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# Scale Invariant Quantum Field Theories

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# Summary


## Abstract

Scale symmetry usually is explicitly broken by quantum corrections due to the necessity of Regularisation and Renormalisation of quantum corrections. However, this can be avoided by using a manifestly scale invariant Regularisation where the Renormalisation scale is replaced by a dynamical Dilaton-dependent Renormalisation function, i.e. $\mu \longrightarrow \mu(\sigma)$. In this case, scale invariance is only broken spontaneously by the non-vanishing VEV of the Dilaton and all scales, including the Renormalisation scale, are generated dynamically via SSB. In this thesis, the concept and implications of spontaneously broken quantum scale invariance (QSI), realised via scale invariant dimensional Regularisation (SIDReg), are discussed for different theories. The QSI 2 Scalar Model is considered up to the 2-loop level. Moreover, it is also discussed in the framework of gauge theories, with particular emphasis on a consistent formulation and the physically relevant scattering process $e^- \, e^+ \longrightarrow \mu^- \, \mu^+$ at the 1-loop level. This scattering process is especially analysed w.r.t. new finite and divergent quantum corrections due to spontaneously broken QSI and the IR-finiteness of the corresponding cross section. Finally, a complete QSI Standard Model is presented as a potential candidate for BSM physics and its effective potential is determined at the 1-loop level.

## Kurzdarstellung

Skalensymmetrie wird für gewöhnlich durch Quantenkorrekturen explizit gebrochen, aufgrund der Notwendigkeit zur Regularisierung und Renormierung dieser Quantenkorrekturen. Dies kann jedoch vermieden werden, indem eine manifest-skaleninvariante Regularisierung gewählt wird, bei welcher die Renormierungsskala durch eine dynamische Dilaton-abhängige Renormierungsfunktion ersetzt wird, d.h. $\mu \longrightarrow \mu(\sigma)$. In diesem Fall wird Skalensymmetrie durch den nicht-verschwindenden VEV des Dilatons lediglich spontan gebrochen und alle Skalen, einschließlich der Renormierungsskala, werden dynamisch via SSB erzeugt. In dieser Masterarbeit werden das Konzept und die Konsequenzen von spontan gebrochener Quantenskaleninvarianz (QSI), realisiert über skaleninvariante dimensionale Regularisierung (SIDReg), für verschiedene Theorien diskutiert. Das QSI 2 Scalar Model wird bis auf 2-Schleifen Niveau betrachtet. Außerdem, wird es auch im Rahmen von Eichtheorien besprochen. Besonders in Bezug auf eine konsistente Formulierung und den physikalisch relevanten Streuprozess $e^- \, e^+ \longrightarrow \mu^- \, \mu^+$ auf 1-Schleifen Niveau. Dieser Streuprozess wird insbesondere bzgl. neuer endlicher und divergenter Quantenkorrekturen aufgrund von spontan gebrochener QSI und der IR-Endlichkeit des zugehörigen Streuquerschnitts untersucht. Abschließend wird ein komplettes QSI Standardmodell als ein möglicher Kandidat für BSM Physik vorgestellt und dessen effektives Potential wird auf 1-Schleifen Niveau bestimmt.

# Contents

# 1. Introduction

Scale symmetry is often used to address the hierarchy and the cosmological constant problem. Additionally, it is also a key feature of universality. In particular, scale invariance plays an important role in the framework of statistical field theories in order to describe phase transitions, and thus is not only of interest for particle physics but also for condensed matter physics. Moreover, as a subset of conformal symmetry, it also has a connection to conformal field theories, and thus AdS/CFT-correspondence. However, quantum corrections usually spoil scale invariance. The reason for this are divergences emerging in loop-calculations that need to be regularised and the fact that every Regularisation introduces a dimensionful parameter, the Renormalisation scale. In other words, quantum corrections require Regularisation and Renormalisation, and thus break scale symmetry explicitly, due to the introduction of the Renormalisation scale, which is referred to as anomalous breaking of scale symmetry.

The anomalous breaking of scale symmetry can be avoided by using a manifestly scale invariant Regularisation, as originally proposed in [9], and discussed, inter alia, in [2, 11, 12, 13, 14, 17, 20, 21, 28, 34]. In this thesis a scale invariant version of dimensional Regularisation (DReg) is defined as scale invariant dimensional Regularisation (SIDReg) in section 2.1 and used throughout the thesis. In particular, this is achieved by replacing the Renormalisation scale in DReg with a dynamical and Dilaton-dependent Renormalisation function in SIDReg, i.e. $\mu \longrightarrow \mu(\sigma)$. The "usual" Renormalisation scale as well as all other mass scales in the theory, such as particle masses, are dynamically generated via spontaneous symmetry breaking (SSB) of scale symmetry with the Dilaton $\sigma$ as associated Goldstone boson. Thus, there is no initial mass scale in theory that could explicitly break scale invariance implying the absence of anomalous scale symmetry breaking. In other words, a classically scale invariant theory regularised using SIDReg admits *spontaneously broken quantum scale symmetry*.

The purpose of this thesis is to provide a self consistent introduction to spontaneously broken quantum scale invariance (QSI), model building, i.e. providing several QSI theories, and the calculation of Green functions with non-vanishing external momenta, such as self energies and scattering amplitudes as well as a cross section as an actual physical observable, in the context of QSI theories. In particular, a 2 Scalar Model, two different variations of QSI Quantum Electrodynamics (QED) as well as a full QSI Standard Model are discussed, and beside the calculation of the effective potential, the self-energies in the QSI 2 Scalar Model are computed up to the 2-loop level. Furthermore, the scattering process $e^- e^+ \longrightarrow \mu^- \mu^+$, including the associated cross section, is considered at the 1-loop level in the framework of a QSI QED. Note that in this thesis global scale symmetry is considered.

In chapter 2, the concept of quantum scale invariance (QSI), its realisation as well as its implications are introduced and discussed. Particular emphasis is put on the 2 Scalar Model in the context of QSI, as it the simplest model for dynamical SSB of quantum scale symmetry, and therefore is an excellent model to illustrate the concepts of spontaneously broken QSI. Further, it is the major part of the Higgs sector in a full QSI Standard Model, and thus is physically of particular relevance. In chapter 3, the effective potential of the 2 Scalar Model is determined up to the 2-loop level and it is shown that it is indeed manifestly quantum scale invariant as well as the corresponding counterterms. Moreover, the self energies are evaluated at the 1-loop and the 2-loop level, working in the broken phase of the theory, and it is shown that one still obtains the same manifestly QSI counterterms, which has not been done in the literature so far.

Chapter 4 provides a detailed discussion w.r.t. the consistent formulation of Gauge theories in the context of quantum scale symmetry. Furthermore, two variations of a QSI QED are introduced as they are needed for the consideration of muon production in the next chapter. In chapter 5, the well-known QED scattering process $e^- \, e^+ \longrightarrow \mu^- \, \mu^+$ is discussed at the 1-loop level in the framework of a quantum scale invariant QED, which has not been done for QSI theories so far. In this context, a conjecture about IR-divergences in the framework of spontaneously broken quantum scale symmetry, as well as new quantum corrections arising from these IR-divergences and evanescent interactions due to QSI, is formulated and exemplarily proven.

A full quantum scale invariant Standard Model is introduced in chapter 6. The Higgs potential of a QSI Standard Model has already been discussed in [13], however, a complete QSI Standard Model has not been provided to full extent in the literature so far. Subsequently, the 1-loop effective potential is determined and discussed in chapter 7. In contrast to [13], this is done in a more Feynman diagrammatic approach using the background field method.

A concluding summary and outlook is to be found in the last chapter of this thesis.

# 2. Quantum Scale Symmetry

In this chapter, the concept of Quantum Scale Invariance (QSI) and its realisation via a manifestly scale invariant Regularisation, as originally proposed in [9], and discussed, inter alia, in [2, 11, 12, 13, 14, 17, 20, 21, 28, 34] is introduced. Because scale symmetry is not observed in the real world, i.e. scale symmetry is broken in the real world, in this thesis only spontaneously broken (quantum) scale symmetry is considered. In order to illustrate the realisation of spontaneously broken QSI and its implications, it is exemplarily discussed for the 2 Scalar Model. This model has not only already been discussed in the context of spontaneously broken QSI in [11, 14, 21] but is also of great interest for physically relevant models of the real world since it is the major part of the Higgs sector in a QSI Standard Model, as discussed in section 2.2, chapter 6 and [13]. Thus, a detailed discussion of the 2 Scalar Model (at tree-level) and its Renormalisation in the framework of QSI is provided in this chapter. The purpose of this chapter is to provide a more or less complete and self-consistent introduction to the theoretical concepts of spontaneously broken quantum scale symmetry and its implications.

## 2.1. Quantum Scale Invariance

First, consider (global) scale symmetry transformations which are given by

$$\begin{aligned} x^\mu &\longmapsto x'^\mu = s\, x^\mu = e^{-\lambda}\, x^\mu \\ \phi(x) &\longmapsto \phi'(x) = s^{-\Delta_\phi}\, \phi(s^{-1}x) = e^{\lambda\,\Delta_\phi}\, \phi(e^\lambda x) \end{aligned} \tag{2.1}$$

where $\phi$ is a scalar field.

A theory, given by an action $S[\phi]$, is said to be scale invariant iff $S[\phi]$ is invariant under the above scaling transformations (2.1).

Let $S$ be the action for a real scalar field $\phi$, as discussed in [42], given by

$$S[\phi] = \int d^D x \left( \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \lambda_\phi\, \phi^p \right) \tag{2.2}$$

Considering the kinetic term first in order to determine the scaling dimension $\Delta_\phi$ of scalar fields $\phi$, one obtains under scaling transformations (2.1)

$$\begin{aligned} \frac{\partial\phi(x)}{\partial x^\mu} &\longmapsto \frac{\partial\phi'(x)}{\partial x^\mu} = s^{-\Delta_\phi}\, \frac{\partial\phi(s^{-1}x)}{\partial x^\mu} = s^{-\Delta_\phi - 1}\, \frac{\partial\phi(y)}{\partial y^\mu} \\ \Longrightarrow \int d^D x \frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) &\longmapsto s^{D-2(\Delta_\phi+1)} \int d^D y \frac{1}{2}\partial_\mu\phi(y)\partial^\mu\phi(y) \\ &= s^{D-2(\Delta_\phi+1)} \int d^D x \frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) \end{aligned}$$

where $y^\mu := s^{-1}\, x^\mu$ and $d^D x = s^D\, d^D y$.
Hence, the kinetic term of the action $S$ is scale invariant iff

$$\Delta_\phi = \frac{D-2}{2} \xrightarrow[D\to 4]{} 1 \tag{2.3}$$

For the mass term, however, one obtains

$$\begin{aligned} -\frac{1}{2}\int d^D x\; m^2\, \phi^2(x) \longmapsto &-\frac{1}{2}\, s^{D-2\Delta_\phi} \int d^D y\; m^2\, \phi^2(y) \\ =&-\frac{1}{2}\, s^{D-2\Delta_\phi} \int d^D x\; m^2\, \phi^2(x) \\ =&-\frac{1}{2}\, s^{2} \int d^D x\; m^2\, \phi^2(x) \end{aligned}$$

where (2.3) has been used in the last equality. It can be seen that the mass term is not scale invariant, and thus scale invariance requires $m \equiv 0$, which is not a surprising result since a scale invariant theory *must not* contain any dimensionful quantities that could serve as a scale. Without any mass terms or dimensionful quantities, however, there is no (absolute) reference scale, and thus physics is equivalent on all scales, i.e. scale invariant [8]. This can also be seen from the 4-divergence of the corresponding Noether-current, as discussed in [8].

For the remaining interaction term in (2.2), one finds

$$-\int d^D x\; \lambda_\phi\, \phi^p(x) \longmapsto -s^{D-p\,\Delta_\phi} \int d^D y\; \lambda_\phi\, \phi^p(y) = -s^{D-p\,\Delta_\phi} \int d^D x\; \lambda_\phi\, \phi^p(x)$$

which is scale invariant iff $p = \frac{D}{\Delta_\phi} = \frac{2D}{D-2} \xrightarrow[D\to 4]{} 4$.

However, in a reasonable 4 dimensional theory $p$ is set to $p = 4$ and remains at this value even when the theory is analytically continued to $D = 4 - 2\epsilon$ dimensions in DReg. Hence, for $p = 4$, one obtains

$$-\int d^D x\; \lambda_\phi\, \phi^4(x) \longmapsto -s^{4-D} \int d^D x\; \lambda_\phi\, \phi^4(x)$$

which is not scale invariant in $D \neq 4$ dimensions. The reason for this is that in $D$ dimensions the coupling constant $\lambda_\phi$ is not dimensionless anymore, but has (an anomalous) mass dimension $[\lambda_\phi] = 4 - D$. In DReg, a Renormalisation scale $\mu_0$ is introduced as $\lambda_\phi \longrightarrow \mu_0^{4-D}\, \lambda_\phi$ in order to keep coupling constants dimensionless. This, however, does not solve the problem since $\mu_0$ is a fixed (mass) scale that spoils scale symmetry. Hence, the Renormalisation scale in DReg explicitly breaks scale invariance in $D \neq 4$ dimensions. Naively, one might think that this is not a problem since ultimately one goes back to $D = 4$ dimensions after Renormalisation. However, scale invariance will still be broken explicitly, even for $D \to 4$, due to divergences emerging in loop-calculations that need to be regularised and the fact that every Regularisation (not only DReg) needs a dimensionful parameter. In [39] it has explicitly been shown for 2-point Green functions at the 1-loop level in QED that scale invariance is explicitly broken ∀ Renormalisation schemes. Thus, quantum corrections explicitly break scale symmetry which is called anomalous breaking of scale symmetry because it is a symmetry of the classical theory / action that is explicitly broken at the quantum level. How this problem can be resolved will be discussed below in this section.

Before this is done, however, the Dilatation current, i.e. the Noether current associated with scale symmetry is derived. Consider infinitesimal scaling transformations

$$s = e^{-\lambda} = 1 - \lambda + \mathcal{O}\left(\lambda^2\right), \quad \text{where} \quad \lambda \ll 1 \tag{2.4}$$

acting on scalar fields

$$\phi_i(x) \longmapsto \phi_i'(x) = s^{-\Delta_\phi}\, \phi_i(s^{-1}x) = \phi_i(x) + \lambda\, \left(\Delta_\phi + x^\mu \partial_\mu\right) \phi_i(x) + \mathcal{O}\left(\lambda^2\right)$$

and on a generic Lagrangian $\mathcal{L}$

$$\mathcal{L}(x) \longmapsto s^{-\Delta_l}\, \mathcal{L}(s^{-1}x) = \mathcal{L}(x) + \lambda\, \left(\Delta_l + x^\mu \partial_\mu\right) \mathcal{L}(x) + \mathcal{O}\left(\lambda^2\right)$$

where $\Delta_l$ is the scaling dimension of the Lagrangian which is determined by demanding

$$\int d^D x\; \mathcal{L}(x) \longmapsto s^{D-\Delta_l} \int d^D x\; \mathcal{L}(x)$$

to be scale invariant, which leads to $\Delta_l = D$. Thus, one obtains

$$\begin{aligned} \delta\phi_i &= \left(\Delta_\phi + x^\mu \partial_\mu\right) \phi_i \\ \delta\mathcal{L} &= \left(D + x^\mu \partial_\mu\right) \mathcal{L} = \partial_\mu \left(x^\mu \mathcal{L}\right) \end{aligned} \tag{2.5}$$

The corresponding Noether current associated with scale symmetry, i.e. the Dilatation current, is then given by

$$\mathcal{D}^\mu = \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi_i\right)} \left(\Delta_\phi + x^\nu \partial_\nu\right) \phi_i - x^\mu \mathcal{L} \tag{2.6}$$

According to Noether's theorem, the Dilatation current is conserved, i.e. $\partial_\mu \mathcal{D}^\mu = 0$, if the theory is symmetric under scaling transformations, i.e. scale invariant. The divergence of the Dilatation current is given by

$$\partial_\mu \mathcal{D}^\mu = \left(1 + \Delta_\phi\right) \left(\partial_\mu \phi_i\right) \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi_i\right)} + \Delta_\phi\, \phi_i \frac{\partial \mathcal{L}}{\partial \phi_i} - D\, \mathcal{L} \tag{2.7}$$

For a Lagrangian of the form $\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi_i\right)\left(\partial^\mu \phi_i\right) - V\left(\phi_1, \ldots, \phi_m\right)$ and $\Delta_\phi = \frac{D-2}{2}$, one finds

$$\partial_\mu \mathcal{D}^\mu = D\, V - \frac{D-2}{2}\, \phi_i \frac{\partial V}{\partial \phi_i} = \left(D - \Delta_\phi\, \phi_i \frac{\partial}{\partial \phi_i}\right) V \tag{2.8}$$

Hence, the potential $V$ of the theory has to satisfy

$$\left(D - \Delta_\phi\, \phi_i \frac{\partial}{\partial \phi_i}\right) V = 0 \tag{2.9}$$

in order to be scale invariant.

As discussed in the introduction, this thesis is about quantum scale invariance (QSI) that is only broken spontaneously. Hence, the theory still has to be scale invariant at the quantum level, or in other words, quantum corrections must not break scale invariance explicitly as it is usually the case and as discussed above. However, the Dilatation current in (2.6), and thus its derivative (2.8) have been derived from the classical Lagrangian $\mathcal{L}$, or equivalently the classical action $S$. For this reason, relation (2.9) is a classical relation for the tree-level potential. In order to obtain an expression for scale invariance that is valid at the quantum level, i.e. for quantum scale invariance, one should consider the behaviour of $N$-point Green functions $G^{(N)}(x_1, \ldots, x_N)$ under the action of scaling transformations as it was done in [21, 39]. Recall that $N$-point Green functions are given by

$$\begin{aligned} G^{(N)}(x_1, \ldots, x_N) &= \langle \Omega \,| T\left\{\phi(x_1), \ldots, \phi(x_N)\right\} | \, \Omega \rangle \\ &= \frac{\int \mathcal{D}\phi \; \phi(x_1) \cdots \phi(x_N) \, e^{iS[\phi]}}{\int \mathcal{D}\phi \; e^{iS[\phi]}} \end{aligned} \tag{2.10}$$

Generalised for $m$ different kinds of scalar quantum fields $\{\phi_i\}_{i=1}^m$

$$\begin{aligned} &G^{(N_1, \ldots, N_m)}(x_1^{(1)}, \ldots, x_{N_1}^{(1)}; \ldots; x_1^{(m)}, \ldots, x_{N_m}^{(m)}) \\ &\quad = \frac{\int \mathcal{D}\phi_1 \cdots \mathcal{D}\phi_m \; \phi_1(x_1^{(1)}) \cdots \phi_1(x_{N_1}^{(1)}) \cdots \phi_m(x_1^{(m)}) \cdots \phi_m(x_{N_m}^{(m)}) \, e^{iS[\phi_1, \ldots, \phi_m]}}{\int \mathcal{D}\phi_1 \cdots \mathcal{D}\phi_m \; e^{iS[\phi_1, \ldots, \phi_m]}} \end{aligned} \tag{2.11}$$

Further, note that (infinitesimal) scaling transformations, as in (2.5), can be derived from the action of the dilatation generator $\hat{D}$, the Noether charge corresponding to the dilatation current (2.6), e.g. $\delta\phi_j = i\,[\hat{D}, \phi_j] = i\,\hat{D}\,\phi_j = (\Delta_\phi + x^\mu \partial_\mu)\,\phi_j$. Massive parameters, such as masses, are not charged under the action of $\hat{D}$, i.e. do not transform (non-trivially) under the action of $\hat{D}$, such that $\delta M = i\,[\hat{D}, M] = 0$, for some generic mass $M$. The dilatation generator can be used to extend the Poincare algebra, as discussed in [21] and appendix A of this thesis.

In order to proceed with the investigation of quantum scale invariance, it is necessary to explicitly define what is meant by QSI.

**Definition 2.1** (Quantum Scale Invariance)**.**
A theory described by the action $S = S[\phi_1, \ldots, \phi_m]$, with field spectrum $\{\phi_i\}_{i=1}^m$, is *quantum scale invariant* (QSI) if its quantum effective action $\Gamma = \Gamma[\phi_1, \ldots, \phi_m]$ is scale invariant, i.e. invariant under the scale symmetry transformations (2.1), or equivalently if the theory's Green functions (2.11) are scale invariant, i.e. satisfy

$$\begin{aligned} 0 &= i\,[\hat{D}, G^{(N_1, \ldots, N_m)}] \\ &= i\,\hat{D}\,G^{(N_1, \ldots, N_m)}(x_1^{(1)}, ..., x_{N_1}^{(1)}; ...; x_1^{(m)}, ..., x_{N_m}^{(m)}) \\ &= \sum_{k=1}^{m} \left( N_k\,\Delta_\phi + \sum_{j=1}^{N_k} x_j^{(k),\mu} \frac{\partial}{\partial x_j^{(k),\mu}} \right) G^{(N_1, \ldots, N_m)}(x_1^{(1)}, ..., x_{N_1}^{(1)}; ...; x_1^{(m)}, ..., x_{N_m}^{(m)}) \end{aligned} \tag{2.12}$$

exactly, where $\hat{D}$ is the Dilatation generator (A.4).

Now, this definition can be used to find a quantum generalisation of relation (2.9).

**Proposition 2.1.**
*The effective potential $V_{\text{eff}}$ of a quantum scale invariant model satisfies*

$$\left(D - \Delta_\phi \, \phi_i \, \frac{\partial}{\partial \phi_i}\right) V_{\text{eff}} = 0, \tag{2.13}$$

*which is the quantum generalisation of (2.9), and thus the QSI effective potential $V_{\text{eff}}$ has to be a homogeneous function of the fields, i.e. has to satisfy*

$$V_{\text{eff}}(\alpha\, \phi_1, \ldots, \alpha\, \phi_m) = \alpha^{\frac{2\,D}{D-2}}\, V_{\text{eff}}(\phi_1, \ldots, \phi_m) \tag{2.14}$$

*for some dimensionless parameter $\alpha$.*

*Proof.* (Sketch)
Starting with the definition of QSI via Green functions (2.12), this requirement can be expressed in momentum space via Fourier transformation of (2.12) and can further equivalently be expressed via other types of Green functions, as stated in [19, 21]. Thus, given (2.12), the 1PI connected amputated Green function $\Gamma^{(N)}(p_1, \ldots, p_N)$ whose generating functional is the quantum effective action $\Gamma = \Gamma[\phi]$ satisfies

$$0 = \left(D - N\, \Delta_\phi - \sum_{j=1}^{N} p_j^\mu \, \frac{\partial}{\partial p_j^\mu}\right) \Gamma^{(N)}(p_1, \ldots, p_N) \tag{2.15}$$

as discussed in [19, 21].
Evaluating this at zero (external) momentum gives

$$0 = (D - N\, \Delta_\phi)\, \Gamma^{(N)}\big|_{p=0} = \left(D - \Delta_\phi\, \phi \, \frac{\partial}{\partial \phi}\right) \Gamma^{(N)}\big|_{p=0}$$

Noting that the effective potential $V_{\text{eff}}$ is given by the sum of all momentum-independent 1PI diagrams and generalising this to $m$ different kinds of quantum fields $\{\phi_i\}_{i=1}^m$, one finds

$$\left(D - \Delta_\phi \, \phi_i \, \frac{\partial}{\partial \phi_i}\right) V_{\text{eff}} = 0$$

which is equation (2.13).
Now, using the homogeneity condition (2.14), differentiating this w.r.t. $\alpha$ and then taking $\alpha \to 1$, one obtains the relation

$$\frac{2\,D}{D-2} V_{\text{eff}} = \phi_i \, \frac{\partial V_{\text{eff}}}{\partial \phi_i}$$

satisfied by the effective potential. Using the explicit expression for the scaling dimension of scalar fields (2.3) it can be seen that this is exactly the relation (2.13). Conversely, given a differentiable function $V_{\text{eff}}$ that satisfies the relation (2.13), one can define a function $f(\alpha) \coloneqq V_{\text{eff}}(\alpha\, \phi_1, \ldots, \alpha\, \phi_m)$ with initial condition $f(1) = V_{\text{eff}}(\phi_1, \ldots, \phi_m)$, which obeys a 1. order ODE in $\alpha$ due to relation (2.13). The solution of this ODE gives the homogeneity condition (2.14), which is unique due to the theorem of Picard-Lindelöf. Thus, the effective potential $V_{\text{eff}}$ of a quantum scale invariant model has to be a homogeneous function of the fields. □

Note that relation (2.13) is an exact, i.e. non-perturbative, statement, and thus has to be valid / fulfilled at all orders of perturbation theory in order for a theory to be quantum scale invariant. The question, however, remains how scale invariance can be realised at the quantum level, i.e. how the explicit breaking of scale symmetry by the necessary Regularisation of divergences from loop-calculations can be avoided.

As originally proposed in [9] and used, inter alia, in [2, 11, 12, 13, 14, 17, 20, 21, 28, 34], this can be achieved, i.e. scale invariance can be maintained even at the quantum level, if a Regularisation that respects scale symmetry, or in other words a manifestly scale invariant Regularisation, is used. It has been shown that such a Regularisation implies the absence of anomalous scale symmetry breaking. Thus, in order to obtain a quantum scale invariant theory a

- classically scale invariant theory,
- and a manifestly scale invariant Regularisation

are required. As in the above mentioned articles, the scale invariant Regularisation used in this thesis is a scale invariant version of DReg, i.e. SIDReg, which is defined as follows.

**Definition 2.2** (Scale Invariant Dimensional Regularisation)**.**
*Scale Invariant Dimensional Regularisation* (SIDReg) is analogously defined as (usual) Dimensional Regularisation (DReg), with the difference that the usual Renormalisation scale is replaced by a Dilaton-dependent *Renormalisation function* $\mu = \mu(\sigma)$, i.e. $\mu \longrightarrow \mu = \mu(\sigma)$, which is charged under the transformation generated by $\hat{D}$, i.e. transforms non-trivially under dilatations, with scaling dimension $\Delta_\mu = 1$, such that

$$\delta\mu = i\,[\hat{D}, \mu(\sigma)] = i\,\frac{\partial\mu}{\partial\sigma}\,[\hat{D}, \sigma] = (\Delta_\mu + x^\mu\partial_\mu)\,\mu(\sigma) = (1 + x^\mu\partial_\mu)\,\mu(\sigma) \tag{2.16}$$

in order to obtain a manifestly scale invariant Regularisation. The Dilaton $\sigma$ is a dynamical scalar field with a non-vanishing vacuum expectation value (VEV) $w := \langle\sigma\rangle$. The usual Renormalisation scale, henceforth denoted as $\mu_0$, is generated dynamically after spontaneous symmetry breaking (SSB) of scale symmetry, such that $\mu_0 = \mu(\langle\sigma\rangle)$.

**Remark.**

(i) Since the Renormalisation scale is replaced by a Renormalisation function $\mu = \mu(\sigma)$, no (fixed) mass scale enters the action that could spoil scale invariance (in $D = 4 - 2\epsilon$ or at the quantum level).

(ii) The Renormalisation function $\mu(\sigma)$ is defined such that it transforms non-trivially under scaling transformations (2.16) with scaling dimension $\Delta_\mu = 1$, which ensures scale invariance in $D = 4 - 2\epsilon$ and at the quantum level, i.e. QSI, as can be seen below.

(iii) Scaling dimension $\Delta_\mu = 1$ implies that the Renormalisation function $\mu(\sigma)$ has mass dimension $[\mu(\sigma)] = 1$ in every spacetime dimension, i.e. even in $D = 4 - 2\epsilon$ dimensions.

(iv) The Dilaton acquires a VEV, and thus scale symmetry is spontaneously broken with the Dilaton $\sigma$ as its associated Goldstone boson. This is not only intended as scale symmetry has not been observed in the real world, as discussed above, but also necessary, as discussed in [28].

- Mathematically, the Dilaton and $\mu(\sigma)$ can, otherwise, not be used for perturbative Renormalisation or computations, because $\mu(\sigma)$ does not have a (polynomial) power series / Taylor expansion if $\sigma$ does not have a VEV about which can be expanded
- Physically, the VEV of the Dilaton is necessary to generate the massive Renormalisation scale $\mu_0 = \mu(\langle\sigma\rangle)$ after SSB of scale symmetry, needed to reproduce the running of the couplings [11, 12, 14, 41]

(v) The Renormalisation function, and thus SIDReg requires a dynamical field, the Dilaton $\sigma$. If the Dilaton is not initially part of the theory, the theory's field spectrum needs to be extended by the Dilaton, and therefore the theory acquires an additional degree of freedom. Note that kinetic terms for the Dilaton are required since $\sigma$ is dynamical.

