input $(\mathcal{I}, \delta)$ to a pointed interpretation that is strongly related to $(\mathcal{I}_{C,\mathcal{T}}, C)$.

N-Rule We define $(\mathcal{I}, \delta) \to_{\mathsf{N}} \mathsf{reduce}(\mathcal{I}, \delta)$ if $(\mathcal{I}, \delta)$ is not reduced.

Soundness as well as termination in polynomial time is then still guaranteed. Furthermore, it is possible to prove that the reduction of $(\mathcal{I}_{C,\mathcal{T}}, C)$ is isomorphic to an induced subinterpretation of $(\mathcal{I}_{C,\mathcal{T}}, C)$. However, it remains an open problem how to make use of the fact that $(\mathcal{I}_{C,\mathcal{T}}, C)$ has an acyclic reduction for showing that also the input has an acyclic reduction, i.e., that the reduction system is complete.

5 Lattice of $\mathcal{EL}$ Concept Descriptions

The goal of this section is to explore the properties of the lattice of $\mathcal{EL}$ concept descriptions ordered by subsumption with respect to the empty TBox. We have already seen on Page 39 that the finitary infimum operation corresponds with the finitary conjunction operation $\bigsqcap$ and that the supremum operation is the least common subsumer operation $\bigvee$.

In the next Section 5.1, we will consider the neighborhood relation induced by the subsumption relation. We do this not only for empty TBoxes, but also for cycle-restricted and general TBoxes. In particular, we demonstrate that $\mathcal{EL}$ is neighborhood generated if the underlying TBox is empty or cycle-restricted and we show how all upper as well as all lower neighbors of a given concept description can be computed. Furthermore, it is proven that $\mathcal{EL}$ is not neighborhood generated for general TBoxes and that $\mathcal{EL}^{\bot}$ and $\mathcal{EL}_{\mathsf{si}}$ are both not neighborhood generated. The possibility of enumerating all lower or upper neighbors is useful when it comes to deciding whether a given concept description is most specific or most general with respect to some monotonic property.

As next step we show in Section 5.2 that the lattice of $\mathcal{EL}$ concept descriptions is distributive. We further prove that, for all concept descriptions C and D with $C \sqsubseteq_{\emptyset} D$, there cannot exist any infinite chain between C and D, i.e., the lattice is of locally finite length. According to Blyth [Bly05, Chapters 4 and 5], we then obtain as an immediate corollary that also the Jordan-Dedekind chain condition is satisfied, which states that for each pair $C \sqsubseteq_{\emptyset} D$, all maximal chains in the interval $[C, D]$ have the same length. Furthermore, we demonstrate in Sections 5.3 and 5.4 that this length can be utilized to define a distance between C and D, and in particular to measure a distance from each concept description C to the top concept description $\top$, which we call the rank of C. Section 5.5 is concerned with constructing a similarity measure between $\mathcal{EL}$ concept descriptions that satisfies the triangle inequality. The computational complexity of computing ranks and distances is investigated in Section 5.6.

We close this chapter with demonstrating in Section 5.7 that the lattice of $\mathcal{EL}$ concept descriptions has relative pseudo-complements, i.e., it is a residuated lattice.

5.1 Neighborhood of Concept Descriptions

In this section we consider the *neighborhood problem* for $\mathcal{EL}$ and its variants. We have already seen that the set of concept descriptions ordered by the subsumption relation constitutes a lattice. In this section we now consider the question whether the subsumption order is discrete in the sense that it is generated by its induced neighborhood relation. Note that for an order relation $\leq$ on some set P its *neighborhood relation* or *transitive reduction* is defined as

$$\prec := \{\, (p, q) \mid p \lneq q \text{ and there exists no } x \text{ such that } p \lneq x \lneq q \,\} = \lneq \setminus (\lneq \circ \lneq).$$

Clearly, if P is finite, then the transitive closure $\prec^+$ equals the irreflexive part $\lneq$. However, there are infinite ordered sets where this does not hold true; even worse, there are cases where $\prec^+$ is empty. Consider, for instance, the set $\mathbb{R}$ of real numbers with their usual ordering $\leq$. It is well-known that $\mathbb{R}$ is dense in itself, that is, for each pair $x \lneq y$, there is another real number z such that $x \lneq z \lneq y$—thus, there are no neighboring real numbers. In general, we say that $\leq$ is *neighborhood generated* if $\prec^+ = \lneq$ is satisfied. Clearly, $\leq$ is a neighborhood generated order relation if, and only if, there is a finite path $p = x_0 \prec x_1 \prec \ldots \prec x_n = q$ for each pair $p \leq q$. An alternative formulation is the following. $\leq$ is not neighborhood generated if, and only if, there exists some pair $p \lneq q$ such that every finite path $p = x_0 \lneq x_1 \lneq \ldots \lneq x_n = q$ can be refined, that is, there is some index i and an element y such that $x_i \lneq y \lneq x_{i+1}$. Of course, if the order relation $\leq$ is *bounded*, i.e., for each element $p \in P$, there exists a finite upper bound on the lengths of $\lneq$-paths issuing from p, then $\leq$ is neighborhood generated.

Although boundedness of a poset $(P, \leq)$ is sufficient for neighborhood generatedness, it is not necessary. The following result of Ganter [Gan18] immediately implies that any unbounded poset can be order-embedded into some neighborhood generated poset, which then must be unbounded as well.

[Gan18]. *Any poset $(P, \leq)$ is order-embeddable into some neighborhood generated poset.*

Proof. For some given poset $(P, \leq)$, we define another poset $(\leq, \sqsubseteq)$ where

$$(a, b) \sqsubseteq (c, d) \quad \text{if, and only if,} \quad (a, b) = (c, d) \text{ or } b \leq c.$$

As one quickly verifies, $\sqsubseteq$ is indeed reflexive, antisymmetric, and transitive. In the following, we shall denote the neighborhood relation of $\sqsubseteq$ by $\prec$.

We first show that $a \lneq b$ implies $(a, a) \prec (a, b)$. Let $a \lneq b$. Of course, we have that $(a, a) \sqsubsetneq (a, b)$. Now consider some pair $c \leq d$ such that $(a, a) \sqsubseteq (c, d) \sqsubsetneq (a, b)$. Then, it follows that $a \leq c \leq d \leq a$, which shows that $(c, d) = (a, a)$.

Analogously, we infer that $a \lneq b$ implies $(a, b) \prec (b, b)$. We conclude that $(a, a) \sqsubsetneq (b, b)$ always implies $(a, a) \prec (a, b) \prec (b, b)$.

Eventually, assume that $a \lneq b \lneq c \lneq d$, i.e., $(a, b) \sqsubsetneq (b, c) \sqsubsetneq (c, d)$ is satisfied. Applying the above yields that $(a, b) \prec (b, b) \prec (b, c) \prec (c, c) \prec (c, d)$. Consequently, we have that $\sqsubsetneq \subseteq \prec^+$. The converse inclusion is trivial. Thus, $(\leq, \sqsubseteq)$ is neighborhood generated.

It remains to show that there is an order-embedding of $(P, \leq)$ into $(\leq, \sqsubseteq)$. For this purpose, define the mapping $f \colon P \to \leq,\ p \mapsto (p, p)$. It is readily verified that f is order-preserving as well as order-reflecting, which immediately implies that f is injective as well. As a corollary, we obtain that f is an order-embedding. □

In the sequel of this section, we shall address the neighborhood problem from different perspectives. We first consider the general problem of existence of neighbors, and then provide means for the computation of all upper neighbors and of all lower neighbors, respectively, in the cases where these exist. As it will turn out, neighbors only exist for all concept descriptions in the description logic $\mathcal{EL}$ without any TBox or in $\mathcal{EL}$ with respect to acyclic or cycle-restricted TBoxes. The presence of either a non-cycle-restricted TBox or of the bottom concept description $\bot$ prevents the existence of neighbors for some concept descriptions. Furthermore, the extensions of $\mathcal{EL}$ with greatest fixed-point semantics also allow for the construction of concept descriptions that do not possess neighbors.

Furthermore, a complexity analysis shows that, for the case of an empty TBox, deciding neighborhood in $\mathcal{EL}$ is in **P**, and that all upper neighbors of an $\mathcal{EL}$ concept description can be computed in deterministic polynomial time, while there exists some $\mathcal{EL}$ concept description that has exponentially many lower neighbors, and the sizes of reduced forms of lower neighbors are always polynomial. As an application, we show in Section 5.1.2 how upper bounds on the computational complexity of deciding whether a concept description is most specific or most general with respect to some computable property can easily be obtained. For acyclic or cycle-restricted TBoxes, deciding neighborhood is in **coNP**, all lower neighbors of an $\mathcal{EL}$ concept description can be enumerated in exponential time, and all upper neighbors of some $\mathcal{EL}$ concept description can be effectively computed.

Eventually, in Section 5.1.9 we compare the notion of lower neighbors with so-called downward refinement operators, in particular with the downward refinement operator for $\mathcal{ELH}$ that has been devised by Lehmann [LH09; Leh10].

5.1.1 Definition. Consider a signature Σ, let $\mathcal{T}$ be a TBox over Σ, and further assume that C and D are concept descriptions over Σ. Then, C is a *lower neighbor* or a *most general strict subsumee* of D with respect to $\mathcal{T}$, denoted as $\mathcal{T} \models C \prec D$ or $C \prec_{\mathcal{T}} D$, if the following statements hold true.

1. $C \sqsubsetneq_{\mathcal{T}} D$
2. For each concept description E over Σ, it holds true that $C \sqsubseteq_{\mathcal{T}} E \sqsubseteq_{\mathcal{T}} D$ implies $E \equiv_{\mathcal{T}} C$ or $E \equiv_{\mathcal{T}} D$.

Additionally, we then also say that D is an *upper neighbor* or a *most specific strict subsumer* of C with respect to $\mathcal{T}$, and we may also write $\mathcal{T} \models D \succ C$ or $D \succ_{\mathcal{T}} C$. △

Obviously, $\top$ does not have any upper neighbors, and dually $\bot$ does not have any lower neighbors.

We first observe that neighborhood of concept descriptions is not preserved by the concept constructors. It is easy to see that $A \sqcap B \prec_{\emptyset} A$. However, it holds true that $\exists\, r.\,(A \sqcap B) \sqsubsetneq_{\emptyset} \exists\, r.\, A \sqcap \exists\, r.\, B \sqsubsetneq_{\emptyset} \exists\, r.\, A$, which shows $\exists\, r.\,(A \sqcap B) \not\prec_{\emptyset} \exists\, r.\, A$. Furthermore, we have that $A \sqcap B \sqcap (A \sqcap B) \equiv_{\emptyset} A \sqcap (A \sqcap B)$, and consequently $A \sqcap B \sqcap (A \sqcap B) \not\prec_{\emptyset} A \sqcap (A \sqcap B)$. There are according counterexamples when neighborhood with respect to a non-empty TBox is considered.

It is easily verified that neighborhood with respect to the empty TBox $\emptyset$ does not coincide with neighborhood w.r.t. a non-empty TBox $\mathcal{T}$. For instance, $A \prec_{\emptyset} \top$ holds true, but $\{\top \sqsubseteq A\} \models A \equiv \top$. For the converse direction, consider the counterexample where $\{A \sqsubseteq B,\ B \sqsubseteq A\} \models A \sqcap B \prec \top$ and $A \sqcap B \sqsubsetneq_{\emptyset} A \sqsubsetneq_{\emptyset} \top$. More details can be found in Section 5.1.8.

5.1.1 The Empty TBox

Since Baader and Morawska [BM10, Proof of Proposition 3.5] showed that $\sqsubseteq_{\emptyset}$ is bounded,[1] we can immediately draw the following conclusion.

[1]Note that it also follows that $\sqsupseteq_{\emptyset}$ is well-founded.

5.1.2 Proposition. *For any signature Σ, the subsumption relation $\sqsubseteq_\emptyset$ on $\mathcal{EL}(\Sigma)$ is neighborhood generated.* □

After this first promising result, we continue with describing the neighborhood relation $\prec_\emptyset$. As an immediate consequence of $\sqsubseteq_\emptyset$ being neighborhood generated, we can deduce that neighbors in arbitrary *directions* exist. More specifically, whenever $C \sqsubsetneq_\emptyset D$ holds true, there are U and L such that $C \prec_\emptyset U \sqsubseteq_\emptyset D$ as well as $C \sqsubseteq_\emptyset L \prec_\emptyset D$. We then also say that U is an upper neighbor of C *in direction* D and, dually, that L is some lower neighbor of D *in direction* C.

5.1.3 Lemma. *Let C and D be $\mathcal{EL}$ concept descriptions over a signature Σ. Then $C \prec_\emptyset D$ holds true only if* $\mathsf{rd}(C) \in \{\mathsf{rd}(D),\ \mathsf{rd}(D)+1\}$.[2]

Proof. Assume that C is a lower neighbor of D with respect to $\emptyset$. In particular, $C \sqsubseteq_\emptyset D$ follows, and so there is a simulation from the tree-shaped interpretation $(\mathcal{I}_D, D)$ to the tree-shaped interpretation $(\mathcal{I}_C, C)$. The mere existence of such a simulation yields that the depth of the tree $(\Delta^{\mathcal{I}_D}, \bigcup\{\, r^{\mathcal{I}_D} \mid r \in \Sigma_\mathsf{R} \,\})$ is bounded by the depth of the tree $(\Delta^{\mathcal{I}_C}, \bigcup\{\, r^{\mathcal{I}_C} \mid r \in \Sigma_\mathsf{R} \,\})$, that is, it must hold true that $\mathsf{rd}(D) \leq \mathsf{rd}(C)$.

Finally, assume that $\mathsf{rd}(C) > \mathsf{rd}(D) + 1$. Then $C \sqsubsetneq_\emptyset C\restriction_{\mathsf{rd}(D)+1} \sqsubsetneq_\emptyset D$. ↯ □

A Necessary Condition

There is a well-known recursive characterization of $\sqsubseteq_\emptyset$ as follows: $C \sqsubseteq_\emptyset D$ if, and only if, $A \in \mathsf{Conj}(D)$ implies $A \in \mathsf{Conj}(C)$ for each concept name A, and for each $\exists\, r.\, F \in \mathsf{Conj}(D)$, there is some $\exists\, r.\, E \in \mathsf{Conj}(C)$ such that $E \sqsubseteq_\emptyset F$. With the help of that we can prove that there is the following necessary condition for neighboring concept descriptions.

5.1.4 Lemma. *Let C and D be some reduced $\mathcal{EL}$ concept descriptions over a signature Σ. If $C \prec_\emptyset D$, then exactly one of the following statements holds true.*

1. *There is exactly one concept name $A \in \mathsf{Conj}(C)$ such that $C \equiv_\emptyset D \sqcap A$.*
2. *There is exactly one existential restriction $\exists\, r.\, E \in \mathsf{Conj}(C)$ such that $C \equiv_\emptyset D \sqcap \exists\, r.\, E$.*

Proof. Consider two reduced $\mathcal{EL}$ concept descriptions C and D over Σ such that C is a lower neighbor of D with respect to $\emptyset$. It follows that $C \sqsubseteq_\emptyset D$, which means that $A \in \mathsf{Conj}(D)$ implies $A \in \mathsf{Conj}(C)$ for any concept name $A \in \Sigma_\mathsf{C}$ and further that, for each existential restriction $\exists\, r.\, F \in \mathsf{Conj}(D)$, there is some $\exists\, r.\, E \in \mathsf{Conj}(C)$ such that $E \sqsubseteq_\emptyset F$.

If there exist two distinct concept names $A, B \in \Sigma_\mathsf{C}$ satisfying $\{A, B\} \subseteq \mathsf{Conj}(C) \setminus \mathsf{Conj}(D)$, then it would immediately follow that

$$C \sqsubseteq_\emptyset D \sqcap A \sqcap B \sqsubsetneq_\emptyset D \sqcap A \sqsubsetneq_\emptyset D,$$

which contradicts our assumption that $C \prec_\emptyset D$. ↯ Consequently, only one of the following two mutually exclusive cases can occur: either there is exactly one concept name $A \in \Sigma_\mathsf{C}$ such that $\{A\} = (\mathsf{Conj}(C) \setminus \mathsf{Conj}(D)) \cap \Sigma_\mathsf{C}$, or it holds true that $\mathsf{Conj}(C) \cap \Sigma_\mathsf{C} = \mathsf{Conj}(D) \cap \Sigma_\mathsf{C}$. We proceed with a case analysis.

[2]Note that the role depth $\mathsf{rd}(C)$ of a concept description C has been defined on Page 35.

1. Assume that $\{A\} = (\mathsf{Conj}(C) \setminus \mathsf{Conj}(D)) \cap \Sigma_\mathsf{C}$ holds true for some concept name A. It follows that $C \sqsubseteq_\emptyset D \sqcap A \sqsubsetneq_\emptyset D$, and so $C \prec_\emptyset D$ implies $C \equiv_\emptyset D \sqcap A$.

2. Now let $\mathsf{Conj}(C) \cap \Sigma_\mathsf{C} = \mathsf{Conj}(D) \cap \Sigma_\mathsf{C}$. Since $C \not\sqsupseteq_\emptyset D$ holds true by assumption, there must exist some existential restriction $\exists\, r.\, E \in \mathsf{Conj}(C)$ such that $E \not\sqsupseteq_\emptyset F$ for any $\exists\, r.\, F \in \mathsf{Conj}(D)$. In particular, we have that $D \not\sqsubseteq_\emptyset \exists\, r.\, E$. Now suppose that there are two such existential restrictions $\exists\, r.\, E$ and $\exists\, s.\, F$ on the top-level conjunction of C. Since C is assumed to be reduced, $\exists\, r.\, E$ and $\exists\, s.\, F$ are incomparable w.r.t. $\emptyset$. It follows that

$$C \sqsubseteq_\emptyset D \sqcap \exists\, r.\, E \sqcap \exists\, s.\, F \sqsubsetneq_\emptyset D \sqcap \exists\, r.\, E \sqsubsetneq_\emptyset D,$$

which obviously contradicts our assumption that $C \prec_\emptyset D$. As a consequence we obtain that there exists exactly one such existential restriction $\exists\, r.\, E \in \mathsf{Conj}(C)$ with $D \not\sqsubseteq_\emptyset \exists\, r.\, E$. It is now straightforward to conclude that $C \sqsubseteq_\emptyset D \sqcap \exists\, r.\, E \sqsubsetneq_\emptyset D$ is satisfied, which together with the precondition $C \prec_\emptyset D$ implies that $C \equiv_\emptyset D \sqcap \exists\, r.\, E$. □

Upper Neighborhood

5.1.5 Proposition. *Let C be a reduced $\mathcal{EL}$ concept description over some signature Σ, and recursively define*

$$\begin{aligned}\mathsf{Upper}(C) := \{\, &\textstyle\bigsqcap \mathsf{Conj}(C) \setminus \{A\} \mid A \in \mathsf{Conj}(C)\,\} \\ &\cup \{\, \textstyle\bigsqcap \mathsf{Conj}(C) \setminus \{\exists\, r.\, D\} \sqcap \bigsqcap \{\, \exists\, r.\, E \mid E \in \mathsf{Upper}(D)\,\} \mid \exists\, r.\, D \in \mathsf{Conj}(C)\,\}.\end{aligned}$$

Then $\mathsf{Upper}(C)$ contains, modulo equivalence, exactly all upper neighbors of C; more specifically, for each $\mathcal{EL}$ concept description D over Σ, it holds true that

$$C \prec_\emptyset D \quad \text{if, and only if,} \quad \mathsf{Upper}(C) \ni D' \text{ for some } D' \text{ with } D \equiv_\emptyset D'.$$

Proof. We show the claim by induction on the role depth of C. The *induction base* where $\mathsf{rd}(C) = 0$ is obvious. For the *induction step* let now $\mathsf{rd}(C) > 0$.

Soundness. It is easily verified that, for any concept name $A \in \mathsf{Conj}(C)$, the concept description $\bigsqcap \mathsf{Conj}(C) \setminus \{A\}$ is an upper neighbor of C. Now fix some existential restriction $\exists\, r.\, E \in \mathsf{Conj}(C)$ and let

$$D := \textstyle\bigsqcap \mathsf{Conj}(C) \setminus \{\exists\, r.\, E\} \sqcap \bigsqcap \{\, \exists\, r.\, F \mid F \in \mathsf{Upper}(E)\,\},$$

i.e., $D \in \mathsf{Upper}(C)$. We shall demonstrate that $C \prec_\emptyset D$.

1. It is easily verified that $C \sqsubseteq_\emptyset D$.

2. We proceed with proving that $C \not\sqsupseteq_\emptyset D$. In particular, we are going to show that there is no existential restriction $\exists\, r.\, F \in \mathsf{Conj}(D)$ such that $E \sqsupseteq_\emptyset F$. Assume that there was some such F. Since C is reduced, we infer that $\exists\, r.\, F \notin \mathsf{Conj}(C)$, and hence $E \prec_\emptyset F$. ↯

3. Let X be a concept description such that $C \sqsubsetneq_\emptyset X \sqsubseteq_\emptyset D$. We need to show that $X \sqsupseteq_\emptyset D$.

a) According to the definition of D, it holds true that $\mathsf{Conj}(C) \cap \Sigma_\mathsf{C} = \mathsf{Conj}(D) \cap \Sigma_\mathsf{C}$. Furthermore, the precondition $C \sqsubsetneq_\emptyset X \sqsubseteq_\emptyset D$ implies that $\mathsf{Conj}(C) \cap \Sigma_\mathsf{C} \supseteq \mathsf{Conj}(X) \cap \Sigma_\mathsf{C} \supseteq \mathsf{Conj}(D) \cap \Sigma_\mathsf{C}$. We conclude that $A \in \mathsf{Conj}(D)$ implies $A \in \mathsf{Conj}(X)$ for any concept name $A \in \Sigma_\mathsf{C}$.

b) Now consider an existential restriction $\exists\, s.\, Y \in \mathsf{Conj}(X)$. Then, $C \sqsubseteq_\emptyset X$ yields some $\exists\, s.\, G \in \mathsf{Conj}(C)$ satisfying $G \sqsubseteq_\emptyset Y$. If $s \neq r$ or $G \neq E$, then by definition of D we have that $\exists\, s.\, G \in \mathsf{Conj}(D)$ as well.

   Eventually, we consider the case where $s = r$ and $G = E$. As $C \not\sqsupseteq_\emptyset X$, there exists some $\exists\, t.\, K \in \mathsf{Conj}(C)$ such that $K \not\sqsupseteq_\emptyset Z$ for all $\exists\, t.\, Z \in \mathsf{Conj}(X)$. If $t \neq r$ or $K \neq E$, then $\exists\, t.\, K \in \mathsf{Conj}(D)$ follows and immediately yields a contradiction to $X \sqsubseteq_\emptyset D$. As a corollary it follows that $E \sqsubsetneq_\emptyset Y$, and so there must be an upper neighbor F of E with $F \sqsubseteq_\emptyset Y$. The induction hypothesis ensures the existence of some concept description F' with $F \equiv_\emptyset F'$ and $\exists\, r.\, F' \in \mathsf{Conj}(D)$.

Summing up, we have shown that $C \sqsubsetneq_\emptyset D$, and furthermore that, for each concept description X, it holds true that $C \sqsubsetneq_\emptyset X \sqsubseteq_\emptyset D$ implies $X \equiv_\emptyset D$. Hence, C is a lower neighbor of D with respect to $\emptyset$.

Completeness. Vice versa, consider a concept description D such that $C \prec_\emptyset D$. Without loss of generality suppose that both C and D are reduced. We have to show that, up to equivalence, $\mathsf{Upper}(C) \ni D$. In accordance with Lemma 5.1.4 we shall only consider two cases. In the first case, if $C \equiv_\emptyset D \sqcap A$ for some unique concept name $A \in \mathsf{Conj}(C)$, the claim is trivial.

In the second case, there exists exactly one existential restriction $\exists\, r.\, E \in \mathsf{Conj}(C)$ such that $C \equiv_\emptyset D \sqcap \exists\, r.\, E$. We have already proven that $C \prec_\emptyset D'$ where

$$D' := \bigsqcap \mathsf{Conj}(C) \setminus \{\exists\, r.\, E\} \sqcap \bigsqcap \{\, \exists\, r.\, F \mid F \in \mathsf{Upper}(E)\, \}.$$

We proceed with demonstrating that $D' \sqsubseteq_\emptyset D$, which then immediately yields that $D' \equiv_\emptyset D$, and thus $D \in \mathsf{Upper}(C)$ modulo equivalence.

If $A \in \mathsf{Conj}(D)$, then it follows that $A \in \mathsf{Conj}(C)$ and further that $A \in \mathsf{Conj}(D')$. Now fix some existential restriction $\exists\, s.\, H \in \mathsf{Conj}(D)$. Then, there exists some $\exists\, s.\, G \in \mathsf{Conj}(C)$ satisfying $G \sqsubseteq_\emptyset H$. In case $s \neq r$ or $G \neq E$ we have that $\exists\, s.\, G \in \mathsf{Conj}(D')$ as well. Otherwise, $E \sqsubseteq_\emptyset H$ holds true. If $H \sqsubseteq_\emptyset E$ would hold true too, then it would follow that $D \sqsubseteq_\emptyset D \sqcap \exists\, r.\, E$, yielding $D \sqsubseteq_\emptyset C$—a contradiction to our assumption that $C \prec_\emptyset D$. So, we conclude that $E \sqsubsetneq_\emptyset H$. Thus, there exists an F with $E \prec_\emptyset F \sqsubseteq_\emptyset H$. The induction hypothesis shows the existence of some $F' \in \mathsf{Upper}(E)$ with $F' \equiv_\emptyset F$, and then $\exists\, r.\, F' \in \mathsf{Conj}(D')$ is satisfied. □

For instance, consider the concept description $A \sqcap \exists\, r.\, B \sqcap \exists\, s.\, (A \sqcap B)$. It is in reduced form and has three upper neighbors, namely $\exists\, r.\, B \sqcap \exists\, s.\, (A \sqcap B)$, $A \sqcap \exists\, r.\, \top \sqcap \exists\, s.\, (A \sqcap B)$, and $A \sqcap \exists\, r.\, B \sqcap \exists\, s.\, A \sqcap \exists\, s.\, B$.

According to Proposition 5.1.5, each top-level conjunct D of some concept description C has exactly one upper neighbor $D^\uparrow$. If C is reduced, then replacing D with $D^\uparrow$ yields one upper neighbor of C, and (an equivalent concept description of) each upper neighbor of C can be generated in this manner. We denote the concept description that is produced from C by replacing

D with $D^{\uparrow}$ by $C^{\uparrow D}$. It then holds true that

$$\mathsf{Upper}(C) = \{\, C^{\uparrow D} \mid D \in \mathsf{Conj}(C) \,\}$$

modulo equivalence. Furthermore, there is a bijection between $\mathsf{Conj}(C)$ and $\mathsf{Upper}(C)$, cf. the next proposition.

5.1.6 Proposition. *Let C be some reduced $\mathcal{EL}$ concept description. The mapping*

$$\begin{aligned} \upsilon_C \colon \mathsf{Conj}(C) &\to \mathsf{Upper}(C) \\ D &\mapsto C^{\uparrow D} \end{aligned}$$

is bijective, that is, $|\mathsf{Conj}(C)| = |\mathsf{Upper}(C)|$ holds true.

Proof. Apparently, υ_C is surjective. We proceed with demonstrating that it is injective as well. It is readily verified that removing one of the concept names on the top-level conjunction of C yields one unique upper neighbor, that is, if $A, B \in \mathsf{Conj}(C)$ with $A \neq B$, then $C^{\uparrow A} = \bigsqcap \mathsf{Conj}(C) \setminus \{A\}$ and $C^{\uparrow B} = \bigsqcap \mathsf{Conj}(C) \setminus \{B\}$ are non-equivalent upper neighbors of C. Analogous statements obviously hold true for top-level conjuncts A and $\exists\, r.\, D$, or $\exists\, r.\, D$ and $\exists\, s.\, E$ where $r \neq s$.

Eventually, assume that $\exists\, r.\, D$ and $\exists\, r.\, E$ are top-level conjuncts of C. Since we have assumed C to be reduced, D and E are incomparable, i.e., it holds true that $D \not\sqsubseteq_\emptyset E$ as well as $E \not\sqsubseteq_\emptyset D$. These two conjuncts induce the following upper neighbors.

$$\begin{aligned} C^{\uparrow \exists\, r.\, D} &= \bigsqcap \mathsf{Conj}(C) \setminus \{\exists\, r.\, D\} \sqcap \bigsqcap \{\, \exists\, r.\, F \mid F \in \mathsf{Upper}(D) \,\} \\ C^{\uparrow \exists\, r.\, E} &= \bigsqcap \mathsf{Conj}(C) \setminus \{\exists\, r.\, E\} \sqcap \bigsqcap \{\, \exists\, r.\, F \mid F \in \mathsf{Upper}(E) \,\} \end{aligned}$$

For proving that $C^{\uparrow \exists\, r.\, D}$ and $C^{\uparrow \exists\, r.\, E}$ are incomparable, we assume the contrary, i.e., let $C^{\uparrow \exists\, r.\, D} \sqsubseteq_\emptyset C^{\uparrow \exists\, r.\, E}$. Since $\exists\, r.\, D \in \mathsf{Conj}(C^{\uparrow \exists\, r.\, E})$, there must exist some $\exists\, r.\, G \in \mathsf{Conj}(C^{\uparrow \exists\, r.\, D})$ satisfying $G \sqsubseteq_\emptyset D$. As C is reduced, it cannot be the case that $\exists\, r.\, G \in \mathsf{Conj}(C) \setminus \{\exists\, r.\, D\}$; it can henceforth only happen that $\exists\, r.\, G \in \{\, \exists\, r.\, F \mid F \in \mathsf{Upper}(D) \,\}$, i.e., $G = F$ for some $F \in \mathsf{Upper}(D)$. Thus, we have that $F = G \sqsubseteq_\emptyset D \prec_\emptyset F$—a contradiction. ↯ □

5.1.7 Lemma. *Let $C \in \mathcal{EL}(\Sigma)$ be a concept description. For each set $\mathbf{D}$ containing only upper neighbors of C and at least two incomparable upper neighbors of C, it holds true that $C \equiv_\emptyset \bigsqcap \mathbf{D}$.*

Proof. Without loss of generality let C be reduced. Assume that $\mathbf{D}$ consists of upper neighbors of C only and further contains two incomparable upper neighbors D and E of C. In particular, there must exist incomparable top-level conjuncts $X, Y \in \mathsf{Conj}(C)$ such that $D \equiv_\emptyset C^{\uparrow X}$ and $E \equiv_\emptyset C^{\uparrow Y}$. Obviously, it now follows that

$$C \sqsubseteq_\emptyset \bigsqcap \mathbf{D} \sqsubseteq_\emptyset D \sqcap E \equiv_\emptyset C^{\uparrow X} \sqcap C^{\uparrow Y} \equiv_\emptyset C. \qquad \square$$

Lower Neighborhood: A First Characterization

5.1.8 Proposition. *For an $\mathcal{EL}$ concept description C over some signature Σ, let*

$$\begin{aligned}\mathsf{Lower}(C) := {} & \{\, C \sqcap A \mid A \in \Sigma_{\mathsf{C}} \text{ and } C \not\sqsubseteq_{\emptyset} A \,\} \\ & \cup \{\, C \sqcap \exists\, r.\, D \mid r \in \Sigma_{\mathsf{R}},\ C \not\sqsubseteq_{\emptyset} \exists\, r.\, D, \text{ and } C \sqsubseteq_{\emptyset} \exists\, r.\, E \text{ for all } E \text{ with } D \prec_{\emptyset} E \,\}.\end{aligned}$$

Then $\mathsf{Lower}(C)$ *contains, modulo equivalence, exactly all lower neighbors of C; more specifically, for each $\mathcal{EL}$ concept description D over Σ, it holds true that*

$$D \prec_{\emptyset} C \quad \text{if, and only if,} \quad D' \in \mathsf{Lower}(C) \text{ for some } D' \text{ with } D \equiv_{\emptyset} D'.$$

Proof. Soundness. We begin with proving *soundness*. Thus, fix some $L \in \mathsf{Lower}(C)$ and, without loss of generality, let C be reduced. If $L = C \sqcap A$ for some concept name A with $C \not\sqsubseteq_{\emptyset} A$, then it is apparent that L is a lower neighbor of C. Henceforth, suppose $L = C \sqcap \exists\, r.\, D$ for some role name r and a concept description D which satisfies $C \not\sqsubseteq_{\emptyset} \exists\, r.\, D$ as well as $C \sqsubseteq_{\emptyset} \exists\, r.\, E$ for each upper neighbor E of D. Then, it follows that $L \sqsubsetneq_{\emptyset} C$. Furthermore, C is obviously equivalent to the concept description

$$C' := C \sqcap \bigsqcap \{\, \exists\, r.\, E \mid E \in \mathsf{Upper}(D) \,\},$$

and it is readily verified that $C' \in \mathsf{Upper}(L)$. Proposition 5.1.5 shows that $L \prec_{\emptyset} C'$ holds true, which yields $L \prec_{\emptyset} C$.

Completeness. We continue with showing *completeness*. For this purpose, consider a lower neighbor L of C. Without loss of generality, assume that both C and L are reduced. According to Lemma 5.1.4, two mutually exclusive cases can occur. In the first case there exists a concept name A such that $L \equiv_{\emptyset} C \sqcap A$. Clearly, $C \not\sqsubseteq_{\emptyset} A$ must hold true, as otherwise $L \equiv_{\emptyset} C \sqcap A \equiv_{\emptyset} C$. ↯ We conclude that $C \sqcap A \in \mathsf{Lower}(C)$. In the second case, there is exactly one existential restriction $\exists\, r.\, D \in \mathsf{Conj}(L)$ such that $L \equiv_{\emptyset} C \sqcap \exists\, r.\, D$. Since $L \prec_{\emptyset} C$ holds true and Proposition 5.1.5 yields that

$$L \equiv_{\emptyset} C \sqcap \exists\, r.\, D \prec_{\emptyset} C \sqcap \bigsqcap \{\, \exists\, r.\, E \mid E \in \mathsf{Upper}(D) \,\} \sqsubseteq_{\emptyset} C,$$

it follows that $C \not\sqsubseteq_{\emptyset} \exists\, r.\, D$ as well as $C \equiv_{\emptyset} C \sqcap \bigsqcap \{\, \exists\, r.\, E \mid E \in \mathsf{Upper}(D) \,\}$, or equivalently, that $C \sqsubseteq_{\emptyset} \exists\, r.\, E$ for all E with $D \prec_{\emptyset} E$. Summing up, we have shown that $C \sqcap \exists\, r.\, D \in \mathsf{Lower}(C)$. □

While the recursive characterization of Upper in Proposition 5.1.5 immediately yields a procedure for enumerating all upper neighbors of a given concept description, the situation is not that apparent for lower neighbors. We can, however, formulate a procedure for computing lower neighbors by means of Proposition 5.1.8. Let C be an $\mathcal{EL}$ concept description over some signature Σ. Proceed as follows.

1. For each concept name $A \in \Sigma_{\mathsf{C}}$ with $C \not\sqsubseteq_{\emptyset} A$, output $C \sqcap A$ as a lower neighbor of C.
2. For each role name $r \in \Sigma_{\mathsf{R}}$, recursively proceed as follows.
   a) Let $D := \top$.

b) While $C \sqsubseteq_\emptyset \exists r. D$, replace D with a lower neighbor of D.

c) If $C \sqsubseteq_\emptyset \exists r. E$ for all E with $D \prec_\emptyset E$, then output $C \sqcap \exists r. D$ as a lower neighbor of C.

As we shall infer from the results in Section 5.6, the above algorithm always terminates but has non-elementary time complexity. Thus, we are going to develop a more efficient procedure for enumerating all lower neighbors of a given $\mathcal{EL}$ concept description in the next section. A complexity analysis shows that the proposed procedure needs only non-deterministic polynomial time or deterministic exponential time, and that there indeed exist $\mathcal{EL}$ concept descriptions with an exponential number of lower neighbors.

Lower Neighborhood: A More Efficient Characterization

According to Proposition 5.1.8, we can enumerate all lower neighbors of the form $C \sqcap A$ by simply iterating through the set of concept names while checking, for each such $A \in \Sigma_\mathsf{C}$, whether $C \not\sqsubseteq_\emptyset A$ or, equivalently, whether $A \notin \mathsf{Conj}(C)$ is satisfied and if so, then output $C \sqcap A$ as a lower neighbor of C. Clearly, this can be done in polynomial time with respect to the size of C plus the size of Σ.

Let $C \in \mathcal{EL}(\Sigma)$ be some reduced concept description and consider a role name $r \in \Sigma_\mathsf{R}$. We define the set of *r-successors* of C as

$$\mathsf{Succ}(C, r) := \{\, D \mid \exists r. D \in \mathsf{Conj}(C) \,\}.$$

Then, for each subset $\mathbf{S} \subseteq \mathsf{Succ}(C, r)$, we define a mapping $\mathsf{Choices}_\mathbf{S} \colon \mathbf{S} \to \wp(\mathcal{EL}(\Sigma))$ as follows.

$$\mathsf{Choices}_\mathbf{S} \colon F \mapsto \{\, X \mid X \in \mathcal{EL}(\Sigma) \text{ such that } F \sqcap X \prec_\emptyset F \text{ and } F' \sqsubseteq_\emptyset X \text{ for each } F' \in \mathbf{S} \setminus \{F\} \,\}$$

According to Proposition 5.1.8, each such set $\mathsf{Choices}_\mathbf{S}(F)$ contains only atomic concept descriptions, i.e., concept descriptions that are either a concept name or some existential restriction. In the following, we consider choice functions in $\bigtimes \mathsf{Choices}_\mathbf{S} := \bigtimes \{\, \mathsf{Choices}_\mathbf{S}(F) \mid F \in \mathbf{S} \,\}$. We call some such choice function $\chi \in \bigtimes \mathsf{Choices}_\mathbf{S}$ *admissible* if $C \not\sqsubseteq_\emptyset \exists r. \bigsqcap \mathsf{Ran}(\chi)$.

5.1.9 Lemma. *A choice function $\chi \in \bigtimes \mathsf{Choices}_\mathbf{S}$ is admissible if, and only if, $\overline{F} \not\sqsubseteq_\emptyset \bigsqcap \mathsf{Ran}(\chi)$ for each $\overline{F} \in \mathsf{Succ}(C, r) \setminus \mathbf{S}$.*

Proof. Fix some $\chi \in \bigtimes \mathsf{Choices}_\mathbf{S}$. The *only if* direction is obvious. We continue with proving the *if* direction, for which it suffices to show that $D \not\sqsubseteq_\emptyset \bigsqcap \mathsf{Ran}(\chi)$ holds true for any $D \in \mathsf{Succ}(C, r)$. By assumption, this is satisfied for each $D \in \mathsf{Succ}(C, r) \setminus \mathbf{S}$. Furthermore, the above definition shows that $F \not\sqsubseteq_\emptyset \chi(F)$ for each $F \in \mathbf{S}$, which immediately implies that $F \not\sqsubseteq_\emptyset \bigsqcap \mathsf{Ran}(\chi)$ for any $F \in \mathbf{S}$, and we are done. □

5.1.10 Proposition. *A concept description $C \sqcap \exists r. D$ is a lower neighbor of C if there is some subset $\mathbf{S} \subseteq \mathsf{Succ}(C, r)$ as well as an admissible choice function $\chi \in \bigtimes \mathsf{Choices}_\mathbf{S}$ such that $D \equiv_\emptyset \bigsqcap \mathsf{Ran}(\chi)$.*

Proof. Since χ is admissible, we have that $C \not\sqsubseteq_\emptyset \exists r. D$. Thus, in order to show that $C \sqcap \exists r. D$ is a lower neighbor of C it remains to prove that $C \sqsubseteq_\emptyset \exists r. E$ for any upper neighbor E of D, cf. Proposition 5.1.8.

We proceed with showing that all concept descriptions in $\mathsf{Ran}(\chi)$ are mutually incomparable, which implies that, modulo equivalence, the upper neighbors of $\bigsqcap \mathsf{Ran}(\chi)$ are exactly the concept descriptions $\bigsqcap \mathsf{Ran}(\chi)^{\uparrow \chi(F)}$ for $F \in \mathbf{S}$. If $F, F' \in \mathbf{S}$ are incomparable, then X and X' are incomparable as well for any $X \in \mathsf{Choices}_{\mathbf{S}}(F)$ and for any $X' \in \mathsf{Choices}_{\mathbf{S}}(F')$: otherwise it would hold true that $F' \sqsubseteq_\emptyset X \sqsubseteq_\emptyset X'$ or $F \sqsubseteq_\emptyset X' \sqsubseteq_\emptyset X$, which both yields a contradiction. ↯ Consequently, all top-level conjuncts in $\bigsqcap \mathsf{Ran}(\chi)$ must be mutually incomparable, and so there are bijections between $\mathbf{S}$, $\mathsf{Ran}(\chi)$, and $\mathsf{Upper}(\bigsqcap \mathsf{Ran}(\chi))$.

Fix some $F \in \mathbf{S}$, i.e., $\chi(F)$ is a top-level conjunct in $\bigsqcap \mathsf{Ran}(\chi)$ and $\bigsqcap \mathsf{Ran}(\chi)^{\uparrow \chi(F)}$ is an upper neighbor of $\bigsqcap \mathsf{Ran}(\chi)$. Then, we have that $F \sqcap \chi(F) \prec_\emptyset F$, and $F \sqsubseteq_\emptyset \chi(F')$ for each $F' \in \mathbf{S} \setminus \{F\}$. Furthermore, it holds true that $F \sqcap \chi(F) \sqsubseteq_\emptyset F \sqcap \chi(F)^\uparrow \sqsubseteq_\emptyset F$. Now assume that $F \sqcap \chi(F)^\uparrow \sqsubseteq_\emptyset F \sqcap \chi(F)$ would be satisfied, which would imply that $F \sqcap \chi(F)^\uparrow \sqsubseteq_\emptyset \chi(F)$. Since $\chi(F)^\uparrow \sqsubseteq_\emptyset \chi(F)$ cannot hold true, we would infer that $F \sqsubseteq_\emptyset \chi(F)$. However, this yields the contradiction $F \equiv_\emptyset F \sqcap \chi(F) \prec_\emptyset F$. ↯ We conclude that $F \sqcap \chi(F) \sqsubsetneq_\emptyset F \sqcap \chi(F)^\uparrow \sqsubseteq_\emptyset F$, which together with the precondition $F \sqcap \chi(F) \prec_\emptyset F$ implies that $F \sqcap \chi(F)^\uparrow \equiv_\emptyset F$. Clearly, this implies that $F \sqsubseteq_\emptyset \bigsqcap \mathsf{Ran}(\chi)^{\uparrow \chi(F)}$ and, thus, $C \sqsubseteq_\emptyset \exists\, r. \bigsqcap \mathsf{Ran}(\chi)^{\uparrow \chi(F)}$. □

5.1.11 Lemma. *Let $C \sqcap \exists\, r. D$ be a lower neighbor of C where both C and D are reduced. Then, there is a mapping $\phi\colon \mathsf{Conj}(D) \to \mathsf{Succ}(C, r)$ with the following properties.*

1. *$\phi(X) \sqsubseteq_\emptyset D^{\uparrow X}$ for each $X \in \mathsf{Conj}(D)$*
2. *ϕ is injective*
3. *$\phi(X) \not\sqsubseteq_\emptyset X$ for any top-level conjunct $X \in \mathsf{Conj}(D)$*
4. *$\phi(Y) \sqsubseteq_\emptyset X$ for any two mutually distinct $X, Y \in \mathsf{Conj}(D)$*

Proof. Fix reduced concept descriptions $C, D \in \mathcal{EL}^\bot(\Sigma)$ and some role name $r \in \Sigma_{\mathsf{R}}$ such that $C \sqcap \exists\, r. D \prec_\emptyset C$. An application of Proposition 5.1.8 yields that $C \not\sqsubseteq_\emptyset \exists\, r. D$ and $C \sqsubseteq_\emptyset \exists\, r. E$ for each upper neighbor E of D.

1. We start with defining such a mapping $\phi\colon \mathsf{Conj}(D) \to \mathsf{Succ}(C, r)$. Fix some top-level conjunct $X \in \mathsf{Conj}(D)$. Then, $D^{\uparrow X}$ is an upper neighbor of D. Since $C \sqsubseteq_\emptyset \exists\, r. D^{\uparrow X}$ is satisfied according to the preconditions, we conclude that there exists some successor $F_X \in \mathsf{Succ}(C, r)$ such that $F_X \sqsubseteq_\emptyset D^{\uparrow X}$. Thus, we can set $\phi(X) := F_X$.
2. We now show that ϕ is injective. Assume the contrary, i.e., there are two non-equivalent top-level conjuncts $X, Y \in \mathsf{Conj}(D)$ such that $\phi(X) = \phi(Y)$. It then holds true that $\phi(X) \sqsubseteq_\emptyset D^{\uparrow X} \sqcap D^{\uparrow Y}$. Now Lemma 5.1.7 implies that $D^{\uparrow X} \sqcap D^{\uparrow Y} \equiv_\emptyset D$, which contradicts the assumption that $C \not\sqsubseteq_\emptyset \exists\, r. D$. ↯
3. Assume to the contrary that $\phi(X) \sqsubseteq_\emptyset X$ is satisfied. Of course, it then immediately follows that $\phi(X) \sqsubseteq_\emptyset X \sqcap D^{\uparrow X} \equiv_\emptyset D$ would be satisfied, which contradicts the assumption that $C \not\sqsubseteq_\emptyset \exists\, r. D$. ↯
4. Let X be a top-level conjunct of D. It then follows that $D^{\uparrow X}$ is some upper neighbor of D and, for each upper neighbor E of D that is incomparable to $D^{\uparrow X}$, it holds true that $E \sqsubseteq_\emptyset X$, cf. Proposition 5.1.5. Fix a further top-level conjunct $Y \in \mathsf{Conj}(D)$ that is incomparable to X. Of course, $D^{\uparrow Y}$ is incomparable to $D^{\uparrow X}$, since $\upsilon_D\colon Z \mapsto D^{\uparrow Z}$ is a bijection between $\mathsf{Conj}(D)$ and $\mathsf{Upper}(D)$, cf. Proposition 5.1.6. We conclude that $\phi(Y) \sqsubseteq_\emptyset D^{\uparrow Y} \sqsubseteq_\emptyset X$. □

As a corollary we obtain that $|\mathsf{Conj}(D)| = |\mathsf{Upper}(D)| \leq |\mathsf{Succ}(C, r)|$ holds true.

5.1.12 Proposition. *A concept description $C \sqcap \exists r. D$ is a lower neighbor of C only if there is some subset $\mathbf{S} \subseteq \mathsf{Succ}(C, r)$ as well as an admissible choice function $\chi \in \bigtimes \mathsf{Choices}_{\mathbf{S}}$ such that $D \equiv_\emptyset \bigsqcap \mathsf{Ran}(\chi)$.*

Proof. We know that there is some injective mapping $\phi\colon \mathsf{Conj}(D) \to \mathsf{Succ}(C, r)$ with all the properties stated in Lemma 5.1.11. Set $\mathbf{S} := \mathsf{Ran}(\phi)$, and define a mapping χ by $\chi(F) := X$ if $F = \phi(X)$.

We proceed with showing that $\chi(F) \in \mathsf{Choices}_{\mathbf{S}}(F)$ for each $F \in \mathbf{S}$, and for this purpose we have to show that $F \sqcap \chi(F) \prec_\emptyset F$ and $F' \sqsubseteq_\emptyset \chi(F)$ for each $F' \in \mathbf{S} \setminus \{F\}$. Fix some $F \in \mathbf{S}$. Then, $\chi(F) = X$ if, and only if, $F = \phi(X)$.

- We have that $\phi(X) \sqsubseteq_\emptyset D^{\uparrow X}$. In particular, this implies that $\phi(X) \sqsubseteq_\emptyset X^\uparrow$, that is, $F \sqsubseteq_\emptyset \chi(F)^\uparrow$. From $\phi(X) \not\sqsubseteq_\emptyset X$, we immediately infer that $F \not\sqsubseteq_\emptyset \chi(F)$. It follows that $F \sqcap \chi(F) \sqsubsetneq_\emptyset F$ and, since $(F \sqcap \chi(F))^{\uparrow \chi(F)} = F \sqcap \chi(F)^\uparrow \equiv_\emptyset F$ is an upper neighbor of $F \sqcap \chi(F)$, we conclude that $F \sqcap \chi(F)$ is a lower neighbor of F as claimed.
- Furthermore, we have that $\phi(Y) \sqsubseteq_\emptyset X$ for each $Y \in \mathsf{Conj}(D) \setminus \{X\}$. If we now consider some $F' \in \mathbf{S} \setminus \{F\}$, then there is some $Y \in \mathsf{Conj}(D) \setminus \{X\}$ satisfying $\chi(F') = Y$ or, equivalently, $F' = \phi(Y)$. We conclude that $F' \sqsubseteq_\emptyset \chi(F)$.

Summing up, we have that χ is a choice function in $\bigtimes \mathsf{Choices}_{\mathbf{S}}$. Obviously, it holds true that $D \equiv_\emptyset \bigsqcap \mathsf{Ran}(\chi)$ and, thus, χ is admissible. □

5.1.13 Corollary. *Let $C \in \mathcal{EL}(\Sigma)$ be some reduced concept description and define the following.*

$$\begin{aligned} \mathsf{Lower}^*(C) = \{\, C \sqcap A \mid A \in \Sigma_{\mathsf{C}} \text{ and } C \not\sqsubseteq_\emptyset A \,\} \\ \cup \left\{ C \sqcap \exists r. \bigsqcap \mathsf{Ran}(\chi) \;\middle|\; \begin{array}{l} r \in \Sigma_{\mathsf{R}} \text{ and there exists some } \mathbf{S} \subseteq \mathsf{Succ}(C, r) \\ \text{such that } \chi \in \bigtimes \mathsf{Choices}^*_{\mathbf{S}} \text{ and } \chi \text{ is admissible} \end{array} \right\} \end{aligned}$$

Note that, for each subset $\mathbf{S} \subseteq \mathsf{Succ}(C, r)$, we define the mapping $\mathsf{Choices}^*_{\mathbf{S}}\colon \mathbf{S} \to \wp(\mathcal{EL}(\Sigma))$ slightly different from $\mathsf{Choices}_{\mathbf{S}}$, namely as follows.

$$\mathsf{Choices}^*_{\mathbf{S}}\colon F \mapsto \{\, X \mid X \in \mathcal{EL}(\Sigma) \text{ where } F \sqcap X \in \mathsf{Lower}^*(F) \text{ and } F' \sqsubseteq_\emptyset X \text{ for each } F' \in \mathbf{S} \setminus \{F\} \,\}$$

Then $\mathsf{Lower}^*(C)$ contains, modulo equivalence, exactly all lower neighbors of C; more specifically, for each $\mathcal{EL}$ concept description D over Σ, it holds true that

$$D \prec_\emptyset C \quad \text{if, and only if,} \quad D' \in \mathsf{Lower}^*(C) \text{ for some } D' \text{ with } D \equiv_\emptyset D'.$$ □

5.1.14 Proposition. *For a fixed concept description C as well as a fixed role name r, all admissible choice functions are incomparable with respect to $\subseteq$. In particular, if $\mathbf{S} \subsetneq \mathbf{T} \subseteq \mathsf{Succ}(C, r)$, then there does not exist admissible choice functions $\chi \in \bigtimes \mathsf{Choices}^*_{\mathbf{S}}$ and $\psi \in \bigtimes \mathsf{Choices}^*_{\mathbf{T}}$ such that $\chi \subseteq \psi$.*

Proof. Consider some $G \in \mathbf{T} \setminus \mathbf{S}$. Further assume that $\chi \in \mathsf{Choices}^*_{\mathbf{S}}$ is admissible, i.e., it follows that $G \not\sqsubseteq_\emptyset \bigsqcap \mathsf{Ran}(\chi)$, which shows that there exists some $F \in \mathbf{S}$ such that $G \not\sqsubseteq_\emptyset \chi(F)$. Consequently, we cannot extend χ to some (admissible) choice function ψ in $\bigtimes \mathsf{Choices}_{\mathbf{T}}$. □

5.1.15 Corollary. *For any reduced $\mathcal{EL}$ concept description C, it holds true that all lower neighbors in $\mathsf{Lower}^*(C)$ are mutually incomparable.* □

Computational Complexity

Eventually, we finish our investigations of $\prec_\emptyset$ with analyzing the computational complexity of three problems related to the neighborhood of $\mathcal{EL}$ concept descriptions. In particular, we shall prove the following results.

- $\prec_\emptyset$ is in **P**.
- Upper can be computed in deterministic quadratic time. In particular, each upper neighbor in $\mathsf{Upper}(C)$ has a quadratic size, and $\mathsf{Upper}(C)$ has a linear cardinality.
- Lower^* can be computed in deterministic exponential time. Furthermore, any lower neighbor in $\mathsf{Lower}^*(C)$ has a quadratic size, and $\mathsf{Lower}^*(C)$ has an exponential cardinality.
- There is a non-deterministic polynomial time procedure which on input C has one (successful) computation path that returns a concept description equivalent to L for any lower neighbor L of C.

Enumerating all Upper Neighbors

5.1.16 Proposition. *The mapping* Upper *can be computed in deterministic polynomial time. More specifically,* $\mathsf{Upper}(C)$ *can be enumerated in deterministic quadratic time w.r.t.* $||C||$ *for each reduced $\mathcal{EL}$ concept description C.*

Proof. We could try to prove the claim by induction on the role depth of C. However, the straightforward attempt to do so would only yield that $\mathsf{Upper}(C)$ is computable in deterministic time $\mathcal{O}(||C||^{\mathsf{rd}(C)+2})$. Thus, we shall follow a more sophisticated approach.

For a finite set $\mathbf{C}$ of reduced $\mathcal{EL}$ concept descriptions, its size is defined by $||\mathbf{C}|| := \sum(\, ||C|| \mid C \in \mathbf{C}\,)$; further let

$$\begin{aligned} \mathsf{Upper}(\mathbf{C})\colon \mathbf{C} &\to \wp(\mathcal{EL}(\Sigma)) \\ C &\mapsto \mathsf{Upper}(C), \end{aligned}$$

and the size of $\mathsf{Upper}(\mathbf{C})$ is defined as $||\mathsf{Upper}(\mathbf{C})|| := \sum(\, ||\mathsf{Upper}(C)|| \mid C \in \mathbf{C}\,)$. More generally, we shall show by induction on the maximal role depth $\mathsf{rd}(\mathbf{C}) := \bigvee\{\, \mathsf{rd}(C) \mid C \in \mathbf{C}\,\}$ that $\mathsf{Upper}(\mathbf{C})$ can be computed in deterministic time $\mathcal{O}(||\mathbf{C}||^2)$, which implies that $||\mathsf{Upper}(\mathbf{C})|| \in \mathcal{O}(||\mathbf{C}||^2)$.

The induction base where $\mathsf{rd}(\mathbf{C}) = 0$ is obvious. For the induction step assume $\mathsf{rd}(\mathbf{C}) > 0$. For computing a single $\mathsf{Upper}(C)$ we can proceed as follows. For each top-level conjunct of C, create a fresh copy of C. Clearly, the number of copies is bounded by $||C||$, and creating these copies hence takes time quadratic in $||C||$. From some of those copies one concept name is removed, which reduces the size of that copy, and one removal needs constant time. The sequence of these removal operations thus requires time linear in $||C||$. Furthermore, for some other copies, a top-level conjunct $\exists\, r.\, D$ is replaced by $\bigsqcap\{\, \exists\, r.\, E \mid E \in \mathsf{Upper}(D)\,\}$. Let

$\mathsf{Succ}(C) := \bigcup\{\, \mathsf{Succ}(C,r) \mid r \in \Sigma_\mathsf{R} \,\} = \{\, D \mid \exists\, r.\, D \in \mathsf{Conj}(C) \,\}$. By induction hypothesis, the object $\mathsf{Upper}(\mathsf{Succ}(C))$ can be computed in deterministic time $\mathcal{O}(||\mathsf{Succ}(C)||^2)$ and has size $\mathcal{O}(||\mathsf{Succ}(C)||^2)$. It is apparent that $||\mathsf{Succ}(C)|| \leq ||C||$, and henceforth $\mathsf{Upper}(\mathsf{Succ}(C))$ can be computed in time $\mathcal{O}(||C||^2)$ and has size $\mathcal{O}(||C||^2)$. For each top-level conjunct $\exists\, r.\, D \in \mathsf{Conj}(C)$, we choose a distinct and so far untouched copy of C, remove $\exists\, r.\, D$, which takes time linear in $||C||$, and add $\exists\, r.\, E$ as new top-level conjunct for each $E \in \mathsf{Upper}(D)$, which takes constant time for finding $\mathsf{Upper}(D)$ within $\mathsf{Upper}(\mathsf{Succ}(C))$ if $\mathsf{Upper}(\mathsf{Succ}(C))$ is computed as a function like above, and requires constant time for adding each $\exists\, r.\, E$ for $E \in \mathsf{Upper}(D)$ as a new top-level conjunct, since E is already computed and we only need to link it to the copy we are editing. Since the number of top-level conjuncts of C which are existential restrictions is bounded by $||C||$, and each replacement takes linear time in $||C||$, as the number of concept descriptions in each $\mathsf{Upper}(D)$ is bounded by $|\mathsf{Conj}(C)| \leq ||C||$, we conclude that only quadratic time in $||C||$ is necessary for the replacement of the existential restrictions. Furthermore, the size of $\mathsf{Upper}(C)$ is quadratic in $||C||$ too, since in the set of the in $||C||$ linearly many copies of C we have removed some nodes and edges, and have added existential restrictions the fillers of which are from the in $||C||$ quadratically sized $\mathsf{Upper}(\mathsf{Succ}(C))$. Finally, if we consider the task of computing $\mathsf{Upper}(\mathbf{C})$, then we can compute, for each $C \in \mathbf{C}$, the set $\mathsf{Upper}(C)$ in time $\mathcal{O}(||C||^2)$ and collect the results in a function. Clearly, this takes $\mathcal{O}(\sum(\, ||C||^2 \mid C \in \mathbf{C}\,)) = \mathcal{O}(||\mathbf{C}||^2)$ time, and $||\mathsf{Upper}(\mathbf{C})||$ can similarly be bounded. □

Deciding Neighborhood

5.1.17 Theorem. *It holds true that* $\prec_\emptyset \in \mathbf{P}$. *More specifically, we can decide in deterministic polynomial time w.r.t.* $||C|| + ||D||$ *whether* C *is a lower neighbor of* D *for any* $\mathcal{EL}$ *concept descriptions* C *and* D.

Proof. We leave out picky details like the encoding of $\mathcal{EL}$ concept descriptions, and recognizing correctly encoded $\mathcal{EL}$ concept descriptions. So, assume that C and D are $\mathcal{EL}$ concept descriptions. We want to show the existence of a procedure which, given C and D as input, decides in deterministic polynomial time whether C is a lower neighbor of D with respect to $\emptyset$. Such a procedure can, e.g., work as follows for input concept descriptions C and D.

1. Reduce C.
2. Compute $\mathsf{Upper}(C)$.
3. Check whether there is some concept description $E \in \mathsf{Upper}(C)$ such that $D \equiv_\emptyset E$. If yes, accept (C, D), and otherwise reject (C, D).

Step 1 needs polynomial time in $||C||$. Step 2 also needs polynomial time in $||C||$, cf. Proposition 5.1.16. Since $\sqsubseteq_\emptyset \in \mathbf{P}$ holds true, $|\mathsf{Upper}(C)| \leq |\mathsf{Conj}(C)| \leq ||C||$ is satisfied, and $||E|| \leq ||\mathsf{Upper}(C)|| \in \mathcal{O}(||C||^2)$ for each $E \in \mathsf{Upper}(C)$, we infer that, for some n that is the exponent for deciding $\sqsubseteq_\emptyset$, Step 3 requires deterministic time in $\mathcal{O}(||C|| \cdot (||C||^2 + ||D||)^n)$, which clearly is polynomial in $||C|| + ||D||$. □

Enumerating all Lower Neighbors

5.1.18 Lemma. *Let C be some reduced $\mathcal{EL}$ concept description over the signature Σ. Then, it holds true that*

$$|\mathsf{Lower}^*(C)| \leq |\Sigma| \cdot (|\Sigma| \cdot ||C|| \cdot 2^{||C||-1})^{\mathsf{rd}(C)}.$$

Proof. We show the claim by induction on the role depth of C. If $\mathsf{rd}(C) = 0$, then it holds true that $|\mathsf{Lower}^*(C)| \leq |\Sigma|$, simply because any lower neighbor in $\mathsf{Lower}^*(C)$ is either of the form $C \sqcap A$ for some concept name $A \in \Sigma_\mathsf{C}$ or of the form $C \sqcap \exists\, r. \top$ for a role name $r \in \Sigma_\mathsf{R}$.

Now assume that $\mathsf{rd}(C) > 0$. Then, we have the following.

$$|\mathsf{Lower}^*(C)| \leq |\Sigma_\mathsf{C}| + \sum_{r \in \Sigma_\mathsf{R}} \sum_{\mathbf{S} \subseteq \mathsf{Succ}(C,r)} |\bigtimes \mathsf{Choices}^*_\mathbf{S}|$$

Furthermore, we can estimate an upper bound for each $|\bigtimes \mathsf{Choices}^*_\mathbf{S}|$.

$$\begin{aligned} |\bigtimes \mathsf{Choices}^*_\mathbf{S}| &\leq |\mathbf{S}| \cdot \mathsf{max}_{F \in \mathbf{S}} |\mathsf{Choices}^*_\mathbf{S}(F)| \\ &\leq |\mathbf{S}| \cdot \mathsf{max}_{F \in \mathbf{S}} |\mathsf{Lower}^*(F)| \end{aligned}$$

As next step, we apply the induction hypothesis to each $F \in \mathbf{S}$.

$$\begin{aligned} |\mathsf{Lower}^*(F)| &\leq |\Sigma| \cdot (|\Sigma| \cdot ||F|| \cdot 2^{||F||-1})^{\mathsf{rd}(F)} \\ &\leq |\Sigma| \cdot (|\Sigma| \cdot ||C|| \cdot 2^{||C||-1})^{\mathsf{rd}(C)-1} \end{aligned}$$

We now sum up our above results.

$$|\mathsf{Lower}^*(C)| \leq |\Sigma_\mathsf{C}| + \sum_{r \in \Sigma_\mathsf{R}} \sum_{\mathbf{S} \subseteq \mathsf{Succ}(C,r)} |\mathbf{S}| \cdot |\Sigma| \cdot (|\Sigma| \cdot ||C|| \cdot 2^{||C||-1})^{\mathsf{rd}(C)-1}$$

It is easy to verify that $\sum_{k=0}^{n} \binom{n}{k} \cdot k = n \cdot 2^{n-1}$, and so we can continue with the following.

$$\begin{aligned} \sum_{\mathbf{S} \subseteq \mathsf{Succ}(C,r)} |\mathbf{S}| &= \sum_{k=0}^{|\mathsf{Succ}(C,r)|} \binom{|\mathsf{Succ}(C,r)|}{k} \cdot k \\ &= |\mathsf{Succ}(C,r)| \cdot 2^{|\mathsf{Succ}(C,r)|-1} \\ &\leq ||C|| \cdot 2^{||C||-1} \end{aligned}$$

Finally, we put the last two results together.

$$\begin{aligned} |\mathsf{Lower}^*(C)| &\leq |\Sigma_\mathsf{C}| + |\Sigma_\mathsf{R}| \cdot (||C|| \cdot 2^{||C||-1}) \cdot |\Sigma| \cdot (|\Sigma| \cdot ||C|| \cdot 2^{||C||-1})^{\mathsf{rd}(C)-1} \\ &= |\Sigma_\mathsf{C}| + |\Sigma_\mathsf{R}| \cdot (|\Sigma| \cdot ||C|| \cdot 2^{||C||-1})^{\mathsf{rd}(C)} \\ &\leq |\Sigma| \cdot (|\Sigma| \cdot ||C|| \cdot 2^{||C||-1})^{\mathsf{rd}(C)} \end{aligned}$$

□

5.1.19 Proposition. *Fix some reduced concept description $C \in \mathcal{EL}(\Sigma)$. Then, for each lower neighbor $D \in \mathsf{Lower}^*(C)$, it holds true that the size of D is quadratic in the size of C.*

Proof. We show the claim by induction on the role depth of C—more specifically, we prove that any lower neighbor $D \in \mathsf{Lower}^*(C)$ satisfies $||D|| \leq (3 + \mathsf{rd}(C)) \cdot ||C|| + 1$.

Clearly, if $\mathsf{rd}(C) = 0$, then each $D \in \mathsf{Lower}(C)$ must be of the form $C \sqcap A$ for some concept name $A \in \Sigma_\mathsf{C}$ or of the form $C \sqcap \exists\, r. \top$ for some role name $r \in \Sigma_\mathsf{R}$. Obviously, this shows that $||D|| \leq ||C|| + 3 \leq 3 \cdot ||C|| + 1$.

Now assume that $\mathsf{rd}(C) > 0$. If D is of the form $C \sqcap A$, we again have that $||D|| \leq 3 \cdot ||C|| + 1 \leq (3 + \mathsf{rd}(C)) \cdot ||C|| + 1$. Thus, we continue with the non-trivial case where D has a form $C \sqcap \exists\, r. \bigsqcap \mathsf{Ran}(\chi)$ for some role name $r \in \Sigma_\mathsf{R}$ and a subset $\mathbf{S} \subseteq \mathsf{Succ}(C, r)$ such that $\chi \in \bigtimes \mathsf{Choices}^*_\mathbf{S}$ is an admissible choice function. It follows that $F \sqcap \chi(F) \in \mathsf{Lower}^*(F)$ for each $F \in \mathbf{S}$. An application of the induction hypothesis yields that $||F \sqcap \chi(F)|| \leq (3 + \mathsf{rd}(F)) \cdot ||F|| + 1$, and so we infer that $||\chi(F)|| \leq (2 + \mathsf{rd}(F)) \cdot ||F||$. Summing up, we have that

$$\begin{aligned}
&||\textstyle\bigsqcap \mathsf{Ran}(\chi)|| \\
&= |\mathsf{Ran}(\chi)| - 1 + \textstyle\sum(\, ||\chi(F)|| \mid F \in \mathbf{S}\,) \\
&\leq ||C|| - 1 + \textstyle\sum(\, (2 + \mathsf{rd}(F)) \cdot ||F|| \mid F \in \mathbf{S}\,) \\
&\leq ||C|| - 1 + \textstyle\sum(\, (2 + \mathsf{rd}(C) - 1) \cdot ||F|| \mid F \in \mathbf{S}\,) \\
&\leq ||C|| - 1 + (2 + \mathsf{rd}(C) - 1) \cdot \textstyle\sum(\, ||F|| \mid F \in \mathbf{S}\,) \\
&\leq ||C|| - 1 + (2 + \mathsf{rd}(C) - 1) \cdot ||C|| \\
&= (2 + \mathsf{rd}(C)) \cdot ||C|| - 1
\end{aligned}$$

and hence $||C \sqcap \exists\, r. \bigsqcap \mathsf{Ran}(\chi)|| \leq (3 + \mathsf{rd}(C)) \cdot ||C|| + 1$ holds true. □

5.1.20 Corollary. *For each reduced $\mathcal{EL}$ concept description C over some signature Σ, it holds true that the size of an (efficient) encoding of* $\mathsf{Lower}^*(C)$ *is exponential in* $||C|| + |\Sigma|$. □

5.1.21 Proposition. *The mapping* Lower^* *can be computed in deterministic exponential time. More specifically, for any reduced $C \in \mathcal{EL}(\Sigma)$, the set* $\mathsf{Lower}^*(C)$ *is computable in deterministic exponential time with respect to* $||C|| + |\Sigma|$.

Proof. Using arguments from the proof of Lemma 5.1.18, the fact that subsumption in $\mathcal{EL}$ can be decided in polynomial time, and Proposition 5.1.19, we see that enumerating all admissible choice functions as required in Corollary 5.1.13 takes at most exponential time with respect to $||C|| + |\Sigma|$. This shows the claim. □

As a further result regarding the computational complexity of computing lower neighbors, we have the following. While it shows a lower complexity for the problem of generating one lower neighbor of some given $\mathcal{EL}$ concept description, one can obviously not expect the proposed procedure to outperform algorithms that efficiently implement Corollary 5.1.13. However, it would be not to hard to suitably adapt the deterministic manner of these algorithms to let them work in a non-deterministic fashion. That way, we can significantly decrease the number of failing computation paths.

5.1.22 Proposition. *For any $\mathcal{EL}$ concept description C, we can compute one lower neighbor of C in non-deterministic polynomial time with respect to* $||C|| + |\Sigma|$. *More specifically, there is a* *non-deterministic polynomial time procedure such that, for any lower neighbor L of C, it has a (successful) computation path that returns some concept description equivalent to L, when started on C as input.*

Proof. The claim essentially is a consequence of Theorem 5.1.17 and Proposition 5.1.19, and the well-known fact that any $\mathcal{EL}$ concept description can be reduced in polynomial time. In particular, a suitable algorithm could work as follows on an input C.

1. Reduce C.
2. Guess some $\mathcal{EL}$ concept description L such that $||L|| \leq (3 + \mathsf{rd}(C)) \cdot ||C|| + 1$ is satisfied.
3. Check whether L is a lower neighbor of C. If yes, then return L; otherwise fail. □

The next lemma's aim is to show that the means of enumerating all lower neighbors from Corollary 5.1.13 is optimal in terms of computational complexity. In particular, each efficient algorithmization of Corollary 5.1.13 runs in exponential time, cf. the above proposition, and there is some example showing that $\mathcal{EL}$ concept descriptions can indeed have exponentially many lower neighbors, cf. the below lemma.

5.1.23 Lemma. *There is a sequence of signatures Σ_n and concept descriptions $C_n \in \mathcal{EL}(\Sigma_n)$ such that, for any $n \in \mathbb{N}$, the set $\mathsf{Lower}^*(C_n)$ of (representatives of) lower neighbors of C_n has a cardinality that is exponential in the size of Σ_n plus the size of C_n.*

Proof. We define a sequence of signatures Σ_n and concept descriptions $C_n \in \mathcal{EL}(\Sigma_n)$ as follows. Fix some $n \in \mathbb{N}$ such that $n \geq 2$. Set $(\Sigma_n)_\mathsf{C} := \{ A_i, B_i \mid i \in \{1, \ldots, n\} \}$ and $(\Sigma_n)_\mathsf{R} := \{r\}$. Furthermore, let

$$C_n := \bigsqcap \{ \exists r. D_n^i \mid i \in \{1, \ldots, n\} \} \quad \text{where} \quad D_n^i := \bigsqcap \{ A_j, B_j \mid j \in \{1, \ldots, n\} \setminus \{i\} \}.$$

If we now set $\mathbf{S} := \mathsf{Succ}(C_n, r)$, then it obviously holds true that $\mathsf{Choices}_\mathbf{S}^*(D_n^i) = \{A_i, B_i\}$ for any index $i \in \{1, \ldots, n\}$. It is easy to verify that any choice function $\chi \in \bigtimes \mathsf{Choices}_\mathbf{S}^*$ is admissible and further that there are exponentially many such choice functions, i.e., C_n has $\Omega(2^n)$ mutually incomparable lower neighbors while the size of C_n is $\mathcal{O}(n^2)$ and the size of Σ_n is $\mathcal{O}(n)$. □

5.1.2 Computational Complexity of Deciding Minimality and Maximality for a Property

Often there are situations where we want to decide whether a concept description is most specific or most general for some decidable property. Since we have shown in the previous Section 5.1.1 how all upper as well as all lower neighbors of a given $\mathcal{EL}$ concept description can be enumerated, we can use these procedures as subroutines in a procedure that decides the problems $\mathsf{Max}_\emptyset(\Xi)$ and $\mathsf{Min}_\emptyset(\Xi)$ for any decidable problem Ξ of $\mathcal{EL}$ concept descriptions. With our complexity results for Upper and Lower^*, it is also immediate to provide upper complexity bounds for these two types of decision problems.

Note that the *polynomial hierarchy* is defined as follows.

$$\mathbf{\Delta}_0^\mathsf{P} := \mathbf{\Sigma}_0^\mathsf{P} := \mathbf{\Pi}_0^\mathsf{P} := \mathsf{P}$$

$$\Delta_{n+1}^{\mathsf{P}} := \mathsf{P}^{\Sigma_n^{\mathsf{P}}}$$
$$\Sigma_{n+1}^{\mathsf{P}} := \mathsf{NP}^{\Sigma_n^{\mathsf{P}}}$$
$$\Pi_{n+1}^{\mathsf{P}} := (\mathsf{co}\,\mathsf{NP})^{\Sigma_n^{\mathsf{P}}}$$
$$\mathsf{PH} := \bigcup\{\, \Delta_n^{\mathsf{P}} \mid n \in \mathbb{N} \,\}$$

In particular, it holds true that $\Delta_1^{\mathsf{P}} = \mathsf{P}$, $\Sigma_1^{\mathsf{P}} = \mathsf{NP}$, and $\Pi_1^{\mathsf{P}} = \mathsf{co}\,\mathsf{NP}$.

5.1.24 Proposition. *Let $\Xi \subseteq \mathcal{EL}(\Sigma)$ be a problem that is closed under subsumees, that is, $C \in \Xi$ and $C \sqsupseteq_\emptyset D$ implies $D \in \Xi$. We consider the problem $\mathsf{Max}_\emptyset(\Xi)$, which consists of all most general elements of Ξ.*

1. *$\Xi \in \mathsf{C}$ implies $\mathsf{Max}_\emptyset(\Xi) \in \mathsf{P}^{\mathsf{C}}$ for each complexity class C.*
2. *$\Xi \in \mathsf{P}$ implies $\mathsf{Max}_\emptyset(\Xi) \in \mathsf{P}$*
3. *$\Xi \in \Sigma_n^{\mathsf{P}}$ implies $\mathsf{Max}_\emptyset(\Xi) \in \Delta_{n+1}^{\mathsf{P}}$ for any number $n \in \mathbb{N}$.*
4. *$\Xi \in \mathsf{C}$ implies $\mathsf{Max}_\emptyset(\Xi) \in \mathsf{C}$ for any complexity class C such that $\mathsf{P\,Space} \subseteq \mathsf{C}$.*
5. *$\Xi \in \mathsf{P\,Space}$ implies $\mathsf{Max}_\emptyset(\Xi) \in \mathsf{P\,Space}$*

Proof. We only prove Statement 1; the others are then obtained as corollaries. In particular, for Statement 4 we need that $\mathsf{P}^{\mathsf{C}} \subseteq \mathsf{C}$ holds true for any complexity class C such that $\mathsf{P\,Space} \subseteq \mathsf{C}$. In case $\mathsf{C} = \mathsf{P\,Space}$ this follows from

$$\mathsf{P\,Space} \subseteq \mathsf{P}^{\mathsf{P\,Space}} \subseteq \mathsf{NP}^{\mathsf{P\,Space}} \subseteq \mathsf{NP\,Space} \subseteq \mathsf{P\,Space},$$

cf. [Pap94, Proof of Theorem 14.4]. With similar arguments, we see that $\mathsf{P}^{\mathsf{C}} \subseteq \mathsf{C}$ holds true as well for the general case $\mathsf{P\,Space} \subseteq \mathsf{C}$, since each polynomial time TURING machine with C-oracle can be "recompiled" to a C-TURING machine.

A deterministic procedure that decides $\mathsf{Max}_\emptyset(\Xi)$ could work as follows when given some $\mathcal{EL}$ concept description C as input.

1. Check if $C \in \Xi$. If not, then reject C.
2. Enumerate all upper neighbors of C.
3. If there exists some upper neighbor D of C with $D \in \Xi$, then reject C; otherwise accept C.

According to Proposition 5.1.16, Step 2 requires polynomial time. We conclude that this procedure shows that $\mathsf{Max}_\emptyset(\Xi) \in \mathsf{P}^{\mathsf{C}}$ holds true. □

5.1.25 Proposition. *Let $\Xi \subseteq \mathcal{EL}(\Sigma)$ be some problem that is closed under subsumers, that is, $C \in \Xi$ and $C \sqsubseteq_\emptyset D$ implies $D \in \Xi$. We consider the problem $\mathsf{Min}_\emptyset(\Xi)$, which consists of all most specific elements of Ξ.*

1. *$\Xi \in \mathsf{C}$ implies $\mathsf{Min}_\emptyset(\Xi) \in \mathsf{co}\,(\mathsf{NP}^{\mathsf{C}})$ for each complexity class C.*
2. *$\Xi \in \mathsf{P}$ implies $\mathsf{Min}_\emptyset(\Xi) \in \mathsf{co}\,\mathsf{NP}$*
3. *$\Xi \in \Sigma_n^{\mathsf{P}}$ implies $\mathsf{Min}_\emptyset(\Xi) \in \Pi_{n+1}^{\mathsf{P}}$ for each number $n \in \mathbb{N}$.*

4. $\Xi \in \mathbf{C}$ *implies* $\mathsf{Min}_\emptyset(\Xi) \in \mathbf{co}\,\mathbf{C}$ *for any complexity class* $\mathbf{C}$ *such that* $\mathbf{P\,Space} \subseteq \mathbf{C}$.
5. $\Xi \in \mathbf{P\,Space}$ *implies* $\mathsf{Min}_\emptyset(\Xi) \in \mathbf{P\,Space}$

Proof. It is sufficient to show Statement 1, since the others are then obtained as immediate consequences. For Statement 4 we use the fact that $\mathbf{co}\,(\mathbf{NP}^{\mathbf{C}}) \subseteq \mathbf{C}$ is satisfied for each complexity class $\mathbf{C}$ satisfying $\mathbf{P\,Space} \subseteq \mathbf{C}$. If $\mathbf{C} = \mathbf{P\,Space}$, then this follows from $\mathbf{NP}^{\mathbf{P\,Space}} \subseteq \mathbf{NP\,Space} \subseteq \mathbf{P\,Space}$, cf. [Pap94, Proof of Theorem 14.4], since we can conclude that $\mathbf{co}\,(\mathbf{NP}^{\mathbf{P\,Space}}) \subseteq \mathbf{co\,P\,Space} = \mathbf{P\,Space}$. More generally if $\mathbf{P\,Space} \subseteq \mathbf{C}$, we can “recompile” any non-deterministic polynomial time TURING machine with $\mathbf{C}$-oracle to some deterministic polynomial space TURING machine with $\mathbf{C}$-oracle, which itself can be “recompiled” to a $\mathbf{C}$-TURING machine.

The following non-deterministic procedure decides the complement of $\mathsf{Min}_\emptyset(\Xi)$. Let C be an $\mathcal{EL}$ concept description that is given as input.

1. Check whether $C \in \Xi$. If not, then accept C.
2. Guess some lower neighbor D of C.
3. If $D \in \Xi$, then accept C; otherwise reject C.

Now Proposition 5.1.22 implies that the above is a procedure that needs non-deterministic polynomial time, and since it uses a $\mathbf{C}$-oracle to decide Ξ, we conclude that the complement of $\mathsf{Min}_\emptyset(\Xi)$ is in $\mathbf{NP}^{\mathbf{C}}$, which implies that $\mathsf{Min}_\emptyset(\Xi) \in \mathbf{co}\,(\mathbf{NP}^{\mathbf{C}})$. □

5.1.3 The Bottom Concept Description

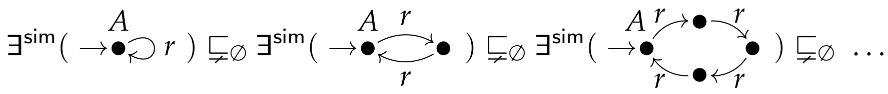
Now consider the extension of $\mathcal{EL}$ with the *bottom concept description* $\bot$ the semantics of which is defined as $\bot^{\mathcal{I}} := \emptyset$ for any interpretation $\mathcal{I}$. Then $\sqsubseteq_\emptyset$ is not bounded, since the following infinite chain exists.

$$\bot \sqsubsetneq_\emptyset \ldots \sqsubsetneq_\emptyset \exists\, r^{n+1}.\,\top \sqsubsetneq_\emptyset \exists\, r^{n}.\,\top \sqsubsetneq_\emptyset \ldots \sqsubsetneq_\emptyset \exists\, r^{2}.\,\top \sqsubsetneq_\emptyset \exists\, r.\,\top \sqsubsetneq_\emptyset \top$$

However, $\sqsupseteq_\emptyset$ is still well-founded, since whenever a chain starts with $\bot$, then the second element must be a satisfiable concept description, that is, some C with $C \not\equiv_\emptyset \bot$, after which the chain can only have a bounded number of elements. Furthermore, $\sqsubseteq_\emptyset$ is not neighborhood generated, as $\bot$ does not have any upper neighbors. To see this, consider a concept description C such that $\bot \sqsubsetneq_\emptyset C$; it then follows that $\bot \sqsubsetneq_\emptyset C \sqcap \exists\, r.\, C \sqsubsetneq_\emptyset C$. Anyway, $\bot$ is the only concept description that causes problems here: for each satisfiable $\mathcal{EL}^\bot$ concept description, that is, for any $C \in \mathcal{EL}^\bot(\Sigma)$ such that $C \not\equiv_\emptyset \bot$, we can enumerate all upper and lower neighbors with the same techniques as in Section 5.1.1. This is due to the fact that an $\mathcal{EL}^\bot$ concept description is satisfiable if, and only if, it does not contain $\bot$ as a subconcept.

5.1.4 Greatest Fixed-Point Semantics

Unfortunately, the situation is also not rosy for the extension $\mathcal{EL}_{\mathsf{si}}$ of $\mathcal{EL}$ with *greatest fixed-point semantics*, see Section 3.4. It then holds true that $\sqsubseteq_\emptyset$ is neither bounded nor neighborhood generated, and $\sqsupset_\emptyset$ is not well-founded. The following infinite chain justifies that $\sqsubseteq_\emptyset$ is not bounded and further that $\sqsupset_\emptyset$ is not well-founded.[3]



5.1.26 Proposition. *If the signature Σ contains at least one role name, then the subsumption relation $\sqsubseteq_\emptyset$ on $\mathcal{EL}_{\mathsf{si}}(\Sigma)$ is not neighborhood generated.*

Proof. Let $r \in \Sigma_{\mathsf{R}}$ be a role name and define the cyclic $\mathcal{EL}_{\mathsf{si}}$ concept description (the *r-loop*)

$$\exists\, r^\infty.\top := \exists^{\mathsf{sim}}(\ \to\!\bullet\!\circlearrowleft r\ )$$

for which the extension contains objects from which an infinite r-path issues. We shall prove that $\exists\, r^\infty.\top$ does not have upper neighbors. Consider some $\mathcal{EL}_{\mathsf{si}}$ concept description $\exists^{\mathsf{sim}}(\mathcal{I},\delta)$ where $\exists\, r^\infty.\top \sqsubsetneq_\emptyset \exists^{\mathsf{sim}}(\mathcal{I},\delta)$, i.e., there is a simulation from $(\mathcal{I},\delta)$ to the r-loop but not in the converse direction. It follows that extensions of names different from r must be empty in $\mathcal{I}$, and further that (the connected component containing δ of) $\mathcal{I}$ is acyclic (more specifically, tree-shaped with root δ). If n is the length of a longest path issuing from δ, then $\exists^{\mathsf{sim}}(\mathcal{I},\delta)$ is equivalent to $\exists\, r^n.\top$.

We can easily conclude that $\exists\, r^\infty.\top \sqsubsetneq_\emptyset \exists\, r^{n+1}.\top \sqsubsetneq \exists\, r^n.\top \equiv_\emptyset \exists^{\mathsf{sim}}(\mathcal{I},\delta)$. Furthermore, we can approximate $\exists\, r^\infty.\top$ from above with the following sequence.

$$\exists\, r^\infty.\top \sqsubsetneq_\emptyset \ldots \sqsubsetneq_\emptyset \exists\, r^{n+1}.\top \sqsubsetneq_\emptyset \exists\, r^n.\top \sqsubsetneq_\emptyset \ldots \sqsubsetneq_\emptyset \exists\, r^2.\top \sqsubsetneq_\emptyset \exists\, r.\top \sqsubsetneq_\emptyset \top \qquad \square$$

5.1.5 Cycle-Restricted TBoxes

Recall that, according to BAADER, BORGWARDT, and MORAWSKA [BBM12b; BBM12a, Definition 2], a TBox $\mathcal{T}$ is called *cycle-restricted* if there does not exist a word $w \in \Sigma_{\mathsf{R}}^+$ and a concept description $C \in \mathcal{EL}(\Sigma)$ such that $C \sqsubseteq_{\mathcal{T}} \exists\, w.C$. Furthermore, deciding whether a TBox is cycle-restricted can be done in polynomial time. In Proposition 4.3.6 we have seen that most specific consequences with respect to cycle-restricted TBoxes always exist in $\mathcal{EL}$ (without greatest fixed-point semantics). Thus, we can utilize our results on neighborhood in $\mathcal{EL}$ without any TBox to constitute procedures for deciding neighborhood and for enumerating all neighbors in $\mathcal{EL}$ with respect to cycle-restricted TBoxes.

5.1.27 Proposition. *For each cycle-restricted TBox $\mathcal{T}$, the subsumption relation $\sqsubseteq_{\mathcal{T}}$ is neighborhood generated.*

Proof. By means of Proposition 4.3.39 it is readily verified that $C \prec_{\mathcal{T}} D$ if, and only if, $C^{\mathcal{T}} \sqsubsetneq_\emptyset D^{\mathcal{T}}$ and there is no most specific consequence $E^{\mathcal{T}}$ such that $C^{\mathcal{T}} \sqsubsetneq_\emptyset E^{\mathcal{T}} \sqsubsetneq_\emptyset D^{\mathcal{T}}$. According to Proposition 4.3.6, all most specific consequences of $\mathcal{T}$ exist in $\mathcal{EL}$. Furthermore, we know that $\sqsubseteq_\emptyset$ is bounded, cf. [BM10, Proof of Proposition 3.5]. Of course, if we now restrict the subsumption relation $\sqsubseteq_\emptyset$ to the most specific consequences of $\mathcal{T}$, that is, if we consider

[3] The pointed interpretations are depicted by labeled, directed graphs where the distinguished objects are each marked with an arrow.

the relation $\sqsubseteq_\emptyset \cap \mathsf{MSS}(\mathcal{T}) \times \mathsf{MSS}(\mathcal{T})$ where $\mathsf{MSS}(\mathcal{T}) := \{ C^\mathcal{T} \mid C \in \mathcal{EL}(\Sigma) \}$, then this relation must also be bounded. Now since there exists an order isomorphism $[C]_\mathcal{T} \mapsto [C^\mathcal{T}]_\emptyset$ between $(\mathcal{EL}(\Sigma), \sqsubseteq_\mathcal{T})/\mathcal{T}$ and $(\mathsf{MSS}(\mathcal{T}), \sqsubseteq_\emptyset \cap \mathsf{MSS}(\mathcal{T}) \times \mathsf{MSS}(\mathcal{T}))/\emptyset$, we conclude that $\sqsubseteq_\mathcal{T}$ is bounded as well and is, thus, neighborhood generated. □

5.1.28 Theorem. *Fix some cycle-restricted $\mathcal{EL}$ TBox $\mathcal{T}$ as well as two $\mathcal{EL}$ concept descriptions C and D. It then holds true that $C \prec_\mathcal{T} D$ if, and only if, $C \sqsubsetneq_\mathcal{T} D$ and $C \sqsubseteq_\mathcal{T} L$ implies $C \equiv_\mathcal{T} L$ for any $L \in \mathsf{Lower}^*(D^\mathcal{T})$. Furthermore, it holds true that $\prec_\mathcal{T} \in$* **coNP**, *i.e., non-neighborhood of two $\mathcal{EL}$ concept descriptions is decidable in non-deterministic polynomial time w.r.t. $||C|| + ||D|| + ||\mathcal{T}|| + |\Sigma|$.*

Proof. We start with proving the *if* statement. Let $C \sqsubseteq_\mathcal{T} X \sqsubsetneq_\mathcal{T} D$, that is, $C^\mathcal{T} \sqsubseteq_\emptyset X^\mathcal{T} \sqsubsetneq_\emptyset D^\mathcal{T}$. Now there is some $L \in \mathsf{Lower}^*(D^\mathcal{T})$ such that $X^\mathcal{T} \sqsubseteq_\emptyset L$, and it follows that $X \sqsubseteq_\mathcal{T} L$. We conclude that $C \equiv_\mathcal{T} L$ holds true, which implies $C \equiv_\mathcal{T} X$.

We proceed with the *only if* direction. Assume $C \prec_\mathcal{T} D$, which immediately yields that $C \sqsubsetneq_\mathcal{T} D$, and further let $L \in \mathsf{Lower}^*(D^\mathcal{T})$ such that $C \sqsubseteq_\mathcal{T} L$. The very definition of a most specific consequence shows that $L \prec_\emptyset D^\mathcal{T}$ implies $L \sqsubsetneq_\mathcal{T} D$. Eventually, our assumption yields that $C \equiv_\mathcal{T} L$.

The complexity result can be obtained as a corollary of the following facts.

- Subsumption in $\mathcal{EL}$ can be decided in polynomial time.
- Most specific consequences w.r.t. cycle-restricted TBoxes always exist in $\mathcal{EL}$ and can be computed in polynomial time.
- Lower neighbors of an $\mathcal{EL}$ concept description can be guessed in polynomial time, cf. Proposition 5.1.22. □

5.1.29 Proposition. *Let $\mathcal{T}$ be a cycle-restricted $\mathcal{EL}$ TBox and C an $\mathcal{EL}$ concept description. Then the set*

$$\mathsf{Lower}_\mathcal{T}(C) := \mathsf{Max}_\mathcal{T}(\mathsf{Lower}^*(C^\mathcal{T}))$$

contains exactly all lower neighbors of C with respect to $\mathcal{T}$ modulo equivalence and can further be computed in exponential time w.r.t. $||C|| + ||\mathcal{T}|| + |\Sigma|$.

Proof. Soundness. Let $L \in \mathsf{Lower}_\mathcal{T}(C)$ and assume that $L \sqsubseteq_\mathcal{T} X \sqsubsetneq_\mathcal{T} C$. It then follows that $L^\mathcal{T} \sqsubseteq_\emptyset X^\mathcal{T} \sqsubsetneq_\emptyset C^\mathcal{T}$ and, thus, there is some M such that $L^\mathcal{T} \sqsubseteq_\emptyset X^\mathcal{T} \sqsubseteq_\emptyset M \prec_\emptyset C^\mathcal{T}$. We conclude that $L \sqsubseteq_\mathcal{T} X \sqsubseteq_\mathcal{T} M$. As L is $\sqsubseteq_\mathcal{T}$-maximal in $\mathsf{Lower}^*(C^\mathcal{T})$, we conclude that $L \equiv_\mathcal{T} M$, which shows that $X \equiv_\mathcal{T} L$, that is, $L \prec_\mathcal{T} C$.

Completeness. Vice versa, assume that $L \prec_\mathcal{T} C$. We infer that $L \sqsubsetneq_\mathcal{T} C$ and further that $L^\mathcal{T} \sqsubsetneq_\emptyset C^\mathcal{T}$. According to Corollary 5.1.13, there exists some lower neighbor $M \in \mathsf{Lower}^*(C^\mathcal{T})$ satisfying $L^\mathcal{T} \sqsubseteq_\emptyset M \prec_\emptyset C^\mathcal{T}$. Thus, it follows that $L \equiv_\mathcal{T} L^\mathcal{T} \sqsubseteq_\mathcal{T} M \sqsubsetneq_\mathcal{T} C^\mathcal{T} \equiv_\mathcal{T} C$, which yields $L \equiv_\mathcal{T} M$. It remains to prove that M is $\sqsubseteq_\mathcal{T}$-maximal. If $M \sqsubsetneq_\mathcal{T} N$ for some $N \in \mathsf{Lower}^*(C^\mathcal{T})$, then $M \sqsubsetneq_\mathcal{T} N \prec_\emptyset C^\mathcal{T}$ immediately implies the contradiction $M \sqsubsetneq_\mathcal{T} N \sqsubsetneq_\mathcal{T} C$.

Complexity. The complexity result can be obtained as a corollary of the following facts.

- Subsumption in $\mathcal{EL}$ can be decided in polynomial time.

- Most specific consequences w.r.t. cycle-restricted TBoxes always exist in $\mathcal{EL}$ and can be computed in polynomial time.
- (Representatives of) all lower neighbors of some $\mathcal{EL}$ concept description can be enumerated in exponential time, cf. Proposition 5.1.21. □

5.1.30 Proposition. *Fix some cycle-restricted $\mathcal{EL}$ TBox $\mathcal{T}$ and consider an $\mathcal{EL}$ concept description C. Then the set*

$$\mathsf{Upper}_{\mathcal{T}}(C) := \mathsf{Min}_{\emptyset}(\bigcup\{\, \mathsf{Upper}(X) \mid X \in \mathsf{Max}_{\emptyset}([C]_{\mathcal{T}}) \,\})$$

contains exactly all upper neighbors of C with respect to $\mathcal{T}$ modulo equivalence.

Proof. Soundness. Assume that $C \prec_{\mathcal{T}} D$ holds true. It follows that $C^{\mathcal{T}} \sqsubsetneq_{\emptyset} D^{\mathcal{T}}$. Now consider some concept description E such that $C^{\mathcal{T}} \sqsubseteq_{\emptyset} E \sqsubseteq_{\emptyset} D^{\mathcal{T}}$. According to the properties of most specific consequences, we can infer that $C \sqsubseteq_{\mathcal{T}} E \sqsubseteq_{\mathcal{T}} D$, which yields that either $E \equiv_{\mathcal{T}} C$ or $E \equiv_{\mathcal{T}} D$. Formulated alternatively, we have that $E \equiv_{\mathcal{T}} C$ if, and only if, $E \not\equiv_{\mathcal{T}} D$.

It is readily verified that some E satisfying $C^{\mathcal{T}} \sqsubseteq_{\emptyset} E \sqsubseteq_{\emptyset} D^{\mathcal{T}}$ and $E \equiv_{\mathcal{T}} C$ exists, namely $E = C^{\mathcal{T}}$. We now fix some such E that is most general (w.r.t. $\emptyset$) such that $C^{\mathcal{T}} \sqsubseteq_{\emptyset} E \sqsubseteq_{\emptyset} D^{\mathcal{T}}$ and $E \equiv_{\mathcal{T}} C$. Then, we immediately conclude that $E \not\equiv_{\mathcal{T}} D$ as well as $E \sqsubsetneq_{\emptyset} D^{\mathcal{T}}$, and furthermore we have that $C \not\equiv_{\mathcal{T}} F \equiv_{\mathcal{T}} D$ for any F with $E \prec_{\emptyset} F \sqsubseteq_{\emptyset} D^{\mathcal{T}}$. In particular, at least one such upper neighbor F of E must exist and we infer that $F \equiv_{\emptyset} D^{\mathcal{T}}$ holds true. Summing up, we have shown that $D \equiv_{\mathcal{T}} F$ for some $F \in \bigcup\{\, \mathsf{Upper}(X) \mid X \in \mathsf{Max}_{\emptyset}([C]_{\mathcal{T}}) \,\}$.

It remains to show that F is most specific w.r.t. $\emptyset$. Assume the contrary, i.e., let $E' \in \mathsf{Max}_{\emptyset}([C]_{\mathcal{T}})$ and $F' \in \mathsf{Upper}(E')$ such that $F' \sqsubsetneq_{\emptyset} F$ and $D \equiv_{\mathcal{T}} F'$. It then follows that $D \sqsubseteq_{\mathcal{T}} F'$, which implies $D^{\mathcal{T}} \sqsubseteq_{\emptyset} F'$. Putting everything together yields the contradiction $D^{\mathcal{T}} \sqsubseteq_{\emptyset} F' \sqsubsetneq_{\emptyset} F \equiv_{\emptyset} D^{\mathcal{T}}$. ↯

Completeness. Vice versa, assume that there are two concept descriptions X and D such that $X \in \mathsf{Max}_{\emptyset}([C]_{\mathcal{T}})$, $D \in \mathsf{Upper}(X)$, and where D is most specific with respect to these two properties, that is, there does not exist any $X' \in \mathsf{Max}_{\emptyset}([C]_{\mathcal{T}})$ and some $D' \in \mathsf{Upper}(X')$ with $D' \sqsubsetneq_{\emptyset} D$. We claim that then $C \prec_{\mathcal{T}} D$ holds true. Before we proceed with proving this, we show the following auxiliary claim.

Claim. *For each Y such that $C \sqsubseteq_{\mathcal{T}} Y \sqsubsetneq_{\emptyset} D$, we have $C \equiv_{\mathcal{T}} Y$.*

Proof. Let $C \sqsubsetneq_{\mathcal{T}} Y \sqsubsetneq_{\emptyset} D$. Then, there must exist some $Z \in \mathsf{Max}_{\emptyset}([C]_{\mathcal{T}})$ such that $C^{\mathcal{T}} \sqsubseteq_{\emptyset} Z \sqsubsetneq_{\emptyset} Y$. Thus, there is some $U \in \mathsf{Upper}(Z)$ such that $C^{\mathcal{T}} \sqsubseteq_{\emptyset} Z \prec_{\emptyset} U \sqsubseteq_{\emptyset} Y$. It follows that $U \sqsubsetneq_{\emptyset} D$ where $U \in \mathsf{Upper}(Z)$ and $Z \in \mathsf{Max}_{\emptyset}([C]_{\mathcal{T}})$. ↯ □

From $D \succ_{\emptyset} X \sqsupseteq_{\emptyset} X^{\mathcal{T}} \equiv_{\emptyset} C^{\mathcal{T}}$ we infer that $C^{\mathcal{T}} \sqsubsetneq_{\emptyset} D$, which immediately implies that $C \sqsubseteq_{\mathcal{T}} D$. Apparently, $C \equiv_{\mathcal{T}} D$ would contradict the precondition that X is most general in $[C]_{\mathcal{T}}$; we conclude that $C \sqsubsetneq_{\mathcal{T}} D$.

Furthermore, it holds true that $D \equiv_{\emptyset} D^{\mathcal{T}}$. To see this, assume the contrary, i.e., let $D^{\mathcal{T}} \sqsubsetneq_{\emptyset} D$. Since $C \sqsubseteq_{\mathcal{T}} D$ and $D \equiv_{\mathcal{T}} D^{\mathcal{T}}$, an application of the above lemma would yield the contradiction $C \equiv_{\mathcal{T}} D^{\mathcal{T}}$. ↯

According to Theorem 5.1.28, it suffices to check whether $C \sqsubseteq_{\mathcal{T}} L$ implies $C \equiv_{\mathcal{T}} L$ for any $L \in \mathsf{Lower}^*(D^{\mathcal{T}})$. However, this is an immediate consequence of the above lemma: if $C \sqsubseteq_{\mathcal{T}} L \prec_{\emptyset} D^{\mathcal{T}}$, then due to $D \equiv_{\emptyset} D^{\mathcal{T}}$ we infer that $C \sqsubseteq_{\mathcal{T}} L \sqsubsetneq_{\emptyset} D$, and so it follows that $C \equiv_{\mathcal{T}} L$. □

5.1.6 Acyclic TBoxes

A *concept definition* is an expression of the form $A \equiv C$ where $A \in \Sigma_{\mathsf{C}}$ is a concept name and where $C \in \mathcal{EL}(\Sigma)$ is a concept description. We then also say that A is a *defined* concept name and C is its *defining* concept description. A concept name that is not defined is called *primitive*. An *acyclic TBox* is a finite set of concept definitions that contains at most one concept definition $A \equiv C$ for each concept name $A \in \Sigma_{\mathsf{C}}$, and for which the following directed graph, called the *dependency graph* of $\mathcal{T}$, is acyclic.

$$(\Sigma_{\mathsf{C}}, \{\, (A, B) \mid A, B \in \Sigma_{\mathsf{C}},\ \exists\, C\colon A \equiv C \in \mathcal{T} \text{ and } B \in \mathsf{Sub}(C)\,\}^{+})$$

The *expansion* of an $\mathcal{EL}$ concept description C with respect to an acyclic TBox $\mathcal{T}$ is obtained from C by exhaustively replacing each defined concept name with its defining concept description. It is easy to see that the expansion of C w.r.t. $\mathcal{T}$ equals the most specific consequence $C^{\mathcal{T}}$.

Of course, any acyclic TBox is cycle-restricted. Thus, we can simply pull and apply the results from the preceding section and, in particular, conclude that the subsumption relation $\sqsubseteq_{\mathcal{T}}$ is neighborhood generated for any acyclic TBox $\mathcal{T}$.

5.1.7 General TBoxes

A similar situation as for greatest fixed-point semantics arises when considering subsumption with respect to a non-cycle-restricted TBox $\mathcal{T}$.

5.1.31 Proposition. *There is an $\mathcal{EL}$ TBox $\mathcal{T}$ over some signature Σ such that $\sqsubseteq_{\mathcal{T}}$ is not bounded and $\sqsupseteq_{\mathcal{T}}$ is not well-founded.*

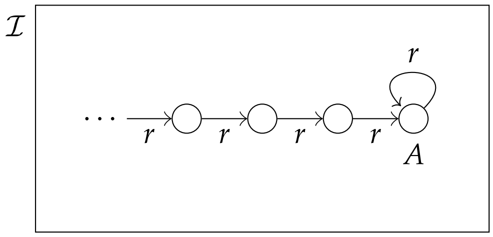
Proof. We demonstrate the claim by giving a counterexample. Define the TBox $\mathcal{T} := \{A \sqsubseteq \exists\, r.\, A\}$ over the signature Σ where $\Sigma_{\mathsf{C}} := \{A\}$ and $\Sigma_{\mathsf{R}} := \{r\}$. Apparently, then the following infinite chain exists.

$$A \sqsubsetneq_{\mathcal{T}} \exists\, r.\, A \sqsubsetneq_{\mathcal{T}} \exists\, r^2.\, A \sqsubsetneq_{\mathcal{T}} \exists\, r^3.\, A \sqsubsetneq_{\mathcal{T}} \ldots$$

The model $\mathcal{I}$ of $\mathcal{T}$ depicted in Figure 5.1.32 shows that the subsumptions in the chain are indeed strict. □

5.1.33 Proposition. *There is an $\mathcal{EL}$ TBox $\mathcal{T}$ over some signature Σ and an $\mathcal{EL}$ concept description C over Σ that strictly subsumes some other $\mathcal{EL}$ concept description w.r.t. $\mathcal{T}$, but does not have lower neighbors with respect to $\mathcal{T}$.*

Proof. We consider a simple signature with exactly one concept name and exactly one role name, i.e., let Σ be given by $\Sigma_{\mathsf{C}} := \{A\}$ and $\Sigma_{\mathsf{R}} := \{r\}$. We are going to show that $\top$ does not



5.1.32 Figure. A model

have lower neighbors with respect to the TBox

$$\mathcal{T} := \{\top \sqsubseteq \exists\, r.\top,\ A \sqsubseteq \exists\, r.A\}.$$

For this purpose, we first prove the validity of the following two statements.

1. If C does not contain the concept name A as a subconcept, then $C \equiv_{\mathcal{T}} \top$.
2. If in the canonical model of an $\mathcal{EL}(\Sigma)$ concept description C with respect to $\mathcal{T}$ the shortest path from the vertex C to a vertex labeled with A has length n, then $C \equiv_{\mathcal{T}} \exists\, r^n.A$.

As a corollary, we then obtain that

$$\mathcal{EL}(\Sigma)/\mathcal{T} = \{\, [\exists\, r^n.A]_{\mathcal{T}} \mid n \in \mathbb{N}\,\} \cup \{[\top]_{\mathcal{T}}\},$$

and furthermore that the subsumption ordering of these concept descriptions is as follows.

$$A \sqsubsetneq_{\mathcal{T}} \exists\, r.A \sqsubsetneq_{\mathcal{T}} \exists\, r^2.A \sqsubsetneq_{\mathcal{T}} \exists\, r^3.A \sqsubsetneq_{\mathcal{T}} \ldots \sqsubsetneq_{\mathcal{T}} \top$$

The interpretation $\mathcal{I}$ in Figure 5.1.32 is a model of $\mathcal{T}$ and witnesses the strictness of the above mentioned subsumptions. We may now safely conclude that $\top$ indeed does not have any lower neighbors with respect to $\mathcal{T}$.

However, we still have to prove the two statements above, with which we proceed now.

1. Let C be an $\mathcal{EL}(\Sigma)$ concept description which does not contain A as a subconcept. It is easy to verify that in the canonical model $\mathcal{I}_{C,\mathcal{T}}$ there is an r-edge from C to $\top$, and the latter has an r-loop. Thus, $(\mathcal{I}_{C,\mathcal{T}}, C)$ and $(\mathcal{I}_{\top,\mathcal{T}}, \top)$ are equi-similar, whence $C \equiv_{\mathcal{T}} \top$.
2. As supposed, let n be the length of a shortest path $\vec{p}$ from C to a vertex D with label A within the canonical model $\mathcal{I}_{C,\mathcal{T}}$. In particular, A is an r-successor of D and A is an r-successor of itself. Henceforth, the other r-paths starting with D can already be simulated in the r-loop of A. All other r-paths starting with C may also be simulated by means of $\vec{p}$ and the r-loop of A. Thus, we conclude that $(\mathcal{I}_{C,\mathcal{T}}, C)$ and the canonical model of $\exists\, r^n.A$ w.r.t. $\mathcal{T}$ are equi-similar, that is, $C \equiv_{\mathcal{T}} \exists\, r^n.A$. □

5.1.34 Proposition. *There is an $\mathcal{EL}$ TBox $\mathcal{T}$ over some signature Σ and an $\mathcal{EL}$ concept description C over Σ that is strictly subsumed by another $\mathcal{EL}$ concept description w.r.t. $\mathcal{T}$, but does not have upper neighbors with respect to $\mathcal{T}$.*

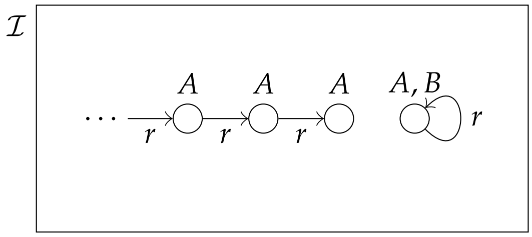
Proof. We try to keep things simple, and consider a rather small signature, namely Σ defined by $\Sigma_\mathsf{C} := \{A, B\}$ and $\Sigma_\mathsf{R} := \{r\}$. Furthermore, in order to find a suitable counterexample, we define a TBox by

$$\mathcal{T} := \{\exists\, r.\, A \sqsubseteq A,\ B \sqsubseteq A,\ B \equiv \exists\, r.\, B\}.$$

From the very definition of $\mathcal{T}$ it follows that the following subsumptions hold true.

$$\begin{aligned} B \equiv_{\mathcal{T}} \exists\, r^n.\, B \sqsubsetneq_{\mathcal{T}} \exists\, r^n.\, A \\ \ldots \sqsubsetneq_{\mathcal{T}} \exists\, r^{n+1}.\, A \sqsubsetneq_{\mathcal{T}} \exists\, r^n.\, A \sqsubsetneq_{\mathcal{T}} \ldots \sqsubsetneq_{\mathcal{T}} \exists\, r^2.\, A \sqsubsetneq_{\mathcal{T}} \exists\, r.\, A \sqsubsetneq_{\mathcal{T}} A \end{aligned}$$

The interpretation $\mathcal{I}$ shown in Figure 5.1.35 is a model of $\mathcal{T}$ and justifies the strictness of the subsumptions above. Let $C_n := \exists\, r^n.\, A$ for $n \in \mathbb{N}$. According to the previous observations, the



5.1.35 Figure. A model

following infinite chain exists.

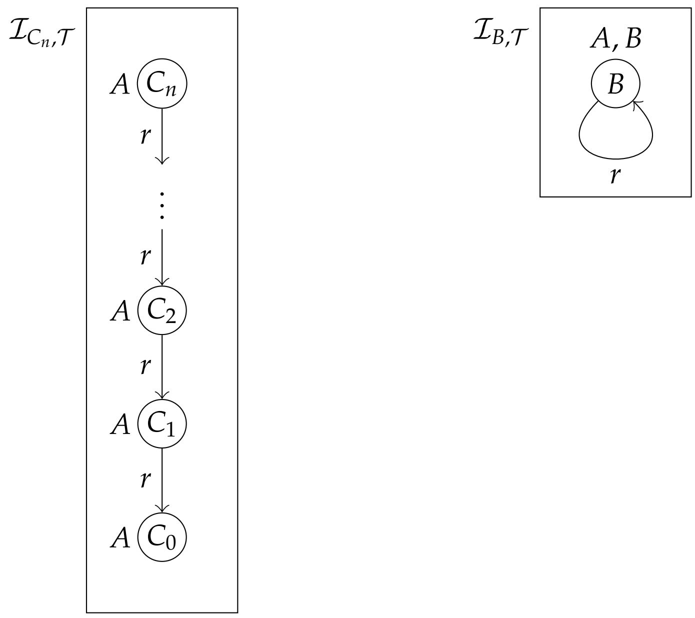
$$B \sqsubsetneq_{\mathcal{T}} \ldots \sqsubsetneq_{\mathcal{T}} C_{n+1} \sqsubsetneq_{\mathcal{T}} C_n \sqsubsetneq_{\mathcal{T}} \ldots \sqsubsetneq_{\mathcal{T}} C_2 \sqsubsetneq_{\mathcal{T}} C_1 \sqsubsetneq_{\mathcal{T}} C_0 = A$$

The canonical models $\mathcal{I}_{C_n,\mathcal{T}}$ and $\mathcal{I}_{B,\mathcal{T}}$ are depicted in Figure 5.1.36. It is readily verified that for each $n \in \mathbb{N}$, there exists a simulation from $(\mathcal{I}_{C_n,\mathcal{T}}, C_n)$ to $(\mathcal{I}_{B,\mathcal{T}}, B)$, but there is no simulation in the converse direction, i.e., it indeed holds true that $B \sqsubsetneq_{\mathcal{T}} C_n$.

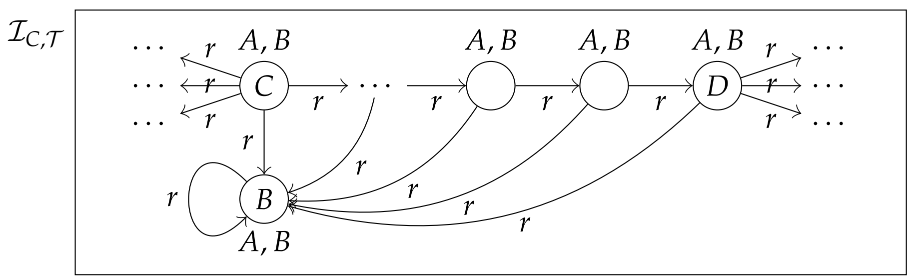
Let $C \in \mathcal{EL}(\Sigma)$. We proceed with a case distinction on whether C contains B as a subconcept.

1. Assume that B is a subconcept of C. We are going to show that then $C \equiv_{\mathcal{T}} B$. The canonical model $\mathcal{I}_{C,\mathcal{T}}$ contains an r-path from the vertex C to some vertex D which has label B. Since $B \equiv \exists\, r.\, B \in \mathcal{T}$, the very definition of canonical models yields that each vertex on this path must be labeled with B, and hence each of these vertices has B as an r-successor. Furthermore, B is an r-successor of itself. We conclude that the canonical model $\mathcal{I}_{C,\mathcal{T}}$ has the structure as depicted in Figure 5.1.37. It is not hard to see that $(\mathcal{I}_{B,\mathcal{T}}, B)$ and $(\mathcal{I}_{C,\mathcal{T}}, C)$ are equi-similar, and thus $B \equiv_{\mathcal{T}} C$.

2. Now let B be no subconcept of C, and consider only the connected component of the canonical model $\mathcal{I}_{C,\mathcal{T}}$ which contains the vertex C. Then this part must be tree-shaped, and each vertex may either have label A or no labels at all. Furthermore, if in a branch of this tree there is a vertex D with label A, then all ancestors of D must also have label A due to the presence of the concept inclusion $\exists\, r.\, A \sqsubseteq A$ in $\mathcal{T}$. If we set n to the length of a longest path in this tree, then $(\mathcal{I}_{C,\mathcal{T}}, C)$ can be simulated in $(\mathcal{I}_{C_n,\mathcal{T}}, C_n)$, i.e., $C_n \sqsubseteq_{\mathcal{T}} C$.



5.1.36 Figure. Two canonical models



5.1.37 Figure. A canonical model

Furthermore, there exists a simulation from $(\mathcal{I}_{C_m,\mathcal{T}}, C_m)$ to $(\mathcal{I}_{C,\mathcal{T}}, C)$ where within the tree m is the length of a longest path all vertices of which are labeled with A. Hence, $C \sqsubseteq_{\mathcal{T}} C_m$.

We conclude that each $\mathcal{EL}(\Sigma)$ concept description C either is equivalent to B w.r.t. $\mathcal{T}$ or there exists an $n \in \mathbb{N}$ such that $B \sqsubsetneq_{\mathcal{T}} C_{n-1} \sqsubsetneq C$, i.e., B does not have upper neighbors with respect to $\mathcal{T}$. □

5.1.38 Corollary. *There is some $\mathcal{EL}$ TBox $\mathcal{T}$ over some signature Σ for which the subsumption relation $\sqsubseteq_{\mathcal{T}}$ is not neighborhood generated.* □

5.1.8 Relationships between ∅-neighbors and $\mathcal{T}$-neighbors

This section's goal is to explore relationships between neighbors w.r.t. ∅, neighbors w.r.t. $\mathcal{T}$, and most specific consequences. For this purpose, let $\mathcal{T}$ be some $\mathcal{EL}$ TBox, let C, D, E be $\mathcal{EL}$ concept descriptions, and let r be some role name.

1. We have that $C \prec_{\mathcal{T}} D$ does not imply $\exists\, r.\, C \prec_{\mathcal{T}} \exists\, r.\, D$. As a counterexample define $\mathcal{T} := \{\exists\, r.\, A \equiv \exists\, r.\, \top\}$. Then it holds true that $A \prec_{\mathcal{T}} \top$, but $\exists\, r.\, A \not\prec_{\mathcal{T}} \exists\, r.\, \top$.

2. It does not hold true that $C \prec_{\mathcal{T}} D$ implies $C \sqcap E \prec_{\mathcal{T}} D \sqcap E$. Consider the counterexample $\mathcal{T} := \{A \sqcap B \equiv B\}$: it holds true that $A \prec_{\mathcal{T}} \top$, but $A \sqcap B \not\prec_{\mathcal{T}} B$.
3. $C \prec_{\mathcal{T}} D$ is equivalent to $C^{\mathcal{T}} \prec_{\mathcal{T}} D^{\mathcal{T}}$, since $C \equiv_{\mathcal{T}} C^{\mathcal{T}}$ holds true for all $\mathcal{EL}$ TBoxes $\mathcal{T}$.
4. $C \prec_{\emptyset} D$ does not imply $C \prec_{\mathcal{T}} D$. As a simple counterexample consider $C := A$, $D := \top$, and $\mathcal{T} := \{\top \sqsubseteq A\}$.
5. $C^{\mathcal{T}} \prec_{\emptyset} D^{\mathcal{T}}$ implies $C \prec_{\mathcal{T}} D$.

   Proof. Assume that $C^{\mathcal{T}}$ is a lower neighbor of $D^{\mathcal{T}}$ with respect to the empty TBox $\emptyset$. We shall immediately conclude that $C^{\mathcal{T}} \sqsubseteq_{\emptyset} D^{\mathcal{T}}$ as well as $C^{\mathcal{T}} \not\sqsupseteq_{\emptyset} D^{\mathcal{T}}$. Applying Proposition 4.3.39 yields that $C \sqsubseteq_{\mathcal{T}} D$ and $C \not\sqsupseteq_{\mathcal{T}} D$, i.e., Statement 1 of Definition 5.1.1 are satisfied. Now consider an $\mathcal{EL}$ concept description E such that $C \sqsubseteq_{\mathcal{T}} E \sqsubseteq_{\mathcal{T}} D$, i.e., by means of Proposition 4.3.39 this is equivalent to $C^{\mathcal{T}} \sqsubseteq_{\emptyset} E^{\mathcal{T}} \sqsubseteq D^{\mathcal{T}}$. By assumption, we may conclude that $E^{\mathcal{T}} \equiv_{\emptyset} C^{\mathcal{T}}$ or $E^{\mathcal{T}} \equiv_{\emptyset} D^{\mathcal{T}}$, i.e., $E \equiv_{\mathcal{T}} C$ or $E \equiv_{\mathcal{T}} D$. Consequently, also Statement 2 of Definition 5.1.1 holds true, and thus $C \prec_{\mathcal{T}} D$ as claimed. □

6. We have that $C \prec_{\emptyset} D$ does not always imply $C^{\mathcal{T}} \prec_{\emptyset} D^{\mathcal{T}}$.
7. Furthermore, $C \prec_{\mathcal{T}} D$ does not imply $C \prec_{\emptyset} D$.

   Proof. Consider the signature Σ where $\Sigma_{\mathsf{C}} := \emptyset$ and $\Sigma_{\mathsf{R}} := \{r\}$. Define $\mathcal{T} := \{\exists\, r.\top \sqsubseteq \exists\, r.\exists\, r.\top\}$, and consider the concept descriptions $C := \exists\, r.\exists\, r.\top$ and $D := \top$. Obviously, modulo equivalence w.r.t. $\mathcal{T}$ there are only two distinct concept descriptions, namely $\exists\, r.\top$ and $\top$. In particular, $[\top]_{\mathcal{T}} = \{\top\}$ and $[\exists\, r.\top]_{\mathcal{T}} = \mathcal{EL}(\Sigma) \setminus \{\top\}$. We conclude that $C \prec_{\mathcal{T}} D$. However, $C \sqsubsetneq_{\emptyset} \exists\, r.\top \sqsubsetneq_{\emptyset} D$, and thus $C \not\prec_{\emptyset} D$. □

8. Eventually, $C \prec_{\mathcal{T}} D$ does not imply $C^{\mathcal{T}} \prec_{\emptyset} D^{\mathcal{T}}$.

   Proof. Let the signature Σ be defined by $\Sigma_{\mathsf{C}} := \{A_1, A_2, B_1, B_2\}$ and $\Sigma_{\mathsf{R}} := \emptyset$, and consider the TBox $\mathcal{T} := \{A_1 \equiv A_2,\ B_1 \equiv B_2\}$. Then, modulo equivalence w.r.t. $\mathcal{T}$ there exist exactly four distinct $\mathcal{EL}(\Sigma)$ concept descriptions, which are $\top$, A_1, B_1, and $A_1 \sqcap B_1$.

   Now define $C := A_1 \sqcap B_1$ and $D := A_1$. The corresponding most specific consequences satisfy $C^{\mathcal{T}} \equiv_{\emptyset} A_1 \sqcap A_2 \sqcap B_1 \sqcap B_2$ and $D^{\mathcal{T}} \equiv_{\emptyset} A_1 \sqcap A_2$, respectively. It is apparent that $A_1 \sqcap A_2 \sqcap B_1$ is strictly between $C^{\mathcal{T}}$ and $D^{\mathcal{T}}$ with respect to the empty TBox, i.e., $C^{\mathcal{T}} \not\prec_{\emptyset} D^{\mathcal{T}}$.

   Eventually, we can readily verify that $C \prec_{\mathcal{T}} D$. □

5.1.9 Comparison with Downward Refinement Operators

There is a connection between the notion of neighbors of concept descriptions and ideal downward refinement operators [LH09; Leh10]. For $\mathcal{EL}$, a *downward refinement operator* with respect to some $\mathcal{EL}$ TBox $\mathcal{T}$ is a mapping $\rho\colon \mathcal{EL}(\Sigma) \to \wp(\mathcal{EL}(\Sigma))$ such that $D \in \rho(C)$ implies $D \sqsubseteq_{\mathcal{T}} C$. Furthermore, ρ is *ideal* if it satisfies the following three conditions.

1. ρ is *finite*, i.e., $\rho(C)$ is finite for each C.
2. ρ is *proper*, i.e., $D \in \rho(C)$ implies $D \not\equiv_{\mathcal{T}} C$.
3. ρ is *complete*, i.e., $D \sqsubsetneq_{\mathcal{T}} C$ implies the existence of concept descriptions $E_1, \ldots, E_n$ such that $E_1 \in \rho(C)$, $E_2 \in \rho(E_1)$, …, $E_n \in \rho(E_{n-1})$, and $E_n \equiv_{\mathcal{T}} D$.

It is easy to see that, if ρ is complete, each set $\rho(C)$ must contain (representatives of) all lower neighbors of C w.r.t. $\mathcal{T}$. Furthermore, Corollary 5.1.13 implies that $\rho_{\mathsf{Lower}} \colon C \mapsto \mathsf{Lower}^*(C)$ is an ideal downward refinement operator w.r.t. $\emptyset$ and, more generally, defining $\rho_{\mathsf{Lower},\mathcal{T}} \colon C \mapsto \mathsf{Lower}_{\mathcal{T}}(C)$ for a cycle-restricted TBox $\mathcal{T}$ yields an ideal downward refinement operator w.r.t. $\mathcal{T}$, cf. Proposition 5.1.29. Our previous results in Section 5.1.7 also show that ideal downward refinement operators cannot exist for general, non-cycle-restricted TBoxes.

Note that [LH09; Leh10] considers subsumption with respect to an $\mathcal{ELH}$ ontology that may only contain axioms of the forms $A \sqsubseteq B$, $A \sqcap B \equiv \bot$, $r \sqsubseteq s$, $\mathsf{domain}(r) = A$, and $\mathsf{range}(r) = A$ for concept names A, B and role names r. If we ignore the role axioms, then the remaining ontology is always cycle-restricted. It then makes sense to compare the above ideal downward refinement operator $\rho_{\mathsf{Lower},\mathcal{T}}$ with the operator ρ^* from [LH09, Theorem 1].

The refinement operator ρ^* is defined by means of another refinement operator ρ, which constructs refinements by applying one of four operations to the syntax tree of the concept description: add a concept name as label to some node, refine a concept name that is a label of a node (i.e., if A labels node v and $B \sqsubseteq A$ is in $\mathcal{T}$, then A can be replaced by B), refine a role name that is a label of an edge, and attach a new subtree to some node. ρ is only weakly complete, i.e., the above condition for completeness is only satisfied for $C = \top$. The refinements of some concept description C w.r.t. ρ^* are then defined as the w.r.t. $\mathcal{T}$ most general concept descriptions D satisfying the following properties.

- There is a sequence of ρ-refinements starting with $\top$ and ending with D.
- $D \sqsubsetneq_{\mathcal{T}} C$
- $\mathsf{rd}(D) \leq \mathsf{rd}(C) + 1$

Since ρ is weakly complete, the first property is redundant. It is now obvious that D must be a lower neighbor of C w.r.t. $\mathcal{T}$, i.e., the set inclusion $\rho^*(C) \subseteq \mathsf{Lower}_{\mathcal{T}}(C)$ must be satisfied modulo $\mathcal{T}$. As we have already argued above, completeness of ρ^* implies that $\rho^*(C) \supseteq \mathsf{Lower}_{\mathcal{T}}(C)$ modulo $\mathcal{T}$.

Lehmann states in [Leh10, Section 2.2] that “refinement operators are used to structure a search process for concepts.” While this argument holds true in theory, one should be cautious when utilizing the ideal downward refinement operators ρ^*, ρ_{Lower}, and $\rho_{\mathsf{Lower},\mathcal{T}}$ for practical applications as the results in the upcoming Section 5.6 suggest. The search for an $\mathcal{EL}$ concept description may need a non-elementary number of steps—more specifically, for constructing the concept description $\exists\, r^n. (A_1 \sqcap \ldots \sqcap A_k)$ where $k \geq 3$ from $\top$ using one of the three ideal downward refinement operators (where $\mathcal{T} = \emptyset$) the number of necessary consecutive refinement steps is asymptotically bounded above and below by

$$\underbrace{2^{2^{\cdot^{\cdot^{2^{2^k}}}}}}_{n \text{ times}}.$$

5.2 Distributivity

On Page 39 we have seen that the least common subsumer $C \vee D$ can be computed, modulo $\emptyset$, by means of the following recursive formula.

$$\begin{aligned} C \vee D = & \bigsqcap(\Sigma_{\mathsf{C}} \cap \mathsf{Conj}(C) \cap \mathsf{Conj}(D)) \\ & \sqcap \bigsqcap\{\, \exists\, r.\,(E \vee F) \mid r \in \Sigma_{\mathsf{R}},\ \exists\, r.\, E \in \mathsf{Conj}(C), \text{ and } \exists\, r.\, F \in \mathsf{Conj}(D)\,\} \end{aligned}$$

5.2.1 Proposition. *For each signature Σ, the lattice $\boldsymbol{\mathcal{EL}}(\Sigma)$ is distributive, i.e., for all concept descriptions $C, D, E \in \mathcal{EL}(\Sigma)$, it holds true that*

$$\begin{aligned} & C \sqcap (D \vee E) \equiv_{\emptyset} (C \sqcap D) \vee (C \sqcap E), \\ \textit{and} \quad & C \vee (D \sqcap E) \equiv_{\emptyset} (C \vee D) \sqcap (C \vee E). \end{aligned}$$

Proof. We first show that the concept names occurring on the top level are the same for both concept descriptions $C \sqcap (D \vee E)$ and $(C \sqcap D) \vee (C \sqcap E)$. For this purpose we use the fact that the powerset lattice is distributive. Fix some concept name $A \in \Sigma_{\mathsf{C}}$. The following equivalences hold true.

$$\begin{aligned} & A \in \mathsf{Conj}(C \sqcap (D \vee E)) \\ \text{if, and only if, } & A \in \mathsf{Conj}(C) \cup \mathsf{Conj}(D \vee E) \\ \text{if, and only if, } & A \in \mathsf{Conj}(C) \cup (\mathsf{Conj}(D) \cap \mathsf{Conj}(E)) \\ \text{if, and only if, } & A \in (\mathsf{Conj}(C) \cup \mathsf{Conj}(D)) \cap (\mathsf{Conj}(C) \cup \mathsf{Conj}(E)) \\ \text{if, and only if, } & A \in \mathsf{Conj}(C \sqcap D) \cap \mathsf{Conj}(C \sqcap E) \\ \text{if, and only if, } & A \in \mathsf{Conj}((C \sqcap D) \vee (C \sqcap E)) \end{aligned}$$

Now consider an existential restriction $\exists\, r.\, Y \in \mathsf{Conj}((C \sqcap D) \vee (C \sqcap E))$, i.e., there must exist $\exists\, r.\, Y_1 \in \mathsf{Conj}(C \sqcap D)$ and $\exists\, r.\, Y_2 \in \mathsf{Conj}(C \sqcap E)$ such that $Y = Y_1 \vee Y_2$. We need to show that there is some $\exists\, r.\, X \in \mathsf{Conj}(C \sqcap (D \vee E))$ with $X \sqsubseteq_{\emptyset} Y$. If $\exists\, r.\, Y_i \in \mathsf{Conj}(C)$ for some $i \in \{1, 2\}$, then choose $X := Y_i$. Otherwise it must hold true that $\exists\, r.\, Y_1 \in \mathsf{Conj}(D)$ and $\exists\, r.\, Y_2 \in \mathsf{Conj}(E)$, which implies $\exists\, r.\,(Y_1 \vee Y_2) \in \mathsf{Conj}(D \vee E)$, and hence we may choose $X := Y_1 \vee Y_2$.

Vice versa, let $\exists\, r.\, X \in \mathsf{Conj}(C \sqcap (D \vee E))$. If $\exists\, r.\, X \in \mathsf{Conj}(C)$, then $\exists\, r.\, X \in \mathsf{Conj}((C \sqcap D) \vee (C \sqcap E))$. If otherwise $\exists\, r.\, X \in \mathsf{Conj}(D \vee E)$, there exist $\exists\, r.\, X_1 \in \mathsf{Conj}(D) \subseteq \mathsf{Conj}(C \sqcap D)$ and $\exists\, r.\, X_2 \in \mathsf{Conj}(E) \subseteq \mathsf{Conj}(C \sqcap E)$ such that $X = X_1 \vee X_2$. Thus, it follows that $\exists\, r.\, X \in \mathsf{Conj}((C \sqcap D) \vee (C \sqcap E))$ too. □

5.2.2 Proposition. *For each signature Σ, the lattice $\boldsymbol{\mathcal{EL}}(\Sigma)$ is* of locally finite length*, that is, for all concept descriptions C and D with $C \sqsubseteq_{\emptyset} D$, every chain in the interval $[C, D]$ has a finite length.*

Proof. The claim is an immediate consequence of the boundedness of $\sqsubseteq_{\emptyset}$, which Baader and Morawska showed in [BM10, Proof of Proposition 3.5]. □

According to Blyth [Bly05, Chapters 4 and 5], the following statements are obtained as immediate consequences of Propositions 5.2.1 and 5.2.2.

5.2.3 Corollary. *1. For each signature Σ, the lattice $\boldsymbol{\mathcal{EL}}(\Sigma)$ is modular, i.e., for all concept descriptions $C, D, E \in \mathcal{EL}(\Sigma)$, it holds true that*

$$\begin{aligned} (C \sqcap D) \vee (C \sqcap E) &\equiv_\emptyset C \sqcap (D \vee (C \sqcap E)), \\ (C \vee D) \sqcap (C \vee E) &\equiv_\emptyset C \vee (D \sqcap (C \vee E)), \\ C \sqsubseteq_\emptyset D \quad \textit{implies} \quad C \vee (E \sqcap D) &\equiv_\emptyset (C \vee E) \sqcap D, \\ \textit{and} \quad C \sqsupseteq_\emptyset D \quad \textit{implies} \quad C \sqcap (E \vee D) &\equiv_\emptyset (C \sqcap E) \vee D. \end{aligned}$$

2. *For each signature Σ, the lattice $\boldsymbol{\mathcal{EL}}(\Sigma)$ is both upper and lower semi-modular, i.e., for all concept descriptions $C, D \in \mathcal{EL}(\Sigma)$, it holds true that*

$$C \sqcap D \prec_\emptyset C \quad \textit{if, and only if,} \quad D \prec_\emptyset C \vee D.$$

3. *For each signature Σ, the lattice $\boldsymbol{\mathcal{EL}}(\Sigma)$ satisfies the* Jordan-Dedekind chain condition*, i.e., for all concept descriptions $C, D \in \boldsymbol{\mathcal{EL}}(\Sigma)$ with $C \sqsubsetneq_\emptyset D$, it holds true that all maximal chains in the interval $[C, D]$ have the same length.* □

The above conclusions that $\boldsymbol{\mathcal{EL}}(\Sigma)$ is of locally finite length and satisfies the Jordan-Dedekind chain condition are used in the following two sections to define distances between $\mathcal{EL}$ concept descriptions using a standard construction from lattice theory. In Section 5.3 the rank of a concept description C is defined as the length of some chain of neighbors from C to $\top$. Then, in Section 5.4 these ranks are further used to define distances between arbitrary concept descriptions.

5.3 Rank Functions

The notion of a rank function can be defined for ordered sets. The following definition specifically tailors this notion for the lattice $\boldsymbol{\mathcal{EL}}(\Sigma)$.

5.3.1 Definition. An $\mathcal{EL}$ *rank function* is a mapping $|\cdot|\colon \mathcal{EL}(\Sigma) \to \mathbb{N}$ with the following properties.

1. $|\top| = 0$
2. $C \equiv_\emptyset D$ implies $|C| = |D|$ *(equivalence closed)*
3. $C \sqsubsetneq_\emptyset D$ implies $|C| \gneq |D|$ *(strictly order preserving)*
4. $C \prec_\emptyset D$ implies $|C| + 1 = |D|$ *(neighborhood preserving)*

For an $\mathcal{EL}$ concept description C, we say that $|C|$ is the *rank* of C. △

5.3.2 Proposition. (Special case of [Bly05, Theorem 4.5]) *For each $C \in \mathcal{EL}(\Sigma)$, let $|C| := 0$ if $C \equiv_\emptyset \top$, and otherwise define*

$$|C| := \max\{\, n+1 \mid \exists\, D_1, \ldots, D_n \in \mathcal{EL}(\Sigma)\colon C \prec_\emptyset D_1 \prec_\emptyset \ldots \prec_\emptyset D_n \prec_\emptyset \top \,\}.$$

Then, $|\cdot|$ is an $\mathcal{EL}$ rank function.

Proof. It is readily verified that $|\cdot|$ satisfies Statements 1 and 2 of Definition 5.3.1. We proceed with proving that Statement 4 holds true for $|\cdot|$, which implies the validity of Statement 3 for $|\cdot|$. Consider $\mathcal{EL}$ concept descriptions C and D such that $C \prec_\emptyset D$. Clearly, if we consider a maximal chain from D to $\top$, and add C as prefix, then we have a maximal chain from C to $\top$. It is thus immediate to conclude $|C| + 1 = |D|$. □

Since $\boldsymbol{\mathcal{EL}}(\Sigma)$ satisfies the JORDAN-DEDEKIND chain condition, we infer that in order to compute the rank $|C|$ of an $\mathcal{EL}$ concept description C over Σ with $C \not\equiv_\emptyset \top$, we simply need to find *one* chain $C \prec_\emptyset D_1 \prec_\emptyset D_2 \prec_\emptyset \ldots \prec_\emptyset D_n \prec_\emptyset \top$, and then it follows that $|C| = n+1$. Furthermore, $|C| = 0$ if $C \equiv_\emptyset \top$.

A *graded lattice* is a lattice for which a rank function exists, cf. [Grä02]. Thus, we can draw the following conclusion.

5.3.3 Corollary. *For each signature Σ, the lattice $\boldsymbol{\mathcal{EL}}(\Sigma)$ is graded.* □

5.3.4 Lemma. *For all $\mathcal{EL}$ concept descriptions C and D over some signature Σ, the following equation holds true.*

$$|C| + |D| = |C \sqcap D| + |C \vee D|$$

Proof. follows from Proposition 5.2.2, Corollary 5.2.3, and [Bly05, Theorem 4.6]. □

The next proposition generalizes Lemma 5.3.4 to the case of conjunctions of arbitrary size.