(vi) The exact value or the order of magnitude of the VEV of the Dilaton $w = \langle\sigma\rangle$ has not yet been determined. The Standard Model (SM) is just a low-energy effective field theory (EFT) and "new physics" beyond the SM could arise at higher energy scales (e.g. the Planck scale $M_{Pl}$), where it is generally expected that there will be "new physics" until or at least at the Planck scale due to several phenomena that cannot be explained by the SM. Hence, scale symmetry could be spontaneously broken at such an higher energy scale which fixes the Dilaton's VEV. Further, in a theory that includes (conformal) gravity, e.g. Brans-Dicke-Jordan gravity, the VEV is related to the Planck scale $M_{Pl}$ [11, 13]. Thus, as discussed in [11, 12, 13, 14], the VEV of the Dilaton $\langle\sigma\rangle$ is expected to be large and of the order of the Planck scale, i.e. $\langle\sigma\rangle \sim M_{Pl}$. Assuming $\langle\sigma\rangle \sim M_{Pl}$ is further motivated by experimental observations and phenomenological reasons. In a theory that contains a Higgs-like boson $\phi$ representing the "visible sector" and the Dilaton $\sigma$ representing the "hidden sector", such as the 2 Scalar Model (discussed below), one can use a large Dilaton VEV with $\langle\phi\rangle \ll \langle\sigma\rangle$ to ensure a very weak coupling between the visible and the hidden sector [13]. New corrections are then suppressed by $\langle\sigma\rangle$ to negative powers which is necessary for the theory to be valid with experiments.

(vii) In DReg and SIDReg, the theory is analytically continued to $D = 4-2\epsilon$ dimensions. In SIDReg, the Renormalisation function $\mu(\sigma)$ is required to have scaling dimension $\Delta_\mu = 1$, and thus mass dimension $[\mu(\sigma)] = 1$, even in $D = 4 - 2\epsilon$ dimensions. In order to ensure this, the Dilaton $\sigma$, which has mass dimension $[\sigma] = \frac{D-2}{2} = 1 - \epsilon$, enters the Renormalisation function to the power of $\frac{2}{D-2} = \frac{1}{1-\epsilon}$. Further, the Renormalisation function itself enters the Lagrangian to the power of $n\epsilon$, for some $n \in \mathbb{Z}$, as can be seen later in this thesis. This leads to the appearance of $\sigma^{\frac{n\,\epsilon}{1-\epsilon}}$. Expanding the Lagrangian in $\epsilon$ leads to infinitely many new *evanescent* interaction terms in the Lagrangian as defined in Def. 2.3.

**Definition 2.3** (Evanescent Interactions)**.**
*Evanescent Interactions* are interaction terms in the Lagrangian $\mathcal{L}$ proportional to $\epsilon^k$, for some $k \in \mathbb{N}$.

There are 3 main consequences affecting physics that are introduced by the Dilaton-dependent Renormalisation function $\mu(\sigma)$ in SIDReg:

**Remark** (3 main consequences of SIDReg)**.**

(i) The Renormalisation scale $\mu_0$ and all other mass scales / massive parameters are generated dynamically after SSB of scale symmetry.
$\Rightarrow$ there is no initial scale in the theory at all
$\Rightarrow$ there is no scale anomaly / anomalous breaking of scale symmetry anymore

(ii) New finite and divergent quantum corrections arising from evanescent interactions $\sim \epsilon^k$ are generated by (quantum) scale invariance of the action in $D = 4 - 2\epsilon$ dimensions due to $\mu(\sigma)$, i.e. $\mu = \mu(\sigma) = \mu_0\,(1 + \text{Int.})$. This is possible due to evanescent interactions $\sim \epsilon^k$ multiplying with $\epsilon$ - poles of an appropriate power emerging in loop-calculations, which leads to a cancellation of $\epsilon$, and thus to new finite or divergent contributions (depending on the powers of $\epsilon$), even for $\epsilon \to 0$.

(iii) Non-Renormalisability due to infinitely many new evanescent interactions introduced by the Renormalisation function in a manifestly scale invariant Regularisation, which has been discussed in [11, 12, 14, 20, 21, 33]. Further, this is exemplarily discussed in more detail for the 2 Scalar Model in section 2.3 and chapter 3.

In SIDReg, a generic interaction term $\lambda_\phi\,\phi^4(x)$, as discussed above, takes the form $\lambda_\phi\,\phi^4(x) \longrightarrow \mu^{4-D}(\sigma)\,\lambda_\phi\,\phi^4(x)$ if analytically continued to $D = 4 - 2\epsilon$ dimensions, where the coupling constant $\lambda_\phi$ is kept dimensionless. Using the scaling properties of $\mu(\sigma)$, as defined above, such an interaction term now transforms under dilatations, i.e. under (2.1), as

$$
\begin{aligned}
-\int d^D x\; \mu^{4-D}(\sigma(x))\,\lambda_\phi\,\phi^4(x) \longmapsto &- s^{D-4\,\Delta_\phi-(4-D)\,\Delta_\mu} \int d^D y\; \mu^{4-D}(\sigma(y))\,\lambda_\phi\,\phi^4(y) \\
&= - s^{D-4\,\frac{D-2}{2}-(4-D)} \int d^D x\; \mu^{4-D}(\sigma(x))\,\lambda_\phi\,\phi^4(x) \\
&= - \int d^D x\; \mu^{4-D}(\sigma(x))\,\lambda_\phi\,\phi^4(x)
\end{aligned}
$$

where the scaling dimension of scalar fields $\Delta_\phi = \frac{D-2}{2}$ and of the Renormalisation function $\Delta_\mu = 1$ have been used. It can be seen that such interactions terms are now (manifestly) scale invariant, even in $D = 4 - 2\epsilon$. Further, a potential $V = V(\phi_1, \ldots, \phi_m, \sigma)$, and analogously in $D = 4 - 2\epsilon$ dimensions $\widetilde{V}(\phi_1, \ldots, \phi_m, \sigma) = \mu^{4-D}(\sigma)\, V(\phi_1, \ldots, \phi_m, \sigma)$, consisting of several such interactions terms now satisfies relation (2.9), and thus the Noether current (2.6) is conversed, not only in 4 but also in $D = 4 - 2\epsilon$ dimensions. In contrast to this, the same potential regularised with conventional DReg satisfies (2.9) only in 4 dimensions. Therefore, SIDReg indeed preserves scale symmetry in $D = 4 - 2\epsilon$ dimensions. However, it still needs to be explicitly shown that SIDReg also preserves scale symmetry at the quantum level as expected, which is done in chapter 3 for the 2 Scalar Model.

Now that SIDReg is defined and some technical details, as well as the major implications have (shortly) been discussed, an Ansatz for an explicit expression of the Dilaton-dependent Renormalisation function $\mu = \mu(\sigma)$ is given.

**Ansatz.**

$$\mu(\sigma) = z\, \sigma^{\frac{2}{D-2}} = z\, \sigma^{\frac{1}{1-\epsilon}} \tag{2.17}$$

**Remark.**

(i) Using this Ansatz (2.17), the Renormalisation scale is given by $\mu_0 = \mu(\langle\sigma\rangle) = z\, \langle\sigma\rangle^{\frac{1}{1-\epsilon}} \equiv z\, w^{\frac{1}{1-\epsilon}}$, generated dynamically after SSB of scale symmetry.

(ii) $z$ is an arbitrary dimensionless parameter, the Renormalisation point parameter or RG flow parameter, and the dependence of $z$ is equivalent to the dependence of the Renormalisation scale $\mu_0$ in conventional DReg. Note that in SIDReg the Renormalisation scale $\mu_0$ is given by $\mu_0 = \mu(\langle\sigma\rangle) = z\, \langle\sigma\rangle^{\frac{1}{1-\epsilon}} \equiv z\, w^{\frac{1}{1-\epsilon}}$, as stated above. Since $w = \langle\sigma\rangle$ is the VEV of the Dilaton, and thus a constant parameter of the theory, the only "running object" in this Renormalisation scale is the parameter $z$. Hence, $z$ keeps track of the dependence of the Renormalisation scale $\mu_0$ and is necessary in order to discuss Renormalisation Group Eqautions (RGEs) [20, 41].

(iii) A more general Renormalisation function, depending on other scalar fields has already been ruled out, as discussed in [11]. For instance, in a 2 Scalar Model, containing a Higgs-like boson $\phi$ and the Dilaton $\sigma$, one could think of a more general Renormalisation function $\mu = \mu(\phi, \sigma)$ depending on both scalar fields, instead of (2.17). However, this Ansatz for $\mu$ would introduce non-decoupling quantum effects between the visible sector ($\phi$) and the hidden sector ($\sigma$), even in their classical decoupling limit, i.e. both sectors would still interact at the quantum level even if they are classically decoupled. Moreover, this Ansatz would introduce terms in $V_{\text{eff}}$ at the quantum level which are unbounded from below, leading to the fact that the potential can be destabilised by small fluctuations about a critical point. This can only be avoided if the Renormalisation function depends on the Dilaton $\sigma$ alone, i.e. $\mu \sim \sigma$. Hence, the Ansatz (2.17) will be used in this thesis. For more details regarding this issue, the reader is referred to [11].

Using Ansatz (2.17) and expanding about the Dilaton's VEV, i.e. $\sigma = \widetilde{\sigma} + \langle\sigma\rangle \equiv \mathfrak{D} + w$, where $w := \langle\sigma\rangle$ is the VEV of the Dilaton, as defined before, and $\mathfrak{D} := \widetilde{\sigma}$ are (small) field fluctuations (about the vacuum), one obtains

$$\mu(\sigma) = z\,\sigma^{\frac{1}{1-\epsilon}} = z\,(w + \mathfrak{D})^{\frac{1}{1-\epsilon}} = z\,w^{\frac{1}{1-\epsilon}} \left(1 + \frac{\mathfrak{D}}{w}\right)^{\frac{1}{1-\epsilon}} = \mu_0 \left(1 + \frac{\mathfrak{D}}{w}\right)^{\frac{1}{1-\epsilon}} \tag{2.18}$$

As can be seen, the Dilaton enters the Lagrangian to anomalous, i.e. non-integer, powers due to the Renormalisation function. In order to derive Feynman rules the Lagrangian needs to be expanded w.r.t. $\epsilon$ and $\mathfrak{D}/w$. Such expansions of the Renormalisation function to some appropriate powers of $\epsilon$, which will enter the Lagrangian, are to be found in equations (B.2) in the appendix.

Actual calculations using SIDReg can be rather subtle and not as trivial as one might expect. For this reason, it is useful to provide a prescription for the calculation of Green functions in a theory with *spontaneously broken* quantum scale invariance using SIDReg. This is in particular true for Green functions with non-zero external momenta, i.e. calculations other than effective potential calculations.

**Prescription** (QSI Approach using SIDReg)**.**

(1) Starting point is a classically scale invariant Lagrangian $\mathcal{L}$ in $D = 4$ dimensions. Since SIDReg necessarily introduces a new field, the Dilaton $\sigma$, which needs to be dynamical, and thus needs kinetic terms, it is already part of the field spectrum of the theory from the beginning. Hence, the considered theory contains the Dilaton as an additional degree of freedom.

(2) Analytically continue the theory to $D = 4 - 2\epsilon$ dimensions using SIDReg.

(3) In order to keep the couplings dimensionless $\mu$ is introduced in the "usual" way, this time, however, as a Dilaton-dependent Renormalisation function $\mu = \mu(\sigma)$. Let $\lambda_i$, $g_i$ and $y_i$ be scalar potential, gauge and Yukawa couplings, respectively. Then, $\mu = \mu(\sigma)$ enters the action as follows:

$$\begin{aligned} \lambda_i &\longrightarrow \mu^{4-D}\,\lambda_i = \mu^{2\epsilon}\,\lambda_i \\ g_i &\longrightarrow \mu^{\frac{4-D}{2}}\,g_i = \mu^{\epsilon}\,g_i \\ y_i &\longrightarrow \mu^{\frac{4-D}{2}}\,y_i = \mu^{\epsilon}\,y_i \end{aligned}$$

which can be deduced by dimensional analysis, noting that, in $D = 4 - 2\epsilon$ dimensions, the mass dimensions of scalar fields $\phi$, gauge fields $A_\mu$ and Dirac spinors $\psi$ are given by $[\phi] = 1 - \epsilon$, $[A_\mu] = 1 - \epsilon$ and $[\psi] = \frac{3}{2} - \epsilon$, respectively.

(4) Perform (multiplicative) Renormalisation, i.e. obtain $S \longrightarrow S_0 = S_{\text{ren}} + S_{\text{ct}}$, using the Lagrangian in the unbroken phase. Note that higher dimensional operators might necessarily be introduced as counterterms depending on the loop-order due to non-Renormalisability.

(5) Expand the Lagrangian about the scalar fields VEVs, $\epsilon$, and the ratio $\mathfrak{D}/w$ in order to obtain a Lagrangian polynomial in fields, i.e. a Lagrangian that only contains fields to integer powers, which is necessary to derive Feynman rules (in the usual way). This Lagrangian then displays the theory in the broken phase. Note that this expansion must not be truncated since this would violate scale invariance explicitly. Further, note that the Renormalisation scale $\mu_0^{n\epsilon}$, for some $n \in \mathbb{Z}$ and generated dynamically after SSB, must not be expanded w.r.t. $\epsilon$ at the level of the Lagrangian since it is used in loop-integrals as in conventional DReg.

(6) In a theory with Higgs sector, apply the minimalisation conditions of the Higgs potential (optional).

(7) Transform to mass eigenstates (optional).

(8) Derive Feynman rules, including the new evanescent interactions.

(9) Use these Feynman rules to calculate Feynman diagrams, and thus Green functions.

(10) Renormalise loop-divergences using counterterm diagrams. However, do not yet go back to $D \to 4$ dimensions. The reason for this is that, e.g. in a scattering process, evanescent terms in the tree-level diagram can still "meet" an $\epsilon$-pole when interference terms between the tree-level and loop diagrams are calculated in order to obtain the squared amplitude. Hence, the limit $D \to 4 \Leftrightarrow \epsilon \to 0$ should be taken only after it can be ensured that no evanescent term will be multiplied with an $\epsilon$-pole anymore.

**Remark.**

(i) In the case of the effective potential $V_{\text{eff}}$, one wants to obtain an explicitly and manifestly scale invariant potential which satisfies relation (2.13). Thus, one does not work in the broken phase of the theory, i.e. the Lagrangian is not expanded about the scalar fields VEVs. Instead, a field shift $\phi_k \longrightarrow \phi_k + \phi_{k,0}$ is applied to the scalar fields, where $\phi_{k,0}$ are background fields. These background fields $\phi_{k,0}$ are charged under the transformation generated by $\hat{D}$, i.e. transform non-trivially under dilatations, and thus the resulting Lagrangian is still manifestly scale invariant. Hence, scale invariance is maintained manifest at all steps of the calculation. After the scale invariant effective potential is calculated, one can then expand about the scalar fields VEVs and go to the broken phase of the theory. An exemplary calculation of the effective potential up to 2-loop order in the 2 Scalar Model is to be found in chapter 3.

(ii) Step (5), i.e. working in the broken phase of the theory does not spoil spontaneously broken QSI, i.e. does not explicitly break scale symmetry, as long as the power series is not truncated and the Renormalisation transformation in step (4) is performed before going to the broken phase of the theory. The reason for this is that in general, if a symmetry is realised, it is realised independently of the coordinates that are used. Here, the difference between the coordinates $\sigma$ and $\mathfrak{D}$ is that the symmetry is manifest for $\sigma$, while it is non-linearly realised for $\mathfrak{D}$. Thus, the Goldstone boson $\mathfrak{D}$ does not transform multiplicatively, but rather shifts, i.e. is associated with shift symmetries, as any other Goldstone boson under the action of the corresponding symmetry [32]. Moreover, in [21] it has explicitly been shown for a 2 Scalar Model, containing the fields $\{\phi, \sigma\}$, that the fields VEVs $\langle\phi\rangle$ and $\langle\sigma\rangle$ are *spurious* w.r.t. the fields $\phi$ and $\sigma$, respectively, and further that, using SIDReg, the Renormalisation scale $\mu_0 = \mu(\langle\sigma\rangle)$ is *spurious* w.r.t. $\sigma$ as well. Where, in [21], a dimensionful parameter was defined to be *spurious* if it is "absorbable in the backgrounds", and thus "effectively vanishing from the functional" (meaning the quantum effective action). In addition to that, in chapter 3, it is exemplarily shown that the same scale invariant counterterms can be obtained from $N$-point Green functions using the broken phase Lagrangian and following the prescription above as from the manifestly scale invariant Lagrangian, i.e. the unbroken phase, and the manifestly scale invariant effective potential. Therefore, working in the broken phase of the theory does not explicitly, but only spontaneously, break quantum scale symmetry and the correct scale invariant counterterms can be obtained from the broken phase Lagrangian.

(iii) As mentioned in (ii), if the Lagrangian (including the Renormalisation function) is expanded w.r.t. the scalar fields VEVs, $\epsilon$, and the ratio $\mathfrak{D}/w$, i.e. working in the broken phase of the theory, the power series must not be truncated since this would otherwise break (quantum) scale invariance explicitly. However, in real calculations, depending on the considered loop order and the number of external particles, only terms of the power series in fields and $\epsilon$ that actually contribute need to be considered. Hence, terms that will definitely not contribute (or will lead only to evanescent corrections that vanish in the limit $\epsilon \to 0$) can practically be neglected.

(iv) As mentioned above, the Renormalisation scale $\mu_0^{n\epsilon} = z^{n\,\epsilon}\,\langle\sigma\rangle^{\frac{n\,\epsilon}{1-\epsilon}}$, for some $n \in \mathbb{Z}$, is not expanded w.r.t. $\epsilon$ at the level of the Lagrangian since it is used in loop-integrals as in conventional DReg, and thus will then (after having integrated over the loop-momenta) be expanded, e.g. giving rise to terms such as $\sim \log\left(m^2/\mu_0^2\right)$. However, there are more subtleties to this. In particular w.r.t. factors of $\mu_0^{n\epsilon}$ in front of mass terms, dynamically generated by SSB. First, consider a generic mass term of the form $\frac{1}{2}\,m^2\,\phi^2$. In $D = 4 - 2\epsilon$ dimensions the scalar field has mass dimension $[\phi] = \frac{D-2}{2} = 1 - \epsilon$, leading to the fact that the mass $m$ itself always has mass dimension $[m] = 1$ in order to ensure that the mass term has mass dimension $D$. Now, consider a mass term $\frac{1}{2}\,\mu_0^{2\epsilon}\,\lambda_\phi\,c\,v^2\,\phi^2$ that was dynamically generated by SSB, for some VEV $v$ and some dimensionless constants $c$, $\lambda_\phi$. Again, in $D = 4 - 2\epsilon$ dimensions the scalar field and the VEV have the same mass dimension, i.e. $[v] = [\phi] = \frac{D-2}{2} = 1 - \epsilon$. Since $c$ and $\lambda_\phi$ are dimensionless, one finds $[\lambda_\phi\,c\,v^2] = 2 - 2\epsilon$. Hence, the squared mass is identified to be $m^2 = \mu_0^{2\epsilon}\,\lambda_\phi\,c\,v^2$

in order to obtain a mass parameter $m$ of mass dimension $[m] = 1$, meaning that $\mu_0^{2\epsilon}$ is part of the definition of the mass parameter $m$. The treatment of these factors of $\mu_0$ in loop-momentum integrals is discussed in appendix D.

Note that beyond quantum scale invariance, one might also want to consider quantum special conformal invariance, i.e. $i\,[\hat{K}_\mu, G^{(N_1,\dots,N_m)}] = i\,\hat{K}_\mu\, G^{(N_1,\dots,N_m)} = 0$, where the generators of special conformal transformations $\hat{K}_\mu$ are defined in appendix A. In [21] it was shown that quantum special conformal invariance and quantum scale invariance are concurrently realised due to the vanishing trace anomaly, i.e.

$$i\,[\hat{D}, G^{(N_1,\dots,N_m)}] = 0 \iff i\,[\hat{K}_\mu, G^{(N_1,\dots,N_m)}] = 0 \tag{2.19}$$

In this thesis, however, solely quantum scale invariance is considered.

## 2.2. The 2 Scalar Model

The 2 Scalar Model is used to exemplarily illustrate the realisation of spontaneously broken quantum scale invariance and its implications. Moreover, it is of great interest for physically relevant models of the real world since it is the major part of the Higgs sector in a quantum scale invariant Standard Model, as discussed later and in [13].

As the name suggests, the 2 Scalar Model consists of 2 real scalar fields $\{\phi, \sigma\}$, where $\phi$ denotes a Higgs-like boson and $\sigma$ the Dilaton. Hence, the Lagrangian in $D = 4$ dimensions is given by

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - V(\phi,\sigma) \tag{2.20}$$

The theory is supposed to be scale invariant, and thus as discussed in the previous section the potential has to be a homogeneous function, i.e. satisfies (2.9). Therefore, the potential may be written as [11]

$$V(\phi,\sigma) = \sigma^4\, W\,(\phi/\sigma) \tag{2.21}$$

The extremum conditions for the potential V

$$\frac{\partial V}{\partial \phi} = 0, \qquad \frac{\partial V}{\partial \sigma} = 0 \tag{2.22}$$

can then equivalently be expressed in terms of $W\,(\phi/\sigma)$

$$W(\chi) = 0, \qquad W'(\chi) = 0 \tag{2.23}$$

if $\langle\phi\rangle, \langle\sigma\rangle \neq 0$ is assumed, where

$$\chi \coloneqq \frac{\phi}{\sigma}, \qquad \chi_0 \coloneqq \frac{\langle\phi\rangle}{\langle\sigma\rangle} \tag{2.24}$$

as stated in [11, 13, 14, 20]. The assumption $\langle\phi\rangle, \langle\sigma\rangle \neq 0$ is not only valid, but also necessary for spontaneously broken QSI and the consistent usage of SIDReg for perturbative Renormalisation, as discussed in the previous section. The ratio of the VEVs $\chi_0 = \langle\phi\rangle/\langle\sigma\rangle$ is fixed by one condition in (2.23), e.g. $W'(\chi_0) = 0$, in terms of dimensionless coupling constants, whereas $W(\chi_0) = 0$ implies vanishing vacuum energy $V(\langle\phi\rangle, \langle\sigma\rangle) = 0$, as stated in [14]. Given that $\chi_0 = \langle\phi\rangle/\langle\sigma\rangle$ is a solution to (2.23), then $\langle\phi\rangle \sim \langle\sigma\rangle$, and thus $\exists$ a flat direction in the theory, with $\phi/\sigma = \chi_0$ in the $(\phi, \sigma)$ - plane [11, 14]. This implies the existence of a Goldstone mode, leading to a (massless) Goldstone boson in the theory, which turns out to be the Dilaton, as expected in the case of a spontaneously broken symmetry.

An example for a scale invariant potential that satisfies (2.9) and further allows for non-trivial solutions of (2.22) is given by

$$V(\phi,\sigma) = \frac{\lambda_\phi}{4!}\,\phi^4 + \frac{\lambda_m}{4}\,\phi^2\,\sigma^2 + \frac{\lambda_\sigma}{4!}\,\sigma^4 \tag{2.25}$$

For this particular choice of $V$, one obtains

$$W(\chi) = \frac{\lambda_\phi}{4!}\,\chi^4 + \frac{\lambda_m}{4}\,\chi^2 + \frac{\lambda_\sigma}{4!}$$

The minimalisation conditions (2.22) for this potential (2.25) are then provided by

$$\begin{aligned} \left.\frac{\partial V}{\partial \phi}\right|_{\substack{\phi=\langle\phi\rangle \\ \sigma=\langle\sigma\rangle}} = 0 \iff \langle\phi\rangle \left(\frac{\lambda_\phi}{6}\,\langle\phi\rangle^2 + \frac{\lambda_m}{2}\,\langle\sigma\rangle^2\right) = 0 \\ \left.\frac{\partial V}{\partial \sigma}\right|_{\substack{\phi=\langle\phi\rangle \\ \sigma=\langle\sigma\rangle}} = 0 \iff \langle\sigma\rangle \left(\frac{\lambda_m}{2}\,\langle\phi\rangle^2 + \frac{\lambda_\sigma}{6}\,\langle\sigma\rangle^2\right) = 0 \end{aligned} \tag{2.26}$$

Solving these conditions, assuming $\langle\phi\rangle, \langle\sigma\rangle \neq 0$, yields

$$\begin{aligned} \chi_0^2 &= \frac{\langle\phi\rangle^2}{\langle\sigma\rangle^2} = -\,3\,\frac{\lambda_m}{\lambda_\phi} \;>\; 0 \\ \lambda_m^2 &= \frac{1}{9}\,\lambda_\phi\,\lambda_\sigma \;\Leftrightarrow\; \lambda_\sigma = 9\,\frac{\lambda_m^2}{\lambda_\phi} \end{aligned} \tag{2.27}$$

which implies $\lambda_m < 0$ and $\lambda_\sigma > 0$, if $\lambda_\phi$ is chosen to be $\lambda_\phi > 0$. Using (2.27), all couplings can be expressed in terms of $\lambda_\phi$ and $\chi_0$, thus

$$\begin{aligned} \lambda_m &= -\,\frac{1}{3}\,\lambda_\phi\,\chi_0^2 \\ \lambda_\sigma &= \lambda_\phi\,\chi_0^4 \end{aligned} \tag{2.28}$$

Further, the potential (2.25) can then be written as

$$V(\phi,\sigma) = \frac{\lambda_\phi}{4!}\,\left(\phi^2 - \varrho\,\sigma^2\right)^2 \tag{2.29}$$

where $\varrho \coloneqq \chi_0^2 = \langle\phi\rangle^2/\langle\sigma\rangle^2$.

Now, before the theory is analytically extended to $D = 4 - 2\epsilon$ dimensions using SIDReg, there are two aspects that should be discussed prior to that, the masses and mass eigenstates of the particles, as well as the connection of this model to the Standard Model Higgs sector. First, consider the expansion of the fields $\phi, \sigma$ about their VEVs

$$\begin{aligned} \phi &= \widetilde{\phi} + \langle\phi\rangle \equiv h + v \\ \sigma &= \widetilde{\sigma} + \langle\sigma\rangle \equiv \mathfrak{D} + w \end{aligned} \tag{2.30}$$

where $h \coloneqq \widetilde{\phi}$ are (small) field fluctuations (about the vacuum) and $v \coloneqq \langle\phi\rangle$ is the VEV of the Higgs-like boson, whereas $\mathfrak{D} = \widetilde{\sigma}$ and $w = \langle\sigma\rangle$ for the Dilaton, as defined in the previous section. Using (2.30), the mass terms of $h$ and $\mathfrak{D}$, generated dynamically after SSB, are then given by

$$\begin{aligned} V\Big|_{\text{bilinear}} &= \frac{1}{2}\,(h,\,\mathfrak{D})\;\mathcal{M}^2_{\phi\sigma}\begin{pmatrix} h \\ \mathfrak{D} \end{pmatrix} \\ &= \frac{1}{2}\,(H,\,S)\;\mathcal{R}_\beta^\top\,\mathcal{M}^2_{\phi\sigma}\,\mathcal{R}_\beta\begin{pmatrix} H \\ S \end{pmatrix} = \frac{1}{2}\,(H,\,S)\;\mathcal{M}^2_{\text{Diag}}\begin{pmatrix} H \\ S \end{pmatrix} \end{aligned} \tag{2.31}$$

where in the second line the flavour eigenstates $h, \mathfrak{D}$ have been transformed to the mass eigenstates $H, S$ using the rotation matrix $\mathcal{R}_\beta$, i.e.

$$\begin{aligned} \begin{pmatrix} H \\ S \end{pmatrix} &= \mathcal{R}_\beta^\top\begin{pmatrix} h \\ \mathfrak{D} \end{pmatrix} = \begin{pmatrix} c_\beta\,h - s_\beta\,\mathfrak{D} \\ s_\beta\,h + c_\beta\mathfrak{D} \end{pmatrix} \\ \begin{pmatrix} h \\ \mathfrak{D} \end{pmatrix} &= \mathcal{R}_\beta\begin{pmatrix} H \\ S \end{pmatrix} = \begin{pmatrix} c_\beta\,H + s_\beta\,S \\ -s_\beta\,H + c_\beta S \end{pmatrix} \end{aligned} \tag{2.32}$$

with

$$\mathcal{R}_\beta \coloneqq \begin{pmatrix} c_\beta & s_\beta \\ -s_\beta & c_\beta \end{pmatrix} \tag{2.33}$$

For the given potential (2.25), the matrix of squared masses $\mathcal{M}^2_{\phi\sigma}$ takes the form

$$\mathcal{M}^2_{\phi\sigma} = \begin{pmatrix} \frac{\lambda_\phi}{2}\,v^2 + \frac{\lambda_m}{2}\,w^2 & \lambda_m\,v\,w \\ \lambda_m\,v\,w & \frac{\lambda_m}{2}\,v^2 + \frac{\lambda_\sigma}{2}\,w^2 \end{pmatrix} = \frac{1}{3}\,\lambda_\phi\,v^2\begin{pmatrix} 1 & -\chi_0 \\ -\chi_0 & \chi_0^2 \end{pmatrix} \tag{2.34}$$

where the minimalisation conditions (2.28) have been used in the second step. Given the explicit form of $\mathcal{M}^2_{\phi\sigma}$ in (2.34), the mixing angle $\beta$ is determined by

$$t_\beta \equiv \tan(\beta) = \frac{v}{w}, \quad s_\beta \equiv \sin(\beta) = \frac{v}{\sqrt{v^2 + w^2}}, \quad c_\beta \equiv \cos(\beta) = \frac{w}{\sqrt{v^2 + w^2}} \tag{2.35}$$

Note that the minimalisation conditions (2.28) have been assumed in (2.35). Otherwise, the expressions would be more complicated. The diagonal matrix of the squared masses $\mathcal{M}^2_{\text{Diag}}$ containing the eigenvalues of $\mathcal{M}^2_{\phi\sigma}$, and thus the squared masses of the particles $H$ and $S$, is given by

$$\mathcal{M}^2_{\text{Diag}} \coloneqq \mathcal{R}_\beta^\top\,\mathcal{M}^2_{\phi\sigma}\,\mathcal{R}_\beta = \begin{pmatrix} M_H^2 & 0 \\ 0 & M_S^2 \end{pmatrix} = \begin{pmatrix} \frac{1}{3}\,\lambda_\phi\,v^2\,(1 + \chi_0^2) & 0 \\ 0 & 0 \end{pmatrix} \tag{2.36}$$

having used the minimalisation conditions (2.28). It can be seen that, after SSB, the mass eigenstate of the Higgs-like boson $H$ obtains a non-vanishing "Higgs mass" $M_H^2 = \frac{1}{3}\lambda_\phi v^2 \, (1+\chi_0^2)$, whereas the mass eigenstate of the Dilaton $S$ is massless, i.e. $M_S^2 = 0$, and thus is the Goldstone boson associated to spontaneously broken scale symmetry, as expected.