5.3.5 Proposition. *Let $\mathbf{C}$ be a set of n $\mathcal{EL}$ concept descriptions over Σ. Then, the following equation holds true.*[4]

$$|\bigsqcap \mathbf{C}| = \sum_{i=1}^{n} (-1)^{i+1} \cdot \sum_{\mathbf{D} \in \binom{\mathbf{C}}{i}} |\bigvee \mathbf{D}|$$

Proof. We show the claim by induction on n. The induction base where $n \in \{0,1\}$ is trivial, and for $n = 2$ has been shown in Lemma 5.3.4. For the induction step let now $n > 2$. Using the equation from Lemma 5.3.4, we infer the following for each $C \in \mathbf{C}$.

$$|C| + |\bigsqcap \mathbf{C} \setminus \{C\}| = |\bigsqcap \mathbf{C}| + |C \vee \bigsqcap \mathbf{C} \setminus \{C\}|$$

By means of the finitely generalized distributivity law we conclude that the following equation holds true for each $C \in \mathbf{C}$.

$$|C| + |\bigsqcap \mathbf{C} \setminus \{C\}| = |\bigsqcap \mathbf{C}| + |\bigsqcap \{\, C \vee D \mid D \in \mathbf{C} \setminus \{C\} \,\}|$$

The induction hypothesis allows for replacing the ranks of the $(n-1)$-ary conjunctions, and thus yields the following equation for each $C \in \mathbf{C}$.

$$|C| + \sum_{j=1}^{n-1} \sum_{\mathbf{D} \in \binom{\mathbf{C} \setminus \{C\}}{j}} (-1)^{j+1} \cdot |\bigvee \mathbf{D}| = |\bigsqcap \mathbf{C}| + \sum_{k=1}^{n-1} \sum_{\mathbf{E} \in \binom{\{\, C \vee D \mid D \in \mathbf{C} \setminus \{C\} \,\}}{k}} (-1)^{k+1} \cdot |\bigvee \mathbf{E}|$$

[4]For some set X and some number $n \in \mathbb{N}$, we define $\binom{X}{n}$ as the set of all subsets of X that have exactly n elements.

$$= |\bigsqcap \mathbf{C}| + \sum_{k=1}^{n-1} \sum_{\mathbf{E} \in \binom{\mathbf{C} \setminus \{C\}}{k}} (-1)^{k+1} \cdot |C \vee \bigvee \mathbf{E}|$$

If we sum up the n equations, then we see that on the left hand side there are exactly n occurrences of $|C|$ for each $C \in \mathbf{C}$, and furthermore that for each $j \in \{2, \ldots, n-1\}$ and for each $\mathbf{D} \in \binom{\mathbf{C}}{j}$, there exist exactly $n - j$ occurrences of the summand $(-1)^{j+1} \cdot |\bigvee \mathbf{D}|$. On the right hand side, there are, obviously, n occurrences of $|\bigsqcap \mathbf{C}|$. Furthermore, for each $k \in \{2, \ldots, n\}$ and for each $\mathbf{E} \in \binom{\mathbf{C}}{k}$, there are exactly k occurrences of the summand $(-1)^k \cdot |\bigvee \mathbf{E}|$. Rearranging and then dividing by n eventually yields the induction claim for n. □

Initially, the asymptotic bounds on the rank function in Section 5.6.1 were unknown. The remainder of this section is concerned with an attempt on computing the rank function more efficiently than the naïve approach, which calculates the rank $|C|$ by determining some generalizing chain of neighbors from C to $\top$ and then measuring its length.

Let $C = A_1 \sqcap \ldots \sqcap A_m \sqcap \exists\, r_1.\, C_1 \sqcap \ldots \sqcap \exists\, r_n.\, C_n$ be a reduced $\mathcal{EL}$ concept description. By means of Proposition 5.3.5, we can compute the rank as follows.

$$\begin{aligned} |C| &= |A_1 \sqcap \ldots \sqcap A_m \sqcap \exists\, r_1.\, C_1 \sqcap \ldots \sqcap \exists\, r_n.\, C_n| \\ &= |A_1 \sqcap \ldots \sqcap A_m| + |\exists\, r_1.\, C_1 \sqcap \ldots \sqcap \exists\, r_n.\, C_n| - |\top| \\ &= m + |\exists\, r_1.\, C_1 \sqcap \ldots \sqcap \exists\, r_n.\, C_n| \end{aligned}$$

Furthermore, it holds true that $\exists\, r.\, C \vee \exists\, s.\, D \equiv_\emptyset \top$ if $r \neq s$. It follows that we can further simplify the rank computation as follows.

$$\begin{aligned} |\exists\, r_1.\, C_1 \sqcap \ldots \sqcap \exists\, r_n.\, C_n| &= |\bigsqcap\{ \bigsqcap\{ \exists\, r_i.\, C_i \mid i \in \{1, \ldots, n\} \text{ and } r_i = r \} \mid r \in \Sigma_\mathsf{R} \}| \\ &= \sum_{r \in \Sigma_\mathsf{R}} |\bigsqcap\{ \exists\, r_i.\, C_i \mid i \in \{1, \ldots, n\} \text{ and } r_i = r \}| \end{aligned}$$

The rank of the conjunction of existential restrictions can be computed by means of Proposition 5.3.5, and finally it is readily verified that the rank of one existential restriction $\exists\, r.\, C$ satisfies the following equation.

$$|\exists\, r.\, C| = 1 + |\bigsqcap\{ \exists\, r.\, D \mid C \prec_\emptyset D \}|$$

Next, we relate the rank of an existential restriction $\exists\, r.\, C$ to the rank of C. More specifically, we show that $|\exists\, r.\, C|$ can be bounded from below by $1 + |C|$ and from above by $1 + |C|^{1+|C|}$.

5.3.6 Lemma. *Let $\exists\, r.\, C$ be an $\mathcal{EL}$ concept description over some signature Σ. Then the following inequalities hold true.*

$$1 + |C| \leq |\exists\, r.\, C| \leq 1 + \sum_{i=1}^{|C|} \prod_{j=0}^{i-2} (|C| - j) \leq 1 + |C| \cdot |C|! \leq 1 + |C|^{1+|C|}.$$

Proof. For each natural number n with $n \leq |C|$, let

$$X_n := \bigsqcap \{ \exists r.D \mid C \prec_\emptyset^n D \}.$$

Clearly, it then holds true that $\exists r.C \equiv_\emptyset X_0 \sqsubsetneq_\emptyset X_1 \sqsubsetneq_\emptyset X_2 \sqsubsetneq_\emptyset \ldots \sqsubsetneq_\emptyset X_{|C|} \equiv_\emptyset \exists r.\top \prec_\emptyset \top$, i.e., $|\exists r.C| \geq 1 + |C|$. As a further step, we infer that $|\exists r.C| = 1 + \sum_{i=1}^{|C|} \mathsf{d}(X_{i-1}, X_i)$, and the distances[5] $\mathsf{d}(X_{i-1}, X_i)$ can be approximated as follows. Beforehand note that $|\mathsf{Upper}(Y)| \leq |\mathsf{Conj}(Y)| \leq |Y|$ holds true for all $\mathcal{EL}$ concept descriptions Y.

Apparently, $\mathsf{d}(X_0, X_1) = 1$ holds true.

In order to construct a chain of neighbors from X_1 to X_2, we could simply iterate over all top-level conjuncts of X_1 and replace each with its *unique* upper neighbor. Of course, the number of top-level conjuncts of X_1 is bounded by the number of upper neighbors of C, is henceforth bounded by the number of top-level conjuncts of C, and thus we obtain that $\mathsf{d}(X_1, X_2) \leq |\mathsf{Conj}(X_1)| \leq |\mathsf{Upper}(C)| \leq |\mathsf{Conj}(C)| \leq |C|$.

The distance between the next two concept descriptions can be approximated as follows.

$$\mathsf{d}(X_2, X_3) \leq |\{ \exists r.E \mid C \prec_\emptyset^2 E \}| \leq \sum_{C \prec_\emptyset D} \underbrace{|\{ \exists r.E \mid D \prec_\emptyset E \}|}_{\leq |\mathsf{Conj}(D)| \leq |D| \leq |C| - 1} \leq |C| \cdot (|C| - 1)$$

Continuing the approach, we infer the following upper bound for the distance between X_3 and X_4.

$$\begin{aligned} \mathsf{d}(X_3, X_4) &\leq |\{ \exists r.F \mid C \prec_\emptyset^3 F \}| \leq \sum_{C \prec_\emptyset^2 E} \underbrace{|\{ \exists r.F \mid E \prec_\emptyset F \}|}_{\leq |\mathsf{Conj}(E)| \leq |E| \leq |C| - 2} \\ &\leq \sum_{C \prec_\emptyset D} \sum_{D \prec_\emptyset E} (|C| - 2) \leq \sum_{C \prec_\emptyset D} \underbrace{|D|}_{= |C| - 1} \cdot (|C| - 2) \leq |C| \cdot (|C| - 1) \cdot (|C| - 2) \end{aligned}$$

In general, for each $i \in \mathbb{N} \cap [1, |C|]$, we observe the following.

$$\begin{aligned} d(X_{i-1}, X_i) &\leq |\{ \exists r.D \mid C \prec_\emptyset^{i-1} D \}| \\ &\leq \sum_{C \prec_\emptyset Y_1} \sum_{Y_1 \prec_\emptyset Y_2} \cdots \sum_{Y_{i-2} \prec_\emptyset Y_{i-1}} 1 \\ &\leq |C| \cdot (|C| - 1) \cdot \ldots \cdot (|C| - (i-2)) \\ &= \prod_{j=0}^{i-2} (|C| - j). \end{aligned}$$

Eventually, we conclude that the following inequalities are satisfied.

$$|\exists r.C| = 1 + \sum_{i=1}^{|C|} \mathsf{d}(X_{i-1}, X_i)$$

[5] For comparable concept descriptions $C \sqsubseteq_\emptyset D$, we define the distance $\mathsf{d}(C, D)$ as the length of a generalizing chain of neighbors from C to D. We will elaborate on such distances in more detail in the next Section 5.4.

$$
\begin{aligned}
&\leq 1 + \sum_{i=1}^{|C|} \prod_{j=0}^{i-2} (|C| - j) \\
&\leq 1 + \sum_{i=1}^{|C|} |C|! \\
&= 1 + |C| \cdot |C|! \\
&\leq 1 + |C|^{1+|C|}
\end{aligned}
$$

□

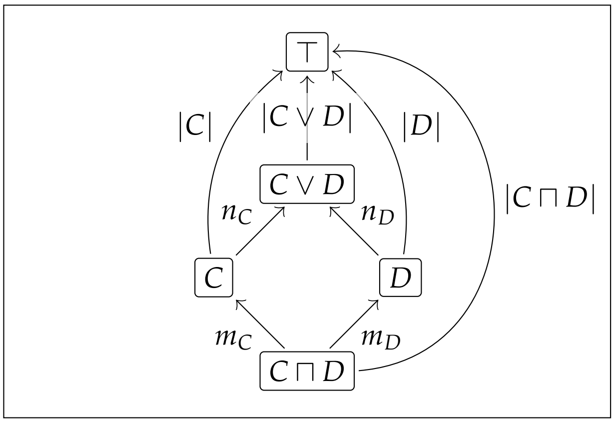
5.4 Distance Functions

5.4.1 Definition. An $\mathcal{EL}$ *metric* or $\mathcal{EL}$ *distance function* is a mapping $\mathsf{d}\colon \mathcal{EL}(\Sigma) \times \mathcal{EL}(\Sigma) \to \mathbb{N}$ with the following properties.

1. $\mathsf{d}(C, D) \geq 0$ *(non-negative)*
2. $\mathsf{d}(C, D) = 0$ if, and only if, $C \equiv_\emptyset D$ *(equivalence closed)*
3. $\mathsf{d}(C, D) = \mathsf{d}(D, C)$ *(symmetric)*
4. $\mathsf{d}(C, E) \leq \mathsf{d}(C, D) + \mathsf{d}(D, E)$ *(triangle inequality)*

We then also say that $\mathsf{d}(C, D)$ is the *distance* between C and D. △

Proposition 5.3.5 for the case $n = 2$ yields that in the rectangle shown in Figure 5.4.2 opposite edges have the same length, where length means length of a maximal chain between the end-points. It is easy to see that $|C \sqcap D| = |C| + m_C = |D| + m_D$ and $|C \vee D| = |C| - n_C = |D| - n_D$. Thus, we infer that $m_C = |C \sqcap D| - |C| = |D| - |C \vee D| = n_D$, and similarly that $m_D = n_C$. Consequently, we can define an $\mathcal{EL}$ distance function in the following way.



5.4.2 Figure. Obtaining a distance function from the rank function

5.4.3 Proposition. (Special case of [Bly05, Exercise 4.25]) *For all $C, D \in \mathcal{EL}(\Sigma)$, define*

$$\mathsf{d}(C, D) := |C \sqcap D| - |C \vee D|.$$

Then, d *is an* $\mathcal{EL}$ *metric.*

Proof. Statements 1 to 3 are obvious. We proceed with proving the triangle inequality in Statement 4. Fix some $\mathcal{EL}$ concept descriptions C, D, and E. Firstly, we observe that

$$|D| \leq |D \sqcap (C \vee E)| = |(C \sqcap D) \vee (D \sqcap E)|,$$
$$\text{and} \quad |C \sqcap E| \leq |D \sqcap (C \sqcap E)| = |(C \sqcap D) \sqcap (D \sqcap E)|.$$

Since $|X \vee Y| + |X \sqcap Y| = |X| + |Y|$ for all $\mathcal{EL}$ concept descriptions X, Y, Z, we infer that

$$|C \sqcap E| + |D| \leq |C \sqcap D| + |D \sqcap E|.$$

Multiplying the inequality with 2, adding some summands, and rearranging now yields

$$\begin{aligned} &|C \sqcap E| - (|C| + |E| - |C \sqcap E|) \\ &\leq |C \sqcap D| - (|C| + |D| - |C \sqcap D|) + |D \sqcap E| - (|D| + |E| - |D \sqcap E|). \end{aligned}$$

Finally, using the identity $|X \vee Y| = |X| + |Y| - |X \sqcap Y|$ we conclude that

$$(|C \sqcap E| - |C \vee E|) \leq (|C \sqcap D| - |C \vee D|) + (|D \sqcap E| - |D \vee E|). \qquad \square$$

The next proposition justifies the name of a distance function. Indeed, if we consider the graph of $\mathcal{EL}$ concept descriptions such that edges exist exactly between neighboring concept descriptions, then the distance $\mathsf{d}(C, D)$ is the length of a shortest path between C and D in this graph.

5.4.4 Proposition. *In the graph $(\mathcal{EL}(\Sigma), \prec_\emptyset \cup \succ_\emptyset)$ it holds true that $\mathsf{d}(C, D)$ is the length of a shortest path from C to D for all $C, D \in \mathcal{EL}(\Sigma)$.*

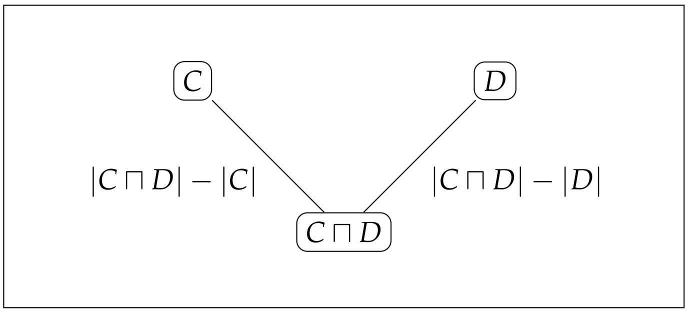
Proof. Set $\sim_\emptyset := \prec_\emptyset \cup \succ_\emptyset$. Firstly, we show by induction over ℓ that for all $\mathcal{EL}$ concept descriptions C and D over Σ and all paths from C to D of length ℓ, it holds true that $\mathsf{d}(C, D) \leq \ell$. The induction base where $\ell \in \{0, 1\}$ is obvious. For the induction step now let $C \sim_\emptyset \ldots \sim_\emptyset E \sim_\emptyset D$ be a path of length $\ell > 1$. In particular, the prefix $C \sim_\emptyset \ldots \sim_\emptyset E$ is a path of length $\ell - 1$ from C to E, and the induction hypothesis yields that $\mathsf{d}(C, E) \leq \ell - 1$. With the triangle inequality we can infer that

$$\mathsf{d}(C, D) \leq \mathsf{d}(C, E) + \mathsf{d}(E, D) \leq (\ell - 1) + 1 = \ell.$$

It remains to show that for all $C, D \in \mathcal{EL}(\Sigma)$, there exists a path $C \sim_\emptyset \ldots \sim_\emptyset D$ of length $\mathsf{d}(C, D)$. We have already seen that $\mathsf{d}(C, D) = (|C \sqcap D| - |C|) + (|C \sqcap D| - |D|)$, and there exists a path from C to $C \sqcap D$ of length $|C \sqcap D| - |C|$ as well as a path from $C \sqcap D$ to D of length $|C \sqcap D| - |D|$. Conjoining these two paths obviously yields a path from C to D of length $\mathsf{d}(C, D)$, see Figure 5.4.5. $\square$

5.4.6 Corollary. *$\mathcal{EL}(\Sigma)$ is a metric lattice, i.e., a lattice which is also a metric space.* $\square$

Furthermore, this metric space is complete, that is, every Cauchy sequence converges modulo equivalence. All subsets of $(\mathcal{EL}(\Sigma)/\emptyset, \mathsf{d})$ are open, since for each $\mathbf{C} \subseteq \mathcal{EL}(\Sigma)/\emptyset$ and each



5.4.5 Figure. Constructing a path from C to D

$[C]_\emptyset \in \mathbf{C}$, it holds true that $\mathsf{B}_{\frac{1}{2}}([C]_\emptyset) = \{[C]_\emptyset\} \subseteq \mathbf{C}$. Consequently, all subsets of $(\mathcal{EL}(\Sigma)/\emptyset, \mathsf{d})$ are closed too. It follows that, for all metric spaces (X, d'), all mappings $f \colon \mathcal{EL}(\Sigma)/\emptyset \to X$ are continuous.

$(\mathcal{EL}(\Sigma)/\emptyset, \mathsf{d})$ is not bounded, i.e., there is no $\varepsilon \in \mathbb{R}$ such that $\mathsf{d}(C, D) \leq \varepsilon$ for all $C, D \in \mathcal{EL}(\Sigma)$. It is also not precompact or totally bounded, as there do not exist finitely many open balls of radius $\frac{1}{2}$ the union of which covers $\mathcal{EL}(\Sigma)/\emptyset$. Furthermore, this metric space of $\mathcal{EL}$ concept descriptions is not compact; the sequence $(\, \exists\, r^n.\top \mid n \in \mathbb{N}\,)$ does contain a converging subsequence. However, it is locally compact, since for each point $[C]_\emptyset$ its neighborhood $\mathsf{B}_{\frac{1}{2}}([C]_\emptyset)$ clearly is compact. If the signature Σ is finite, then each closed ball $\{\, [D]_\emptyset \mid \mathsf{d}(C, D) \leq \varepsilon \,\}$ is finite and thus compact; it then follows that $(\mathcal{EL}(\Sigma)/\emptyset, \mathsf{d})$ is proper. We have already shown that all subsets of $\mathcal{EL}(\Sigma)/\emptyset$ are clopen, and hence this metric space is neither connected nor path connected. It is well known that $\mathcal{EL}(\Sigma)/\emptyset$ is countable, and so it is separable.

In a canonical way, the metric space of $\mathcal{EL}$ concept descriptions over some signature Σ induces a topological space τ_d the base of which is the set of open balls $\mathsf{B}_\varepsilon([C]_\emptyset)$ for $\varepsilon \in \mathbb{R}$ and $C \in \mathcal{EL}(\Sigma)$. In particular, τ_d is the smallest subset of $\wp(\mathcal{EL}(\Sigma)/\emptyset)$ which contains all open balls, and satisfies the following conditions.

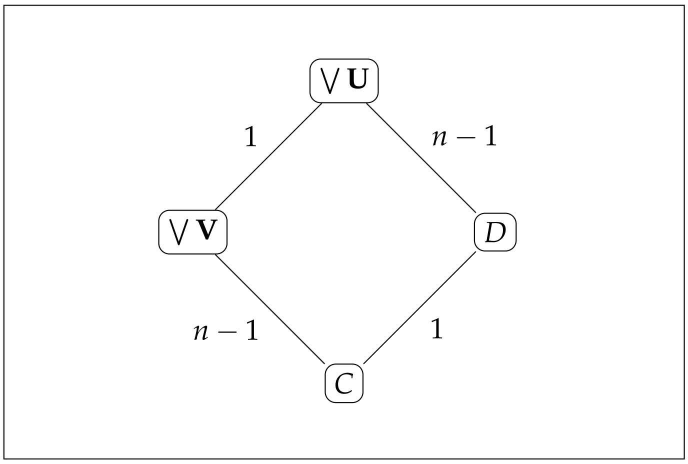
1. $\{\emptyset, \mathcal{EL}(\Sigma)/\emptyset\} \subseteq \tau_\mathsf{d}$.
2. $\bigcup \mathbf{C} \in \tau_\mathsf{d}$ for all $\mathbf{C} \subseteq \tau_\mathsf{d}$.
3. $\bigcap \mathbf{C} \in \tau_\mathsf{d}$ for all finite $\mathbf{C} \subseteq \tau_\mathsf{d}$.

Since all singletons $\{[C]_\emptyset\}$ are open balls $\mathsf{B}_{\frac{1}{2}}([C]_\emptyset)$, the induced topology τ_d contains all these singletons. Due to the $\bigcup$-closedness we conclude that $\tau_\mathsf{d} = \wp(\mathcal{EL}(\Sigma)/\emptyset)$.

It is readily verified that all pairs of distinct points have disjoint neighborhoods, and thus τ_d is a Hausdorff space, separated space, or T_2 space. Since all topological spaces the base of which are the open balls of some metric space are perfectly normal Hausdorff or T_6, we conclude that τ_d is even a T_6 space. This means that all disjoint (closed) subsets $\mathbf{C}$ and $\mathbf{D}$ of $\mathcal{EL}(\Sigma)/\emptyset$ can be precisely separated by a continuous function $f \colon \mathcal{EL}(\Sigma)/\emptyset \to \mathbb{R}$, i.e., $f^{-1}(\{0\}) = \mathbf{C}$ and $f^{-1}(\{1\}) = \mathbf{D}$.

5.4.7 Proposition. *Let* $C \in \mathcal{EL}(\Sigma)$, *then* $\mathsf{d}(C, \bigvee \mathsf{Upper}(C)) = |\mathsf{Upper}(C)|$ *modulo equivalence.*

Proof. We show by induction on n that if $\mathbf{U} \subseteq \mathsf{Upper}(C)$ with $|\mathbf{U}| = n > 0$, then $\mathsf{d}(C, \bigvee \mathbf{U}) = n$. The induction base where $n = 1$ is trivial. For the induction step now assume that $n > 1$, and let $\mathbf{U} = \mathbf{V} \cup \{D\}$ such that $D \notin \mathbf{V}$. Clearly, $\mathsf{d}(C, D) = 1$, and the induction hypothesis yields that $\mathsf{d}(C, \bigvee \mathbf{V}) = n - 1$. Since the conjunction of two non-equivalent upper neighbors of C is equivalent to C, it follows that an analogous statement holds true for an arbitrary, but greater than 2, number of upper neighbors, i.e., $C \equiv_\emptyset D \sqcap \bigvee \mathbf{V}$, and hence we can infer by means of Lemma 5.3.4 that opposite sides in the rectangle shown in Figure 5.4.8 have the same length. We conclude that $\mathsf{d}(C, \bigvee \mathbf{U}) = n$. □



5.4.8 Figure. Relating distance and upper neighbors

According the the previous proposition, we can compute the rank of an $\mathcal{EL}$ concept description C as follows.

1. Let $D := C$ and $r := 0$.
2. While $D \not\equiv_\emptyset \top$, compute the set $\mathsf{Upper}(D)$ of upper neighbors of D, set $r := r + |\mathsf{Upper}(D)|$ and $D := \bigvee \mathsf{Upper}(D)$.
3. Return r.

5.5 Similarity Functions

In [EPT15] Ecke, Peñaloza, and Turhan defined the notion of a concept similarity measure as a function of type $\mathcal{EL}(\Sigma) \times \mathcal{EL}(\Sigma) \to [0, 1]$, and then considered so-called *relaxed instances* of concept descriptions with respect to ontologies. Simply speaking, a is a relaxed instance of C if there is a concept that is similar enough to C and has a as an instance. It is straightforward to consider these relaxed instances also with respect to the distance function we have just introduced. More formally, we define them as follows.

5.5.1 Definition. Consider an interpretation $\mathcal{I}$ over some signature Σ and a concept description $C \in \mathcal{EL}(\Sigma)$, and let $n \in \mathbb{N}$. Then, the expression $\text{Ɑ} \leq n. C$ is called a *relaxed concept description*, and its extension is defined by

$$(\text{Ɑ} \leq n. C)^\mathcal{I} := \bigcup \{\, D^\mathcal{I} \mid D \in \mathcal{EL}(\Sigma) \text{ and } \mathsf{d}(C, D) \leq n \,\}.$$

Suppose that $\mathcal{O}$ is an ontology over some signature Σ, and further let $a \in \Sigma_\mathsf{I}$ be an individual name, $C \in \mathcal{EL}(\Sigma)$ a concept description, and $n \in \mathbb{N}$. We then say that a is a *relaxed instance* of C with respect to $\mathcal{O}$ and distance threshold n, denoted as $\mathcal{O} \models a \sqsubseteq \mathsf{C}{\leq}\, n.\, C$, if it holds true that $a^\mathcal{I} \in (\mathsf{C}{\leq}\, n.\, C)^\mathcal{I}$ for each model $\mathcal{I}$ of $\mathcal{O}$. △

For transforming our distance function d into a similarity function $\mathsf{s}\colon \mathcal{EL}(\Sigma) \times \mathcal{EL}(\Sigma) \to [0,1]$ we can proceed as follows. We begin with transforming d into a metric with range $[0,1)$. For that purpose, we choose an order-preserving, sub-additive[6] function $f\colon [0,\infty) \to [0,1)$ with $\mathsf{ker}(f) = \{0\}$.[7] Then $f \circ \mathsf{d}$ is such a metric with range $[0,1)$. Suitable functions are the following.

- $f\colon x \mapsto \frac{x}{1+x}$ or more generally $f\colon x \mapsto (\frac{x}{1+x})^y$ for $y > 0$
- $f\colon x \mapsto 1 - \frac{1}{2^x}$ or more generally $f\colon x \mapsto 1 - y^x$ for $y \in (0,1)$

Then, $\mathsf{s} := 1 - f \circ \mathsf{d}$ is a similarity function on $\mathcal{EL}(\Sigma)$. It is easy to verify that s satisfies the following properties which have been defined by Lehmann and Turhan in [LT12], for all $\mathcal{EL}$ concept descriptions C, D, E over Σ.

1. $\mathsf{s}(C,D) = \mathsf{s}(D,C)$ *(symmetric)*
2. $1 + \mathsf{s}(C,D) \geq \mathsf{s}(C,E) + \mathsf{s}(E,D)$ *(triangle inequality)*
3. $C \equiv_\emptyset D$ implies $\mathsf{s}(C,E) = \mathsf{s}(D,E)$ *(equivalence invariant)*
4. $C \equiv_\emptyset D$ if, and only if, $\mathsf{s}(C,D) = 1$ *(equivalence closed)*
5. $C \sqsubseteq_\emptyset D \sqsubseteq_\emptyset E$ implies $\mathsf{s}(C,D) \geq \mathsf{s}(C,E)$ *(subsumption preserving)*
6. $C \sqsubseteq_\emptyset D \sqsubseteq_\emptyset E$ implies $\mathsf{s}(C,E) \leq \mathsf{s}(D,E)$ *(reverse subsumption preserving)*

However, as it turns out such a similarity measure $1 - f \circ \mathsf{d}$ does not satisfy the property of *structural dependence*, i.e., there exist concept descriptions C and D as well as a sequence $(\, E_n \mid n \in \mathbb{N}\,)$ of concept names and existential restrictions such that $m \neq n$ implies $E_m \not\sqsubseteq_\emptyset E_n$ and where

$$\lim_{n\to\infty} (1 - f \circ \mathsf{d})(C \sqcap \bigsqcap\{\, E_\ell \mid \ell \leq n \,\}, D \sqcap \bigsqcap\{\, E_\ell \mid \ell \leq n \,\}) \neq 1.$$

For instance, consider a signature Σ without role names and where $\Sigma_C := \{A\} \cup \{\, B_n \mid n \in \mathbb{N}\,\}$. It is now readily verified that

$$(1 - f \circ \mathsf{d})(A \sqcap \bigsqcap\{\, B_\ell \mid \ell \leq n \,\}, \bigsqcap\{\, B_\ell \mid \ell \leq n \,\}) = 1 - f(1)$$

for all $n \in \mathbb{N}$, and since $f(1) > 0$ we conclude that the sequence does not converge to 1 for $n \to \infty$.

For extending our rank function $|\cdot|$ and our distance function d to $\mathcal{EL}^\bot$, we can simply define $|\bot| := \infty$, $\mathsf{d}(\bot,\bot) := 0$, and $\mathsf{d}(\bot, C) := \mathsf{d}(C,\bot) := \infty$ for $C \not\equiv_\emptyset \bot$. When transforming the extended metric into a similarity measure then two concept descriptions have a similarity of 0 if, and only if, exactly one of them is unsatisfiable. In $\mathcal{EL}$ without the bottom concept description $\bot$, a similarity of 0 can never occur when utilizing the above construction.

[6] A function $f\colon [0,\infty) \to \mathbb{R}$ is *sub-additive* if $f(x+y) \leq f(x) + f(y)$ is satisfied for all x and y, which is for instance satisfied if $f'' < 0$ and $f(0) = 0$.

[7] The *kernel* of f is $\mathsf{ker}(f) := \{\, x \mid f(x) = 0 \,\}$.

5.6 Computational Complexity

We continue with some investigations on the computational complexity of the rank function from Proposition 5.3.2. In Section 5.6.1 we show that the rank function cannot be bounded by an elementary function. We further consider some decision problems related to the rank function in Section 5.6.2.

Knuth's Up-Arrow Notation

For better readability, we use Knuth's *up-arrow notation*, that is, we set

$$x \uparrow\uparrow n := \underbrace{x^{x^{\cdot^{\cdot^{\cdot^{x^{x}}}}}}}_{n \text{ times}}$$

and further we define the following syntactic sugar as another abbreviation.

$$(x,y) \uparrow\uparrow n := \underbrace{x^{x^{\cdot^{\cdot^{\cdot^{x^{x^{y}}}}}}}}_{n \text{ times}}$$

We further use the following notions the origins of which have been described by Knuth [Knu76]. Let $g\colon \mathbb{N} \to \mathbb{N}$ be a function. Then, the sets $\mathcal{O}(g)$, $\Omega(g)$, and $\Theta(g)$ are defined as follows.

$$\mathcal{O}(g) := \{\, f \mid f\colon \mathbb{N} \to \mathbb{N} \text{ and } \exists\, c \in \mathbb{R}_+ \;\exists\, n_0 \in \mathbb{N}\;\forall\, n \geq n_0\colon f(n) \leq c \cdot g(n) \,\}$$
$$\Omega(g) := \{\, f \mid f\colon \mathbb{N} \to \mathbb{N} \text{ and } \exists\, c \in \mathbb{R}_+ \;\exists\, n_0 \in \mathbb{N}\;\forall\, n \geq n_0\colon f(n) \geq c \cdot g(n) \,\}$$
$$\Theta(g) := \{\, f \mid f\colon \mathbb{N} \to \mathbb{N} \text{ and } \exists\, c,d \in \mathbb{R}_+ \;\exists\, n_0 \in \mathbb{N}\;\forall\, n \geq n_0\colon c \cdot g(n) \leq f(n) \leq d \cdot g(n) \,\}$$

Obviously, we have that $g \in \mathcal{O}(f)$ is equivalent to $f \in \Omega(g)$, and further that $\Theta(f) = \mathcal{O}(f) \cap \Omega(f)$ holds true. It is not hard to verify that $f \in \mathcal{O}(g)$ is equivalent to $\limsup_{n\to\infty} \frac{f(n)}{g(n)} < \infty$, and dually $f \in \Omega(g)$ is equivalent to $\liminf_{n\to\infty} \frac{f(n)}{g(n)} > 0$.

Furthermore, we write $f \preceq g$ and say that f is *asymptotically bounded above* by g if $f \in \mathcal{O}(g)$, we write $f \succeq g$ and say that f is *asymptotically bounded below* by g if $f \in \Omega(g)$, and we write $f \asymp g$ and say that f is *asymptotically bounded above and below* by g if $f \in \Theta(g)$. Another notation that we use within this document is the following. We write $f \sim g$ and say that f *asymptotically equals* g if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$. Clearly, $f \sim g$ implies $f \asymp g$.

5.6.1 Computing the Rank Function

As it turns out, the rank of $\mathcal{EL}$ concept descriptions can be asymptotically $\mathsf{rd}(C)$-exponential in the size of C or, more formally, the rank can be asymptotically bounded above and below by $(2, ||C||) \uparrow\uparrow \mathsf{rd}(C)$. This vast growth makes it practically impossible to compute the rank function.

It is easy to prove that in the case where $\mathsf{rd}(C) = 1$, the rank of C can be at least exponential with respect to the size of C. To see this, consider the concept description $C^k :=$

$\exists\, r. \bigsqcap\{A_1, \ldots, A_k\}$ for each $k \in \mathbb{N}$. It is well-known that there are exponentially many subsets of $\{A_1, \ldots, A_k\}$ with $\lfloor \frac{k}{2} \rfloor$ elements; let $X_1, \ldots, X_\ell$ be an enumeration of these, and define $D_m :=$ $\bigsqcap\{\, \exists\, r. \bigsqcap X_i \mid i \in \{m, \ldots, \ell\}\,\}$. Clearly, then $C^k \sqsubsetneq_\emptyset D_1 \sqsubsetneq_\emptyset D_2 \sqsubsetneq_\emptyset \ldots \sqsubsetneq_\emptyset D_\ell \sqsubsetneq_\emptyset \top$ is an exponentially long chain of strict subsumptions. We conclude that $|C^k|$ is at least exponential in n.

For the general case where the role depth is arbitrarily big, we start with defining the signature Σ_k by $(\Sigma_k)_\mathsf{C} := \{A_1, \ldots, A_k\}$ for each $k \in \mathbb{N}$ with $k \geq 3$ and $(\Sigma_k)_\mathsf{R} := \{r\}$. Note that the precondition $k \geq 3$ is essential, as we will discuss later. Our aim for the sequel of this section is to demonstrate that the $\mathcal{EL}$ concept description

$$C_n^k := \exists\, r^n. \bigsqcap\{A_1, \ldots, A_k\}$$

over Σ_k has a rank that is asymptotically bounded below by $(2, k) \uparrow\uparrow n$, i.e., the following holds true.

$$|C_n^k| \succeq \underbrace{2^{2^{\cdot^{\cdot^{2^{2^k}}}}}}_{n \text{ times}}$$

However, we shall not do this directly, but rather translate our setting into order theory. For this purpose, it is necessary to introduce some notions.

As usual, a *partially ordered set* (abbrv. poset) $\mathbb{P}$ is a pair $(P, \leq)$ consisting of a set P and binary relation $\leq$ on P that is reflexive, antisymmetric, and transitive. An *ideal* in $\mathbb{P}$ is some subset of P that is closed under $\leq$, that is, a subset $I \subseteq P$ such that $p \leq i$ implies $p \in I$ for each $i \in I$ and for any $p \in P$. The *prime ideal* of some element $p \in P$ is the ideal

$$\downarrow p := \{\, q \mid q \in P \text{ and } q \leq p \,\}.$$

Obviously, any ideal is a union of prime ideals. As further abbreviation, let $\downarrow Q := \bigcup\{\, \downarrow q \mid q \in Q \,\}$. We denote the set of all ideals in $\mathbb{P}$ by $\mathsf{Ideals}(\mathbb{P})$.

Two elements $p, q \in P$ are *comparable* in $\mathbb{P}$ if either $p \leq q$ or $q \leq p$ holds true. A *chain* in $\mathbb{P}$ is a subset $C \subseteq P$ such that any two elements in C are comparable, while an *antichain* in $\mathbb{P}$ is a subset $A \subseteq P$ such that no two (distinct) elements in A are comparable. The *height* of $\mathbb{P}$ is defined as the supremum over all cardinalities of chains in $\mathbb{P}$, denoted by $\mathsf{height}(\mathbb{P})$. The *width* of $\mathbb{P}$ is defined as the supremum over all cardinalities of antichains in $\mathbb{P}$, denoted by $\mathsf{width}(\mathbb{P})$. DILWORTH's theorem [Dil50] states that there is always a partition of P into n disjoint chains if $\mathbb{P}$ has width n. In particular, we have that $|P| \leq \mathsf{height}(\mathbb{P}) \cdot \mathsf{width}(\mathbb{P})$. A further important theorem that connects the aforementioned notions is the following. According to STEINER [Ste93], for each partially ordered set $\mathbb{P} := (P, \leq)$, it holds true that

$$2^{\mathsf{width}(\mathbb{P})} + |P| - \mathsf{width}(\mathbb{P}) \leq |\mathsf{Ideals}(\mathbb{P})| \leq \left(\frac{|P| + \mathsf{width}(\mathbb{P})}{\mathsf{width}(\mathbb{P})}\right)^{\mathsf{width}(\mathbb{P})}.$$

Let $\mathbb{P} := (P, \leq)$ and $\mathbb{Q} := (Q, \sqsubseteq)$ be partially ordered sets. We call some mapping $f \colon P \to Q$ *order-preserving* if $x \leq y$ implies $f(x) \sqsubseteq f(y)$ for any elements $x, y \in P$. Furthermore, f is *order-reflecting* if $f(x) \sqsubseteq f(y)$ implies $x \leq y$ for any elements $x, y \in P$. It is easy to verify that f and g are both order-reflecting, if $g \circ f = \mathsf{id}_P$ and $f \circ g = \mathsf{id}_Q$ holds true, and further both f

and g are order-preserving. Another immediate corollary is that any order-reflecting mapping is injective. We call some such mapping $f\colon P \to Q$ an *order-isomorphism* from $\mathbb{P}$ to $\mathbb{Q}$, denoted as $f\colon \mathbb{P} \simeq \mathbb{Q}$, if it is bijective, order-preserving, and order-reflecting. We conclude that in order to prove that two partially ordered sets $\mathbb{P}$ and $\mathbb{Q}$ are isomorphic, it suffices to find two mutually inverse mappings between P and Q which are both order-preserving.

Let now $P_0^k := (\Sigma_k)_\mathsf{C}$ and inductively define the posets $\mathbb{P}_n^k := (P_n^k, \subseteq)$ as follows.

$$P_1^k := \wp(P_0^k)$$
$$P_{n+1}^k := \mathsf{Ideals}(\mathbb{P}_n^k) \quad \text{for any } n \in \mathbb{N}_+$$

Note that, if we set $\mathbb{P}_0^k := (P_0^k, \emptyset)$, then $P_1^k = \mathsf{Ideals}(\mathbb{P}_0^k)$ is satisfied as well. Furthermore, we define the following posets $\mathbb{E}_n^k := (E_n^k, \sqsupseteq_\emptyset)$.

$$E_1^k := \{ \textstyle\bigsqcap \mathbf{A} \mid \mathbf{A} \subseteq (\Sigma_k)_\mathsf{C} \}$$
$$E_{n+1}^k := \{ \textstyle\bigsqcap \{ \exists\, r.\, C \mid C \in \mathbf{C} \} \mid \mathbf{C} \subseteq E_n^k \} \quad \text{for any } n \in \mathbb{N}_+$$

To ease readability, we shall not distinguish between equivalence classes of $\mathbb{E}_n^k$ w.r.t. $\emptyset$ and their representatives, i.e., we identify $\mathbb{E}_n^k$ and $\mathbb{E}_n^k/\emptyset$.

Figure 5.6.1 displays the poset $\mathbb{P}_2^3$ and Figure 5.6.3 shows the poset $\mathbb{E}_2^3$; both are isomorphic to $\mathsf{FCD}(3)$, the free distributive lattice on three generators.

5.6.2 Lemma. *$\mathbb{P}_n^k$ and $\mathbb{E}_n^k$ are isomorphic for any $n \in \mathbb{N}_+$.*

Proof. We are going to prove the claim by induction on n. For the induction base let $n = 1$. It is readily verified that

$$\iota_1^k\colon \mathbb{P}_1^k \simeq \mathbb{E}_1^k$$
$$\mathbf{A} \mapsto \textstyle\bigsqcap \mathbf{A}$$

is an isomorphism from $\mathbb{P}_1^k$ to $\mathbb{E}_1^k$, and has the following inverse isomorphism.

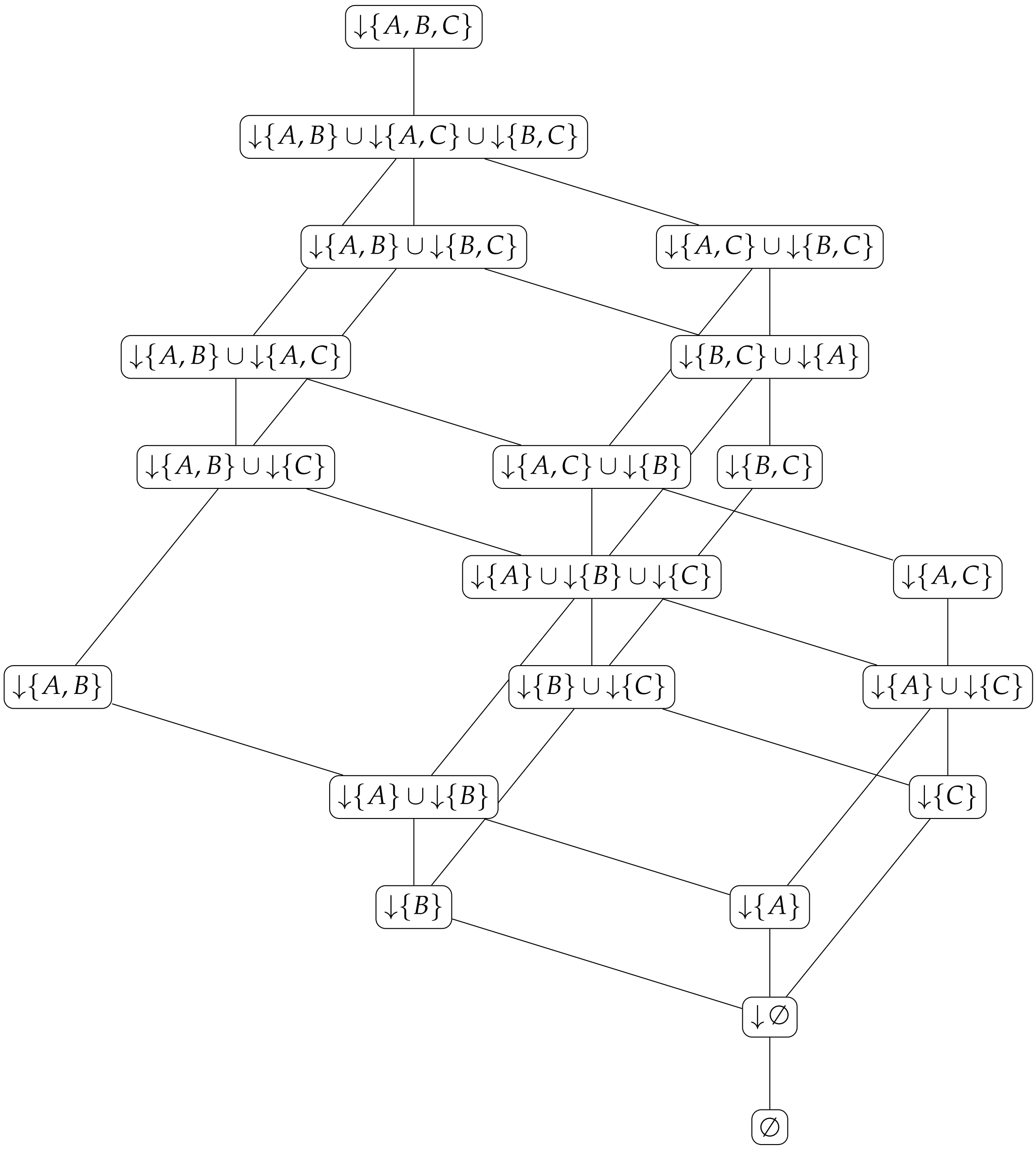
$$\kappa_1^k\colon \quad \mathbb{E}_1^k \simeq \mathbb{P}_1^k$$
$$\textstyle\bigsqcap \mathbf{A} \mapsto \mathbf{A}$$

Regarding the induction step assume that $n > 1$. We now show that

$$\iota_{n+1}^k\colon \qquad \mathbb{P}_{n+1}^k \simeq \mathbb{E}_{n+1}^k$$
$$\{p_1, \ldots, p_m\} \mapsto \exists\, r.\, \iota_n^k(p_1) \sqcap \cdots \sqcap \exists\, r.\, \iota_n^k(p_m)$$

is an isomorphism from $\mathbb{P}_{n+1}^k$ to $\mathbb{E}_{n+1}^k$, and that its inverse isomorphism is as follows.

$$\kappa_{n+1}^k\colon \qquad \mathbb{E}_{n+1}^k \simeq \mathbb{P}_{n+1}^k$$
$$\exists\, r.\, C_1 \sqcap \cdots \sqcap \exists\, r.\, C_m \mapsto {\downarrow}\, \kappa_n^k(C_1) \cup \cdots \cup {\downarrow}\, \kappa_n^k(C_m)$$



5.6.1 Figure. The ordered set $\mathbb{P}_2^3$

- Consider $\{p_1, \ldots, p_\ell\}$, $\{p_1, \ldots, p_m\} \in P^k_{n+1}$ where it holds true that $\ell \leq m$, that is, $\{p_1, \ldots, p_\ell\} \subseteq \{p_1, \ldots, p_m\}$. In particular, both $\{p_1, \ldots, p_\ell\}$ and $\{p_1, \ldots, p_m\}$ are ideals in $\mathbb{P}^k_n$. It is obvious that

$$\iota^k_{n+1}(\{p_1, \ldots, p_\ell\}) \sqsupseteq_\emptyset \iota^k_{n+1}(\{p_1, \ldots, p_m\})$$

  holds true.

- Furthermore, we have the following.

$$\begin{aligned}
&(\kappa^k_{n+1} \circ \iota^k_{n+1})(\{p_1, \ldots, p_m\}) \\
&= \kappa^k_{n+1}(\exists\, r.\, \iota^k_n(p_1) \sqcap \cdots \sqcap \exists\, r.\, \iota^k_n(p_m)) \\
&= \downarrow \kappa^k_n(\iota^k_n(p_1)) \cup \cdots \cup \downarrow \kappa^k_n(\iota^k_n(p_m)) \\
&= \downarrow p_1 \cup \cdots \cup \downarrow p_m \\
&= \{p_1, \ldots, p_m\}
\end{aligned}$$

  The penultimate equation follows from the induction hypothesis, while the last equation is true, since $\{p_1, \ldots, p_m\}$ is already an ideal.

- Assume that $\exists\, r.\, C_1 \sqcap \cdots \sqcap \exists\, r.\, C_\ell \sqsupseteq_\emptyset \exists\, r.\, D_1 \sqcap \cdots \sqcap \exists\, r.\, D_m$. We shall now prove that $\downarrow \kappa^k_n(C_1) \cup \cdots \cup \downarrow \kappa^k_n(C_\ell)$ is a subset of $\downarrow \kappa^k_n(D_1) \cup \cdots \cup \downarrow \kappa^k_n(D_m)$. So, consider some element p of the former, i.e., $p \subseteq \kappa^k_n(C_i)$ for some index $i \in \{1, \ldots, \ell\}$, which is equivalent to $\iota^k_n(p) \sqsupseteq_\emptyset C_i$. We now have that there exists some index $j \in \{1, \ldots, m\}$ such that $C_i \sqsupseteq_\emptyset D_j$, and we infer that $p \subseteq \kappa^k_n(D_j)$.

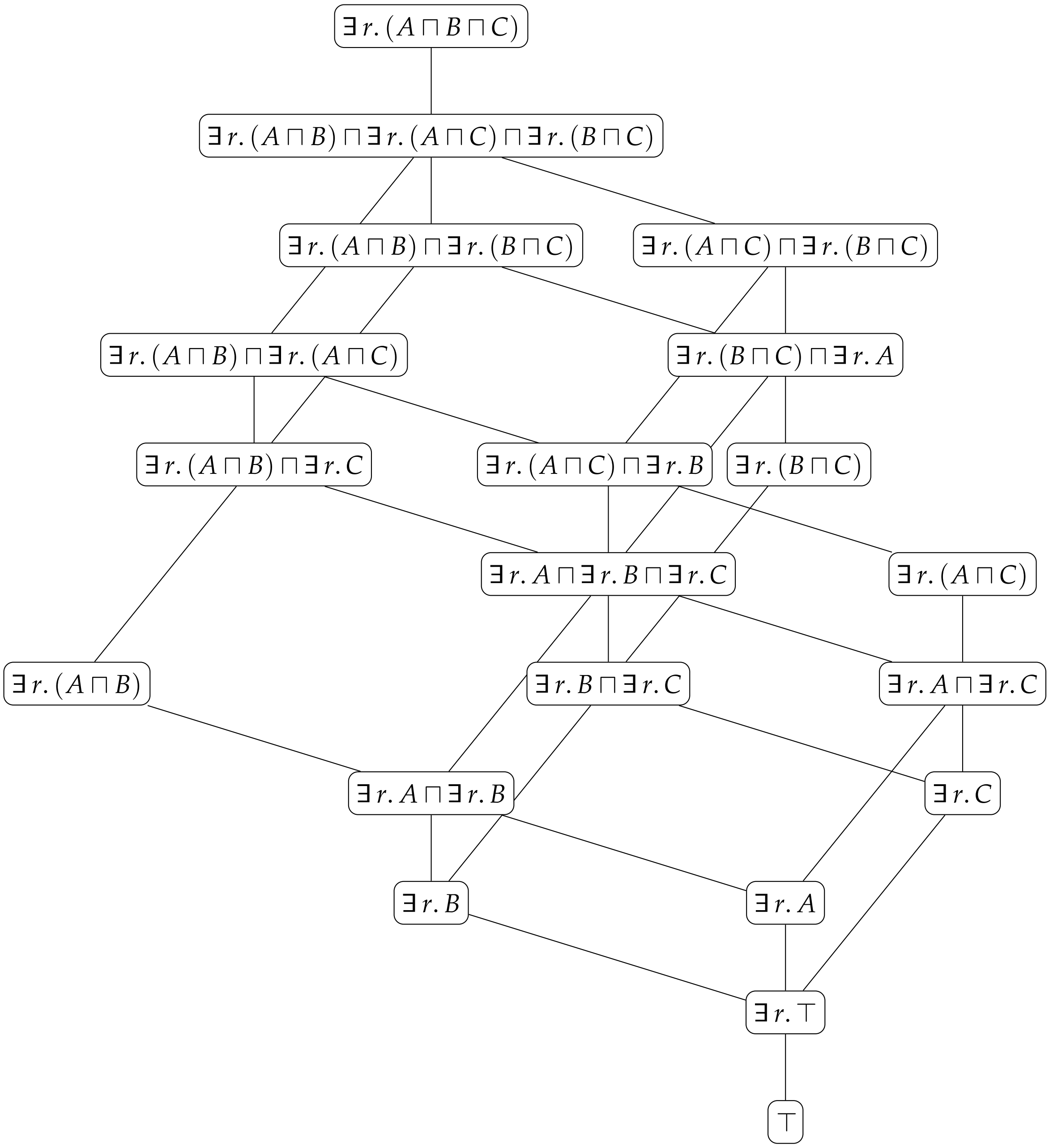
- Eventually, we prove that $\iota^k_{n+1} \circ \kappa^k_{n+1} = \mathsf{id}$.

$$\begin{aligned}
&(\iota^k_{n+1} \circ \kappa^k_{n+1})(\exists\, r.\, C_1 \sqcap \cdots \sqcap \exists\, r.\, C_m) \\
=\;& \iota^k_{n+1}(\downarrow \kappa^k_n(C_1) \cup \cdots \cup \downarrow \kappa^k_n(C_m)) \\
=\;& \bigsqcap\{\, \exists\, r.\, \iota^k_n(p) \mid p \in \downarrow \kappa^k_n(C_1) \cup \cdots \cup \downarrow \kappa^k_n(C_m) \,\} \\
\equiv_\emptyset\;& \bigsqcap\{\, \exists\, r.\, \iota^k_n(p) \mid p \in \{\kappa^k_n(C_1), \ldots, \kappa^k_n(C_m)\} \,\} \\
=\;& \exists\, r.\, \iota^k_n(\kappa^k_n(C_1)) \sqcap \cdots \sqcap \exists\, r.\, \iota^k_n(\kappa^k_n(C_m)) \\
\equiv_\emptyset\;& \exists\, r.\, C_1 \sqcap \cdots \sqcap \exists\, r.\, C_m
\end{aligned}$$

  The first equivalence follows from the fact that $p \subseteq q$ implies $\iota^k_n(p) \sqsupseteq_\emptyset \iota^k_n(q)$ and, consequently, $\exists\, r.\, \iota^k_n(p) \sqsupseteq_\emptyset \exists\, r.\, \iota^k_n(q)$. The second equivalence is an immediate consequence of our induction hypothesis. □

5.6.4 Lemma. $|C^k_n| \geq \mathsf{height}(\mathbb{P}^k_{n+1})$ *holds true for each* $n \in \mathbb{N}$.

Proof. Fix some $n \in \mathbb{N}$. In the previous lemma we have shown that $\mathbb{P}^k_{n+1}$ and $\mathbb{E}^k_{n+1}$ are isomorphic, and so it follows that $\mathsf{height}(\mathbb{P}^k_{n+1}) = \mathsf{height}(\mathbb{E}^k_{n+1})$. Apparently, $\emptyset$ is the smallest element of $\mathbb{P}^k_{n+1}$ and P^k_n is the greatest element of $\mathbb{P}^k_{n+1}$. Since ι^k_{n+1} is an order-isomorphism, it follows that (the equivalence class with representative) $\iota^k_{n+1}(\emptyset)$ is the smallest element of $\mathbb{E}^k_{n+1}$ and that (the equivalence class with representative) $\iota^k_{n+1}(P^k_n)$ is the greatest element of



5.6.3 Figure. The ordered set $\mathbb{E}_2^3$

$\mathbb{E}^k_{n+1}$. It obviously holds true that $\iota^k_{n+1}(\emptyset) \equiv_\emptyset \top$. Furthermore, we show by induction on n that $\iota^k_{n+1}(P^k_n) \equiv_\emptyset C^k_n$ is satisfied. If $n = 0$, then $\iota^k_1(P^k_0) = \iota^k_1((\Sigma_k)_\mathsf{C}) = \bigsqcap(\Sigma_k)_\mathsf{C} \equiv_\emptyset \exists\, r^0. \bigsqcap(\Sigma_k)_\mathsf{C} = C^k_0$ holds true. Now let $n > 0$. Then we have the following.

$$\begin{aligned}
& \iota^k_{n+2}(P^k_{n+1}) \\
= \;& \bigsqcap\{\, \exists\, r.\, \iota^k_{n+1}(p) \mid p \in P^k_{n+1} \,\} \\
\equiv_\emptyset \;& \exists\, r.\, \iota^k_{n+1}(P^k_n) \\
\equiv_\emptyset \;& \exists\, r.\, C^k_n \\
= \;& C^k_{n+1}
\end{aligned}$$

The second equality follows from the fact that P^k_n is the greatest element within P^k_{n+1}, which means that $\iota^k_{n+1}(p) \sqsupseteq_\emptyset \iota^k_{n+1}(P^k_n)$ is satisfied for any $p \in P^k_{n+1}$. The penultimate equation is a consequence of the induction hypothesis. Eventually, we conclude that the rank of C^k_n must be greater than or equal to the height of $\mathbb{P}^k_{n+1}$. □

The next lemma shows that each antichain A in $\mathbb{P}^k_n$ induces an antichain in $\mathbb{P}^k_{n+1}$ the cardinality of which is exponential in $|A|$. In particular, this implies that the width of $\mathbb{P}^k_{n+1}$ is exponential in the width of $\mathbb{P}^k_n$. Since the width of $\mathbb{P}^k_1$, the powerset lattice of $\{A_1, \ldots, A_k\}$, obviously is asymptotically bounded above and below by 2^k, it now follows by induction that the width of $\mathbb{P}^k_n$ is asymptotically bounded above and below by $(2, k) \uparrow\uparrow n$, that is,

$$\mathsf{width}(\mathbb{P}^k_n) \asymp (2, k) \uparrow\uparrow n.$$

5.6.5 Lemma. *Let $n > 0$ and consider some antichain A in $\mathbb{P}^k_n$ such that $|A| = 2 \cdot \ell$. Then, the following A' is an antichain in $\mathbb{P}^k_{n+1}$.*

$$A' := \{\, \downarrow a_1 \cup \cdots \cup \downarrow a_\ell \mid \{a_1, \ldots, a_\ell\} \in \tbinom{A}{\ell} \,\}$$

Proof. Consider two mutually distinct $\{a_1, \ldots, a_\ell\}$ and $\{b_1, \ldots, b_\ell\}$ in $\binom{A}{\ell}$. It is readily verified that $\downarrow a_1 \cup \cdots \cup \downarrow a_\ell$ and $\downarrow b_1 \cup \cdots \cup \downarrow b_\ell$ are elements of $\mathbb{P}^k_{n+1}$, and we shall now show that these are incomparable with respect to $\subseteq$.

From the assumption $\{a_1, \ldots, a_\ell\} \neq \{b_1, \ldots, b_\ell\}$ it follows that, without loss of generality, $a_1 \notin \{b_1, \ldots, b_\ell\}$, that is, $a_1 \neq b_i$ for any index $i \in \{1, \ldots, \ell\}$. Now the precondition that A is an antichain yields that $a_1 \not\subseteq b_i$ for each index $i \in \{1, \ldots, \ell\}$, and we infer that $a_1 \notin \downarrow b_i$ for each i, which means that

$$a_1 \notin \downarrow b_1 \cup \cdots \cup \downarrow b_\ell.$$

Furthermore, it is trivial that $a_1 \in \downarrow a_1$, and consequently we have that

$$a_1 \in \downarrow a_1 \cup \cdots \cup \downarrow a_\ell.$$

We conclude that $\downarrow a_1 \cup \cdots \cup \downarrow a_\ell \not\subseteq \downarrow b_1 \cup \cdots \cup \downarrow b_\ell$. The converse direction follows analogously, and so we have that $\downarrow a_1 \cup \cdots \cup \downarrow a_\ell$ and $\downarrow b_1 \cup \cdots \cup \downarrow b_\ell$ are indeed not comparable. □

5.6.6 Proposition. $\mathsf{width}(\mathbb{P}_n^k)$ *is asymptotically bounded above and below by* $(2,k) \uparrow\uparrow n$ *for any* $n \in \mathbb{N}_+$.

Proof. We prove the statement by induction on n. For the induction base where $n = 1$, SPERNER's theorem [Spe28] yields that

$$\mathsf{width}(\mathbb{P}_1^k) = \binom{k}{\lfloor \frac{k}{2} \rfloor}.$$

Furthermore, for the *central binomial coefficients* it is well-known that $\binom{2\cdot k}{k} \sim \frac{4^k}{\sqrt{\pi \cdot k}}$ is satisfied. We infer that

$$\mathsf{width}(\mathbb{P}_1^k) \sim \frac{2^k}{\sqrt{\frac{\pi}{2} \cdot k}}$$

or more simplified that $\mathsf{width}(\mathbb{P}_1^k) \asymp 2^k$.

Now for the induction step let $n > 1$. The induction hypothesis states that there exists some antichain in $\mathbb{P}_n^k$ that has n-exponential cardinality in k. An application of Lemma 5.6.5 yields an antichain in $\mathbb{P}_{n+1}^k$ that has $(n+1)$-exponential cardinality in k, which implies that $\mathsf{width}(\mathbb{P}_{n+1}^k)$ is at least $(n+1)$-exponential in k. Since $|P_{n+1}^k| \leq (2,k) \uparrow\uparrow (n+1)$ holds true, we conclude that $\mathsf{width}(\mathbb{P}_{n+1}^k)$ is at most $(n+1)$-exponential in k as well. □

5.6.7 Proposition. $\mathsf{height}(\mathbb{P}_{n+1}^k) \geq \mathsf{width}(\mathbb{P}_n^k)$ *for each* $n \in \mathbb{N}_+$.

Proof. We show that any antichain A in $\mathbb{P}_n^k$ induces a chain C in $\mathbb{P}_{n+1}^k$ such that $|A| = |C|$, which obviously implies our claim. Thus, consider some such antichain $A = \{a_1, \ldots, a_\ell\}$ in $\mathbb{P}_n^k$. We define C as follows.

$$C := \{c_1, \ldots, c_\ell\} \quad \text{where} \quad c_i := \downarrow a_1 \cup \cdots \cup \downarrow a_i$$

It is apparent that C consists of ideals of $\mathbb{P}_n^k$, that is, $C \subseteq P_{n+1}^k$ is satisfied. Furthermore, we can readily verify that any two elements in C are comparable with respect to $\subseteq$. We conclude that C is a chain. It remains to prove that $|A| = |C|$. Of course, $|C| \leq |A|$ follows from the very definition of C. We show the converse inequality by demonstrating that no two elements of C are equal. Let $1 \leq i_1 < i_2 \leq \ell$. Of course, it then holds true that $c_{i_1} \subseteq c_{i_2}$. Now consider some a_j where $j \in \{i_1 + 1, \ldots, i_2\}$. Since A is an antichain, it follows that a_j is $\subseteq$-incomparable to each a_h for $h \in \{1, \ldots, i_1\}$, which implies that $a_j \notin \downarrow a_h$ although $a_j \in \downarrow a_j$. Consequently, the set inclusion is strict. □

5.6.8 Corollary. $\mathsf{height}(\mathbb{P}_{n+1}^k)$ *is asymptotically bounded above and below by* $(2,k) \uparrow\uparrow n$ *for any* $n \in \mathbb{N}_+$*, that is,*

$$\mathsf{height}(\mathbb{P}_{n+1}^k) \asymp (2,k) \uparrow\uparrow n.$$ □

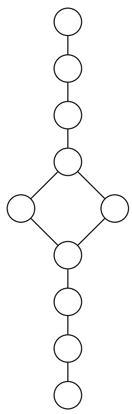
With the results obtained so far we can now conclude that our initial claim on the rank of the concept description C_n^k is true. More specifically, Corollary 5.6.8 and Lemma 5.6.4 yield the following fact.

5.6.9 Corollary. *$|C_n^k|$ is asymptotically bounded above and below by $(2,k) \uparrow\uparrow n$ for any $n \in \mathbb{N}_+$, that is,*

$$|C_n^k| \asymp (2,k) \uparrow\uparrow n. \qquad \square$$

We close this section with a justification that the precondition $k \geq 3$ is crucial for the non-elementary growth of the ranks of the concept descriptions C_n^k. The proof of this asymptotic behavior heavily relies on the fact that the width of $\mathbb{P}_{n+1}^k$ is exponential in $\mathbb{P}_n^k$. However, this does not hold true in case $k < 3$. To see this, reconsider the proof of Lemma 5.6.5. For $k < 3$ any antichain in $\mathbb{P}_1^k$ has at most two elements. Since for the central binomial coefficient it holds true that $\binom{2}{1} = 2$, we can only infer that there must exist some antichain in $\mathbb{P}_2^k$ with two elements and so on and so forth. In fact, we can easily verify that $\mathsf{width}(\mathbb{P}_n^k) \leq 2$ is always satisfied. In particular, we have the following.

- Each $\mathbb{P}_n^0$ is a chain of height n.
- Each $\mathbb{P}_n^1$ is a chain of height $n+1$.
- Each $\mathbb{P}_n^2$ consists of two chains of height n that are connected by two incomparable elements as displayed in Figure 5.6.10.



5.6.10 Figure. A line diagram of $\mathbb{P}_n^2$

In fact, the upcoming Proposition 5.6.12 shows that the ranks of the concept descriptions $\exists\, r^n.\, \top$, $\exists\, r^n.\, A$, and $\exists\, r^n.\, (A \sqcap B)$ are all linear in n.

5.6.11 Lemma. *For each $n \in \mathbb{N}$ and each reduced concept description C, it holds true that*

$$\mathsf{Upper}(\exists\, r^{n+1}.\, C) = \{\exists\, r^n.\, \bigsqcap\{\, \exists\, r.\, D \mid C \prec_\emptyset D \,\}\}.$$

Proof. We show the claim by induction over n. The base case where $n = 0$ has been proven in Proposition 5.1.5. For the inductive case let $n > 0$. According to Proposition 5.1.5 and the induction hypothesis, it holds true that

$$\mathsf{Upper}(\exists\, r^{n+2}.\, C) = \{\exists\, r.\, \exists\, r^{n+1}.\, C\}$$

$$= \{ \exists r. E \mid E \in \mathsf{Upper}(\exists r^{n+1}. C) \}$$
$$= \{\exists r. \exists r^n. \bigsqcap\{ \exists r. D \mid C \prec_\emptyset D \}\}$$
$$= \{\exists r^{n+1}. \bigsqcap\{ \exists r. D \mid C \prec_\emptyset D \}\}. \qquad \square$$

5.6.12 Proposition. *Let $n \in \mathbb{N}$. Then the following equalities hold true.*

1. $|\exists r^n. \top| = n$
2. $|\exists r^n. A| = n + 1$
3. $|\exists r^n. (A \sqcap B)| = 2 \cdot (n + 1)$

Proof. 1. We show the claim by induction on n. If $n = 0$, then $\exists r^n. \top = \top$ and we can thus immediately conclude that $|\exists r^n. \top| = 0$. Let $n > 0$. Since it holds true that $\exists r^{n+1}. C \prec_\emptyset \exists r^n. \bigsqcap\{ \exists r. D \mid C \prec_\emptyset D \}$, we infer that $|\exists r^{n+1}. \top| = 1 + |\exists r^n. \top| = 1 + n$.

2. We know that $\exists r^n. A \prec_\emptyset \exists r^n. \top$, and so we infer that $|\exists r^n. A| = 1 + |\exists r^n. \top| = 1 + n$.

3. We first observe that the following neighboring subsumptions hold true.

$$\begin{aligned}
\exists r^n. (A \sqcap B) &\prec_\emptyset \exists r^{n-1}. (\exists r. A \sqcap \exists r. B) \\
&\prec_\emptyset \exists r^{n-2}. (\exists r^2. A \sqcap \exists r^2. B) \\
&\prec_\emptyset \ldots \\
&\prec_\emptyset \exists r^{n-j}. (\exists r^j. A \sqcap \exists r^j. B) \\
&\prec_\emptyset \ldots \\
&\prec_\emptyset \exists r^n. A \sqcap \exists r^n. B \\
&\prec_\emptyset \exists r^n. A \sqcap \exists r^n. \top \\
&\equiv_\emptyset \exists r^n. A
\end{aligned}$$

We conclude that $|\exists r^n. (A \sqcap B)| = (n + 1) + |\exists r^n. A| = 2 \cdot (n + 1)$. $\square$

Of course, for the border case where $k = 3$, Lemma 5.6.5 does not immediately show a start of the non-elementary growth either. This is due to the fact that then $\mathbb{P}_1^3$ has width 3, and the central binomial coefficient $\binom{3}{1} = \binom{3}{2}$ evaluates to 3, i.e., an application of Lemma 5.6.5 does not induce a bigger antichain in $\mathbb{P}_2^3$. However, we have seen in Figure 5.6.1 that $\mathbb{P}_2^3$ has an antichain of cardinality 4. Now the sequence $(x_n \mid n \in \mathbb{N} \text{ and } n \geq 2)$ where $x_2 := 4$ and $x_{n+1} := \binom{x_n}{\lfloor \frac{x_n}{2} \rfloor}$ grows non-elementarily, cf. Table 5.6.13, and each $\mathbb{P}_n^3$ contains an antichain of cardinality x_n.

n	2	3	4	5	6	7	8
x_n	4	$\binom{4}{2} = 6$	$\binom{6}{3} = 20$	$\binom{20}{10} = 184756$	$\binom{184756}{92378} \approx 2.33 \cdot 10^{55614}$	$\approx \binom{2.33 \cdot 10^{55614}}{1.16 \cdot 10^{55614}}$	?

5.6.13 Table. Initial values of the sequence with $x_2 := 4$ and $x_{n+1} := \binom{x_n}{\lfloor \frac{x_n}{2} \rfloor}$

Eventually, we have run some experiments in which we tried to compute ranks of concept descriptions of the form $\exists\, r^n.\,(A_1 \sqcap \cdots \sqcap A_k)$. The result are listed in Table 5.6.14. As expected, this only works for sufficiently small values of n and k and henceforth we have provided lower bounds, namely lower bounds of the widths of $\mathbb{P}^k_n$, for these ranks if possible. Please note the anomaly in Table 5.6.14 for the case where $k = 3$ and $n = 2$ as explained above.

$k\backslash n$	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	7
2	2	4	6	8	10	12	14
3	3	$8 \geq \binom{3}{1}$	$20 \geq 4$	$84 \geq \binom{4}{2}$	$8573 \geq \binom{6}{3}$	$? \geq \binom{20}{10}$	$? \geq \binom{184756}{92378}$
4	4	$16 \geq \binom{4}{2}$	$168 \geq \binom{6}{3}$	$? \geq \binom{20}{10}$	$? \geq \binom{184756}{92378}$	$? \gtrapprox \binom{2.33\cdot 10^{55614}}{1.16\cdot 10^{55614}}$	?
5	5	$32 \geq \binom{5}{2}$	$7581 \geq \binom{10}{5}$	$? \geq \binom{252}{126}$	$? \gtrapprox \binom{3.63\cdot 10^{74}}{1.82\cdot 10^{74}}$	?	?
6	6	$64 \geq \binom{6}{3}$	$? \geq \binom{20}{10}$	$? \geq \binom{184756}{92378}$	$? \gtrapprox \binom{2.33\cdot 10^{55614}}{1.16\cdot 10^{55614}}$	?	?
7	7	$128 \geq \binom{7}{3}$	$? \geq \binom{35}{17}$	$? \gtrapprox \binom{4.54\cdot 10^{9}}{2.27\cdot 10^{9}}$	?	?	?
8	8	$256 \geq \binom{8}{4}$	$? \geq \binom{70}{35}$	$? \gtrapprox \binom{1.12\cdot 10^{20}}{5.61\cdot 10^{19}}$	?	?	?
9	9	$512 \geq \binom{9}{4}$	$? \geq \binom{126}{63}$	$? \gtrapprox \binom{6.03\cdot 10^{36}}{3.02\cdot 10^{36}}$	?	?	?
10	10	$1024 \geq \binom{10}{5}$	$? \geq \binom{252}{126}$	$? \gtrapprox \binom{3.63\cdot 10^{74}}{1.82\cdot 10^{74}}$	?	?	?

5.6.14 Table. Some ranks of C^k_n and corresponding lower bounds of widths of $\mathbb{P}^k_n$.

5.6.2 Decision Problems related to the Rank Function

There are three decision problems tightly related with the rank function on $\mathcal{EL}$ concept descriptions: for some given $\mathcal{EL}$ concept description C and a number $n \in \mathbb{N}$, the first one asks if the rank of C equals n, the second one asks whether n is an upper bound for $|C|$, and the third one asks if $|C| \geq n$. In particular, we define these three decision problems as follows.

$$\mathbf{P}_{\mathcal{EL}\text{-Rank}} := \{\, (C, n) \mid C \in \mathcal{EL}(\Sigma),\ n \in \mathbb{N},\ \text{and}\ |C| = n \,\}$$
$$\mathbf{P}_{\mathcal{EL}\text{-Rank-Upper-Bound}} := \{\, (C, n) \mid C \in \mathcal{EL}(\Sigma),\ n \in \mathbb{N},\ \text{and}\ |C| \leq n \,\}$$
$$\mathbf{P}_{\mathcal{EL}\text{-Rank-Lower-Bound}} := \{\, (C, n) \mid C \in \mathcal{EL}(\Sigma),\ n \in \mathbb{N},\ \text{and}\ |C| \geq n \,\}$$

In the following, we shall investigate the relationships between these problems in terms of reducibility and we shall provide bounds for their complexities.

5.6.15 Proposition. *The following Turing reductions exist.*

1. $\mathbf{P}_{\mathcal{EL}\text{-Rank}} \leq^{\mathsf{P}}_T \mathbf{P}_{\mathcal{EL}\text{-Rank-Upper-Bound}}$
2. $\mathbf{P}_{\mathcal{EL}\text{-Rank}} \leq^{\mathsf{P}}_T \mathbf{P}_{\mathcal{EL}\text{-Rank-Lower-Bound}}$
3. $\mathbf{P}_{\mathcal{EL}\text{-Upper-Bound-Rank}} \leq^{\mathsf{P}}_T \mathbf{P}_{\mathcal{EL}\text{-Rank}}$ *if numbers are unarily encoded.*

4. $\mathbf{P}_{\mathcal{EL}\text{-Upper-Bound-Rank}} \leq_T^{\mathbf{PSpace}} \mathbf{P}_{\mathcal{EL}\text{-Rank}}$ *for binary encoding of numbers*

Proof. 1. It is easy to see that $(C,n) \in \mathbf{P}_{\mathcal{EL}\text{-Rank}}$ holds true if, and only if, $(C,n) \in \mathbf{P}_{\mathcal{EL}\text{-Rank-Upper-Bound}}$ as well as $(C,n-1) \notin \mathbf{P}_{\mathcal{EL}\text{-Rank-Upper-Bound}}$ are satisfied. This shows that in order to construct some Turing machine that decides $\mathbf{P}_{\mathcal{EL}\text{-Rank}}$, we could query an oracle Turing machine for $\mathbf{P}_{\mathcal{EL}\text{-Rank-Upper-Bound}}$ twice. Indeed, this Turing machine runs in polynomial time for the following reason. If n is unarily encoded, then its predecessor $n-1$ can be computed in constant time. Otherwise if n is efficiently encoded, i.e. without loss of generality, if n is binarily encoded, then its predecessor $n-1$ can be computed in linear time (with respect to the size of n, i.e., the length of an encoding of n) as follows.

- Find the lowest bit in n that is 1. (In particular, all lower bits in n are then 0.)
- Flip this bit and all lower ones.

2. Analogously as for Statement 1, since $(C,n) \in \mathbf{P}_{\mathcal{EL}\text{-Rank}}$ holds true if, and only if, $(C,n) \in \mathbf{P}_{\mathcal{EL}\text{-Rank-Lower-Bound}}$ as well as $(C,n+1) \notin \mathbf{P}_{\mathcal{EL}\text{-Rank-Lower-Bound}}$ are satisfied.
3. It is apparent that $(C,n) \in \mathbf{P}_{\mathcal{EL}\text{-Upper-Bound-Rank}}$ is satisfied if, and only if, $(C,m) \in \mathbf{P}_{\mathcal{EL}\text{-Rank}}$ holds true for some $m \leq n$. It follows that we can construct a Turing machine deciding $\mathbf{P}_{\mathcal{EL}\text{-Upper-Bound-Rank}}$ which uses an oracle for $\mathbf{P}_{\mathcal{EL}\text{-Rank}}$. Obviously, if n is unarily encoded, then the number of queries to an oracle for $\mathbf{P}_{\mathcal{EL}\text{-Rank}}$ is linear in n.
4. In case of efficient encodings, i.e., if n is binarily encoded, then it might be the case that we need to pose an exponential number of queries to the oracle for $\mathbf{P}_{\mathcal{EL}\text{-Rank}}$, which implies that the oracle Turing machine constructed for Statement 3 now has an exponential time complexity. However, to generate and store these queries only polynomial space is required. Thus, we can now only infer that there must exist some polynomial space Turing reduction. □

5.6.16 Corollary. *The following statements are satisfied for each complexity class* $\mathbf{C}$.

1. $\mathbf{P}_{\mathcal{EL}\text{-Rank-Upper-Bound}} \in \mathbf{C}$ *implies* $\mathbf{P}_{\mathcal{EL}\text{-Rank}} \in \mathbf{P}^{\mathbf{C}}$
2. $\mathbf{P}_{\mathcal{EL}\text{-Rank-Lower-Bound}} \in \mathbf{C}$ *implies* $\mathbf{P}_{\mathcal{EL}\text{-Rank}} \in \mathbf{P}^{\mathbf{C}}$
3. *If* $\mathbf{P}_{\mathcal{EL}\text{-Rank}} \in \mathbf{C}$ *and numbers are in unary encoding, then* $\mathbf{P}_{\mathcal{EL}\text{-Rank-Upper-Bound}} \in \mathbf{P}^{\mathbf{C}}$.
4. *If* $\mathbf{P}_{\mathcal{EL}\text{-Rank}} \in \mathbf{C}$ *and numbers are in binary encoding, then* $\mathbf{P}_{\mathcal{EL}\text{-Rank-Upper-Bound}} \in \mathbf{PSpace}^{\mathbf{C}}$. □

5.6.17 Proposition. 1. *If numbers are in unary encoding, then* $\mathbf{P}_{\mathcal{EL}\text{-Rank-Upper-Bound}} \in \mathbf{2EXP}$*, otherwise* $\mathbf{P}_{\mathcal{EL}\text{-Rank-Upper-Bound}} \in \mathbf{3EXP}$.

2. *If n is fixed, then we can decide whether* $(C,n) \in \mathbf{P}_{\mathcal{EL}\text{-Rank-Upper-Bound}}$ *in deterministic polynomial time w.r.t.* $||C||$.
3. *The same upper complexity bounds hold true for* $\mathbf{P}_{\mathcal{EL}\text{-Rank-Lower-Bound}}$ *and* $\mathbf{P}_{\mathcal{EL}\text{-Rank}}$.

Proof. 1. In order to decide whether $(C,n) \in \mathbf{P}_{\mathcal{EL}\text{-Rank-Upper-Bound}}$, we can use the following procedure.

a) Set $D := C$ and $i := 0$.

b) While $i < n$, replace D with an upper neighbor of D if it exists, and increment i.

c) If $D \equiv_\emptyset \top$, then accept (C, n), otherwise reject.

Each computation of an upper neighbor of D requires quadratic time w.r.t. $||D||$, and the size of such an upper neighbor is also quadratic in $||D||$. Since the loop is executed n times, we conclude that it needs time in $\mathcal{O}(||C||^2 + (||C||^2)^2 + \ldots + ||C||^{2^n})$. If n is unarily encoded, this is obviously double exponential. Otherwise if n is binarily encoded, then the length of the encoding of n is $\log_2(n)$, and thus we infer that the procedure needs triple exponential time w.r.t. the size of the encoding of (C, n).

2. In case of n being fixed, the above procedure is polynomial in $||C||$ with exponent $2 + 2^2 + \cdots + 2^{2^n}$.
3. There are obvious variations of the above algorithm with similar complexities for the other decision problems $\mathbf{P}_{\mathcal{EL}\text{-Rank-Lower-Bound}}$ and $\mathbf{P}_{\mathcal{EL}\text{-Rank}}$. □

5.7 Most General Differences

$\mathcal{EL}$ as a sub-Boolean description logic does not provide the full set of Boolean operations on concept descriptions. In particular, only the conjunction operation is available, which corresponds to the finitary infimum operator in the lattice of $\mathcal{EL}$ concept descriptions. Furthermore, least common subsumers are the suprema in that lattice. However, there is no way to express negations of concept descriptions.

When working with sets, there is the *set difference* operator: the difference $X \setminus Y$ consists of all elements in X that are not contained in Y. Of course, it holds true that $X \setminus Y = X \cap Y^{\complement}$. In terms of lattice theory [Bly05, Section 7.1; Bir40, Section IX.12] $X \setminus Y$ is the pseudo-complement of Y relative to X, since $X \setminus Y = \bigcap\{ Z \mid Y \cup Z \supseteq X \}$ is always satisfied, and the existence of these relative pseudo-complements shows that all powerset lattices (ordered by set inclusion $\subseteq$) are *Brouwerian lattices*.

In Boolean description logics like $\mathcal{ALC}$ it is easy to define a difference operator on concept descriptions: we can simply set $C \setminus D := C \sqcap \neg D$. For defining a difference operator in $\mathcal{EL}$, we first give an equivalent characterization of set differences that uses only the operations $\cup$ and $\cap$ and the relation $\supseteq$, which have all an analog in $\mathcal{EL}$. Let $X \supseteq Y$. Then we can also characterize the set difference as the $\subseteq$-smallest set Z satisfying $X \supseteq Z$ and $X = Y \cup Z$. In case $X \not\supseteq Y$ we have $X \setminus Y = X \setminus (X \cap Y)$. Eventually, replacing $\supseteq$ with $\sqsubseteq_\emptyset$, replacing $\cup$ with $\sqcap$, and replacing $\cap$ with $\vee$ leads to the following definition of a difference operator on $\mathcal{EL}$ concept descriptions. Note that we (implicitly) make use of these most general differences in Corollary 5.1.13.

5.7.1 Definition. Let $C, D \in \mathcal{EL}(\Sigma)$ be two concept descriptions such that $C \sqsubseteq_\emptyset D$. Then, some concept description $E \in \mathcal{EL}(\Sigma)$ is called *most general difference* (abbrv. MGD) of C with respect to D (or, alternatively, *complement of D relative to C*) if it satisfies the following conditions.

1. $C \sqsubseteq_\emptyset E$
2. $C \equiv_\emptyset D \sqcap E$
3. $C \sqsubseteq_\emptyset F$ and $C \equiv_\emptyset D \sqcap F$ implies $F \sqsubseteq_\emptyset E$ for any $F \in \mathcal{EL}(\Sigma)$.

Furthermore, if $C \not\sqsubseteq_\emptyset D$, then a most general difference of C w.r.t. D is defined as a most general difference of C w.r.t. $C \vee D$. $\triangle$

It is an immediate consequence from the above definition that all most general differences of C with respect to D are equivalent. Thus, we shall denote *the* most general difference by $C \setminus D$ if it exists. Of course, in the extension $\mathcal{EL}^\perp$ of $\mathcal{EL}$ with the bottom concept description $\perp$ most general differences cannot exist, since $\perp \setminus C$ must be equivalent to the negation $\neg C$, which is a concept description that cannot be expressed in $\mathcal{EL}^\perp$ if $\perp \not\equiv_\emptyset C \not\equiv_\emptyset \top$.

We continue our investigations by considering the question whether such most general differences always exist. For this purpose, we first define a so-called syntactic difference and then show that it coincides with the most general difference.

5.7.2 Definition. For two concept descriptions $C, D \in \mathcal{EL}^\perp(\Sigma)$, the *syntactic difference* of C with respect to D is defined modulo $\emptyset$ as the following concept description.

$$C \mathbin{\setminus\!\!\setminus} D := \bigsqcap \mathsf{Conj}(C) \setminus \{ E \mid D \sqsubseteq_\emptyset E \} \qquad \triangle$$

5.7.3 Proposition. *Most general differences always exist in $\mathcal{EL}$ and can be computed in deterministic polynomial time. In particular, $C \setminus D \equiv_\emptyset C \mathbin{\setminus\!\!\setminus} D$ holds true for any two concept descriptions $C, D \in \mathcal{EL}(\Sigma)$ satisfying $C \sqsubseteq_\emptyset D$.*

Proof. It is obvious that $C \sqsubseteq_\emptyset C \mathbin{\setminus\!\!\setminus} D$. This fact together with the precondition $C \sqsubseteq_\emptyset D$ implies that $C \sqsubseteq_\emptyset D \sqcap (C \mathbin{\setminus\!\!\setminus} D)$ is satisfied as well. Now fix some $X \in \mathsf{Conj}(C)$. In case $D \sqsubseteq_\emptyset X$ it immediately follows that $D \sqcap (C \mathbin{\setminus\!\!\setminus} D) \sqsubseteq_\emptyset X$. Otherwise if $D \not\sqsubseteq_\emptyset X$, then $X \in \mathsf{Conj}(C \mathbin{\setminus\!\!\setminus} D)$ holds true, which implies $D \sqcap (C \mathbin{\setminus\!\!\setminus} D) \sqsubseteq_\emptyset X$ as well. We conclude that $D \sqcap (C \mathbin{\setminus\!\!\setminus} D) \sqsubseteq_\emptyset C$ is satisfied.

Eventually, let $E \in \mathcal{EL}(\Sigma)$ such that $C \sqsubseteq_\emptyset E$ and $C \equiv_\emptyset D \sqcap E$. We shall show that $E \sqsubseteq_\emptyset C \mathbin{\setminus\!\!\setminus} D$. Fix some $Y \in \mathsf{Conj}(C \mathbin{\setminus\!\!\setminus} D)$, that is, $Y \in \mathsf{Conj}(C)$ such that $D \not\sqsubseteq_\emptyset Y$. Since $C \equiv_\emptyset D \sqcap E$, it follows that $D \sqcap E \sqsubseteq_\emptyset Y$ and so $D \not\sqsubseteq_\emptyset Y$ implies $E \sqsubseteq_\emptyset Y$.

It is clear that $C \mathbin{\setminus\!\!\setminus} D$ can be computed in polynomial time, since the subsumption problem for $\mathcal{EL}$ is in **P**. $\square$

Example. If we consider the concept descriptions $C := \exists r.(A \sqcap B)$ and $D := \exists r.A \sqcap \exists r.B$, then we see that $C \setminus D \equiv_\emptyset C$ holds true. $\triangle$

Reformulating the general definition of a relative pseudo-complement to the case of $\boldsymbol{\mathcal{EL}}(\Sigma)$, the lattice of $\mathcal{EL}$ concept descriptions over Σ, leads to the following definition. For $\mathcal{EL}$ concept descriptions C and D, the *relative pseudo-complement* of D w.r.t. C is defined modulo $\emptyset$ as follows.

$$D \to C := \bigvee \{ E \mid D \sqcap E \sqsubseteq_\emptyset C \}$$

A lattice in which relative pseudo-complements always exist is called *Brouwerian lattice*, *implicative lattice*, or *residuated lattice*.

5.7.4 Proposition. *For all $\mathcal{EL}$ concept descriptions C and D, the most general difference of C w.r.t. D and the relative pseudo-complement of D w.r.t. C are always equivalent modulo $\emptyset$, i.e.,* *the following equivalence is satisfied.*

$$C \setminus D \equiv_\emptyset D \to C$$

Proof. Statement 2 yields that

$$C \equiv_\emptyset (C \vee D) \sqcap (C \setminus D) \sqsupseteq_\emptyset D \sqcap (C \setminus D).$$

We infer that $C \setminus D$ is an element of $\{\, E \mid D \sqcap E \sqsubseteq_\emptyset C \,\}$. It remains to show that $C \setminus D$ is most general in this set. Thus, consider a concept description E such that $C \setminus D \sqsubsetneq_\emptyset E$ and $D \sqcap E \sqsubseteq_\emptyset C$. Then we have the following.

$$\begin{aligned} C &\equiv_\emptyset (C \vee D) \sqcap (C \setminus D) \\ &\equiv_\emptyset (C \sqcap (C \setminus D)) \vee (D \sqcap (C \setminus D)) \\ &\equiv_\emptyset C \vee (D \sqcap (C \setminus D)) \\ &\sqsubseteq_\emptyset C \vee (D \sqcap E) \\ &\sqsubseteq_\emptyset C \vee C \\ &\equiv_\emptyset C. \end{aligned}$$

Further note that $C \vee (D \sqcap E) \equiv_\emptyset (C \sqcap E) \vee (D \sqcap E) \equiv_\emptyset (C \vee D) \sqcap E$ where the latter equivalence is obtained by an application of the distributivity law from Proposition 5.2.1 and the former follows from $C \sqsubseteq_\emptyset C \setminus D \sqsubseteq_\emptyset E$. Thus, we have $C \sqsubseteq_\emptyset E$ as well as $C \equiv_\emptyset (C \vee D) \sqcap E$, and so Statement 3 implies that $E \sqsubseteq_\emptyset C \setminus D$. ↯ □

5.7.5 Corollary. $\boldsymbol{\mathcal{EL}}(\Sigma)$ *is a Brouwerian lattice.* □

In the remainder of this section, we explore some mathematical laws that are satisfied for most general differences.

5.7.6 Proposition. *The following statements hold true for any concept descriptions* $C, D, E, F \in \mathcal{EL}(\Sigma)$.

1. $C \setminus D \sqsubseteq_\emptyset E \setminus F$ *if* $C \sqsubseteq_\emptyset E$ *and* $D \sqsupseteq_\emptyset F$
2. $C \setminus \top \equiv_\emptyset C$ *or, more generally,* $C \setminus D \equiv_\emptyset C$ *if* $C \perp_\emptyset D$
3. $(C \sqcap D) \setminus E \equiv_\emptyset (C \setminus E) \sqcap (D \setminus E)$
4. $(C \vee D) \setminus E \sqsupseteq_\emptyset (C \setminus E) \vee (D \setminus E)$
5. $(C \setminus D) \setminus E \equiv_\emptyset C \setminus (D \sqcap E)$

Proof. Statements 1 and 2 easily follow from Proposition 5.7.3. Since $\mathsf{Conj}(C \sqcap D) = \mathsf{Conj}(C) \cup \mathsf{Conj}(D)$ holds true, Statement 3 follows from Proposition 5.7.3 as well. We shall now prove Statement 4.

It is well-known that $\mathsf{Conj}(C \vee D) = \{\, X \vee Y \mid X \in \mathsf{Conj}(C) \text{ and } Y \in \mathsf{Conj}(D) \,\}$ holds true. It then follows according to Proposition 5.7.3 that

$$\mathsf{Conj}((C \vee D) \setminus E) = \{\, X \vee Y \mid X \in \mathsf{Conj}(C) \text{ and } Y \in \mathsf{Conj}(D) \text{ such that } E \not\sqsubseteq_\emptyset X \vee Y \,\}$$

and likewise

$$\begin{aligned}&\mathsf{Conj}((C \setminus E) \vee (D \setminus E))\\ &= \{\, X \vee Y \mid X \in \mathsf{Conj}(C) \text{ and } Y \in \mathsf{Conj}(D) \text{ such that } E \not\sqsubseteq_{\emptyset} X \text{ and } E \not\sqsubseteq_{\emptyset} Y \,\}.\end{aligned}$$

Clearly, we have that $\mathsf{Conj}((C \vee D) \setminus E) \subseteq \mathsf{Conj}((C \setminus E) \vee (D \setminus E))$, which yields the claim.

Statement 5 is again immediately clear due to Proposition 5.7.3. □

Note that the converse direction of Statement 2 does not hold true: as a counterexample one can consider the concept descriptions $C := \exists\, r.\,(A \sqcap B)$ and $D := \exists\, r.\, A \sqcap \exists\, r.\, B$ again. Furthermore, a counterexample against the converse direction of Statement 4 is $C := \exists\, r.\,(A \sqcap B_1)$, $D := \exists\, r.\,(A \sqcap B_2)$, and $E := \exists\, r.\, A$: it then holds true that $(C \vee D) \setminus E \equiv_{\emptyset} \top$ and $(C \setminus E) \vee (D \setminus E) \equiv_{\emptyset} E$.

We say that C is *strongly not subsumed* by D, denoted as $\emptyset \models C \not\sqsubseteq\!\!\!\!/ \; D$ or, alternatively, as $C \not\sqsubseteq\!\!\!\!/_{\emptyset} D$, if $C \not\sqsubseteq_{\emptyset} E$ for *each* $E \in \mathsf{Conj}(D)$. Note that $C \not\sqsubseteq_{\emptyset} D$ holds true if, and only if, there is *some* $E \in \mathsf{Conj}(D)$ such that $C \not\sqsubseteq_{\emptyset} E$.

5.7.7 Proposition. *Let $C, D, E \in \mathcal{EL}(\Sigma)$ be concept descriptions. If $C \sqsubseteq_{\emptyset} E$ and $D \not\sqsubseteq\!\!\!\!/_{\emptyset} E$, then $C \setminus D \sqsubseteq_{\emptyset} E$.*

Proof. Let $Z \in \mathsf{Conj}(E)$. Then, $C \sqsubseteq_{\emptyset} E$ implies there is some $X \in \mathsf{Conj}(C)$ such that $X \sqsubseteq_{\emptyset} Z$. Furthermore, from $D \not\sqsubseteq\!\!\!\!/_{\emptyset} E$ it follows that $D \not\sqsubseteq_{\emptyset} Z$. We infer that $D \not\sqsubseteq_{\emptyset} X$, that is, $X \in \mathsf{Conj}(C \setminus D)$ holds true as well. □

6 Axiomatization of $\mathcal{EL}$ Concept Inclusions from Closure Operators

In this chapter we show how concept inclusions can be axiomatized from datasets described in the form of a closure operator. This is not an audacious assumption, since we have already seen in Theorem 4.1.9 that each finite interpretation $\mathcal{I}$ induces the closure operator $\phi_{\mathcal{I}}$ and further Proposition 4.3.35 shows that every $\mathcal{EL}^{\bot}_{\mathsf{si}}$ TBox $\mathcal{T}$ induces the closure operator $\phi_{\mathcal{T}}$. Closure operators can then further be obtained by means of the supremum and infimum operation in the lattice of closure operators as explained in Section 1.5. We will also show that each simple ABox $\mathcal{A}$ induces a closure operator $\phi_{\mathcal{A}}$ under a slight restriction of the semantics such that concept inclusions can be entailed by $\mathcal{A}$. Section 6.2 reformulates the definition of a closure operator for the lattice of $\mathcal{EL}^{\bot}_{\mathsf{si}}$ concept descriptions and further defines the notion of *compatibility* as an important property. Section 6.3 considers infima of $\mathcal{EL}^{\bot}_{\mathsf{si}}$ closure operators, shows that the notion of an infimum corresponds to the *logical intersection* of TBoxes, and explains problems that might occur for such infima. In particular, there are examples of infima that cannot be axiomatized in form of a finite TBox or, equivalently, there are examples of TBoxes for which the logical intersection must be infinite. The suprema of $\mathcal{EL}^{\bot}_{\mathsf{si}}$ closure operators are investigated in Section 6.4. As it turns out, these are rather unproblematic in most cases. Then, Section 6.5 shows how closure operators can be "tamed" by restricting the role depth. For instance, with such restrictions it is then possible to approximate logical intersections with arbitrary precision. Eventually, Section 6.6 demonstrates how methods from Formal Concept Analysis can be utilized for computing *concept inclusion bases* for closure operators that are finite and compatible. More specifically, we reduce the problem of computing a concept inclusion base of an $\mathcal{EL}^{\bot}_{\mathsf{si}}$ closure operator to the problem of computing an implication base of a set closure operator. Additionally, *existing knowledge* in form of valid concept inclusions can be incorporated in the computation, and it is proven the resulting concept inclusion base has *minimal cardinality*.

Some applications are finally presented in Section 6.8. By iterating the construction with step-wise increasing role depth, we can compute concept inclusion bases in which role depths are only as large as necessary and cyclic $\mathcal{EL}^{\bot}_{\mathsf{si}}$ concept descriptions are avoided if possible. Another application is concerned with situations where a stream of interpretations shall be transformed in a corresponding stream of concept inclusion bases, e.g., if new observations in form of an interpretations are available on a regular basis. We also show how logical intersections of TBoxes can be approximated. Furthermore, we explain how a dataset in form of an ABox can be axiomatized if we slightly modify the semantics. A further application describes how an existing TBox can be used for filtering erroneous objects from a dataset such that concept inclusions are only axiomatized that are valid for the compatible objects. Finally, we show how our results on axiomatizing closure operators can be used to constitute an interactive, gentle repair algorithm for TBoxes.

Since in Section 6.6 and also later in Sections 7.1 and 7.2 we shall expand on DISTEL's existing approach for axiomatizing concept inclusions from interpretations [Dis11, Chapter 5], we briefly introduce it in the following Section 6.1. Furthermore, we will prove a non-trivial upper bound on the computational complexity of computing concept inclusion bases from interpretations. The role-depth-bounded variant in [BDK16] is also mentioned, and we show that there is a certain depth for which we obtain a role-depth-bounded concept inclusion base that is complete for *all* $\mathcal{EL}^{\bot}_{\mathsf{si}}$ concept inclusions valid in the input interpretation and has minimal cardinality.

6.1 Bases of $\mathcal{EL}^{\bot}_{\mathsf{si}}$ Concept Inclusions for Interpretations

A *concept inclusion base* for an interpretation $\mathcal{I}$ is a TBox $\mathcal{T}$ such that, for each concept inclusion $C \sqsubseteq D$, it holds true that $\mathcal{I} \models C \sqsubseteq D$ if, and only if, $\mathcal{T} \models C \sqsubseteq D$.

DISTEL has shown in [Dis11, Corollary 5.13 and Theorem 5.18] that, for each finite interpretation $\mathcal{I}$, there is a minimal base of $\mathcal{EL}^{\bot}_{\mathsf{si}}$ concept inclusions[1] for $\mathcal{I}$: the *canonical base* $\mathsf{Can}_{\mathcal{EL}^{\bot}_{\mathsf{si}}}(\mathcal{I})$. The construction is built upon the notion of a model-based most specific concept description, see Section 4.1, and is as follows. From the given finite interpretation $\mathcal{I}$, the *induced formal context* $\mathbb{K}_{\mathcal{I}} := (\Delta^{\mathcal{I}}, \mathbf{M}, I)$ [Dis11, Definitions 4.2 and 5.3] is constructed where

$$\mathbf{M} := \{\bot\} \cup \Sigma_{\mathsf{C}} \cup \{\,\exists\, r.\, X^{\mathcal{I}} \mid r \in \Sigma_{\mathsf{R}} \text{ and } \emptyset \neq X \subseteq \Delta^{\mathcal{I}}\,\}$$

and $I := \{\,(\delta, C) \mid \delta \in C^{\mathcal{I}}\,\}$. Then the canonical base is defined as

$$\mathsf{Can}_{\mathcal{EL}^{\bot}_{\mathsf{si}}}(\mathcal{I}) := \{\, \textstyle\bigsqcap \mathbf{P} \sqsubseteq \bigsqcap \mathbf{P}^{II} \mid \mathbf{P} \in \mathsf{PsInt}(\mathbb{K}_{\mathcal{I}}, \mathcal{S})\,\}$$

where the background knowledge $\mathcal{S}$ contains all implications $\{C\} \to \{D\}$ over $\mathbf{M}$ where $C \sqsubseteq_{\emptyset} D$ is satisfied. Note that the operator $\bigsqcap$ is defined on Page 36.

Similarly, BORCHMANN, DISTEL, and KRIEGEL have proven in [BDK16, Theorem 4.32] that each finite interpretation $\mathcal{I}$ has a minimal base of $\mathcal{EL}^{\bot}_{d}$ concept inclusions as well, namely the *canonical base* $\mathsf{Can}_{\mathcal{EL}^{\bot}}(\mathcal{I}, d)$. The construction of $\mathsf{Can}_{\mathcal{EL}^{\bot}}(\mathcal{I}, d)$ in [BDK16] is essentially the same except that in $\mathbf{M}$ each existential restriction $\exists\, r.\, X^{\mathcal{I}}$ is replaced by $\exists\, r.\, X^{\mathcal{I}_{d-1}}$.

Additionally, DISTEL explained in [Dis11, Section 5.3] that the canonical base $\mathsf{Can}_{\mathcal{EL}^{\bot}_{\mathsf{si}}}(\mathcal{I})$ can be unraveled up to a depth of $d_{\mathcal{I}} := |\Delta^{\mathcal{I}}|^{|\Delta^{\mathcal{I}}|+1}$ such that adding some further concept inclusions with a role depth $d_{\mathcal{I}} + 1$ yields a base of $\mathcal{EL}^{\bot}$ concept inclusions for $\mathcal{I}$. Although this base only contains $\mathcal{EL}^{\bot}$ concept inclusions, it is also complete for the $\mathcal{EL}^{\bot}_{\mathsf{si}}$ concept inclusions that are valid in $\mathcal{I}$. However, the unraveled base is not of minimal cardinality.

If we now directly compute the canonical base $\mathsf{Can}_{\mathcal{EL}^{\bot}}(\mathcal{I}, d_{\mathcal{I}} + 1)$ then it must be equivalent to DISTEL's unraveling, that is, it is sound and complete for *all* valid concept inclusions of $\mathcal{I}$. Furthermore, it enjoys the property of *minimal cardinality*.

6.1.1 Proposition. *The canonical base $\mathsf{Can}_{\mathcal{EL}^{\bot}}(\mathcal{I}, d_{\mathcal{I}} + 1)$, which contains only $\mathcal{EL}^{\bot}$ concept inclusions, is sound and complete for all $\mathcal{EL}^{\bot}_{\mathsf{si}}$ concept inclusions that are valid in $\mathcal{I}$, and it has*

[1] The results in [Dis11] are formulated in the description logic $\mathcal{EL}^{\bot}_{\mathsf{gfp}}$. However, we have seen in Section 3.4.2 that $\mathcal{EL}_{\mathsf{gfp}}$ and $\mathcal{EL}_{\mathsf{si}}$ are polynomially equivalent. Thus, we reformulate the results in $\mathcal{EL}^{\bot}_{\mathsf{si}}$.

minimal cardinality among all concept inclusion bases for $\mathcal{I}$. □

So far, the complexity of computing concept inclusion bases has not been determined. Using simple arguments, one could only infer that the canonical base $\mathsf{Can}(\mathcal{I})$ can be computed in double exponential time with respect to $\mathcal{I}$. However, we give an answer to this open question in the following proposition. As it turns out, $\mathsf{Can}(\mathcal{I})$ can always be computed in (single) exponential time, and further there exist interpretations $\mathcal{I}$ for which all concept inclusion bases must have sizes that are at least exponential w.r.t. $|\Delta^{\mathcal{I}}|$, that is, for which a concept inclusion base cannot be encoded in polynomial space.

6.1.2 Theorem. *For each finite interpretation $\mathcal{I}$, its canonical base $\mathsf{Can}_{\mathcal{EL}^{\bot}_{\mathsf{si}}}(\mathcal{I})$ can be computed in exponential time with respect to $\mathcal{I}$. Furthermore, there are finite interpretations $\mathcal{I}$ for which a concept inclusion base cannot be encoded in polynomial space w.r.t. $|\Delta^{\mathcal{I}}|$.*

Proof. We start the proof with citing an important result on sizes of canonical bases of formal contexts: Albano [Alb17, Theorem 3.2.1] has shown that, for any formal context $\mathbb{K} := (G, M, I)$, it holds true that $|\mathsf{Can}(\mathbb{K})| \leq |M| \cdot |\mathsf{Int}(\mathbb{K})|$. We have seen above that the premises of the concept inclusions in $\mathsf{Can}(\mathcal{I})$ correspond to the pseudo-intents of the induced formal context $\mathbb{K}_{\mathcal{I}}$ and, more specifically, for every pseudo-intent $\mathbf{P}$ of $\mathbb{K}_{\mathcal{I}}$, its conjunction $\bigsqcap \mathbf{P}$ is such a premise in $\mathsf{Can}(\mathcal{I})$. It follows that the number of concept inclusions in $\mathsf{Can}(\mathcal{I})$ is bounded by the number of implications in $\mathsf{Can}(\mathbb{K}_{\mathcal{I}})$. Applying Albano's result yields that the number of concept inclusions in $\mathsf{Can}(\mathcal{I})$ cannot be greater than $|\mathbf{M}| \cdot |\mathsf{Int}(\mathbb{K}_{\mathcal{I}})|$. Analyzing the definition of the attribute set $\mathbf{M}$ shows that its number of elements is bounded by $1 + |\Sigma_{\mathsf{C}}| + |\Sigma_{\mathsf{R}}| \cdot (2^{|\Delta^{\mathcal{I}}|} - 1)$, that is, the cardinality of $\mathbf{M}$ is at most exponential in the cardinality of the domain $\Delta^{\mathcal{I}}$. Furthermore, for every formal context $\mathbb{K} := (G, M, I)$, the number of intents of $\mathbb{K}$ is bounded by the minimum of $2^{|G|}$ and $2^{|M|}$. Consequently, the cardinality of $\mathsf{Int}(\mathbb{K}_{\mathcal{I}})$ cannot exceed $2^{|\Delta^{\mathcal{I}}|}$. Summing up, the cardinality of $\mathsf{Can}(\mathcal{I})$ is at most exponential in $|\Delta^{\mathcal{I}}|$.

In Section 4.1 we have seen that the model-based most specific concept descriptions can always be computed in exponential time: either one computes the powering $\wp(\mathcal{I})$ and then gets the MMSC $X^{\mathcal{I}}$ as $\exists^{\mathsf{sim}}(\wp(\mathcal{I}), X)$, or one computes the product $\bigtimes\{ (\mathcal{I}, \delta) \mid \delta \in X \}$ and then $\exists^{\mathsf{sim}}(\bigtimes\{ (\mathcal{I}, \delta) \mid \delta \in X \})$ equals the MMSC $X^{\mathcal{I}}$. Consequently, there is always an encoding of the attribute set $\mathbf{M}$ that has an exponential size w.r.t. $|\Delta^{\mathcal{I}}|$. Furthermore, $\mathsf{Can}(\mathcal{I})$ consists of at most exponentially many concept inclusions, the premises and conclusions in $\mathsf{Can}(\mathcal{I})$ have at most exponentially many top-level conjuncts, and each of these top-level conjuncts has an exponential size. In summary, the size of an (efficient) encoding of $\mathsf{Can}(\mathcal{I})$ has exponential size.

We proceed with demonstrating that we can compute the canonical base $\mathsf{Can}(\mathcal{I})$ in exponential time w.r.t. $|\Delta^{\mathcal{I}}|$. We divide this computation task into three steps.

Computing the attribute set $\mathbf{M}$. We have already argued that each model-based most specific concept description can be computed in exponential time, and since there are at most exponentially many model-based most specific concept descriptions, we conclude that $\mathbf{M}$ can be computed in exponential time too.

Computing the induced context $\mathbb{K}_{\mathcal{I}}$. It remains to compute the incidence relation of $\mathbb{K}_{\mathcal{I}}$. For that purpose, we consider each object $\delta \in \Delta^{\mathcal{I}}$ and each attribute $C \in \mathbf{M}$, and check if $\delta \in C^{\mathcal{I}}$ holds true. Since each such check requires time polynomial in $|\Delta^{\mathcal{I}}| + ||C||$, that is, time exponential in $|\Delta^{\mathcal{I}}|$, and exponentially many such checks are necessary, we conclude that the incidence relation of the induced context can be computed in exponential time. Including the aforementioned result shows that the induced context can be computed in exponential time.

Computing the canonical base $\mathsf{Can}(\mathcal{I})$. We consider the algorithm *NextClosures* from Section 2.1. Since $\mathbb{K}_{\mathcal{I}}$ has at most $2^{|\Delta^{\mathcal{I}}|}$ intents, there are at most $2^{|\Delta^{\mathcal{I}}|} \cdot |\mathbf{M}|$ fresh candidates during the algorithm's run on $\mathbb{K}_{\mathcal{I}}$ as input. We have already argued that the cardinality of $\mathbf{M}$ is exponential in $|\Delta^{\mathcal{I}}|$, and it follows that each fresh candidate will at most exponentially many times be closed against $\mathcal{L}^*$ (where $\mathcal{L}$ denotes the approximation of $\mathsf{Can}(\mathbb{K}_{\mathcal{I}})$ during the algorithm's run, which will satisfy $\mathcal{L} = \mathsf{Can}(\mathbb{K}_{\mathcal{I}})$ after termination). Computing the closure of a subset $C \subseteq \mathbf{M}$ against $\mathcal{L}^*$ takes time bounded by $|\mathcal{L}|^2 \cdot (|\mathbf{M}|^2 + |\mathbf{M}|)$, since for computing this closure we need to loop at most $|\mathcal{L}|$ times and within each loop iteration it is necessary to check, for each implication $X \to Y \in \mathcal{L}$, whether $X \subsetneq C$ holds true and if so, we add all elements of Y to C. Consequently, closing a subset of $\mathbf{M}$ against $\mathcal{L}^*$ requires exponential time with respect to $|\Delta^{\mathcal{I}}|$.

Summing up, during a run of *NextClosures* on an induced context $\mathbb{K}_{\mathcal{I}}$ at most exponentially many fresh candidates will be computed, each of these candidates will at most exponentially many times be closed against $\mathcal{L}^*$ for the current $\mathcal{L} \subseteq \mathsf{Can}(\mathbb{K}_{\mathcal{I}})$, and each of these closures can be computed in exponential time. Consequently, *NextClosures* runs in exponential time on the input $\mathbb{K}_{\mathcal{I}}$. Eventually, the transformation from $\mathsf{Can}(\mathbb{K}_{\mathcal{I}})$ to $\mathsf{Can}(\mathcal{I})$ is trivial, does not notably increase the size of an encoding, and needs only one traversal through $\mathsf{Can}(\mathbb{K}_{\mathcal{I}})$, that is, $\mathsf{Can}(\mathcal{I})$ can be computed from $\mathsf{Can}(\mathbb{K}_{\mathcal{I}})$ in exponential time as well.

We conclude that, using *NextClosures*, the canonical base of an interpretation can always be computed in deterministic exponential time.

Kuznetsov and Obiedkov [KO08, Theorem 4.1] have shown that the number of implications in the canonical base $\mathsf{Can}(\mathbb{K})$ of a formal context $\mathbb{K} := (G, M, I)$ can be exponential in $|G| \cdot |M|$. Their proof shows that we can even ignore the size of the attribute set M, since the considered formal contexts (G, M, I) are such that the size of M is linear in the size of G and the corresponding canonical bases contain exponentially many implications also with respect to the size of the object set G. As a consequence, we obtain that there exist formal contexts (G, M, I) for which an implication base cannot be encoded in space polynomial in $|G|$, as the canonical base of a formal context $\mathbb{K}$ has a minimal number of implications among all implication bases for $\mathbb{K}$. Since each formal context can be treated as an interpretation over a signature without role names, this important result immediately transfers from the Formal Concept Analysis setting to the Description Logic setting, and we conclude that there exist interpretations $\mathcal{I}$ for which a concept inclusion base cannot be encoded in polynomial space with respect to the cardinality of the domain $\Delta^{\mathcal{I}}$. □

Similarly to the above Theorem 6.1.2, we can prove the following result. Note that in Section 4.1 we have shown that the size of a role-depth-bounded MMSC $X^{\mathcal{I}_d}$ is exponential in $\mathcal{I}$ and d.

6.1.3 Proposition. *For each finite interpretation $\mathcal{I}$ and role depth bound $d \in \mathbb{N}$, its canonical* *base* $\mathsf{Can}_{\mathcal{EL}^{\bot}}(\mathcal{I}, d)$ *can be computed in exponential time with respect to* $\mathcal{I}$ *and* d*. Furthermore, there are finite interpretations* $\mathcal{I}$ *for which a concept inclusion base cannot be encoded in polynomial space w.r.t.* $|\Delta^{\mathcal{I}}|$. □

6.2 Closure Operators and Their Properties

In this section, we specialize the notion of a closure operator from Section 1.5 to the lattice of $\mathcal{EL}^{\bot}_{\mathsf{si}}$ concept descriptions. Since our overall goal for this chapter is to develop a method for axiomatizing concept inclusions from such closure operators, we shall also formulate a condition that must be satisfied in order to be axiomatizable.

A *closure operator* in (the dual of) $\boldsymbol{\mathcal{EL}}^{\bot}_{\mathsf{si}}(\Sigma)$ is a mapping $\phi\colon \mathcal{EL}^{\bot}_{\mathsf{si}}(\Sigma) \to \mathcal{EL}^{\bot}_{\mathsf{si}}(\Sigma)$ that satisfies the following conditions for all $\mathcal{EL}^{\bot}_{\mathsf{si}}$ concept descriptions C and D.

1. $C^{\phi} \sqsubseteq_{\emptyset} C$ *(extensive)*
2. $C \sqsubseteq_{\emptyset} D$ implies $C^{\phi} \sqsubseteq_{\emptyset} D^{\phi}$ *(monotonic)*
3. $C^{\phi\phi} \equiv_{\emptyset} C^{\phi}$ *(idempotent)*

Note that we write C^{ϕ} instead of $\phi(C)$, and such a concept description C^{ϕ} is called a *closure* of ϕ. We shall denote by $\mathsf{Clo}(\phi)$ the set of all (equivalence classes of) closures of ϕ. A closure operator ϕ is *finite* if $\mathsf{Clo}(\phi)$ is finite, and *infinite* otherwise.

Furthermore, an $\mathcal{EL}^{\bot}_{\mathsf{si}}$ concept inclusion $C \sqsubseteq D$ is *valid* for ϕ, denoted as $\phi \models C \sqsubseteq D$ or $C \sqsubseteq_{\phi} D$, if $E^{\phi} \sqsubseteq_{\emptyset} C$ implies $E^{\phi} \sqsubseteq_{\emptyset} D$ for every $\mathcal{EL}^{\bot}_{\mathsf{si}}$ concept description E. It is readily verified that $C \sqsubseteq_{\phi} D$ holds true if, and only if, $C^{\phi} \sqsubseteq_{\emptyset} D$.

6.2.1 Corollary. *Let* ϕ *be some closure operator in* $\boldsymbol{\mathcal{EL}}^{\bot}_{\mathsf{si}}(\Sigma)$*, and further assume that* C *as well as* D *are* $\mathcal{EL}^{\bot}_{\mathsf{si}}$ *concept descriptions. Then, the following statements hold true.*

1. $(C \sqcap D)^{\phi} \sqsubseteq_{\emptyset} C^{\phi} \sqcap D^{\phi}$
2. $(C \sqcap D)^{\phi} \equiv_{\emptyset} (C^{\phi} \sqcap D^{\phi})^{\phi}$
3. $C^{\phi} \vee D^{\phi} \sqsubseteq_{\emptyset} (C \vee D)^{\phi}$
4. $C^{\phi} \vee D^{\phi} \equiv_{\emptyset} (C^{\phi} \vee D^{\phi})^{\phi}$

Proof. The statements are obtained as corollaries of Section 1.5. □

6.2.2 Definition. A closure operator ϕ in $\boldsymbol{\mathcal{EL}}^{\bot}_{\mathsf{si}}(\Sigma)$ is *compatible* if $C^{\phi} \sqsubseteq_{\emptyset} \exists\, r.\, D$ implies $C^{\phi} \sqsubseteq_{\emptyset} \exists\, r.\, D^{\phi}$ for all $C, D \in \mathcal{EL}^{\bot}_{\mathsf{si}}(\Sigma)$. △

6.2.3 Lemma. *A closure operator* ϕ *in* $\boldsymbol{\mathcal{EL}}^{\bot}_{\mathsf{si}}(\Sigma)$ *is* compatible *if, and only if, for each successor-reduced* $\mathcal{EL}_{\mathsf{si}}$ *concept description* $\exists^{\mathsf{sim}}(\mathcal{I}, \delta)$ *that is a closure of* ϕ*, also the concept description* $\exists^{\mathsf{sim}}(\mathcal{I}, \epsilon)$ *is a closure of* ϕ *for each object* $\epsilon \in \Delta^{\mathcal{I}}$ *that is reachable from* δ*.*

Proof. Let ϕ be compatible and consider some successor-reduced $\mathcal{EL}_{\mathsf{si}}$ concept description $\exists^{\mathsf{sim}}(\mathcal{I}, \delta)$ that is a closure of ϕ. We prove the claim by induction on the reachability relation on $\Delta^{\mathcal{I}}$. Fix some object ϵ such that $(\delta, \epsilon) \in r^{\mathcal{I}}$, i.e., $(\exists^{\mathsf{sim}}(\mathcal{I}, \delta))^{\phi} = \exists^{\mathsf{sim}}(\mathcal{I}, \delta) \sqsubseteq_{\emptyset} \exists\, r.\, \exists^{\mathsf{sim}}(\mathcal{I}, \epsilon)$ holds true. Since ϕ is compatible, we conclude that $(\exists^{\mathsf{sim}}(\mathcal{I}, \delta))^{\phi} = \exists^{\mathsf{sim}}(\mathcal{I}, \delta) \sqsubseteq_{\emptyset}$

$\exists r.(\exists^{\mathsf{sim}}(\mathcal{I},\epsilon))^{\phi}$, i.e., there must exist some object ζ such that $(\delta,\zeta) \in r^{\mathcal{I}}$ and $\exists^{\mathsf{sim}}(\mathcal{I},\zeta) \sqsubseteq_{\emptyset} (\exists^{\mathsf{sim}}(\mathcal{I},\epsilon))^{\phi}$. This implies $\exists^{\mathsf{sim}}(\mathcal{I},\zeta) \sqsubseteq_{\emptyset} \exists^{\mathsf{sim}}(\mathcal{I},\epsilon)$ and, since $\exists^{\mathsf{sim}}(\mathcal{I},\delta)$ is successor-reduced, Definition 3.4.13 shows that $\epsilon = \zeta$. We conclude that $\exists^{\mathsf{sim}}(\mathcal{I},\epsilon)$ must be a closure of ϕ.

Vice versa, let $\exists^{\mathsf{sim}}(\mathcal{I},\delta)$ and $\exists^{\mathsf{sim}}(\mathcal{J},\epsilon)$ be $\mathcal{EL}_{\mathsf{si}}$ concept descriptions such that $\exists^{\mathsf{sim}}(\mathcal{I},\delta)$ is a closure of ϕ and $\exists^{\mathsf{sim}}(\mathcal{I},\delta) \sqsubseteq \exists r.\exists^{\mathsf{sim}}(\mathcal{J},\epsilon)$ is satisfied. Since each concept description is equivalent to its successor-reduction, we can without loss of generality assume that $\exists^{\mathsf{sim}}(\mathcal{I},\delta)$ is successor-reduced. Then there must exist some ζ with $(\delta,\zeta) \in r^{\mathcal{I}}$ such that $\exists^{\mathsf{sim}}(\mathcal{I},\zeta) \sqsubseteq_{\emptyset} \exists^{\mathsf{sim}}(\mathcal{J},\epsilon)$ holds true. By assumption, $\exists^{\mathsf{sim}}(\mathcal{I},\zeta)$ is a closure of ϕ, which implies $\exists^{\mathsf{sim}}(\mathcal{I},\zeta) \sqsubseteq_{\emptyset} (\exists^{\mathsf{sim}}(\mathcal{J},\epsilon))^{\phi}$. Eventually, this shows that $\exists^{\mathsf{sim}}(\mathcal{I},\delta) \sqsubseteq_{\emptyset} \exists r.(\exists^{\mathsf{sim}}(\mathcal{J},\epsilon))^{\phi}$ as claimed. □

6.2.4 Lemma. *Let ϕ be a closure operator in $\mathcal{EL}^{\bot}_{\mathsf{si}}(\Sigma)$. Then the following statements hold true.*

1. *For every $\mathcal{EL}^{\bot}_{\mathsf{si}}$ concept inclusion $C \sqsubseteq D$, it holds true that $\phi \models C \sqsubseteq D$ implies $\phi \models C \sqcap E \sqsubseteq D \sqcap E$ for all $\mathcal{EL}^{\bot}_{\mathsf{si}}$ concept descriptions E.*
2. *If ϕ is compatible, then for every $\mathcal{EL}^{\bot}_{\mathsf{si}}$ concept inclusion $C \sqsubseteq D$, it holds true that $\phi \models C \sqsubseteq D$ implies $\phi \models \exists r.C \sqsubseteq \exists r.D$ for all role names $r \in \Sigma_{\mathsf{R}}$.*

Proof.
1. Let $C^{\phi} \sqsubseteq_{\emptyset} D$. This implies $C^{\phi} \sqcap E \sqsubseteq_{\emptyset} D \sqcap E$. According to the mathematical laws for closure operators cited in Section 1.5, we have that $C^{\phi} \sqcap E \sqsupseteq_{\emptyset} (C \sqcap E)^{\phi}$, which then yields the claim.
2. Let $C^{\phi} \sqsubseteq_{\emptyset} D$. As ϕ is extensive, we know that $(\exists r.C)^{\phi} \sqsubseteq_{\emptyset} \exists r.C$ must be satisfied. Using compatibility yields that $(\exists r.C)^{\phi} \sqsubseteq_{\emptyset} \exists r.C^{\phi}$ holds true. We can now conclude that $(\exists r.C)^{\phi} \sqsubseteq_{\emptyset} \exists r.D$ as needed. □

If a closure operator ϕ is not compatible, then we could define the closure operator $\widehat{\phi}$ that exhaustively applies ϕ until no more changes occur. However, it cannot always be guaranteed that this process would finish after finitely many steps, i.e., an ill-formed infinite concept description might be constructed. In general, this approach can only succeed if almost all concept descriptions are closures of ϕ, i.e., there are only finitely many concept descriptions being not closed for ϕ. In this case, it is possible to show that $\widehat{\phi}$ is the $\trianglelefteq$-maximal compatible closure operator $\trianglelefteq$-below ϕ.

6.2.5 Definition. A closure operator ϕ in $\mathcal{EL}^{\bot}_{\mathsf{si}}(\Sigma)$ is called

1. *representable* if there is a set $\mathsf{Prem}(\phi)$ of $\mathcal{EL}^{\bot}_{\mathsf{si}}$ concept descriptions
2. *finitely representable* if there is a finite set $\mathsf{Prem}(\phi)$ of $\mathcal{EL}^{\bot}_{\mathsf{si}}$ concept descriptions
3. *acyclically representable* if there is a set $\mathsf{Prem}(\phi)$ of $\mathcal{EL}^{\bot}$ concept descriptions
4. *finitely acyclically representable* if there is a finite set $\mathsf{Prem}(\phi)$ of $\mathcal{EL}^{\bot}$ concept descriptions

such that, for each concept inclusion α, the (possibly infinite) TBox $\{\, C \sqsubseteq C^{\phi} \mid C \in \mathsf{Prem}(\phi) \,\}$ entails α if, and only if, α is valid in ϕ. △

6.2.6 Definition. A closure operator ϕ in $\mathcal{EL}^{\bot}_{\mathsf{si}}(\Sigma)$ is

1. *axiomatizable* if there exists a (possibly infinite) $\mathcal{EL}^{\bot}_{\mathsf{si}}$ TBox $\mathcal{T}$

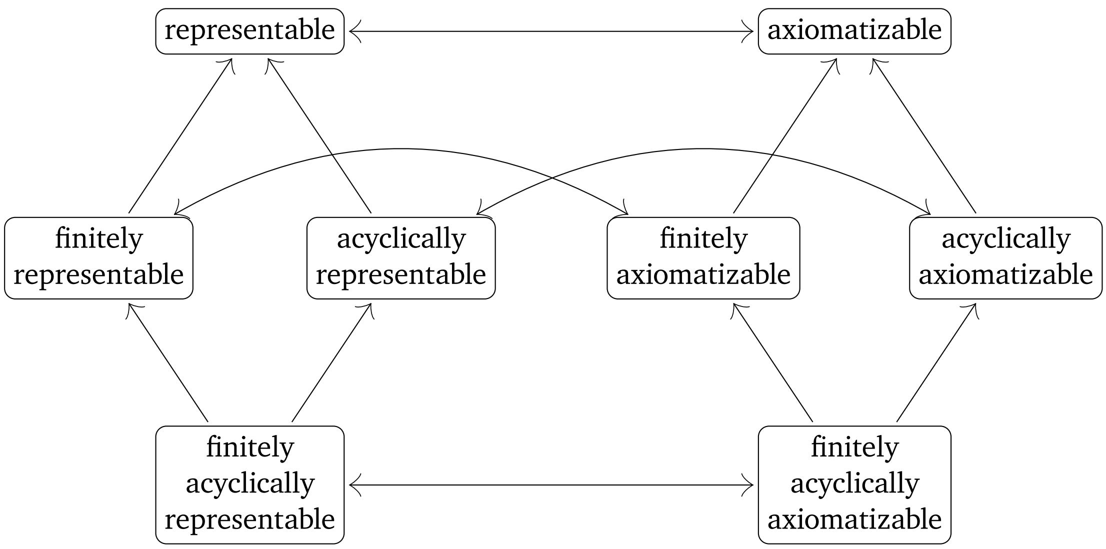
2. *finitely axiomatizable* if there exists an $\mathcal{EL}^{\bot}_{\mathsf{si}}$ TBox[2] $\mathcal{T}$
3. *acyclically axiomatizable* if there exists a (possibly infinite) TBox $\mathcal{T}$ in which all premises are $\mathcal{EL}^{\bot}$ concept descriptions and all conclusions are $\mathcal{EL}^{\bot}_{\mathsf{si}}$ concept descriptions
4. *finitely acyclically axiomatizable* if there exists a TBox $\mathcal{T}$ in which all premises are $\mathcal{EL}^{\bot}$ concept descriptions and all conclusions are $\mathcal{EL}^{\bot}_{\mathsf{si}}$ concept descriptions

such that, for any $\mathcal{EL}^{\bot}_{\mathsf{si}}$ concept inclusion α, it holds true that α is valid in ϕ if, and only if, α is entailed by $\mathcal{T}$. △

6.2.7 Lemma. *Fix some adverb $x \in \{\epsilon,$ finitely, acyclically, finitely acyclically$\}$ and let ϕ be a closure operator in $\boldsymbol{\mathcal{EL}}^{\bot}_{\mathsf{si}}(\Sigma)$. Then, ϕ is x representable if, and only if, ϕ is x axiomatizable.*

Proof. If ϕ is x representable, then $\{\, C \sqsubseteq C^{\phi} \mid C \in \mathsf{Prem}(\phi) \,\}$ x axiomatizes ϕ.

Vice versa, assume that ϕ is x axiomatized by some TBox $\mathcal{T}$. Define $\mathsf{Prem}(\phi) := \mathsf{Prem}(\mathcal{T})$; we show that $\mathcal{T}$ and $\{\, C \sqsubseteq C^{\phi} \mid C \in \mathsf{Prem}(\phi) \,\}$ are equivalent. Consider some concept inclusion $C \sqsubseteq D$ in $\mathcal{T}$. Then it must be valid in ϕ as well, i.e., $C^{\phi} \sqsubseteq_{\emptyset} D$ holds true. We conclude that $\{C \sqsubseteq C^{\phi}\}$ entails $C \sqsubseteq D$. Since each concept inclusion $C \sqsubseteq C^{\phi}$ is valid in ϕ for each premise $C \in \mathsf{Prem}(\phi)$, it follows that $\mathcal{T} \models C \sqsubseteq C^{\phi}$. □



6.2.8 Figure. Implications between properties of closure operators

The implications between properties of closure operators are compactly displayed in Figure 6.2.8.

6.2.9 Proposition. *Each axiomatizable closure operator is compatible.*

Proof. Consider some possibly infinite $\mathcal{T}$ where $\phi \models \alpha$ is equivalent to $\mathcal{T} \models \alpha$ for each concept inclusion α.

[2] Note that TBoxes are always finite, cf. Page 34.

If $C^\phi \sqsubseteq_\emptyset \exists\, r.\, D$, then $C \sqsubseteq \exists\, r.\, D$ is valid in ϕ. We infer that $C \sqsubseteq \exists\, r.\, D$ is entailed by $\mathcal{T}$. It is further trivial that the concept inclusion $D \sqsubseteq D^\phi$ is valid in ϕ, i.e., $\mathcal{T}$ entails $D \sqsubseteq D^\phi$ as well. We conclude that $\mathcal{T}$ entails $C \sqsubseteq \exists\, r.\, D^\phi$, and thus we conclude that $C^\phi \sqsubseteq_\emptyset \exists\, r.\, D^\phi$. □

6.2.10 Definition. $\mathcal{EL}^\bot_{\mathsf{si}}$ is *compact* if, for each infinite $\mathcal{EL}^\bot_{\mathsf{si}}$ TBox $\mathcal{T}$ that entails an $\mathcal{EL}^\bot_{\mathsf{si}}$ concept inclusion $C \sqsubseteq D$, there is a finite subset $\mathcal{F} \subseteq_{\mathsf{fin}} \mathcal{T}$ such that $\mathcal{F} \models C \sqsubseteq D$. △

[Lut19]. *$\mathcal{EL}^\bot_{\mathsf{si}}$ is not compact for infinite signatures.*

Proof. Consider the following TBox $\mathcal{T}$ where we use $\exists\, r^\infty.\, \top$ as an abbreviation for the concept description $\exists^{\mathsf{sim}}(\mathcal{I}, \delta)$ where $\Delta^\mathcal{I} := \{\delta\}$ and $r^\mathcal{I} := \{(\delta, \delta)\}$

$$\mathcal{T} := \{\, A \sqsubseteq \exists\, r.\, X_1,\ X_n \sqsubseteq \exists\, r.\, X_{n+1},\ \exists\, r^\infty.\, \top \sqsubseteq B \mid n \in \mathbb{N}_+ \,\}$$

It is easy to see that $\mathcal{T}$ entails the concept inclusion $A \sqsubseteq B$, but no finite subset of $\mathcal{T}$ does so. □

It is currently unclear whether $\mathcal{EL}^\bot_{\mathsf{si}}$ is compact for finite signatures. If this happens to be true, then we can prove as follows that the properties compatibility and axiomatizability coincide.

6.2.11 Proposition. *Assume that the signature Σ is finite and $\mathcal{EL}^\bot_{\mathsf{si}}$ is compact for finite signatures. Each compatible closure operator is axiomatizable.*

Proof. If ϕ is compatible, then we choose $\mathcal{T} := \{\, \alpha \mid \phi \models \alpha \,\}$. It is then obvious that $\phi \models \alpha$ implies $\mathcal{T} \models \alpha$ for each concept inclusion α.

It remains to prove that $\mathcal{T} \models \alpha$ also implies $\phi \models \alpha$. Thus let $C \sqsubseteq D$ be some concept inclusion that is entailed from $\mathcal{T}$. Compactness yields a finite sub-TBox $\mathcal{F} \subseteq_{\mathsf{fin}} \mathcal{T}$ which entails our considered concept inclusion $C \sqsubseteq D$, that is, $C^\mathcal{F} \sqsubseteq_\emptyset D$ holds true.

First assume that C is satisfiable w.r.t. $\mathcal{F}$. If $C^\phi \equiv_\emptyset \bot$, then $C^\phi \sqsubseteq_\emptyset D$ is clearly satisfied. Now let $C^\phi \not\equiv_\emptyset \bot$. Proposition 4.3.31 yields that $C^\mathcal{F}$ is equivalent to $C^{\mathcal{F}_{\mathsf{sat}}}$. We continue with proving that $C^\phi \sqsubseteq_\emptyset C^\mathcal{F}$, which then implies the claim.

We know that we can construct the most specific consequence $C^{\mathcal{F}_{\mathsf{sat}}}$ step-wise with a rule-based approach. More specifically, there is a sequence of ABoxes

$$\{C \sqsubseteq C\} =: \mathcal{A}_0 \to \mathcal{A}_1 \to \mathcal{A}_2 \to \ldots \to \mathcal{A}_n := \mathcal{A}'_{C,\mathcal{F}_{\mathsf{sat}}}$$

such that each $\mathcal{A}_{i+1}$ is obtained from $\mathcal{A}_i$ by applying the $\sqcap$-rule, the $\exists$-rule, or the $\sqsubseteq'$-rule. Without loss of generality, assume that the $\sqsubseteq'$-rule is only applied if the other two rules are not applicable, i.e., the $\sqsubseteq'$-rule is only applied to $\sqcap$-$\exists$-complete ABoxes. Furthermore, let $\mathcal{J}_k$ be the induced interpretation of $\mathcal{A}_k$ for each suitable index k. We shall now show by induction that, for each index k, the relation

$$\mathfrak{S}_k := \{\, (D, \epsilon) \mid D \sqsupseteq_\emptyset \exists^{\mathsf{sim}}(\mathcal{I}, \epsilon) \,\}$$

is a simulation from $(\mathcal{J}_k, C)$ to $(\mathcal{I}, \delta)$ where $C^\phi = \exists^{\mathsf{sim}}(\mathcal{I}, \delta)$ and, without loss of generality, the concept description $\exists^{\mathsf{sim}}(\mathcal{I}, \delta)$ is reduced and the domain $\Delta^\mathcal{I}$ contains only objects that are reachable from δ. Since $\exists^{\mathsf{sim}}(\mathcal{I}, \delta) = C^\phi \sqsubseteq_\emptyset C$ must be satisfied due to extensitivity of ϕ, it follows that $(C, \delta) \in \mathfrak{S}_k$ for each k. Further note that during the construction of the above sequence of ABoxes individual names can only be added to the ABoxes but never be deleted, that is, $\mathfrak{S}_k \subseteq \mathfrak{S}_{k+1}$ holds true for every k.

The induction base for $k = 0$ is obvious. For the induction step we proceed with a case distinction on the rule that has been applied for the step $\mathcal{A}_k \to \mathcal{A}_{k+1}$. Fix some pair $(D, \epsilon) \in \mathfrak{S}_{k+1}$, i.e., $D \sqsupseteq_\emptyset \exists^{\mathsf{sim}}(\mathcal{I}, \epsilon)$ holds true. In particular, we also have $(D, \epsilon) \in \mathfrak{S}_\ell$ for each index $\ell \leq k$ if D already occurs as an individual name in $\mathcal{A}_\ell$.

$\sqcap$-*rule.* By assumption there is some axiom $D \sqsubseteq E$ in $\mathcal{A}_k$ such that

$$\mathcal{A}_{k+1} = \mathcal{A}_k \cup \{\, D \sqsubseteq F \mid F \in \mathsf{Conj}(E) \,\}.$$

The only differences in the induced interpretations $\mathcal{J}_k$ and $\mathcal{J}_{k+1}$ arise from the new assertions $D \sqsubseteq A$ for a concept name $A \in \mathsf{Conj}(E)$, since it holds true that $A^{\mathcal{J}_{k+1}} = A^{\mathcal{J}_k} \cup \{D\}$ for each such $A \in \mathsf{Conj}(E)$ and all other extensions of names do not differ between $\mathcal{J}_k$ and $\mathcal{J}_{k+1}$.

The case $D = E$ is easy, as it then immediately follows that $D \sqsubseteq_\emptyset A$, which shows that $\exists^{\mathsf{sim}}(\mathcal{I}, \epsilon) \sqsubseteq_\emptyset A$, i.e., $\epsilon \in A^{\mathcal{I}}$.

Otherwise, the axiom $D \sqsubseteq E$ must have been created by an application of the $\sqsubseteq'$-rule in a previous step, i.e., there is some index $\ell < k$ such that $\mathcal{A}_{\ell+1} = \mathcal{A}_\ell \cup \{D \sqsubseteq E\}$ and there is some F such that $F \sqsubseteq E$ is a concept inclusion in $\mathcal{F}_{\mathsf{sat}}$ where $D \sqsubseteq_{\mathcal{A}_\ell} F$. By definition, $F \sqsubseteq E$ must be valid in ϕ.

Since $\mathcal{A}_\ell$ is $\sqcap$-$\exists$-complete, $D \sqsubseteq_{\mathcal{A}_\ell} F$ implies $D \in F^{\mathcal{J}_\ell}$. By induction hypothesis, $\mathfrak{S}_\ell$ is a simulation from $\mathcal{J}_\ell$ to $\mathcal{I}$ and contains (D, ϵ), which implies $\epsilon \in F^{\mathcal{I}}$, i.e., $\exists^{\mathsf{sim}}(\mathcal{I}, \epsilon) \sqsubseteq_\emptyset F$ holds true. We conclude that $\exists^{\mathsf{sim}}(\mathcal{I}, \epsilon) \sqsubseteq_\phi E$. Thus, it follows that $(\exists^{\mathsf{sim}}(\mathcal{I}, \epsilon))^\phi \sqsubseteq_\emptyset E$, and compatibility of ϕ in conjunction with the fact that $\exists^{\mathsf{sim}}(\mathcal{I}, \delta)$ is a closure of ϕ implies $\exists^{\mathsf{sim}}(\mathcal{I}, \epsilon) \sqsubseteq_\emptyset E$. In particular, this means that $\epsilon \in A^{\mathcal{I}}$ for each concept name $A \in \mathsf{Conj}(E)$.