Next, the connection of the 2 Scalar Model to the Standard Model Higgs sector is considered. The Standard Model Lagrangian is almost classically scale invariant. Only the Higgs mass term $\mathcal{L} \supset -\mu^2 \, \Phi^\dagger \Phi$ breaks scale symmetry explicitly. Therefore, as discussed in [20], the Standard Model can be scale invariant if the Higgs mass term emerges from spontaneous scale symmetry breaking at a higher energy scale (e.g. the Planck scale $M_{Pl}$). Note that the Standard Model Higgs potential can be written as

$$V_{\text{Higgs}}^{\text{SM}}(\Phi) = \lambda_{\text{SM}} \left( \Phi^\dagger \Phi + \frac{\mu^2}{2\,\lambda_{\text{SM}}} \right)^2$$

for $\mu^2 < 0$. This potential could be replaced as follows [20]

$$\lambda_{\text{SM}} \left( \Phi^\dagger \Phi + \frac{\mu^2}{2\,\lambda_{\text{SM}}} \right)^2 \longrightarrow \frac{\lambda_\phi}{3!} \left( \Phi^\dagger \Phi - \frac{\varrho}{2}\,\sigma^2 \right)^2$$

which is exactly the potential of the 2 Scalar Model in (2.29), i.e. already having used the minimalisation conditions (2.28), with $\phi \to \sqrt{2}\,\Phi$. The original Standard Model Higgs potential can then be reproduced if the Dilaton is replaced by its VEV, i.e. $\sigma \to \langle\sigma\rangle$, and $\lambda_\phi \to 6\,\lambda_{\text{SM}}$. The reason for this is

$$\frac{\mu^2}{2\,\lambda_{\text{SM}}} = -\frac{v^2}{2} \longleftrightarrow -\frac{\varrho}{2}\,\langle\sigma\rangle^2 = -\frac{v^2}{2}$$

having used the well-known minimalisation condition for the Standard Model Higgs potential $\mu^2 = -\lambda_{\text{SM}}\, v^2$, as well as $\varrho = \langle\phi\rangle^2/\langle\sigma\rangle^2 \equiv v^2/w^2$. Further, one might conclude that spontaneous scale symmetry breaking implies spontaneous electroweak symmetry breaking, with vanishing vacuum energy $V(\langle\phi\rangle, \langle\sigma\rangle) = 0$, and thus all scales are dynamically generated by $\langle\sigma\rangle$ [11, 13]. Note that it was assumed that $v = \langle\phi\rangle$ is equal to the Standard Model Higgs VEV, i.e. $v = \langle\phi\rangle \sim 10^2\,\text{GeV}$. On the other hand side, the exact value for the Dilaton VEV $w = \langle\sigma\rangle$ is unknown, but expected to be large $v \ll w$, as motivated in the previous section. A further motivation for this is given by the tree-level Higgs mass $M_H$. In the Standard Model, the squared Higgs mass is given by $M_{H,\text{SM}}^2 = 2\,\lambda_{\text{SM}}\, v^2$ at tree-level, whereas here in the 2 Scalar Model, the tree-level squared Higgs mass is given by $M_H^2 = \frac{1}{3}\lambda_\phi v^2 \, (1+\chi_0^2)$, as discussed above. Using $\lambda_\phi = 6\,\lambda_{\text{SM}}$ and $\chi_0 = v/w$, one finds

$$\begin{aligned} M_H^2 &= \frac{1}{3}\,\lambda_\phi\, v^2 \left(1 + \frac{v^2}{w^2}\right) = 2\,\lambda_{\text{SM}}\, v^2 \left(1 + \frac{v^2}{w^2}\right) \\ \Longrightarrow \delta M_H^2 &:= M_H^2 - M_{H,\text{SM}}^2 = 2\,\lambda_{\text{SM}}\, v^2\, \frac{v^2}{w^2} \end{aligned} \tag{2.37}$$

which should better be a vanishingly small correction. Choosing the Dilaton VEV to be of the order of the Planck scale, i.e. $w = \langle\sigma\rangle \sim M_{Pl} \sim 10^{18}\,\text{GeV}$, does not only guarantee this, but also leads to a very weak coupling between the visible ($\phi$) and the hidden sector ($\sigma$) due to (2.28), c.f. previous section. Thus, choosing $v = \langle\phi\rangle \sim 10^2\,\text{GeV}$ and $w = \langle\sigma\rangle \sim M_{Pl} \sim 10^{18}\,\text{GeV}$ leads to

$$\begin{aligned}
\chi_0 &= \frac{v}{w} \sim \mathcal{O}(10^{-16}) \\
\varrho &= \chi_0^2 = \frac{v^2}{w^2} \sim \mathcal{O}(10^{-32}) \\
\delta M_H^2 &= 2\,\lambda_{\text{SM}}\,v^2\,\frac{v^2}{w^2} \sim \mathcal{O}\left(10^{-28}\,\text{GeV}^2\right) \\
\lambda_\phi &= 6\,\lambda_{\text{SM}} \sim \mathcal{O}(1) \\
\lambda_m &= -\frac{1}{3}\,\lambda_\phi\,\frac{v^2}{w^2} \sim -\,\mathcal{O}(10^{-32}) \\
\lambda_\sigma &= \lambda_\phi\,\frac{v^4}{w^4} \sim \mathcal{O}(10^{-64}) \\
\Longrightarrow &\lambda_\sigma \ll \|\lambda_m\| \ll \lambda_\phi
\end{aligned} \tag{2.38}$$

Note that, as discussed in [11, 13, 34] and mentioned in [20], a classical hierarchy of couplings at tree-level, as shown in (2.38), and thus a mass hierarchy with a light Higgs mass $M_H \sim \langle\phi\rangle \sim \mathcal{O}\,(10^2\,\text{GeV}) \ll \langle\sigma\rangle$ is radiatively stable, i.e. stabel against quantum corrections, in a theory with spontaneously broken QSI (at least at the 1-loop level). Due to *only* spontaneously broken QSI, achieved by using a Regularisation that respects scale symmetry (SIDReg), this is, however, expected to be true at all orders of perturbation theory [13]. Therefore, no additional fine-tuning at the quantum level is necessary, suggesting a possible solution to the *hierarchy problem* [11]. Moreover, given a Dilaton VEV of the order of the Planck scale one obtains the following relations for the mixing angle and the mass eigenstates

$$\left.\begin{aligned}
t_\beta &= \frac{v}{w} \sim \mathcal{O}(10^{-16}) \\
s_\beta &= \frac{v}{\sqrt{v^2+w^2}} \sim \mathcal{O}(10^{-16}) \\
c_\beta &= \frac{w}{\sqrt{v^2+w^2}} \sim \mathcal{O}(1)
\end{aligned}\right\} \Longrightarrow \quad \beta \sim \mathcal{O}(10^{-16}) \tag{2.39}$$
$$\Longrightarrow H = c_\beta\,h - s_\beta\,\mathfrak{D} \approx h, \qquad S = s_\beta\,h + c_\beta\,\mathfrak{D} \approx \mathfrak{D}$$

Thus, it can be seen that the particles $h$ and $\mathfrak{D}$ are then *almost* mass eigenstates.

Now, that the 2 Scalar Model was discussed sufficiently in $D = 4$ dimensions, the analytical continuation of the theory to $D = 4 - 2\epsilon$ dimensions using SIDReg is considered. As explained in the previous section, in SIDReg, the scalar coupling constants are rescaled as $\lambda_j \to \mu^{2\epsilon}(\sigma)\,\lambda_j$, and thus $V(\phi,\sigma) \to \widetilde{V}(\phi,\sigma) = \mu^{2\epsilon}(\sigma)\,V(\phi,\sigma)$ for the potential in $D = 4 - 2\epsilon$ dimensions. Therefore, the Lagrangian of the 2 Scalar Model in $D = 4 - 2\epsilon$ dimensions then reads as

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - \widetilde{V}(\phi,\sigma) \tag{2.40}$$

with potential

$$\widetilde{V}(\phi,\sigma) = \mu^{2\epsilon}(\sigma)\, V(\phi,\sigma) = \mu^{2\epsilon}(\sigma)\, \left(\frac{\lambda_\phi}{4!}\,\phi^4 + \frac{\lambda_m}{4}\,\phi^2\,\sigma^2 + \frac{\lambda_\sigma}{4!}\,\sigma^4\right) \tag{2.41}$$

Using the expansion of the fields about their VEVs as shown in (2.30), and further expanding w.r.t. $\epsilon$ and $\mathfrak{D}/w$, one obtains

$$\begin{aligned}\widetilde{V}(h+w,\mathfrak{D}+w) = \widetilde{V}(v,w) &+ \widetilde{T}_{\varphi_i}\,\varphi_i + \frac{1}{2}\,\widetilde{M}^2_{ij}\,\varphi_i\,\varphi_j + \frac{1}{3!}\,\widetilde{\mathcal{V}}_{ijk}\,\varphi_i\,\varphi_j\,\varphi_k \\ &+ \frac{1}{4!}\,\widetilde{\mathcal{V}}_{ijkl}\,\varphi_i\,\varphi_j\,\varphi_k\,\varphi_l + \frac{1}{5!}\,\widetilde{\mathcal{V}}_{ijklm}\,\varphi_i\,\varphi_j\,\varphi_k\,\varphi_l\,\varphi_m \\ &+ \frac{1}{6!}\,\widetilde{\mathcal{V}}_{ijklmn}\,\varphi_i\,\varphi_j\,\varphi_k\,\varphi_l\,\varphi_m\,\varphi_n + \ldots\end{aligned} \tag{2.42}$$

where $\{\varphi_i\}_{i=1}^2 = \{h,\mathfrak{D}\}$. Further, the ellipsis in (2.42) denotes infinitely many terms consisting of $\geq 7$ scalar fields $\varphi_i$. These non-Renormalisable higher order terms are introduced by the Renormalisation function $\mu^{2\epsilon}(\sigma)$ and the expansion w.r.t. $\epsilon$ and $\mathfrak{D}/w$ in order to obtain a Lagrangian consisting only of fields of integer powers, as mentioned in the previous section. The tadpoles $\widetilde{T}_{\varphi_i}$, the squared masses $\widetilde{M}^2_{ij}$ and the coefficients $\widetilde{\mathcal{V}}_{ijk\cdots}$ in (2.42) are given by

$$\begin{aligned}\widetilde{T}_{\varphi_i} &\coloneqq \left.\frac{\partial\widetilde{V}}{\partial\varphi_i}\right|_{\varphi_s=0,\,\forall\,s} \\ \widetilde{M}^2_{ij} &\coloneqq \left.\frac{\partial^2\widetilde{V}}{\partial\varphi_i\partial\varphi_j}\right|_{\varphi_s=0,\,\forall\,s} \\ \widetilde{\mathcal{V}}_{ijk\cdots} &\coloneqq \left.\frac{\partial^a\widetilde{V}}{\partial\varphi_i\partial\varphi_j\partial\varphi_k\cdots}\right|_{\varphi_s=0,\,\forall\,s}\end{aligned} \tag{2.43}$$

where $a$ is determined by the number of indices of $\widetilde{\mathcal{V}}_{ijk\cdots}$, i.e. $a \coloneqq \text{length}[(i,j,k,\ldots)]$. $\widetilde{M}^2_{ij}$ and $\widetilde{\mathcal{V}}_{ijk\cdots}$ are symmetric under index exchange by definition. Explicit expressions of the coefficients in (2.43) for the given potential (2.41) are to be found in appendix C in equations (C.2) to (C.37). It can be seen that these coefficients (2.43) contain $\epsilon$ -dependent, i.e. *evanescent*, terms due to $\mu^{2\epsilon}(\sigma)$, as discussed in the previous section. As in 4 dimensions, the matrix of squared masses $\widetilde{\mathcal{M}}^2_{\phi\sigma}$, given in (C.5), is a non-diagonal matrix and so is the associated propagator for the fields $h$ and $\mathfrak{D}$

$$\widetilde{D}_p = p^2 - \widetilde{\mathcal{M}}^2_{\phi\sigma} \tag{2.44}$$

The inverse propagator is obtained by inverting (2.44) and can be written as

$$\widetilde{D}_p^{-1} = \frac{\widetilde{A}}{p^2 - \widetilde{M}^2_H} + \frac{\widetilde{B}}{p^2 - \widetilde{M}^2_S} \tag{2.45}$$

where $\widetilde{M}_H^2$ and $\widetilde{M}_S^2$ are the eigenvalues of $\widetilde{\mathcal{M}}_{\phi\sigma}^2$, given in (C.17) & (C.18). Further, $\widetilde{A}$ and $\widetilde{B}$ are symmetric matrices whose properties and explicit expressions are given in (C.28) to (C.37).

The minimalisation conditions for the potential (2.41) in $D = 4 - 2\epsilon$ dimensions are provided by

$$\begin{aligned} \widetilde{T}_h &= \frac{\partial \widetilde{V}}{\partial h}\bigg|_{\substack{h=0\\ \mathfrak{D}=0}} \equiv \frac{\partial \widetilde{V}}{\partial \phi}\bigg|_{\substack{\phi=v\\ \sigma=w}} = 0 \;\Leftrightarrow\; v\left(\frac{\lambda_\phi}{6}\,v^2 + \frac{\lambda_m}{2}\,w^2\right) = 0 \\ \widetilde{T}_{\mathfrak{D}} &= \frac{\partial \widetilde{V}}{\partial \mathfrak{D}}\bigg|_{\substack{h=0\\ \mathfrak{D}=0}} \equiv \frac{\partial \widetilde{V}}{\partial \sigma}\bigg|_{\substack{\phi=v\\ \sigma=w}} = 0 \;\Leftrightarrow\; \frac{w}{1-\epsilon}\left(\frac{\epsilon\,\lambda_\phi}{12}\,\frac{v^4}{w^2} + \frac{\lambda_m}{2}\,v^2 + \frac{2-\epsilon}{12}\,\lambda_\sigma\,w^2\right) = 0 \end{aligned} \tag{2.46}$$

where the second relation has not been expanded w.r.t. $\epsilon$, as it is done in (C.3). While the first condition in (2.46) is the same as in the 4-dimensional case, it can be seen that the second one in (2.46) is different compared to the 4-dimensional case in (2.26), i.e. obtains $\epsilon$ - corrections in $D = 4 - 2\epsilon$ dimensions. Nonetheless, solving (2.46), provides the same $\epsilon$ - independent minimalisation conditions for the coupling constants as in (2.28), i.e.

$$\begin{aligned} \lambda_m &= -\frac{1}{3}\,\lambda_\phi\,\frac{v^2}{w^2} \\ \lambda_\sigma &= \lambda_\phi\,\frac{v^4}{w^4} \end{aligned} \tag{2.47}$$

Using these conditions (2.47) yields not only vanishing tadpoles $\widetilde{T}_{\varphi_i} = 0\;\forall\, i$, but also a vanishing vacuum energy $\widetilde{V}(v, w) = 0$, and further reduces $\widetilde{\mathcal{M}}_{\phi\sigma}^2$, given in (C.5) & (C.6), and its eigenvalues $\widetilde{M}_H^2$ and $\widetilde{M}_S^2$, given in (C.17) & (C.18), to the same results as in the 4-dimensional case in (2.34) and (2.36) multiplied by $\mu_0^{2\epsilon}$, respectively. Then, this also leads to a reduction of the relations for the mixing angle $\beta$ to the same as in 4 dimensions, i.e. to (2.35). Thus, even in $D = 4 - 2\epsilon$ dimensions one obtains the same matrix of squared masses after applying (2.47)

$$\widetilde{\mathcal{M}}_{\phi\sigma}^2 \xrightarrow[(2.47)]{} \widetilde{\mathcal{M}}_{\phi\sigma}^2 = \mathcal{M}_{\phi\sigma}^2 = \frac{1}{3}\,\mu_0^{2\epsilon}\,\lambda_\phi\,v^2 \begin{pmatrix} 1 & -\frac{v}{w} \\ -\frac{v}{w} & \frac{v^2}{w^2} \end{pmatrix}, \tag{2.48}$$

as well as the same squared masses

$$\begin{aligned} \widetilde{M}_H^2 &\xrightarrow[(2.47)]{} \widetilde{M}_H^2 = M_H^2 = \frac{1}{3}\,\mu_0^{2\epsilon}\,\lambda_\phi\,v^2\left(1 + \frac{v^2}{w^2}\right) \\ \widetilde{M}_S^2 &\xrightarrow[(2.47)]{} \widetilde{M}_S^2 = M_S^2 = 0 \end{aligned} \tag{2.49}$$

for the same mass eigenstates $H$ and $S$ with the same mixing angle $\beta$ as in the 4-dimensional case. The reason for this is that zero vacuum energy and tadpoles lead to the fact that new evanescent corrections due to $\mu^{2\epsilon}(\sigma) = \mu_0^{2\epsilon}\,(1 + \text{corrections})$ can then only arise above the bilinear mass terms, i.e. at terms with 3 fields and more, since these evanescent corrections are always accompanied by at least 1 additional power of the field $\mathfrak{D}$, c.f. (B.2). Note that if one does not use the minimalisation conditions (2.47) the relations for the mixing angle $\beta$ would otherwise be far more complicated containing $\epsilon$ - dependent corrections due to evanescent corrections in the mass terms (C.5) & (C.6), and therefore would make the transition to mass eigenstates more complicated.

Now, using first the minimalisation conditions (2.47) and then transforming to mass eigenstates $H$ and $S$, using (2.32), the potential, given in (2.41) & (2.42), can be written as

$$\begin{aligned} \widetilde{V}(H,S) = \frac{1}{2}\,\widetilde{M}_H^2\,H^2 + \frac{1}{3!}\,\widetilde{\lambda}_{ijk}\,\rho_i\,\rho_j\,\rho_k + \frac{1}{4!}\,\widetilde{\lambda}_{ijkl}\,\rho_i\,\rho_j\,\rho_k\,\rho_l \\ + \frac{1}{5!}\,\widetilde{\lambda}_{ijklm}\,\rho_i\,\rho_j\,\rho_k\,\rho_l\,\rho_m + \frac{1}{6!}\,\widetilde{\lambda}_{ijklmn}\,\rho_i\,\rho_j\,\rho_k\,\rho_l\,\rho_m\,\rho_n + \dots \end{aligned} \tag{2.50}$$

where $\{\rho_i\}_{i=1}^2 = \{H,S\}$. Again, the ellipsis denotes infinitely many terms with $\geq 7$ scalar fields $\rho_i$ and explicit expressions for the symmetric coefficients $\widetilde{\lambda}_{ijk\cdots}$ are to be found in appendix C in equations (C.40) to (C.48).

For further details regarding the 2 Scalar Model the reader is referred to [11, 13, 14, 20].

## 2.3. Renormalisation of the QSI 2 Scalar Model

In this section the Renormalisation of the QSI 2 Scalar Model (2.40), introduced in the previous section, is discussed. As already mentioned in section 2.1 and discussed in [11, 12, 14, 20, 21, 33], a quantum scale invariant theory is non-Renormalisable due to the Renormalisation function $\mu(\sigma)$, which introduces infinitely many new *evanescent interactions* to the action in $D = 4 - 2\epsilon$ dimensions. These evanescent interactions can give rise to finite and divergent quantum corrections due to divergences in loop-calculations (c.f. section 2.1), and thus higher dimensional non-polynomial operators of the form

$$\frac{\phi^{4+2\,p}}{\sigma^{2\,p}}\,, \qquad p = 1,2,3,\dots \tag{2.51}$$

emerge at loop-level, up to $p \leq L$ for $L$ loops. This is explicitly shown in chapter 3 for the QSI 2 Scalar Model.

For this reason, these higher order terms need to be included at least as counterterms in order to renormalise the theory. Thus, one actually needs to consider the following (more general) potential

$$\begin{aligned} V(\phi,\sigma) &= \frac{\lambda_\phi}{4!}\,\phi^4 + \frac{\lambda_m}{4}\,\phi^2\,\sigma^2 + \frac{\lambda_\sigma}{4!}\,\sigma^4 + \sum_{n=1}^{\infty} \frac{\lambda_{4+2\,n}}{4+2\,n}\,\frac{\phi^{4+2\,n}}{\sigma^{2\,n}} \\ \widetilde{V}(\phi,\sigma) &= \mu^{2\epsilon}(\sigma)\,V(\phi,\sigma) \end{aligned} \tag{2.52}$$

instead of (2.41). However, these higher order coupling constants $\lambda_{4+2\,n}$ can be set to zero at tree-level, i.e.

$$\lambda_{4+2\,n} \equiv 0 \;\; \forall\, n \tag{2.53}$$

which is done in this thesis (except something else is stated), such that these higher dimensional operators (2.51) only need to be included as counterterms and only up to the corresponding loop-order that is considered.

**Remark.**

(i) At the 1-loop-level only new *finite* quantum corrections due to evanescent interactions can emerge, because there are only simple $\epsilon$ - poles at the 1-loop-level. Consequently, evanescent interactions $\sim \epsilon^k$, $k \in \mathbb{N}$, can then only give rise to

- new finite quantum corrections, for $k = 1$
- still evanescent quantum corrections, for $k \geq 2$

Note that evanescent corrections will ultimately not contribute as $\epsilon \to 0$.

$\Rightarrow$ The divergence structure remain unchanged at the 1-loop-level, and thus 1-loop counterterms in a QSI theory, i.e. a SIDReg-regularised theory, are the same as in the usual DReg case.

$\Rightarrow$ There are no higher dimensional counterterms of the form (2.51) needed at the 1-loop-level. This is explicitly shown in chapter 3. There it can bee seen that a higher dimensional term of the form $\phi^6/\sigma^2$ indeed emerges at the 1-loop-level in the effective potential due to evanescent interactions, however, it emerges as a finite quantum correction, and thus does not need to be renormalised (with a counterterm).

(ii) At the 2-loop-level, however, there are not only new *finite* but also new *divergent* quantum corrections due to evanescent interactions. The reason for this is that at the 2-loop-level there are simple $\epsilon$ - poles and $\epsilon$ - poles of the order 2. Therefore, evanescent interactions $\sim \epsilon^k$, $k \in \mathbb{N}$, can then give rise to

- new finite quantum corrections, for $k = 1$ and $k = 2$ meeting a $1/\epsilon$ - pole and $1/\epsilon^2$ - pole, respectively
- new divergent quantum corrections, for $k = 1$ meeting a $1/\epsilon^2$ - pole
- still evanescent quantum corrections, in the other cases

Again, evanescent corrections do ultimately not contribute as $\epsilon \to 0$.

$\Rightarrow$ The divergence structure does change at the 2-loop-level in a QSI theory, and thus the 2-loop counterterms are different in the SIDReg-regularised theory compared to the usual DReg case.

$\Rightarrow$ Higher dimensional counterterms of the form (2.51) are needed at the 2-loop-level, which is explicitly shown in chapter 3. There it can be seen that higher dimensional terms of the form $\phi^6/\sigma^2$ and $\phi^8/\sigma^4$ emerge as divergent quantum corrections at the 2-loop-level in the effective potential due to evanescent interactions, i.e. with $1/\epsilon$ - divergence, and thus need to be renormalised with *non-polynomial* counterterms.

Moreover, note that non-Renormalisability due to quantum scale invariance is not necessarily a problem, since gravity is non-renormalisable as well and every theory of nature must ultimately contain gravity. Further, as discussed in [20], the usual renormalisable theory is a low energy effective theory valid below the scale of spontaneous (quantum) scale symmetry breaking $\langle\sigma\rangle$.

Now, consider the Renormalisation transformation for the QSI 2 Scalar Model (2.40)

$$\begin{aligned} S &\longrightarrow S_0 = S_{\text{ren}} + S_{\text{ct}} \\ \mathcal{L} &\longrightarrow \mathcal{L}_0 = \mathcal{L}_{\text{ren}} + \mathcal{L}_{\text{ct}} \\ \phi &\longrightarrow \phi_0 = \sqrt{Z_\phi}\,\phi \\ \sigma &\longrightarrow \sigma_0 = \sqrt{Z_\sigma}\,\sigma \\ \lambda_k &\longrightarrow \lambda_{k,B} = \mu^{2\epsilon}(\sigma)\,\lambda_{k,0} = \mu^{2\epsilon}(\sigma)\,Z_{\lambda_k}\,\lambda_k \end{aligned} \tag{2.54}$$

The counterterm Lagrangian is then given by

$$\begin{aligned} \mathcal{L}_{\text{ct}} &= (Z_\phi - 1)\,\frac{1}{2}\,\partial_\mu\phi\partial^\mu\phi + (Z_\sigma - 1)\,\frac{1}{2}\,\partial_\mu\sigma\partial^\mu\sigma \\ &\quad - \mu^{2\epsilon}(\sigma)\,\Bigg(\left(Z_{\lambda_\phi}\,Z_\phi^2 - 1\right)\,\frac{\lambda_\phi}{4!}\,\phi^4 + (Z_{\lambda_m}\,Z_\phi\,Z_\sigma - 1)\,\frac{\lambda_m}{4}\,\phi^2\,\sigma^2 \\ &\qquad + \left(Z_{\lambda_\sigma}\,Z_\sigma^2 - 1\right)\,\frac{\lambda_\sigma}{4!}\,\sigma^4 + \frac{\delta\lambda_6^{(2)}}{6}\,\frac{\phi^6}{\sigma^2} + \frac{\delta\lambda_8^{(2)}}{8}\,\frac{\phi^8}{\sigma^4} + \ldots\Bigg) \end{aligned} \tag{2.55}$$

including the non-polynomial 2-loop counterterms $\delta\lambda_6^{(2)}$ and $\delta\lambda_8^{(2)}$ which are necessary for the Renormalisation of the theory at the 2-loop-level, as explained above. It is convenient to further define

$$\begin{aligned} Z_{V_\phi} &:= Z_{\lambda_\phi}\,Z_\phi^2 \\ Z_{V_m} &:= Z_{\lambda_m}\,Z_\phi\,Z_\sigma \\ Z_{V_\sigma} &:= Z_{\lambda_\sigma}\,Z_\sigma^2 \end{aligned} \tag{2.56}$$

Expanding these counterterms up to $\mathcal{O}\left(\hbar^2\right)$ yields

$$\begin{aligned} Z_{V_\phi} &= 1 + \delta Z_{V_\phi}^{(1)} + \delta Z_{V_\phi}^{(2)} + \mathcal{O}\left(\hbar^3\right) \\ Z_{V_m} &= 1 + \delta Z_{V_m}^{(1)} + \delta Z_{V_m}^{(2)} + \mathcal{O}\left(\hbar^3\right) \\ Z_{V_\sigma} &= 1 + \delta Z_{V_\sigma}^{(1)} + \delta Z_{V_\sigma}^{(2)} + \mathcal{O}\left(\hbar^3\right) \end{aligned} \tag{2.57}$$

with

$$\begin{aligned} \delta Z_{V_\phi}^{(1)} &= \delta Z_{\lambda_\phi}^{(1)} + 2\,\delta Z_\phi^{(1)} \\ \delta Z_{V_\phi}^{(2)} &= \delta Z_{\lambda_\phi}^{(2)} + 2\,\delta Z_\phi^{(2)} + \left(\delta Z_\phi^{(1)}\right)^2 + 2\,\delta Z_{\lambda_\phi}^{(1)}\,\delta Z_\phi^{(1)} \\ \delta Z_{V_m}^{(1)} &= \delta Z_{\lambda_m}^{(1)} + \delta Z_\phi^{(1)} + \delta Z_\sigma^{(1)} \\ \delta Z_{V_m}^{(2)} &= \delta Z_{\lambda_m}^{(2)} + \delta Z_\phi^{(2)} + \delta Z_\sigma^{(2)} + \delta Z_{\lambda_m}^{(1)}\,\delta Z_\phi^{(1)} + \delta Z_{\lambda_m}^{(1)}\,\delta Z_\sigma^{(1)} + \delta Z_\phi^{(1)}\,\delta Z_\sigma^{(1)} \\ \delta Z_{V_\sigma}^{(1)} &= \delta Z_{\lambda_\sigma}^{(1)} + 2\,\delta Z_\sigma^{(1)} \\ \delta Z_{V_\sigma}^{(2)} &= \delta Z_{\lambda_\sigma}^{(2)} + 2\,\delta Z_\sigma^{(2)} + \left(\delta Z_\sigma^{(1)}\right)^2 + 2\,\delta Z_{\lambda_\sigma}^{(1)}\,\delta Z_\sigma^{(1)} \end{aligned} \tag{2.58}$$

Thus, the counterterm Lagrangian up to the 2-loop-level may be written as

$$\begin{aligned}
\mathcal{L}_{\text{ct}} &= \mathcal{L}_{\text{ct1}} + \mathcal{L}_{\text{ct2}} + \mathcal{O}\left(\hbar^3\right) \\
&= \frac{1}{2}\,\delta Z_\phi^{(1)}\,\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}\,\delta Z_\sigma^{(1)}\,\partial_\mu\sigma\partial^\mu\sigma \\
&- \mu^{2\epsilon}(\sigma)\left(\delta Z_{V_\phi}^{(1)}\,\frac{\lambda_\phi}{4!}\,\phi^4 + \delta Z_{V_m}^{(1)}\,\frac{\lambda_m}{4}\,\phi^2\,\sigma^2 + \delta Z_{V_\sigma}^{(1)}\,\frac{\lambda_\sigma}{4!}\,\sigma^4\right) \\
&+ \frac{1}{2}\,\delta Z_\phi^{(2)}\,\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}\,\delta Z_\sigma^{(2)}\,\partial_\mu\sigma\partial^\mu\sigma \\
&- \mu^{2\epsilon}(\sigma)\left(\delta Z_{V_\phi}^{(2)}\frac{\lambda_\phi}{4!}\phi^4 + \delta Z_{V_m}^{(2)}\frac{\lambda_m}{4}\phi^2\sigma^2 + \delta Z_{V_\sigma}^{(2)}\frac{\lambda_\sigma}{4!}\sigma^4 + \frac{\delta\lambda_6^{(2)}}{6}\frac{\phi^6}{\sigma^2} + \frac{\delta\lambda_8^{(2)}}{8}\frac{\phi^8}{\sigma^4}\right) \\
&+ \mathcal{O}\left(\hbar^3\right)
\end{aligned} \tag{2.59}$$

Note that in (2.55) and (2.59) it has already been used that $\delta\lambda_6^{(1)} \equiv 0$ and $\delta\lambda_8^{(1)} \equiv 0$ at the 1-loop-level, as discussed above.

For 1-loop calculations (e.g. for scattering processes or decays) it will be useful to express the 1-loop counterterm Lagrangian $\mathcal{L}_{\text{ct1}}$ in terms of mass eigenstates $H$ and $S$, and with the minimalisation conditions (2.47) being used. Expanding about the fields VEVs and w.r.t. $\epsilon$, as well as using (2.47) and (2.32) the 1-loop counterterm Lagrangian is then given by

$$\begin{aligned}
\mathcal{L}_{\text{ct1}} &= \frac{1}{2}\,\delta Z_H\,\partial_\mu H\partial^\mu H + \frac{1}{2}\,\delta Z_S\,\partial_\mu S\partial^\mu S + \delta Z_{HS}\,\partial_\mu H\partial^\mu S \\
&- \mu_0^{2\epsilon}\,\delta V_0 - \mu_0^{2\epsilon}\,(\delta T_H + \epsilon\,\delta Y_1)\,H - \mu_0^{2\epsilon}\,(\delta T_S + \epsilon\,\delta Y_2)\,S \\
&- \frac{1}{2}\,(\delta Z_H + \delta Z_{M_H} + \epsilon\,\delta Y_{11})\,M_H^2\,H^2 - \frac{1}{2}\,\mu_0^{2\epsilon}\left(\delta M_S^2 + \epsilon\,\delta Y_{22}\right)S^2 \\
&- \frac{1}{2}\,\mu_0^{2\epsilon}\left(\delta M_{HS}^2 + \epsilon\,\delta Y_{12}\right)H\,S - \mu_0^{2\epsilon}\,(\delta Z_{111} + \epsilon\,\delta Y_{111})\,\frac{\lambda_\phi}{3!}\,v\,H^3 \\
&- \mu_0^{2\epsilon}\,(\delta Z_{112} + \epsilon\,\delta Y_{112})\,\frac{\lambda_\phi}{2}\,v\,H^2\,S - \mu_0^{2\epsilon}\,(\delta Z_{122} + \epsilon\,\delta Y_{122})\,\frac{\lambda_\phi}{2}\,v\,H\,S^2 \\
&- \mu_0^{2\epsilon}\,(\delta Z_{222} + \epsilon\,\delta Y_{222})\,\frac{\lambda_\phi}{3!}\,v\,S^3 - \mu_0^{2\epsilon}\,(\delta Z_{1111} + \epsilon\,\delta Y_{1111})\,\frac{\lambda_\phi}{4!}\,H^4 \\
&- \mu_0^{2\epsilon}\,(\delta Z_{1112} + \epsilon\,\delta Y_{1112})\,\frac{\lambda_\phi}{3!}\,H^3\,S - \mu_0^{2\epsilon}\,(\delta Z_{1122} + \epsilon\,\delta Y_{1122})\,\frac{\lambda_\phi}{4}\,H^2\,S^2 \\
&- \mu_0^{2\epsilon}\,(\delta Z_{1222} + \epsilon\,\delta Y_{1222})\,\frac{\lambda_\phi}{3!}\,H\,S^3 - \mu_0^{2\epsilon}\,(\delta Z_{2222} + \epsilon\,\delta Y_{2222})\,\frac{\lambda_\phi}{4!}\,S^4 + \cdots
\end{aligned} \tag{2.60}$$

where the ellipsis denotes infinitely many terms of higher orders in the fields as well as in $\epsilon$ and explicit expressions for the counterterms in (2.60) that are not multiplied by $\epsilon$ are provided in (C.60) & (C.61).

**Remark.**

(i) Important for the Renormalisation transformation is that it makes the theory finite, which is achieved by (2.54), as shown in chapter 3.

(ii) The coefficients of the expanded 1-loop counterterm Lagrangian $\mathcal{L}_{\text{ct1}}$ in terms of flavour eigenstates $\{h, \mathfrak{D}\}$ and in terms of mass eigenstates $\{H, S\}$ are given in appendix C.

(iii) Note that the factor of $\mu_0^{2\epsilon}$ in the $H^2$ mass term is implicitly contained in (2.60) via the squared mass $M_H^2 = \frac{1}{3}\,\mu_0^{2\epsilon}\,\lambda_\phi\,v^2\,\left(1 + \frac{v^2}{w^2}\right)$.

(iv) The counterterm superscripts "$(k)$" indicating the $k$-th counterterm order are suppressed in (2.60) for simplicity since all counterterms in (2.60) are of 1-loop order.

(v) Note finite contributions in (2.60) from $\epsilon\,\delta Y_{ij\ldots}$, as the 1-loop counterterms contain $1/\epsilon$ - poles.

# 3. Scale Invariant Effective Potential

In this chapter the effective potential for the quantum scale invariant 2 Scalar Model (2.40), discussed in section 2.2, is determined up to the 2-loop level. This is done in order to show that there is indeed no anomalous breaking of scale symmetry due to quantum corrections, i.e. that scale invariance is maintained at the quantum level, if SIDReg is used to regularise the theory, and thus that the effective potential is a homogeneous function satisfying (2.13). Moreover, it is shown that non-vanishing $\beta$-functions, and thus running couplings are obtained despite quantum scale symmetry, as well as the fact that the quantum scale invariant effective potential satisfies the Callan-Symanzik equations. These evaluations have been done in [11, 13] at the 1-loop level and in [14] at the 2-loop level. However, some typos have been spotted in the 2-loop results for the finite contributions to the effective potential in [14]. Furthermore, the divergent parts of the self energies for $\phi$ and $\sigma$ in the QSI 2 Scalar Model are determined explicitly at the 2-loop level, which has not been done in the literature so far. In addition to that, in the last section of this chapter it is shown that the same scale invariant counterterms are obtained from $N$-point Green functions with non-vanishing external momenta using the expanded Lagrangian of the broken phase of the theory compared to those counterterms obtained from the manifestly scale invariant effective potential. This is done up to the 2-loop level as a consistency check that working in the broken phase of theory in the context of quantum scale symmetry does not explicitly break scale symmetry, but leads to the same scale invariant counterterms as in a manifestly (quantum) scale invariant approach via the effective potential, and also has not yet been done explicitly in the literature so far.

As mentioned above, the Lagrangian of the theory that is considered in this chapter is given in (2.40). In this chapter (except for the last section) the Lagrangian is not expanded about the fields VEVs, and thus the theory is considered in its unbroken and not its broken phase. However, a field shift of the form

$$\begin{aligned} \phi &\longrightarrow \phi + \phi_0 \\ \sigma &\longrightarrow \sigma + \sigma_0 \end{aligned} \tag{3.1}$$

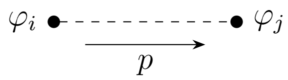
where $\phi_0$ and $\sigma_0$ are background fields, is applied in order to determine the effective potential, as explained in [39, 43]. Due to this field shift (3.1), all quantities are derived from the shifted Lagrangian $\mathcal{L}\left(\phi + \phi_0, \sigma + \sigma_0\right)$, and thus all equations from (C.1) to (C.38), as well as (C.49) to (C.58), can still be used if the replacement $h \to \phi$, $v \to \phi_0$, $\mathfrak{D} \to \sigma$ and $w \to \sigma_0$ is made, with the difference that the masses and coefficients are then field dependent quantities. Note that these background fields $\phi_0$ and $\sigma_0$ are charged under dilatations, such that the Lagrangian is still manifestly scale invariant, i.e. in the unbroken phase of theory, as mentioned in the last remark of section 2.1.

## 3.1. 1-Loop Effective Potential

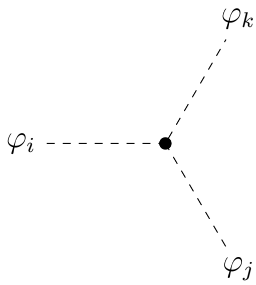
The 1-loop contribution to the effective potential $V_{\text{eff}}$ is given by

$$V_{1\text{L}} = -\frac{i}{2} \int \frac{d^D k}{(2\,\pi)^D}\, \text{Tr}\Big[\log\Big(k^2 - \widetilde{\mathcal{M}}^2_{\phi\sigma}\Big)\Big] \tag{3.2}$$

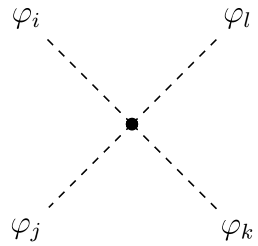
as stated in [11, 14]. However, this relation should be re-derived for the quantum scale invariant case as a consistency check whether (3.2) still holds in a QSI theory regularised using SIDReg with a dynamical Renormalisation function $\mu(\sigma)$. First, consider the relevant Feynman rules for the QSI 2 Scalar Model




$$= i\left(\widetilde{D}_p^{-1}\right)_{ij}$$




$$= -\,i\,\widetilde{\mathcal{V}}_{ijk},$$




$$= -\,i\,\widetilde{\mathcal{V}}_{ijkl} \tag{3.3}$$

where $\{\varphi_i\}_{i=1}^2 = \{\phi, \sigma\}$, derived from the Lagrangian (2.40) with the potential (2.41), this time, however, with the field shift (3.1) being used, i.e. $\mathcal{L}\,(\phi + \phi_0, \sigma + \sigma_0)$ and $\widetilde{V} = \widetilde{V}(\phi + \phi_0, \sigma + \sigma_0)$, such that the propagator and the coefficients $\widetilde{\mathcal{V}}_{ijk\cdots}$ in (3.3) are field dependent, i.e. dependent on $\phi_0$ and $\sigma_0$. Next, the following relations between masses and interaction coefficients are necessary for the derivation of (3.2).

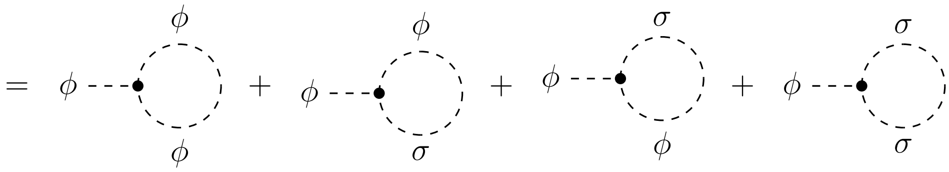
$$\begin{aligned}
\frac{\partial \widetilde{M}^2_{11}}{\partial \phi_0} &= \widetilde{\mathcal{V}}_{111}, & \frac{\partial \widetilde{M}^2_{11}}{\partial \sigma_0} &= \widetilde{\mathcal{V}}_{112} \\
\frac{\partial \widetilde{M}^2_{12}}{\partial \phi_0} &= \widetilde{\mathcal{V}}_{112}, & \frac{\partial \widetilde{M}^2_{12}}{\partial \sigma_0} &= \widetilde{\mathcal{V}}_{122} \\
\frac{\partial \widetilde{M}^2_{22}}{\partial \phi_0} &= \widetilde{\mathcal{V}}_{122}, & \frac{\partial \widetilde{M}^2_{22}}{\partial \sigma_0} &= \widetilde{\mathcal{V}}_{222}
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
\frac{\partial \widetilde{M}^2_{H}}{\partial \phi_0} &= \widetilde{\mathcal{V}}_{111}\,\widetilde{A}_{11} + 2\,\widetilde{\mathcal{V}}_{112}\,\widetilde{A}_{12} + \widetilde{\mathcal{V}}_{122}\,\widetilde{A}_{22} \\
\frac{\partial \widetilde{M}^2_{S}}{\partial \phi_0} &= \widetilde{\mathcal{V}}_{111}\,\widetilde{B}_{11} + 2\,\widetilde{\mathcal{V}}_{112}\,\widetilde{B}_{12} + \widetilde{\mathcal{V}}_{122}\,\widetilde{B}_{22} \\
\frac{\partial \widetilde{M}^2_{H}}{\partial \sigma_0} &= \widetilde{\mathcal{V}}_{112}\,\widetilde{A}_{11} + 2\,\widetilde{\mathcal{V}}_{122}\,\widetilde{A}_{12} + \widetilde{\mathcal{V}}_{222}\,\widetilde{A}_{22} \\
\frac{\partial \widetilde{M}^2_{S}}{\partial \sigma_0} &= \widetilde{\mathcal{V}}_{112}\,\widetilde{B}_{11} + 2\,\widetilde{\mathcal{V}}_{122}\,\widetilde{B}_{12} + \widetilde{\mathcal{V}}_{222}\,\widetilde{B}_{22}
\end{aligned} \tag{3.5}$$

These relations can be derived by direct calculation using (C.6) to (C.10), as well as (C.17) and (C.29), with the replacement $v \to \phi_0$ and $w \to \sigma_0$ being used.

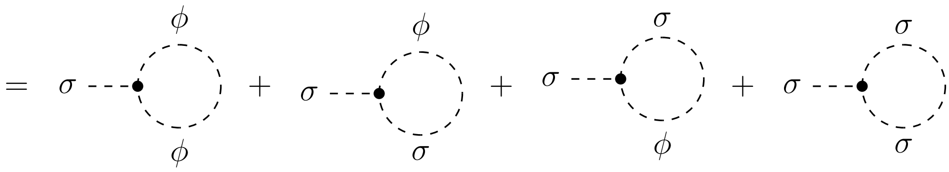
As explained in [39], the derivative of the effective potential is given by the 1PI 1-point Green function, calculated using the shifted Lagrangian $\mathcal{L}\left(\phi+\phi_0, \sigma+\sigma_0\right)$, or in other words the Feynman rules (3.3). Thus, at the 1-loop-level one obtains

$$-i\,\frac{\partial V_{1\mathrm{L}}}{\partial \phi_0} = i\,\frac{\delta \Gamma_{1\mathrm{L}}}{\delta \phi}\bigg|_{\substack{\phi=\phi_0\\ \sigma=\sigma_0}}$$




$$\begin{aligned}
&= \frac{1}{2}\left(-i\,\widetilde{\mathcal{V}}_{1jk}\right)\int \frac{d^D q}{(2\,\pi)^D}\, i\left(\widetilde{D}_q^{-1}\right)_{jk} \qquad (3.6)\\
&= \frac{1}{2}\int \frac{d^D q}{(2\,\pi)^D}\left(\widetilde{\mathcal{V}}_{111}\left(\widetilde{D}_q^{-1}\right)_{11} + 2\,\widetilde{\mathcal{V}}_{112}\left(\widetilde{D}_q^{-1}\right)_{12} + \widetilde{\mathcal{V}}_{122}\left(\widetilde{D}_q^{-1}\right)_{22}\right)\\
&= -\frac{1}{2}\int \frac{d^D q}{(2\,\pi)^D}\left(\frac{1}{q^2-\widetilde{M}_H^2}\left(-\frac{\partial \widetilde{M}_H^2}{\partial \phi_0}\right) + \frac{1}{q^2-\widetilde{M}_S^2}\left(-\frac{\partial \widetilde{M}_S^2}{\partial \phi_0}\right)\right)\\
&= -\frac{1}{2}\int \frac{d^D q}{(2\,\pi)^D}\,\frac{\partial}{\partial \phi_0}\left(\log\left(q^2-\widetilde{M}_H^2\right) + \log\left(q^2-\widetilde{M}_S^2\right)\right)
\end{aligned}$$

$$-i\,\frac{\partial V_{1\mathrm{L}}}{\partial \sigma_0} = i\,\frac{\delta \Gamma_{1\mathrm{L}}}{\delta \sigma}\bigg|_{\substack{\phi=\phi_0\\ \sigma=\sigma_0}}$$




$$\begin{aligned}
&= \frac{1}{2}\left(-i\,\widetilde{\mathcal{V}}_{2jk}\right)\int \frac{d^D q}{(2\,\pi)^D}\, i\left(\widetilde{D}_q^{-1}\right)_{jk} \qquad (3.7)\\
&= \frac{1}{2}\int \frac{d^D q}{(2\,\pi)^D}\left(\widetilde{\mathcal{V}}_{112}\left(\widetilde{D}_q^{-1}\right)_{11} + 2\,\widetilde{\mathcal{V}}_{122}\left(\widetilde{D}_q^{-1}\right)_{12} + \widetilde{\mathcal{V}}_{222}\left(\widetilde{D}_q^{-1}\right)_{22}\right)\\
&= -\frac{1}{2}\int \frac{d^D q}{(2\,\pi)^D}\left(\frac{1}{q^2-\widetilde{M}_H^2}\left(-\frac{\partial \widetilde{M}_H^2}{\partial \sigma_0}\right) + \frac{1}{q^2-\widetilde{M}_S^2}\left(-\frac{\partial \widetilde{M}_S^2}{\partial \sigma_0}\right)\right)\\
&= -\frac{1}{2}\int \frac{d^D q}{(2\,\pi)^D}\,\frac{\partial}{\partial \sigma_0}\left(\log\left(q^2-\widetilde{M}_H^2\right) + \log\left(q^2-\widetilde{M}_S^2\right)\right)
\end{aligned}$$

where in the penultimate step of (3.6) and (3.7) the relations (3.4) and (3.5) have been used. Thus, from (3.6) and (3.7) it can be seen that the 1-loop effective potential is given by

$$V_{1\mathrm{L}} = -\frac{i}{2}\int \frac{d^D k}{(2\,\pi)^D}\,\left(\log\left(k^2 - \widetilde{M}_H^2\right) + \log\left(k^2 - \widetilde{M}_S^2\right)\right) \tag{3.8}$$

where the loop-momentum has been renamed $q \to k$. Formula (3.8) agrees with (3.2), given above, which can be seen by explicit calculation.

$$\begin{aligned}\mathrm{Tr}\Big[\log\left(k^2 - \widetilde{\mathcal{M}}_{\phi\sigma}^2\right)\Big] &= \log\left(\det\left(k^2 - \widetilde{\mathcal{M}}_{\phi\sigma}^2\right)\right) = \log\left(\left(k^2 - \widetilde{M}_H^2\right)\left(k^2 - \widetilde{M}_S^2\right)\right) \\ &= \log\left(k^2 - \widetilde{M}_H^2\right) + \log\left(k^2 - \widetilde{M}_S^2\right)\end{aligned} \tag{3.9}$$

Using (D.4), the 1-loop effective potential is given by

$$\begin{aligned} V_{1\mathrm{L}} &= -\frac{i}{2}\int \frac{d^D k}{(2\,\pi)^D}\,\mathrm{Tr}\Big[\log\left(k^2 - \widetilde{\mathcal{M}}_{\phi\sigma}^2\right)\Big] \\ &= -\frac{i}{2}\int \frac{d^D k}{(2\,\pi)^D}\,\left(\log\left(k^2 - \widetilde{M}_H^2\right) + \log\left(k^2 - \widetilde{M}_S^2\right)\right) \\ &= -\frac{\mu^{2\epsilon}(\sigma_0)}{64\,\pi^2}\sum_{k=1}^{2}\hat{M}_{\rho_k}^4\left[\frac{1}{\epsilon} + \frac{3}{2} - \log\left(\frac{M_{\rho_k}^2}{4\,\pi\,\mu^2(\sigma_0)}\,e^{\gamma_E}\right) + 2\,c_{\rho_k}^{(1)}\right] + \mathcal{O}\left(\epsilon\right) \\ &= -\frac{\mu^{2\epsilon}(\sigma_0)}{64\,\pi^2}\sum_{k=1}^{2}\hat{M}_{\rho_k}^4\left[\frac{1}{\epsilon} + \frac{3}{2} - \log\left(\frac{M_{\rho_k}^2}{4\,\pi\,\mu^2(\sigma_0)}\,e^{\gamma_E}\right)\right] + \Delta U_{1\mathrm{L}} + \mathcal{O}\left(\epsilon\right) \end{aligned} \tag{3.10}$$

where $\{\rho_k\}_{k=1}^2 = \{H, S\}$.

**Remark.**

(i) Note that $\hat{M}_{\rho_k}^2 := \mu^{-2\epsilon}(\sigma_0)\,M_{\rho_k}^2$, as defined and explained in the appendix D.1. Hence, $\hat{M}_{\rho_k}^2$ is an eigenvalue of $\mathcal{M}_{\phi\sigma}^2$ without the factor $\mu^{2\epsilon}(\sigma_0)$, c.f. (C.19). Consequently, the mass dimensions are given by $[\hat{M}_{\rho_k}^2] = 2 - 2\epsilon$ and $[M_{\rho_k}^2] = 2$.

(ii) As explained in the appendix D.1, the expression (3.10) is consistent w.r.t. mass dimensions, as $[V_{1\mathrm{L}}] = D = 4 - 2\epsilon$, $[\mu] = 1$ and the masses as above.

(iii) In order to compare (3.10) with the result in [11, 14] the following definition need to be made

$$N_{ij} := \left[\mu\left(\frac{\partial\mu}{\partial\varphi_i}\frac{\partial V}{\partial\varphi_j} + \frac{\partial\mu}{\partial\varphi_j}\frac{\partial V}{\partial\varphi_i}\right) + \left(\mu\,\frac{\partial^2\mu}{\partial\varphi_i\,\partial\varphi_j} - \frac{\partial\mu}{\partial\varphi_i}\frac{\partial\mu}{\partial\varphi_j}\right)V\right]_{\substack{\phi=\phi_0\\ \sigma=\sigma_0}} \tag{3.11}$$

where $\{\varphi_i\}_{i=1}^2 = \{\phi, \sigma\}$, and thus

$$\begin{aligned}\widetilde{M}_{ij}^2 = \frac{\partial^2\widetilde{V}}{\partial\varphi_i\,\partial\varphi_j}\bigg|_{\substack{\phi=\phi_0\\ \sigma=\sigma_0}} &= M_{ij}^2 + 2\,\epsilon\,\mu^{2\epsilon}\,\frac{N_{ij}}{\mu^2} + 4\,\epsilon^2\,\mu^{2\epsilon}\,\mu^{-2}\,\frac{\partial\mu}{\partial\varphi_i}\frac{\partial\mu}{\partial\varphi_j}\,V\,\bigg|_{\substack{\phi=\phi_0\\ \sigma=\sigma_0}} \\ &= M_{ij}^2 + 2\,\epsilon\,\mu^{2\epsilon}(\sigma_0)\,\frac{N_{ij}\big|_{\epsilon=0}}{\mu^2(\sigma_0)} + \mathcal{O}\left(\epsilon^2\right)\end{aligned} \tag{3.12}$$

Note that $N_{ij}$ needs to be evaluated at $\epsilon = 0$ for a consistent power series in $\epsilon$, because $N_{ij}$ itself also depends on $\epsilon$. The factor of $\mu^{2\epsilon}(\sigma_0)$ is not expanded because it is also contained in $\widetilde{M}_{ij}^2$ and $M_{ij}^2$ as part of the definition of squared masses, c.f. (C.6) and (C.8), respectively. In [11, 14] the factor $\mu^{2\epsilon}(\sigma_0)$ is only contained in $\widetilde{M}_{ij}^2$ but not in $M_{ij}^2$, which leads to a slightly different form of intermediate results that, however, are ultimately equal.
Now, comparing (3.12) with (C.6) leads to

$$N_{ij}\big|_{\epsilon=0} = \frac{1}{4}\,\mu^2(\sigma_0)\,\phi_0^2\,u_{ij}^{(1)} \qquad \Longleftrightarrow \qquad u_{ij}^{(1)} = 4\,\frac{N_{ij}\big|_{\epsilon=0}}{\mu^2(\sigma_0)\,\phi_0^2} \tag{3.13}$$

Using (3.13) and (C.8), one obtains

$$\begin{aligned} 4\,\frac{\mathrm{Tr}\left(\mathcal{M}_{\phi\sigma}^2\,N\big|_{\epsilon=0}\right)}{\mu^2(\sigma_0)} &= 4\,\frac{M_{ij}^2\,N_{ji}\big|_{\epsilon=0}}{\mu^2(\sigma_0)} = \frac{1}{2}\,\mu_0^{2\epsilon}(\sigma_0)\,\phi_0^4\,u_{ij}\,u_{ji}^{(1)} \\ &= \mu_0^{2\epsilon}(\sigma_0)\,2\left(\hat{M}_H^4\,c_H^{(1)} + \hat{M}_S^4\,c_S^{(1)}\right) \\ &= \mu_0^{2\epsilon}(\sigma_0)\sum_{k=1}^{2} 2\,\hat{M}_{\rho_k}^4\,c_{\rho_k}^{(1)} \end{aligned} \tag{3.14}$$

Using (3.14), it can be seen that (3.10) is equal to the results in [11, 14].

(iv) Beside the usual Coleman-Weinberg contribution, a new finite quantum correction

$$\begin{aligned} \Delta U_{1\mathrm{L}} &:= -\,\frac{\mu^{2\epsilon}(\sigma_0)}{32\,\pi^2}\sum_{k=1}^{2}\hat{M}_{\rho_k}^4\,c_{\rho_k}^{(1)} = -\,\frac{1}{16\,\pi^2}\,\frac{\mathrm{Tr}\left(\mathcal{M}_{\phi\sigma}^2\,N\big|_{\epsilon=0}\right)}{\mu^2(\sigma_0)} \\ &= \frac{\mu^{2\epsilon}(\sigma_0)}{64\,\pi^2}\left[\frac{\lambda_\phi\,\lambda_m}{12}\,\frac{\phi_0^6}{\sigma_0^2} - \left(\frac{4}{3}\,\lambda_\phi\,\lambda_m + \frac{3}{2}\,\lambda_m^2 - \frac{\lambda_\phi\,\lambda_\sigma}{12}\right)\phi_0^4 \right. \\ &\qquad \left. -\,\frac{\lambda_m}{12}\,(48\,\lambda_m + 25\,\lambda_\sigma)\,\phi_0^2\,\sigma_0^2 - \frac{7}{12}\,\lambda_\sigma^2\,\sigma_0^4\right] \end{aligned} \tag{3.15}$$

is obtained due to evanescent interactions, as discussed in chapter 2. It can be seen that this new finite quantum correction indeed contains a non-polynomial contribution of the form

$$\Delta U_{1\mathrm{L}} \supset \frac{\lambda_\phi\,\lambda_m}{12}\,\frac{\phi_0^6}{\sigma_0^2} \tag{3.16}$$

as mentioned in section 2.3.

(v) The background fields $\{\phi_0, \sigma_0\}$ in (3.10) can be replaced by the fields $\{\phi, \sigma\}$ in order to finally obtain the 1-loop contribution to the effective potential as a function of the fields $\{\phi, \sigma\}$, similar to the tree-level potential, i.e. $\mu = \mu(\sigma)$, $M_{\rho_k}^2 = M_{\rho_k}^2(\phi, \sigma)$, $c_{\rho_k}^{(1)} = c_{\rho_k}^{(1)}(\phi, \sigma)$, and thus $V_{1\mathrm{L}} = V_{1\mathrm{L}}(\phi, \sigma)$.