$\exists$-*rule.* Assume that there is some assertion $D \sqsubseteq \exists\, r.\, E$ in $\mathcal{A}_k$ such that

$$\mathcal{A}_{k+1} = \mathcal{A}_k \cup \{(D, E) \sqsubseteq r,\ E \sqsubseteq E\}.$$

If E is not already an individual name occurring in $\mathcal{A}_k$, then we have that

- $\Delta^{\mathcal{J}_{k+1}} = \Delta^{\mathcal{J}_k} \cup \{E\}$,
- $A^{\mathcal{J}_{k+1}} = A^{\mathcal{J}_k} \cup \{E\}$ if $E = A$,
- $A^{\mathcal{J}_{k+1}} = A^{\mathcal{J}_k}$ otherwise, and
- extensions of other concept names do not differ between $\mathcal{J}_{k+1}$ and $\mathcal{J}_k$;

otherwise $\Delta^{\mathcal{J}_{k+1}} = \Delta^{\mathcal{J}_k}$ holds true and the extensions of each concept name do not differ between $\mathcal{J}_{k+1}$ and $\mathcal{J}_k$. Furthermore, we have $r^{\mathcal{J}_{k+1}} = r^{\mathcal{J}_k} \cup \{(D, E)\}$, and extensions of other role names do not differ between $\mathcal{J}_{k+1}$ and $\mathcal{J}_k$.

If $\exists\, r.\, E \in \mathsf{Conj}(D)$, then $\exists^{\mathsf{sim}}(\mathcal{I}, \epsilon) \sqsubseteq_\emptyset \exists\, r.\, E$ follows. We conclude that there is some $\zeta \in \Delta^{\mathcal{I}}$ such that $(\epsilon, \zeta) \in r^{\mathcal{I}}$ and $\exists^{\mathsf{sim}}(\mathcal{I}, \zeta) \sqsubseteq_\emptyset E$, i.e., $(E, \zeta) \in \mathfrak{S}_{k+1}$.

Otherwise, the assertion $D \in \exists r. E$ must have been created by an application of the $\sqsubseteq'$-rule (possibly followed by the $\sqcap$-rule) in some previous step, that is, we have some index $\ell < k$ and some concept inclusion $F \sqsubseteq G$ in $\mathcal{F}_{\mathsf{sat}}$ where $D \in_{\mathcal{A}_\ell} F$, and $\mathcal{A}_{\ell+1} = \mathcal{A}_\ell \cup \{D \in G\}$, and $\exists r. E \in \mathsf{Conj}(G)$. Note that $F \sqsubseteq G$ is valid in ϕ.

Since $\mathcal{A}_\ell$ is $\sqcap$-$\exists$-complete, $D \in_{\mathcal{A}_\ell} F$ implies $D \in F^{\mathcal{J}_\ell}$. According to the induction hypothesis the relation $\mathfrak{S}_\ell$ is a simulation from $\mathcal{J}_\ell$ to $\mathcal{I}$ containing (D, ϵ), and we hence conclude that $\epsilon \in F^{\mathcal{I}}$, i.e., $\exists^{\mathsf{sim}}(\mathcal{I}, \epsilon) \sqsubseteq_\emptyset F$. We further infer that $\exists^{\mathsf{sim}}(\mathcal{I}, \epsilon) \sqsubseteq_\phi G \sqsubseteq_\emptyset \exists r. E$, which implies $(\exists^{\mathsf{sim}}(\mathcal{I}, \epsilon))^\phi \sqsubseteq_\emptyset \exists r. E$. Since ϕ is compatible and $\exists^{\mathsf{sim}}(\mathcal{I}, \delta)$ is a closure of ϕ, it follows that $\exists^{\mathsf{sim}}(\mathcal{I}, \epsilon) \sqsubseteq_\emptyset \exists r. E$, and so we can conclude as above that there exists some $\zeta \in \Delta^{\mathcal{I}}$ such that $(\epsilon, \zeta) \in r^{\mathcal{I}}$ and $(E, \zeta) \in \mathfrak{S}_{\ell+1} \subseteq \mathfrak{S}_{k+1}$.

It remains to demonstrate that $\zeta \in A^{\mathcal{I}}$ is satisfied for the case where $E = A$. However, this follows from $\exists^{\mathsf{sim}}(\mathcal{I}, \zeta) \sqsubseteq_\emptyset E$, which is satisfied in both above cases.

$\sqsubseteq'$-rule. Eventually, consider the case where there exists some concept inclusion $E \sqsubseteq F$ in $\mathcal{F}_{\mathsf{sat}}$ such that $D \in_{\mathcal{A}_k} E$ and

$$\mathcal{A}_{k+1} = \mathcal{A}_k \cup \{D \in F\}.$$

If F is not a concept name, then the interpretations $\mathcal{J}_{k+1}$ and $\mathcal{J}_k$ are equal and we are done. Otherwise for $F = A$ the only difference between the interpretations $\mathcal{J}_{k+1}$ and $\mathcal{J}_k$ is possibly that $A^{\mathcal{J}_{k+1}} = A^{\mathcal{J}_k} \cup \{D\}$. As above, $D \in_{\mathcal{A}_k} E$ implies $D \in E^{\mathcal{J}_k}$. It follows that $\epsilon \in E^{\mathcal{I}}$ and thus $\exists^{\mathsf{sim}}(\mathcal{I}, \epsilon) \sqsubseteq_\emptyset E$. We conclude that $\exists^{\mathsf{sim}}(\mathcal{I}, \epsilon) \sqsubseteq_\phi F$ and thus $(\exists^{\mathsf{sim}}(\mathcal{I}, \epsilon))^\phi \sqsubseteq_\emptyset F$. Now using compatibility of ϕ together with the precondition that $\exists^{\mathsf{sim}}(\mathcal{I}, \delta)$ is a closure of ϕ yields $\exists^{\mathsf{sim}}(\mathcal{I}, \epsilon) \sqsubseteq F$, i.e., $\epsilon \in A^{\mathcal{I}}$.

Finally, assume that C is satisfiable w.r.t. $\mathcal{F}$. We continue with a proof by contradiction and show that that the closure C^ϕ must be equivalent to $\bot$. Thus, let $C^\phi \not\equiv_\emptyset \bot$. Like in the proof of Proposition 4.3.27, let $\mathcal{U}$ be the subset of $\mathcal{F}$ not containing any concept inclusions $E \sqsubseteq \bot$. Since $\mathcal{U}$ is an $\mathcal{EL}_{\mathsf{si}}$ TBox, we can use the same proof as above to conclude that $C^\phi \sqsubseteq_\emptyset C^{\mathcal{U}}$ holds true. According to arguments in the proof of Proposition 4.3.27, there must exist some role word $w \in \Sigma_{\mathsf{R}}^*$ as well as some concept inclusion $E \sqsubseteq \bot \in \mathcal{F} \setminus \mathcal{U}$ such that $C^{\mathcal{U}} \sqsubseteq_\emptyset \exists w. E$. It follows that $C^\phi \sqsubseteq_\emptyset \exists w. E$. We now use the compatibility of ϕ inductively to conclude that $C^\phi \sqsubseteq_\emptyset \exists w. E^\phi$. Since $E \sqsubseteq \bot$ is valid for ϕ, we infer that $C^\phi \sqsubseteq \exists w. \bot \equiv_\emptyset \bot$. ↯ □

We close this section with providing some examples of compatible closure operators.

6.2.12 Proposition. *For each finite interpretation $\mathcal{I}$, the mapping*

$$\begin{aligned} \phi_{\mathcal{I}} \colon \mathcal{EL}_{\mathsf{si}}^{\bot}(\Sigma) &\to \mathcal{EL}_{\mathsf{si}}^{\bot}(\Sigma) \\ C &\mapsto C^{\mathcal{II}} \end{aligned}$$

is a compatible, finitely acyclically representable closure operator.

Proof. Theorem 4.1.9 shows that $\phi_{\mathcal{I}}$ is a closure operator. Propositions 3.4.4 and 4.1.6 yields that $\phi_{\mathcal{I}}$ is compatible. Furthermore, Proposition 6.1.1 implies that $\phi_{\mathcal{I}}$ is finitely acyclically representable. □

6.2.13 Proposition. *For each $\mathcal{EL}^{\bot}_{\mathsf{si}}$ TBox $\mathcal{T}$, the mapping*

$$\begin{aligned} \phi_{\mathcal{T}} \colon \mathcal{EL}^{\bot}_{\mathsf{si}}(\Sigma) &\to \mathcal{EL}^{\bot}_{\mathsf{si}}(\Sigma) \\ C &\mapsto C^{\mathcal{T}} \end{aligned}$$

is a compatible, finitely representable closure operator. If all premises in $\mathcal{T}$ are $\mathcal{EL}^{\bot}$ concept descriptions, then $\phi_{\mathcal{T}}$ is finitely acyclically representable.

Proof. The fact that $\phi_{\mathcal{T}}$ is a closure operator has been proven in Proposition 4.3.35. Compatibility is demonstrated in Corollary 4.3.42. The statements on representability are obvious. □

It is easy to verify that, for each subset $\Gamma \subseteq \Sigma$, the below mapping ϕ_{Γ} is a compatible closure operator.

$$\begin{aligned} \phi_{\Gamma} \colon \mathcal{EL}^{\bot}_{\mathsf{si}}(\Sigma) &\to \mathcal{EL}^{\bot}_{\mathsf{si}}(\Sigma) \\ \exists^{\mathsf{sim}}(\mathcal{I}, \delta) &\mapsto \begin{cases} \exists^{\mathsf{sim}}(\mathcal{I}, \delta) & \text{if } \sigma^{\mathcal{I}} = \emptyset \text{ for each } \sigma \in \Sigma \setminus \Gamma \\ \bot & \text{otherwise} \end{cases} \end{aligned}$$

For each $\mathcal{EL}_{\mathsf{si}}$ concept description C that is not equivalent to some $\mathcal{EL}$ concept description, we define its rank as $|C| \coloneqq \infty$. Now fix some number $\ell \in \mathbb{N}$. Then, the following mapping is a compatible closure operator.

$$\begin{aligned} \phi_{\ell} \colon \mathcal{EL}^{\bot}_{\mathsf{si}}(\Sigma) &\to \mathcal{EL}^{\bot}_{\mathsf{si}}(\Sigma) \\ C &\mapsto \begin{cases} C & \text{if } |C| \leq \ell \\ \bot & \text{otherwise} \end{cases} \end{aligned}$$

Note that in Section 6.8.4 we will also show that each (simple) ABox $\mathcal{A}$ induces a closure operator $\phi_{\mathcal{A}}$, which is compatible and finitely acyclically representable.

6.3 The Infimum of Closure Operators

This section is concerned with the infimum operation on closure operators. For two closure operators ϕ and ψ, the infimum $\phi \bigtriangleup \psi$ satisfies the equivalence $C^{\phi \bigtriangleup \psi} \equiv_{\emptyset} C^{\phi} \vee C^{\psi}$ for each $\mathcal{EL}_{\mathsf{si}}$ concept description C, cf. Section 1.5. We first show some general statements for such infima and later provide some applications that utilize the canonical base of an infimum. We define the notion of a logical intersection of two TBoxes and show how it corresponds to the infimum of the corresponding closure operators. As it turns out, it is not always possible to finitely represent the logical intersection, since there are examples of pairs of TBoxes for which the logical intersection would need to contain infinitely many concept inclusions and can thus not exist. However, it is possible to *approximate* the logical intersection up to a predefined role depth by computing a canonical base for the depth restriction of the infimum of the corresponding closure operators. A further application considers a setting where a (possibly infinite) sequence of interpretations is available. For instance, these interpretations could contain observations that are made accessible on a regular basis. The overall goal is then to constitute an incremental procedure that can axiomatize the concept inclusions being valid up to the current time point, i.e., being valid in all interpretations that have been processed until the current time point. A last application varies the restrictions on the input data set: we replace interpretations by simple ABoxes, and thus switching from closed-world assumption to open-world assumption. Basically, such simple ABoxes can be seen as a three-mode data structure; we can not only express that some object belongs to some concept name or not, or is connected to some other object by some role name or not, but we can also express that it is unknown.

6.3.1 Proposition. *If ϕ and ψ are finite closure operators, then the infimum $\phi \vartriangle \psi$ is finite too.*

Proof. The claim is an obvious consequence of the fact that $C^{\phi \vartriangle \psi} \equiv_\emptyset C^\phi \vee C^\psi$ holds true for each $\mathcal{EL}_{\mathsf{si}}$ concept description C. □

6.3.2 Proposition. *If ϕ and ψ are compatible closure operators in $\boldsymbol{\mathcal{EL}}^\bot_{\mathsf{si}}(\Sigma)$, then their infimum $\phi \vartriangle \psi$ is compatible as well.*

Proof. Consider some $\mathcal{EL}^\bot_{\mathsf{si}}$ concept descriptions C and D such that $C^{\phi \vartriangle \psi} \sqsubseteq_\emptyset \exists r.\, D$. Since $C^{\phi \vartriangle \psi} \equiv_\emptyset C^\phi \vee C^\psi$ holds true, we conclude that $C^\phi \sqsubseteq_\emptyset \exists r.\, D$ and $C^\psi \sqsubseteq_\emptyset \exists r.\, D$. Compatibility of ϕ and ψ implies $C^\phi \sqsubseteq_\emptyset \exists r.\, D^\phi$ and $C^\psi \sqsubseteq_\emptyset \exists r.\, D^\psi$. Since we have that $D^\phi \sqsubseteq_\emptyset D^\phi \vee D^\psi$ and likewise $D^\psi \sqsubseteq_\emptyset D^\phi \vee D^\psi$, we conclude that both C^ϕ and C^ψ are subsumed by $\exists r.\, D^{\phi \vartriangle \psi}$ w.r.t. $\emptyset$, i.e., $C^{\phi \vartriangle \psi} \sqsubseteq_\emptyset \exists r.\, D^{\phi \vartriangle \psi}$ follows. □

Since we always consider finitely representable closure operators in order to be able to treat these with a computer, we have that, for each such closure operator ϕ, there is always a TBox $\mathcal{T}$ with the property that a concept inclusion is valid for ϕ if, and only if, it is entailed by $\mathcal{T}$, cf. Section 6.2. In the following, we show that the notion of an infimum of two finitely representable closure operators can also be equivalently treated as a notion of a *logical intersection* of two TBoxes. In particular, we shall demonstrate that a concept inclusion is simultaneously entailed by two TBoxes $\mathcal{T}_1$ and $\mathcal{T}_2$ if, and only if, it is valid for the infimum $\phi_{\mathcal{T}_1} \vartriangle \phi_{\mathcal{T}_2}$ of the corresponding closure operators.

6.3.3 Proposition. *Let $\mathcal{T}_1 \cup \mathcal{T}_2 \cup \{C \sqsubseteq D\}$ be an $\mathcal{EL}^\bot_{\mathsf{si}}$ TBox. Then, the following statements are equivalent.*

1. *$C \sqsubseteq_{\mathcal{T}_1} D$ and $C \sqsubseteq_{\mathcal{T}_2} D$*
2. *$C^{\mathcal{T}_1} \sqsubseteq_\emptyset D$ and $C^{\mathcal{T}_2} \sqsubseteq_\emptyset D$*
3. *$C^{\mathcal{T}_1} \vee C^{\mathcal{T}_2} \sqsubseteq_\emptyset D$*
4. *$\phi_{\mathcal{T}_1} \models C \sqsubseteq D$ and $\phi_{\mathcal{T}_2} \models C \sqsubseteq D$*
5. *$\phi_{\mathcal{T}_1} \vartriangle \phi_{\mathcal{T}_2} \models C \sqsubseteq D$*

Proof. Statements 1 and 2 are equivalent by Proposition 4.3.39. The very definition of least common subsumers yields that Statements 2 and 3 are equivalent. Furthermore, Corollary 4.3.40 implies the equivalence of Statements 1 and 4. Eventually, Section 1.5, or alternatively [Kri16b, Section 3.1], shows the equivalence of Statements 4 and 5. □

We conclude that, if ϕ and ψ are closure operators with respective finite representations $\mathcal{T}$ and $\mathcal{U}$, then a finite representation of the infimum $\phi \vartriangle \psi$ is a *logical intersection* of $\mathcal{T}$ and $\mathcal{U}$, which is defined in the following way.

6.3.4 Definition. Let $\mathcal{DL}_1$ and $\mathcal{DL}_2$ denote description logics and fix two $\mathcal{DL}_1$ terminological boxes $\mathcal{T}$ and $\mathcal{U}$. Then, some $\mathcal{DL}_2$ terminological box $\mathcal{V}$ is called *logical intersection* of $\mathcal{T}$ and $\mathcal{U}$ in $\mathcal{DL}_2$ if it satisfies the following conditions.

1. $\mathcal{T} \models \mathcal{V}$ and $\mathcal{U} \models \mathcal{V}$ hold true.
2. For any $\mathcal{DL}_2$ terminological box $\mathcal{W}$, if $\mathcal{T} \models \mathcal{W}$ and $\mathcal{U} \models \mathcal{W}$, then $\mathcal{V} \models \mathcal{W}$.

It is immediate to conclude that, if existent, the logical intersection of $\mathcal{T}$ and $\mathcal{U}$ is unique modulo equivalence—hence, we shall denote it by $\mathcal{T} \vartriangle \mathcal{U}$. △

Note that the logical intersection must be finite, since terminological boxes are always finite. If we would drop this constraint, then the logical intersection of terminological boxes $\mathcal{T}$ and $\mathcal{U}$ would exist in every case, as it could simply contain all concept inclusions that are entailed by both $\mathcal{T}$ and $\mathcal{U}$, i.e.,

$$\mathcal{T} \vartriangle \mathcal{U} \equiv \{\, C \sqsubseteq D \mid \mathcal{T} \models C \sqsubseteq D \text{ and } \mathcal{U} \models C \sqsubseteq D \,\}.$$

For our purposes, we only consider the cases where the description logics $\mathcal{DL}_1$ and $\mathcal{DL}_2$ are one of $\mathcal{EL}$, $\mathcal{EL}^{\bot}$, $\mathcal{EL}_d$, $\mathcal{EL}_d^{\bot}$, $\mathcal{EL}_{\mathsf{si}}$, or $\mathcal{EL}_{\mathsf{si}}^{\bot}$.

6.3.5 Proposition. *The logical intersection of the $\mathcal{EL}$ TBoxes*

$$\begin{aligned} \mathcal{T} &:= \{A \sqsubseteq \exists\, r.\, B_1,\ B_1 \sqsubseteq \exists\, r.\, B_1\} \\ \textit{and}\quad \mathcal{U} &:= \{A \sqsubseteq \exists\, r.\, B_2,\ B_2 \sqsubseteq \exists\, r.\, B_2\} \end{aligned}$$

does not exist in $\mathcal{EL}$, i.e., there is no (finite) $\mathcal{EL}$ TBox that axiomatizes the infimum $\phi_{\mathcal{T}} \vartriangle \phi_{\mathcal{U}}$.

Proof. It is easy to see that both $\mathcal{T}$ and $\mathcal{U}$ entails $A \sqsubseteq \exists\, r^n.\top$ for each number $n \in \mathbb{N}$. Now assume that $\mathcal{T} \vartriangle \mathcal{U}$ exists in $\mathcal{EL}$, i.e., $\mathcal{T} \vartriangle \mathcal{U} \models A \sqsubseteq \exists\, r^n.\top$ for each $n \in \mathbb{N}$. We infer that $A^{\mathcal{T}\vartriangle\mathcal{U}} \sqsubseteq_{\emptyset} \exists\, r^n.\top$ for each $n \in \mathbb{N}$. We conclude that the most specific consequence $A^{\mathcal{T}\vartriangle\mathcal{U}}$ must be a cyclic $\mathcal{EL}_{\mathsf{si}}$ concept description, i.e., the logical intersection $\mathcal{T} \vartriangle \mathcal{U}$ is not cycle-restricted and the canonical model $\mathcal{I}_{A,\mathcal{T}\vartriangle\mathcal{U}}$ is not tree-shaped. More specifically, there is a concept description C in the domain of $\mathcal{I}_{A,\mathcal{T}\vartriangle\mathcal{U}}$ and role words $v \in \Sigma_{\mathsf{R}}^{*}$, $w \in \Sigma_{\mathsf{R}}^{+}$ such that $A \sqsubseteq_{\mathcal{T}\vartriangle\mathcal{U}} \exists\, v.\, C$ and $C \sqsubseteq_{\mathcal{T}\vartriangle\mathcal{U}} \exists\, w.\, C$.

Claim. *If $\mathcal{T}$ entails $C \sqsubseteq D$, then* $\mathsf{Sig}(\mathcal{T}) \cup \mathsf{Sig}(C) \supseteq \mathsf{Sig}(D)$.[3]

Proof. Fix some name $\sigma \in \mathsf{Sig}(D) \setminus (\mathsf{Sig}(\mathcal{T}) \cup \mathsf{Sig}(C))$. On Page 83 we have seen that the canonical model $\mathcal{I}_{C,\mathcal{T}}$ is a model of $\mathcal{T}$ and satisfies $C^{\mathcal{I}_{C,\mathcal{T}}} \neq \emptyset$. Define $\mathcal{J} := \mathcal{I}_{C,\mathcal{T}}$ except for $\sigma^{\mathcal{J}} := \emptyset$. Then $\mathcal{J}$ is also a model of $\mathcal{T}$ and $C^{\mathcal{J}} \neq \emptyset$, but $D^{\mathcal{J}} = \emptyset$. We conclude that $\mathcal{T}$ does not entail $C \sqsubseteq D$. □

[3]By $\mathsf{Sig}(X)$ we denote the set of all concept and role names occurring in X.

Since in $\mathcal{T}$ and $\mathcal{U}$ only the role name r occurs, we infer that v and w cannot contain any role names different from r. Furthermore, we conclude that $\mathsf{Sig}(C) \subseteq \{A, r\}$. If A occurs in C, then there is some number $m \in \mathbb{N}$ such that $C \sqsubseteq_\emptyset \exists\, r^m.\, A$—it follows that both $\mathcal{T}$ and $\mathcal{U}$ entail $A \sqsubseteq \exists\, v.\, \exists\, r^m.\, A$, a contradiction. ↯ As a consequence, C must be of the form $\exists\, r^n.\, \top$ for some $n \in \mathbb{N}$. However, this means that $\exists\, r^n.\, \top \sqsubseteq \exists\, w.\, \exists\, r^n.\, \top$ must be entailed by both $\mathcal{T}$ and $\mathcal{U}$, which yields a contradiction as well. ↯ Thus, the logical intersection $\mathcal{T} \triangle \mathcal{U}$ cannot exist in $\mathcal{EL}$. □

6.3.6 Proposition. *If the signature Σ contains some role name, then the infimum $\phi_\mathcal{S} \triangle \phi_\mathcal{T}$ is not finitely acyclically representable for the TBoxes $\mathcal{S} := \{A \sqsubseteq B_1\}$ and $\mathcal{T} := \{A \sqsubseteq B_2\}$, although both closure operators $\phi_\mathcal{S}$ and $\phi_\mathcal{T}$ are finitely acyclically representable.*

Proof. Fix a role name $r \in \Sigma_\mathsf{R}$. It is easy to see that both $\mathcal{S}$ and $\mathcal{T}$ entail the concept inclusion

$$\exists\, r^n.\, (A \sqcap B_1) \sqcap \exists\, r^n.\, (A \sqcap B_2) \sqsubseteq \exists\, r^n.\, (A \sqcap B_1 \sqcap B_2)$$

for each number $n \in \mathbb{N}$, i.e., each of these concept inclusions must be valid in the infimum $\phi_\mathcal{S} \triangle \phi_\mathcal{T}$. Furthermore, note that $C^{\mathcal{S} \triangle \mathcal{T}} \equiv_\emptyset C^\mathcal{S} \vee C^\mathcal{T}$ holds true for each concept description $C \in \mathcal{EL}^\bot_\mathsf{si}(\Sigma)$.

Now assume that there is some finite set $\mathbf{P}$ of $\mathcal{EL}^\bot$ concept descriptions such that $\phi_\mathcal{S} \triangle \phi_\mathcal{T} \equiv \mathcal{P}$ is satisfied for $\mathcal{P} := \{\, P \sqsubseteq P^\mathcal{S} \vee P^\mathcal{T} \mid P \in \mathbf{P}\,\}$. This implies that $\mathcal{P}$ entails the above concept inclusion for each $n \in \mathbb{N}$. Obviously, the above concept inclusion is no tautology, i.e., not valid in all interpretations. An application of Lemma 4.3.19 shows that, for each $n \in \mathbb{N}$, there must exist some premise $P_n \in \mathbf{P}$ such that

1. either $\exists\, r^n.\, (A \sqcap B_1) \sqcap \exists\, r^n.\, (A \sqcap B_2) \sqsubseteq_\emptyset P_n$
   and $\exists\, r^n.\, (A \sqcap B_1) \sqcap \exists\, r^n.\, (A \sqcap B_2) \not\sqsubseteq_\emptyset P_n^\mathcal{S} \vee P_n^\mathcal{T}$,
2. or $\exists\, r^k.\, (A \sqcap B_i) \sqsubseteq_\emptyset P_n$ and $\exists\, r^k.\, (A \sqcap B_i) \not\sqsubseteq_\emptyset P_n^\mathcal{S} \vee P_n^\mathcal{T}$
   for some $k \in \{0, \ldots, n\}$ and $i \in \{1, 2\}$.

Let $d := \mathsf{max}\{\, \mathsf{rd}(P) \mid P \in \mathbf{P}\,\}$. Then the first option cannot be satisfied for $n = d + 1$ and there must exist some $k \in \{0, \ldots, d\}$ and some $i \in \{1, 2\}$ such that $\exists\, r^k.\, (A \sqcap B_i) \sqsubseteq_\emptyset P_{d+1}$ holds true. This means that P_{d+1} must be equivalent to a concept description either of the form $\exists\, r^j.\, \top$ where $j \leq k$ or of the form $\exists\, r^k.\, X$ where $X \in \{A, B_i, A \sqcap B_i\}$. It is readily verified that in every case P_{d+1} must be equivalent to $P_{d+1}^\mathcal{S} \vee P_{d+1}^\mathcal{T}$ w.r.t. $\emptyset$, which yields a contradiction. ↯ □

6.3.7 Proposition. *If the signature Σ contains some role name, then the infimum $\phi_\mathcal{S} \triangle \phi_\mathcal{T}$ is not finitely representable for the TBoxes $\mathcal{S} := \{A \sqsubseteq B_1\}$ and $\mathcal{T} := \{A \sqsubseteq B_2\}$.*

Proof. The proof is similar to the above proof of Proposition 6.3.6. However, the set $\mathbf{P}$ of premises can now also contain $\mathcal{EL}^\bot_\mathsf{si}$ concept descriptions. Without loss of generality, assume that $\mathbf{P}$ does not contain any acyclic $\mathcal{EL}^\bot_\mathsf{si}$ concept descriptions. Now let $\mathbf{P} = \mathbf{P}_1 \uplus \mathbf{P}_2$ be a partition such that $\mathbf{P}_1$ contains all $\mathcal{EL}^\bot$ concept descriptions in $\mathbf{P}$, and such that $\mathbf{P}_2$ contains all (cyclic) $\mathcal{EL}^\bot_\mathsf{si}$ concept descriptions in $\mathbf{P}$.

According to the comment below Lemma 4.3.19, we then get that, for each $n \in \mathbb{N}$, there must exist some premise $P_n \in \mathbf{P}_1$ satisfying one of the properties in the above proof. We can then similarly construct a contradiction by considering $d := \mathsf{max}\{\, \mathsf{rd}(P) \mid P \in \mathbf{P}_1\,\}$. □

It is currently an open problem whether the infimum $\phi \bigtriangleup \psi$ is acyclically representable if the closure operators ϕ and ψ are both finitely acyclically representable. Since our overall goal is to compute finite representations of closure operators if this is possible, a solution to this question is not directly necessary. However, an affirmative answer would then show that we can start an enumeration of a then acyclic representation of $\phi \bigtriangleup \psi$, but which could continue infinitely long. In such an enumeration, we could step-wise increase the role depths of the premises and compute a concept inclusion base for the current role depth relative to the union of the bases for smaller role depths, as explained in Section 6.8.1.

The author believes that the above statement holds true, and that it might help to show that, for any $d \in \mathbb{N}$, there exists some $e \in \mathbb{N}$ such that $(C\upharpoonright_e)^{\mathcal{T}} \sqsubseteq_\emptyset C^{\mathcal{T}}\upharpoonright_d$.

6.3.8 Corollary. *If ϕ and ψ are finite, compatible closure operators, then the infimum $\phi \bigtriangleup \psi$ is finite and compatible as well.*

6.4 The Supremum of Closure Operators

Note that, for two closure operators ϕ and ψ, the supremum closure $C^{\phi \bigtriangledown \psi}$ of each $\mathcal{EL}^\bot_{\mathsf{si}}$ concept description C is defined as the most general $\mathcal{EL}^\bot_{\mathsf{si}}$ concept description that is subsumed by C and is a closure of both ϕ and ψ, i.e., it is formally defined as follows.[4]

$$C^{\phi \bigtriangledown \psi} := \bigvee \{\, D \mid D \in \mathcal{EL}^\bot_{\mathsf{si}}(\Sigma) \text{ and } D \sqsubseteq_\emptyset C \text{ and } D \equiv_\emptyset D^\phi \equiv_\emptyset D^\psi \,\}$$

It would be straightforward to claim that such a supremum closure $C^{\phi \bigtriangledown \psi}$ can always be obtained as the fixed point of the sequence

$$C, \quad C^\phi, \quad (C^\phi)^\psi, \quad ((C^\phi)^\psi)^\phi, \quad (((C^\phi)^\psi)^\phi)^\psi, \quad \ldots$$

However, this need not be true if we do not impose constraints on the signature or on the closure operators. For instance, let the signature Σ contain the countably many concept names A_0, A_1, A_2, ... and further assume that ϕ and ψ have the following mappings.

$$A_0 \overset{\phi}{\mapsto} A_0 \sqcap A_1 \overset{\psi}{\mapsto} A_0 \sqcap A_1 \sqcap A_2 \overset{\phi}{\mapsto} A_0 \sqcap A_1 \sqcap A_2 \sqcap A_3 \overset{\psi}{\mapsto} \ldots$$

It is apparent that exhaustive alternating applications of ϕ and ψ starting from A_0 would construct the infinite conjunction $\bigsqcap\{\, A_n \mid n \in \mathbb{N} \,\}$, i.e., the sequence does not reach a *fixed point* after finitely many iterations. In contrast, we have $A_0^{\phi \bigtriangledown \psi} \equiv_\emptyset \bot$.

As we have already argued earlier, we only consider finitely representable closure operators to be able to load these into some computing device. Under this restriction we can now show that our above claim holds true.

[4]We do not know whether $\mathcal{EL}^\bot_{\mathsf{si}}(\Sigma)$ is complete and can thus not be sure that the closures $C^{\phi \bigtriangledown \psi}$ always exist. Of course, we could construct the least common subsumer using products of interpretations, but the problem here is that we possibly have infinitely many operands, yielding an infinite interpretation that cannot (directly) be used within an $\mathcal{EL}^\bot_{\mathsf{si}}$ concept description. However, Proposition 6.4.1 shows that we can ignore this problem if the involved closure operators are finitely representable.

6.4.1 Proposition. *Let ϕ and ψ be finitely representable closure operators. For each $\mathcal{EL}^{\bot}_{\mathsf{si}}$ concept description C, the sequence*

$$C, \quad C^{\phi}, \quad (C^{\phi})^{\psi}, \quad ((C^{\phi})^{\psi})^{\phi}, \quad (((C^{\phi})^{\psi})^{\phi})^{\psi}, \quad \ldots$$

is ultimately constant modulo $\emptyset$, i.e., it has a fixed point, and the fixed point is the supremum closure $C^{\phi \triangledown \psi}$ modulo $\emptyset$.

Proof. Since ϕ and ψ are finitely representable, there are (finite) $\mathcal{EL}_{\mathsf{si}}$ TBoxes $\mathcal{S}$ and $\mathcal{T}$ such that $\phi \equiv \mathcal{S}$ and $\psi \equiv \mathcal{T}$. This implies that $D^{\phi} \equiv_{\emptyset} D^{\mathcal{S}}$ and $D^{\psi} \equiv_{\emptyset} D^{\mathcal{T}}$ for each $\mathcal{EL}_{\mathsf{si}}$ concept description D. According to the characterization in Theorem 4.3.26 together with the arguments in the proof of Proposition 4.3.27, computing $D^{\mathcal{S}}$ or $D^{\mathcal{T}}$ can be done by saturating D with subconcepts occurring in $\mathcal{S}$ or in $\mathcal{T}$, respectively. Since both TBoxes $\mathcal{S}$ and $\mathcal{T}$ are finite and all concept descriptions occurring in both TBoxes are finite as well, we see that the process of exhaustively saturating C alternatingly w.r.t. $\mathcal{S}$ and w.r.t. $\mathcal{T}$ must stagnate after a finite number of steps, i.e., the above sequence must have a fixed point.

Clearly, the fixed point F must be a closure of both ϕ and ψ. Since ϕ and ψ are extensive, we also obtain that $F \sqsubseteq_{\emptyset} C$. We thus conclude that $F \sqsubseteq_{\emptyset} C^{\phi \triangledown \psi}$. It remains to prove that the converse subsumption is satisfied as well.

Since each closure operator is extensive, we know that $C^{\phi \triangledown \psi} \sqsubseteq_{\emptyset} C$ must hold true. Applying the fact that ϕ is monotonous yields that $C^{(\phi \triangledown \psi)\phi} \sqsubseteq_{\emptyset} C^{\phi}$. As the supremum closure $C^{\phi \triangledown \psi}$ is already a closure of ϕ, we obtain that $C^{\phi \triangledown \psi} \sqsubseteq_{\emptyset} C^{\phi}$. This argumentation can now be continued inductively with alternating between both closure operators ϕ and ψ, demonstrating that the supremum closure is more specific than each concept description of the above sequence. Eventually, this shows that $C^{\phi \triangledown \psi} \sqsubseteq_{\emptyset} F$. □

We continue this section with providing some general statements on suprema of closure operators. Afterwards, some applications utilizing the notion of a supremum are presented. In particular, we show how an error-tolerant axiomatization of concept inclusions from a given interpretation can be achieved. To do so, we need a closure operator ψ such that being non-closed for ψ indicates the presence of an error. Then axiomatizing the supremum $\phi_{\mathcal{I}} \triangledown \psi$ for some finite interpretation $\mathcal{I}$ filters the errors and axiomatizes only the error-free part of $\mathcal{I}$.

6.4.2 Proposition. *Let ϕ and ψ be closure operators such that ϕ is finite. Then the supremum $\phi \triangledown \psi$ is finite as well.*

Proof. Note that the supremum closure $C^{\phi \triangledown \psi}$ is the largest concept description that is more specific than C and is a closure of both ϕ and ψ. The claim is now obvious. □

6.4.3 Proposition. *If ϕ and ψ are finitely representable closure operators in $\boldsymbol{\mathcal{EL}^{\bot}_{\mathsf{si}}}(\Sigma)$, then their supremum $\phi \triangledown \psi$ is compatible.*

Proof. First note that ϕ and ψ are compatible. Fix some $\mathcal{EL}^{\bot}_{\mathsf{si}}$ concept descriptions C and D such that $C^{\phi \triangledown \psi} \sqsubseteq_{\emptyset} \exists\, r.\, D$. We shall demonstrate that then also $C^{\phi \triangledown \psi} \sqsubseteq_{\emptyset} \exists\, r.\, D^{\phi \triangledown \psi}$ is satisfied. We furthermore know that the closure $D^{\phi \triangledown \psi}$ can be obtained as the fixed point of the sequence D, D^{ϕ}, $(D^{\phi})^{\psi}$, $((D^{\phi})^{\psi})^{\phi}$, $(((D^{\phi})^{\psi})^{\phi})^{\psi}, \ldots$ modulo $\emptyset$. An induction along this sequence now shows the claim. Beforehand note that $C^{\phi \triangledown \psi}$ is a closure of both ϕ and ψ, i.e., it holds true that $C^{\phi \triangledown \psi} \equiv_\emptyset (C^{\phi \triangledown \psi})^\phi \equiv_\emptyset (C^{\phi \triangledown \psi})^\psi$. Thus, the assumption $C^{\phi \triangledown \psi} \sqsubseteq_\emptyset \exists\, r.\, D$ together with compatibility of ϕ yields that $(C^{\phi \triangledown \psi})^\phi \sqsubseteq_\emptyset \exists\, r.\, D^\phi$. Afterwards, compatibility of ψ implies that $(C^{\phi \triangledown \psi})^\phi \sqsubseteq_\emptyset \exists\, r.\, (D^\phi)^\psi$. Continuing this argumentation shows that $C^{\phi \triangledown \psi} \sqsubseteq_\emptyset \exists\, r.\, D^{\phi \triangledown \psi}$ indeed holds true. □

6.4.4 Proposition. *If ϕ and ψ are finitely acyclically representable closure operators, then their supremum $\phi \triangledown \psi$ is finitely acyclically representable as well.*

Proof. Assume that ϕ is finitely acyclically representable by means of a finite set $\mathsf{Prem}(\phi) \subseteq \mathcal{EL}^\bot(\Sigma)$, and likewise let ψ be finitely acyclically representable by means of a finite set $\mathsf{Prem}(\psi) \subseteq \mathcal{EL}^\bot(\Sigma)$. Furthermore, define $\mathcal{S} := \{\, P \sqsubseteq P^\phi \mid P \in \mathsf{Prem}(\phi) \,\}$ and $\mathcal{T} := \{\, P \sqsubseteq P^\psi \mid P \in \mathsf{Prem}(\psi) \,\}$. In the following, we show that the supremum $\phi \triangledown \psi$ is finitely acyclically representable as well, namely by means of the set $\mathsf{Prem}(\phi \triangledown \psi) := \mathsf{Prem}(\phi) \cup \mathsf{Prem}(\psi)$. Set $\mathcal{U} := \{\, P \sqsubseteq P^{\phi \triangledown \psi} \mid P \in \mathsf{Prem}(\phi \triangledown \psi) \,\}$.

Fix some $\mathcal{EL}^\bot_{\mathsf{si}}$ concept description C. We know that the closure $C^{\phi \triangledown \psi}$ can be obtained as the fixed point of the sequence C, C^ϕ, $(C^\phi)^\psi$, $((C^\phi)^\psi)^\phi$, $(((C^\phi)^\psi)^\phi)^\psi$, … modulo $\emptyset$. By assumption, we have the following.

- $\mathcal{S}$ entails $C \sqsubseteq C^\phi$.
- $\mathcal{T}$ entails $C^\phi \sqsubseteq (C^\phi)^\psi$.
- $\mathcal{S}$ entails $(C^\phi)^\psi \sqsubseteq ((C^\phi)^\psi)^\phi$.
- $\mathcal{T}$ entails $((C^\phi)^\psi)^\phi \sqsubseteq (((C^\phi)^\psi)^\phi)^\psi$.
- …

We conclude that the union $\mathcal{S} \cup \mathcal{T}$ entails $C \sqsubseteq C^{\phi \triangledown \psi}$.

Eventually, we show that $\mathcal{U}$ entails the union $\mathcal{S} \cup \mathcal{T}$, which then yields our claim. Consider some $P \in \mathsf{Prem}(\phi)$. Then $\mathcal{U}$ contains the concept inclusion $P \sqsubseteq P^{\phi \triangledown \psi}$, and further $\phi \trianglelefteq \phi \triangledown \psi$ implies $P^\phi \sqsupseteq_\emptyset P^{\phi \triangledown \psi}$. We conclude that $\mathcal{U}$ entails each concept inclusion $P \sqsubseteq P^\phi$ in $\mathcal{S}$. With analogous arguments we obtain that $\mathcal{U}$ entails $\mathcal{T}$ as well. □

6.4.5 Proposition. *If ϕ and ψ are finitely representable closure operators, then their supremum $\phi \triangledown \psi$ is finitely representable as well.*

Proof. The proof is essentially the same as for Proposition 6.4.4, except that the premises are now $\mathcal{EL}^\bot_{\mathsf{si}}$ concept descriptions instead of $\mathcal{EL}^\bot$ concept descriptions. □

6.4.6 Corollary. *If ϕ and ψ are finitely representable closure operators where ϕ is furthermore finite, then the supremum $\phi \triangledown \psi$ is finite and compatible.*

6.5 Role-Depth-Bounded Closure Operators

6.5.1 Definition. For a closure operator ϕ and some number $d \in \mathbb{N}$, let the *restriction* of ϕ to role depth d be defined as the closure operator $\phi\!\upharpoonright_d$ in $\boldsymbol{\mathcal{EL}^\bot_d}(\Sigma)$ where $C^{\phi\upharpoonright_d} := C^\phi\!\upharpoonright_d$ for each $\mathcal{EL}^\bot$ concept description C where $\mathsf{rd}(C) \leq d$. △

All of the properties defined for closure operators in $\boldsymbol{\mathcal{EL}}^{\bot}_{\mathsf{si}}(\Sigma)$ with the exception of compatibility can be easily treated as properties for closure operators in $\boldsymbol{\mathcal{EL}}^{\bot}_{d}(\Sigma)$ by replacing $\mathcal{EL}^{\bot}_{\mathsf{si}}$ with $\mathcal{EL}^{\bot}_{d}$ in the respective definitions. In particular, this applies to the following: closure, finite, infinite, validity (of a concept inclusion), representable, finitely representable, acyclically representable, finitely acyclically representable, axiomatizable, finitely axiomatizable, acyclically axiomatizable, and finitely acyclically axiomatizable. We say that a restriction $\phi\restriction_d$ is *compatible* if $C^{\phi\restriction_d} \sqsubseteq_\emptyset \exists\, r.\, D$ implies $C^{\phi\restriction_d} \sqsubseteq_\emptyset \exists\, r.\, D^{\phi\restriction_{d-1}}$ for all $C \in \mathcal{EL}^{\bot}_{d}(\Sigma)$ and $D \in \mathcal{EL}^{\bot}_{d-1}(\Sigma)$.

Of course, a restriction $\phi\restriction_d$ is finite if the underlying signature Σ is finite, since then only finitely many $\mathcal{EL}^{\bot}$ concept descriptions with a role depth not exceeding d exist. It is further easy to see that the properties representable, finitely representable, acyclically representable, finitely acyclically representable, axiomatizable, finitely axiomatizable, acyclically axiomatizable, and finitely acyclically axiomatizable are all equivalent for each restriction $\phi\restriction_d$.

6.5.2 Proposition. *Let ϕ be a closure operator and $C \sqsubseteq D$ be some $\mathcal{EL}^{\bot}_{d}$ concept inclusion. $C \sqsubseteq D$ is valid for ϕ if, and only if, $C \sqsubseteq D$ is valid for $\phi\restriction_d$.*

Proof. It is apparent that the following equivalences hold true.

$$\begin{aligned} & C \sqsubseteq_\phi D \\ \text{if, and only if,}\quad & C^\phi \sqsubseteq_\emptyset D \\ \text{if, and only if,}\quad & C^\phi\restriction_d \sqsubseteq_\emptyset D \\ \text{if, and only if,}\quad & C^{\phi\restriction_d} \sqsubseteq_\emptyset D \\ \text{if, and only if,}\quad & C \sqsubseteq_{\phi\restriction_d} D \end{aligned}$$

□

6.5.3 Proposition. *If ϕ is a compatible closure operator in $\boldsymbol{\mathcal{EL}}^{\bot}_{\mathsf{si}}(\Sigma)$, then each restriction $\phi\restriction_d$ is a finite, compatible closure operator in $\boldsymbol{\mathcal{EL}}^{\bot}_{d}(\Sigma)$.*

Proof. We have already argued above that each restriction $\phi\restriction_d$ is finite. We further show that $\phi\restriction_d$ is compatible. Thus, assume that $C^{\phi\restriction_d} \sqsubseteq_\emptyset \exists\, r.\, D$ holds true for $\mathcal{EL}^{\bot}$ concept descriptions C and D such that $\mathsf{rd}(C) \leq d$ and $\mathsf{rd}(D) \leq d-1$. We know that C^ϕ must be more specific than $C^{\phi\restriction_d}$ modulo $\emptyset$, which implies that $C^\phi \sqsubseteq_\emptyset \exists\, r.\, D$ holds true. Since ϕ is compatible, we infer that $C^\phi \sqsubseteq_\emptyset \exists\, r.\, D^\phi$ is satisfied, and we conclude that $C^\phi \sqsubseteq_\emptyset \exists\, r.\, D^{\phi\restriction_{d-1}}$. The latter concept description $\exists\, r.\, D^{\phi\restriction_{d-1}}$ has a role depth of at most d, and so it follows that $C^\phi\restriction_d \sqsubseteq_\emptyset \exists\, r.\, D^{\phi\restriction_{d-1}}$. Of course, $C^\phi\restriction_d$ and $C^{\phi\restriction_d}$ are equivalent modulo $\emptyset$ by definition of a restriction, and we are done. □

6.6 Bases of $\mathcal{EL}^{\bot}_{\mathsf{si}}$ Concept Inclusions for Closure Operators

6.6.1 Definition. Fix a compatible closure operator ϕ as well as some TBox $\mathcal{T}$ containing background knowledge that is valid in ϕ. A *concept inclusion base for ϕ relative to $\mathcal{T}$* is a TBox $\mathcal{B}$ such that, for any concept inclusion α, it holds true that $\phi \models \alpha$ if, and only if, $\mathcal{B} \cup \mathcal{T} \models \alpha$. △

In the following, we show how concept inclusion bases of closure operators can be computed. In particular, a unified, generalized theory of the results in [Dis11, Chapter 5; Kri15c, Section 4; BDK16, Sections 4.2 and 4.4; Kri19e, Section 8.4] is provided. This also implies that statements, proofs, and proof ideas can be similar to those used in the aforementioned documents.

Since we want to reduce the problem of computing a concept inclusion base for a DL closure operator to the problem of computing an implication base for an FCA closure operator, we need some finite set $\mathbf{M}$ of attributes on which we define the latter; otherwise infinite closures might occur, which cannot be handled in a computing device. To be able to suitably define such a finite attribute set $\mathbf{M}$, the reduction presented in the sequel of this section requires the given DL closure operator to be finite, i.e., to have only finitely many closures, and further to be compatible. Some exemplary cases of finite, compatible closure operators are as follows.

- $\phi_{\mathcal{I}}$ for some finite interpretation $\mathcal{I}$
- $\phi_{\mathcal{A}}$ for some simple ABox $\mathcal{A}$, cf. Section 6.8.4
- $\phi_{\mathcal{T}}$ for some $\mathcal{EL}^{\bot}_{\mathsf{si}}$ TBox $\mathcal{T}$ where it is known that only finitely many closures exist, e.g., if $\mathcal{T}$ is a concept inclusion base of some finite interpretation
- $\phi{\restriction_d}$ for some compatible closure operator ϕ
- $\phi \vartriangle \psi$ for finite, compatible closure operators ϕ and ψ
- $\phi \triangledown \psi$ for finitely representable closure operators ϕ and ψ where ϕ is finite (according to Corollary 6.6.9 it suffices to require that ϕ is finite and compatible, and ψ is finitely representable)

Throughout the whole section, assume that the signature Σ is finite and fix some finite, compatible closure operator ϕ. Further let $\mathcal{T}$ be some TBox containing background knowledge that is valid for ϕ. Without loss of generality we assume that $\mathcal{T}$ only contains concept inclusions of the form $E \sqsubseteq E^{\phi}$.[5] Sometimes it is necessary to distinguish between the *unrestricted case* of a closure operator ϕ in $\boldsymbol{\mathcal{EL}}^{\bot}_{\mathsf{si}}(\Sigma)$ and the *role-depth-bounded case* of a restricted closure operator $\phi{\restriction_d}$ in $\boldsymbol{\mathcal{EL}}^{\bot}_{d}(\Sigma)$. This will be explicitly mentioned; otherwise both ϕ and $\phi{\restriction_d}$ can be treated in the same way. More specifically, if no case distinction is made, then the statement or proof is formulated for the unrestricted case and the corresponding statement or proof for the role-depth-bounded case is obtained by replacing ϕ with $\phi{\restriction_d}$ and $\mathcal{EL}^{\bot}_{\mathsf{si}}$ with $\mathcal{EL}^{\bot}_{d}$. For our given closure operator and valid background knowledge this means the following.

unrestricted case. ϕ is a finite, compatible closure operator in $\boldsymbol{\mathcal{EL}}^{\bot}_{\mathsf{si}}(\Sigma)$ and $\mathcal{T}$ is an $\mathcal{EL}^{\bot}_{\mathsf{si}}$ TBox containing only concept inclusions of the form $E \sqsubseteq E^{\phi}$.

role-depth-bounded case. For some compatible closure operator ϕ in $\boldsymbol{\mathcal{EL}}^{\bot}_{\mathsf{si}}(\Sigma)$, we only consider the restriction $\phi{\restriction_d}$, which is a finite, compatible closure operator in $\boldsymbol{\mathcal{EL}}^{\bot}_{d}(\Sigma)$, and we further assume that $\mathcal{T}$ is an $\mathcal{EL}^{\bot}_{d}$ TBox containing only concept inclusions of the form $E \sqsubseteq E^{\phi\restriction_d}$.

We now define the following set of concept descriptions.

$$\mathbf{M} := \{\bot\} \cup \Sigma_{\mathsf{C}} \cup \{\, \exists\, r.\, C \mid r \in \Sigma_{\mathsf{R}} \text{ and } C \in \mathsf{Clo}(\phi)\,\} \qquad \textit{(unrestricted case)}$$

$$\mathbf{M} := \{\bot\} \cup \Sigma_{\mathsf{C}} \cup \{\, \exists\, r.\, C \mid r \in \Sigma_{\mathsf{R}} \text{ and } C \in \mathsf{Clo}(\phi{\restriction_{d-1}})\,\} \qquad \textit{(role-depth-bounded case)}$$

[5] If this is not the case, then replace each CI $E \sqsubseteq F$ in $\mathcal{T}$ by $E \sqsubseteq E^{\phi}$. The modified TBox is then still valid for ϕ, but it may have more consequences.

Note that we can also replace the existential restrictions $\exists\, r.\, C$ in $\mathbf{M}$ where C is a closure of ϕ by an existential restriction $\exists\, r.\, D$ where C and D are equivalent modulo $\mathcal{T}$. For instance, if we have some method for reducing $\mathcal{EL}^{\bot}_{\mathsf{si}}$ concept descriptions w.r.t. some $\mathcal{EL}^{\bot}_{\mathsf{si}}$ TBox, then such a concept description D can be a reduction of C w.r.t. $\mathcal{T}$. By a reduction we mean a concept description that is smaller or less complex; for instance, sometimes it might be possible that a cyclic $\mathcal{EL}^{\bot}_{\mathsf{si}}$ concept description can be reduced to an acyclic $\mathcal{EL}^{\bot}$ concept description w.r.t. a TBox. We say that some concept description $C \in \mathcal{EL}^{\bot}_{\mathsf{si}}(\Sigma)$ is *expressible in terms of* $\mathbf{M}$ if there is a subset $\mathbf{X} \subseteq \mathbf{M}$ such that $C \equiv_{\mathcal{T}} \bigsqcap \mathbf{X}$ holds true.

It is easy to see that in order to generate the attribute set $\mathbf{M}$, it suffices to have a procedure that can enumerate the finite set $\mathsf{Clo}(\phi)$ of closures. For the cases of finite, compatible closure operators listed above, we demonstrate how the set of all closures can be computed.

- For a closure operator $\phi_{\mathcal{I}}$ induced by a finite interpretation $\mathcal{I}$, we can simply iterate through all subsets X of the domain $\Delta^{\mathcal{I}}$ and return the respective model-based most specific concept description $X^{\mathcal{I}}$. Note that in order to ease computation, we can beforehand reduce the interpretation $\mathcal{I}$ with the methods described in Section 3.4.4, which only need polynomial time.
- A closure operator $\phi_{\mathcal{A}}$ induced by some simple ABox $\mathcal{A}$, which is defined in Section 6.8.4, can be treated as in the former case, since $\phi_{\mathcal{A}}$ equals the closure operator that is induced by the canonical Σ_{I}-model $\mathcal{I}_{\mathcal{A}}^{\Sigma_{\mathsf{I}}}$ of $\mathcal{A}$.
- If $\phi_{\mathcal{T}}$ is a closure operator induced by a TBox $\mathcal{T}$ and this TBox $\mathcal{T}$ is a concept inclusion base of some finite interpretation $\mathcal{I}$, then Corollary 4.3.52 shows that $\phi_{\mathcal{T}}$ equals $\phi_{\mathcal{I}}$. Thus, we can treat $\phi_{\mathcal{T}}$ as in the first case.
- The finitely many closures of a restriction $\phi\!\restriction_{d-1}$ can be enumerated as follows.
  1. Set $C := \top$.
  2. If the role depth of C does not exceed $d-1$, compute the closure $C := C^{\phi\restriction_{d-1}}$, and then output C.
  3. Compute the set of lower neighbors of C, e.g., by means of Corollary 5.1.13, and for each such lower neighbor L, set $C := L$ and go to Statement 2.
- Let ϕ and ψ be finite, compatible closure operators. Since $C^{\phi \vartriangle \psi} \equiv_{\emptyset} C^{\phi} \vee C^{\psi}$ holds true, we can simply enumerate the sets $\mathsf{Clo}(\phi)$ and $\mathsf{Clo}(\psi)$ and then compute all least common subsumers $C \vee D$ where $C \in \mathsf{Clo}(\phi)$ and $D \in \mathsf{Clo}(\psi)$. To see this, we formally prove that $\mathsf{Clo}(\phi \vartriangle \psi) = \{\, C \vee D \mid C \in \mathsf{Clo}(\phi) \text{ and } D \in \mathsf{Clo}(\psi) \,\}$ holds true modulo $\emptyset$. If C is a closure of $\phi \vartriangle \psi$, then $C \equiv_{\emptyset} C^{\phi} \vee C^{\psi}$, which shows the set inclusion $\subseteq$. Now consider a closure C of ϕ as well as a closure D of ψ. Then

  $$C^{\phi} \vee D^{\psi} \sqsupseteq_{\emptyset} (C^{\phi} \vee D^{\psi})^{\phi \vartriangle \psi} \sqsupseteq_{\emptyset} (C^{\phi})^{\phi \vartriangle \psi} \vee (D^{\psi})^{\phi \vartriangle \psi} \equiv_{\emptyset} C^{\phi} \vee D^{\psi}$$

  holds true, where the first subsumption follows from the extensitivity of $\phi \vartriangle \psi$, the second subsumption is implied by Corollary 6.2.1, and the third subsumption follows from $\phi \vartriangle \psi \trianglelefteq \phi$ and $\phi \vartriangle \psi \trianglelefteq \psi$, since according to Section 1.5 this implies $(\phi \vartriangle \psi) \circ \phi = \phi$ as well as $(\phi \vartriangle \psi) \circ \psi = \psi$. Then $C \vee D \equiv_{\emptyset} C^{\phi} \vee D^{\psi}$ yields that $C \vee D$ is a closure of $\phi \vartriangle \psi$.

- The last case is concerned with the supremum of two closure operators ϕ and ψ where ϕ is finite and compatible, and ψ is finitely representable. In order to enumerate all closures of $\phi \triangledown \psi$, we can generate all closures of ϕ and then compute the respective closures w.r.t. $\phi \triangledown \psi$. To justify this, we show that $\mathsf{Clo}(\phi \triangledown \psi) = \{ C^{\phi \triangledown \psi} \mid C \in \mathsf{Clo}(\phi) \}$ is satisfied modulo $\emptyset$. The set inclusion $\supseteq$ is obvious. If C is a closure of the supremum $\phi \triangledown \psi$, then C must also be a closure of ϕ, i.e., C is equivalent to both $C^{\phi \triangledown \psi}$ and C^{ϕ} modulo $\emptyset$. We conclude that $C \equiv_{\emptyset} (C^{\phi})^{\phi \triangledown \psi}$.

Furthermore, we define a projection mapping $\pi\colon \mathcal{EL}^{\bot}_{\mathsf{si}}(\Sigma) \to \wp(\mathbf{M})$ by

$$\pi(C) := \{ D \mid D \in \mathbf{M} \text{ and } C \sqsubseteq_{\mathcal{T}} D \}.$$

We then also call $\lceil C \rceil := \bigsqcap \pi(C)$ the *upper approximation* of C in terms of $\mathbf{M}$.

6.6.2 Lemma. (Generalization of [Bor14, Lemma 4.2.6]) *The pair $(\pi, \bigsqcap)$ is a Galois connection between $(\mathcal{EL}^{\bot}_{\mathsf{si}}(\Sigma), \sqsubseteq_{\mathcal{T}})/{\equiv_{\mathcal{T}}}$ and $(\wp(\mathbf{M}), \subseteq)$, i.e., the following statements hold true for all subsets $\mathbf{X}, \mathbf{Y} \subseteq \mathbf{M}$ and for all concept descriptions $C, D \in \mathcal{EL}^{\bot}_{\mathsf{si}}(\Sigma)$.*

1. *$\mathbf{X} \subseteq \pi(C)$ if, and only if, $\bigsqcap \mathbf{X} \sqsupseteq_{\mathcal{T}} C$*
2. *$\mathbf{X} \subseteq \mathbf{Y}$ implies $\bigsqcap \mathbf{X} \sqsupseteq_{\mathcal{T}} \bigsqcap \mathbf{Y}$*
3. *$\mathbf{X} \subseteq \pi(\bigsqcap \mathbf{X})$*
4. *$\bigsqcap \mathbf{X} \equiv_{\mathcal{T}} \bigsqcap \pi(\bigsqcap \mathbf{X})$*
5. *$C \sqsubseteq_{\mathcal{T}} D$ implies $\pi(C) \supseteq \pi(D)$*
6. *$C \sqsubseteq_{\mathcal{T}} \bigsqcap \pi(C)$*
7. *$\pi(C) = \pi(\bigsqcap \pi(C))$*

Proof. It suffices to show Statement 1; the others are then obtained as corollaries. Fix $\mathbf{X} \subseteq \mathbf{M}$ and $C \in \mathcal{EL}^{\bot}_{\mathsf{si}}(\Sigma)$. By definition, $\mathbf{X} \subseteq \pi(C)$ is equivalent to $C \sqsubseteq_{\mathcal{T}} X$ for each $X \in \mathbf{X}$. Obviously, the latter statement is equivalent to $C \sqsubseteq_{\mathcal{T}} \bigsqcap \mathbf{X}$, and we are done. □

6.6.3 Lemma. *If C is expressible in terms of $\mathbf{M}$, then C is equivalent to its upper approximation $\bigsqcap \pi(C)$ modulo $\mathcal{T}$.*

Proof. Since C is expressible in terms of $\mathbf{M}$, there is some subset $\mathbf{X} \subseteq \mathbf{M}$ such that $C \equiv_{\mathcal{T}} \bigsqcap \mathbf{X}$ holds true. The claim is now an immediate consequence of Statement 4 in Lemma 6.6.2. □

For each $\mathcal{EL}_{\mathsf{si}}$ concept description C, we define its *lower approximation* of C in terms of $\mathbf{M}$ as follows.

$$\lfloor C \rfloor := \bigsqcap \{ A \mid A \in \mathsf{Conj}(C) \} \sqcap \bigsqcap \{ \exists\, r.\, D^{\phi} \mid \exists\, r.\, D \in \mathsf{Conj}(C) \} \qquad \textit{(unrestricted case)}$$

$$\lfloor C \rfloor := \bigsqcap \{ A \mid A \in \mathsf{Conj}(C) \} \sqcap \bigsqcap \{ \exists\, r.\, D^{\phi\restriction_{d-1}} \mid \exists\, r.\, D \in \mathsf{Conj}(C) \} \qquad \textit{(role-depth-b. case)}$$

Additionally, we define $\lfloor \bot \rfloor := \bot$. Obviously, each lower approximation is expressible in terms of $\mathbf{M}$.

6.6.4 Proposition. *Each closure of ϕ is expressible in terms of $\mathbf{M}$.*

Proof. Fix some $\mathcal{EL}_{\mathsf{si}}$ concept description C. It is easy to see that $\lfloor C \rfloor \sqsubseteq_{\emptyset} C$ always holds true. We can further show that $C^{\phi} \sqsubseteq_{\emptyset} \lfloor C \rfloor$ is satisfied. If $A \in \mathsf{Conj}(C)$, then we have $C^{\phi} \sqsubseteq_{\emptyset} C \sqsubseteq_{\emptyset} A$.

Now let $\exists\, r.\, D \in \mathsf{Conj}(C)$. Then we have $C^\phi \sqsubseteq_\emptyset C \sqsubseteq_\emptyset \exists\, r.\, D$. Since ϕ is compatible, we conclude that $C^\phi \sqsubseteq \exists\, r.\, D^\phi$.

Summing up, we have shown that $C^\phi \sqsubseteq_\emptyset \lfloor C \rfloor \sqsubseteq_\emptyset C$ holds true for each $\mathcal{EL}_{\mathsf{si}}$ concept description C, and it follows that

$$C^\phi \equiv_\emptyset C^{\phi\phi} \sqsubseteq_\emptyset \lfloor C^\phi \rfloor \sqsubseteq_\emptyset C^\phi.$$

The case for a closure $\bot$ is trivial. We infer that each closure of ϕ is expressible in terms of $\mathbf{M}$. □

6.6.5 Proposition. (Generalization of [Dis11, Theorem 5.10]) *The following TBox $\mathcal{B}_{\mathbf{M}}$ is sound and complete for ϕ relative to $\mathcal{T}$.*

$$\mathcal{B}_{\mathbf{M}} \coloneqq \{\, \textstyle\bigsqcap \mathbf{X} \sqsubseteq (\bigsqcap \mathbf{X})^\phi \mid \mathbf{X} \subseteq \mathbf{M} \,\}$$

Proof. Soundness is obvious, since each concept inclusion from $\mathcal{B}_{\mathbf{M}}$ is valid in ϕ.

unrestricted case. For proving completeness, we show that $\mathcal{B}_{\mathbf{M}}$ entails the concept inclusion $C \sqsubseteq C^\phi$ for each $\mathcal{EL}_{\mathsf{si}}$ concept description C. Fix some such $C \in \mathcal{EL}_{\mathsf{si}}(\Sigma)$. Since the lower approximation $\lfloor C \rfloor$ is expressible in terms of $\mathbf{M}$, the concept inclusion $\lfloor C \rfloor \sqsubseteq \lfloor C \rfloor^\phi$ is entailed by $\mathcal{B}_{\mathbf{M}} \cup \mathcal{T}$.[6] Of course, $\lfloor C \rfloor$ is more specific than C modulo $\emptyset$, and so we infer using monotonicity of ϕ that $\mathcal{B}_{\mathbf{M}} \cup \mathcal{T} \models \lfloor C \rfloor \sqsubseteq C^\phi$. We already know that C^ϕ is more specific than the lower approximation $\lfloor C \rfloor$ modulo $\emptyset$, it follows that $\lfloor C \rfloor \equiv_{\mathcal{B}_{\mathbf{M}} \cup \mathcal{T}} C^\phi$.

It remains to prove that $\mathcal{B}_{\mathbf{M}} \cup \mathcal{T}$ entails $C \sqsubseteq \lfloor C \rfloor$. According to Lemma 3.4.6, we can equivalently show that $\mathcal{B}_{\mathbf{M}} \cup \mathcal{T} \models C \sqsubseteq \lfloor C \rfloor \restriction_d$ for each number $d \in \mathbb{N}$. We do this by induction on d. The induction base where $d = 0$ is obvious. Regarding the induction step we have

$$\begin{aligned}
\lfloor C \rfloor \restriction_{d+1} &\equiv_\emptyset && \textstyle\bigsqcap \{\, A \mid A \in \mathsf{Conj}(C) \,\} \sqcap \bigsqcap \{\, \exists\, r.\, D^\phi \restriction_d \mid \exists\, r.\, D \in \mathsf{Conj}(C) \,\} \\
&\equiv_{\mathcal{B}_{\mathbf{M}} \cup \mathcal{T}} && \textstyle\bigsqcap \{\, A \mid A \in \mathsf{Conj}(C) \,\} \sqcap \bigsqcap \{\, \exists\, r.\, \lfloor D \rfloor \restriction_d \mid \exists\, r.\, D \in \mathsf{Conj}(C) \,\} && \text{(6.6.A)} \\
&\sqsupseteq_{\mathcal{B}_{\mathbf{M}} \cup \mathcal{T}} && \textstyle\bigsqcap \{\, A \mid A \in \mathsf{Conj}(C) \,\} \sqcap \bigsqcap \{\, \exists\, r.\, D \mid \exists\, r.\, D \in \mathsf{Conj}(C) \,\} && \text{(6.6.B)} \\
&\equiv_\emptyset && C
\end{aligned}$$

where Equation (6.6.A) follows from the above justified fact that D^ϕ and $\lfloor D \rfloor$ are equivalent modulo $\mathcal{B}_{\mathbf{M}} \cup \mathcal{T}$, and Equation (6.6.B) is a consequence of the induction hypothesis.

role-depth-bounded case. For demonstrating completeness in the role-depth-bounded case, we show by structural induction that $\mathcal{B}_{\mathbf{M}}$ entails the concept inclusion $C \sqsubseteq C^{\phi \restriction_d}$ for each $\mathcal{EL}$ concept description C with a role depth not exceeding d.

- For $C = \top$, the conjunction $\bigsqcap \emptyset$ is equivalent to $\top$ and the concept inclusion $\bigsqcap \emptyset \sqsubseteq (\bigsqcap \emptyset)^\phi$ is in $\mathcal{B}_{\mathbf{M}}$.
- The cases $C = \bot$ and $C = A$ for some concept name $A \in \Sigma_{\mathsf{C}}$ are satisfied by construction, since $\{\bot\} \cup \Sigma_{\mathsf{C}} \subseteq \mathbf{M}$.

[6] The union with $\mathcal{T}$ is needed here, since we allowed that the concept descriptions in $\mathbf{M}$ may be replaced by $\mathcal{T}$-equivalent ones.