In order to renormalise the theory, and thus obtain a finite result, counterterms need to be determined. For this reason, the 1-loop counterterm Lagrangian $\mathcal{L}_{\text{ct1}}$ in (2.59) is used. The divergent part in (3.10) is given by

$$\begin{aligned} V_{\text{1L}}\Big|_{\text{div}} &= -\,\frac{\mu^{2\epsilon}(\sigma)}{64\,\pi^2}\,\sum_{k=1}^{2}\,\Big(\hat{M}^2_{\rho_k}(\phi,\sigma)\Big)^2\,\frac{1}{\epsilon} \\ &= -\,\frac{\mu^{2\epsilon}(\sigma)}{64\,\pi^2}\,\bigg[\Big(\hat{M}^2_{H}(\phi,\sigma)\Big)^2+\Big(\hat{M}^2_{S}(\phi,\sigma)\Big)^2\bigg]\,\frac{1}{\epsilon} \\ &= -\,\frac{\mu^{2\epsilon}(\sigma)}{64\,\pi^2}\,\bigg[\frac{1}{4}\,\left(\lambda_\phi^2+\lambda_m^2\right)\,\phi^4+\frac{1}{2}\,\lambda_m\,\left(\lambda_\phi+4\,\lambda_m+\lambda_\sigma\right)\,\phi^2\,\sigma^2 \\ &\qquad\qquad +\frac{1}{4}\,\left(\lambda_m^2+\lambda_\sigma^2\right)\,\sigma^4\bigg]\,\frac{1}{\epsilon} \end{aligned} \tag{3.17}$$

whereas the tree-level counterterm potential at the 1-loop order, c.f. (2.59), reads

$$\widetilde{V}_{\text{tree,ct1}} = \mu^{2\epsilon}(\sigma)\,\left(\delta Z^{(1)}_{V_\phi}\,\frac{\lambda_\phi}{4!}\,\phi^4+\delta Z^{(1)}_{V_m}\,\frac{\lambda_m}{4}\,\phi^2\,\sigma^2+\delta Z^{(1)}_{V_\sigma}\,\frac{\lambda_\sigma}{4!}\,\sigma^4\right) \tag{3.18}$$

Hence, the 1-loop counterterms in the MS-scheme are given by

$$\begin{aligned} \delta Z^{(1)}_{V_\phi} &= \frac{3}{32\,\pi^2}\,\frac{\lambda_\phi^2+\lambda_m^2}{\lambda_\phi}\,\frac{1}{\epsilon} \\ \delta Z^{(1)}_{V_m} &= \frac{1}{32\,\pi^2}\,\left(\lambda_\phi+4\,\lambda_m+\lambda_\sigma\right)\,\frac{1}{\epsilon} \\ \delta Z^{(1)}_{V_\sigma} &= \frac{3}{32\,\pi^2}\,\frac{\lambda_m^2+\lambda_\sigma^2}{\lambda_\sigma}\,\frac{1}{\epsilon} \end{aligned} \tag{3.19}$$

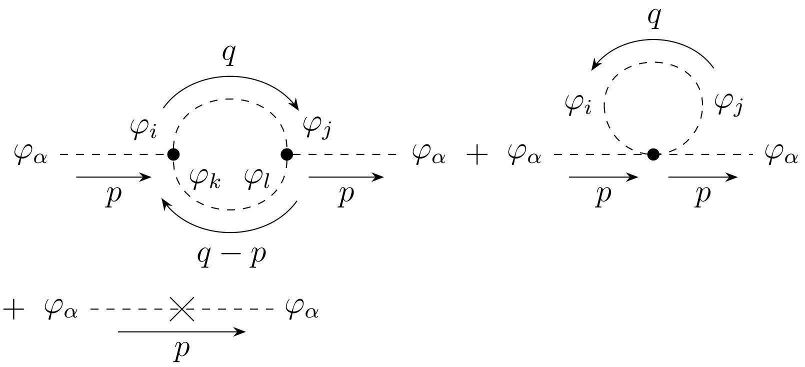
There are no higher dimensional non-polynomial operators in the divergent part of $V_{\text{1L}}$, c.f. (3.17). At the 1-loop-level, such higher order non-polynomial terms only emerge as new finite quantum corrections, as shown in (3.15) and (3.16). Consequently, no higher dimensional non-polynomial counterterms are necessary to renormalise the theory, i.e. $\delta\lambda_6^{(1)} = \cdots = 0$, as already discussed in section 2.3. Now, the (1-loop) wave function Renormalisation coefficients $\delta Z^{(1)}_\phi$ and $\delta Z^{(1)}_\sigma$ need to be determined. For a purely scalar theory, as considered here, the wave function Renormalisation coefficients are expected to be zero at the 1-loop-level. In order to show this explicitly, the following counterterm Feynman rules, again derived from the shifted Lagrangian $\mathcal{L}(\phi+\phi_0,\sigma+\sigma_0)$, are needed.

$$\begin{aligned} \phi\;\underset{\xrightarrow[\;p\;]{}}{\text{-----}\times\text{-----}}\;\phi &= i\,p^2\,\delta Z^{(1)}_\phi - i\,\mu^{2\epsilon}(\sigma_0)\,\left(\frac{\lambda_\phi}{2}\,\delta Z^{(1)}_{V_\phi}\,\phi_0^2+\frac{\lambda_m}{2}\,\delta Z^{(1)}_{V_m}\,\sigma_0^2\right) \\ \sigma\;\underset{\xrightarrow[\;p\;]{}}{\text{-----}\times\text{-----}}\;\sigma &= i\,p^2\,\delta Z^{(1)}_\sigma - i\,\mu^{2\epsilon}(\sigma_0)\,\left(\frac{\lambda_m}{2}\,\delta Z^{(1)}_{V_m}\,\phi_0^2+\frac{\lambda_\sigma}{2}\,\delta Z^{(1)}_{V_\sigma}\,\sigma_0^2\right) \end{aligned} \tag{3.20}$$

Note that terms of the order $\mathcal{O}\left(\epsilon^0\right)$, i.e. evanescent interaction times counterterm, have been neglected here, because such terms do not contribute to the MS-counterterms. Moreover, the mass dimensions are $[p^2] = 2$ and $[\phi_0^2] = [\sigma_0^2] = 2 - 2\epsilon$.

Using (D.2) & (D.6), the (renormalised) self-energies of $\phi$ and $\sigma$ are then provided by

$$-i\,\Sigma^{(1\mathrm{L})}_{\varphi_\alpha,\mathrm{ren}} =$$




$$\begin{aligned}
&= \frac{1}{2}\,\widetilde{\mathcal{V}}_{\alpha ik}\,\widetilde{\mathcal{V}}_{\alpha jl} \int \frac{d^D q}{(2\,\pi)^D} \left(\widetilde{D}^{-1}_{q}\right)_{ij} \left(\widetilde{D}^{-1}_{q-p}\right)_{lk} + \frac{1}{2}\,\widetilde{\mathcal{V}}_{\alpha\alpha ij} \int \frac{d^D q}{(2\,\pi)^D} \left(\widetilde{D}^{-1}_{q}\right)_{ji} \\
&\quad + i\,p^2\,\delta Z^{(1)}_{\varphi_\alpha} - i\,\mu^{2\epsilon}(\sigma_0) \left( \frac{\lambda_{\varphi_\alpha}}{2}\,\delta Z^{(1)}_{V_{\varphi_\alpha}}\,\varphi^2_{\alpha,0} + \frac{\lambda_m}{2}\,\delta Z^{(1)}_{V_m}\,\varphi^2_{\beta,0} \right) \\
&= \frac{i}{32\,\pi^2}\,\mu^{-2\epsilon}(\sigma_0)\,\widetilde{\mathcal{V}}_{\alpha ik}\,\widetilde{\mathcal{V}}_{\alpha jl} \left( \widetilde{A}_{ij} + \widetilde{B}_{ij} \right) \left( \widetilde{A}_{lk} + \widetilde{B}_{lk} \right) \frac{1}{\epsilon} \\
&\quad + \frac{i}{32\,\pi^2}\,\widetilde{\mathcal{V}}_{\alpha\alpha ij} \left( \widetilde{A}_{ji}\,\hat{M}^2_H + \widetilde{B}_{ji}\,\hat{M}^2_S \right) \frac{1}{\epsilon} + \cdots \\
&\quad + i\,p^2\,\delta Z^{(1)}_{\varphi_\alpha} - i\,\mu^{2\epsilon}(\sigma_0) \left( \frac{\lambda_{\varphi_\alpha}}{2}\,\delta Z^{(1)}_{V_{\varphi_\alpha}}\,\varphi^2_{\alpha,0} + \frac{\lambda_m}{2}\,\delta Z^{(1)}_{V_m}\,\varphi^2_{\beta,0} \right) \\
&= \frac{i}{32\,\pi^2}\,\mu^{2\epsilon}(\sigma_0) \left[ \lambda^2_{\varphi_\alpha}\,\varphi^2_{\alpha,0} + \lambda^2_m \left( \varphi^2_{\alpha,0} + 2\,\varphi^2_{\beta,0} \right) \right] \frac{1}{\epsilon} \\
&\quad + \frac{i}{64\,\pi^2}\,\mu^{2\epsilon}(\sigma_0) \left[ \left( \lambda^2_{\varphi_\alpha} + \lambda^2_m \right) \varphi^2_{\alpha,0} + \lambda_m \left( \lambda_{\varphi_\alpha} + \lambda_{\varphi_\beta} \right) \varphi^2_{\beta,0} \right] \frac{1}{\epsilon} \\
&\quad + i\,p^2\,\delta Z^{(1)}_{\varphi_\alpha} - i\,\mu^{2\epsilon}(\sigma_0) \left( \frac{\lambda_{\varphi_\alpha}}{2}\,\delta Z^{(1)}_{V_{\varphi_\alpha}}\,\varphi^2_{\alpha,0} + \frac{\lambda_m}{2}\,\delta Z^{(1)}_{V_m}\,\varphi^2_{\beta,0} \right) + \mathcal{O}\left(\epsilon^0\right) \\
&\overset{!}{=} \text{finite}
\end{aligned} \tag{3.21}$$

where $\{\varphi_i\}_{i=1}^2 = \{\phi, \sigma\}$, $i, j, k, l, \alpha, \beta \in \{1, 2\}$, it is implicitly summed over the roman indices $\{i, j, k, l\}$, but *not* over the greek indices $\{\alpha, \beta\}$, which are fixed to be either 1 or 2, and further $\alpha \neq \beta$, i.e. either $(\alpha, \beta) = (1, 2)$ for $\Sigma^{(1\mathrm{L})}_{\phi,\,\mathrm{ren}}$ or $(\alpha, \beta) = (2, 1)$ for $\Sigma^{(1\mathrm{L})}_{\sigma,\,\mathrm{ren}}$. Using (3.19) completely cancels all divergencies in (3.21), and thus one obtains

$$\delta Z^{(1)}_{\phi} = 0, \qquad \delta Z^{(1)}_{\sigma} = 0 \tag{3.22}$$

as expected.

**Remark** (MS to $\overline{\mathrm{MS}}$ scheme)**.**
The counterterms in (3.19) and (3.22) are given in the MS-scheme so far. However, all results can (and will) be expressed in the $\overline{\mathrm{MS}}$-scheme by absorbing $4\,\pi$ and $e^{\gamma_E}$ in $\mu$, i.e.

$$\mu^2 \longrightarrow \mu^2\,\frac{e^{\gamma_E}}{4\,\pi} \tag{3.23}$$

which is especially useful for multi-loop calculations, as discussed in [39].

Finally, after Renormalisation and then going back to 4 dimensions, i.e. $\epsilon \to 0$, the result for the effective potential up to the 1-loop level is given by

$$\begin{aligned} V_{\text{eff}}(\phi,\sigma) &= V_{\text{tree}}(\phi,\sigma) + V_{\text{tree,ct1}}(\phi,\sigma) + V_{\text{1L}}(\phi,\sigma) + \mathcal{O}\left(\hbar^2\right) \\ &= V_{\text{tree}}(\phi,\sigma) + V_{\text{1L,reg}}(\phi,\sigma) + \Delta U_{\text{1L}}(\phi,\sigma) + \mathcal{O}\left(\hbar^2\right) \end{aligned} \tag{3.24}$$

with

$$V_{\text{tree}}(\phi,\sigma) \equiv V(\phi,\sigma) = \frac{\lambda_\phi}{4!}\,\phi^4 + \frac{\lambda_m}{4}\,\phi^2\,\sigma^2 + \frac{\lambda_\sigma}{4!}\,\sigma^4, \tag{3.25}$$

and $V_{\text{1L,ren}} = V_{\text{tree,ct1}} + V_{\text{1L}} = V_{\text{1L,reg}} + \Delta U_{\text{1L}}$, i.e. in the $\overline{\text{MS}}$-scheme

$$\begin{aligned} V^{\overline{\text{MS}}}_{\text{1L,ren}}(\phi,\sigma) &= \frac{1}{64\,\pi^2}\sum_{k=1}^{2}\left(M^2_{\rho_k}(\phi,\sigma)\right)^2\left[\log\left(\frac{M^2_{\rho_k}(\phi,\sigma)}{\mu^2(\sigma)}\right) - \frac{3}{2} - 2\,c^{(1)}_{\rho_k}(\phi,\sigma)\right] \\ &= \frac{1}{64\,\pi^2}\sum_{k=1}^{2}\left(M^2_{\rho_k}(\phi,\sigma)\right)^2\left[\log\left(\frac{M^2_{\rho_k}(\phi,\sigma)}{\mu^2(\sigma)}\right) - \frac{3}{2}\right] + \Delta U_{\text{1L}}(\phi,\sigma) \end{aligned} \tag{3.26}$$

where $\{\rho_k\}_{k=1}^2 = \{H, S\}$ and $\Delta U_{\text{1L}}(\phi,\sigma)$ defined in (3.15).

**Remark.**

(i) It has been used that in 4 dimensions, i.e. in the limit $\epsilon \to 0$, $\hat{M}^2_{\rho_k}$ is identical to $M^2_{\rho_k}$. Explicit expressions for $M^2_{\rho_k}$ and $c^{(1)}_{\rho_k}$ are to be found in (C.19) and (C.20), respectively, with the replacement $v \to \phi$, $w \to \sigma$ being used.

(ii) As already mentioned, it can be seen that, beside the regular Coleman-Weinberg term $V_{\text{1L,reg}}$, a new finite quantum correction $\Delta U_{\text{1L}}$ is obtained due to evanescent interactions introduced by the Renormalisation function, i.e. as a result of QSI. This new quantum correction contains a higher dimensional non-polynomial operator as shown in (3.15) and (3.16).

(iii) The 1-loop effective potential (3.24) with (3.25) and (3.26) is a homogeneous function of the fields, and thus satisfies (2.13). No massive parameters are introduced at the quantum level due to the usage of SIDReg with a dynamical Renormalisation function $\mu(\sigma)$ instead of DReg. Hence, the theory indeed is scale invariant at the quantum level (at least at the 1-loop-level), i.e. quantum scale invariant, as intended.

(iv) The counterterms, given in (3.19) and (3.22), respect the symmetries of the theory, especially scale symmetry. Moreover, the 1-loop counterterms are the same as for the usual DReg case, i.e. they do not obtain corrections due to QSI. The reason for this is that the divergence structure remain unchanged at the 1-loop-level, since evanescent interactions can only lead to new *finite* quantum corrections at the 1-loop-level, as explained in section 2.3.

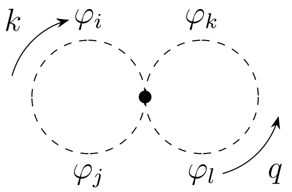
(v) The results in this section are in perfect agreement with the results presented in [11, 14].

For more details w.r.t. to the 1-loop effective potential of the QSI 2 Scalar Model the reader is referred to [11, 14, 20].

## 3.2. 2-Loop Effective Potential

The 2-loop contribution to the effective potential can be computed using the Feynman rules (3.3) and is basically given by 2 different kinds of Feynman diagrams, the snowman diagram and the sunset diagram given below. In addition to $\mathcal{L}_{\mathrm{ct2}}$ with $V_{\mathrm{tree,ct2}}$ given in (2.59), 1-loop diagrams with counterterm insertions giving rise to $V_{\mathrm{1L,ct1}}$ are needed for the Subrenormalisation of non-local divergences.

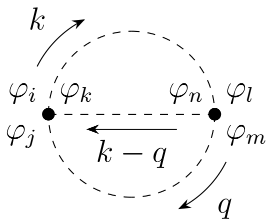
First, contributions from snowman diagrams are considered

$V_{\mathrm{2L}}^{(a)} = i$ 


$$
\begin{aligned}
&= -\frac{1}{8}\,\widetilde{\mathcal{V}}_{ijkl}\int\frac{d^Dk}{(2\,\pi)^D}\int\frac{d^Dq}{(2\,\pi)^D}\left(\widetilde{D}_k^{-1}\right)_{ij}\left(\widetilde{D}_q^{-1}\right)_{kl} \qquad (3.27)\\
&= \frac{1}{8}\,\frac{1}{\left(16\,\pi^2\right)^2}\,\mu^{2\epsilon}(\sigma_0)\,\widehat{\mathcal{V}}_{ijkl}\Big[\widetilde{A}_{ij}\,\widetilde{A}_{kl}\,J\left(\widetilde{M_H^2},\widetilde{M_H^2}\right)+\widetilde{A}_{ij}\,\widetilde{B}_{kl}\,J\left(\widetilde{M_H^2},\widetilde{M_S^2}\right)\\
&\qquad +\widetilde{B}_{ij}\,\widetilde{A}_{kl}\,J\left(\widetilde{M_S^2},\widetilde{M_H^2}\right)+\widetilde{B}_{ij}\,\widetilde{B}_{kl}\,J\left(\widetilde{M_S^2},\widetilde{M_S^2}\right)\Big]\\
&=: V_{\mathrm{2L},\,1/\epsilon^2}^{(a)}+V_{\mathrm{2L},\,1/\epsilon}^{(a)}+\Delta U_{\mathrm{2L},\,1/\epsilon}^{(a)}+V_{\mathrm{2L},\,\mathrm{fin}}^{(a)}+\Delta U_{\mathrm{2L},\,\mathrm{fin}}^{(a)}+\mathcal{O}\left(\epsilon\right)
\end{aligned}
$$

where $\widehat{\mathcal{V}}_{ijkl}$ is $\widetilde{\mathcal{V}}_{ijkl}$ without the factor of $\mu^{2\epsilon}(\sigma_0)$ as shown in (C.11), the 2-loop function $J(\widetilde{x},\widetilde{y})$ is given in (D.8) and explicit expressions for the quantities in the last line are given below.

Contributions from sunset diagrams are given by

$V_{\mathrm{2L}}^{(b)} = i$ 

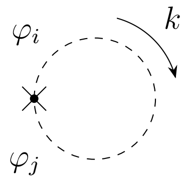

$$
\begin{aligned}
&= -\frac{1}{12}\,\widetilde{\mathcal{V}}_{ijk}\,\widetilde{\mathcal{V}}_{lmn}\int\frac{d^Dk}{(2\,\pi)^D}\int\frac{d^Dq}{(2\,\pi)^D}\left(\widetilde{D}_k^{-1}\right)_{il}\left(\widetilde{D}_q^{-1}\right)_{jm}\left(\widetilde{D}_{k-q}^{-1}\right)_{kn} \qquad (3.28)\\
&= -\frac{\mu^{4\epsilon}(\sigma_0)\,\widehat{\mathcal{V}}_{ijk}\,\widehat{\mathcal{V}}_{lmn}}{12\left(16\,\pi^2\right)^2}\Big[\widetilde{A}_{il}\,\widetilde{A}_{jm}\,\widetilde{A}_{kn}\,I_{HHH}+\widetilde{A}_{il}\,\widetilde{A}_{jm}\,\widetilde{B}_{kn}\,I_{HHS}\\
&\qquad +\widetilde{A}_{il}\,\widetilde{B}_{jm}\,\widetilde{A}_{kn}\,I_{HSH}+\widetilde{B}_{il}\,\widetilde{A}_{jm}\,\widetilde{A}_{kn}\,I_{SHH}+\widetilde{A}_{il}\,\widetilde{B}_{jm}\,\widetilde{B}_{kn}\,I_{HSS}\\
&\qquad +\widetilde{B}_{il}\,\widetilde{A}_{jm}\,\widetilde{B}_{kn}\,I_{SHS}+\widetilde{B}_{il}\,\widetilde{B}_{jm}\,\widetilde{A}_{kn}\,I_{SSH}+\widetilde{B}_{il}\,\widetilde{B}_{jm}\,\widetilde{B}_{kn}\,I_{SSS}\Big]\\
&=: V_{\mathrm{2L},\,1/\epsilon^2}^{(b)}+V_{\mathrm{2L},\,1/\epsilon}^{(b)}+\Delta U_{\mathrm{2L},\,1/\epsilon}^{(b)}+V_{\mathrm{2L},\,\mathrm{fin}}^{(b)}+\Delta U_{\mathrm{2L},\,\mathrm{fin}}^{(b)}+\mathcal{O}\left(\epsilon\right)
\end{aligned}
$$

where

$$I_{\alpha\beta\gamma} := I\left(\widetilde{M}^2_\alpha, \widetilde{M}^2_\beta, \widetilde{M}^2_\gamma\right)$$

with $\alpha, \beta, \gamma \in \{H, S\}$. Again, $\widehat{\mathcal{V}}_{ijk}$ is $\widetilde{\mathcal{V}}_{ijk}$ without the factor of $\mu^{2\epsilon}(\sigma_0)$ as shown in (C.9), the 2-loop function $I(\widetilde{x}, \widetilde{y}, \widetilde{z})$ is given in (D.10) and explicit expressions for the quantities in the last line are given below.

Finally, contributions from 1-loop diagrams with counterterm insertions are

$V_{\text{1L,ct1}} = i$ 


$$\begin{aligned} &= \frac{i}{2}\, \delta\widetilde{\mathcal{V}}_{ij} \int \frac{d^D k}{(2\,\pi)^D} \left(\widetilde{D}_k^{-1}\right)_{ij} \\ &= -\frac{1}{32\,\pi^2}\, \delta\widetilde{\mathcal{V}}_{ij} \left(\widetilde{A}_{ij}\, A_0\left(\widetilde{M}^2_H\right) + \widetilde{B}_{ij}\, A_0\left(\widetilde{M}^2_S\right)\right) \\ &=: V^{1/\epsilon^2}_{\text{1L,ct1}} + V^{1/\epsilon}_{\text{1L,ct1}} + \Delta U^{1/\epsilon}_{\text{1L,ct1}} + V^{\text{fin}}_{\text{1L,ct1}} + \Delta U^{\text{fin}}_{\text{1L,ct1}} + \mathcal{O}(\epsilon) \end{aligned} \tag{3.29}$$

where (3.22) has been used, $\delta\widetilde{\mathcal{V}}_{ij}$ is to be found in (C.62), the $A_0$ - function is given in (D.2) and explicit expressions for the quantities in the last line are given below.

Note that the $\Delta U$ - quantities are new divergent and finite quantum corrections due to evanescent interactions, as discussed in section 2.3. Further, note that in the expressions above the background fields $\{\phi_0, \sigma_0\}$ can again be replaced by the fields $\{\phi, \sigma\}$ in order to finally obtain the 2-loop contribution to the effective potential as a function of the fields $\{\phi, \sigma\}$, as done in the previous section for the 1-loop case.

Now, the explicit results for the above expressions are given as follows. In particular, poles of second order in $\epsilon$ are provided by

$$\begin{aligned} V^{(a)}_{\text{2L},\,1/\epsilon^2} = \frac{\mu^{2\epsilon}(\sigma)}{32\,(16\,\pi^2)^2} \Big[ &\left(\lambda_\phi^3 + 2\,\lambda_\phi\,\lambda_m^2 + \lambda_m^2\,\lambda_\sigma\right)\phi^4 + 2\,\lambda_m\,(\lambda_\phi^2 + 9\,\lambda_m^2 \\ &+ \lambda_\phi\,\lambda_\sigma + \lambda_\sigma^2)\,\phi^2\,\sigma^2 + \left(\lambda_\phi\,\lambda_m^2 + 2\,\lambda_m^2\,\lambda_\sigma + \lambda_\sigma^3\right)\sigma^4\Big]\,\frac{1}{\epsilon^2} \end{aligned} \tag{3.30}$$

$$\begin{aligned} V^{(b)}_{\text{2L},\,1/\epsilon^2} = \frac{\mu^{2\epsilon}(\sigma)}{16\,(16\,\pi^2)^2} \Big[ &\left(\lambda_\phi^3 + \lambda_\phi\,\lambda_m^2 + 2\,\lambda_m^3\right)\phi^4 + \lambda_m\,(\lambda_\phi^2 + 6\,\lambda_\phi\,\lambda_m + 10\,\lambda_m^2 \\ &+ 6\,\lambda_m\,\lambda_\sigma + \lambda_\sigma^2)\,\phi^2\,\sigma^2 + \left(2\,\lambda_m^3 + \lambda_m^2\,\lambda_\sigma + \lambda_\sigma^3\right)\sigma^4\Big]\,\frac{1}{\epsilon^2} \end{aligned} \tag{3.31}$$

$$\begin{aligned} V^{1/\epsilon^2}_{\text{1L,ct1}} = -\frac{\mu^{2\epsilon}(\sigma)}{16\,(16\,\pi^2)^2} \Big[ &\left(3\,\lambda_\phi^3 + 4\,\lambda_\phi\,\lambda_m^2 + 4\,\lambda_m^3 + \lambda_m^2\,\lambda_\sigma\right)\phi^4 \\ &+ \lambda_m\left(4\,\lambda_\phi^2 + 12\,\lambda_\phi\,\lambda_m + 2\,\lambda_\phi\,\lambda_\sigma + 38\,\lambda_m^2 + 12\,\lambda_m\,\lambda_\sigma + 4\,\lambda_\sigma^2\right)\phi^2\,\sigma^2 \\ &+ \left(\lambda_\phi\,\lambda_m^2 + 4\,\lambda_m^3 + 4\,\lambda_m^2\,\lambda_\sigma + 3\,\lambda_\sigma^3\right)\sigma^4\Big]\,\frac{1}{\epsilon^2} \end{aligned} \tag{3.32}$$

Hence, the subrenormalised sum of the 3 contributions to the pole of second order in $\epsilon$ is then given by

$$\begin{aligned}
V_{\text{2L,SR}}^{1/\epsilon^2} &\coloneqq V_{\text{2L},\,1/\epsilon^2}^{(a)} + V_{\text{2L},\,1/\epsilon^2}^{(b)} + V_{\text{1L,ct1}}^{1/\epsilon^2} \\
&= -\frac{\mu^{2\epsilon}(\sigma)}{32\,(16\,\pi^2)^2}\Big[\left(3\,\lambda_\phi^3 + 4\,\lambda_\phi\,\lambda_m^2 + 4\,\lambda_m^3 + \lambda_m^2\,\lambda_\sigma\right)\,\phi^4 \\
&\quad + 2\,\lambda_m\,\left(2\,\lambda_\phi^2 + 6\,\lambda_\phi\,\lambda_m + \lambda_\phi\,\lambda_\sigma + 19\,\lambda_m^2 + 6\,\lambda_m\,\lambda_\sigma + 2\,\lambda_\sigma^2\right)\,\phi^2\,\sigma^2 \\
&\quad + \left(\lambda_\phi\,\lambda_m^2 + 4\,\lambda_m^3 + 4\,\lambda_m^2\,\lambda_\sigma + 3\,\lambda_\sigma^3\right)\,\sigma^4\Big]\,\frac{1}{\epsilon^2}
\end{aligned} \tag{3.33}$$

The subrenormalised sum of the 3 contributions to the simple pole in $\epsilon$ that would also be obtained in usual DReg is provided by

$$\begin{aligned}
V_{\text{2L,SR}}^{1/\epsilon} &\coloneqq V_{\text{2L},\,1/\epsilon}^{(a)} + V_{\text{2L},\,1/\epsilon}^{(b)} + V_{\text{1L,ct1}}^{1/\epsilon} \\
&= \frac{\mu^{2\epsilon}(\sigma)}{16\,(16\,\pi^2)^2}\Big[\left(\lambda_\phi^3 + \lambda_\phi\,\lambda_m^2 + 2\,\lambda_m^3\right)\,\phi^4 \\
&\qquad + \lambda_m\,\left(\lambda_\phi^2 + 6\,\lambda_\phi\,\lambda_m + 10\,\lambda_m^2 + 6\,\lambda_m\,\lambda_\sigma + \lambda_\sigma^2\right)\,\phi^2\,\sigma^2 \\
&\qquad + \left(2\,\lambda_m^3 + \lambda_m^2\,\lambda_\sigma + \lambda_\sigma^3\right)\,\sigma^4\Big]\,\frac{1}{\epsilon}
\end{aligned} \tag{3.34}$$

whereas the subrenormalised sum of the 3 new contributions to the simple pole in $\epsilon$ due to evanescent interactions, i.e. only in SIDReg, is

$$\begin{aligned}
\Delta U_{\text{2L,SR}}^{1/\epsilon} &\coloneqq \Delta U_{\text{2L},\,1/\epsilon}^{(a)} + \Delta U_{\text{2L},\,1/\epsilon}^{(b)} + \Delta U_{\text{1L,ct1}}^{1/\epsilon} \\
&= \frac{\mu^{2\epsilon}(\sigma)}{16\,(16\,\pi^2)^2}\Bigg[\bigg(\frac{20}{3}\,\lambda_\phi^2\,\lambda_m + \frac{7}{6}\,\lambda_\phi\,\lambda_m^2 - 2\,\lambda_m^3 - \frac{1}{2}\,\lambda_\phi^2\,\lambda_\sigma \\
&\qquad + \frac{1}{4}\,\lambda_\phi\,\lambda_\sigma^2 - \frac{4}{3}\,\lambda_\phi\,\lambda_m\,\lambda_\sigma + \frac{7}{12}\,\lambda_m^2\,\lambda_\sigma\bigg)\,\phi^4 \\
&\qquad + \lambda_m\left(8\lambda_\phi\lambda_m + \lambda_\phi\lambda_\sigma + \frac{41}{2}\lambda_m^2 + \frac{43}{3}\lambda_m\lambda_\sigma + \frac{1}{2}\lambda_\sigma^2\right)\phi^2\sigma^2 \\
&\qquad + \left(4\,\lambda_m^3 + \frac{1}{3}\,\lambda_m^2\,\lambda_\sigma + \frac{7}{4}\,\lambda_\sigma^3\right)\,\sigma^4 \\
&\qquad - \lambda_m\,\left(\frac{7}{6}\,\lambda_\phi^2 - \frac{7}{3}\,\lambda_\phi\,\lambda_m + \frac{1}{6}\,\lambda_\phi\,\lambda_\sigma\right)\,\frac{\phi^6}{\sigma^2} - \frac{1}{4}\,\lambda_\phi\,\lambda_m^2\,\frac{\phi^8}{\sigma^4}\Bigg]\,\frac{1}{\epsilon}
\end{aligned} \tag{3.35}$$