- Assume that $C = D \sqcap E$ is a conjunction. The induction hypothesis yields that $\mathcal{B}_{\mathbf{M}}$ entails $D \sqsubseteq D^{\phi\upharpoonright_d}$ and $E \sqsubseteq E^{\phi\upharpoonright_d}$, which implies that $\mathcal{B}_{\mathbf{M}}$ entails $D \sqcap E \sqsubseteq D^{\phi\upharpoonright_d} \sqcap E^{\phi\upharpoonright_d}$ as well. Furthermore, both closures $D^{\phi\upharpoonright_d}$ and $E^{\phi\upharpoonright_d}$ are expressible in terms of $\mathbf{M}$, i.e., there exist subsets $\mathbf{X}$ and $\mathbf{Y}$ of $\mathbf{M}$ such that $D^{\phi\upharpoonright_d} \equiv_{\mathcal{T}} \bigsqcap \mathbf{X}$ and $E^{\phi\upharpoonright_d} \equiv_{\mathcal{T}} \bigsqcap \mathbf{Y}$ hold true. Thus, $\mathcal{B}_{\mathbf{M}}$ contains the concept inclusion $\bigsqcap(\mathbf{X} \cup \mathbf{Y}) \sqsubseteq (\bigsqcap(\mathbf{X} \cup \mathbf{Y}))^{\phi\upharpoonright_d}$, which means that $\mathcal{B}_{\mathbf{M}} \cup \mathcal{T}$ entails $D^{\phi\upharpoonright_d} \sqcap E^{\phi\upharpoonright_d} \sqsubseteq (D^{\phi\upharpoonright_d} \sqcap E^{\phi\upharpoonright_d})^{\phi\upharpoonright_d}$. Since $\phi\upharpoonright_d$ is extensive and monotonic, we also have that $(D^{\phi\upharpoonright_d} \sqcap E^{\phi\upharpoonright_d})^{\phi\upharpoonright_d} \sqsubseteq_{\emptyset} (D \sqcap E)^{\phi\upharpoonright_d}$. In summary, it follows that $\mathcal{B}_{\mathbf{M}} \cup \mathcal{T}$ entails $D \sqcap E \sqsubseteq (D \sqcap E)^{\phi\upharpoonright_d}$.
- Finally, fix some existential restriction $C = \exists\, r.\, D$. By induction hypothesis, $\mathcal{B}_{\mathbf{M}}$ entails $D \sqsubseteq D^{\phi\upharpoonright_d}$. The subsumption $D^{\phi\upharpoonright_d} \sqsubseteq_{\emptyset} D^{\phi\upharpoonright_{d-1}}$ is obvious. Furthermore, $\exists\, r.\, D^{\phi\upharpoonright_{d-1}}$ is equivalent to an element of $\mathbf{M}$ modulo $\mathcal{T}$, which implies that $\mathcal{B}_{\mathbf{M}} \cup \mathcal{T}$ entails $\exists\, r.\, D^{\phi\upharpoonright_{d-1}} \sqsubseteq (\exists\, r.\, D^{\phi\upharpoonright_{d-1}})^{\phi\upharpoonright_d}$. Utilizing extensitivity of $\phi\upharpoonright_{d-1}$ and monotonicity of $\phi\upharpoonright_d$ shows that $(\exists\, r.\, D^{\phi\upharpoonright_{d-1}})^{\phi\upharpoonright_d}$ is subsumed by $(\exists\, r.\, D)^{\phi\upharpoonright_d}$ modulo $\emptyset$. We conclude that $\exists\, r.\, D \sqsubseteq (\exists\, r.\, D)^{\phi\upharpoonright_d}$ is a consequence of $\mathcal{B}_{\mathbf{M}} \cup \mathcal{T}$. □

Now we define the *induced closure operator* $\widehat{\phi}$ on the set $\mathbf{M}$ as follows.

$$\mathbf{X}^{\widehat{\phi}} := \{\, C \mid C \in \mathbf{M} \text{ and } \bigsqcap \mathbf{X} \sqsubseteq_{\phi} C \,\}$$

It is easy to see that, for arbitrary subsets $\mathbf{X}, \mathbf{Y} \subseteq \mathbf{M}$, the implication $\mathbf{X} \to \mathbf{Y}$ is valid for $\widehat{\phi}$ if, and only if, the concept inclusion $\bigsqcap \mathbf{X} \sqsubseteq \bigsqcap \mathbf{Y}$ is valid for ϕ.

$$\begin{aligned} \widehat{\phi} \models \mathbf{X} \to \mathbf{Y} &\text{ if, and only if, } \mathbf{Y} \subseteq \mathbf{X}^{\widehat{\phi}} \\ &\text{ if, and only if, } \phi \models \bigsqcap \mathbf{X} \sqsubseteq Y \text{ for each } Y \in \mathbf{Y} \\ &\text{ if, and only if, } \phi \models \bigsqcap \mathbf{X} \sqsubseteq \bigsqcap \mathbf{Y} \end{aligned}$$

6.6.6 Lemma. $(\bigsqcap \mathbf{X})^{\phi}$ *and* $\bigsqcap \mathbf{X}^{\widehat{\phi}}$ *are equivalent modulo* $\mathcal{T}$ *for each subset* $\mathbf{X} \subseteq \mathbf{M}$.

Proof. We start with proving that $\pi((\bigsqcap \mathbf{X})^{\phi}) = \mathbf{X}^{\widehat{\phi}}$.

$$\begin{aligned} \pi((\bigsqcap \mathbf{X})^{\phi}) &= \{\, C \mid C \in \mathbf{M} \text{ and } (\bigsqcap \mathbf{X})^{\phi} \sqsubseteq_{\mathcal{T}} C \,\} \\ &= \{\, C \mid C \in \mathbf{M} \text{ and } ((\bigsqcap \mathbf{X})^{\phi})^{\mathcal{T}} \sqsubseteq_{\emptyset} C \,\} \\ &= \{\, C \mid C \in \mathbf{M} \text{ and } (\bigsqcap \mathbf{X})^{\phi} \sqsubseteq_{\emptyset} C \,\} \\ &= \{\, C \mid C \in \mathbf{M} \text{ and } \bigsqcap \mathbf{X} \sqsubseteq_{\phi} C \,\} \\ &= \mathbf{X}^{\widehat{\phi}} \end{aligned}$$

Note that $((\bigsqcap \mathbf{X})^{\phi})^{\mathcal{T}}$ and $(\bigsqcap \mathbf{X})^{\phi}$ are equivalent modulo $\emptyset$, since $\phi \models \mathcal{T}$ holds true by our initial assumption, i.e., each closure of ϕ is a most specific consequence for $\mathcal{T}$.

We conclude that $\bigsqcap \pi((\bigsqcap \mathbf{X})^{\phi}) \equiv_{\mathcal{T}} \bigsqcap \mathbf{X}^{\widehat{\phi}}$ holds true. Since $(\bigsqcap \mathbf{X})^{\phi}$ is a closure of ϕ, it is expressible in terms of $\mathbf{M}$. We conclude that $\bigsqcap \pi((\bigsqcap \mathbf{X})^{\phi})$ and $(\bigsqcap \mathbf{X})^{\phi}$ are equivalent modulo $\mathcal{T}$. Summing up, we obtain that $(\bigsqcap \mathbf{X})^{\phi} \equiv_{\mathcal{T}} \bigsqcap \mathbf{X}^{\widehat{\phi}}$. □

In the following, we show how implication bases for $\widehat{\phi}$ can be used to construct concept inclusion bases for ϕ. Since we do not want to have tautological concept inclusions in the result and we also want to incorporate the background knowledge $\mathcal{T}$ in order to achieve that the result does not contain any concept inclusion that is already entailed by $\mathcal{T}$, we define the following implication set $\widehat{\mathcal{T}}$ that we use as background knowledge during the axiomatization of $\widehat{\phi}$.

$$\widehat{\mathcal{T}} := \{\, \mathbf{X} \to \{C\} \mid \mathbf{X} \cup \{C\} \subseteq \mathbf{M} \text{ and } \bigsqcap \mathbf{X} \sqsubseteq_{\mathcal{T}} C \,\}$$

This implication set is strongly related to the projection operator π defined earlier. Since $\bigsqcap \mathbf{X} \sqsubseteq_{\mathcal{T}} C$ holds true if, and only if, $C \in \pi(\bigsqcap \mathbf{X})$ is satisfied, $\widehat{\mathcal{T}}$ is equivalent to the implication set $\{\, \mathbf{X} \to \pi(\bigsqcap \mathbf{X}) \mid \mathbf{X} \subseteq \mathbf{M} \,\}$. We can furthermore show that each closure $\mathbf{X}^{\widehat{\mathcal{T}}}$ is equal to $\pi(\bigsqcap \mathbf{X})$. The inclusion $\mathbf{X}^{\widehat{\mathcal{T}}} \supseteq \pi(\bigsqcap \mathbf{X})$ follows from the fact that $\widehat{\mathcal{T}}$ entails the implication $\mathbf{X} \to \pi(\bigsqcap \mathbf{X})$. It is further easy to see that $\bigsqcap \mathbf{X} \sqsubseteq_{\mathcal{T}} \bigsqcap \mathbf{X}^{\widehat{\mathcal{T}}}$ and so Lemma 6.6.2 implies $\mathbf{X}^{\widehat{\mathcal{T}}} \subseteq \pi(\bigsqcap \mathbf{X})$.

Note that we make use of the background implication set $\widehat{\mathcal{T}}$ only virtually. In particular, we would not pre-compute it in its full form, but rather devise a procedure that is able to compute closures for it. This is due to the fact that $\widehat{\mathcal{T}}$ can have an exponential size, and using it explicitly would severely impact the performance of our approach of axiomatizing ϕ.

We have shown above that $\mathbf{X}^{\widehat{\mathcal{T}}}$ equals $\pi(\bigsqcap \mathbf{X})$. Thus, for computing a closure $\mathbf{X}^{\widehat{\mathcal{T}}}$ *on-demand* for some subset $\mathbf{X} \subseteq \mathbf{M}$, we can use the following procedure.

1. Initialize a set $\mathbf{Y} := \emptyset$.
2. Iterate over $\mathbf{M}$ and check for each element C whether $\bigsqcap \mathbf{X} \sqsubseteq_{\mathcal{T}} C$ holds true. If yes, then add C to $\mathbf{Y}$.
3. Eventually, return $\mathbf{X} \cup \mathbf{Y}$.

Since deciding subsumption w.r.t. some TBox can be done in polynomial time, we conclude that the above procedure runs in time $|\mathbf{M}| \cdot (|\mathbf{M}| \cdot s + ||\mathcal{T}||)^n$, where s is the largest size of some concept description in $\mathbf{M}$ and n is the exponent for deciding subsumption, that is in polynomial time w.r.t. $|\mathbf{M}|$, s, and $||\mathcal{T}||$. In contrast, computing closures for $\widehat{\mathcal{T}}$ in the usual way from an explicit representation of $\widehat{\mathcal{T}}$ would first require the computation of $\widehat{\mathcal{T}}$, which can have a size that is exponential in $\mathbf{M}$, and we conclude that taking this way of computation needs exponential time w.r.t. $\mathbf{M}$.

6.6.7 Theorem. (Generalization of [Dis11, Theorem 5.12]) *If $\mathcal{L}$ is an implication base for $\widehat{\phi}$ w.r.t. the background knowledge $\widehat{\mathcal{T}}$, then*

$$\mathcal{B}_{\mathcal{L}} := \{\, \bigsqcap \mathbf{U} \sqsubseteq (\bigsqcap \mathbf{U})^{\phi} \mid \mathbf{U} \to \mathbf{V} \in \mathcal{L} \text{ for some } \mathbf{V} \,\}$$

is sound and complete for ϕ relative to $\mathcal{T}$.

Proof. Soundness is obvious. We proceed with proving that $\mathcal{B}_{\mathcal{L}} \cup \mathcal{T}$ entails $\mathcal{B}_{\mathbf{M}}$ from Proposition 6.6.5, which then yields completeness. For this purpose fix some model $\mathcal{I}$ of $\mathcal{B}_{\mathcal{L}} \cup \mathcal{T}$. Our goal is to show that $\mathcal{I}$ is a model of $\mathcal{B}_{\mathbf{M}}$ as well. We define the formal context $\mathbb{K} := (\Delta^{\mathcal{I}}, \mathbf{M}, I)$ where $I := \{\, (\delta, C) \mid \delta \in C^{\mathcal{I}} \,\}$, and we now show that all implications from $\mathcal{L}$ and from $\widehat{\mathcal{T}}$ are valid in $\mathbb{K}$.

Consider some such implication $\mathbf{U} \to \mathbf{V}$ in $\mathcal{L}$. Without loss of generality assume that $\mathbf{V} = \mathbf{U}^{\widehat{\phi}}$. By definition, $\mathcal{B}_{\mathcal{L}}$ contains the concept inclusion $\bigsqcap \mathbf{U} \sqsubseteq (\bigsqcap \mathbf{U})^{\phi}$, i.e., it must be valid in $\mathcal{I}$. We conclude that the implication $\mathbf{U} \to \mathbf{U}^{\widehat{\phi}}$ is valid in $\mathbb{K}$, since the following set inclusions are satisfied.

$$\mathbf{U}^{I} = (\bigsqcap \mathbf{U})^{\mathcal{I}} \subseteq (\bigsqcap \mathbf{U})^{\phi\mathcal{I}} = (\bigsqcap \mathbf{U}^{\widehat{\phi}})^{\mathcal{I}} = \mathbf{U}^{\widehat{\phi}I}$$

Further let $\mathbf{Y} \to \mathbf{Z}$ be an implication in $\widehat{\mathcal{T}}$, that is, $\bigsqcap \mathbf{Y} \sqsubseteq_{\mathcal{T}} \bigsqcap \mathbf{Z}$ is satisfied. Since $\mathcal{I}$ is a model of $\mathcal{T}$, we have that the concept inclusion $\bigsqcap \mathbf{Y} \sqsubseteq \bigsqcap \mathbf{Z}$ is valid in $\mathcal{I}$. This implies that

$$\mathbf{Y}^{I} = (\bigsqcap \mathbf{Y})^{\mathcal{I}} \subseteq (\bigsqcap \mathbf{Z})^{\mathcal{I}} = \mathbf{Z}^{I},$$

which means that the implication $\mathbf{Y} \to \mathbf{Z}$ is valid in $\mathbb{K}$.

If we now consider an arbitrary subset $\mathbf{X} \subseteq \mathbf{M}$, then the implication $\mathbf{X} \to \mathbf{X}^{\widehat{\phi}}$ is, of course, valid for $\widehat{\phi}$. This immediately implies that it must be a consequence of $\mathcal{L} \cup \widehat{\mathcal{T}}$ and further that it is valid in $\mathbb{K}$. We conclude that the concept inclusion $\bigsqcap \mathbf{X} \sqsubseteq \bigsqcap \mathbf{X}^{\widehat{\phi}}$ is valid in $\mathcal{I}$. We further have that the concept descriptions $\bigsqcap \mathbf{X}^{\widehat{\phi}}$ and $(\bigsqcap \mathbf{X})^{\phi}$ are equivalent modulo $\emptyset$, which yields that $\bigsqcap \mathbf{X} \sqsubseteq (\bigsqcap \mathbf{X})^{\phi}$ is valid in $\mathcal{I}$. Since the considered subset $\mathbf{X}$ is arbitrary, we conclude that $\mathcal{I}$ is indeed a model of $\mathcal{B}_{\mathbf{M}}$. □

6.6.8 Corollary. (Generalization of [Dis11, Corollary 5.13]) *The following TBox* $\mathsf{Can}(\phi, \mathcal{T})$*, called* canonical base *for* ϕ *relative to* $\mathcal{T}$*, is sound and complete for* ϕ *relative to* $\mathcal{T}$*.*

$$\mathsf{Can}(\phi, \mathcal{T}) := \{ \bigsqcap \mathbf{P} \sqsubseteq \bigsqcap \mathbf{P}^{\widehat{\phi}} \mid \mathbf{P} \text{ is a pseudo-closure of } \widehat{\phi} \text{ relative to } \widehat{\mathcal{T}} \}$$

6.6.9 Corollary. *Each finite, compatible closure operator is finitely representable.*

Note that each finitely representable closure operator is compatible as shown in Section 6.2. However, not every finitely representable closure operator is also finite. A counterexample is the closure operator induced by the empty TBox: $\emptyset$ is a finite axiomatization of $\phi_{\emptyset}$, but the set of closures of $\phi_{\emptyset}$ contains *all* $\mathcal{EL}^{\bot}_{\mathsf{si}}$ concept descriptions.

6.6.10 Theorem. (Generalization of [Dis11, Theorem 5.18]) $\mathsf{Can}(\phi, \mathcal{T})$ *is of minimal cardinality among all TBoxes that are sound and complete for* ϕ *relative to* $\mathcal{T}$*.*

Proof. Consider some TBox $\mathcal{B}$ that is sound and complete for ϕ relative to $\mathcal{T}$. Without loss of generality assume that $\mathcal{B}$ only contains concept inclusions of the form $E \sqsubseteq E^{\phi}$. Further note that we have assumed the very same for the background knowledge $\mathcal{T}$: it only contains concept inclusions of the form $E \sqsubseteq E^{\phi}$ as well. We already know that, by construction, $|\mathsf{Can}(\phi, \mathcal{T})| \leq |\mathsf{Can}(\widehat{\phi}, \widehat{\mathcal{T}})|$ is satisfied, and we are now going to devise an implication set $\mathcal{L}$ that satisfies

$$|\mathsf{Can}(\widehat{\phi}, \widehat{\mathcal{T}})| \leq |\mathcal{L}| \leq |\mathcal{B}|,$$

which then implies the claim. In particular, we choose the following.

$$\mathcal{L} := \{ \pi(\lfloor E \rfloor) \to \pi(E^{\phi}) \mid E \in \mathsf{Prem}(\mathcal{B}) \}$$

The second inequality $|\mathcal{L}| \leq |\mathcal{B}|$ is obviously satisfied. To prove the first inequality we show that $\mathcal{L}$ is sound and complete for $\widehat{\phi}$ relative to $\widehat{\mathcal{T}}$.

Soundness. We know that $\lfloor E \rfloor \sqsubseteq_\emptyset E$ is always satisfied, which implies $\lfloor E \rfloor^\phi \sqsubseteq_\emptyset E^\phi$ due to monotonicity of ϕ. Since the approximation $\lfloor E \rfloor$ is expressible in terms of $\mathbf{M}$, we obtain $\lfloor E \rfloor \equiv_\mathcal{T} \bigsqcap \pi(\lfloor E \rfloor)$. It follows that $\lfloor E \rfloor^\phi \equiv_\mathcal{T} (\bigsqcap \pi(\lfloor E \rfloor))^\phi \equiv_\mathcal{T} \bigsqcap (\pi(\lfloor E \rfloor))^{\widehat{\phi}}$. Summing up shows that $\bigsqcap (\pi(\lfloor E \rfloor))^{\widehat{\phi}} \sqsubseteq_\mathcal{T} E^\phi$. Utilizing the laws of a Galois connection yields $(\pi(\lfloor E \rfloor))^{\widehat{\phi}} \supseteq \pi(E^\phi)$.

Completeness, unrestricted case. We already know that $\mathcal{L}$ is complete for $\widehat{\phi}$ relative to $\widehat{\mathcal{T}}$ if, and only if, $\mathsf{Mod}(\mathcal{L} \cup \widehat{\mathcal{T}}) \subseteq \mathsf{Clo}(\widehat{\phi})$ holds true. Thus, consider some subset $\mathbf{U} \subseteq \mathbf{M}$ that is a model of $\mathcal{L} \cup \widehat{\mathcal{T}}$.

We have already seen before that $\mathbf{U}^{\widehat{\mathcal{T}}}$ equals $\pi(\bigsqcap \mathbf{U})$, and so we conclude that $\mathbf{U} = \mathbf{U}^{\widehat{\mathcal{T}}}$ implies $\mathbf{U} = \pi(\bigsqcap \mathbf{U})$.

We further show that $\mathbf{U} = \mathbf{U}^{\widehat{\mathcal{T}}}$ implies that $\bigsqcap \mathbf{U} \equiv_\emptyset (\bigsqcap \mathbf{U})^\mathcal{T}$. Since all existentially quantified subconcepts of $\bigsqcap \mathbf{U}$ are closures of ϕ, it suffices to prove that the root $\bigsqcap \mathbf{U}$ is saturated for $\mathcal{T}$, i.e., we show that $\bigsqcap \mathbf{U} \sqsubseteq_\emptyset E$ implies $\bigsqcap \mathbf{U} \sqsubseteq_\emptyset E^\phi$ for each concept inclusion $E \sqsubseteq E^\phi \in \mathcal{T}$. Thus, fix some $E \sqsubseteq E^\phi \in \mathcal{T}$ and let $\bigsqcap \mathbf{U} \sqsubseteq_\emptyset E$. It follows that $\bigsqcap \mathbf{U} \sqsubseteq_\mathcal{T} E^\phi$. Since E^ϕ is expressible in terms of $\mathbf{M}$, we can w.l.o.g. assume that $\mathsf{Conj}(E^\phi) \subseteq \mathbf{M}$. As a consequence, we obtain that $\bigsqcap \mathbf{U} \sqsubseteq_\mathcal{T} Y$ for each $Y \in \mathsf{Conj}(E^\phi)$, which means that $\mathsf{Conj}(E^\phi) \subseteq \mathbf{U}$ as $\mathbf{U}$ is a model of $\widehat{\mathcal{T}}$.

Since $\mathbf{U}$ is a model of $\mathcal{L}$, we obtain that $\pi(\lfloor E \rfloor) \subseteq \mathbf{U}$ implies $\pi(E^\phi) \subseteq \mathbf{U}$ for each concept inclusion $E \sqsubseteq E^\phi \in \mathcal{B}$. The premise can be transformed into an equivalent statement in the following way.

$$
\begin{aligned}
& & \pi(\lfloor E \rfloor) &\subseteq \mathbf{U} \\
&\text{if, and only if,} & \pi(\lfloor E \rfloor) &\subseteq \pi(\textstyle\bigsqcap \mathbf{U}) \\
&\text{if, and only if,} & \textstyle\bigsqcap \pi(\lfloor E \rfloor) &\sqsupseteq_\mathcal{T} \textstyle\bigsqcap \mathbf{U} \\
&\text{if, and only if,} & \lfloor E \rfloor &\sqsupseteq_\mathcal{T} \textstyle\bigsqcap \mathbf{U} \\
&\text{if, and only if,} & \lfloor E \rfloor &\sqsupseteq_\emptyset (\textstyle\bigsqcap \mathbf{U})^\mathcal{T} \\
&\text{if, and only if,} & \lfloor E \rfloor &\sqsupseteq_\emptyset \textstyle\bigsqcap \mathbf{U} \\
&\text{if, and only if,} & E &\sqsupseteq_\emptyset \textstyle\bigsqcap \mathbf{U}
\end{aligned}
$$

All of the above equivalences except the last one are either implied by Lemma 6.6.2 or by the results from Section 4.3. Let $\bigsqcap \mathbf{U} \sqsubseteq_\emptyset E$. Consider some top-level conjunct $\exists\, r.\, F \in \mathsf{Conj}(E)$, then there must exist some $\exists\, r.\, G^\phi \in \mathbf{U}$ such that $G^\phi \sqsubseteq_\emptyset F$. We can immediately conclude that $G^\phi \sqsubseteq_\emptyset F^\phi$, since ϕ is monotonic and idempotent. Thus, we can safely infer that $\bigsqcap \mathbf{U} \sqsubseteq_\emptyset \lfloor E \rfloor$.

We further transform the above conclusion into an equivalent statement as follows.

$$
\begin{aligned}
& & \pi(E^\phi) &\subseteq \mathbf{U} \\
&\text{if, and only if,} & \pi(E^\phi) &\subseteq \pi(\textstyle\bigsqcap \mathbf{U}) \\
&\text{if, and only if,} & \textstyle\bigsqcap \pi(E^\phi) &\sqsupseteq_\mathcal{T} \textstyle\bigsqcap \mathbf{U}
\end{aligned}
$$

$$
\begin{aligned}
&\text{if, and only if,} && E^{\phi} \sqsupseteq_{\mathcal{T}} \textstyle\bigsqcap \mathbf{U} \\
&\text{if, and only if,} && E^{\phi} \sqsupseteq_{\emptyset} (\textstyle\bigsqcap \mathbf{U})^{\mathcal{T}} \\
&\text{if, and only if,} && E^{\phi} \sqsupseteq_{\emptyset} \textstyle\bigsqcap \mathbf{U}
\end{aligned}
$$

We conclude that $\bigsqcap \mathbf{U} \sqsubseteq_{\emptyset} E$ implies $\bigsqcap \mathbf{U} \sqsubseteq_{\emptyset} E^{\phi}$ for each concept inclusion $E \sqsubseteq E^{\phi} \in \mathcal{B}$, i.e., the root $\bigsqcap \mathbf{U}$ is saturated for $\mathcal{B}$. Since all existentially quantified subconcepts of $\bigsqcap \mathbf{U}$ are closures of ϕ, these are saturated for $\mathcal{B}$ as well. Consequently, $\bigsqcap \mathbf{U}$ must be a most specific consequence for $\mathcal{B}$.

It remains to show that $\mathbf{U}$ is a closure of $\widehat{\phi}$. For this purpose, fix some $C \in \mathbf{U}^{\widehat{\phi}}$, i.e., it holds true that $\bigsqcap \mathbf{U} \sqsubseteq_{\phi} C$ and so $\bigsqcap \mathbf{U} \sqsubseteq_{\mathcal{B} \cup \mathcal{T}} C$. Of course, the latter statement is equivalent to $(\bigsqcap \mathbf{U})^{\mathcal{B} \cup \mathcal{T}} \sqsubseteq_{\emptyset} C$. As we have shown above that $\bigsqcap \mathbf{U}$ is a most specific consequence of both $\mathcal{B}$ and $\mathcal{T}$, it holds true that $(\bigsqcap \mathbf{U})^{\mathcal{B} \cup \mathcal{T}}$ is equivalent to $\bigsqcap \mathbf{U}$ modulo $\emptyset$. It thus follows that $\bigsqcap \mathbf{U} \sqsubseteq_{\emptyset} C$. In particular, this implies $\bigsqcap \mathbf{U} \sqsubseteq_{\mathcal{T}} C$, and since $\mathbf{U}$ is a model of $\widehat{\mathcal{T}}$, we infer further that $C \in \mathbf{U}$ must be satisfied.

Completeness, role-depth-bounded case. The proof for the role-depth-bounded case is essentially the same as for the unrestricted case, except that we cannot prove that $\bigsqcap \mathbf{U}$ is a most specific consequence for both $\mathcal{B}$ and $\mathcal{T}$. Instead, we suitably consider the concept description $\bigsqcap \mathbf{V}$ where

$$\mathbf{V} := \{\, A \mid A \in \mathbf{U} \,\} \cup \{\, \exists\, r.\, F^{\phi} \mid \exists\, r.\, F^{\phi\restriction_{d-1}} \in \mathbf{U} \,\}.$$

Of course, $\bigsqcap \mathbf{V}$ is more specific than $\bigsqcap \mathbf{U}$ modulo $\emptyset$, and $(\bigsqcap \mathbf{V})\restriction_{d} \equiv_{\emptyset} \bigsqcap \mathbf{U}$ holds true.

It is now possible to prove that $\bigsqcap \mathbf{V}$ is a most specific consequence of both $\mathcal{B}$ and $\mathcal{T}$. We must only make use of the fact that $\bigsqcap \mathbf{V} \sqsubseteq_{\emptyset} C$ is equivalent to $\bigsqcap \mathbf{U} \sqsubseteq_{\emptyset} C$ for each concept description C with a role depth not exceeding d.

Finally, if we consider some $C \in \mathbf{U}^{\widehat{\phi}}$, then we have $\bigsqcap \mathbf{U} \sqsubseteq_{\mathcal{B} \cup \mathcal{T}} C$ as in the unrestricted case. Of course, this implies that $\bigsqcap \mathbf{V} \sqsubseteq_{\mathcal{B} \cup \mathcal{T}} C$, and we can conclude that $\bigsqcap \mathbf{V} \sqsubseteq_{\emptyset} C$, since $\bigsqcap \mathbf{V}$ is a most specific consequence of both $\mathcal{B}$ and $\mathcal{T}$. As argued above, $\mathsf{rd}(C) \leq d$ yields $\bigsqcap \mathbf{U} \sqsubseteq_{\emptyset} C$, and we are done. □

6.6.11 Corollary. *The canonical base* $\mathsf{Can}(\phi, \mathcal{T})$ *does not contain any concept inclusion that is entailed by* $\mathcal{T}$*.*

Proof. Assume to the contrary that there is some concept inclusion $C \sqsubseteq D$ in $\mathsf{Can}(\phi, \mathcal{T})$ which is entailed by $\mathcal{T}$. Now define the TBox $\mathcal{B} := \mathsf{Can}(\phi, \mathcal{T}) \setminus \{C \sqsubseteq D\}$. Then $\mathcal{B}$ is still sound for ϕ. Since $\mathcal{T}$ entails $C \sqsubseteq D$, we conclude that $\mathcal{B} \cup \mathcal{T}$ entails $\mathsf{Can}(\phi, \mathcal{T}) \cup \mathcal{T}$, which shows that $\mathcal{B} \cup \mathcal{T}$ is complete for ϕ. However, since $|\mathcal{B}| \lneq |\mathsf{Can}(\phi, \mathcal{T})|$ holds true, we get a contradiction to the minimality shown in Theorem 6.6.10. □

6.7 Comments and Remarks

The original intention was to use a generalized version of the algorithm *NextClosures* for enumerating canonical bases of $\mathcal{EL}_{\mathsf{si}}^{\bot}$ closure operators. For this purpose, the author published in [Kri16b] such a variant that is capable of handling closure operators in lattices. However, several problems occurred.

1. It had been unknown whether and how lower neighbors of $\mathcal{EL}_{\mathsf{si}}^{\bot}$ concept descriptions can be computed. Such a computation step is necessary to generate the next candidates after a closure was found. As we have seen in Section 5.1, the lattice $\boldsymbol{\mathcal{EL}}_{\mathsf{si}}^{\bot}(\Sigma)$ is not always neighborhood generated, and so this step is impossible.
2. It had also not been clear whether there is a quasi-rank function on the lattice $\boldsymbol{\mathcal{EL}}_{\mathsf{si}}^{\bot}(\Sigma)$. Note that a quasi-rank function is a function that satisfies Statements 1 to 3 of Definition 5.3.1. The example in the proof of Proposition 5.1.26 shows that such a quasi-rank function cannot exist.
3. A further issue was that computing closures w.r.t. implication sets only saturates an element at its root. Thus, the notion of a most specific consequence w.r.t. a TBox as the DL generalization of closures w.r.t. implication sets was defined and investigated in Section 4.3. With the help of that, we could now reformulate the notion of a pseudo-closure for the case of closure operators in $\boldsymbol{\mathcal{EL}}_{\mathsf{si}}^{\bot}(\Sigma)$ as follows: a *pseudo-closure* of ϕ is a concept description P such that $P \not\equiv_\emptyset P^\phi$ and P is a most specific consequence of $\{ Q \sqsubseteq Q^\phi \mid Q$ is a pseudo-closure such that $Q \sqsupset_\emptyset P$ or $Q \sqsupseteq_\emptyset U$ for some $\exists\, r.\, U \in \mathsf{Sub}(P) \}$. It is straightforward to show that then $\{ P \sqsubseteq P^\phi \mid P$ is a pseudo-closure $\}$ is sound and complete for ϕ, but we do not know whether it is finite (and hence a TBox) and how to actually compute it.

As a consequence, a reduction to the enumeration of implication bases of set closure operators, i.e., to the existing methods from Formal Concept Analysis, has been devised in Section 6.6.

We could only take the above described approach for axiomatizing concept inclusions from role-depth-bounded closure operators as defined in Section 6.5, since we then only have to deal with $\mathcal{EL}$ concept descriptions and $\bot$. The next lemma shows that enumerating the (role-depth-bounded) pseudo-closures along the the rank function is suitable, that is, with respect to increasing rank. The reason is that, if we want to decide whether a current candidate concept description C is a pseudo-closure, then we already know all pseudo-closures Q where $Q \sqsupset_\emptyset C$ or $Q \sqsupseteq_\emptyset U$ for some $\exists\, r.\, U \in \mathsf{Sub}(C)$.

6.7.1 Lemma. *If* $D \in \mathsf{Sub}(C)$, *then* $|D| \leq |C|$.

Proof. The proof is by induction on the depth of D within C. If D is in the top level of C, then $C \sqsubseteq_\emptyset D$ follows, and so Definition 5.3.1 implies $|C| \geq |D|$. Otherwise, there must exist some top-level conjunct $\exists\, r.\, E$ of C such that $D \in \mathsf{Sub}(E)$. Of course, we then have that $|C| \geq |\exists\, r.\, E|$, and the induction hypothesis yields $|D| \leq |E|$. Lemma 5.3.6 shows that $|E| < |\exists\, r.\, E|$, and in summary we obtain $|C| > |D|$. □

Future research could investigate how the attribute set $\mathbf{M}$ on Page 185 can be replaced by a smaller set. For instance, we can partition the set of closures of some closure operator ϕ as follows. A closure C is *reachable* if there is some concept description P such that $P \sqsubseteq_\phi C$ and $P \not\sqsubseteq_\emptyset C$, and otherwise *unreachable*. It is not hard to show that then the set $\mathbf{M} := \bigcup\{\, \mathsf{Conj}(C) \mid C \text{ is a reachable closure} \,\}$ can be used as well (at least until the minimality proof). However, the notion of reachability is still too general. If we consider the closure operator induced by the TBox $\{A \sqsubseteq B\}$, then it would obviously suffice to have $\mathbf{M} = \{A, B\}$. We see that we need to define when a reachable closure is *essential*, and in the example only $A \sqcap B$ should be essential but not $\exists\, r.\,(A \sqcap B)$.

6.8 Applications

6.8.1 Small Role Depths in a Concept Inclusion Base

For axiomatizing concept inclusions valid in a given closure operator ϕ, we consider the following three use cases and propose respective computing approaches that keep the role depths as small as possible.

1. In the first use case the goal is to compute a base of $\mathcal{EL}^\bot$ concept inclusions for $\phi\restriction_{d_{\mathsf{max}}}$. This can be achieved as follows.

   Initially, compute a base $\mathcal{B}_0$ of concept inclusions for $\phi\restriction_0$.

   The subsequent bases are then computed inductively until the predefined role depth bound d_{max} has been reached. More specifically, compute a base $\mathcal{B}_{d+1}$ of concept inclusions for $\phi\restriction_{d+1}$ relative to $\mathcal{B}_0 \cup \ldots \cup \mathcal{B}_d$.

   Eventually, return $\mathcal{B}_0 \cup \ldots \cup \mathcal{B}_{d_{\mathsf{max}}}$.
2. In the second use case the goal is to compute a base of $\mathcal{EL}^\bot$ concept inclusions for ϕ if it is finitely acyclically representable. Clearly, this is a special case of the first use case when defining $d_{\mathsf{max}} := d_\phi + 1$ where the number d_ϕ is the greatest role depth of a premise in the existing finite acyclic representation of ϕ, i.e., we define the following.

$$d_\phi := \min\{\, \max\{\, \mathsf{rd}(C) \mid C \sqsubseteq D \in \mathcal{T} \,\} \mid \mathcal{T} \text{ is a finite acyclic representation for } \phi \,\}$$

3. In the third use case the goal is to compute a base of concept inclusions for ϕ which contains $\mathcal{EL}^\bot$ concept inclusions only up to a role depth d_{max} and may further contain $\mathcal{EL}^\bot_{\mathsf{si}}$ concept inclusions. This can be achieved as follows.

   Start the computation as for the first use case. Additionally, compute a base $\mathcal{B}_\infty$ of $\mathcal{EL}^\bot_{\mathsf{si}}$ concept inclusions for ϕ relative to $\mathcal{B}_0 \cup \ldots \cup \mathcal{B}_{d_{\mathsf{max}}}$. Then, return $\mathcal{B}_0 \cup \ldots \cup \mathcal{B}_{d_{\mathsf{max}}} \cup \mathcal{B}_\infty$.

Now let $\mathcal{B} := \mathcal{B}_0 \cup \ldots \cup \mathcal{B}_{d_{\mathsf{max}}}$ be a result for the first use case. By construction, $\mathcal{B}$ satisfies the following two properties.

1. For each role depth bound $d \leq d_{\mathsf{max}}$, the subset $\{\, \alpha \mid \alpha \in \mathcal{B} \text{ and } \mathsf{rd}(\alpha) \leq d \,\}$ is a base for $\phi\restriction_d$.

2. For each role depth bound $d \leq d_{\mathsf{max}}$, it holds true that $\mathcal{B}$ contains a minimal number of concept inclusions α with $\mathsf{rd}(\alpha) = d$ among all TBoxes satisfying the above condition.

6.8.2 Axiomatization of Concept Inclusions from Sequences of Interpretations

Consider a setting where a sequence $(\, \mathcal{I}_n \mid n \in \mathbb{N}\,)$ of interpretations can be observed and, for each time point $n \in \mathbb{N}$, a terminological box $\mathcal{T}_n$ shall be constructed that entails exactly those concept inclusions which are simultaneously valid in all previously observed interpretations $\mathcal{I}_0, \mathcal{I}_1, \ldots, \mathcal{I}_n$, that is, such that for each concept inclusion $C \sqsubseteq D$, it holds true that $\mathcal{T}_n \models C \sqsubseteq D$ if, and only if, $\mathcal{I}_k \models C \sqsubseteq D$ for all previous time points $k \leq n$. For the initial moment $n = 0$, we can simply compute $\mathcal{T}_0$ as a concept inclusion base for $\mathcal{I}_0$ utilizing the approaches from Distel [Dis11], or from Borchmann, Distel, and Kriegel [BDK16]. Of course, for the following moments $n \geq 1$, we could construct a concept inclusion base for the disjoint union of the interpretations $\mathcal{I}_0, \mathcal{I}_1, \ldots, \mathcal{I}_n$. However, since the aforementioned methods require the construction of so-called *induced contexts* the size of which may be exponential in the cardinality of the interpretation's domain, this technique could possibly be infeasible for late time points. Furthermore, it would require the storing of all interpretations observed so far. We shall present another technique for solving the above mentioned task. Please note that this problem has already been addressed by Kriegel [Kri15c] for the case where $\mathcal{I}_{n+1} \models \mathcal{T}_n$ for all time points $n \in \mathbb{N}$. Herein, we propose a solution that circumvents this rather restrictive precondition.

The following proposition states that the concept inclusions that are both valid in an interpretation $\mathcal{I}$ and entailed by some TBox $\mathcal{T}$ are exactly those which are valid for the infimum $\phi_{\mathcal{I}} \vartriangle \phi_{\mathcal{T}}$ of the induced closure operators.

6.8.1 Proposition. *Let $\mathcal{I}$ be an interpretation, $\mathcal{T}$ an $\mathcal{EL}^{\bot}_{\mathsf{si}}$ TBox, and $C \sqsubseteq D$ an $\mathcal{EL}^{\bot}_{\mathsf{si}}$ concept inclusion. Then, the following statements are equivalent:*

1. *$C \sqsubseteq_{\mathcal{I}} D$ and $C \sqsubseteq_{\mathcal{T}} D$*
2. *$C^{\mathcal{I}\mathcal{I}} \sqsubseteq_{\emptyset} D$ and $C^{\mathcal{T}} \sqsubseteq_{\emptyset} D$*
3. *$C^{\mathcal{I}\mathcal{I}} \vee C^{\mathcal{T}} \sqsubseteq_{\emptyset} D$*
4. *$\phi_{\mathcal{I}} \models C \sqsubseteq D$ and $\phi_{\mathcal{T}} \models C \sqsubseteq D$*
5. *$\phi_{\mathcal{I}} \vartriangle \phi_{\mathcal{T}} \models C \sqsubseteq D$*

Proof. The proof is similar to the proof of Proposition 6.3.3. □

Consequently, we can outline the following incremental procedure for computing concept inclusion bases from some sequence of interpretations $\mathcal{I}_0, \mathcal{I}_1, \ldots, \mathcal{I}_n$.[7]

1. Upon availability of the first observed interpretation $\mathcal{I}_0$, compute its canonical base $\mathcal{B}_0 := \mathsf{Can}(\phi_{\mathcal{I}_0}, \emptyset)$ using the results from Section 6.6 or, equivalently, using the results from [Dis11, Chapter 5].
2. For each subsequent interpretation $\mathcal{I}_{k+1}$, compute the canonical base $\mathcal{B}_{k+1} := \mathsf{Can}(\phi_{\mathcal{I}_{k+1}} \vartriangle \phi_{\mathcal{B}_k}, \emptyset)$ using the results from Section 6.6.

[7]The sequence need not be finite; the procedure could continue indefinitely as well.

It is readily verified that—by construction—for each time point $k \in \mathbb{N}$, the TBox $\mathcal{B}_k$ entails an $\mathcal{EL}^{\bot}_{\mathsf{si}}$ concept inclusion $C \sqsubseteq D$ if, and only if, $C \sqsubseteq D$ is valid in all interpretations $\mathcal{I}_0, \ldots, \mathcal{I}_k$. Note that each result $\mathcal{B}_k$ is equivalent to the canonical base of the disjoint union $\mathcal{I}_0 \uplus \ldots \uplus \mathcal{I}_k$ computed by means of the results in [Dis11, Chapter 5], and both have the same cardinality, cf. [Dis11, Theorem 5.18] and Theorem 6.6.10.

With the methods that are developed in Section 6.6, we can also solve the problem in a slightly different way such that we do not throw away each previous TBox when computing the next one. More specifically, we devise an incremental procedure which shows in each step the changes in the concept inclusion base induced by integrating the observations from the new interpretation.

1. Initially, compute the canonical base $\mathcal{B}_0 := \mathsf{Can}(\phi_{\mathcal{I}_0}, \emptyset)$ for the first interpretation $\mathcal{I}_0$.
2. When the next interpretation $\mathcal{I}_{k+1}$ is available, we can either remove from $\mathcal{B}_k$ all concept inclusions that are not valid in $\mathcal{I}_{k+1}$ or replace each concept inclusion $C \sqsubseteq D$ in $\mathcal{B}_k$ that is not valid in $\mathcal{I}_{k+1}$ by the modified concept inclusion $C \sqsubseteq D \vee C^{\mathcal{I}_{k+1}\mathcal{I}_{k+1}}$. The resulting TBox is called the *adjustment* of $\mathcal{B}_k$ w.r.t. $\mathcal{I}_{k+1}$ and is denoted by $\mathcal{B}^*_k$.[8] It then holds true that each interpretation $\mathcal{I}_0, \ldots, \mathcal{I}_k, \mathcal{I}_{k+1}$ is a model of the adjustment $\mathcal{B}^*_k$. This is obvious when simply deleting CIs; otherwise we have the following. Fix some $\ell \in \{0, \ldots, k\}$. Then $C \sqsubseteq D$ is valid in $\mathcal{I}_\ell$, i.e., $C^{\mathcal{I}_\ell\mathcal{I}_\ell} \sqsubseteq_\emptyset D$ holds true. We immediately conclude that $C^{\mathcal{I}_\ell\mathcal{I}_\ell} \sqsubseteq_\emptyset D \vee C^{\mathcal{I}_{k+1}\mathcal{I}_{k+1}}$ is satisfied as well, which means that $C \sqsubseteq D \vee C^{\mathcal{I}_{k+1}\mathcal{I}_{k+1}}$ is valid in $\mathcal{I}_\ell$. Furthermore, it is obvious that $C^{\mathcal{I}_{k+1}\mathcal{I}_{k+1}}$ is more specific than $D \vee C^{\mathcal{I}_{k+1}\mathcal{I}_{k+1}}$ modulo $\emptyset$, and it follows that $C \sqsubseteq D \vee C^{\mathcal{I}_{k+1}\mathcal{I}_{k+1}}$ is valid in $\mathcal{I}_{k+1}$ as well.

   Furthermore, we then compute the next TBox $\mathcal{B}_{k+1} := \mathsf{Can}(\phi_{\mathcal{I}_{k+1}} \triangle \phi_{\mathcal{B}_k}, \mathcal{B}^*_k) \cup \mathcal{B}^*_k$. Since $\mathcal{B}^*_k$ is sound for $\mathcal{I}_0, \ldots, \mathcal{I}_{k+1}$ and $\mathcal{B}_k$ is both sound and complete for $\mathcal{I}_0, \ldots, \mathcal{I}_k$, we conclude that $\mathcal{B}_{k+1}$ must be sound and complete for all interpretations $\mathcal{I}_0, \ldots, \mathcal{I}_{k+1}$ that we have observed so far.

We have seen that the methods in Section 6.6 can only be applied to finite[9] closure operators. Both above proposed procedures require the computation of a canonical base for an infimum $\phi_{\mathcal{I}} \triangle \phi_{\mathcal{T}}$ where $\mathcal{I}$ is some interpretation and $\mathcal{T}$ is some TBox. Unfortunately, it holds true that, as one quickly verifies, the infimum $\phi_{\mathcal{I}} \triangle \phi_{\mathcal{T}}$ is infinite in general. In contrast, closure operators of the form $\phi_{\mathcal{I}}$ for some finite interpretation $\mathcal{I}$ have only finitely many closures, namely all those of the form $X^{\mathcal{I}}$ for some $X \subseteq \Delta^{\mathcal{I}}$. However, for the case of axiomatizing a sequence of finite interpretations $\mathcal{I}_0, \mathcal{I}_1, \ldots, \mathcal{I}_n$ we find the following.

6.8.2 Proposition. *The initial closure operator $\phi_{\mathcal{I}_0}$ as well as all subsequent closure operators $\phi_{\mathcal{I}_{k+1}} \triangle \phi_{\mathcal{B}_k}$ for $k \in \{0, \ldots, n-1\}$ are finite.*

Proof. Since each interpretation $\mathcal{I}_k$ is finite, its domain $\Delta^{\mathcal{I}_k}$ must be finite. Thus, there are only finitely many MMSCs for $\mathcal{I}_k$, which means that the closure operator $\phi_{\mathcal{I}_k}$ is finite. We show the other statement by induction on k.

induction base. Since $\mathcal{B}_0$ is a concept inclusion base for $\mathcal{I}_0$, Corollary 4.3.52 shows that the closures of the induced closure operators $\phi_{\mathcal{I}_0}$ and $\phi_{\mathcal{B}_0}$ coincide, which means that

[8] Alternatively, an adjustment can be obtained by replacing each concept inclusion $C \sqsubseteq D$ by $C \sqsubseteq C^{\mathcal{B}_k} \vee C^{\mathcal{I}_{k+1}\mathcal{I}_{k+1}}$.

[9] Recall that we call a closure operator ϕ *finite* if $\mathsf{Clo}(\phi)$ is finite, and *infinite* otherwise.

$\phi_{\mathcal{B}_0}$ is finite. We conclude that the closure operator $\phi_{\mathcal{I}_1} \triangle \phi_{\mathcal{B}_0}$ must be finite, since $C^{\phi_{\mathcal{I}_1} \triangle \phi_{\mathcal{B}_0}} = C^{\phi_{\mathcal{I}_1}} \vee C^{\phi_{\mathcal{B}_0}}$ holds true and both closure operators $\phi_{\mathcal{I}_1}$ and $\phi_{\mathcal{B}_0}$ are finite.

induction step. Since $\mathcal{B}_k$ is a concept inclusion base for $\phi_{\mathcal{I}_k} \triangle \phi_{\mathcal{B}_{k-1}}$, we have $\phi_{\mathcal{B}_k} = \phi_{\mathcal{I}_k} \triangle \phi_{\mathcal{B}_{k-1}}$. Furthermore, since $\phi_{\mathcal{I}_k} \triangle \phi_{\mathcal{B}_{k-1}}$ is finite by induction hypothesis, we conclude that this also holds true for $\phi_{\mathcal{B}_k}$. Now we have that each closure of $\phi_{\mathcal{I}_{k+1}} \triangle \phi_{\mathcal{B}_k}$ is of the form $C^{\phi_{\mathcal{I}_{k+1}}} \vee C^{\phi_{\mathcal{B}_k}}$, and since $\phi_{\mathcal{I}_{k+1}}$ is finite it follows that $\phi_{\mathcal{I}_{k+1}} \triangle \phi_{\mathcal{B}_k}$ must be finite as well. □

6.8.3 Merging Terminological Boxes

We have already seen in the beginning of Section 6.3 that there is a strong correspondence between the logical intersection of two TBoxes $\mathcal{T}$ and $\mathcal{U}$ and the infimum $\phi_{\mathcal{T}} \triangle \phi_{\mathcal{U}}$ of the induced closure operators. Thus, a use case that utilizes the notion of an infimum of closure operators is the *merging* of TBoxes, i.e., the computation of a logical intersection. In theory, this could be tackled by computing the canonical base for such an infimum. Unfortunately, this would not practically work, since Propositions 6.3.5 – 6.3.7 show that the logical intersection $\mathcal{T} \triangle \mathcal{U}$ does not need to exist in $\mathcal{EL}^{\bot}$ or in $\mathcal{EL}^{\bot}_{\mathsf{si}}$, and it also follows that such an infimum $\phi_{\mathcal{T}} \triangle \phi_{\mathcal{U}}$ need not be finite.

What we can do, however, is to set some upper bound d on the role depths and then only axiomatize the logical intersection of $\mathcal{T}$ and $\mathcal{U}$ up to depth d. This can be achieved by considering the restriction $(\phi_{\mathcal{T}} \triangle \phi_{\mathcal{U}})\upharpoonright_d$, which has only finitely many closures.

6.8.4 Axiomatization of Concept Inclusions from ABoxes

Assume that $\mathcal{A}$ is a *simple* ABox which may not only contain positive assertions, but also negative assertions, i.e., $\mathcal{A}$ may consist of axioms of the forms

$$a \sqsubseteq A,\ a \not\sqsubseteq A,\ (a,b) \sqsubseteq r,\ (a,b) \not\sqsubseteq r,$$

where $a, b \in \Sigma_{\mathsf{I}}$ are individual names, $A \in \Sigma_{\mathsf{C}}$ is a concept name, and $r \in \Sigma_{\mathsf{R}}$ is a role name. We further require $\mathcal{A}$ to be consistent, that is, it has a model. Apparently, $\mathcal{A}$ is consistent if, and only if, it does not contain $a \sqsubseteq A$ and $a \not\sqsubseteq A$ at the same time, and similarly it does not simultaneously contain $(a,b) \sqsubseteq r$ and $(a,b) \not\sqsubseteq r$.

ABoxes entail only tautological concept inclusions with respect to the default semantics, i.e., when we only adopt the *Unique Name Assumption* (abbrv. *UNA*), i.e., different individual names address different individuals, and the *Open World Assumption* (abbrv. *OWA*), i.e., an axiom may be true in the domain of interest irrespective of it being entailed by the ABox, or alternatively, there may be axioms the validity of which cannot be decided with only the information contained in the ABox. In particular, if an ABox entails $C \sqsubseteq D$, then $C \sqsubseteq_{\emptyset} D$ must hold true. We can show this fact as follows. Assume that $C \sqsubseteq D$ is no tautology, i.e., there exists some interpretation $\mathcal{I}$ containing a counterexample against $C \sqsubseteq D$. Now the interpretation $\mathcal{I}_{\mathcal{A}}$ from Page 92 is a model of $\mathcal{A}$, and we define $\mathcal{J}$ as the disjoint union of $\mathcal{I}$ and $\mathcal{I}_{\mathcal{A}}$ such that individual names are interpreted by $\mathcal{I}_{\mathcal{A}}$. Clearly, $\mathcal{J}$ is then a model of $\mathcal{A}$ which contains a counterexample against $C \sqsubseteq D$, i.e., $C \not\sqsubseteq_{\mathcal{A}} D$. Put simply, we can extend each model of an ABox by a counterexample against some non-tautological concept inclusion.

Consequently, when we aim at learning terminological boxes from assertional boxes as above, we have to impose further restrictions on the allowed models. An idea which would probably perform well in practice would be to further require the *Domain Closure Assumption* (abbrv. *DCA*), i.e., all individuals/objects of the domain of interest are known. The DCA is also utilized in *Database Theory*, where it is assumed that every individual/object which occurs in the domain of interest also occurs in the data set. Applying the DCA to the *Description Logic* setting, we would enforce that there are no individuals except explicitly described in the signature or used in the ABox, or when applying it to interpretations $\mathcal{I}$, the restriction of the extension function to Σ_I is surjective. Analogously, requiring the UNA to hold true for interpretations $\mathcal{I}$ implies that the restriction of the extension function to Σ_I is injective. Thus, for interpretations $\mathcal{I}$ satisfying both the UNA and DCA, the mapping $\cdot^\mathcal{I}\restriction_{\Sigma_\mathsf{I}}$ is bijective, and w.l.o.g. we shall hence simply assume that $\Delta^\mathcal{I} = \Sigma_\mathsf{I}$.[10]

In particular, we then restrict the semantics as follows. A Σ_I-*interpretation* is an interpretation $\mathcal{I}$ where $\Delta^\mathcal{I} := \Sigma_\mathsf{I}$ and where $a^\mathcal{I} := a$ for all individual names $a \in \Sigma_\mathsf{I}$.[11] Furthermore, we call a Σ_I-interpretation $\mathcal{I}$ a Σ_I-*model* of $\mathcal{A}$ if $\mathcal{I}$ is a model of $\mathcal{A}$, and we shall then write $\mathcal{I} \models^{\Sigma_\mathsf{I}} \mathcal{A}$. The ABox $\mathcal{A}$ Σ_I-*entails* a concept inclusion $C \sqsubseteq D$ if, for each Σ_I-interpretation $\mathcal{I}$, it holds true that $\mathcal{I} \models^{\Sigma_\mathsf{I}} \mathcal{A}$ implies $\mathcal{I} \models C \sqsubseteq D$, and we denote this as $\mathcal{A} \models^{\Sigma_\mathsf{I}} C \sqsubseteq D$. If Σ_I is finite, then reasoning with respect to Σ_I-semantics can be reduced to reasoning with respect to default semantics when we further allow for the use of nominals. A *nominal* is a concept description of the form $\{a\}$ where $a \in \Sigma_\mathsf{I}$ is an individual name, and its extension is defined by

$$\{a\}^\mathcal{I} := \{a^\mathcal{I}\}$$

for each interpretation $\mathcal{I}$. It is easy to verify that, for any concept inclusion $C \sqsubseteq D$, it holds true that $\mathcal{A} \models^{\Sigma_\mathsf{I}} C \sqsubseteq D$ if, and only if, $\mathcal{A} \cup \mathcal{T}_{\Sigma_\mathsf{I}} \models C \sqsubseteq D$ where the TBox $\mathcal{T}_{\Sigma_\mathsf{I}}$ encodes the Σ_I-semantics, i.e., that objects do not have multiple names (UNA) and that all objects are known, i.e., named (DCA). In particular, $\mathcal{T}_{\Sigma_\mathsf{I}}$ is defined as follows.[12]

$$\mathcal{T}_{\Sigma_\mathsf{I}} := \{\, \{a\} \sqcap \{b\} \sqsubseteq \bot \mid a, b \in \Sigma_\mathsf{I} \text{ and } a \neq b \,\} \cup \{\, \top \sqsubseteq \bigsqcup \{\, \{a\} \mid a \in \Sigma_\mathsf{I} \,\}\,\}$$

Note that the above Σ_I-semantics are a special case of the semantics for fixed-domain reasoning [GRS16].

Our goal now is to find a technique for the axiomatization of assertional boxes with respect to Σ_I-semantics, that is, to compute a concept inclusion base for a given ABox $\mathcal{A}$ that is sound and complete for all $C \sqsubseteq D$ satisfying $\mathcal{A} \models^{\Sigma_\mathsf{I}} C \sqsubseteq D$. Before we investigate the technical details, we first demonstrate that using an ABox (with UNA, DCA, OWA) as input yields indeed different results than using an interpretation (with UNA, DCA, CWA[13]). Both the ABox and each of its

[10] Since we want to develop a technique for axiomatizing concept inclusions from ABoxes that can be transformed into an implementation, we require that the input is finite, which in particular means that the set Σ_I of individual names must be finite.

[11] Note that this somehow corresponds to the HERBRAND universe of a FO-theory.

[12] The binary disjunction operator $\sqcup$ and its finitary generalization $\bigsqcup$ are defined dually to the conjunction operators $\sqcap$ and $\bigsqcap$ from Pages 34 and 36. In particular, $(C \sqcup D)^\mathcal{I} := C^\mathcal{I} \cup D^\mathcal{I}$.

[13] The *Closed World Assumption* (abbrv. CWA) is the converse of the Open World Assumption. In particular, for the case of an interpretation $\mathcal{I}$ it means that an axiom holds true in the domain of interest described by $\mathcal{I}$ if, and

Σ_{I}-model have in common that the set of individuals/objects is fully known. However, an ABox allows for the presence of unknown facts, i.e., by leaving out both assertions $a \mathrel{\varepsilon} A$ and $a \not\mathrel{\varepsilon} A$ we leave it open whether a is an instance of A, simply because we do not know it. This degree of freedom is not possible in an interpretation $\mathcal{I}$: either an object $\delta \in \Delta^{\mathcal{I}}$ belongs to an extension $A^{\mathcal{I}}$ or not; there is no means to express that it is not known. Consequently, utilizing ABoxes as input data to learn from allows for more practical use cases.

For instance, define $\mathcal{A} := \{a \mathrel{\varepsilon} A,\ a \mathrel{\varepsilon} B\}$ over the signature Σ with $\Sigma_{\mathsf{C}} := \{A, B\}$, $\Sigma_{\mathsf{R}} := \emptyset$, and $\Sigma_{\mathsf{I}} := \{a, b\}$. Then, the concept inclusion $A \sqsubseteq B$ is no consequence of $\mathcal{A}$, but it would be if we consider $\mathcal{A}$ as an interpretation—more specifically, $A \sqsubseteq B$ is valid in the canonical model $\mathcal{I}_{\mathcal{A}}$ that has been defined on Page 92. Note that it is defined as follows.

$$\Delta^{\mathcal{I}_{\mathcal{A}}} := \Sigma_{\mathsf{I}}$$
$$\cdot^{\mathcal{I}_{\mathcal{A}}} \colon \begin{cases} A \mapsto \{\, a \mid a \mathrel{\varepsilon} A \in \mathcal{A} \,\} & \text{for each } A \in \Sigma_{\mathsf{C}} \\ r \mapsto \{\, (a,b) \mid (a,b) \mathrel{\varepsilon} r \in \mathcal{A} \,\} & \text{for each } r \in \Sigma_{\mathsf{R}} \end{cases}$$

The *dual canonical model* $\mathcal{I}_{\mathcal{A}}^{\partial}$ is given as follows.

$$\Delta^{\mathcal{I}_{\mathcal{A}}^{\partial}} := \Sigma_{\mathsf{I}}$$
$$\cdot^{\mathcal{I}_{\mathcal{A}}^{\partial}} \colon \begin{cases} A \mapsto \{\, a \mid a \not\mathrel{\varepsilon} A \notin \mathcal{A} \,\} & \text{for each } A \in \Sigma_{\mathsf{C}} \\ r \mapsto \{\, (a,b) \mid (a,b) \not\mathrel{\varepsilon} r \notin \mathcal{A} \,\} & \text{for each } r \in \Sigma_{\mathsf{R}} \end{cases}$$

Clearly, both $\mathcal{I}_{\mathcal{A}}$ and $\mathcal{I}_{\mathcal{A}}^{\partial}$ are Σ_{I}-models of $\mathcal{A}$. Furthermore, it holds true that any Σ_{I}-model of $\mathcal{A}$ is *between* these two canonical models and, more specifically, for each Σ_{I}-model $\mathcal{I}$ of $\mathcal{A}$, it holds true that

$$A^{\mathcal{I}_{\mathcal{A}}} \subseteq A^{\mathcal{I}} \subseteq A^{\mathcal{I}_{\mathcal{A}}^{\partial}} \text{ for any } A \in \Sigma_{\mathsf{C}}$$
$$\text{and } r^{\mathcal{I}_{\mathcal{A}}} \subseteq r^{\mathcal{I}} \subseteq r^{\mathcal{I}_{\mathcal{A}}^{\partial}} \text{ for each } r \in \Sigma_{\mathsf{R}}.$$

As an immediate consequence we obtain that, for each finite signature Σ, there are only finitely many Σ_{I}-models of a simple ABox.

The following proposition states some equivalent characterizations of Σ_{I}-entailment.

6.8.3 Proposition. *Let $\mathcal{A}$ be an ABox, and assume that $C \sqsubseteq D$ is a concept inclusion. Then, the following statements are equivalent:*

1. $\mathcal{A} \models^{\Sigma_{\mathsf{I}}} C \sqsubseteq D$
2. $\mathcal{I} \models^{\Sigma_{\mathsf{I}}} \mathcal{A}$ *implies* $\mathcal{I} \models C \sqsubseteq D$ *for each interpretation* $\mathcal{I}$.
3. $\mathcal{I} \models^{\Sigma_{\mathsf{I}}} \mathcal{A}$ *implies* $\phi_{\mathcal{I}} \models C \sqsubseteq D$ *for any interpretation* $\mathcal{I}$.
4. $\bigtriangleup\{\, \phi_{\mathcal{I}} \mid \mathcal{I} \models^{\Sigma_{\mathsf{I}}} \mathcal{A} \,\} \models C \sqsubseteq D$
5. $\emptyset \models \bigvee\{\, C^{\mathcal{I}\mathcal{I}} \mid \mathcal{I} \models^{\Sigma_{\mathsf{I}}} \mathcal{A} \,\} \sqsubseteq D$.

only if, it is valid in $\mathcal{I}$.

Proof. Statements 1 and 2 are equivalent by the very definition of Σ_I-semantics. An application of Theorem 4.1.9 or of [Dis11, Lemma 4.1] shows that Statements 2 and 3 are equivalent too. The equivalence of Statements 3 to 5 follows immediately from Section 1.5. □

We define the mapping $\phi_\mathcal{A} \colon \mathcal{EL}^\bot_\mathsf{si}(\Sigma) \to \mathcal{EL}^\bot_\mathsf{si}(\Sigma)$ induced by some simple ABox $\mathcal{A}$ as above by

$$\phi_\mathcal{A} \colon C \mapsto C^\mathcal{A} := \bigvee \{\, C^{\mathcal{I}\mathcal{I}} \mid \mathcal{I} \models^{\Sigma_\mathsf{I}} \mathcal{A} \,\}.$$

It then holds true that $\phi_\mathcal{A} = \bigtriangleup \{\, \phi_\mathcal{I} \mid \mathcal{I} \models^{\Sigma_\mathsf{I}} \mathcal{A} \,\}$, and so $\phi_\mathcal{A}$ is a closure operator in the dual of $\boldsymbol{\mathcal{EL}}^\bot_\mathsf{si}(\Sigma)$. Furthermore, the interpretation $\mathcal{I}^{\Sigma_\mathsf{I}}_\mathcal{A}$, called *canonical* Σ_I*-model* of $\mathcal{A}$, is defined as the disjoint union of all Σ_I-models of $\mathcal{A}$, that is, we set

$$\mathcal{I}^{\Sigma_\mathsf{I}}_\mathcal{A} := \biguplus \{\, \mathcal{I} \mid \mathcal{I} \models^{\Sigma_\mathsf{I}} \mathcal{A} \,\}.$$

It follows that, for any concept inclusion $C \sqsubseteq D$,

$$\mathcal{A} \models^{\Sigma_\mathsf{I}} C \sqsubseteq D \text{ if, and only if, } \mathcal{I}^{\Sigma_\mathsf{I}}_\mathcal{A} \models C \sqsubseteq D,$$

and so the closure operators $\phi_\mathcal{A}$ and $\phi_{\mathcal{I}^{\Sigma_\mathsf{I}}_\mathcal{A}}$ coincide. The canonical Σ_I-model $\mathcal{I}^{\Sigma_\mathsf{I}}_\mathcal{A}$ is finite if the signature Σ is finite.

Returning back to our initial goal of axiomatizing concept inclusions from some such finite simple ABox $\mathcal{A}$, we can now provide a solution for doing so, namely we suggest to compute some concept inclusion base of this newly introduced closure operator $\phi_\mathcal{A}$ or, equivalently, to compute a concept inclusion base of the canonical Σ_I-model $\mathcal{I}^{\Sigma_\mathsf{I}}_\mathcal{A}$. For the latter, we could also apply the existing procedures from Distel [Dis11] and Borchmann, Distel, and Kriegel [BDK16].

6.8.5 Error-Tolerant Axiomatization of Concept Inclusions from Interpretations

Assume that an interpretation $\mathcal{I}$ as well as a TBox $\mathcal{T}$ are given such that $\mathcal{I}$ contains observations that could possibly be faulty due to inaccurate generation methods, and that $\mathcal{T}$ is certainly valid in the domain of interest, e.g., as it has been hand-crafted by experts. In particular, we assume that $\mathcal{I}$ is not a model of $\mathcal{T}$, i.e., that at least one domain element in $\mathcal{I}$ exists which serves as a counterexample against at least one concept inclusion in $\mathcal{T}$. However, we are expected to axiomatize terminological knowledge from $\mathcal{I}$ which is valid in the domain of interest. As a solution, we suggest to construct the concept inclusion base of the supremum of the closure operators that are induced by $\mathcal{I}$ and by $\mathcal{T}$, respectively. It is then ensured that only concept inclusions are axiomatized which are valid for all those domain elements of $\mathcal{I}$ that respect the concept inclusions in $\mathcal{T}$, i.e., we axiomatize concept inclusions from $\mathcal{I}$ that are compatible with the axioms contained in $\mathcal{T}$. In a certain sense this yields a method for an error correction in $\mathcal{I}$ when learning concept inclusions. We will define a short motivating example as follows. Define the signature Σ by $\Sigma_\mathsf{C} := \{\mathsf{Person}, \mathsf{Car}, \mathsf{Wheel}\}$ and $\Sigma_\mathsf{R} := \{\mathsf{child}\}$, and consider the following interpretation $\mathcal{I}$.

$\mathcal{I}$:


Consider the concept inclusion $\mathsf{Car} \sqsubseteq \exists\,\mathsf{child}.\mathsf{Wheel}$. Of course, it is valid in $\mathcal{I}$ and, thus, it is entailed by the canonical base for $\mathcal{I}$ when applying the construction of Distel [Dis11] or of Borchmann, Distel, and Kriegel [BDK16]. In the real world, however, this concept inclusion does not make sense. We already know that only persons have children, and that nothing is both a person and a car. This existing, verified knowledge can be formulated as the following $\mathcal{EL}^{\bot}$ TBox $\mathcal{T}$.

$$\mathcal{T} := \{\exists\,\mathsf{child}.\top \sqsubseteq \mathsf{Person},\ \mathsf{Person} \sqcap \mathsf{Car} \sqsubseteq \bot\}$$

It is easy to see that the object δ is not compatible with $\mathcal{T}$—in contrast to the other objects ϵ, ζ, and η. If we now consider the supremum $\phi_{\mathcal{I}} \triangledown \phi_{\mathcal{T}}$ instead, then we effectively filter out all information on the incompatible object δ.

Note that the closure of Car with respect to $\phi_{\mathcal{I}} \triangledown \phi_{\mathcal{T}}$ is the least common subsumer of all those concept descriptions that are closures of both $\phi_{\mathcal{I}}$ and $\phi_{\mathcal{T}}$ and that are subsumed by Car. According to Proposition 6.4.1, this closure can also be computed by an exhaustive alternating application of both closure operators until a fixed point is reached. As we shall see below, the fixed point $\bot$ is reached after the first iteration, and hence $\bot$ is the closure.

$$\mathsf{Car}^{\mathcal{I}\mathcal{I}} \equiv \mathsf{Car} \sqcap \exists\,\mathsf{child}.\mathsf{Wheel}$$
$$(\mathsf{Car} \sqcap \exists\,\mathsf{child}.\mathsf{Wheel})^{\mathcal{T}} \equiv \mathsf{Car} \sqcap \exists\,\mathsf{child}.\mathsf{Wheel} \sqcap \mathsf{Person} \sqcap \bot \equiv \bot$$

It follows that the concept inclusion $\mathsf{Car} \sqsubseteq \bot$ is valid for $\phi_{\mathcal{I}} \triangledown \phi_{\mathcal{T}}$, and hence the canonical base contains the axiom expressing the non-existence of cars. This is obviously a correct result, since the only object being a Car has been filtered out from $\mathcal{I}$ by $\mathcal{T}$.

In the following, we show explicitly that $\phi_{\mathcal{I}} \triangledown \phi_{\mathcal{T}}$ defines a filtering of $\mathcal{I}$ by specifying an interpretation for which the induced closure operator coincides with $\phi_{\mathcal{I}} \triangledown \phi_{\mathcal{T}}$. Beforehand, we show how closures of suprema can be computed if one of the closure operators is induced by an interpretation.