Finally, the sum of the 3 regular contributions to the finite part that would also be obtained in usual DReg reads as

$$V_{\text{2L,SR}}^{\text{fin}} \coloneqq V_{\text{2L},\,\text{fin}}^{(a)} + V_{\text{2L},\,\text{fin}}^{(b)} + V_{\text{1L,ct1}}^{\text{fin}} \tag{3.36}$$

with

$$\begin{aligned}
V^{(a)}_{2\mathrm{L,\,fin}} = \frac{\mu^{2\epsilon}(\sigma)}{8\,(16\,\pi^2)^2}\,k_{ijkl}\Bigg\{ & \left(A_{ij}\,\hat{M}_H^2 + B_{ij}\,\hat{M}_S^2\right)\left(A_{kl}\,\hat{M}_H^2 + B_{kl}\,\hat{M}_S^2\right)\left(1+\frac{\pi^2}{6}\right) \\
& + \hat{M}_H^2\,\hat{M}_S^2\,A_{ij}\,B_{kl}\left(\overline{\log}\left(M_H^2\right) - \overline{\log}\left(M_S^2\right)\right)^2 \\
& + 2\Big[A_{ij}\,\hat{M}_H^2\Big(\overline{\log}\left(M_H^2\right)-1\Big) + B_{ij}\,\hat{M}_S^2\Big(\overline{\log}\left(M_S^2\right)-1\Big)\Big] \\
& \times\Big[A_{kl}\,\hat{M}_H^2\Big(\overline{\log}\left(M_H^2\right)-1\Big) + B_{kl}\,\hat{M}_S^2\Big(\overline{\log}\left(M_S^2\right)-1\Big)\Big]\Bigg\}
\end{aligned} \tag{3.37}$$

$$\begin{aligned}
V^{(b)}_{2\mathrm{L,\,fin}} = {} & \frac{\mu^{2\epsilon}(\sigma)}{4\,(16\,\pi^2)^2}\,k_{ijk}\,k_{lmn}\,\phi^2 \\
& \times\Bigg\{\Big[A_{il}\,\hat{M}_H^2\Big(\overline{\log}\left(M_H^2\right)-1\Big) + B_{il}\,\hat{M}_S^2\Big(\overline{\log}\left(M_S^2\right)-1\Big)\Big] \\
& \times\Big[A_{jm}\,A_{kn}\Big(\overline{\log}\left(M_H^2\right)-1\Big) + B_{jm}\,B_{kn}\Big(\overline{\log}\left(M_S^2\right)-1\Big)\Big] \\
& + 2\Big[A_{il}\,\hat{M}_H^2\Big(\overline{\log}\left(M_H^2\right)-1\Big)^2 + B_{il}\,\hat{M}_S^2\Big(\overline{\log}\left(M_S^2\right)-1\Big)^2\Big]\,A_{jm}\,B_{kn} \\
& + \frac{1}{2}\Big[B_{il}\,\hat{M}_H^2 - A_{il}\,\hat{M}_S^2\Big]\Big[\overline{\log}^2\left(M_H^2\right) - \overline{\log}^2\left(M_S^2\right) \\
& \qquad\qquad - 2\Big(\overline{\log}\left(M_H^2\right) - \overline{\log}\left(M_S^2\right)\Big)\Big]\,A_{jm}\,B_{kn} \\
& - \Big[A_{il}\,\hat{M}_H^2\Big(\overline{\log}\left(M_H^2\right)-1\Big) + B_{il}\,\hat{M}_S^2\Big(\overline{\log}\left(M_S^2\right)-1\Big)\Big] \\
& \times\Big[A_{jm}+B_{jm}\Big]\Big[A_{kn}+B_{kn}\Big] \\
& + \Big[A_{il}\,\hat{M}_H^2 + B_{il}\,\hat{M}_S^2\Big]\left(A_{jm}+B_{jm}\right)\left(A_{kn}+B_{kn}\right)\left(\frac{3}{2}+\frac{\pi^2}{12}\right) \\
& + \frac{1}{6}\,A_{il}\,A_{jm}\Big[A_{kn}\,\xi\left(\hat{M}_H^2,\hat{M}_H^2,\hat{M}_H^2\right) + B_{kn}\Big\{\xi\left(\hat{M}_H^2,\hat{M}_H^2,\hat{M}_S^2\right) \\
& \qquad\qquad + \xi\left(\hat{M}_H^2,\hat{M}_S^2,\hat{M}_H^2\right) + \xi\left(\hat{M}_S^2,\hat{M}_H^2,\hat{M}_H^2\right)\Big\}\Big] \\
& + \frac{1}{6}\,B_{il}\,B_{jm}\Big[B_{kn}\,\xi\left(\hat{M}_S^2,\hat{M}_S^2,\hat{M}_S^2\right) + A_{kn}\Big\{\xi\left(\hat{M}_H^2,\hat{M}_S^2,\hat{M}_S^2\right) \\
& \qquad\qquad + \xi\left(\hat{M}_S^2,\hat{M}_H^2,\hat{M}_S^2\right) + \xi\left(\hat{M}_S^2,\hat{M}_S^2,\hat{M}_H^2\right)\Big\}\Big]\Bigg\}
\end{aligned} \tag{3.38}$$

and

$$V^{\text{fin}}_{\text{1L,ct1}} = -\frac{\mu^{2\epsilon}(\sigma)}{4\,(16\,\pi^2)^2}\,\delta u_{ij}\left\{A_{ij}\,\hat{M}_H^2\left(\overline{\log}^2\left(M_H^2\right) - 2\,\overline{\log}\left(M_H^2\right) + 2 + \frac{\pi^2}{6}\right)\right.$$
$$\left. + B_{ij}\,\hat{M}_S^2\left(\overline{\log}^2\left(M_S^2\right) - 2\,\overline{\log}\left(M_S^2\right) + 2 + \frac{\pi^2}{6}\right)\right\} \tag{3.39}$$

The sum of the 3 new contributions to the finite part due to evanescent interactions, i.e. only in SIDReg, is

$$\Delta U^{\text{fin}}_{\text{2L,SR}} \coloneqq \Delta U^{(a)}_{\text{2L, fin}} + \Delta U^{(b)}_{\text{2L, fin}} + \Delta U^{\text{fin}}_{\text{1L,ct1}} \tag{3.40}$$

with

$$\Delta U^{(a)}_{\text{2L, fin}} = \frac{\mu^{2\epsilon}(\sigma)}{8\,(16\,\pi^2)^2}\left\{k^{(2)}_{ijkl}\left(A_{ij}\,\hat{M}_H^2 + B_{ij}\,\hat{M}_S^2\right)\left(A_{kl}\,\hat{M}_H^2 + B_{kl}\,\hat{M}_S^2\right)\right.$$
$$- 2\,k^{(1)}_{ijkl}\left[\left(A_{ij}\,\hat{M}_H^2 + B_{ij}\,\hat{M}_S^2\right)\right.$$
$$\times\left\{A_{kl}\,\hat{M}_H^2\left(\overline{\log}\left(M_H^2\right) - 1\right) + B_{kl}\,\hat{M}_S^2\left(\overline{\log}\left(M_S^2\right) - 1\right)\right.$$
$$\left.\left. - \left[\left(A_{kl}\,c_H^{(1)} + A_{kl}^{(1)}\right)\hat{M}_H^2 + \left(B_{kl}\,c_S^{(1)} + B_{kl}^{(1)}\right)\hat{M}_S^2\right]\right\}\right]$$
$$- 2\,k_{ijkl}\left\{2\,A_{ij}\left(A_{kl}\,c_H^{(1)} + A_{kl}^{(1)}\right)\hat{M}_H^4\left(\overline{\log}\left(M_H^2\right) - 1\right)\right.$$
$$+ 2\,B_{ij}\left(B_{kl}\,c_S^{(1)} + B_{kl}^{(1)}\right)\hat{M}_S^4\left(\overline{\log}\left(M_S^2\right) - 1\right)$$
$$+ \left[A_{ij}\left(B_{kl}\,c_S^{(1)} + B_{kl}^{(1)}\right) + B_{ij}\left(A_{kl}\,c_H^{(1)} + A_{kl}^{(1)}\right)\right]$$
$$\times\hat{M}_H^2\,\hat{M}_S^2\left(\overline{\log}\left(M_H^2\right) + \overline{\log}\left(M_S^2\right) - 2\right)$$
$$- \left(A_{ij}\,\hat{M}_H^2 + B_{ij}\,\hat{M}_S^2\right)$$
$$\times\left[\left(A_{kl}\left(c_H^{(2)} - c_H^{(1)}\right) + A_{kl}^{(1)}\,c_H^{(1)} + A_{kl}^{(2)}\right)\hat{M}_H^2\right.$$
$$\left. + \left(B_{kl}\left(c_S^{(2)} - c_S^{(1)}\right) + B_{kl}^{(1)}\,c_S^{(1)} + B_{kl}^{(2)}\right)\hat{M}_S^2\right]$$
$$- \frac{1}{2}\left[\left(A_{ij}\,c_H^{(1)} + A_{ij}^{(1)}\right)\hat{M}_H^2 + \left(B_{ij}\,c_S^{(1)} + B_{ij}^{(1)}\right)\hat{M}_S^2\right]$$
$$\left.\left.\times\left[\left(A_{kl}\,c_H^{(1)} + A_{kl}^{(1)}\right)\hat{M}_H^2 + \left(B_{kl}\,c_S^{(1)} + B_{kl}^{(1)}\right)\hat{M}_S^2\right]\right\}\right\} \tag{3.41}$$

$$
\begin{aligned}
\Delta U^{(b)}_{2\mathrm{L,\,fin}} = \frac{\mu^{2\epsilon}(\sigma)}{4\,(16\,\pi^2)^2}\,\phi^2 \Bigg\{ & \left(k_{ijk}\,k^{(2)}_{lmn} + \frac{1}{2}\,k^{(1)}_{ijk}\,k^{(1)}_{lmn}\right)\left(A_{il}\,\hat{M}^2_H + B_{il}\,\hat{M}^2_S\right) \\
& \times (A_{jm}+B_{jm})\,(A_{kn}+B_{kn}) \\
+ k_{ijk}\,k^{(1)}_{lmn} & \Bigg\{\Bigg[\hat{M}^2_H\,\left(A_{il}\,\left(3 - 2\,\overline{\log}\,(M^2_H) + c^{(1)}_H\right) + A^{(1)}_{il}\right) \\
& + \hat{M}^2_S\,\left(B_{il}\,\left(3 - 2\,\overline{\log}\,(M^2_S) + c^{(1)}_S\right) + B^{(1)}_{il}\right)\Bigg]\,(A_{jm}+B_{jm}) \\
& + 2\,\left(A_{il}\,\hat{M}^2_H + B_{il}\,\hat{M}^2_S\right)\left(A^{(1)}_{jm} + B^{(1)}_{jm}\right)\Bigg\}\,(A_{kn}+B_{kn}) \\
+ \frac{1}{2}\,k_{ijk}\,k_{lmn} & \Bigg\{\Bigg[\hat{M}^2_H\Bigg(A_{il}\,\left(c^{(2)}_H - c^{(1)}_H\right) + A^{(1)}_{il}\,c^{(1)}_H \\
& \quad - 2\Big(\overline{\log}\,(M^2_H) - 1\Big)\left(A_{il}\,c^{(1)}_H + A^{(1)}_{il}\right)\Bigg) \\
& + \hat{M}^2_S\Bigg(B_{il}\,\left(c^{(2)}_S - c^{(1)}_S\right) + B^{(1)}_{il}\,c^{(1)}_S \\
& \quad - 2\Big(\overline{\log}\,(M^2_S) - 1\Big)\left(B_{il}\,c^{(1)}_S + B^{(1)}_{il}\right)\Bigg)\Bigg]\,(A_{jm}+B_{jm}) \\
& + \Bigg[\hat{M}^2_H\,A_{il}\left(c^{(1)}_H - 2\,\overline{\log}\,(M^2_H) + 2\right) \\
& \quad + \hat{M}^2_S\,B_{il}\left(c^{(1)}_S - 2\,\overline{\log}\,(M^2_S) + 2\right)\Bigg]\left(A^{(1)}_{jm} + B^{(1)}_{jm}\right)\Bigg\} \\
& \times (A_{kn}+B_{kn}) \\
+ \frac{1}{2}\,k_{ijk}\,k_{lmn} & \\
\times \Bigg\{\hat{M}^2_H & \Bigg[B_{il}\Big(2\,A^{(1)}_{jm}\,A^{(1)}_{kn} + A^{(2)}_{jm}\,B_{kn} + A^{(1)}_{jm}\,B_{kn} + 2\,A^{(1)}_{jm}\,B^{(1)}_{kn}\Big) \\
& + A_{il}\Big(3\,A^{(1)}_{jm}\,A^{(1)}_{kn} + 4\,A^{(2)}_{jm}\,B_{kn} + 2\,B_{jm}\,B^{(1)}_{kn} \\
& \quad + 4\,A^{(1)}_{jm}\left(B_{kn} + B^{(1)}_{kn}\right) + B^{(1)}_{jm}\,B^{(1)}_{kn} + 2\,B_{jm}\,B^{(2)}_{kn}\Big) \\
& + A_{il}\,A_{jm}\left(3\,\left(A^{(1)}_{kn} + A^{(2)}_{kn}\right) + 2\,\left(B^{(1)}_{kn} + B^{(2)}_{kn}\right)\right)\Bigg] \\
+ \hat{M}^2_S & \Bigg[A_{il}\Big(2\,B^{(1)}_{jm}\,B^{(1)}_{kn} + B^{(2)}_{jm}\,A_{kn} + B^{(1)}_{jm}\,A_{kn} + 2\,B^{(1)}_{jm}\,A^{(1)}_{kn}\Big) \\
& + B_{il}\Big(3\,B^{(1)}_{jm}\,B^{(1)}_{kn} + 4\,B^{(2)}_{jm}\,A_{kn} + 2\,A_{jm}\,A^{(1)}_{kn} \\
& \quad + 4\,B^{(1)}_{jm}\left(A_{kn} + A^{(1)}_{kn}\right) + A^{(1)}_{jm}\,A^{(1)}_{kn} + 2\,A_{jm}\,A^{(2)}_{kn}\Big) \\
& + B_{il}\,B_{jm}\left(3\,\left(B^{(1)}_{kn} + B^{(2)}_{kn}\right) + 2\,\left(A^{(1)}_{kn} + A^{(2)}_{kn}\right)\right)\Bigg]\Bigg\}\Bigg\}
\end{aligned}
\tag{3.42}
$$

and

$$
\begin{aligned}
\Delta U^{\text{fin}}_{\text{1L,ct1}} = -\ \frac{\mu^{2\epsilon}(\sigma)}{2\,(16\,\pi^2)^2} \Bigg\{ & \delta u^{(2)}_{ij} \left( A_{ij}\,\hat{M}_H^2 + B_{ij}\,\hat{M}_S^2 \right) \\
& + \delta u^{(1)}_{ij} \Bigg\{ \hat{M}_H^2 \Big[ A_{ij} \left( c_H^{(1)} - \overline{\log}\left(M_H^2\right) + 1 \right) + A^{(1)}_{ij} \Big] \\
& \qquad + \hat{M}_S^2 \Big[ B_{ij} \left( c_S^{(1)} - \overline{\log}\left(M_S^2\right) + 1 \right) + B^{(1)}_{ij} \Big] \Bigg\} \\
& - \delta u_{ij} \Bigg\{ \hat{M}_H^2 \left( \overline{\log}\left(M_H^2\right) - 1 \right) \left( A_{ij}\, c_H^{(1)} + A^{(1)}_{ij} \right) \\
& \qquad + \hat{M}_S^2 \left( \overline{\log}\left(M_S^2\right) - 1 \right) \left( B_{ij}\, c_S^{(1)} + B^{(1)}_{ij} \right) \\
& \qquad - \hat{M}_H^2 \Big[ A_{ij}\, c_H^{(2)} + \left( A^{(1)}_{ij} - A_{ij} \right) c_H^{(1)} + A^{(2)}_{ij} \Big] \\
& \qquad - \hat{M}_S^2 \Big[ B_{ij}\, c_S^{(2)} + \left( B^{(1)}_{ij} - B_{ij} \right) c_S^{(1)} + B^{(2)}_{ij} \Big] \Bigg\} \Bigg\}
\end{aligned}
\tag{3.43}
$$

where

$$
\overline{\log}\,(x) := \log \left( \frac{x}{4\,\pi\,\mu_0^2}\, e^{\gamma_E} \right) \tag{3.44}
$$

In conclusion, the subrenormalised 2-loop contribution to the effective potential is then given by

$$
\begin{aligned}
V_{\text{2L,SR}} &= V^{(a)}_{\text{2L}} + V^{(b)}_{\text{2L}} + V_{\text{1L,ct1}} \\
&= V^{1/\epsilon^2}_{\text{2L,SR}} + V^{1/\epsilon}_{\text{2L,SR}} + \Delta U^{1/\epsilon}_{\text{2L,SR}} + V^{\text{fin}}_{\text{2L,SR}} + \Delta U^{\text{fin}}_{\text{2L,SR}} + \mathcal{O}\left(\epsilon\right)
\end{aligned}
\tag{3.45}
$$

with explicit expressions as given above and coefficients provided in appendix C, however, with the VEVs $\{v, w\}$ being replaced by the fields $\{\phi, \sigma\}$, as discussed above. In order to completely renormalise the theory, and thus obtain a finite result, the 2-loop counterterms need to be determined. For this reason, the 2-loop counterterm Lagrangian $\mathcal{L}_{\text{ct2}}$ in (2.59) is used, i.e. $\widetilde{V}_{\text{tree,ct2}}$ is needed,

$$
\begin{aligned}
\widetilde{V}_{\text{tree,ct2}} = \mu^{2\epsilon}(\sigma) \Bigg( & \delta Z^{(2)}_{V_\phi}\, \frac{\lambda_\phi}{4!}\, \phi^4 + \delta Z^{(2)}_{V_m}\, \frac{\lambda_m}{4}\, \phi^2\, \sigma^2 + \delta Z^{(2)}_{V_\sigma}\, \frac{\lambda_\sigma}{4!}\, \sigma^4 \\
& + \frac{\delta\lambda_6^{(2)}}{6}\, \frac{\phi^6}{\sigma^2} + \frac{\delta\lambda_8^{(2)}}{8}\, \frac{\phi^8}{\sigma^4} \Bigg)
\end{aligned}
\tag{3.46}
$$

In particular, the divergent contributions given in (3.33), (3.34) and (3.35) need to be cancelled by the 2-loop counterterms. While the second order pole in $\epsilon$ is not corrected by QSI, it can be seen that the simple pole in $\epsilon$ gets a correction in the form of $\Delta U^{1/\epsilon}_{\text{2L,SR}}$ due to QSI. This represents a new divergent quantum correction due to evanescent interactions, i.e. QSI, which changes the divergence structure, and thus leads to new corrections in the 2-loop counterterms. Moreover, in (3.35) it can be seen that these new divergent quantum corrections contain higher dimensional operators of the form (2.51), accompanied by a simple $\epsilon$ - pole, and thus introduce the necessity of appropriate counterterms $\delta\lambda_6^{(2)}, \delta\lambda_8^{(2)}$, as discussed in section 2.3. This stands in contrast to the 1-loop case where only new finite, but no new divergent quantum corrections arise due to evanescent interactions.

The 2-loop counterterms in the MS-scheme may be written as

$$\begin{aligned}
\delta Z_{V_\phi}^{(2)} &= \frac{1}{(16\,\pi^2)^2}\left(\frac{\delta_\phi^{(2)}}{\epsilon^2} + \frac{\delta_\phi^{(1)} + \nu_\phi^{(1)}}{\epsilon}\right)\\
\delta Z_{V_m}^{(2)} &= \frac{1}{(16\,\pi^2)^2}\left(\frac{\delta_m^{(2)}}{\epsilon^2} + \frac{\delta_m^{(1)} + \nu_m^{(1)}}{\epsilon}\right)\\
\delta Z_{V_\sigma}^{(2)} &= \frac{1}{(16\,\pi^2)^2}\left(\frac{\delta_\sigma^{(2)}}{\epsilon^2} + \frac{\delta_\sigma^{(1)} + \nu_\sigma^{(1)}}{\epsilon}\right)\\
\delta\lambda_6^{(2)} &= \frac{1}{(16\,\pi^2)^2}\,\frac{\delta\nu_6^{(1)}}{\epsilon}\\
\delta\lambda_8^{(2)} &= \frac{1}{(16\,\pi^2)^2}\,\frac{\delta\nu_8^{(1)}}{\epsilon}
\end{aligned} \tag{3.47}$$

where the $\delta_i^{(2)}$ and $\delta_i^{(1)}$, $i \in \{\phi, m, \sigma\}$, cancel second order and simple poles in $\epsilon$, respectively, while the $\nu_j^{(1)}$, $i \in \{\phi, m, \sigma, 6, 8\}$, cancel the new simple poles in $\epsilon$ in (3.35), introduced by QSI. Explicitly, these counterterms are given by

$$\begin{aligned}
\delta_\phi^{(2)} &= \frac{3}{4}\left[3\,\lambda_\phi^2 + 4\,\lambda_m^2 + \frac{\lambda_m^2}{\lambda_\phi}\,(4\,\lambda_m + \lambda_\sigma)\right]\\
\delta_\phi^{(1)} &= -\,\frac{3}{2}\left[\lambda_\phi^2 + \lambda_m^2 + 2\,\frac{\lambda_m^3}{\lambda_\phi}\right]\\
\nu_\phi^{(1)} &= \frac{1}{8}\left[\frac{\lambda_m^2}{\lambda_\phi}\,(24\,\lambda_m - 7\,\lambda_\sigma) - \left(14\,\lambda_m^2 - 16\,\lambda_m\,\lambda_\sigma + 3\,\lambda_\sigma^2\right) - \lambda_\phi\,(80\,\lambda_m - 6\,\lambda_\sigma)\right]
\end{aligned} \tag{3.48}$$

$$\begin{aligned}
\delta_m^{(2)} &= \frac{1}{4}\,\left[2\,\lambda_\phi^2 + 6\,\lambda_\phi\,\lambda_m + \lambda_\phi\,\lambda_\sigma + 19\,\lambda_m^2 + 6\,\lambda_m\,\lambda_\sigma + 2\,\lambda_\sigma^2\right]\\
\delta_m^{(1)} &= -\,\frac{1}{4}\,\left[\lambda_\phi^2 + 6\,\lambda_\phi\,\lambda_m + 10\,\lambda_m^2 + 6\,\lambda_m\,\lambda_\sigma + \lambda_\sigma^2\right]\\
\nu_m^{(1)} &= -\,\frac{1}{24}\,\left[6\,\lambda_\phi\,(8\,\lambda_m + \lambda_\sigma) + 123\,\lambda_m^2 + 86\,\lambda_m\,\lambda_\sigma + 3\,\lambda_\sigma^2\right]
\end{aligned} \tag{3.49}$$

$$\begin{aligned}
\delta_\sigma^{(2)} &= \frac{3}{4}\left[\frac{\lambda_m^2}{\lambda_\sigma}\,(\lambda_\phi + 4\,\lambda_m) + 4\,\lambda_m^2 + 3\,\lambda_\sigma^2\right]\\
\delta_\sigma^{(1)} &= -\,\frac{3}{2}\left[\lambda_\sigma^2 + \lambda_m^2 + 2\,\frac{\lambda_m^3}{\lambda_\sigma}\right]\\
\nu_\sigma^{(1)} &= -\,\frac{1}{8}\left[48\,\frac{\lambda_m^3}{\lambda_\sigma} + 4\,\lambda_m^2 + 21\,\lambda_\sigma^2\right]
\end{aligned} \tag{3.50}$$

and

$$\delta\nu_6^{(1)} = \frac{1}{16}\,\lambda_\phi\,\lambda_m\,(7\,\lambda_\phi - 14\,\lambda_m + \lambda_\sigma)\,, \qquad \delta\nu_8^{(1)} = \frac{1}{8}\,\lambda_\phi\,\lambda_m^2 \tag{3.51}$$

Finally, after Renormalisation and then going back to 4 dimensions, i.e. $\epsilon \to 0$, the result for the effective potential up to the 2-loop level is given by

$$\begin{aligned} V_{\text{eff}}(\phi,\sigma) &= V_{\text{tree}}(\phi,\sigma) + V_{\text{tree,ct1}}(\phi,\sigma) + V_{\text{1L}}(\phi,\sigma) \\ &\quad + V_{\text{tree,ct2}}(\phi,\sigma) + V_{\text{2L}}^{(a)}(\phi,\sigma) + V_{\text{2L}}^{(b)}(\phi,\sigma) + V_{\text{1L,ct1}}(\phi,\sigma) + \mathcal{O}\left(\hbar^3\right) \\ &= V_{\text{tree}}(\phi,\sigma) + V_{\text{1L,reg}}(\phi,\sigma) + \Delta U_{\text{1L}}(\phi,\sigma) + V_{\text{2L, reg}}(\phi,\sigma) + \Delta U_{\text{2L}}(\phi,\sigma) \\ &\quad + \mathcal{O}\left(\hbar^3\right) \end{aligned} \tag{3.52}$$

where $V_{\text{tree}}$, $V_{\text{tree,ct1}}$, $V_{\text{tree,ct2}}$ and the sum of the regular 1-loop contribution $V_{\text{1L,reg}}$ and the new finite 1-loop correction $\Delta U_{\text{1L}}$, i.e. $V_{\text{1L,ren}} = V_{\text{tree,ct1}} + V_{\text{1L}} = V_{\text{1L,reg}} + \Delta U_{\text{1L}}$, are given in (3.25), (3.18), (3.46) and (3.26), respectively, where $\Delta U_{\text{1L}}$ is explicitly defined in (3.15). The fully renormalised 2-loop contribution to $V_{\text{eff}}$ is provided by

$$\begin{aligned} V_{\text{2L, ren}}(\phi,\sigma) &= V_{\text{tree,ct2}}(\phi,\sigma) + V_{\text{2L}}^{(a)}(\phi,\sigma) + V_{\text{2L}}^{(b)}(\phi,\sigma) + V_{\text{1L,ct1}}(\phi,\sigma) \\ &= V_{\text{2L, reg}}(\phi,\sigma) + \Delta U_{\text{2L}}(\phi,\sigma) \end{aligned} \tag{3.53}$$

where, in the MS-scheme, or in the $\overline{\text{MS}}$-scheme if the replacement (3.23) is used,

$$\begin{aligned} V_{\text{2L, reg}}(\phi,\sigma) &= \lim_{\epsilon\to 0} V_{\text{2L,SR}}^{\text{fin}} = \lim_{\epsilon\to 0} \left(V_{\text{2L, fin}}^{(a)} + V_{\text{2L, fin}}^{(b)} + V_{\text{1L,ct1}}^{\text{fin}}\right) \\ \Delta U_{\text{2L}}(\phi,\sigma) &= \lim_{\epsilon\to 0} \Delta U_{\text{2L,SR}}^{\text{fin}} = \lim_{\epsilon\to 0} \left(\Delta U_{\text{2L, fin}}^{(a)} + \Delta U_{\text{2L, fin}}^{(b)} + \Delta U_{\text{1L,ct1}}^{\text{fin}}\right) \end{aligned} \tag{3.54}$$

with constituents on the RHS of (3.54) explicitly given in (3.37), (3.38), (3.39), (3.41), (3.42) and (3.43), respectively.

**Remark.**

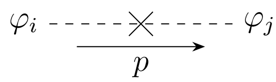
(i) Again it can be seen that, beside the regular 2-loop contribution $V_{\text{2L, reg}}$, a new finite quantum correction $\Delta U_{\text{2L}}$ is obtained due to evanescent interactions introduced by the Renormalisation function, i.e. as a result of QSI. These new quantum corrections contain higher dimensional non-polynomial operators of the form (2.51).

(ii) This time, however, in contrast to the 1-loop case, there is also a new *divergent* quantum correction $\Delta U_{\text{2L,SR}}^{1/\epsilon}$, given in (3.35). In particular, this is a correction to the simple pole in $\epsilon$. Thus, the 2-loop counterterms, given in (3.47) to (3.51), obtain new corrections due to QSI as well, represented by $\nu_i$. Moreover, new counterterms $\delta\lambda_6$ and $\delta\lambda_8$ are necessary for the Renormalisation of the theory due to higher dimensional non-polynomial operators of the form (2.51), here with $p = 1$ and $p = 2$, that emerge as new *divergent* quantum corrections in $\Delta U_{\text{2L,SR}}^{1/\epsilon}$ introduced by evanescent interactions, i.e. as a result of QSI, indicating non-renormalisability, as discussed in section 2.3. However, all counterterms still respect the symmetries of the theory, in particular scale symmetry. Hence, the divergence structure is changed at the 2-loop level and the 2-loop counterterms of the QSI theory are distinguishable from the usual DReg - regularised theory.

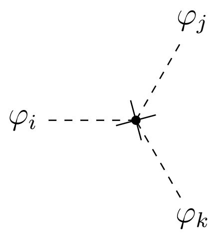
(iii) The 2-loop effective potential (3.52) is a homogeneous function of the fields, and thus satisfies (2.13). No massive parameters are introduced at the quantum level due to the usage of SIDReg with a dynamical Renormalisation function $\mu(\sigma)$ instead of DReg. Hence, the theory indeed is scale invariant at the quantum level, at least up to the 2-loop level, i.e. quantum scale invariant, as intended.

(iv) The results of this section agree with the results presented in [14]. However, some typos w.r.t. the finite 2-loop contributions to $V_{\text{eff}}$, in particular in equations (B-6) and (B-11) of [14], as well as a global minus sign missing in (B-14) and (B-15), have been spotted in [14]. For a comparison with [14], note the slightly different notation in this thesis, especially w.r.t. the definition of $\overline{\log}\,(x)$ given in (3.44).

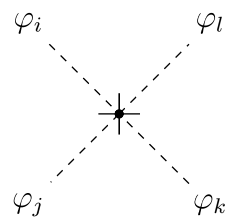
Now, the 2-loop wave function Renormalisation coefficients $\delta Z_\phi^{(2)}$ and $\delta Z_\sigma^{(2)}$ remain to be determined. At the 2-loop level it is expected that they are non-vanishing. Analogous to the 1-loop case, these wave function Renormalisation coefficients are obtained by renormalising the self energies of $\phi$ and $\sigma$. For the 2-loop calculation of these renormalised self energies the following counterterm Feynman rules, again derived from the shifted Lagrangian $\mathcal{L}(\phi+\phi_0, \sigma+\sigma_0)$, are necessary. First, 1-loop counterterm Feynman rules, necessary for Subrenormalisation,



$$= i\,p^2\,\delta_{ij}\,\delta Z_{\varphi_i}^{(1)} - i\,\delta\widetilde{\mathcal{V}}_{ij}$$

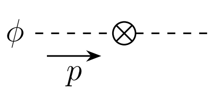


$$= -\,i\,\delta\widetilde{\mathcal{V}}_{ijk},$$

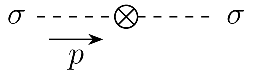


$$= -\,i\,\delta\widetilde{\mathcal{V}}_{ijkl} \tag{3.55}$$

with explicit expressions for $\delta\widetilde{\mathcal{V}}_{ij\cdots}$ given in the appendix C in (C.49) to (C.58), and second, 2-loop counterterm Feynman rules (for propagators)



$$\begin{aligned}
&= i\,p^2\,\delta Z_\phi^{(2)} - i\,\mu^{2\epsilon}(\sigma_0)\left[\frac{\lambda_\phi}{2}\,\delta Z_{V_\phi}^{(2)}\,\phi_0^2 + \frac{\lambda_m}{2}\,\delta Z_{V_m}^{(2)}\,\sigma_0^2 \right.\\
&\qquad \left. + 5\,\delta\lambda_6^{(2)}\,\frac{\phi_0^4}{\sigma_0^2} + 7\,\delta\lambda_8^{(2)}\,\frac{\phi_0^6}{\sigma_0^4}\right]\\
&= i\,p^2\,\delta Z_\phi^{(2)} - \frac{i\,\mu^{2\epsilon}(\sigma_0)}{(16\,\pi^2)^2}\left[\left(\frac{\lambda_\phi}{2}\,\delta_\phi^{(2)}\,\phi_0^2 + \frac{\lambda_m}{2}\,\delta_m^{(2)}\,\sigma_0^2\right)\frac{1}{\epsilon^2}\right.\\
&\qquad + \left(\frac{\lambda_\phi}{2}\,\delta_\phi^{(1)}\,\phi_0^2 + \frac{\lambda_m}{2}\,\delta_m^{(1)}\,\sigma_0^2\right)\frac{1}{\epsilon}\\
&\qquad \left. + \left(\frac{\lambda_\phi}{2}\,\nu_\phi^{(1)}\,\phi_0^2 + \frac{\lambda_m}{2}\,\nu_m^{(1)}\,\sigma_0^2 + 5\,\delta\nu_6^{(1)}\,\frac{\phi_0^4}{\sigma_0^2} + 7\,\delta\nu_8^{(1)}\,\frac{\phi_0^6}{\sigma_0^4}\right)\frac{1}{\epsilon}\right]
\end{aligned} \tag{3.56}$$



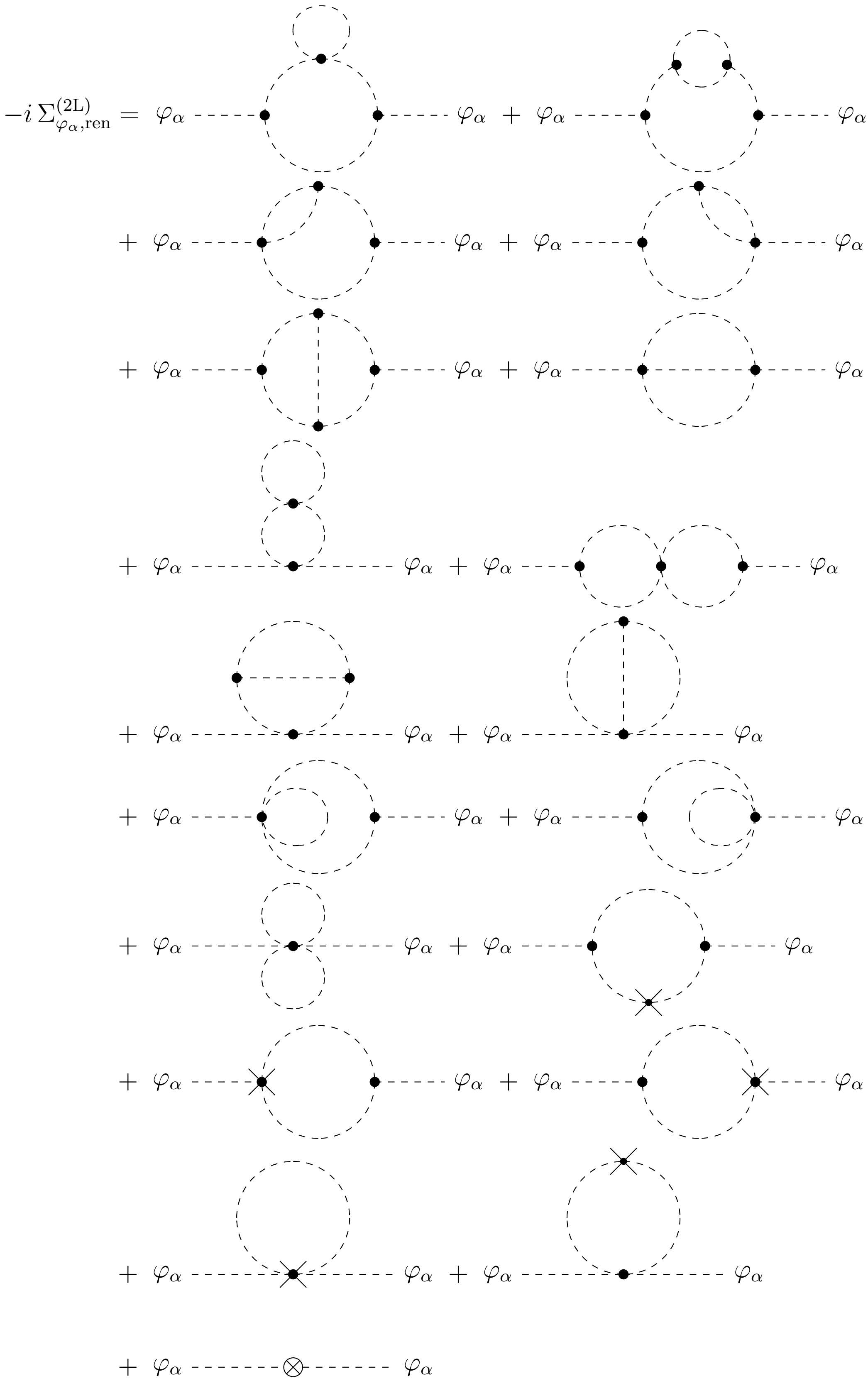
$$
\begin{aligned}
&= i\,p^2\,\delta Z_\sigma^{(2)} - i\,\mu^{2\epsilon}(\sigma_0)\Bigg[\frac{\lambda_\sigma}{2}\,\delta Z_{V_\sigma}^{(2)}\,\sigma_0^2 + \frac{\lambda_m}{2}\,\delta Z_{V_m}^{(2)}\,\phi_0^2 \\
&\quad + \delta\lambda_6^{(2)}\,\frac{\phi_0^6}{\sigma_0^4} + \frac{5}{2}\delta\lambda_8^{(2)}\,\frac{\phi_0^8}{\sigma_0^6} + \epsilon\left(\frac{7}{12}\lambda_\sigma\,\delta Z_{V_\sigma}^{(2)}\,\sigma_0^2 + \frac{3}{2}\lambda_m\,\delta Z_{V_m}^{(2)}\,\phi_0^2\right. \\
&\quad \left. - \frac{1}{12}\,\lambda_\phi\,\delta Z_{V_\phi}^{(2)}\,\frac{\phi_0^4}{\sigma_0^2} - \frac{5}{3}\,\delta\lambda_6^{(2)}\,\frac{\phi_0^6}{\sigma_0^4} - \frac{9}{4}\,\delta\lambda_8^{(2)}\,\frac{\phi_0^8}{\sigma_0^6}\right) + \ldots\Bigg] \\
&= i\,p^2\,\delta Z_\sigma^{(2)} - \frac{i\,\mu^{2\epsilon}(\sigma_0)}{(16\,\pi^2)^2}\Bigg[\left(\frac{\lambda_\sigma}{2}\,\delta_\sigma^{(2)}\,\sigma_0^2 + \frac{\lambda_m}{2}\,\delta_m^{(2)}\,\phi_0^2\right)\frac{1}{\epsilon^2} \\
&\quad + \left(\frac{\lambda_\sigma}{2}\,\delta_\sigma^{(1)}\,\sigma_0^2 + \frac{\lambda_m}{2}\,\delta_m^{(1)}\,\phi_0^2\right)\frac{1}{\epsilon} \\
&\quad + \left(\frac{7}{12}\,\lambda_\sigma\,\delta_\sigma^{(2)}\,\sigma_0^2 + \frac{3}{2}\,\lambda_m\,\delta_m^{(2)}\,\phi_0^2 - \frac{\lambda_\phi}{12}\,\delta_\phi^{(2)}\,\frac{\phi_0^4}{\sigma_0^2} + \frac{\lambda_\sigma}{2}\,\nu_\sigma^{(1)}\,\sigma_0^2\right. \\
&\quad \left. + \frac{\lambda_m}{2}\,\nu_m^{(1)}\,\phi_0^2 + \delta\nu_6^{(1)}\,\frac{\phi_0^6}{\sigma_0^4} + \frac{5}{2}\,\delta\nu_8^{(1)}\,\frac{\phi_0^8}{\sigma_0^6}\right)\frac{1}{\epsilon} + \mathcal{O}\left(\epsilon^0\right)\Bigg]
\end{aligned}
\tag{3.57}
$$

where in the second equality in (3.56) and (3.57) relations (3.47) for the 2-loop counterterms have been used. Note that the last term, i.e. the third term, of the second equality in (3.56) and (3.57), respectively, is a new contribution due to QSI, and thus is not present if the theory is regularised using usual DReg.

Using the above Feynman rules in (3.3), (3.55), (3.56) and (3.57), the renormalised 2-loop contribution $\Sigma^{(2L)}_{\varphi_\alpha,\mathrm{ren}}$ to the self energy of $\{\varphi_\alpha\}_{\alpha=1}^2 = \{\phi,\sigma\}$ is given by the Feynman diagrams in (3.58). Again, as in the 1-loop case in the previous section, it is *not* summed over the Greek index $\alpha$, which is fixed to be either 1 (for $\phi$) or 2 (for $\sigma$), but it is implicitly summed over the Roman indices $i,j,k,l,m,n,\ldots \in \{1,2\}$. However, note that this time the labels of the lines with fields $\varphi_i$ with Roman indices, as well as the momentum labels, are suppressed in (3.58) for readability. Nonetheless, they are still present implicitly, and every Feynman diagram shown in (3.58) need to be understood as a sum over all contributing diagrams of the respective kind consisting of internal lines given by propagators of the form illustrated in (3.3), as in the 1-loop case.

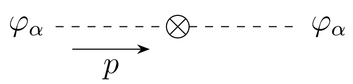
Whereas the first nine Feynman diagrams in (3.58) are the usual 2-loop diagrams in a scalar theory, the four diagrams 10 to 13 contain vertices with more than 4 particles, and thus only emerge in the QSI theory (regularised using SIDReg). Therefore, these four diagrams are purely *evanescent* and contribute solely as new divergent and finite quantum corrections. In particular, diagrams 10 to 12 contain a 5-point vertex, while diagram 13 contains a 6-point vertex, whose Feynman rules are analogously derived as those in (3.3). Diagrams 14 to 18 are (the usual) 1-loop diagrams with 1-loop counterterm insertions, necessary for the subrenormalisation, and the last diagram in (3.58) is the 2-loop counterterm diagram.

Note that of course all diagrams in (3.58) contain evanescent terms, and thus contribute to the new divergent and finite quantum corrections.




(3.58)

$$
\begin{aligned}
\Leftrightarrow\; & - i\, \Sigma^{(2\mathrm{L})}_{\varphi_\alpha,\mathrm{ren}} \\
& = \frac{i}{2}\, \widetilde{\mathcal{V}}_{\alpha ij}\, \widetilde{\mathcal{V}}_{\alpha kl}\, \widetilde{\mathcal{V}}_{mnrs} \int \frac{d^D l_1}{(2\,\pi)^D} \int \frac{d^D l_2}{(2\,\pi)^D} \big(\widetilde{D}^{-1}_{l_1}\big)_{kn} \big(\widetilde{D}^{-1}_{l_1}\big)_{im} \big(\widetilde{D}^{-1}_{l_1-p}\big)_{jl} \big(\widetilde{D}^{-1}_{l_2}\big)_{rs} \\
& + \frac{i}{2}\, \widetilde{\mathcal{V}}_{\alpha ij}\, \widetilde{\mathcal{V}}_{\alpha kl}\, \widetilde{\mathcal{V}}_{mnr}\, \widetilde{\mathcal{V}}_{qst} \\
& \qquad \times \int \frac{d^D l_1}{(2\,\pi)^D} \int \frac{d^D l_2}{(2\,\pi)^D} \big(\widetilde{D}^{-1}_{l_1}\big)_{iq} \big(\widetilde{D}^{-1}_{l_1}\big)_{km} \big(\widetilde{D}^{-1}_{l_1-p}\big)_{jl} \big(\widetilde{D}^{-1}_{l_2-l_1}\big)_{rs} \big(\widetilde{D}^{-1}_{l_2}\big)_{nt} \\
& + \frac{i}{2}\, \widetilde{\mathcal{V}}_{\alpha il}\, \widetilde{\mathcal{V}}_{mnr}\, \widetilde{\mathcal{V}}_{\alpha jks} \int \frac{d^D l_1}{(2\,\pi)^D} \int \frac{d^D l_2}{(2\,\pi)^D} \big(\widetilde{D}^{-1}_{l_1}\big)_{im} \big(\widetilde{D}^{-1}_{l_1-p}\big)_{jl} \big(\widetilde{D}^{-1}_{l_2-l_1}\big)_{kn} \big(\widetilde{D}^{-1}_{l_2}\big)_{rs} \\
& + \frac{i}{2}\, \widetilde{\mathcal{V}}_{\alpha ij}\, \widetilde{\mathcal{V}}_{mns}\, \widetilde{\mathcal{V}}_{\alpha klr} \int \frac{d^D l_1}{(2\,\pi)^D} \int \frac{d^D l_2}{(2\,\pi)^D} \big(\widetilde{D}^{-1}_{l_1}\big)_{im} \big(\widetilde{D}^{-1}_{l_1-p}\big)_{jl} \big(\widetilde{D}^{-1}_{l_2-l_1}\big)_{kn} \big(\widetilde{D}^{-1}_{l_2}\big)_{rs} \\
& + \frac{i}{2}\, \widetilde{\mathcal{V}}_{\alpha ij}\, \widetilde{\mathcal{V}}_{\alpha kl}\, \widetilde{\mathcal{V}}_{mnr}\, \widetilde{\mathcal{V}}_{qst} \\
& \qquad \times \int \frac{d^D l_1}{(2\,\pi)^D} \int \frac{d^D l_2}{(2\,\pi)^D} \big(\widetilde{D}^{-1}_{l_1}\big)_{ir} \big(\widetilde{D}^{-1}_{l_1-p}\big)_{jq} \big(\widetilde{D}^{-1}_{l_2-l_1}\big)_{ms} \big(\widetilde{D}^{-1}_{l_2}\big)_{kn} \big(\widetilde{D}^{-1}_{l_2-p}\big)_{lt} \\
& + \frac{i}{6}\, \widetilde{\mathcal{V}}_{\alpha ijk}\, \widetilde{\mathcal{V}}_{\alpha lmn} \int \frac{d^D l_1}{(2\,\pi)^D} \int \frac{d^D l_2}{(2\,\pi)^D} \big(\widetilde{D}^{-1}_{l_1-p}\big)_{im} \big(\widetilde{D}^{-1}_{l_2-l_1}\big)_{jn} \big(\widetilde{D}^{-1}_{l_2}\big)_{kl} \\
& + \frac{i}{4}\, \widetilde{\mathcal{V}}_{\alpha\alpha ij}\, \widetilde{\mathcal{V}}_{klmn} \int \frac{d^D l_1}{(2\,\pi)^D} \int \frac{d^D l_2}{(2\,\pi)^D} \big(\widetilde{D}^{-1}_{l_1}\big)_{ik} \big(\widetilde{D}^{-1}_{l_1}\big)_{jl} \big(\widetilde{D}^{-1}_{l_2}\big)_{mn} \\
& + \frac{i}{4}\, \widetilde{\mathcal{V}}_{\alpha ij}\, \widetilde{\mathcal{V}}_{\alpha rs}\, \widetilde{\mathcal{V}}_{klmn} \int \frac{d^D l_1}{(2\,\pi)^D} \int \frac{d^D l_2}{(2\,\pi)^D} \big(\widetilde{D}^{-1}_{l_1}\big)_{jl} \big(\widetilde{D}^{-1}_{l_1-p}\big)_{ik} \big(\widetilde{D}^{-1}_{l_2}\big)_{ns} \big(\widetilde{D}^{-1}_{l_2-p}\big)_{mr} \qquad (3.59) \\
& + \frac{i}{4}\, \widetilde{\mathcal{V}}_{\alpha\alpha ij}\, \widetilde{\mathcal{V}}_{klm}\, \widetilde{\mathcal{V}}_{nrs} \int \frac{d^D l_1}{(2\,\pi)^D} \int \frac{d^D l_2}{(2\,\pi)^D} \big(\widetilde{D}^{-1}_{l_1}\big)_{is} \big(\widetilde{D}^{-1}_{l_1}\big)_{jk} \big(\widetilde{D}^{-1}_{l_2-l_1}\big)_{mn} \big(\widetilde{D}^{-1}_{l_2}\big)_{lr} \\
& + \frac{i}{6}\, \widetilde{\mathcal{V}}_{\alpha\alpha ijk}\, \widetilde{\mathcal{V}}_{lmn} \int \frac{d^D l_1}{(2\,\pi)^D} \int \frac{d^D l_2}{(2\,\pi)^D} \big(\widetilde{D}^{-1}_{l_1}\big)_{jl} \big(\widetilde{D}^{-1}_{l_2-l_1}\big)_{kn} \big(\widetilde{D}^{-1}_{l_2}\big)_{im} \\
& + \frac{i}{4}\, \widetilde{\mathcal{V}}_{\alpha kl}\, \widetilde{\mathcal{V}}_{\alpha ijmn} \int \frac{d^D l_1}{(2\,\pi)^D} \int \frac{d^D l_2}{(2\,\pi)^D} \big(\widetilde{D}^{-1}_{l_1}\big)_{jl} \big(\widetilde{D}^{-1}_{l_1-p}\big)_{ik} \big(\widetilde{D}^{-1}_{l_2}\big)_{mn} \\
& + \frac{i}{4}\, \widetilde{\mathcal{V}}_{\alpha ij}\, \widetilde{\mathcal{V}}_{\alpha klmn} \int \frac{d^D l_1}{(2\,\pi)^D} \int \frac{d^D l_2}{(2\,\pi)^D} \big(\widetilde{D}^{-1}_{l_1}\big)_{jl} \big(\widetilde{D}^{-1}_{l_1-p}\big)_{ik} \big(\widetilde{D}^{-1}_{l_2}\big)_{mn} \\
& + \frac{i}{8}\, \widetilde{\mathcal{V}}_{\alpha\alpha ijkl} \int \frac{d^D l_1}{(2\,\pi)^D} \int \frac{d^D l_2}{(2\,\pi)^D} \big(\widetilde{D}^{-1}_{l_1}\big)_{ij} \big(\widetilde{D}^{-1}_{l_2}\big)_{kl} \\
& - \delta\widetilde{\mathcal{V}}_{mn}\, \widetilde{\mathcal{V}}_{\alpha ij}\, \widetilde{\mathcal{V}}_{\alpha kl} \int \frac{d^D l_1}{(2\,\pi)^D} \big(\widetilde{D}^{-1}_{l_1}\big)_{jm} \big(\widetilde{D}^{-1}_{l_1}\big)_{kn} \big(\widetilde{D}^{-1}_{l_1-p}\big)_{il} \\
& + \frac{1}{2} \Big(\delta\widetilde{\mathcal{V}}_{\alpha ij}\, \widetilde{\mathcal{V}}_{\alpha kl} + \delta\widetilde{\mathcal{V}}_{\alpha kl}\, \widetilde{\mathcal{V}}_{\alpha ij}\Big) \int \frac{d^D l_1}{(2\,\pi)^D} \big(\widetilde{D}^{-1}_{l_1}\big)_{jl} \big(\widetilde{D}^{-1}_{l_1-p}\big)_{ik} \\
& + \frac{1}{2}\, \delta\widetilde{\mathcal{V}}_{\alpha\alpha ij} \int \frac{d^D l_1}{(2\,\pi)^D} \big(\widetilde{D}^{-1}_{l_1}\big)_{ij} + \frac{1}{2}\, \delta\widetilde{\mathcal{V}}_{kl}\, \widetilde{\mathcal{V}}_{\alpha\alpha ij} \int \frac{d^D l_1}{(2\,\pi)^D} \big(\widetilde{D}^{-1}_{l_1}\big)_{il} \big(\widetilde{D}^{-1}_{l_1}\big)_{jk}
\end{aligned}
$$

$+$ 

where the 2-loop counterterm in the last line of (3.59) is explicitly given in (3.56) and (3.57) for $\phi$ and $\sigma$, respectively.

In order the evaluate the 2-loop momentum integrals, in particular w.r.t. the IBP-reduction to known scalar integrals, the results of [16] have been used. Further, similar to the effective potential, the 2-loop contribution of the renormalised self energy $\Sigma^{(2L)}_{\varphi_\alpha,\mathrm{ren}}$ may be written as

$$-i\Sigma^{(2L)}_{\varphi_\alpha,\mathrm{ren}} = -i\Sigma^{(2L,SR)}_{\varphi_\alpha,\,1/\epsilon^2} - i\Sigma^{(2L,SR)}_{\varphi_\alpha,\,1/\epsilon} - i\Delta\Sigma^{(2L,SR)}_{\varphi_\alpha,\,1/\epsilon} + \mathcal{O}(\epsilon^0) + \; \varphi_\alpha \text{ - - - - -}\otimes\text{- - - - - } \varphi_\alpha \quad (\xrightarrow{p}) \tag{3.60}$$

where $\Delta\Sigma^{(2L,\,SR)}_{\varphi_\alpha,\,1/\epsilon}$ denotes the new divergent quantum correction to the $1/\epsilon$ - part of the subrenormalised self energy.
The second order pole in $\epsilon$ of the subrenormalised self energy of $\phi$ and $\sigma$ are given by

$$\begin{aligned} -i\,\Sigma^{(2L,\,SR)}_{\phi,\,1/\epsilon^2} &= \frac{i\,\mu^{2\epsilon}(\sigma_0)}{8\,(16\,\pi^2)^2}\Big\{\Big[9\,\lambda_\phi^3 + 12\,\lambda_\phi\,\lambda_m^2 + 3\,\lambda_m^2\,\lambda_\sigma + 12\,\lambda_m^3\Big]\,\phi_0^2 \\ &\quad + \lambda_m\Big[2\,\lambda_\phi^2 + 6\,\lambda_\phi\,\lambda_m + \lambda_\phi\,\lambda_\sigma + 19\,\lambda_m^2 + 6\,\lambda_m\,\lambda_\sigma + 2\,\lambda_\sigma^2\Big]\,\sigma_0^2\Big\}\,\frac{1}{\epsilon^2} \end{aligned} \tag{3.61}$$

$$\begin{aligned} -i\,\Sigma^{(2L,\,SR)}_{\sigma,\,1/\epsilon^2} &= \frac{i\,\mu^{2\epsilon}(\sigma_0)}{8\,(16\,\pi^2)^2}\Big\{\Big[9\,\lambda_\sigma^3 + 12\,\lambda_\sigma\,\lambda_m^2 + 3\,\lambda_m^2\,\lambda_\phi + 12\,\lambda_m^3\Big]\,\sigma_0^2 \\ &\quad + \lambda_m\Big[2\,\lambda_\sigma^2 + 6\,\lambda_\sigma\,\lambda_m + \lambda_\sigma\,\lambda_\phi + 19\,\lambda_m^2 + 6\,\lambda_m\,\lambda_\phi + 2\,\lambda_\phi^2\Big]\,\phi_0^2\Big\}\,\frac{1}{\epsilon^2} \end{aligned} \tag{3.62}$$

The contribution to the simple pole in $\epsilon$ of the subrenormalised self energy of $\phi$ and $\sigma$ that would also be obtained in usual DReg are provided by

$$\begin{aligned} -i\,\Sigma^{(2L,\,SR)}_{\phi,\,1/\epsilon} &= \frac{i}{24\,(16\,\pi^2)^2}\,\left(\lambda_\phi^2 + 3\,\lambda_m^2\right)\,p^2\,\frac{1}{\epsilon} \\ &\quad - \frac{i\,\mu^{2\epsilon}(\sigma_0)}{24\,(16\,\pi^2)^2}\Big\{\Big[18\,\lambda_\phi^3 + 18\,\lambda_\phi\,\lambda_m^2 + 36\,\lambda_m^3\Big]\,\phi_0^2 \\ &\quad + 3\,\lambda_m\Big[\lambda_\phi^2 + 6\,\lambda_\phi\,\lambda_m + 10\,\lambda_m^2 + 6\,\lambda_m\,\lambda_\sigma + \lambda_\sigma^2\Big]\,\sigma_0^2\Big\}\,\frac{1}{\epsilon} \end{aligned} \tag{3.63}$$

$$\begin{aligned} -i\,\Sigma^{(2L,\,SR)}_{\sigma,\,1/\epsilon} &= \frac{i}{24\,(16\,\pi^2)^2}\,\left(\lambda_\sigma^2 + 3\,\lambda_m^2\right)\,p^2\,\frac{1}{\epsilon} \\ &\quad - \frac{i\,\mu^{2\epsilon}(\sigma_0)}{24\,(16\,\pi^2)^2}\Big\{\Big[18\,\lambda_\sigma^3 + 18\,\lambda_\sigma\,\lambda_m^2 + 36\,\lambda_m^3\Big]\,\sigma_0^2 \\ &\quad + 3\,\lambda_m\Big[\lambda_\sigma^2 + 6\,\lambda_\sigma\,\lambda_m + 10\,\lambda_m^2 + 6\,\lambda_m\,\lambda_\phi + \lambda_\phi^2\Big]\,\phi_0^2\Big\}\,\frac{1}{\epsilon} \end{aligned} \tag{3.64}$$

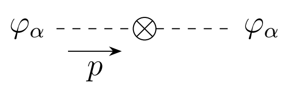
whereas the new contribution to the simple pole in $\epsilon$ of the subrenormalised self energy of $\phi$ and $\sigma$ due to evanescent interactions, i.e. only in SIDReg, are given by

$$\begin{aligned}
-i\,\Delta\Sigma^{(2\mathrm{L},\,\mathrm{SR})}_{\phi,\,1/\epsilon} = -\frac{i\,\mu^{2\epsilon}(\sigma_0)}{48\,(16\,\pi^2)^2}\Bigg\{\Big[&6\,\lambda_\phi\,\lambda_m\,\left(40\,\lambda_\phi + 7\,\lambda_m - 8\,\lambda_\sigma\right) \\
&+ 9\,\lambda_\phi\,\lambda_\sigma\,\left(\lambda_\sigma - 2\,\lambda_\phi\right) - \lambda_m^2\,\left(72\,\lambda_m - 21\,\lambda_\sigma\right)\Big]\,\phi_0^2 \\
+ \lambda_m\Big[6\,\lambda_\phi\,\left(8\,\lambda_m + \lambda_\sigma\right) + 123\,\lambda_m^2 &+ 86\,\lambda_m\,\lambda_\sigma + 3\,\lambda_\sigma^2\Big]\,\sigma_0^2 \\
- 15\,\lambda_\phi\,\lambda_m\Big[7\,\lambda_\phi - 14\,\lambda_m + \lambda_\sigma\Big]\,\frac{\phi_0^4}{\sigma_0^2} &- 42\,\lambda_\phi\,\lambda_m^2\,\frac{\phi_0^6}{\sigma_0^4}\Bigg\}\,\frac{1}{\epsilon}
\end{aligned} \tag{3.65}$$

$$\begin{aligned}
-i\,\Delta\Sigma^{(2\mathrm{L},\,\mathrm{SR})}_{\sigma,\,1/\epsilon} = \frac{i\,\mu^{2\epsilon}(\sigma_0)}{48\,(16\,\pi^2)^2}\Bigg\{\Big[&12\,\lambda_\phi\,\lambda_m\,\left(3\,\lambda_\phi + 5\,\lambda_m + \lambda_\sigma\right) \\
&+ \lambda_m\,\left(219\,\lambda_m^2 + 22\,\lambda_m\,\lambda_\sigma + 33\,\lambda_\sigma^2\right)\Big]\,\phi_0^2 \\
&+ \lambda_m^2\Big[21\,\lambda_\phi - 60\,\lambda_m + 72\,\lambda_\sigma\Big]\,\sigma_0^2 \\
&- \Big[3\,\lambda_\phi\,\left(3\,\lambda_\phi^2 + 4\,\lambda_m^2\right) + 3\,\lambda_m^2\,\left(4\,\lambda_m + \lambda_\sigma\right)\Big]\,\frac{\phi_0^4}{\sigma_0^2} \\
&+ 3\,\lambda_\phi\,\lambda_m\Big[7\,\lambda_\phi - 14\,\lambda_m + \lambda_\sigma\Big]\,\frac{\phi_0^6}{\sigma_0^4} + 15\,\lambda_\phi\,\lambda_m^2\,\frac{\phi_0^8}{\sigma_6}\Bigg\}\,\frac{1}{\epsilon}
\end{aligned} \tag{3.66}$$

In order for the self energy to be fully renormalised in the MS-scheme one needs to demand

$$0 \overset{!}{=} -i\,\Sigma^{(2\mathrm{L},\,\mathrm{SR})}_{\varphi_\alpha,\,1/\epsilon^2} - i\,\Sigma^{(2\mathrm{L},\,\mathrm{SR})}_{\varphi_\alpha,\,1/\epsilon} - i\,\Delta\Sigma^{(2\mathrm{L},\,\mathrm{SR})}_{\varphi_\alpha,\,1/\epsilon} + \tag{3.67}$$




Using the explicit expression for the 2-loop counterterm for $\phi$ and $\sigma$ in (3.56) and (3.57), respectively, the following result is obtained for the wave function Renormalisation coefficients at the 2-loop level

$$\begin{aligned}
\delta Z_\phi^{(2)} &= \frac{1}{(16\,\pi^2)^2}\,\frac{\rho_\phi^{(1)}}{\epsilon} \\
\delta Z_\sigma^{(2)} &= \frac{1}{(16\,\pi^2)^2}\,\frac{\rho_\sigma^{(1)}}{\epsilon}
\end{aligned} \tag{3.68}$$

with

$$\begin{aligned}
\rho_\phi^{(1)} &= -\frac{1}{24}\,\left(\lambda_\phi^2 + 3\,\lambda_m^2\right) \\
\rho_\sigma^{(1)} &= -\frac{1}{24}\,\left(\lambda_\sigma^2 + 3\,\lambda_m^2\right)
\end{aligned} \tag{3.69}$$

Furthermore, renormalising the 2-loop self energy for $\phi$ and $\sigma$ also acts as a consistency check for the other 2-loop counterterms in (3.47) to (3.51), which exactly cancel all other divergences in (3.67).