6.8.4 Lemma. *For each $\mathcal{EL}^{\bot}_{\mathsf{si}}$ concept description C, the following statement holds true.*

$$C^{\phi_{\mathcal{I}} \triangledown \psi} \equiv_{\emptyset} (\bigcup\{\, X \mid X \subseteq C^{\mathcal{I}} \text{ and } X \subseteq X^{\mathcal{I}\psi\mathcal{I}} \,\})^{\mathcal{I}}$$

Proof. We have the following which shall be justified below.

$$\begin{aligned}
C^{\phi_{\mathcal{I}} \triangledown \psi} &\equiv_{\emptyset} \bigvee\{\, D \mid D \in \mathcal{EL}^{\bot}_{\mathsf{si}}(\Sigma) \text{ and } D \equiv_{\emptyset} D^{\mathcal{I}\mathcal{I}} \equiv_{\emptyset} D^{\psi} \sqsubseteq_{\emptyset} C \,\} && (6.8.A)\\
&\equiv_{\emptyset} \bigvee\{\, X^{\mathcal{I}} \mid X \subseteq \Delta^{\mathcal{I}} \text{ and } X^{\mathcal{I}} \equiv_{\emptyset} X^{\mathcal{I}\psi} \sqsubseteq_{\emptyset} C \,\} && (6.8.B)\\
&\equiv_{\emptyset} \bigvee\{\, X^{\mathcal{I}} \mid X \subseteq C^{\mathcal{I}} \text{ and } X \subseteq X^{\mathcal{I}\psi\mathcal{I}} \,\} && (6.8.C)\\
&\equiv_{\emptyset} (\bigcup\{\, X \mid X \subseteq C^{\mathcal{I}} \text{ and } X \subseteq X^{\mathcal{I}\psi\mathcal{I}} \,\})^{\mathcal{I}} && (6.8.D)
\end{aligned}$$

We begin with observing that the equivalence in Equation (6.8.A) is satisfied; it follows from the characterization of suprema of closure operators in Section 1.5. As all concept descriptions D over which the least common subsumer is constructed are model-based most specific concept descriptions for $\mathcal{I}$, we infer that Equation (6.8.B) holds true. Lemma 4.1.2 shows that $X^{\mathcal{I}} \sqsubseteq_{\emptyset} C$

is equivalent to $X \subseteq C^{\mathcal{I}}$. Of course, $X^{\mathcal{I}} \equiv_\emptyset X^{\mathcal{I}\psi}$ is satisfied if, and only if, $X^{\mathcal{I}} \sqsubseteq_\emptyset X^{\mathcal{I}\psi}$ as well as $X^{\mathcal{I}} \sqsupseteq_\emptyset X^{\mathcal{I}\psi}$, where the former is equivalent to $X \subseteq X^{\mathcal{I}\psi\mathcal{I}}$ and the latter is trivially true (since ψ is extensive). We conclude the validity of Equation (6.8.C). Equation (6.8.D) immediately follows from a claim in the proof of Proposition 4.1.4. □

For an interpretation $\mathcal{I}$ and some closure operator ψ, we define the following interpretation $\mathcal{I}_\psi$, called *filtering* of $\mathcal{I}$ w.r.t. ψ.

$$\Delta^{\mathcal{I}_\psi} := \{\, X \mid X \subseteq \Delta^{\mathcal{I}} \text{ and } X^{\mathcal{I}} \equiv_\emptyset (X^{\mathcal{I}})^\psi \,\}$$

$$\cdot^{\mathcal{I}_\psi} : \begin{cases} A \mapsto \{\, X \mid X \subseteq A^{\mathcal{I}} \,\} & \text{for any } A \in \Sigma_{\mathsf{C}} \\ r \mapsto \{\, (X,Y) \mid Y \in \mathsf{Min}(\mathsf{Succ}_\psi(X,r)) \,\} & \text{for any } r \in \Sigma_{\mathsf{R}} \end{cases}$$

The successor sets are defined by

$$\mathsf{Succ}_\psi(X,r) := \mathsf{Succ}(X,r) \cap \Delta^{\mathcal{I}_\psi} = \{\, Y \mid Y^{\mathcal{I}} \in \mathsf{Clo}(\psi),\ Y \subseteq \mathsf{succ}(X,r),\ \text{and } X \subseteq \mathsf{pred}(Y,r) \,\}$$

using the notions from the definition of the powering of $\mathcal{I}$ on Page 72. The next proposition now shows that the supremum $\phi_{\mathcal{I}} \triangledown \psi$ is equal to the induced closure operator of the filtering $\mathcal{I}_\psi$.

6.8.5 Proposition. *It holds true that $C^{\phi_{\mathcal{I}} \triangledown \psi} \equiv_\emptyset C^{\mathcal{I}_\psi \mathcal{I}_\psi}$ for each $\mathcal{EL}^{\bot}_{\mathsf{si}}$ concept description C.*

Proof. The case $C = \bot$ is trivial. Thus, assume that C is an $\mathcal{EL}_{\mathsf{si}}$ concept description. If we show that $\{X\}^{\mathcal{I}_\psi}$ is equivalent to $(X^{\mathcal{I}})^\psi$ for each $X \in \Delta^{\mathcal{I}_\psi}$, i.e., where $X^{\mathcal{I}}$ is a closure of ψ, then we can immediately derive the following. Note that the first equivalence has already been shown in the proof of Lemma 6.8.4 as Equation (6.8.B).

$$\begin{aligned} C^{\phi_{\mathcal{I}} \triangledown \psi} &\equiv_\emptyset \bigvee \{\, X^{\mathcal{I}} \mid X \subseteq \Delta^{\mathcal{I}} \text{ and } X^{\mathcal{I}} \equiv_\emptyset X^{\mathcal{I}\psi} \sqsubseteq_\emptyset C \,\} \\ &\equiv_\emptyset \bigvee \{\, X^{\mathcal{I}} \mid X \in \Delta^{\mathcal{I}_\psi} \text{ and } X^{\mathcal{I}} \sqsubseteq_\emptyset C \,\} \\ &\equiv_\emptyset \bigvee \{\, (X^{\mathcal{I}})^\psi \mid X \in \Delta^{\mathcal{I}_\psi} \text{ and } (X^{\mathcal{I}})^\psi \sqsubseteq_\emptyset C \,\} \\ &\equiv_\emptyset \bigvee \{\, \{X\}^{\mathcal{I}_\psi} \mid X \in \Delta^{\mathcal{I}_\psi} \text{ and } \{X\}^{\mathcal{I}_\psi} \sqsubseteq_\emptyset C \,\} \\ &\equiv_\emptyset \bigvee \{\, \{X\}^{\mathcal{I}_\psi} \mid X \in \Delta^{\mathcal{I}_\psi} \text{ and } X \in C^{\mathcal{I}_\psi} \,\} \\ &\equiv_\emptyset C^{\mathcal{I}_\psi \mathcal{I}_\psi} \end{aligned}$$

It remains to show that the above equivalence $\{X\}^{\mathcal{I}_\psi} \equiv_\emptyset (X^{\mathcal{I}})^\psi$ is satisfied for each X in the domain of $\mathcal{I}_\psi$. Thus, fix some such $X \in \Delta^{\mathcal{I}_\psi}$. We already know from Proposition 4.1.6 that $X^{\mathcal{I}}$ is equivalent to $\exists^{\mathsf{sim}}(\wp(\mathcal{I}), X)$ modulo $\emptyset$. Now let $(\mathcal{J}, X)$ be the successor-reduction of $(\wp(\mathcal{I}), X)$; then $X^{\mathcal{I}}$ is equivalent to $\exists^{\mathsf{sim}}(\mathcal{J}, X)$ modulo $\emptyset$ as well. Since $\exists^{\mathsf{sim}}(\mathcal{J}, X)$ is a closure of ψ and is successor-reduced, an application of Lemma 6.2.3 yields that also $\exists^{\mathsf{sim}}(\mathcal{J}, Y)$ is a closure of ψ for each Y that is reachable from X in $\mathcal{J}$. Without loss of generality assume that $\mathcal{J}$ only contains objects that can be reached from X. It then follows that $Y^{\mathcal{I}}$ is a closure of ψ for each $Y \in \Delta^{\mathcal{J}}$, since $\exists^{\mathsf{sim}}(\mathcal{J}, Y)$ is a closure of ψ and satisfies $\exists^{\mathsf{sim}}(\mathcal{J}, Y) \equiv_\emptyset \exists^{\mathsf{sim}}(\wp(\mathcal{I}), Y) \equiv_\emptyset Y^{\mathcal{I}}$. Consequently, $\Delta^{\mathcal{J}}$ is a subset of $\Delta^{\mathcal{I}_\psi}$.

We proceed with demonstrating that the pointed interpretations $(\mathcal{J}, X)$ and $(\mathcal{I}_\psi, X)$ are equi-similar. We first show that the relation $\mathfrak{S} := \{\, (Y,Y) \mid Y \in \Delta^{\mathcal{J}} \,\}$ is a simulation from $(\mathcal{J}, X)$ to

$(\mathcal{I}_\psi, X)$. Obviously, $(X, X) \in \mathfrak{S}$ is satisfied. Now consider an arbitrary pair $(Y, Y) \in \mathfrak{S}$.

- If $Y \in A^{\mathcal{J}}$, then $Y \in A^{\wp(\mathcal{I})}$ follows. We conclude that $Y \subseteq A^{\mathcal{I}}$ holds true, which implies $Y \in A^{\mathcal{I}_\psi}$.
- Let $(Y, Z) \in r^{\mathcal{J}}$. In particular, then both Y and Z are closures of ψ and $(Y, Z) \in r^{\wp(\mathcal{I})}$. The latter implies $Z \in \mathsf{Min}(\mathsf{Succ}(Y, r))$, and so we conclude that $Z \in \mathsf{Succ}(Y, r) \cap \Delta^{\mathcal{I}_\psi} = \mathsf{Succ}_\psi(Y, r)$. To justify that $(Y, Z) \in r^{\mathcal{I}_\psi}$ is satisfied, we need to verify that Z is minimal in $\mathsf{Succ}_\psi(Y, r)$. Assume the contrary, i.e., there is some $W \in \mathsf{Succ}_\psi(Y, r)$ such that $W \subsetneq Z$. Then W must also be an element of $\mathsf{Succ}(Y, r)$, which contradicts the minimality of Z in $\mathsf{Succ}(Y, r)$. $\lightning$

We conclude that $\{X\}^{\mathcal{I}_\psi} \equiv_\emptyset \exists^{\mathsf{sim}}(\mathcal{I}_\psi, X) \sqsubseteq_\emptyset \exists^{\mathsf{sim}}(\mathcal{J}, X) \equiv_\emptyset (X^{\mathcal{I}})^\psi$.

Eventually, we have to show the converse subsumption $(X^{\mathcal{I}})^\psi \sqsubseteq_\emptyset \{X\}^{\mathcal{I}_\psi}$. According to Lemma 4.1.2 and since $X^{\mathcal{I}}$ is already a closure of ψ, we can equivalently show that X is a subset of $(\{X\}^{\mathcal{I}_\psi})^{\mathcal{I}}$, i.e., that there is a simulation from $(\mathcal{I}_\psi, X)$ to $(\mathcal{I}, \delta)$ for each object $\delta \in X$. Fix some $\delta \in X$ and define the relation $\mathfrak{T} := \{ (Y, \epsilon) \mid Y \ni \epsilon \}$. Clearly, the pair (X, δ) is contained in $\mathfrak{T}$. Now consider an arbitrary pair $(Y, \epsilon) \in \mathfrak{T}$.

- If $Y \in A^{\mathcal{I}_\psi}$, then $Y \subseteq A^{\mathcal{I}}$, and so it follows that $\epsilon \in A^{\mathcal{I}}$.
- If $(Y, Z) \in r^{\mathcal{I}_\psi}$, then $Z \in \mathsf{Min}(\mathsf{Succ}_\psi(Y, r))$, which in particular means that $Y \subseteq \mathsf{pred}(Z, r)$. We infer that $\epsilon \in \mathsf{pred}(Z, r)$, and so there must exist some $\zeta \in Z$ satisfying $(\epsilon, \zeta) \in r^{\mathcal{I}}$. Of course, (Z, ζ) is in $\mathfrak{T}$.

We conclude that $\mathfrak{T}$ is a simulation as needed. □

6.8.6 Interactive, Gentle Ontology Repair

A further application is the *repairing of ontologies*. More specifically, we consider situations where an ontology $\mathcal{O}$ entails an unwanted axiom α and the ontology should be modified such that it does not have α as a consequence anymore. On the one hand, it might be the case that the entailed axiom α is simply wrong in the domain of interest that is described by $\mathcal{O}$ and, on the other hand, we might want to publish the ontology but the consequence α represents sensitive information that should not be accessible. Baader, Kriegel, Nuradiansyah, and Peñaloza [Baa+18a] have introduced a general framework for repairing ontologies. The approach is independent of the concrete logical language that is used to formulate the axioms in the ontology, it only assumes that there is a monotonic consequence operator between sets of axioms. Their *(modified) gentle repair algorithm* computes a so-called *justification*, i.e., a minimal subset of the ontology that entails α, and then *weakens* some axiom in the justification (instead of removing it as for classical repairs). However, the aforementioned step of weakening some axiom in a justification needs to be iterated in order to ensure that the resulting ontology does not have α as a consequence, and furthermore at most exponentially many iterations are sufficient. In [Baa+18a] it is then further shown how suitable *weakening relations* on $\mathcal{EL}$ concept inclusions can be defined, yielding unsupervised repair approaches for $\mathcal{EL}$ TBoxes. We continue with citing the definition of a *repair* for the special case of TBoxes, and then describe how our results from this chapter can be utilized to compute such repairs as well.

6.8.6 Definition. (Special Case of [Baa+18a, Definition 1]) Let $\mathcal{T} = \mathcal{T}_s \cup \mathcal{T}_r$ be an $\mathcal{EL}^{\bot}_{\mathsf{si}}$ TBox consisting of a static and a refutable part, and $C \sqsubseteq D$ a concept inclusion such that $\mathcal{T} \models C \sqsubseteq D$ and $\mathcal{T}_s \not\models C \sqsubseteq D$. The TBox $\mathcal{T}'$ is a *repair* of $\mathcal{T}$ w.r.t. $C \sqsubseteq D$ if $\mathcal{T} \models \mathcal{T}'$ and $\mathcal{T}_s \cup \mathcal{T}' \not\models C \sqsubseteq D$. △

The algorithm *Attribute Exploration* by Ganter [Gan84] allows for exploring the implication theory of a domain of interest in a sound and complete manner. In particular, it interacts with an *expert* by asking whether some implication is valid, and the expert can either confirm the implication or has to provide a counterexample against it. Of course, an ontology repair algorithm can also use expert interaction. We simply ask the expert to specify a counterexample against the concept inclusion, i.e., some interpretation $\mathcal{I}$ such that $\mathcal{I}$ is a model of the static TBox $\mathcal{T}_s$ and $C \sqsubseteq D$ is not valid for $\mathcal{I}$. In order to remove the consequence, we replace the refutable TBox $\mathcal{T}_r$ by the canonical base of the infimum $\phi_{\mathcal{T}_r} \triangle \phi_{\mathcal{I}}$ relative to $\mathcal{T}_s$. Since $C \sqsubseteq D$ is not valid for $\mathcal{I}$, it is not valid for the induced closure operator $\phi_{\mathcal{I}}$, and consequently it is not valid for $\phi_{\mathcal{T}_r} \triangle \phi_{\mathcal{I}}$, which shows that the replacement $\mathsf{Can}(\phi_{\mathcal{T}_r} \triangle \phi_{\mathcal{I}}, \mathcal{T}_s)$ does not entail $C \sqsubseteq D$ anymore. However, the expert should be as general as possible when specifying the counterexample, since the repair also does not entail all other concept inclusions that are not valid for $\mathcal{I}$.

In order to give some more freedom to the expert, a counterexample could also be provided in the form of some simple ABox $\mathcal{A}$ such that $C \sqsubseteq D$ is not Σ_{I}-entailed by $\mathcal{A}$. We then use the closure operator $\phi_{\mathcal{A}}$ instead of $\phi_{\mathcal{I}}$.

7 Axiomatization of $\mathcal{M}$ and of Horn-$\mathcal{M}$ Concept Inclusions from Interpretations

In this chapter we show how concept inclusions in two quite expressive description logics can can axiomatized from interpretations. More specifically, we generalize from the $\mathcal{EL}$ case [BDK16] described in Section 6.1 and show in Section 7.1 how we can generate a base of $\mathcal{M}$ concept inclusions and then adapt the technique to the description logic Horn-$\mathcal{M}$ from Section 3.2. For both cases, we assume that there is a bound on the role depths, i.e., we show how from the input interpretation a TBox can be constructed that is sound and complete for all concept inclusions up to a predefined role depth. Since instance checking for the sublogic Horn-$\mathcal{M}^-$ is **P**-complete for data complexity—and for Horn-$\mathcal{M}$ probably also **P**-complete—as explained in Section 3.2.2, such a Horn-$\mathcal{M}$ TBox can be used as schema in *ontology-based data access* (abbrv. OBDA) applications. Eventually, Section 7.3 demonstrates how role inclusions can be axiomatized in a sound and complete manner and how the resulting base can integrated in the results obtained in Sections 7.1 and 7.2.

7.1 Bases of $\mathcal{M}$ Concept Inclusions

For the sequel of this section, fix some finite interpretation $\mathcal{I}$ over a finite signature Σ as well as some role depth bound $d \in \mathbb{N}$. Whenever we refer to a concept description, we mean an $\mathcal{M}$ concept description with a role depth not exceeding d; likewise we only consider concept inclusions involving $\mathcal{M}_d$ concept inclusions. Note that we have explained in Section 4.2 how model-based most specific $\mathcal{M}$ concept descriptions $X^{\mathcal{I}_d}$ can be computed.

7.1.1 Lemma. (Generalization of [BDK16, Lemma 4.6]) *Let C and D be $\mathcal{M}$ concept descriptions with role depths not exceeding d. If the concept inclusion $C \sqsubseteq D$ is valid in $\mathcal{I}$, then the concept inclusion $C \sqsubseteq C^{\mathcal{I}\mathcal{I}_d}$ is valid in $\mathcal{I}$ too, and furthermore $C \sqsubseteq D$ follows from $\{C \sqsubseteq C^{\mathcal{I}\mathcal{I}_d}\}$.*

Proof. For the concept description C it follows by an application of Statement 6 of Lemma 4.1.2 that $C^{\mathcal{I}} = C^{\mathcal{I}\mathcal{I}_d\mathcal{I}}$, i.e., the CI $C \sqsubseteq C^{\mathcal{I}\mathcal{I}_d}$ is always valid in $\mathcal{I}$.

Now consider a model $\mathcal{J}$ of the CI $C \sqsubseteq C^{\mathcal{I}\mathcal{I}_d}$. Since $\mathcal{I} \models C \sqsubseteq D$, it follows that $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$, and by Statement 1 of Lemma 4.1.2 we conclude that $\emptyset \models C^{\mathcal{I}\mathcal{I}_d} \sqsubseteq D$. In particular, then the last CI is also valid in $\mathcal{J}$, and hence $\mathcal{J} \models C \sqsubseteq D$. Since $\mathcal{J}$ was an arbitrary model, we conclude that $\{C \sqsubseteq C^{\mathcal{I}\mathcal{I}_d}\} \models C \sqsubseteq D$. □

In the case of $\mathcal{EL}^{\bot}_{\mathsf{gfp}}$, Baader and Distel showed that each (unbounded) MMSC of an interpretation $\mathcal{I}$ can be expressed in terms of $\{\bot\} \cup \Sigma_{\mathsf{C}} \cup \{\exists r. X^{\mathcal{I}} \mid r \in \Sigma_{\mathsf{R}} \text{ and } \emptyset \neq X \subseteq \Delta^{\mathcal{I}}\}$, see Section 6.1. Similarly, for the role-depth-bounded case, Borchmann, Distel, and the author showed that each MMSC of $\mathcal{I}$ in $\mathcal{EL}_d^\bot$ is expressible in terms of $\{\bot\} \cup \Sigma_\mathsf{C} \cup \{\, \exists\, r.\, X^{\mathcal{I}_{d-1}} \mid r \in \Sigma_\mathsf{R} \text{ and } \emptyset \neq X \subseteq \Delta^\mathcal{I} \,\}$. As a straightforward extension to $\mathcal{M}$, we define the following set $\mathbf{M}$ and then show that it can express each model-based most specific concept description of $\mathcal{I}$.

$$\mathbf{M} := \{\bot\} \cup \{\, A, \neg A \mid A \in \Sigma_\mathsf{C} \,\} \cup \left\{ \begin{array}{r|c} \exists \geq m.\, r.\, X^{\mathcal{I}_{d-1}}, & r \in \Sigma_\mathsf{R}, \\ \exists \leq n.\, r, & 0 < m \leq |\Delta^\mathcal{I}|, \\ \exists\, r.\, \mathsf{Self}, & 0 \leq n \leq |\Delta^\mathcal{I}|, \\ \forall\, r.\, X^{\mathcal{I}_{d-1}} & \emptyset \neq X \subseteq \Delta^\mathcal{I} \end{array} \right\}$$

Like in Section 6.6 and [Dis11, Chapter 5], we say that an $\mathcal{M}$ concept description C is *expressible in terms of* $\mathbf{M}$ if there is some subset $\mathbf{X}$ of $\mathbf{M}$ such that C is equivalent to the conjunction $\bigsqcap \mathbf{X}$ modulo $\emptyset$. Analogously, we also define a projection mapping $\pi\colon \mathcal{M}_d(\Sigma) \to \wp(\mathbf{M})$ by

$$\pi(C) := \{\, D \mid D \in \mathbf{M} \text{ and } C \sqsubseteq_\emptyset D \,\}.$$

In particular, then similar statements as in Lemmas 6.6.2 and 6.6.3 and Proposition 6.6.4 hold true, which are as follows. Note that now there is no background TBox containing existing knowledge.

1. $(\pi, \bigsqcap)$ is a Galois connection between $(\mathcal{M}_d(\Sigma), \sqsubseteq_\emptyset)/\equiv_\emptyset$ and $(\wp(\mathbf{M}), \subseteq)$.
2. If C is expressible in terms of $\mathbf{M}$, then C is equivalent to $\bigsqcap \pi(C)$ modulo $\emptyset$.
3. All model-based most specific concept descriptions of $\mathcal{I}$ are expressible in terms of $\mathbf{M}$.

The first two statements can be proven in the same way as Lemmas 6.6.2 and 6.6.3. The third statement is an obvious consequence of Theorem 4.2.2.

7.1.2 Proposition. (Generalization of [BDK16, Theorem 4.9]) *The following TBox $\mathcal{B}_\mathbf{M}$ is sound and complete for the $\mathcal{M}$ concept inclusions that are valid in $\mathcal{I}$ and have role depths of at most d.*

$$\mathcal{B}_\mathbf{M} := \{\, \bigsqcap \mathbf{X} \sqsubseteq (\bigsqcap \mathbf{X})^{\mathcal{I}\mathcal{I}_d} \mid \mathbf{X} \subseteq \mathbf{M} \,\} \cup \mathcal{N}$$

The sub-TBox $\mathcal{N}$ is defined by

$$\mathcal{N} := \{\, \top \sqsubseteq \exists \leq |\Delta^\mathcal{I}|.\, r \mid r \in \Sigma_\mathsf{R} \,\}$$

and encodes that, for each role name $r \in \Sigma_\mathsf{R}$, each object in $\mathcal{I}$ cannot have more r-successors than there are objects in $\mathcal{I}$.[1]

Proof. It is obvious that $\mathcal{B}_\mathbf{M}$ is sound for $\mathcal{I}$. If $C \sqsubseteq D$ is some concept inclusion which is valid in $\mathcal{I}$, then Lemma 7.1.1 yields that the CI $C \sqsubseteq C^{\mathcal{I}\mathcal{I}_d}$ is also valid in $\mathcal{I}$, and furthermore the entailment $\{C \sqsubseteq C^{\mathcal{I}\mathcal{I}_d}\} \models C \sqsubseteq D$ holds true. Hence, it suffices to show that our TBox $\mathcal{B}_\mathbf{M}$ entails all CIs of the form $C \sqsubseteq C^{\mathcal{I}\mathcal{I}_d}$. We proceed with a proof by induction on the structure of C.

- If C is $\bot$, a concept name A, a negated concept name $\neg A$, an existential self restriction $\exists\, r.\, \mathsf{Self}$, or an unqualified less-than restriction $\exists \leq n.\, r$ where $n \leq |\Delta^\mathcal{I}|$, then C is an element of $\mathbf{M}$, and so we can immediately conclude that the CI $C \sqsubseteq C^{\mathcal{I}\mathcal{I}_d}$ is in $\mathcal{B}_\mathbf{M}$.

[1] Note that each concept inclusion in $\mathcal{N}$ has the same models as $\exists \geq (|\Delta^\mathcal{I}| + 1).\, r.\, \top \sqsubseteq \bot$.

- For the case $C = \top$ we have that $\top$ is equivalent to $\bigsqcap\emptyset$ modulo $\emptyset$. It follows that $\mathcal{B}_{\mathbf{M}}$ entails the CI $\top \sqsubseteq \top^{\mathcal{I}\mathcal{I}_d}$.

- Consider an unqualified less-than restriction $\exists\leq n.\,r$ such that $n > |\Delta^{\mathcal{I}}|$. We have that $\exists\leq n.\,r \sqsupseteq_\emptyset \exists\leq|\Delta^{\mathcal{I}}|.\,r$, which implies that $\top$ and $\exists\leq n.\,r$ are equivalent modulo $\mathcal{N}$. We have already seen that the CI $\top \sqsubseteq \top^{\mathcal{I}\mathcal{I}_d}$ follows from $\mathcal{B}_{\mathbf{M}}$, and so we conclude that $\mathcal{B}_{\mathbf{M}}$ entails $\exists\leq n.\,r \sqsubseteq (\exists\leq n.\,r)^{\mathcal{I}\mathcal{I}_d}$.

- Consider a conjunction $C = D \sqcap E$. By induction hypothesis, $\mathcal{B}_{\mathbf{M}}$ entails both CIs $D \sqsubseteq D^{\mathcal{I}\mathcal{I}_d}$ and $E \sqsubseteq E^{\mathcal{I}\mathcal{I}_d}$, which implies that $\mathcal{B}_{\mathbf{M}}$ entails $D \sqcap E \sqsubseteq D^{\mathcal{I}\mathcal{I}_d} \sqcap E^{\mathcal{I}\mathcal{I}_d}$. Since all MMSCs are expressible in terms of $\mathbf{M}$, there are subsets $\mathbf{X}$ and $\mathbf{Y}$ of $\mathbf{M}$ such that $D^{\mathcal{I}\mathcal{I}_d} \equiv_\emptyset \bigsqcap\mathbf{X}$ and $E^{\mathcal{I}\mathcal{I}_d} \equiv_\emptyset \bigsqcap\mathbf{Y}$. Of course, then the CI $\bigsqcap(\mathbf{X}\cup\mathbf{Y}) \sqsubseteq (\bigsqcap(\mathbf{X}\cup\mathbf{Y}))^{\mathcal{I}\mathcal{I}_d}$ is in $\mathcal{B}_{\mathbf{M}}$. We conclude that $\mathcal{B}_{\mathbf{M}}$ entails $D^{\mathcal{I}\mathcal{I}_d} \sqcap E^{\mathcal{I}\mathcal{I}_d} \sqsubseteq (D^{\mathcal{I}\mathcal{I}_d} \sqcap E^{\mathcal{I}\mathcal{I}_d})^{\mathcal{I}\mathcal{I}_d}$. Now Statement 5 of Lemma 4.1.2 shows that $(D^{\mathcal{I}\mathcal{I}_d} \sqcap E^{\mathcal{I}\mathcal{I}_d})^{\mathcal{I}\mathcal{I}_d}$ is more specific than $(D \sqcap E)^{\mathcal{I}\mathcal{I}_d}$ modulo $\emptyset$. Summing up yields the claim.

- Assume that $C = \forall\, r.\, D$ is a value restriction. Then the following subsumptions hold true.

$$\begin{aligned}\forall\, r.\, D &\sqsubseteq_{\mathcal{B}_{\mathbf{M}}} \forall\, r.\, D^{\mathcal{I}\mathcal{I}_d}\\ &\sqsubseteq_\emptyset \forall\, r.\, D^{\mathcal{I}\mathcal{I}_{d-1}}\\ &\sqsubseteq_{\mathcal{B}_{\mathbf{M}}} (\forall\, r.\, D^{\mathcal{I}\mathcal{I}_{d-1}})^{\mathcal{I}\mathcal{I}_d}\\ &\sqsubseteq_\emptyset (\forall\, r.\, D)^{\mathcal{I}\mathcal{I}_d}\end{aligned}$$

  The first subsumption is a consequence of the induction hypothesis and the fact that value restrictions are monotonic. For the second subsumption, observe that $D^{\mathcal{I}\mathcal{I}_{d-1}}$ certainly satisfies that $\mathsf{rd}(D^{\mathcal{I}\mathcal{I}_{d-1}}) \leq d$ as well as $D^{\mathcal{I}} \subseteq D^{\mathcal{I}\mathcal{I}_{d-1}\mathcal{I}}$, and so an application of Statement 2 of Definition 4.1.1 yields that $D^{\mathcal{I}\mathcal{I}_d}$ is more specific than $D^{\mathcal{I}\mathcal{I}_{d-1}}$ modulo $\emptyset$. The third subsumption follows from the fact that $\forall\, r.\, D^{\mathcal{I}\mathcal{I}_{d-1}}$ is contained in $\mathbf{M}$. The last subsumption follows from Statement 5 of Lemma 4.1.2.

- Now let $C = \exists\geq n.\, r.\, D$ be a qualified greater-than restriction, and first assume that $n \leq |\Delta^{\mathcal{I}}|$. Then, we may argue similarly as for the value restrictions that the following subsumptions hold true.

$$\begin{aligned}\exists\geq n.\, r.\, D &\sqsubseteq_{\mathcal{B}_{\mathbf{M}}} \exists\geq n.\, r.\, D^{\mathcal{I}\mathcal{I}_d}\\ &\sqsubseteq_\emptyset \exists\geq n.\, r.\, D^{\mathcal{I}\mathcal{I}_{d-1}}\\ &\sqsubseteq_{\mathcal{B}_{\mathbf{M}}} (\exists\geq n.\, r.\, D^{\mathcal{I}\mathcal{I}_{d-1}})^{\mathcal{I}\mathcal{I}_d}\\ &\sqsubseteq_\emptyset (\exists\geq n.\, r.\, D)^{\mathcal{I}\mathcal{I}_d}\end{aligned}$$

  For the remaining case where $n > |\Delta^{\mathcal{I}}|$, we argue as follows.

$$\begin{aligned}\exists\geq n.\, r.\, D &\sqsubseteq_\emptyset \exists\geq n.\, r.\, \top\\ &\sqsubseteq_\emptyset \exists\geq |\Delta^{\mathcal{I}}|+1.\, r.\, \top\\ &\sqsubseteq_{\mathcal{N}} \bot,\end{aligned}$$

Hence, the concept descriptions $\bot$ and $\exists \geq n.r.D$ are equivalent modulo $\mathcal{N}$. Since we have already proven above that $\bot \sqsubseteq \bot^{\mathcal{I}\mathcal{I}_d}$ follows from $\mathcal{B}_{\mathbf{M}}$, also the CI $\exists \geq n.r.D \sqsubseteq (\exists \geq n.r.D)^{\mathcal{I}\mathcal{I}_d}$ is entailed by $\mathcal{B}_{\mathbf{M}}$. □

The *induced formal context* of $\mathcal{I}$ is defined as

$$\mathbb{K}_{\mathcal{I}} := (\Delta^{\mathcal{I}}, \mathbf{M}, I)$$

where the incidence relation is defined by $(\delta, C) \in I$ if $\delta \in C^{\mathcal{I}}$. It is easy to see that $\mathbf{X}^I$ equals $(\bigsqcap \mathbf{X})^{\mathcal{I}}$ for each subset $\mathbf{X} \subseteq \mathbf{M}$. We conclude that, for all subsets $\mathbf{X}, \mathbf{Y} \subseteq \mathbf{M}$, the concept inclusion $\bigsqcap \mathbf{X} \sqsubseteq \bigsqcap \mathbf{Y}$ is valid in $\mathcal{I}$ if, and only if, the implication $\mathbf{X} \to \mathbf{Y}$ is valid in $\mathbb{K}_{\mathcal{I}}$.

Now fix some concept description $C \in \mathbf{M}$. Considering the singleton $\mathbf{X} := \{C\}$ shows that $\{C\}^I$ equals $C^{\mathcal{I}}$. Using the laws of Galois connections we infer that, for each subset $X \subseteq \Delta^{\mathcal{I}}$, the statements $X^{\mathcal{I}_d} \sqsubseteq_\emptyset C$, and $X \subseteq C^{\mathcal{I}}$, and $C \in X^I$ are equivalent. As a consequence, we obtain that X^I equals $\pi(X^{\mathcal{I}_d})$.

As an immediate consequence of the above, we get that $X^{II} = (\bigsqcap X^I)^{\mathcal{I}} = (\bigsqcap \pi(X^{\mathcal{I}_d}))^{\mathcal{I}} = X^{\mathcal{I}_d\mathcal{I}}$ holds true for each subset $X \subseteq \Delta^{\mathcal{I}}$ where the last equality follows from the fact that the MMSC $X^{\mathcal{I}_d}$ is expressible in terms of $\mathbf{M}$. Furthermore, we infer that $\mathbf{X}^{II} = \pi((\mathbf{X}^I)^{\mathcal{I}_d}) = \pi((\bigsqcap \mathbf{X})^{\mathcal{I}\mathcal{I}_d})$, and applying the operator $\bigsqcap$ yields $\bigsqcap \mathbf{X}^{II} \equiv_\emptyset \bigsqcap \pi((\bigsqcap \mathbf{X})^{\mathcal{I}\mathcal{I}_d})$. Since the MMSC $(\bigsqcap \mathbf{X})^{\mathcal{I}\mathcal{I}_d}$ is expressible in terms of $\mathbf{M}$, we conclude that $\bigsqcap \mathbf{X}^{II} \equiv_\emptyset (\bigsqcap \mathbf{X})^{\mathcal{I}\mathcal{I}_d}$.

As final step we use the tautological concept inclusions to define some background knowledge for the computation of the canonical implication base of the induced concept context which is trivial in terms of Description Logics, but not for Formal Concept Analysis, due to their different semantics. In particular, we define the following implication set $\mathcal{S}$ that is used as background knowledge.

$$\mathcal{S} := \{\, \mathbf{X} \to \{C\} \mid \mathbf{X} \cup \{C\} \subseteq \mathbf{M} \text{ and } \bigsqcap \mathbf{X} \sqsubseteq_\emptyset C \,\}.$$

Similarly to what has been explained on Page 190 for the implication set $\widehat{\mathcal{T}}$, we can easily show that $\mathcal{S} = \{\, \mathbf{X} \to \pi(\bigsqcap \mathbf{X}) \mid \mathbf{X} \subseteq \mathbf{M} \,\}$ holds true and further we use the implication set $\mathcal{S}$ only virtually, i.e., we can devise an oracle that is able to compute closures $\mathbf{X}^{\mathcal{S}}$ for subsets $\mathbf{X} \subseteq \mathbf{M}$.

7.1.3 Theorem. (Generalization of [BDK16, Theorem 4.30]) *Let $\mathcal{L}$ be an implication base of the induced formal context $\mathbb{K}_{\mathcal{I}}$ with respect to the background knowledge $\mathcal{S}$. Then, the following TBox $\mathcal{B}_{\mathcal{L}}$ is sound and complete for the $\mathcal{M}$ concept inclusions that are valid in $\mathcal{I}$ and have role depths of at most d.*

$$\mathcal{B}_{\mathcal{L}} := \{\, \bigsqcap \mathbf{X} \sqsubseteq (\bigsqcap \mathbf{X})^{\mathcal{I}\mathcal{I}_d} \mid \mathbf{X} \to \mathbf{Y} \in \mathcal{L} \textit{ for some } \mathbf{Y} \,\} \cup \mathcal{N}$$

Proof. It is obvious that $\mathcal{B}_{\mathcal{L}}$ is sound. For showing completeness for the concept inclusions, it suffices to show that $\mathcal{B}_{\mathcal{L}}$ entails the complete TBox $\mathcal{B}_{\mathbf{M}}$ from Proposition 7.1.2.

Consider a model $\mathcal{J}$ of $\mathcal{B}_{\mathcal{L}}$. We divide the remaining part of this proof in three steps.

1. First, we show that all implications in $\mathcal{L}$ are also valid in the induced formal context $\mathbb{K}_{\mathcal{J}} := (\Delta^{\mathcal{J}}, \mathbf{M}, J)$ where $(\delta, C) \in J$ if $\delta \in C^{\mathcal{J}}$. W.l.o.g. we may assume that $\mathcal{L}$ only contains

implications of the form $\mathbf{X} \to \mathbf{X}^{II}$. Hence, let $\mathbf{X} \to \mathbf{X}^{II} \in \mathcal{L}$, then it follows that $\mathbf{X}^{J} = (\bigsqcap \mathbf{X})^{\mathcal{J}} \subseteq (\bigsqcap \mathbf{X})^{\mathcal{I}\mathcal{I}_d \mathcal{J}} = (\bigsqcap \mathbf{X}^{II})^{\mathcal{J}} = \mathbf{X}^{IIJ}$, i.e., the implication $\mathbf{X} \to \mathbf{X}^{II}$ is valid in $\mathbb{K}_{\mathcal{J}}$.

2. Then, we prove that the background knowledge $\mathcal{S}$ is valid in the induced formal context $\mathbb{K}_{\mathcal{J}}$ too. Consider an implication $\mathbf{X} \to \{C\}$ in $\mathcal{S}$. Since $\bigsqcap \mathbf{X} \sqsubseteq C$ is a tautology, it must be valid in $\mathcal{J}$. It follows that the considered implication must be valid in the induced formal context $\mathbb{K}_{\mathcal{J}}$.
3. Finally, we show that $\mathcal{J}$ is a model of each CI $\bigsqcap \mathbf{X} \sqsubseteq (\bigsqcap \mathbf{X})^{\mathcal{I}\mathcal{I}_d}$ in $\mathcal{B}_{\mathbf{M}}$, which then yields completeness of $\mathcal{B}_{\mathcal{L}}$. Thus, fix some such concept inclusion. Since the implication set $\mathcal{L} \cup \mathcal{S}$ is sound and complete for $\mathbb{K}_{\mathcal{I}}$, and $\mathbf{X} \to \mathbf{X}^{II}$ is valid in $\mathbb{K}_{\mathcal{I}}$, it holds true that $\mathbf{X} \to \mathbf{X}^{II}$ is entailed by $\mathcal{L} \cup \mathcal{S}$. As $\mathbb{K}_{\mathcal{J}}$ is a model of both $\mathcal{L}$ and $\mathcal{S}$, it follows that $\mathbf{X} \to \mathbf{X}^{II}$ is valid in $\mathbb{K}_{\mathcal{J}}$ too. We conclude that the CI $\bigsqcap \mathbf{X} \sqsubseteq \bigsqcap \mathbf{X}^{II}$ is valid in $\mathcal{J}$, and using the equivalence $\bigsqcap \mathbf{X}^{II} \equiv_{\emptyset} (\bigsqcap \mathbf{X})^{\mathcal{I}\mathcal{I}_d}$ finishes the proof. □

7.1.4 Corollary. (Generalization of [BDK16, Theorem 4.32]) *The following TBox, called* canonical base *for $\mathcal{I}$, is sound and complete for the $\mathcal{M}$ concept inclusions that are valid in $\mathcal{I}$ and have role depths not exceeding d.*

$$\mathsf{Can}_{\mathcal{M}}(\mathcal{I}, d) := \{\, \bigsqcap \mathbf{P} \sqsubseteq \bigsqcap \mathbf{P}^{II} \mid \mathbf{P} \textit{ is a pseudo-intent of } \mathbb{K}_{\mathcal{I}} \textit{ relative to } \mathcal{S} \,\} \cup \mathcal{N} \qquad \square$$

The author conjectures that the canonical $\mathcal{M}$ concept inclusion base $\mathsf{Can}_{\mathcal{M}}(\mathcal{I}, d)$ has *minimal cardinality* among all $\mathcal{M}$ concept inclusion bases for $\mathcal{I}$ and d. However, it is not immediately possible to suitably adapt the minimality proof for the $\mathcal{EL}$ case in [BDK16; Dis11]. The crucial point is that we need the validity of the following claim, which resembles [Dis11, Lemma 5.16; BDK16, Lemma A.9] for our case of $\mathcal{M}$.

Claim. *Fix some $\mathcal{M}$ TBox $\mathcal{T} \cup \{C \sqsubseteq D\}$ in which all occurring concept descriptions have role depths not exceeding d. Further assume that $\mathcal{I}$ is a finite model of $\mathcal{T}$ such that, for each subconcept $\exists\, r.\, X$ of C, the filler X is (equivalent to) some model-based most specific concept description of $\mathcal{I}$; more specifically, we assume that $Y \equiv Y^{\mathcal{I}\mathcal{I}_{d-1}}$ is satisfied for each $\exists\, r.\, Y \in \mathsf{Conj}(C)$. If $C \not\sqsubseteq_{\emptyset} D$ and $C \sqsubseteq_{\mathcal{T}} D$, then $C \sqsubseteq_{\emptyset} E$ and $C \not\sqsubseteq_{\emptyset} F$ holds true for some concept inclusion $E \sqsubseteq F$ contained in $\mathcal{T}$.*

It should be possible to prove the claim as soon as there is technique for computing most specific consequences w.r.t. $\mathcal{M}$ TBoxes. However, this remains an open problem here.

With a similar proof as for Proposition 7.2.6 we can justify the following complexity result on computing the canonical base.

7.1.5 Proposition. *The canonical $\mathcal{M}$ concept inclusion base for a finite interpretation $\mathcal{I}$ and role depth bound $d \geq 1$ can be computed in exponential time with respect to $\mathcal{I}$ and d, and further there exist finite interpretations $\mathcal{I}$ for which the canonical $\mathcal{M}$ concept inclusion base cannot be encoded in polynomial space w.r.t. $|\Delta^{\mathcal{I}}|$.* □

7.2 Bases of Horn-$\mathcal{M}$ Concept Inclusions

In Section 1.6 we have introduced the new notion of so-called *joining implications*. More specifically, we have assumed that there are two distinct sets of attributes: the first one containing attributes that may occur in premises of implications, while conclusions must only contain attributes from the second set. A canonical base for the joining implications valid in a given formal context has been developed and it has been proven that it is of minimal cardinality among all such bases. In this chapter, we propose an application to inductive learning in the Horn description logic Horn-$\mathcal{M}$ from Section 3.2. Using the canonical base of joining implications, we show how the Horn-$\mathcal{M}$ concept inclusions valid in a given interpretation can be axiomatized.

Now fix some finite interpretation $\mathcal{I}$ over a signature Σ, and further let $d \in \mathbb{N}$ be a role-depth bound. In the remainder of this section, we show how the techniques from Section 1.6 can be applied to axiomatize Horn-$\mathcal{M}$ concept inclusions valid in $\mathcal{I}$. Note that the proofs are suitable adaptations of those for the case of $\mathcal{M}$ in Section 7.1. Further remark that Horn-$\mathcal{M}$ concept inclusions are of the form $C \sqsubseteq D$ where C is an $\mathcal{EL}^*$ concept description and D is an $\mathcal{M}^{\leq 1}$ concept description, cf. Section 3.2.1.

7.2.1 Lemma (Analog of Lemma 7.1.1). *If the Horn-$\mathcal{M}$ concept inclusion $C \sqsubseteq D$ is valid in $\mathcal{I}$, then the Horn-$\mathcal{M}$ concept inclusion $C \sqsubseteq C^{\mathcal{I}\mathcal{I}_d^{\mathcal{M}^{\leq 1}}}$ is valid in $\mathcal{I}$ too, and furthermore $C \sqsubseteq D$ follows from $\{C \sqsubseteq C^{\mathcal{I}\mathcal{I}_d^{\mathcal{M}^{\leq 1}}}\}$.* □

The characterization of MMSCs in Section 4.2 restricted to the special case of $\mathcal{EL}^*$ now implies that each MMSC $X^{\mathcal{I}_d^{\mathcal{EL}^*}}$ is expressible in terms of $\mathbf{M}_\mathsf{p}$ defined as follows.

$$\mathbf{M}_\mathsf{p} := \{\bot\} \cup \Sigma_\mathsf{C} \cup \{\, \exists\, r.\,\mathsf{Self} \mid r \in \Sigma_\mathsf{R}\,\} \cup \{\, \exists\, r.\, X^{\mathcal{I}_{d-1}^{\mathcal{EL}^*}} \mid r \in \Sigma_\mathsf{R} \text{ and } \emptyset \neq X \subseteq \Delta^{\mathcal{I}}\,\}$$

As next step, we prove that there exists a concept inclusion base for $\mathcal{I}$ in which the premises of concept inclusions are of the form $\bigsqcap \mathbf{C}$ for subsets $\mathbf{C} \subseteq \mathbf{M}_\mathsf{p}$.

7.2.2 Proposition (Analog of Proposition 7.1.2). *The following Horn-$\mathcal{M}$ TBox is sound and complete for the Horn-$\mathcal{M}$ concept inclusions valid in $\mathcal{I}$ and having role depths at most d.*

$$\mathcal{B}_{\mathbf{M}_\mathsf{p}} := \{\, \bigsqcap \mathbf{C} \sqsubseteq (\bigsqcap \mathbf{C})^{\mathcal{I}\mathcal{I}_d^{\mathcal{M}^{\leq 1}}} \mid \mathbf{C} \subseteq \mathbf{M}_\mathsf{p}\,\}$$

Proof. Soundness is obvious. According to Lemma 7.2.1, for completeness it suffices to show that $\mathcal{B}_{\mathbf{M}_\mathsf{p}} \models C \sqsubseteq C^{\mathcal{I}\mathcal{I}_d^{\mathcal{M}^{\leq 1}}}$ for each $\mathcal{EL}^*$ concept description C. We do this by structural induction on $C \in \mathcal{EL}^*(\Sigma)$. The base cases where $C \in \{\bot\} \cup \Sigma_\mathsf{C} \cup \{\, \exists\, r.\,\mathsf{Self} \mid r \in \Sigma_\mathsf{R}\,\}$ are obvious.

Case $C = D \sqcap E$. Two applications of the induction hypothesis yield that $\mathcal{B}_{\mathbf{M}_\mathsf{p}}$ entails $D \sqsubseteq D^{\mathcal{I}\mathcal{I}_d^{\mathcal{M}^{\leq 1}}}$ and $E \sqsubseteq E^{\mathcal{I}\mathcal{I}_d^{\mathcal{M}^{\leq 1}}}$. We infer that

$$D \sqcap E \sqsubseteq_{\mathcal{B}_{\mathbf{M}_\mathsf{p}}} D^{\mathcal{I}\mathcal{I}_d^{\mathcal{M}^{\leq 1}}} \sqcap E^{\mathcal{I}\mathcal{I}_d^{\mathcal{M}^{\leq 1}}}.$$

Since $\mathcal{EL}^*$ is less expressive than $\mathcal{M}^{\leq 1}$, we can show by induction on the role-depth bound d and using the recursive characterizations in Section 4.2 that $X^{\mathcal{I}_d^{\mathcal{M}^{\leq 1}}} \sqsubseteq_\emptyset X^{\mathcal{I}_d^{\mathcal{EL}^*}}$ holds true for each subset $X \subseteq \Delta^{\mathcal{I}}$. Consequently, it follows that

$$D^{\mathcal{I}\mathcal{I}_d^{\mathcal{M}^{\leq 1}}} \sqcap E^{\mathcal{I}\mathcal{I}_d^{\mathcal{M}^{\leq 1}}} \sqsubseteq_\emptyset D^{\mathcal{I}\mathcal{I}_d^{\mathcal{EL}^*}} \sqcap E^{\mathcal{I}\mathcal{I}_d^{\mathcal{EL}^*}}.$$

Now the latter concept description is expressible in terms of $\mathbf{M}_\mathsf{p}$, and so we infer that $\mathcal{B}_{\mathbf{M}_\mathsf{p}}$

must entail the concept inclusion

$$D^{\mathcal{I}\mathcal{I}_d^{\mathcal{EL}^*}} \sqcap E^{\mathcal{I}\mathcal{I}_d^{\mathcal{EL}^*}} \sqsubseteq (D^{\mathcal{I}\mathcal{I}_d^{\mathcal{EL}^*}} \sqcap E^{\mathcal{I}\mathcal{I}_d^{\mathcal{EL}^*}})^{\mathcal{I}\mathcal{I}_d^{\mathcal{M}^{\leq 1}}}.$$

As $D^{\mathcal{I}\mathcal{I}_d^{\mathcal{EL}^*}}$ is subsumed by D and $E^{\mathcal{I}\mathcal{I}_d^{\mathcal{EL}^*}}$ is subsumed by E with respect to $\emptyset$, we conclude that

$$(D^{\mathcal{I}\mathcal{I}_d^{\mathcal{EL}^*}} \sqcap E^{\mathcal{I}\mathcal{I}_d^{\mathcal{EL}^*}})^{\mathcal{I}\mathcal{I}_d^{\mathcal{M}^{\leq 1}}} \sqsubseteq_\emptyset (D \sqcap E)^{\mathcal{I}\mathcal{I}_d^{\mathcal{M}^{\leq 1}}}$$

and we are done.

Case $C = \exists\, r.\, D$. Using the induction hypothesis, we get that $\mathcal{B}_{\mathbf{M}_\mathsf{p}}$ entails $\exists\, r.\, D \sqsubseteq \exists\, r.\, D^{\mathcal{I}\mathcal{I}_d^{\mathcal{M}^{\leq 1}}}$. It is further readily verified that

$$\exists\, r.\, D^{\mathcal{I}\mathcal{I}_d^{\mathcal{M}^{\leq 1}}} \sqsubseteq_\emptyset \exists\, r.\, D^{\mathcal{I}\mathcal{I}_{d-1}^{\mathcal{M}^{\leq 1}}} \sqsubseteq_\emptyset \exists\, r.\, D^{\mathcal{I}\mathcal{I}_{d-1}^{\mathcal{EL}^*}}$$

holds true. Since $\exists\, r.\, D^{\mathcal{I}\mathcal{I}_{d-1}^{\mathcal{EL}^*}}$ is an element of $\mathbf{M}_\mathsf{p}$ (choose $X := D^{\mathcal{I}}$), it must be the case that $\mathcal{B}_{\mathbf{M}_\mathsf{p}}$ entails $\exists\, r.\, D^{\mathcal{I}\mathcal{I}_{d-1}^{\mathcal{EL}^*}} \sqsubseteq (\exists\, r.\, D^{\mathcal{I}\mathcal{I}_{d-1}^{\mathcal{EL}^*}})^{\mathcal{I}\mathcal{I}_d^{\mathcal{M}^{\leq 1}}}$. Eventually, we know that $D^{\mathcal{I}\mathcal{I}_{d-1}^{\mathcal{EL}^*}} \sqsubseteq D$ is a tautology, and we conclude that $\exists\, r.\, D^{\mathcal{I}\mathcal{I}_{d-1}^{\mathcal{EL}^*}} \sqsubseteq (\exists\, r.\, D)^{\mathcal{I}\mathcal{I}_d^{\mathcal{M}^{\leq 1}}}$ follows from $\mathcal{B}_{\mathbf{M}_\mathsf{p}}$. □

Similarly to Sections 6.1 and 7.1, we define the *induced formal context* $\mathbb{K}_{\mathcal{I}} := (\Delta^{\mathcal{I}}, \mathbf{M}, I)$ where $\mathbf{M} := \mathbf{M}_\mathsf{p} \cup \mathbf{M}_\mathsf{c}$ for the premise attribute set $\mathbf{M}_\mathsf{p}$ that is already defined above and for the conclusion attribute set is given as

$$\begin{aligned}\mathbf{M}_\mathsf{c} := \{\bot\} \cup \{\, A, \neg A \mid A \in \Sigma_\mathsf{C} \,\} \cup \{\, \exists\, r.\, \mathsf{Self},\ \exists \leq 1.\, r,\ \forall\, r.\, \bot \mid r \in \Sigma_\mathsf{R} \,\} \\ \cup \left\{ \circlearrowright r.\, X^{\mathcal{I}_{d-1}^{\mathcal{M}^{\leq 1}}} \;\middle|\; \begin{array}{l} \circlearrowright \in \{\, \exists \geq n \mid 1 \leq n \leq |\Delta^{\mathcal{I}}| \,\} \cup \{\forall\}, \\ r \in \Sigma_\mathsf{R},\text{ and } \emptyset \neq X \subseteq \Delta^{\mathcal{I}} \end{array} \right\},\end{aligned}$$

and where $(\delta, C) \in I$ if $\delta \in C^{\mathcal{I}}$. Of course, it holds true that $\bigsqcap \mathbf{X} \sqsubseteq \bigsqcap \mathbf{Y}$ is a Horn-$\mathcal{M}$ concept inclusion for each subset $\mathbf{X} \subseteq \mathbf{M}_\mathsf{p}$ and each subset $\mathbf{Y} \subseteq \mathbf{M}_\mathsf{c}$, and such a concept inclusion is valid in $\mathcal{I}$ if, and only if, the joining implication $\mathbf{X} \to \mathbf{Y}$ is valid in the induced formal context $\mathbb{K}_{\mathcal{I}}$. As we are only interested in axiomatizing those concept inclusions that are valid in $\mathcal{I}$ and are no tautologies, we define the following joining implication set that we shall use as background knowledge on the FCA side.

$$\begin{aligned}\mathcal{S} := \{\, \{C\} \to \{D\} \mid C \in \mathbf{M}_\mathsf{p},\ D \in \mathbf{M}_\mathsf{c},\text{ and } C \sqsubseteq_\emptyset D \,\} \\ \cup \{\, \{C, \exists\, r.\, \mathsf{Self}\} \to \{D\} \mid C \in \mathbf{M}_\mathsf{p},\ r \in \Sigma_\mathsf{R},\ D \in \mathbf{M}_\mathsf{c},\text{ and } C \sqcap \exists\, r.\, \mathsf{Self} \sqsubseteq_\emptyset D \,\}\end{aligned}$$

We will see at the end of this section that the model-based most specific concept descriptions $X^{\mathcal{I}_d}$ can have an exponential size w.r.t. $\mathcal{I}$ and d in $\mathcal{M}^{\leq 1}$. Since the problem of deciding subsumption in Horn-$\mathcal{M}$ is **EXP**-complete, we infer that a naïve approach of computing $\mathcal{S}$ needs double exponential time. However, a more sophisticated analysis yields that most concept inclusions cannot be valid. In particular, a concept description from $\mathbf{M}_\mathsf{p}$ only contains concept names and existential (self-)restrictions and, thus, these can never be subsumed (w.r.t. $\emptyset$) by a concept description from $\mathbf{M}_\mathsf{c}$ containing a negated concept name, a local functionality restriction, a qualified at-least restriction where $n > 1$, or a value restriction. Thus, we conclude from the characterization in Section 4.2 that $\mathcal{S}$ does not contain any implication $\{C\} \to \{D\}$ or $\{C, \exists\, r.\,\mathsf{Self}\} \to \{D\}$ except for the trivial cases where $C = \bot$, $C = D$, or $D = \exists\, r.\,\mathsf{Self}$ (only for the second form), and it can hence be computed in single exponential time. Even in the case where the tautological TBox $\mathcal{S}$ is not that simple, e.g., for another description logic where subsumption is also **EXP**-complete and model-based most specific concept descriptions can have exponential sizes, we could also dispense with the expensive computation of $\mathcal{S}$, since the canonical base can then still be computed in single exponential time with the only drawback that it could contain tautologies.

Furthermore, when computing the MMSC of a conjunction $\bigsqcap \mathbf{C}$ where $\mathbf{C} \subseteq \mathbf{M}_{\mathsf{p}}$ we do not have to do this on the DL side, which is expensive, but it suffices to compute the result $\mathbf{C}^{\mathsf{pc}}$ on the FCA side by applying the derivation operators $\cdot^{\mathsf{p}}$ and $\cdot^{\mathsf{c}}$. The conjunction $\bigsqcap \mathbf{C}^{\mathsf{pc}}$ is then (equivalent to) the MMSC in the DL $\mathcal{M}^{\leq 1}$.

7.2.3 Lemma. *For any subset $\mathbf{C} \subseteq \mathbf{M}_{\mathsf{p}}$, the following equivalence holds true.*

$$\bigsqcap \mathbf{C}^{\mathsf{pc}} \equiv_{\emptyset} (\bigsqcap \mathbf{C})^{\mathcal{I}\mathcal{I}_d^{\mathcal{M}^{\leq 1}}}$$

Proof. The equivalence follows by a suitable variation of the argumentation on Page 210. However, we shall present the full proof in the following. In particular, we show the following subclaims.

1. $\mathbf{C}^{\mathsf{pc}} = \pi_{\mathsf{c}}((\bigsqcap \mathbf{C})^{\mathcal{I}\mathcal{I}_d^{\mathcal{M}^{\leq 1}}})$ where the projection mapping $\pi_{\mathsf{c}} \colon \mathcal{M}^{\leq 1}(\Sigma) \to \wp(\mathbf{M}_{\mathsf{c}})$ is defined by $\pi_{\mathsf{c}}(D) := \{\, E \mid E \in \mathbf{M}_{\mathsf{c}} \text{ and } D \sqsubseteq_{\emptyset} E \,\}$.
2. If C is expressible in terms of $\mathbf{M}_{\mathsf{c}}$, then $C \equiv_{\emptyset} \bigsqcap \pi_{\mathsf{c}}(C)$ holds true.

Statement 2 can be shown in a similar way as Lemma 6.6.3, and the proof of Statement 1 is as follows.

$$\begin{aligned}
\pi_{\mathsf{c}}((\bigsqcap \mathbf{C})^{\mathcal{I}\mathcal{I}_d^{\mathcal{M}^{\leq 1}}}) &= \{\, D \mid D \in \mathbf{M}_{\mathsf{c}} \text{ and } (\bigsqcap \mathbf{C})^{\mathcal{I}\mathcal{I}_d^{\mathcal{M}^{\leq 1}}} \sqsubseteq_{\emptyset} D \,\} \\
&= \{\, D \mid D \in \mathbf{M}_{\mathsf{c}} \text{ and } (\bigsqcap \mathbf{C})^{\mathcal{I}} \subseteq D^{\mathcal{I}} \,\} \\
&= \{\, D \mid D \in \mathbf{M}_{\mathsf{c}} \text{ and } \mathbf{C}^{\mathsf{p}} \subseteq \{D\}^{\mathsf{c}} \,\} \\
&= \{\, D \mid D \in \mathbf{M}_{\mathsf{c}} \text{ and } D \in \mathbf{C}^{\mathsf{pc}} \,\} \\
&= \mathbf{C}^{\mathsf{pc}}
\end{aligned}$$

Our main claim then follows easily in the following way, where the first equivalence follows from Statement 1 and where the second equivalence is a consequence of Statement 2 together with the fact that the concept description $(\bigsqcap \mathbf{C})^{\mathcal{I}\mathcal{I}_d^{\mathcal{M}^{\leq 1}}}$ is expressible in terms of $\mathbf{M}_{\mathsf{c}}$.

$$\bigsqcap \mathbf{C}^{\mathsf{pc}} \equiv_{\emptyset} \bigsqcap \pi_{\mathsf{c}}((\bigsqcap \mathbf{C})^{\mathcal{I}\mathcal{I}_d^{\mathcal{M}^{\leq 1}}}) \equiv_{\emptyset} (\bigsqcap \mathbf{C})^{\mathcal{I}\mathcal{I}_d^{\mathcal{M}^{\leq 1}}}. \qquad \square$$

The main result for inductive learning of Horn-$\mathcal{M}$ concept inclusions is as follows. It states that (the premises of) each pc-implication base of the induced context $\mathbb{K}_{\mathcal{I}}$ give rise to a base of Horn-$\mathcal{M}$ concept inclusions for $\mathcal{I}$.

7.2.4 Theorem (Analog of Theorem 7.1.3)**.** *If $\mathcal{L}$ is a joining implication base for $\mathbb{K}_{\mathcal{I}}$ relative to $\mathcal{S}$, then the following TBox is sound and complete for the* Horn-$\mathcal{M}$ *concept inclusions that are valid in $\mathcal{I}$ and have role depths not exceeding d.*

$$\mathcal{B}_{\mathcal{L}} := \{ \bigsqcap \mathbf{C} \sqsubseteq (\bigsqcap \mathbf{C})^{\mathcal{I}\mathcal{I}_d^{\mathcal{M}\leq 1}} \mid \mathbf{C} \to \mathbf{D} \in \mathcal{L} \}$$

Proof. It suffices to prove that $\mathcal{B}_{\mathcal{L}}$ entails $\mathcal{B}_{\mathbf{M}_{\mathsf{p}}}$, i.e., we show that $\mathcal{B}_{\mathcal{L}}$ entails $\bigsqcap \mathbf{C} \sqsubseteq (\bigsqcap \mathbf{C})^{\mathcal{I}\mathcal{I}_d^{\mathcal{M}\leq 1}}$ for each subset $\mathbf{C} \subseteq \mathbf{M}_{\mathsf{p}}$. Thus, fix some model $\mathcal{J}$ of $\mathcal{B}_{\mathcal{L}}$ and consider some subset $\mathbf{C} \subseteq \mathbf{M}_{\mathsf{p}}$. We define the formal context $\mathbb{K}_{\mathcal{J}} := (\Delta^{\mathcal{J}}, \mathbf{M}, J)$ where $(\delta, C) \in J$ if $\delta \in C^{\mathcal{J}}$.

1. We show that $\mathbb{K}_{\mathcal{J}} \models \mathcal{L}$. Fix some joining implication $\mathbf{C} \to \mathbf{D}$ contained in $\mathcal{L}$. We know that $\mathbf{C}^{J_{\mathsf{p}}} = \mathbf{C}^{J} = (\bigsqcap \mathbf{C})^{\mathcal{J}}$ holds true and, analogously, that $\mathbf{D}^{J_{\mathsf{c}}} = \mathbf{D}^{J} = (\bigsqcap \mathbf{D})^{\mathcal{J}}$ is satisfied. Since

$$\bigsqcap \mathbf{C} \sqsubseteq_{\mathcal{B}_{\mathcal{L}}} (\bigsqcap \mathbf{C})^{\mathcal{I}\mathcal{I}_d^{\mathcal{M}\leq 1}} \equiv_{\emptyset} \bigsqcap \mathbf{C}^{I_{\mathsf{p}} I_{\mathsf{c}}} \sqsubseteq_{\emptyset} \bigsqcap \mathbf{D},$$

   we infer that $(\bigsqcap \mathbf{C})^{\mathcal{J}} \subseteq (\bigsqcap \mathbf{D})^{\mathcal{J}}$. Consequently, we have that $\mathbf{C}^{J_{\mathsf{p}}} \subseteq \mathbf{D}^{J_{\mathsf{c}}}$, that is, the considered joining implication $\mathbf{C} \to \mathbf{D}$ is valid in $\mathbb{K}_{\mathcal{J}}$.

2. We show that $\mathbb{K}_{\mathcal{J}} \models \mathcal{S}$. Let $\{C_1, C_2\} \to \{D\} \in \mathcal{S}$, i.e., $C_1 \sqcap C_2 \sqsubseteq D$ is a tautology. In particular, we infer that $C_1^{\mathcal{J}} \cap C_2^{\mathcal{J}} \subseteq D^{\mathcal{J}}$, which implies $\{C_1\}^{J_{\mathsf{p}}} \cap \{C_2\}^{J_{\mathsf{p}}} \subseteq \{D\}^{J_{\mathsf{c}}}$, i.e., the implication $\{C_1, C_2\} \to \{D\}$ is valid in $\mathbb{K}_{\mathcal{J}}$ as well.

3. We finish the proof by demonstrating that $\bigsqcap \mathbf{C} \sqsubseteq (\bigsqcap \mathbf{C})^{\mathcal{I}\mathcal{I}_d^{\mathcal{M}\leq 1}}$ is valid in $\mathcal{J}$. As $\mathcal{L} \cup \mathcal{S}$ is sound and complete for the joining implications valid in $\mathbb{K}_{\mathcal{I}}$, we have that $\mathcal{L} \cup \mathcal{S} \models \mathbf{C} \to \mathbf{C}^{\mathsf{pc}}$, and so $\mathbf{C} \to \mathbf{C}^{\mathsf{pc}}$ must be valid in $\mathbb{K}_{\mathcal{J}}$ too. Clearly, this shows that the concept inclusion $\bigsqcap \mathbf{C} \sqsubseteq \bigsqcap \mathbf{C}^{\mathsf{pc}}$ is valid in $\mathcal{J}$. Using the equivalence $\bigsqcap \mathbf{C}^{\mathsf{pc}} \equiv_{\emptyset} (\bigsqcap \mathbf{C})^{\mathcal{I}\mathcal{I}_d^{\mathcal{M}\leq 1}}$ yields the claim. □

Instantiating the previous theorem with the canonical pc-implication base from Proposition 1.6.6 now yields the following corollary.

7.2.5 Corollary (Analog of Corollary 7.1.4)**.** *The following* Horn-$\mathcal{M}$ *TBox, called* canonical Horn-$\mathcal{M}$ concept inclusion base *for $\mathcal{I}$ and d, is sound and complete for the* Horn-$\mathcal{M}$ *concept inclusions that are valid in $\mathcal{I}$ and have role depths at most d.*

$$\begin{aligned} \mathsf{Can}_{\mathsf{Horn}\text{-}\mathcal{M}}(\mathcal{I}, d) &:= \{ \bigsqcap \mathbf{C} \sqsubseteq \bigsqcap \mathbf{D} \mid \mathbf{C} \to \mathbf{D} \in \mathsf{Can}_{\mathsf{pc}}(\mathbb{K}_{\mathcal{I}}, \mathcal{S}) \} \\ &= \{ \bigsqcap (\mathbf{P} \cap \mathbf{M}_{\mathsf{p}}) \sqsubseteq \bigsqcap (\mathbf{P} \cap \mathbf{M}_{\mathsf{p}})^{\mathsf{pc}} \mid \mathbf{P} \in \mathsf{PsClo}(\phi^{\mathsf{pc}}_{\mathbb{K}_{\mathcal{I}}}, \mathcal{S}) \} \end{aligned}$$ □

In the sequel of this section, we investigate the computational complexity of computing the canonical Horn-$\mathcal{M}$ concept inclusion base. As it turns out, the complexity is the same as for computing the canonical pc-implication base—both can be obtained in exponential time. Afterwards, we investigate whether we can show that the canonical Horn-$\mathcal{M}$ concept inclusion base has minimal cardinality.

7.2.6 Proposition. *The canonical* Horn-$\mathcal{M}$ *concept inclusion base for a finite interpretation $\mathcal{I}$ and role depth bound $d \geq 1$ can be computed in exponential time with respect to $\mathcal{I}$ and d, and further* *there exist finite interpretations $\mathcal{I}$ for which the canonical* Horn-$\mathcal{M}$ *concept inclusion base cannot be encoded in polynomial space w.r.t.* $|\Delta^{\mathcal{I}}|$.

Proof. The proof is very similar to Theorem 6.1.2 and heavily depends on an argument from Albano [Alb17] too. However, we need to elaborate on the size of model-based most specific concept descriptions as well as on the complexity of computing these. The remaining argumentation is then the same.

We make use of the recursive formula for MMSCs in Section 4.2. Fix some finite interpretation $\mathcal{I}$ such that $\Delta^{\mathcal{I}}$ contains n objects. We start with the $\mathcal{M}^{\leq 1}$ case. In particular, we inductively construct upper estimates[2] u_d such that $||X^{\mathcal{I}_d}|| \preceq u_d$ is satisfied for any subset $X \subseteq \Delta^{\mathcal{I}}$. For $d = 0$, $X^{\mathcal{I}_d}$ can only contain concept names and negations of concept names, i.e., we set $u_0 := 2 \cdot |\Sigma_{\mathsf{C}}| \leq 2 \cdot |\Sigma|$. Furthermore, for computing $X^{\mathcal{I}_d}$ we only need to traverse through Σ_{C} and check, for each concept name A, whether $X \subseteq A^{\mathcal{I}}$ or $X \cap A^{\mathcal{I}} = \emptyset$ is satisfied; clearly, this needs only polynomial time w.r.t. n.

For a depth $d > 0$, $X^{\mathcal{I}_d}$ can contain concept names and negations thereof, one unqualified at-most restriction $\exists \leq n.\, r$ for each role name $r \in \Sigma$ as well as one existential self-restriction $\exists\, r.\, \mathsf{Self}$ for each role name $r \in \Sigma_{\mathsf{R}}$, and $X^{\mathcal{I}_d}$ can further contain qualified at-least restrictions $\exists \geq k.\, r.\, Y^{\mathcal{I}_{d-1}}$ for each $k \in \{1, \ldots, n\}$, $r \in \Sigma_{\mathsf{R}}$, and $Y \subseteq \Delta^{\mathcal{I}}$ as well as universal restrictions $\forall\, r.\, Y^{\mathcal{I}_{d-1}}$ for each role name $r \in \Sigma_{\mathsf{R}}$ and each subset $Y \subseteq \Delta^{\mathcal{I}}$, which yields an upper estimate of $u_d := 2 \cdot |\Sigma_{\mathsf{C}}| + 2 \cdot |\Sigma_{\mathsf{R}}| + (n+1) \cdot |\Sigma_{\mathsf{R}}| \cdot 2^n \cdot u_{d-1}$. Consequently, we obtain the following estimate.

$$||X^{\mathcal{I}_d^{\mathcal{M}^{\leq 1}}}|| \preceq \sum_{k=0}^{d} ((n+1) \cdot |\Sigma_{\mathsf{R}}| \cdot 2^n)^k \cdot 2 \cdot |\Sigma|$$

Furthermore, finding all top-level conjuncts of the form A or $\neg A$ can be done in polynomial time w.r.t. n just like for $d = 0$. Determining all top-level conjuncts of the form $\exists \leq n.\, r$ or $\exists\, r.\, \mathsf{Self}$ can be achieved in polynomial time w.r.t. n as well. For the qualified at-least restrictions and the universal restrictions occurring in the top-level conjunction, the crucial task is to determine the minimal successor sets $\mathsf{Min}(\mathsf{Succ}(X, \mho\, r))$: while for $\mho = \forall$ this is polynomial, it can take exponential time w.r.t. n for $\mho = \exists \geq k$. An induction on d shows that $X^{\mathcal{I}_d}$ can be computed in exponential time w.r.t. n and d.

In order to show that the above exponential bound is tight, we consider the following interpretation $\mathcal{I}_n$ over the signature Σ_n where $(\Sigma_n)_{\mathsf{C}} := \{A_1, \ldots, A_n\}$ and $(\Sigma_n)_{\mathsf{R}} := \{r\}$.

$$\Delta^{\mathcal{I}_n} := \{\delta_1, \ldots, \delta_n\}$$
$$\cdot^{\mathcal{I}_n} : \begin{cases} A_i \mapsto \{\, \delta_j \mid j \in \{1, \ldots, n\} \text{ and } i \neq j \,\} & \text{for each } i \in \{1, \ldots, n\} \\ r \mapsto \{\, (\delta_i, \delta_j) \mid i, j \in \{1, \ldots, n\} \,\} \end{cases}$$

It is then easy to verify that, in $\mathcal{I}_n$, the model-based most specific concept description of $\{\delta_1, \ldots, \delta_n\}$ for the role-depth bound 1 in $\mathcal{M}^{\leq 1}$ contains the mutually $\sqsubseteq_{\emptyset}$-incomparable top-level conjuncts $\exists \geq k.\, r.\, \bigsqcap \mathbf{A}$ for each $k \in \{2, \ldots, n\}$ and each $\mathbf{A} \in \binom{(\Sigma_n)_{\mathsf{C}}}{n-k}$.

[2]Following Knuth [Knu76], we write $f \preceq g$ for $f \in \mathcal{O}(g)$, that is, if $\exists\, c \in \mathbb{R}_+ \; \exists\, n_0 \in \mathbb{N} \; \forall\, n \geq n_0 \colon f(n) \leq c \cdot g(n)$, and then say that f is *asymptotically bounded above* by g.

Similarly for the easier $\mathcal{EL}^*$ case, we can prove by induction on d that we can compute $X^{\mathcal{I}_d}$ in exponential time w.r.t. n and d, and that we get the following estimate.

$$||X^{\mathcal{I}_d^{\mathcal{EL}^*}}|| \preceq \sum_{k=0}^{d} (|\Sigma_\mathsf{R}| \cdot 2^n)^k \cdot |\Sigma|$$

We conclude that model-based most specific concept descriptions always have a size that is at most exponential in n and d and can be computed in exponential time w.r.t. n and d in both description logics $\mathcal{EL}^*$ and $\mathcal{M}^{\leq 1}$. □

Note that in order to save space for representing the model-based most specific concept descriptions, we could also represent them in the form $X^{\mathcal{I}}\restriction_d$ where $X^{\mathcal{I}}$ is the model-based most specific concept description without any bound on the role depth and $E\restriction_d$ denotes the *unraveling* of some concept description E (formulated in a DL with greatest fixed-point semantics) up to role depth d. In general, these unbounded MMSCs $X^{\mathcal{I}}$ only exist in extensions of the considered DL with greatest fixed-point semantics. The advantage is that then the size of $X^{\mathcal{I}}\restriction_d$ is exponential only in $\mathcal{I}$ but not in d.

The author conjectures that, for each finite interpretation $\mathcal{I}$, the canonical Horn-$\mathcal{M}$ concept inclusion base $\mathsf{Can}_{\mathsf{Horn}\text{-}\mathcal{M}}(\mathcal{I}, d)$ has *minimal cardinality* among all Horn-$\mathcal{M}$ concept inclusion bases for $\mathcal{I}$ and d. However, it is not immediately possible to suitably adapt the minimality proof for the $\mathcal{EL}$ case described in [BDK16; Dis11], since not all notions from $\mathcal{EL}$ are available in more expressive description logics. The crucial point is that we need the validity of the following claim, which resembles [Dis11, Lemma 5.16; BDK16, Lemma A.9] for our case of Horn-$\mathcal{M}$.

Claim. *Fix some* Horn-$\mathcal{M}$ *TBox* $\mathcal{T} \cup \{C \sqsubseteq D\}$ *in which all occurring concept descriptions have role depths not exceeding d. Further assume that* $\mathcal{I}$ *is a finite model of* $\mathcal{T}$ *such that, for each subconcept* $\exists\, r.\, X$ *of C, the filler X is (equivalent to) some model-based most specific concept description of* $\mathcal{I}$ *in the description logic* $\mathcal{EL}^*$*; more specifically, we assume that* $Y \equiv Y^{\mathcal{I}\mathcal{I}_{d-1}^{\mathcal{EL}^*}}$ *is satisfied for each* $\exists\, r.\, Y \in \mathsf{Conj}(C)$*. If* $C \not\sqsubseteq_\emptyset D$ *and* $C \sqsubseteq_\mathcal{T} D$*, then* $C \sqsubseteq_\emptyset E$ *and* $C \not\sqsubseteq_\emptyset F$ *holds true for some concept inclusion* $E \sqsubseteq F$ *contained in* $\mathcal{T}$*.*

It might be possible to prove the claim if there is technique for computing most specific consequences w.r.t. Horn-$\mathcal{M}$ TBoxes. However, this is left for future research.

7.3 Bases of Role Inclusions

In Section 3.1 so-called role inclusions have been introduced as expressions of the form $r \sqsubseteq s$ where r and s are role names. Such an axiom is valid in an interpretation $\mathcal{I}$ if $r^{\mathcal{I}}$ is a subset of $s^{\mathcal{I}}$. We shall now describe how we can extend our previous approaches for axiomatizing concept inclusions from interpretations such that also the valid role inclusions are described in a sound and complete manner.

Fix some finite interpretation $\mathcal{I}$. We want to construct a minimal RBox $\mathcal{R}$ from $\mathcal{I}$ that is sound and complete for all role inclusions valid in $\mathcal{I}$. For this purpose, we use the equivalence relation $\equiv_\mathcal{I}$ on Σ_R defined by $r \equiv_\mathcal{I} s$ if $r^{\mathcal{I}} = s^{\mathcal{I}}$ as well as the quasi-order relation $\sqsubseteq_\mathcal{I}$ on Σ_R

defined by $r \sqsubseteq_{\mathcal{I}} s$ if $r^{\mathcal{I}} \subseteq s^{\mathcal{I}}$. Now let Γ_{R} be a set of representatives of the equivalence relation $\equiv_{\mathcal{I}}$, i.e., we have $|\Gamma_{\mathsf{R}} \cap [r]| = 1$ for each role name $r \in \Sigma_{\mathsf{R}}$. Fix some such representative $r \in \Gamma_{\mathsf{R}}$ and consider an arbitrary enumeration $\{r_1, \ldots, r_\ell\}$ of the equivalence class of r. Then define the following RBox.

$$\mathcal{R}_r := \{r_1 \sqsubseteq r_2,\ r_2 \sqsubseteq r_3,\ \ldots,\ r_{\ell-1} \sqsubseteq r_\ell,\ r_\ell \sqsubseteq r_1\}$$

Furthermore, let $\prec_{\mathcal{I}}$ be the neighborhood relation of $\sqsubseteq_{\mathcal{I}}$, that is, $r \prec_{\mathcal{I}} s$ is satisfied if, and only if, $r^{\mathcal{I}}$ is a strict subset of $s^{\mathcal{I}}$ and there is no role name $t \in \Gamma_{\mathsf{R}}$ satisfying $r^{\mathcal{I}} \subsetneq t^{\mathcal{I}} \subsetneq s^{\mathcal{I}}$. The final RBox is then defined as follows.

$$\mathcal{R} := \{\, r \sqsubseteq s \mid r, s \in \Gamma_{\mathsf{R}} \text{ and } r \prec_{\mathcal{I}} s \,\} \cup \bigcup \{\, \mathcal{R}_r \mid r \in \Gamma_{\mathsf{R}} \,\}$$

Obviously, the constructed RBox is sound and complete for $\mathcal{I}$, that is, a role inclusion α is valid in $\mathcal{I}$ if, and only if, $\mathcal{R}$ entails α. It is further easy to see that $\mathcal{R}$ has minimal cardinality among all RBoxes that are sound and complete for $\mathcal{I}$.

For instance, if we want to extend the approach in Section 7.1, then we would only need to modify the definition of the background knowledge $\mathcal{S}$ on Page 210 by replacing the subsumption relation $\sqsubseteq_{\emptyset}$ with $\sqsubseteq_{\mathcal{R}}$, that is, we now define $\mathcal{S}$ as follows.

$$\mathcal{S} := \{\, \mathbf{X} \to \{C\} \mid \mathbf{X} \cup \{C\} \subseteq \mathbf{M} \text{ and } \textstyle\bigsqcap \mathbf{X} \sqsubseteq_{\mathcal{R}} C \,\}.$$

Additionally, we have to add $\mathcal{R}$ to the ontology $\mathcal{B}_{\mathcal{L}}$ in Theorem 7.1.3 and also to the ontology $\mathsf{Can}_{\mathcal{M}}(\mathcal{I}, d)$ in Corollary 7.1.4 and then both ontologies are sound and complete both for the concept inclusions and for the role inclusions valid in $\mathcal{I}$.

8 Axiomatization of $\mathsf{Prob}\text{-}\mathcal{EL}^{\bot}$ Concept Inclusions from Interpretations

Of course, the probabilistic multi-world interpretations from Section 3.6 can be treated as families of directed graphs the vertices and edges of which are labeled and for which there exists a probability measure on this graph family. Thus, results of scientific experiments, e.g., in medicine, psychology, biology, finance, or economy, that are repeated several times can induce probabilistic interpretations in a natural way. Each repetition corresponds to a world, and the results of a particular repetition are encoded in the graph structure of that world.

Within this chapter, we consider the probabilistic variant $\mathsf{Prob}^{>}\mathcal{EL}^{\bot}$ of the description logic $\mathcal{EL}^{\bot}$ from Section 3.6 and develop a suitable axiomatization technique for deducing terminological knowledge from the assertional data given in such probabilistic interpretations. More specifically, we shall provide a method for constructing a set of concept inclusions from probabilistic interpretations in a sound and complete manner. Note that the probabilistic DL $\mathsf{Prob}^{>}\mathcal{EL}^{\bot}$ allows the usage of probability restrictions only with lower probability bounds. This choice shall ease readability; it is not hard to verify that similar results can be obtained when additionally allowing for upper probability bounds. Furthermore, we have chosen $\mathsf{Prob}^{>}\mathcal{EL}^{\bot}$ as a DL that is not closed under Boolean operations in order to prevent the generated ontology from *overfitting*.

For instance, a researcher could collect data on consumption of the drugs ethanol and nicotine as well as on occurrence of serious health effects, e.g., cancer, psychological disorders, pneumonia, etc., such that a world corresponds to a single person and all worlds are equally likely. Then, the resulting probabilistic interpretation could be analyzed with the procedure described in the sequel of this chapter, which produces a *sound and complete axiomatization* of it. In particular, the outcome would then be a *logical-statistical evaluation* of the input data, and could include concept inclusions like the following.[1]

$$\begin{aligned}&\exists\,\mathsf{drinks}.\,(\mathsf{Alcohol}\sqcap\exists\,\mathsf{frequency}.\,\mathsf{TwiceAWeek})\\ \sqsubseteq\;&\mathsf{d}\geq\tfrac{1}{10}.\,\exists\,\mathsf{suffersFrom}.\,\mathsf{Cancer}\sqcap\mathsf{d}\geq\tfrac{1}{5}.\,\exists\,\mathsf{develops}.\,\mathsf{PsychologicalDisorder}\end{aligned}$$

$$\begin{aligned}&\exists\,\mathsf{smokes}.\,\mathsf{Tobacco}\\ \sqsubseteq\;&\mathsf{d}\geq\tfrac{1}{4}.\,\exists\,\mathsf{suffersFrom}.\,\mathsf{Cancer}\sqcap\mathsf{d}\geq\tfrac{1}{3}.\,\exists\,\mathsf{suffersFrom}.\,\mathsf{Pneumonia}\end{aligned}$$

The first one states that any person who drinks alcohol twice a week suffers from cancer with a probability of at least 10 % and develops some psychological disorder with a probability of at

[1] Please note that, although similar statements with adjusted probability bounds do hold true in the real world, the mentioned statements are not taken from any publications in the medical or psychological domain. The author has simply read Wikipedia articles and then wrote down the statements.

least 20 %; the second one expresses that each person smoking tobacco suffers from cancer with a probability of at least 25 % and suffers from pneumonia with a probability of at least $33\frac{1}{3}$ %.

However, one should be cautious when interpreting the results, since the procedure, like any other existing statistical evaluation technique, cannot distinguish between *causality* and *correlation*. It might as well be the case that an application of our procedure yields concept inclusions of the following type.

$$\begin{aligned}&\mathsf{d} \geq \tfrac{1}{2}.\,\exists\,\mathsf{develops}.\,\mathsf{PsychologicalDisorder}\\ \sqsubseteq\ &\mathsf{d} \geq \tfrac{1}{3}.\,\exists\,\mathsf{drinks}.\,(\mathsf{Alcohol} \sqcap \exists\,\mathsf{frequency}.\,\mathsf{Daily})\end{aligned}$$

The above concept inclusion reads as follows: any person who develops a psychological disorder with a probability of at least 50 % drinks alcohol on a daily basis with a probability of at least $33\frac{1}{3}$ %.

It should further be mentioned that for evaluating observations by means of the proposed technique no hypotheses are necessary. Instead, the procedure simply provides a sound and complete axiomatization of the observations, and the output is, on the one hand, not too hard to be understood by humans (at least if, the probability depth is not set too high) and, on the other hand, well-suited to be further processed by a computer.

This chapter also resolves an issue found by Franz Baader with the techniques described by the author in [Kri15a, Sections 5 and 6]. In particular, the concept inclusion base proposed therein in Proposition 2 is only complete with respect to those probabilistic interpretations that are also quasi-uniform with a probability ε of each world. Herein, we describe a more sophisticated axiomatization technique of not necessarily quasi-uniform probabilistic interpretations that ensures completeness of the constructed concept inclusion base with respect to *all* probabilistic interpretations, but which only allows for bounded nesting of probability restrictions. It is not hard to generalize the following results to a more expressive probabilistic description logic, for example to a probabilistic variant Prob-$\mathcal{M}$ of the description logic $\mathcal{M}$, for which an axiomatization technique is available, see Chapter 7. That way, we can regain the same, or even a greater, expressivity as the author has tried to tackle in [Kri15a], but without the possibility to nest probability restrictions arbitrarily deep. A first step for resolving this issue has already been made in [Kri18a] where a nesting of probability restrictions is not supported. As a follow-up, [Kri19d] expands on these results in [Kri18a] with the goal to allow for nesting of probabilistically quantified concept descriptions. In the sequel of this chapter, we shall now present the results from [Kri19d].

8.1 Bases of $\mathsf{Prob}_n^{>}\mathcal{EL}^\bot$ Concept Inclusions

In this section, we shall develop an effective method for axiomatizing $\mathsf{Prob}_n^{>}\mathcal{EL}^\bot$ concept inclusions which are valid in a given finite probabilistic interpretation. After defining the appropriate notion of a *concept inclusion base*, we show how this problem can be tackled using the aforementioned existing results on computing concept inclusion bases in $\mathcal{EL}^\bot$ from Chapter 6 or in [Dis11; BDK16]. More specifically, we devise an extension of the given signature by finitely many probability restrictions $\mathsf{d} \gtrdot p.\,C$ that are treated as additional concept names, and we define so-called *scalings* $\mathcal{I}_n$ of the input probabilistic interpretation $\mathcal{I}$ which are (single-world) interpretations that suitably interpret these new concept names and, furthermore, such that there is a correspondence between $\mathsf{Prob}_n^{>}\mathcal{EL}^{\bot}$ CIs valid in $\mathcal{I}$ and CIs valid in $\mathcal{I}_n$. This very correspondence makes it possible to utilize the above mentioned techniques for axiomatizing CIs in $\mathcal{EL}^{\bot}$.

8.1.1 Definition. A $\mathsf{Prob}_n^{>}\mathcal{EL}^{\bot}$ *concept inclusion base* for a probabilistic interpretation $\mathcal{I}$ is a $\mathsf{Prob}_n^{>}\mathcal{EL}^{\bot}$ terminological box $\mathcal{T}$ which is *sound* for $\mathcal{I}$, that is, $C \sqsubseteq_{\mathcal{T}} D$ implies $C \sqsubseteq_{\mathcal{I}} D$ for each $\mathsf{Prob}_n^{>}\mathcal{EL}^{\bot}$ concept inclusion $C \sqsubseteq D$,[2] and *complete* for $\mathcal{I}$, that is, $C \sqsubseteq_{\mathcal{I}} D$ only if $C \sqsubseteq_{\mathcal{T}} D$ for any $\mathsf{Prob}_n^{>}\mathcal{EL}^{\bot}$ concept inclusion $C \sqsubseteq D$. △

The following definition is to be read inductively, that is, initially some objects are defined for the probability depth $n = 0$, and if the objects are defined for the probability depth n, then these are used to define the next objects for the probability depth $n + 1$.[3]

A first important step is to significantly reduce the possibilities of concept descriptions occurring as a filler in the probability restrictions, that is, of fillers C in expressions $\mathsf{d} \gtrdot p.\, C$. As it turns out, it suffices to consider only those fillers that are model-based most specific concept descriptions of some suitable *scaling* of the given probabilistic interpretation $\mathcal{I}$. We shall demonstrate that there are only finitely many such fillers—provided that the given probabilistic interpretation $\mathcal{I}$ is finite.

As next step, we restrict the probability bounds p occurring in probability restrictions $\mathsf{d} \gtrdot p.\, C$. Apparently, it is sufficient to consider only those values p that can occur when evaluating the extension of $\mathsf{Prob}_{n+1}^{>}\mathcal{EL}^{\bot}$ concept descriptions in $\mathcal{I}$, which, obviously, are the values $\mathbb{P}^{\mathcal{I}}\{\delta \in C^{\mathcal{I}}\}$ for any $\delta \in \Delta^{\mathcal{I}}$ and any $C \in \mathsf{Prob}_n^{>}\mathcal{EL}^{\bot}(\Sigma)$. In the sequel of this section we will see that there are only finitely many such probability bounds if $\mathcal{I}$ is finite.

Having found a finite number of representatives for probability bounds as well as a finite number of fillers to be used in probability restrictions for each probability depth n, we now show that we can treat these finitely many concept descriptions as concept names of a signature Γ_n extending Σ in a way such that any $\mathsf{Prob}_n^{>}\mathcal{EL}^{\bot}$ concept inclusion is valid in $\mathcal{I}$ if, and only if, that concept inclusion projected onto the extended signature Γ_n is valid in a suitable *scaling* of $\mathcal{I}$ that interprets Γ_n.

8.1.2 Definition. Fix some probabilistic interpretation $\mathcal{I}$ over a signature Σ. Then, we define the following objects Γ_n, $\mathcal{I}_n$, and $P_{\mathcal{I},n}$ by simultaneous induction over $n \in \mathbb{N}$.

1. The nth signature Γ_n is inductively defined as follows. We set $(\Gamma_0)_\mathsf{C} := \Sigma_\mathsf{C}$ and $(\Gamma_0)_\mathsf{R} := \Sigma_\mathsf{R}$. The subsequent signatures are then obtained in the following way.

$$(\Gamma_{n+1})_\mathsf{C} := (\Gamma_n)_\mathsf{C} \cup \left\{ \mathsf{d} \geq p.\, X^{\mathcal{I}_n} \;\middle|\; \begin{array}{l} p \in P_{\mathcal{I},n} \setminus \{0\},\; X \subseteq \Delta^{\mathcal{I}} \times \Omega^{\mathcal{I}}, \\ \text{and } \bot \not\equiv_\emptyset X^{\mathcal{I}_n} \not\equiv_\emptyset \top \end{array} \right\}$$

$$(\Gamma_{n+1})_\mathsf{R} := \Sigma_\mathsf{R}$$

2. The *nth scaling* of $\mathcal{I}$ is defined as the interpretation $\mathcal{I}_n$ over Γ_n that has the following

[2] Of course, soundness is equivalent to $\mathcal{I} \models \mathcal{T}$.

[3] The probability depth $\mathsf{pd}(C)$ of a concept description C has been defined on Page 66.

components.

$$\Delta^{\mathcal{I}_n} := \Delta^{\mathcal{I}} \times \Omega^{\mathcal{I}}$$

$$\cdot^{\mathcal{I}_n} \colon \begin{cases} A \mapsto \{\, (\delta,\omega) \mid \delta \in A^{\mathcal{I}(\omega)} \,\} & \text{for each } A \in (\Gamma_n)_{\mathsf{C}} \\ r \mapsto \{\, ((\delta,\omega),(\epsilon,\omega)) \mid (\delta,\epsilon) \in r^{\mathcal{I}(\omega)} \,\} & \text{for each } r \in (\Gamma_n)_{\mathsf{R}} \end{cases}$$

3. The nth set $P_{\mathcal{I},n}$ of probability values for $\mathcal{I}$ is given as follows.

$$P_{\mathcal{I},n} := \{\, \mathbb{P}^{\mathcal{I}}\{\delta \in C^{\mathcal{I}}\} \mid \delta \in \Delta^{\mathcal{I}} \text{ and } C \in \mathsf{Prob}_n^{>}\mathcal{EL}^\bot(\Sigma) \,\}$$

Furthermore, for each $p \in [0,1)$, we define $(p)^+_{\mathcal{I},n}$ as the next value in $P_{\mathcal{I},n}$ above p, that is, we set

$$(p)^+_{\mathcal{I},n} := \bigwedge\{\, q \mid q \in P_{\mathcal{I},n} \text{ and } q > p \,\}. \qquad \triangle$$

Of course, we have that $\{0,1\} \subseteq P_{\mathcal{I},n}$ for each $n \in \mathbb{N}$. Note that $\mathcal{I}_{n+1}$ extends $\mathcal{I}_n$ by also interpreting the additional concept names in $(\Gamma_{n+1})_{\mathsf{C}} \setminus (\Gamma_n)_{\mathsf{C}}$, that is, the restriction $\mathcal{I}_{n+1}\restriction_{\Gamma_n}$ equals $\mathcal{I}_n$. Similarly, $\mathcal{I}_n\restriction_{\Gamma_m}$ and $\mathcal{I}_m$ are equal if $m \leq n$.

As explained earlier, it suffices to only consider fillers in probabilistic restrictions that are model-based most specific concept descriptions. More specifically, the following holds true.

8.1.3 Lemma. *Consider a probabilistic interpretation $\mathcal{I}$ and a concept description $\mathsf{d} \geqslant p.\, C$ such that $C \in \mathcal{EL}^\bot(\Gamma_n)$ for some $n \in \mathbb{N}$. The concept equivalence $\mathsf{d} \geqslant p.\, C \equiv \mathsf{d} \geqslant p.\, C^{\mathcal{I}_n\mathcal{I}_n}$ is valid in $\mathcal{I}$.*

Proof. Using structural induction on $\mathcal{EL}^\bot$ concept descriptions C over Σ, it can be proven that $C^{\mathcal{I}(\omega)} \times \{\omega\} = C^{\mathcal{I}_n} \cap (\Delta^{\mathcal{I}} \times \{\omega\})$ is satisfied for each world $\omega \in \Omega^{\mathcal{I}}$ and for every $n \in \mathbb{N}$, cf. [Kri17d, Lemma 16]. For extending this result to $\mathsf{Prob}_n^{>}\mathcal{EL}^\bot$ concept descriptions that are in $\mathcal{EL}^\bot(\Gamma_n)$, we need to show a further inductive case for the probability restrictions $\mathsf{d} \geqslant p.\, C$. As one quickly verifies, the following equalities hold true for all probability restrictions $\mathsf{d} \geqslant p.\, C \in (\Gamma_n)_{\mathsf{C}}$.

$$(\mathsf{d} \geqslant p.\, C)^{\mathcal{I}(\omega)} \times \{\omega\} = \{\, (\delta,\omega) \mid \mathbb{P}^{\mathcal{I}}\{\delta \in C^{\mathcal{I}}\} \geqslant p \,\} = (\mathsf{d} \geqslant p.\, C)^{\mathcal{I}_n} \cap (\Delta^{\mathcal{I}} \times \{\omega\})$$

It follows that, for any $n \in \mathbb{N}$ and for each concept description $C \in \mathcal{EL}^\bot(\Gamma_n)$, it holds true that $C^{\mathcal{I}(\omega)} = \pi_1(C^{\mathcal{I}_n} \cap (\Delta^{\mathcal{I}} \times \{\omega\}))$ (where π_1 projects pairs to their first components). By applying well-known properties of Galois connections we obtain that $C^{\mathcal{I}(\omega)} = C^{\mathcal{I}_n\mathcal{I}_n\mathcal{I}(\omega)}$, and so $\mathbb{P}^{\mathcal{I}}\{\delta \in C^{\mathcal{I}}\} = \mathbb{P}^{\mathcal{I}}\{\delta \in C^{\mathcal{I}_n\mathcal{I}_n\mathcal{I}}\}$ holds true. $\square$

The above lemma does not hold true for arbitrary fillers C, but only for fillers that can (syntactically) also be seen as $\mathcal{EL}^\bot$ concept descriptions over Γ_n. However, this does not cause any problems, since we can simply project any other filler onto this signature Γ_n. In particular, we define projections of arbitrary $\mathsf{Prob}_n^{>}\mathcal{EL}^\bot$ concept descriptions onto the signature Γ_n in the following manner.

8.1.4 Definition. Fix some $n \in \mathbb{N}$ as well as a probabilistic interpretation $\mathcal{I}$. The *nth projection* $\pi_{\mathcal{I},n}(C)$ of a $\mathsf{Prob}_n^{>}\mathcal{EL}^\bot$ concept description C with respect to $\mathcal{I}$ is obtained from C by replacing subconcepts of the form $\mathsf{d} \gtrdot p.\, D$ with suitable elements from $(\Gamma_n)_{\mathsf{C}}$ and, more specifically, we recursively define it as follows. We set $\pi_{\mathcal{I},0}(C) := C$ for each concept description $C \in \mathcal{EL}^{\bot}(\Sigma)$.[4] The subsequent projections are then given in the following manner.

$$\begin{aligned}
\pi_{\mathcal{I},n+1}(A) &:= A && \text{if } A \in \Sigma_{\mathsf{C}} \cup \{\bot, \top\} \\
\pi_{\mathcal{I},n+1}(C \sqcap D) &:= \pi_{\mathcal{I},n+1}(C) \sqcap \pi_{\mathcal{I},n+1}(D) \\
\pi_{\mathcal{I},n+1}(\exists\, r.\, C) &:= \exists\, r.\, \pi_{\mathcal{I},n+1}(C) \\
\pi_{\mathcal{I},n+1}(\mathsf{d} \gtrdot p.\, C) &:= \begin{cases} \bot & \text{if } {\gtrdot}\, p = {>}\, 1 \\ \top & \text{else if } {\gtrdot}\, p = {\geq}\, 0 \\ \bot & \text{else if } (\pi_{\mathcal{I},n}(C))^{\mathcal{I}_{n+1}\mathcal{I}_{n+1}} \equiv_{\emptyset} \bot \\ \top & \text{else if } (\pi_{\mathcal{I},n}(C))^{\mathcal{I}_{n+1}\mathcal{I}_{n+1}} \equiv_{\emptyset} \top \\ \mathsf{d} \geq p.(\pi_{\mathcal{I},n}(C))^{\mathcal{I}_{n+1}\mathcal{I}_{n+1}} & \text{else if } {\gtrdot} = {\geq} \text{ and } p \in P_{\mathcal{I},n+1} \\ \mathsf{d} \geq (p)^{+}_{\mathcal{I},n+1}.(\pi_{\mathcal{I},n}(C))^{\mathcal{I}_{n+1}\mathcal{I}_{n+1}} & \text{else} \end{cases}
\end{aligned}$$

△

For technical details, we introduce further notation: we denote by $\pi'_{\mathcal{I},n+1}(\mathsf{d} \gtrdot p.\, C)$ and $\pi''_{\mathcal{I},n+1}(\mathsf{d} \gtrdot p.\, C)$ the concept description that is obtained from the projection $\pi_{\mathcal{I},n+1}(\mathsf{d} \gtrdot p.\, C)$ by replacing $(\pi_{\mathcal{I},n}(C))^{\mathcal{I}_{n+1}\mathcal{I}_{n+1}}$ with $\pi_{\mathcal{I},n}(C)$ and C, respectively.

For some of the upcoming proofs we need the following lemma, which expresses the fact that the probabilistic restriction constructor—more specifically, each mapping $C \mapsto \mathsf{d} \gtrdot p.\, C$ for $\gtrdot \in \{\geq, >\}$ and $p \in [0,1] \cap \mathbb{Q}$—is monotonic.

8.1.5 Lemma. *Consider a $\mathsf{Prob}^{>}\mathcal{EL}^{\bot}$ terminological box $\mathcal{T}$ and a $\mathsf{Prob}^{>}\mathcal{EL}^{\bot}$ concept inclusion $C \sqsubseteq D$. Then, $C \sqsubseteq_{\mathcal{T}} D$ implies $\mathsf{d} \gtrdot p.\, C \sqsubseteq_{\mathcal{T}} \mathsf{d} \gtrdot p.\, D$ for any $\gtrdot \in \{\geq, >\}$ and for each $p \in [0,1] \cap \mathbb{Q}$.*

Proof. Fix some model $\mathcal{I}$ of $\mathcal{T}$ and let $\mathbb{P}^{\mathcal{I}}\{\delta \in C^{\mathcal{I}}\} \gtrdot p$ for an object $\delta \in \Delta^{\mathcal{I}}$. From $\mathcal{T} \models C \sqsubseteq D$ we infer that, for each world $\omega \in \Omega^{\mathcal{I}}$, it holds true that $\delta \in C^{\mathcal{I}(\omega)}$ implies $\delta \in D^{\mathcal{I}(\omega)}$. Consequently, we have that $\{\delta \in C^{\mathcal{I}}\} \subseteq \{\delta \in D^{\mathcal{I}}\}$ and, thus, $\mathbb{P}^{\mathcal{I}}\{\delta \in D^{\mathcal{I}}\} \gtrdot p$ due to the monotonicity of the probability measure $\mathbb{P}^{\mathcal{I}}$. □

Usually, projection mappings in mathematics are *idempotent*. It is easy to verify by induction over n that this also holds true for our projection mappings $\pi_{\mathcal{I},n}$ which we have just defined. This justifies the naming choice. Furthermore, we can show that the mappings $\pi_{\mathcal{I},n}$ are *extensive*, i.e., projecting some $\mathsf{Prob}_n^{>}\mathcal{EL}^{\bot}$ concept description C onto the nth signature Γ_n yields a more specific concept description, cf. the next lemma. Furthermore, the mappings $\pi_{\mathcal{I},n}$ are monotonic—a fact that can be proven by induction over n as well. As a corollary, it follows that each mapping $\pi_{\mathcal{I},n}$ is a *closure operator*. However, please just take this as a side note, since we do not need the two additional properties of idempotency and monotonicity within this chapter.

8.1.6 Lemma. *Assume that $\mathcal{I}$ is a probabilistic interpretation, let $n \in \mathbb{N}$, and fix some $\mathsf{Prob}_n^{>}\mathcal{EL}^{\bot}$ concept description C. Then, it holds true that $\pi_{\mathcal{I},n}(C) \sqsubseteq_{\emptyset} C$.*

Proof. We prove by induction over k that $\pi_{\mathcal{I},k}(C) \sqsubseteq_{\emptyset} C$ for any $k \leq n$ and every $C \in \mathsf{Prob}_k^{>}\mathcal{EL}^{\bot}(\Sigma)$. Due to equality of C and its 0th projection $\pi_{\mathcal{I},0}(C)$, the base case for

[4] Note that $\Sigma = \Gamma_0$.

$k = 0$ is obvious. For the inductive step for $k + 1$, we continue with an (inner) induction on the structure of C. All cases, except the case for a probability restriction $\mathsf{d} > p.\, D$, are easy. We claim that $\pi_{\mathcal{I},k+1}(\mathsf{d} > p.\, D) \sqsubseteq_\emptyset \mathsf{d} > p.\, D$. Since $(\pi_{\mathcal{I},k}(D))^{\mathcal{I}_{k+1}\mathcal{I}_{k+1}} \sqsubseteq_\emptyset \pi_{\mathcal{I},k}(D)$, it follows by means of Lemma 8.1.5 that

$$\pi_{\mathcal{I},k+1}(\mathsf{d} > p.\, D) \sqsubseteq_\emptyset \pi'_{\mathcal{I},k+1}(\mathsf{d} > p.\, D). \tag{8.1.A}$$

The induction hypothesis together with Lemma 8.1.5 implies that

$$\pi'_{\mathcal{I},k+1}(\mathsf{d} > p.\, D) \sqsubseteq_\emptyset \pi''_{\mathcal{I},k+1}(\mathsf{d} > p.\, D) \tag{8.1.B}$$

and, furthermore, it is apparent that

$$\pi''_{\mathcal{I},k+1}(\mathsf{d} > p.\, D) \sqsubseteq_\emptyset \mathsf{d} > p.\, D. \tag{8.1.C}$$

In summary, Equations (8.1.A) – (8.1.C) show that $\pi_{\mathcal{I},k+1}(\mathsf{d} > p.\, D)$ is subsumed by $\mathsf{d} > p.\, D$ with respect to the empty TBox. □

As a crucial observation regarding projections, we see that—within our given probabilistic interpretation $\mathcal{I}$—we do not have to distinguish between any $\mathsf{Prob}^{>}_{n}\mathcal{EL}^{\bot}$ concept description C and its nth projection $\pi_{\mathcal{I},n}(C)$, since the upcoming lemma shows that both always possess the same extension in each world of $\mathcal{I}$. Simply speaking, the signatures Γ_n contain enough building bricks to describe anything that happens within $\mathcal{I}$ up to a probability depth of n.

8.1.7 Lemma. *Assume that $\mathcal{I}$ is a probabilistic interpretation, let $n \in \mathbb{N}$, and consider some $\mathsf{Prob}^{>}_{n}\mathcal{EL}^{\bot}$ concept description C. Then, C and its nth projection $\pi_{\mathcal{I},n}(C)$ have the same extension in any world of $\mathcal{I}$.*

Proof. We show the claim by means of an outer induction on n and an inner induction on the structure of C. The outer base case for $n = 0$ is trivial, since then C and its projection $\pi_{\mathcal{I},0}(C)$ are equal. We proceed with the outer inductive case for $n + 1$ and a structural induction on C. Then, according to the definition of an $(n + 1)$st projection, the only non-trivial case considers probabilistic restrictions occurring in C. It is readily verified that $\mathsf{d} > p.\, E$ and $\pi''_{\mathcal{I},n+1}(\mathsf{d} > p.\, E)$ have the same extension in each world of $\mathcal{I}$. Using the fact that E is a $\mathsf{Prob}^{>}_{n}\mathcal{EL}^{\bot}$ concept description together with the outer induction hypothesis, we infer that $\pi''_{\mathcal{I},n+1}(\mathsf{d} > p.\, E)$ and $\pi'_{\mathcal{I},n+1}(\mathsf{d} > p.\, E)$ have the same extension in each world of $\mathcal{I}$ too. An application of Lemma 8.1.3 now yields that, in every world of $\mathcal{I}$, also the extensions of $\pi'_{\mathcal{I},n+1}(\mathsf{d} > p.\, E)$ and $\pi_{\mathcal{I},n+1}(\mathsf{d} > p.\, E)$ are the same. □

As a last important statement on the properties of the projection mappings, we now demonstrate that validity of some concept inclusion $C \sqsubseteq D$ with a probability depth not exceeding n is equivalent to validity of the projected concept inclusion $\pi_{\mathcal{I},n}(C) \sqsubseteq \pi_{\mathcal{I},n}(D)$ in the scaling $\mathcal{I}_n$. This is a key proposition for the upcoming construction of a concept inclusion base for $\mathcal{I}$.

8.1.8 Proposition. *Let $n \in \mathbb{N}$, and consider a probabilistic interpretation $\mathcal{I}$ as well as some $\mathsf{Prob}^{>}_{n}\mathcal{EL}^{\bot}$ concept inclusion $C \sqsubseteq D$. Then, $C \sqsubseteq D$ is valid in $\mathcal{I}$ if, and only if, the nth projected concept inclusion $\pi_{\mathcal{I},n}(C) \sqsubseteq \pi_{\mathcal{I},n}(D)$ is valid in the nth scaling $\mathcal{I}_n$.*

Proof. We start with observing that, according to Lemma 8.1.7, $C \sqsubseteq D$ is valid in $\mathcal{I}$ if, and only if, $\pi_{\mathcal{I},n}(C) \sqsubseteq \pi_{\mathcal{I},n}(D)$ is valid in $\mathcal{I}$. Then, the equivalence of $\mathcal{I} \models \pi_{\mathcal{I},n}(C) \sqsubseteq \pi_{\mathcal{I},n}(D)$ and $\mathcal{I}_n \models \pi_{\mathcal{I},n}(C) \sqsubseteq \pi_{\mathcal{I},n}(D)$ follows from the very definition of the nth scaling $\mathcal{I}_n$ and the fact that the projections $\pi_{\mathcal{I},n}(C)$ and $\pi_{\mathcal{I},n}(D)$ can be interpreted as $\mathcal{EL}^{\bot}$ concept descriptions over Γ_n. □

Now we go on to considering the sets $P_{\mathcal{I},n}$ of essential probability values. As we have already claimed, these sets are always finite—provided that the fixed probabilistic interpretation is finite. In order to prove this, we need the following statement.

8.1.9 Proposition. *For each probabilistic interpretation $\mathcal{I}$ and any $n \in \mathbb{N}$, the following equation is satisfied.*

$$P_{\mathcal{I},n} = \{\, \mathbb{P}^{\mathcal{I}}\{\delta \in X^{\mathcal{I}_n\mathcal{I}}\} \mid \delta \in \Delta^{\mathcal{I}} \text{ and } X \subseteq \Delta^{\mathcal{I}} \times \Omega^{\mathcal{I}} \,\}$$

Proof. Fix some $\mathsf{Prob}_n^{>}\mathcal{EL}^{\bot}$ concept description C. From Lemma 8.1.7 we infer that

$$\mathbb{P}^{\mathcal{I}}\{\delta \in C^{\mathcal{I}}\} = \mathbb{P}^{\mathcal{I}}\{\delta \in (\pi_{\mathcal{I},n}(C))^{\mathcal{I}}\}$$

holds true. In the proof of Lemma 8.1.3 we have shown that $\mathbb{P}^{\mathcal{I}}\{\delta \in D^{\mathcal{I}}\} = \mathbb{P}^{\mathcal{I}}\{\delta \in D^{\mathcal{I}_n\mathcal{I}_n\mathcal{I}}\}$ holds true for each $\mathsf{Prob}^{>}\mathcal{EL}^{\bot}$ concept description D over Σ that (syntactically) is also an $\mathcal{EL}^{\bot}$ concept description over Γ_n. Hence, we can use this identity for $D := \pi_{\mathcal{I},n}(C)$, which yields that

$$\mathbb{P}^{\mathcal{I}}\{\delta \in (\pi_{\mathcal{I},n}(C))^{\mathcal{I}}\} = \mathbb{P}^{\mathcal{I}}\{\delta \in (\pi_{\mathcal{I},n}(C))^{\mathcal{I}_n\mathcal{I}_n\mathcal{I}}\}.$$

Of course, we have that $(\pi_{\mathcal{I},n}(C))^{\mathcal{I}_n} \subseteq \Delta^{\mathcal{I}} \times \Omega^{\mathcal{I}}$.

Since C is an arbitrary concept description, we conclude that $P_{\mathcal{I},n}$ is a subset of $\{\, \mathbb{P}^{\mathcal{I}}\{\delta \in X^{\mathcal{I}_n\mathcal{I}}\} \mid \delta \in \Delta^{\mathcal{I}}$ and $X \subseteq \Delta^{\mathcal{I}} \times \Omega^{\mathcal{I}} \,\}$. The reverse set inclusion is trivial. □

For most, if not all, practical use cases we can argue that the given probabilistic interpretation $\mathcal{I}$ can be assumed as finite. Utilizing some of our previous results then implies that each nth scaling of $\mathcal{I}$ is finite as well. More specifically, the following is satisfied.

8.1.10 Corollary. *If $\mathcal{I}$ is a finite probabilistic interpretation, then it holds true that, for each $n \in \mathbb{N}$, the subset $\Gamma_n \setminus \Sigma$ of the nth signature is finite, the nth scaling $\mathcal{I}_n$ is finite and has a finite active signature, and the nth set $P_{\mathcal{I},n}$ of probability values is finite and satisfies $P_{\mathcal{I},n} \subseteq \mathbb{Q}$.* □

As already mentioned, we want to make use of existing techniques that allow for axiomatizing interpretations in the description logic $\mathcal{EL}^{\bot}$. In order to do so, we need to be sure that the semantics of $\mathcal{EL}^{\bot}$ and its probabilistic sibling $\mathsf{Prob}^{>}\mathcal{EL}^{\bot}$ are not too different, or expressed alternatively, that there is a suitable correspondence between (non-probabilistic) entailment in $\mathcal{EL}^{\bot}$ and (probabilistic) entailment in $\mathsf{Prob}^{>}\mathcal{EL}^{\bot}$. A more sophisticated formulation is presented in the following lemma. Beforehand, note that there might exist $\mathcal{EL}^{\bot}$ concept inclusions over the nth signature Γ_n which, when treated as $\mathsf{Prob}^{>}\mathcal{EL}^{\bot}$ concept inclusions, are tautologies, i.e., are valid in each probabilistic interpretation. As there is no need for such tautologies in the computed base of concept inclusions, we collect these in a TBox (symbolized by $\mathcal{B}$ or $\mathcal{B}_n$ in the following) that is used as background knowledge during the computation.

8.1.11 Lemma. *Let $\mathcal{T}$ be a $\mathsf{Prob}^{>}\mathcal{EL}^\bot$ TBox, and assume that $\mathcal{B}$ is a set that consists of tautological $\mathsf{Prob}^{>}\mathcal{EL}^\bot$ concept inclusions, i.e., $\emptyset \models \mathcal{B}$. If $C \sqsubseteq D$ is a $\mathsf{Prob}^{>}\mathcal{EL}^\bot$ concept inclusion that is entailed by $\mathcal{T} \cup \mathcal{B}$ with respect to non-probabilistic entailment, then $C \sqsubseteq D$ is also entailed by $\mathcal{T}$ with respect to probabilistic entailment.*

Proof. Fix some signature Σ, let $\mathcal{T} \cup \mathcal{B} \models C \sqsubseteq D$ (non-probabilistically), and consider some probabilistic interpretation $\mathcal{I}$ such that $\mathcal{I} \models \mathcal{T}$. Of course, it also holds true that $\mathcal{I} \models \mathcal{B}$. We extend Σ to the signature Γ defined as follows: $\Gamma_\mathsf{C} := \Sigma_\mathsf{C} \cup \{\, \mathsf{d} \gtrdot p.\, C \mid {\gtrdot} \in \{\geq, >\},\ p \in [0,1] \cap \mathbb{Q},\ \text{and } C \in \mathsf{Prob}^{>}\mathcal{EL}^\bot(\Sigma) \,\}$ and $\Gamma_\mathsf{R} := \Sigma_\mathsf{R}$. It is apparent that, syntactically, $\mathcal{EL}^\bot(\Gamma) = \mathsf{Prob}^{>}\mathcal{EL}^\bot(\Sigma)$ holds true. Furthermore, we define the interpretation $\mathcal{J}$ where $\Delta^\mathcal{J} := \Delta^\mathcal{I} \times \Omega^\mathcal{I}$, $A^\mathcal{J} := \{\, (\delta, \omega) \mid \delta \in A^{\mathcal{I}(\omega)} \,\}$ for each $A \in \Gamma_\mathsf{C}$, and $r^\mathcal{J} := \{\, ((\delta, \omega), (\epsilon, \omega)) \mid (\delta, \epsilon) \in r^{\mathcal{I}(\omega)} \,\}$ for each $r \in \Gamma_\mathsf{R}$. We can show with structural induction that $C^\mathcal{J} = \bigcup\{\, C^{\mathcal{I}(\omega)} \times \{\omega\} \mid \omega \in \Omega^\mathcal{I} \,\}$ for any $C \in \mathcal{EL}^\bot(\Gamma)$. Consequently, $\mathcal{I} \models E \sqsubseteq F$ is equivalent to $\mathcal{J} \models E \sqsubseteq F$ for each $\mathsf{Prob}^{>}\mathcal{EL}^\bot$ concept inclusion $E \sqsubseteq F$. It follows that $\mathcal{J} \models \mathcal{T} \cup \mathcal{B}$, and we infer that $\mathcal{J} \models C \sqsubseteq D$, which implies that $\mathcal{I} \models C \sqsubseteq D$. As $\mathcal{I}$ is an arbitrary model of $\mathcal{T}$, we can safely conclude that $\mathcal{T} \models C \sqsubseteq D$ (probabilistically). □

As final step, we show that each concept inclusion base of the probabilistic scaling $\mathcal{I}_n$ induces a $\mathsf{Prob}_n^{>}\mathcal{EL}^\bot$ concept inclusion base of $\mathcal{I}$. While soundness is easily verified, completeness follows from the fact that $C \sqsubseteq_\mathcal{T} \pi_{\mathcal{I},n}(C) \sqsubseteq_\mathcal{T} \pi_{\mathcal{I},n}(D) \sqsubseteq_\emptyset D$ holds true for every valid $\mathsf{Prob}_n^{>}\mathcal{EL}^\bot$ concept inclusion $C \sqsubseteq D$ of $\mathcal{I}$.

8.1.12 Theorem. *Fix a number $n \in \mathbb{N}$ and some finite probabilistic interpretation $\mathcal{I}$. If $\mathcal{T}_n$ is a concept inclusion base for the nth scaling $\mathcal{I}_n$ with respect to some set $\mathcal{B}_n$ of tautological $\mathsf{Prob}_n^{>}\mathcal{EL}^\bot$ concept inclusions used as background knowledge, then the following terminological box $\mathcal{T}$ is a $\mathsf{Prob}_n^{>}\mathcal{EL}^\bot$ concept inclusion base for $\mathcal{I}$.*

$$\mathcal{T} := \mathcal{T}_n \cup \bigcup\{\, \mathcal{U}_{\mathcal{I},\ell} \mid \ell \in \{1, \ldots, n\} \,\} \quad \textit{where}$$
$$\mathcal{U}_{\mathcal{I},\ell} := \{\, \mathsf{d} > p.\, X^{\mathcal{I}_\ell \mathcal{I}_\ell} \sqsubseteq \mathsf{d} \geq (p)^{+}_{\mathcal{I},\ell}.\, X^{\mathcal{I}_\ell \mathcal{I}_\ell} \mid p \in P_{\mathcal{I},\ell} \setminus \{1\} \textit{ and } X \subseteq \Delta^\mathcal{I} \times \Omega^\mathcal{I} \,\}$$

Proof. Soundness is clearly satisfied. We proceed with showing completeness; thus, fix some $\mathsf{Prob}_n^{>}\mathcal{EL}^\bot$ concept inclusion $C \sqsubseteq D$ which is valid in $\mathcal{I}$. We shall demonstrate the validity of the following subsumptions.

$$C \sqsubseteq_\mathcal{T} \pi_{\mathcal{I},n}(C) \sqsubseteq_\mathcal{T} \pi_{\mathcal{I},n}(D) \sqsubseteq_\emptyset D$$

According to Lemma 8.1.6, it holds true that $\pi_{\mathcal{I},n}(D) \sqsubseteq_\emptyset D$. Proposition 8.1.8 immediately yields that $\pi_{\mathcal{I},n}(C) \sqsubseteq \pi_{\mathcal{I},n}(D)$ is valid in the nth scaling $\mathcal{I}_n$. Since $\mathcal{T}_n$ is complete for $\mathcal{I}_n$ relative to $\mathcal{B}_n$, it follows that $\mathcal{T}_n \cup \mathcal{B}_n$ entails $\pi_{\mathcal{I},n}(C) \sqsubseteq \pi_{\mathcal{I},n}(D)$ with respect to non-probabilistic entailment and, thus, $\mathcal{T}$ entails $\pi_{\mathcal{I},n}(C) \sqsubseteq \pi_{\mathcal{I},n}(D)$ with respect to probabilistic entailment.

It remains to show that $C \sqsubseteq_\mathcal{T} \pi_{\mathcal{I},n}(C)$ holds true; we do so by proving with an induction on k that $C \sqsubseteq_\mathcal{T} \pi_{\mathcal{I},k}(C)$ holds true for each $k \leq n$ and $C \in \mathsf{Prob}_k^{>}\mathcal{EL}^\bot(\Sigma)$. The base case where $k = 0$ is obvious, since each $\mathsf{Prob}_0^{>}\mathcal{EL}^\bot$ concept description C equals its 0th projection $\pi_{\mathcal{I},0}(C)$. For the inductive case for $k+1$, we proceed with an inner induction on the structure of C. The only non-trivial case considers probability restrictions $\mathsf{d} \gtrdot p.\, E$. Of course, E is then a $\mathsf{Prob}_k^{>}\mathcal{EL}^\bot$

concept description, and the induction hypothesis yields that $E \sqsubseteq_{\mathcal{T}} \pi_{\mathcal{I},k}(E)$. As an immediate consequence from Lemma 8.1.5 we infer that

$$\mathsf{d} \geqslant p.\, E \sqsubseteq_{\mathcal{T}} \mathsf{d} \geqslant p.\, \pi_{\mathcal{I},k}(E). \tag{8.1.D}$$

Furthermore, the concept inclusion $\pi_{\mathcal{I},k}(E) \sqsubseteq (\pi_{\mathcal{I},k}(E))^{\mathcal{I}_{k+1}\mathcal{I}_{k+1}}$ is valid in $\mathcal{I}_{k+1}$. Since $\mathcal{I}_n\restriction_{\Gamma_{k+1}} = \mathcal{I}_{k+1}$ holds true, and both $\pi_{\mathcal{I},k}(E)$ and $(\pi_{\mathcal{I},k}(E))^{\mathcal{I}_{k+1}\mathcal{I}_{k+1}}$ (syntactically) are $\mathcal{EL}^{\bot}$ concept descriptions over $\Gamma_{k+1} \subseteq \Gamma_n$, we conclude that the considered concept inclusion $\pi_{\mathcal{I},k}(E) \sqsubseteq (\pi_{\mathcal{I},k}(E))^{\mathcal{I}_{k+1}\mathcal{I}_{k+1}}$ is valid in $\mathcal{I}_n$. Consequently, this CI is (non-probabilistically) entailed by $\mathcal{T}_n \cup \mathcal{B}_n$ and, according to Lemma 8.1.11, it is hence (probabilistically) entailed by $\mathcal{T}$. An application of Lemma 8.1.5 now shows that

$$\mathsf{d} \geqslant p.\, \pi_{\mathcal{I},k}(E) \sqsubseteq_{\mathcal{T}} \mathsf{d} \geqslant p.(\pi_{\mathcal{I},k}(E))^{\mathcal{I}_{k+1}\mathcal{I}_{k+1}}. \tag{8.1.E}$$

Obviously, the subset $\mathcal{U}_{\mathcal{I},k+1}$ of $\mathcal{T}$ entails the concept inclusion

$$\mathsf{d} \geqslant p.(\pi_{\mathcal{I},k}(E))^{\mathcal{I}_{k+1}\mathcal{I}_{k+1}} \sqsubseteq \pi''_{\mathcal{I},k+1}(\mathsf{d} \geqslant p.(\pi_{\mathcal{I},k}(E))^{\mathcal{I}_{k+1}\mathcal{I}_{k+1}}),$$

and since the latter concept description is exactly $\pi_{\mathcal{I},k+1}(\mathsf{d} \geqslant p.\, E)$ we infer that

$$\mathsf{d} \geqslant p.(\pi_{\mathcal{I},k}(E))^{\mathcal{I}_{k+1}\mathcal{I}_{k+1}} \sqsubseteq_{\mathcal{T}} \pi_{\mathcal{I},k+1}(\mathsf{d} \geqslant p.\, E). \tag{8.1.F}$$

Putting the results from Equations (8.1.D) – (8.1.F) together now demonstrates the truth of the claim that $\mathsf{d} \geqslant p.\, E$ is subsumed by $\pi_{\mathcal{I},k+1}(\mathsf{d} \geqslant p.\, E)$ with respect to $\mathcal{T}$. □

As already mentioned in Chapter 6 and according to [Kri15c], a suitable such concept inclusion base $\mathcal{T}_n$ for the nth scaling $\mathcal{I}_n$ with respect to background knowledge $\mathcal{B}_n$ exists and can be computed effectively, namely the canonical base $\mathsf{Can}(\mathcal{I}_n, \mathcal{B}_n)$. This enables us to immediately draw the following conclusion.

8.1.13 Corollary. *Let $\mathcal{I}$ be a finite probabilistic interpretation, fix some $n \in \mathbb{N}$, and let $\mathcal{B}_n$ denote the set of all $\mathcal{EL}^{\bot}$ concept inclusions over Γ_n that are tautological with respect to probabilistic entailment, i.e., are valid in every probabilistic interpretation. Then, the* canonical base *for $\mathcal{I}$ and probability depth n that is defined as*

$$\mathsf{Can}(\mathcal{I}, n) := \mathsf{Can}(\mathcal{I}_n, \mathcal{B}_n) \cup \bigcup \{\, \mathcal{U}_{\mathcal{I},\ell} \mid \ell \in \{1, \ldots, n\} \,\}$$

is a $\mathsf{Prob}_n^{>}\mathcal{EL}^{\bot}$ concept inclusion base for $\mathcal{I}$, and it can be computed effectively.

Eventually, we close our investigations with a complexity analysis of the problem of actually computing the canonical base $\mathsf{Can}(\mathcal{I}, n)$. As it turns out, this computation is—in terms of computational complexity—not more expensive than the corresponding axiomatization task in $\mathcal{EL}^{\bot}$, cf. [Kri18c, Proposition 2]; both in $\mathcal{EL}^{\bot}$ and in $\mathsf{Prob}_n^{>}\mathcal{EL}^{\bot}$ concept inclusion bases can be computed in exponential time.

However, this result only holds true if we dispense with the pre-computation of the tautological background knowledge $\mathcal{B}_n$ at all. First of all, Γ_n can have exponential size, and there are d-exponentially many $\mathcal{EL}^{\bot}$ concept descriptions over some fixed signature. Thus, a naïve enumeration of $\mathcal{B}_n$ is too expensive. However, also computing the implicative background knowledge on the FCA side with utilizing some $\mathsf{Prob}^{>}\mathcal{EL}^{\bot}$ reasoner on demand is too expensive as well. This is due to the fact that one needs to enumerate all implications $\mathbf{C} \to \{D\}$ where $\mathbf{C} \cup \{D\}$ is a subset of the attribute set of the induced formal context of $\mathcal{I}_d$. On the one hand, the number of such implications is exponential in the size of the attribute set and this attribute set can contain exponentially many concept descriptions that can each have an exponential size. On the other hand, we have already seen that deciding subsumption in $\mathsf{Prob}^{>}\mathcal{EL}^{\bot}$ is an **EXP**-complete problem. Even a more sophisticated approach—much like the oracle defined on Page 190—that directly uses a $\mathsf{Prob}^{>}\mathcal{EL}^{\bot}$ reasoner to close a pseudo-intent against the tautological $\mathsf{Prob}^{>}\mathcal{EL}^{\bot}$ concept inclusions does not solve this problem due to the exponential size of the attributes of the induced context and subsumption being **EXP**-complete for $\mathsf{Prob}^{>}\mathcal{EL}^{\bot}$.

Hence, if we define $\mathsf{Can}^*(\mathcal{I}, n) := \mathsf{Can}(\mathcal{I}_n) \cup \bigcup\{\, \mathcal{U}_{\mathcal{I},\ell} \mid \ell \in \{1, \ldots, n\}\,\}$, then $\mathsf{Can}^*(\mathcal{I}, n)$ is still a $\mathsf{Prob}_n^{>}\mathcal{EL}^{\bot}$ concept inclusion base for $\mathcal{I}$ but, as a drawback, might contain tautological axioms. Its advantage is that it can always be computed in exponential time.

8.1.14 Proposition. *For any finite probabilistic interpretation $\mathcal{I}$ and any $n \in \mathbb{N}$, the canonical base $\mathsf{Can}^*(\mathcal{I}, n)$ can be computed in deterministic time that is exponential in $|\Delta^{\mathcal{I}}| \cdot |\Omega^{\mathcal{I}}|$ and polynomial in n, i.e., in deterministic time $\mathcal{O}(n^p \cdot 2^{|\Delta^{\mathcal{I}}| \cdot |\Omega^{\mathcal{I}}|})$ for some fixed p. Furthermore, there are finite probabilistic interpretations $\mathcal{I}$ for which a concept inclusion base cannot be encoded in polynomial space with respect to $|\Delta^{\mathcal{I}}| \cdot |\Omega^{\mathcal{I}}| \cdot n$.*

Proof. The statements are obtained as corollaries of Theorem 6.1.2 for the following reasons. The sum of two rational numbers can be computed in polynomial time. This result is necessary for determining the complexity of evaluating a $\mathsf{Prob}^{>}\mathcal{EL}^{\bot}$ concept description C in some world of a probabilistic interpretation $\mathcal{I}$, which is polynomial in $|C| + |\Delta^{\mathcal{I}}| \cdot |\Omega^{\mathcal{I}}|$. For each $n \in \mathbb{N}$, it holds true that the cardinality of $P_{\mathcal{I},n}$ is bounded by $|\Delta^{\mathcal{I}}| \cdot 2^{|\Delta^{\mathcal{I}}| \cdot |\Omega^{\mathcal{I}}|}$, i.e., $|P_{\mathcal{I},n}|$ is exponential in $|\Delta^{\mathcal{I}}| \cdot |\Omega^{\mathcal{I}}|$. For each $n \in \mathbb{N}$, we have that the cardinality of $\Gamma_n \setminus \Sigma$ is bounded by $n \cdot |\Delta^{\mathcal{I}}| \cdot 2^{2 \cdot |\Delta^{\mathcal{I}}| \cdot |\Omega^{\mathcal{I}}|}$, i.e., $|\Gamma_n \setminus \Sigma|$ is exponential in $|\Delta^{\mathcal{I}}| \cdot |\Omega^{\mathcal{I}}|$. Furthermore, each element in $\Gamma_n \setminus \Sigma$ has an encoding of exponential size, and we conclude that $\Gamma_n \setminus \Sigma$ also has an encoding of exponential size. □

We have devised an effective procedure for computing finite axiomatizations of observations that are represented as probabilistic interpretations. More specifically, we have shown how concept inclusion bases—TBoxes that are sound and complete for the input data set—can be constructed in the probabilistic description logic $\mathsf{Prob}^{>}\mathcal{EL}^{\bot}$. In a complexity analysis we found that we can always compute a canonical base in exponential time.

Future research is possible in various directions. One could extend the results to a more expressive probabilistic DL, e.g., to Prob-$\mathcal{M}$, or one could include upper probability bounds. Furthermore, for increasing the practicability of the approach, it could be investigated how the construction of a concept inclusion base can be made *incremental* or *interactive*. It might be the case that there already exists a TBox and there are new observations in form of a probabilistic interpretation; the goal is then to construct a TBox being a base for the CIs that are entailed by the existing knowledge as well as hold true in the new observations. While this would represent a *push* approach of learning, future research could tackle the *pull* approach as well, i.e., equip the procedure with expert interaction such that an exploration of partial observations is made possible.

Additionally, it is worth investigating whether the proposed approach could be optimized; for instance, one could check if equivalent results can be obtained with a subset of Γ_n or with another extended signature. Currently, it is also unknown whether, for each finite probabilistic interpretation $\mathcal{I}$, there is some finite bound n on the probability depth such that each $\mathsf{Prob}_n^{>}\mathcal{EL}^{\bot}$ concept inclusion base for $\mathcal{I}$ is also sound and complete for *all* $\mathsf{Prob}^{>}\mathcal{EL}^{\bot}$ concept inclusions that are valid in $\mathcal{I}$—much like this is the case for the role depth in $\mathcal{EL}^{\bot}$.

8.2 Axiomatization of Concept Inclusions from Probabilistic ABoxes

A *simple probabilistic ABox* is a finite set of axioms of the form $a \sqsubseteq \mathsf{d} \bowtie p.\, A$ or $(a,b) \sqsubseteq \mathsf{d} \bowtie p.\, r$ where $a, b \in \Sigma_\mathsf{I}$ are individual names, $A \in \Sigma_\mathsf{C}$ is a concept name, $r \in \Sigma_\mathsf{R}$ is a role name, $\bowtie \in \{<, \leq, =, \geq, >\}$ is a relation symbol, and $p \in [0,1] \cap \mathbb{Q}$ is a rational probability value. A probabilistic interpretation $\mathcal{I}$ is a model of $a \sqsubseteq \mathsf{d} \bowtie p.\, A$ if $\mathbb{P}^{\mathcal{I}}\{a^{\mathcal{I}} \in A^{\mathcal{I}}\} \bowtie p$ is satisfied, and a model of $(a,b) \sqsubseteq \mathsf{d} \bowtie p.\, r$ if $\mathbb{P}^{\mathcal{I}}\{(a^{\mathcal{I}}, b^{\mathcal{I}}) \in r^{\mathcal{I}}\} \bowtie p$ holds true. We assume that each individual has a fixed extension over all worlds, i.e., $a^{\mathcal{I}(\omega)} = a^{\mathcal{I}(\psi)}$ holds true for all worlds $\omega, \psi \in \Omega^{\mathcal{I}}$.

Much like in Section 6.8.4, we now apply the Open World Assumption, the Unique Name Assumption, and the Domain Closure Assumption in order to be able to derive terminological knowledge from such a simple probabilistic ABox. As for the non-probabilistic case, assuming unique names and a closed domain implies that, for each probabilistic interpretation $\mathcal{I}$, there is always a bijection between the domain $\Delta^{\mathcal{I}}$ and the set Σ_I of individual names. Without loss of generality, we thus assume that $\Delta^{\mathcal{I}} = \Sigma_\mathsf{I}$ is always satisfied, and then $\mathcal{I}$ is called a *probabilistic* Σ_I*-interpretation*. In order to make our methods from axiomatizing probabilistic interpretations applicable to probabilistic ABoxes, we will now describe how to select a suitable Σ_I-model of $\mathcal{A}$.

Given a simple probabilistic ABox $\mathcal{A}$ over a finite signature Σ, it is easy to verify that there are (at most) $|\Sigma_\mathsf{I}| \cdot 2^{|\Sigma_\mathsf{C}|} \cdot |\Sigma_\mathsf{I}|^2 \cdot 2^{|\Sigma_\mathsf{R}|}$ different worlds in a probabilistic Σ_I-interpretation being a model of $\mathcal{A}$. It is straightforward to transform the ABox $\mathcal{A}$ into a *linear program* for which the solutions correspond to Σ_I-models of $\mathcal{A}$. In particular, a solution vector describes a probability measure on the set of worlds.

Entropy is a measure for describing the amount of *randomness* or *uninformativeness* of a particular system, and has variants in different fields, e.g., in *thermodynamics* (where it was defined first), in *statistical mechanics*, but also in *probability theory* (where it describes a lack of *predictability*), and in *information theory* as well (where it is also called *Shannon entropy*, and describes the amount of information contained in a message sent through a channel). In particular, one of the first works on entropy in information theory was published by Shannon and Weaver in 1949 [SW49]. Later, there were several researchers who adapted the notion of entropy to *probabilistic logic*. For example, a motivating and wide introduction can be found in the book *The Uncertain Reasoner's Companion - a Mathematical Perspective* [Par94] written by Paris. A thorough introduction to reasoning under maximum entropy semantics can be found therein on pages 76 ff. Paris shows that reasoning under maximum entropy semantics somehow implements common sense reasoning, i.e., yields conclusions which are expected by humans when dealing with probabilistic theories or data sets. In particular, a knowledge base entails an axiom w.r.t. maximum entropy semantics if, and only if, the axiom is valid in the (unique) model of the knowledge base for which entropy is maximal. Put simply, for a given incomplete knowledge base (e.g., an ABox), the maximum entropy model is a model without any bias on probabilities that cannot be precisely deduced from the knowledge base.

More specifically, we define that a *maximum entropy* Σ_{I}*-model* of $\mathcal{A}$ is some Σ_{I}-model for which the entropy

$$\mathbb{H}(\mathcal{I}) := -\sum(\mathbb{P}^{\mathcal{I}}\{\omega\} \cdot \mathsf{log}(\mathbb{P}^{\mathcal{I}}\{\omega\}) \mid \omega \in \Omega^{\mathcal{I}})$$

is maximal. It can be shown that there is always a unique maximum entropy model, and we thus denote it by $\mathcal{I}_{\mathcal{A}}^{\mathsf{ME}}$. Of course, we can formulate a linear optimization program such that its unique solution describes the probability measure of $\mathcal{I}_{\mathcal{A}}^{\mathsf{ME}}$. Eventually, we can utilize the approach from the previous section and construct a $\mathsf{Prob}^{>}\mathcal{EL}^{\bot}$ concept inclusion base for $\mathcal{I}_{\mathcal{A}}^{\mathsf{ME}}$, yielding a *maximum entropy base* of $\mathsf{Prob}^{>}\mathcal{EL}^{\bot}$ concept inclusions for $\mathcal{A}$.

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