## 3.3. $\beta$ - Functions and Callan-Symanzik Equation for $V_{\text{eff}}$

Before the $\beta$ - functions of the 2 Scalar Model are determined up to the 2-loop level, all counterterms (in MS-scheme) are summarised below in order to obtain an overview.

$$\begin{aligned}
\delta Z_\phi^{(1)} &= 0 & \delta\lambda_6^{(1)} &= 0 \\
\delta Z_\sigma^{(1)} &= 0 & \delta\lambda_8^{(1)} &= 0 \\
\delta Z_{V_\phi}^{(1)} &= \frac{1}{16\,\pi^2}\,\frac{\delta_\phi^{(0)}}{\epsilon} \\
\delta Z_{V_m}^{(1)} &= \frac{1}{16\,\pi^2}\,\frac{\delta_m^{(0)}}{\epsilon} \\
\delta Z_{V_\sigma}^{(1)} &= \frac{1}{16\,\pi^2}\,\frac{\delta_\sigma^{(0)}}{\epsilon}
\end{aligned} \tag{3.70}$$

and

$$\begin{aligned}
\delta Z_\phi^{(2)} &= \frac{1}{(16\,\pi^2)^2}\,\frac{\rho_\phi^{(1)}}{\epsilon} & \delta\lambda_6^{(2)} &= \frac{1}{(16\,\pi^2)^2}\,\frac{\delta\nu_6^{(1)}}{\epsilon} \\
\delta Z_\sigma^{(2)} &= \frac{1}{(16\,\pi^2)^2}\,\frac{\rho_\sigma^{(1)}}{\epsilon} & \delta\lambda_8^{(2)} &= \frac{1}{(16\,\pi^2)^2}\,\frac{\delta\nu_8^{(1)}}{\epsilon} \\
\delta Z_{V_\phi}^{(2)} &= \frac{1}{(16\,\pi^2)^2}\left(\frac{\delta_\phi^{(2)}}{\epsilon^2} + \frac{\delta_\phi^{(1)} + \nu_\phi^{(1)}}{\epsilon}\right) \\
\delta Z_{V_m}^{(2)} &= \frac{1}{(16\,\pi^2)^2}\left(\frac{\delta_m^{(2)}}{\epsilon^2} + \frac{\delta_m^{(1)} + \nu_m^{(1)}}{\epsilon}\right) \\
\delta Z_{V_\sigma}^{(2)} &= \frac{1}{(16\,\pi^2)^2}\left(\frac{\delta_\sigma^{(2)}}{\epsilon^2} + \frac{\delta_\sigma^{(1)} + \nu_\sigma^{(1)}}{\epsilon}\right)
\end{aligned} \tag{3.71}$$

with

$$\begin{aligned}
\delta_\phi^{(0)} &= \frac{3}{2}\,\frac{\lambda_\phi^2 + \lambda_m^2}{\lambda_\phi} \\
\delta_m^{(0)} &= \frac{1}{2}\,(\lambda_\phi + 4\,\lambda_m + \lambda_\sigma) \\
\delta_\sigma^{(0)} &= \frac{3}{2}\,\frac{\lambda_m^2 + \lambda_\sigma^2}{\lambda_\sigma}
\end{aligned} \tag{3.72}$$

and $\delta_i^{(2)}$, $\delta_i^{(1)}$, $\nu_i^{(1)}$, $\delta\nu_j^{(1)}$, $\rho_k^{(1)}$, for $i \in \{\phi, m, \sigma\}$, $j \in \{6, 8\}$, $k \in \{\phi, \sigma\}$, respectively, given in (3.48) to (3.51) and (3.69).

Further, recall the Renormalisation transformation for the coupling constants in (2.54).

$$\begin{aligned}
\lambda_{\phi,B} &= \mu^{2\epsilon}(\sigma)\,\lambda_{\phi,0} = \mu^{2\epsilon}(\sigma)\,Z_{\lambda_\phi}\,\lambda_\phi = \mu^{2\epsilon}(\sigma)\,Z_{V_\phi}\,Z_\phi^{-2}\,\lambda_\phi \\
\lambda_{m,B} &= \mu^{2\epsilon}(\sigma)\,\lambda_{m,0} = \mu^{2\epsilon}(\sigma)\,Z_{\lambda_m}\,\lambda_m = \mu^{2\epsilon}(\sigma)\,Z_{V_m}\,Z_\phi^{-1}\,Z_\sigma^{-1}\,\lambda_m \\
\lambda_{\sigma,B} &= \mu^{2\epsilon}(\sigma)\,\lambda_{\sigma,0} = \mu^{2\epsilon}(\sigma)\,Z_{\lambda_\sigma}\,\lambda_\sigma = \mu^{2\epsilon}(\sigma)\,Z_{V_\sigma}\,Z_\sigma^{-2}\,\lambda_\sigma
\end{aligned} \tag{3.73}$$

For the $\beta$ - functions of $\lambda_6$ and $\lambda_8$, which are set identical to zero at tree-level, it will be assumed that these coupling constants are non-zero during the derivation process and once the $\beta$ - functions are derived the limit $\lambda_{6,8} \to 0$ is taken. Therefore, the following Renormalisation transformations for $\lambda_6$ and $\lambda_8$ are considered

$$\begin{aligned} \lambda_{6,B} &= \mu^{2\epsilon}(\sigma)\,\lambda_{6,0} = \mu^{2\epsilon}(\sigma)\,Z_{\lambda_6}\,\lambda_6 = \mu^{2\epsilon}(\sigma)\,Z_{V_6}\,Z_\phi^{-3}\,Z_\sigma\,\lambda_6 \\ \lambda_{8,B} &= \mu^{2\epsilon}(\sigma)\,\lambda_{8,0} = \mu^{2\epsilon}(\sigma)\,Z_{\lambda_8}\,\lambda_8 = \mu^{2\epsilon}(\sigma)\,Z_{V_8}\,Z_\phi^{-4}\,Z_\sigma^2\,\lambda_8 \end{aligned} \tag{3.74}$$

where $\delta Z_{V_6}^{(i)} := \delta\lambda_6^{(i)}/\lambda_6$ and $\delta Z_{V_8}^{(i)} := \delta\lambda_8^{(i)}/\lambda_8$.

Taking the logarithm of (3.73) and (3.74), one obtains

$$\log\left(\lambda_{j,B}\right) = 2\,\epsilon\,\log\left(\mu(\sigma)\right) + \log\left(\lambda_j\right) + \Theta_j \tag{3.75}$$

with $j \in \{\phi, m, \sigma, 6, 8\}$ and

$$\begin{aligned} \Theta_\phi &:= \log\left(Z_{V_\phi}\right) - 2\,\log\left(Z_\phi\right) \\ &= \delta Z_{V_\phi}^{(1)} - \frac{1}{2}\left(\delta Z_{V_\phi}^{(1)}\right)^2 + \delta Z_{V_\phi}^{(2)} - 2\,\delta Z_\phi^{(2)} + \mathcal{O}\left(\hbar^3\right) \\ \Theta_m &:= \log\left(Z_{V_m}\right) - \log\left(Z_\phi\right) - \log\left(Z_\sigma\right) \\ &= \delta Z_{V_m}^{(1)} - \frac{1}{2}\left(\delta Z_{V_m}^{(1)}\right)^2 + \delta Z_{V_m}^{(2)} - \delta Z_\phi^{(2)} - \delta Z_\sigma^{(2)} + \mathcal{O}\left(\hbar^3\right) \\ \Theta_\sigma &:= \log\left(Z_{V_\sigma}\right) - 2\,\log\left(Z_\sigma\right) \\ &= \delta Z_{V_\sigma}^{(1)} - \frac{1}{2}\left(\delta Z_{V_\sigma}^{(1)}\right)^2 + \delta Z_{V_\sigma}^{(2)} - 2\,\delta Z_\sigma^{(2)} + \mathcal{O}\left(\hbar^3\right) \\ \Theta_6 &:= \log\left(Z_{V_6}\right) - 3\,\log\left(Z_\phi\right) + \log\left(Z_\sigma\right) \\ &= \delta Z_{V_6}^{(2)} - 3\,\delta Z_\phi^{(2)} + \delta Z_\sigma^{(2)} + \mathcal{O}\left(\hbar^3\right) \\ \Theta_8 &:= \log\left(Z_{V_8}\right) - 4\,\log\left(Z_\phi\right) + 2\,\log\left(Z_\sigma\right) \\ &= \delta Z_{V_8}^{(2)} - 4\,\delta Z_\phi^{(2)} + 2\,\delta Z_\sigma^{(2)} + \mathcal{O}\left(\hbar^3\right) \end{aligned} \tag{3.76}$$

where it has already been used that $\delta Z_\phi^{(1)} = \delta Z_\sigma^{(1)} = \delta Z_{V_6}^{(1)} = \delta Z_{V_8}^{(1)} = 0$.
The $\beta$ - function of a generic coupling $\lambda$ is defined by

$$\beta_\lambda := \frac{d\lambda}{d\log\left(z\right)} = z\,\frac{d\lambda}{dz} \tag{3.77}$$

Further, physics, and thus the "bare" couplings $\lambda_{j,B}$, $j \in \{\phi, m, \sigma, 6, 8\}$ are independent of the Renormalisation parameter $z$. For this reason, the derivative of equation (3.75) w.r.t. $\log\left(z\right)$ is demanded to be zero, i.e.

$$\begin{aligned} 0 = \frac{d\log\left(\lambda_{j,B}\right)}{d\log\left(z\right)} &= \left[\frac{\partial}{\partial\log\left(z\right)} + \sum_k \beta_{\lambda_k}\,\frac{\partial}{\partial\lambda_k}\right]\,\log\left(\lambda_{j,B}\right) \\ &= 2\,\epsilon + \left(\frac{1}{\lambda_j} + \frac{\partial\Theta_j}{\partial\lambda_j}\right)\,\beta_{\lambda_j} + \sum_{k\neq j}\frac{\partial\Theta_j}{\partial\lambda_k}\,\beta_{\lambda_k} \end{aligned} \tag{3.78}$$

where the definition of a $\beta$ - function (3.77) and (3.75) have been used and it is *not* summed implicitly over $j$. The last line of (3.78) provides a system of implicit equations for the different $\beta$ - functions, which can be used to determine explicit expressions for these $\beta$ - functions. In general this system is a coupled system of implicit equations. This is also the case here, however, it can be seen that the first three equations for $j \in \{\phi, m, \sigma\}$ decouple from the last two equations, because the corresponding counterterms, and thus the corresponding $\Theta_j$ are independent of $\lambda_6$ and $\lambda_8$, as can be seen in (3.48) to (3.50), (3.69) and (3.72). Therefore, the $\beta$ - functions for $\lambda_\phi, \lambda_m, \lambda_\sigma$ can be determined isolated from those of $\lambda_6, \lambda_8$, which is done in the following.

Analogous to previous results, $\beta$ - functions are given in the following form

$$\beta_{\lambda_j} = \beta_{\lambda_j}^{(1L)} + \beta_{\lambda_j}^{(2L)} + \Delta\beta_{\lambda_j}^{(2L)} + \mathcal{O}\left(\hbar^3\right) \tag{3.79}$$

After solving the first three equations of (3.78) for $\beta_{\lambda_j}$, $j \in \{\phi, m, \sigma\}$, and then taking the limit $\epsilon \to 0$, one obtains for the $\beta$ - function of $\lambda_\phi$

$$\beta_{\lambda_\phi}^{(1L)} = \frac{3}{16\,\pi^2}\left[\lambda_\phi^2 + \lambda_m^2\right] \tag{3.80}$$

$$\beta_{\lambda_\phi}^{(2L)} = -\frac{1}{\left(16\,\pi^2\right)^2}\left[\frac{17}{3}\,\lambda_\phi^3 + 5\,\lambda_\phi\,\lambda_m^2 + 12\,\lambda_m^3\right] \tag{3.81}$$

$$\begin{aligned}\Delta\beta_{\lambda_\phi}^{(2L)} = -\frac{1}{\left(16\,\pi^2\right)^2}\Bigg[&8\,\lambda_\phi\,\lambda_m\,\left(5\,\lambda_\phi - \lambda_\sigma\right) + \frac{3}{2}\,\lambda_\phi\,\lambda_\sigma\,\left(\lambda_\sigma - 2\,\lambda_\phi\right)\\ &+ \frac{7}{2}\,\lambda_m^2\,\left(2\,\lambda_\phi + \lambda_\sigma\right) - 12\,\lambda_m^3\Bigg]\end{aligned} \tag{3.82}$$

for the $\beta$ - function of $\lambda_m$

$$\beta_{\lambda_m}^{(1L)} = \frac{1}{16\,\pi^2}\,\lambda_m\left[\lambda_\phi + 4\,\lambda_m + \lambda_\sigma\right] \tag{3.83}$$

$$\beta_{\lambda_m}^{(2L)} = -\frac{1}{\left(16\,\pi^2\right)^2}\,\frac{\lambda_m}{6}\left[5\,\lambda_\phi^2 + 36\,\lambda_\phi\,\lambda_m + 54\,\lambda_m^2 + 36\,\lambda_m\,\lambda_\sigma + 5\,\lambda_\sigma^2\right] \tag{3.84}$$

$$\Delta\beta_{\lambda_m}^{(2L)} = -\frac{1}{\left(16\,\pi^2\right)^2}\,\frac{\lambda_m}{6}\left[48\,\lambda_\phi\,\lambda_m + 6\,\lambda_\phi\,\lambda_\sigma + 123\,\lambda_m^2 + 86\,\lambda_m\,\lambda_\sigma + 3\,\lambda_\sigma^2\right] \tag{3.85}$$

and for the $\beta$ - function of $\lambda_\sigma$

$$\beta_{\lambda_\sigma}^{(1L)} = \frac{3}{16\,\pi^2}\left[\lambda_\sigma^2 + \lambda_m^2\right] \tag{3.86}$$

$$\beta_{\lambda_\sigma}^{(2L)} = -\frac{1}{\left(16\,\pi^2\right)^2}\left[\frac{17}{3}\,\lambda_\sigma^3 + 5\,\lambda_\sigma\,\lambda_m^2 + 12\,\lambda_m^3\right] \tag{3.87}$$

$$\Delta\beta_{\lambda_\sigma}^{(2L)} = -\frac{1}{\left(16\,\pi^2\right)^2}\left[\frac{21}{2}\,\lambda_\sigma^3 + 2\,\lambda_\sigma\,\lambda_m^2 + 24\,\lambda_m^3\right] \tag{3.88}$$

Now, knowing the $\beta$ - functions of $\lambda_\phi, \lambda_m, \lambda_\sigma$, the $\beta$ - functions of $\lambda_6, \lambda_8$ can separately be determined by solving the last two equations of (3.78) for $\beta_{\lambda_j}$, $j \in \{6, 8\}$, and then taking the limit $\epsilon \to 0$, *as well as* $\lambda_{6,8} \to 0$ (as discussed above). One obtains for the $\beta$ - function of $\lambda_6$

$$\beta^{(1L)}_{\lambda_6} = 0, \qquad \beta^{(2L)}_{\lambda_6} = 0 \tag{3.89}$$

$$\Delta\beta^{(2L)}_{\lambda_6} = \frac{1}{(16\,\pi^2)^2}\,\frac{\lambda_\phi\,\lambda_m}{4}\left[7\,\lambda_\phi - 14\,\lambda_m + \lambda_\sigma\right] \tag{3.90}$$

and for the $\beta$ - function of $\lambda_8$

$$\beta^{(1L)}_{\lambda_8} = 0, \qquad \beta^{(2L)}_{\lambda_8} = 0 \tag{3.91}$$

$$\Delta\beta^{(2L)}_{\lambda_8} = \frac{1}{(16\,\pi^2)^2}\,\frac{\lambda_\phi\,\lambda_m^2}{2} \tag{3.92}$$

**Remark.**

(i) It can be seen that the 1-loop $\beta$ - functions do *not* get new corrections due to QSI, whereas the 2-loop $\beta$ - functions do obtain new quantum corrections, represented by $\Delta\beta^{(2L)}_{\lambda_j}$, due to QSI, i.e. due to evanescent interactions. This is not surprising because there are *no* new *divergent* quantum corrections at the 1-loop level, but at the 2-loop level, as discussed above. This can already be seen by the counterterms, which do not obtain corrections at the 1-loop level, but at the 2-loop level. For this reason, at the 1-loop level the $\beta$ - functions of the QSI theory cannot be distinguished from those of the theory regularised using usual DReg, i.e. they are the same, however, at the 2-loop level the $\beta$ - functions differ from those of the DReg - regularised theory due to new quantum corrections introduced by evanescent interactions, i.e. introduced by QSI. Thus, the 2-loop running of the couplings of the QSI theory with (only) spontaneously broken scale invariance (even at the quantum level) is different from the theory where scale symmetry is *explicitly* broken by quantum corrections, which has also been discussed in [14], and the origin of this difference are evanescent interactions, giving rise to new quantum corrections, due to QSI.

(ii) In (3.89) to (3.92), it can be seen that the couplings $\lambda_6$ and $\lambda_8$ obtain a non-vanishing running at the 2-loop level, even though these couplings are identically zero at tree-level. Further, this running is solely restricted to the new corrections of the $\beta$ - functions, and thus is not present in the DReg - regularised theory. Again, this is not a surprising result because the corresponding counterterms $\delta\lambda_6$ and $\delta\lambda_8$, and thus the contributions to the corresponding $\beta$ - functions, are only caused by evanescent interactions, i.e. by QSI, and are not present in the DReg - regularised theory, i.e. without QSI. This running of couplings of higher dimensional non-polynomial terms shows again that the QSI theory, i.e. the SIDReg - regularised theory, is non-renormalisable in contrast to the DReg - regularised theory due to higher dimensional non-polynomial operators of the form (2.51) introduced by evanescent interactions.

(iii) The results of the $\beta$ - functions for the 2 Scalar Model up to the 2-loop level displayed above are in perfect agreement with those given in [14].

At this stage of this thesis it is worth drawing a first conclusion w.r.t. the vanishing of the scale anomaly in a theory with spontaneously broken quantum scale symmetry, regularised using SIDReg.

**Remark.**
As discussed in chapter 2, scale symmetry is broken explicitly by quantum corrections in a classically scale invariant theory that is regularised using usual DReg ($\mu = \text{const}$). Thus, scale symmetry is lost in $D$ dimensions, as well as at the quantum level and remains only present at tree-level in 4 dimensions due to the massive Renormalisation scale. This anomalous breaking of scale symmetry is often referred to as scale or trace anomaly due to the fact that it manifests itself by a non-vanishing contribution to the trace of the energy-momentum tensor $T^{\mu}_{\mu} \sim \beta_{\lambda_j} \neq 0$ [8, 13, 21].

Using SIDReg to regularise a classically scale invariant theory ensures the absence of the scale anomaly and provides a theory that is scale invariant even at the quantum level, i.e. QSI, where scale symmetry is only broken spontaneously. Nonetheless, the theory still admits non-zero $\beta$ - functions, and thus a running of the couplings, as shown above. Hence, the vanishing of the $\beta$ - functions is *not* necessary for quantum scale invariance, but the Regularisation of the theory, which has to respect scale symmetry, is essential for QSI.

Now, the Callan-Symanzik equation for the (scale invariant) effective potential $V_{\text{eff}}$ is derived [14, 41]. Analogous to the "bare" couplings, the effective potential $V_{\text{eff}}$ is independent of the Renormalisation parameter $z$, and thus the Callan-Symanzik equation for scale invariant theories [14, 41] is given by

$$\begin{aligned} 0 &= \frac{dV_{\text{eff}}}{d\log(z)} \\ &= \left[\frac{\partial}{\partial \log(z)} + \sum_k \frac{d\lambda_k}{d\log(z)}\frac{\partial}{\partial\lambda_k} + \sum_i \frac{d\varphi_i}{d\log(z)}\frac{\partial}{\partial\varphi_i}\right] V_{\text{eff}} \\ &= \left[z\frac{\partial}{\partial z} + \sum_k \beta_{\lambda_k}\frac{\partial}{\partial\lambda_k} - \sum_i \gamma_{\varphi_i}\,\varphi_i\,\frac{\partial}{\partial\varphi_i}\right] V_{\text{eff}} \end{aligned} \tag{3.93}$$

where the definition of the $\beta$ - functions (3.77), as well as

$$\frac{d\varphi_i}{d\log(z)} = \frac{d}{d\log(z)}\, Z_{\varphi_i}^{-1/2}\,\varphi_{i,0} = -\,\frac{1}{2}\,\frac{1}{Z_{\varphi_i}}\,\frac{dZ_{\varphi_i}}{d\log(z)}\,\varphi_i = -\,\gamma_{\varphi_i}\,\varphi_i \tag{3.94}$$

where

$$\gamma_{\varphi_i} \coloneqq \frac{1}{2}\,\frac{1}{Z_{\varphi_i}}\,\frac{dZ_{\varphi_i}}{d\log(z)} \tag{3.95}$$

for $\{\varphi_i\}_{i=1}^{2} = \{\phi, \sigma\}$, have been used. For the (scale invariant) effective potential of the 2 Scalar Model this means

$$0 = \left[ z \, \frac{\partial}{\partial z} + \sum_k \beta_{\lambda_k} \, \frac{\partial}{\partial \lambda_k} - \gamma_\phi \, \phi \, \frac{\partial}{\partial \phi} - \gamma_\sigma \, \sigma \, \frac{\partial}{\partial \sigma} \right] V_{\text{eff}} \left( \phi, \sigma \right) \tag{3.96}$$

with

$$\begin{aligned} \gamma_\phi &= - \frac{2}{\left(16 \, \pi^2\right)^2} \, \rho_\phi^{(1)} + \mathcal{O}\left(\hbar^3\right) \\ \gamma_\sigma &= - \frac{2}{\left(16 \, \pi^2\right)^2} \, \rho_\sigma^{(1)} + \mathcal{O}\left(\hbar^3\right) \end{aligned} \tag{3.97}$$

and the $\beta$ - functions given above. In the present case, all quantities have been determined up to the 2-loop level, and thus the Callan-Symanzik equation for the scale invariant effective potential up to the 2-loop level may then be written as

$$\begin{aligned} 0 = \; & z \, \frac{\partial V_{\text{tree}}}{\partial z} \\ & + z \, \frac{\partial V_{\text{1L,reg}}}{\partial z} + \sum_k \beta_{\lambda_k}^{(\text{1L})} \, \frac{\partial V_{\text{tree}}}{\partial \lambda_k} \\ & + z \, \frac{\partial \Delta U_{\text{1L}}}{\partial z} \\ & + z \, \frac{\partial V_{\text{2L,reg}}}{\partial z} + \sum_k \beta_{\lambda_k}^{(\text{1L})} \frac{\partial V_{\text{1L,reg}}}{\partial \lambda_k} + \left[ \sum_k \beta_{\lambda_k}^{(\text{2L})} \frac{\partial}{\partial \lambda_k} - \gamma_\phi^{(\text{2L})} \phi \frac{\partial}{\partial \phi} - \gamma_\sigma^{(\text{2L})} \sigma \frac{\partial}{\partial \sigma} \right] V_{\text{tree}} \\ & + z \, \frac{\partial \Delta U_{\text{2L}}}{\partial z} + \sum_k \beta_{\lambda_k}^{(\text{1L})} \, \frac{\partial \Delta U_{\text{1L}}}{\partial \lambda_k} + \sum_k \Delta \beta_{\lambda_k}^{(\text{2L})} \, \frac{\partial V_{\text{tree}}}{\partial \lambda_k} \\ & + \mathcal{O}\left(\hbar^3\right) \end{aligned} \tag{3.98}$$

where the effective potential up to the 2-loop level is given in (3.52) and its constituents, used in (3.98), are provided in the previous sections of this chapter. $\gamma_\phi^{(\text{2L})}$ and $\gamma_\sigma^{(\text{2L})}$ are only the 2-loop contribution of the quantities in (3.97). Every line on the RHS of (3.98) is equal to zero by itself. The first line on the RHS of (3.98) is of the order $\mathcal{O}\left(\hbar^0\right)$, the second and third lines are of the order $\mathcal{O}\left(\hbar\right)$, and the last two lines are of the order $\mathcal{O}\left(\hbar^2\right)$. Moreover, lines 1, 2 and 4 are the "usual" contributions, whereas lines 3 and 5 are new contributions due to evanescent interactions, i.e. as a result of QSI.

Hence, the Callan-Symanzik equations for the quantum scale invariant effective potential of the 2 Scalar Model $V_{\text{eff}} = V_{\text{eff}}(\phi, \sigma)$ is verified up to the 2-loop level, which is in agreement with [14].

## 3.4. Working in the broken Phase of the Theory

In this section it is shown that the same scale invariant counterterms are obtained from $N$-point Green functions with non-vanishing external momenta using the expanded Lagrangian in the broken phase of the theory as those obtained from a manifestly scale invariant approach using the scale invariant effective potential. As mentioned at the beginning of this chapter, the field shift (3.1) is *not* used in this section. Instead, the Lagrangian is expanded about the fields VEVs $\{v, w\}$ and w.r.t. $\epsilon$, as shown in (2.42). In order to show the validity of working in the broken phase of theory in the context of quantum scale symmetry, as proposed above and in the prescription in section 2.1, the following steps are conducted as consistency check.

(a) At the 1-loop level, working with the Lagrangian (2.40), with expanded potential (2.42), in flavour eigenstates $\{h, \mathfrak{D}\}$, it is shown that the same scale invariant 1-loop counterterms are obtained from the 2-point Green function, i.e. the self-energy, for $h$ and $\mathfrak{D}$ with external momentum $p$ as those obtained from the manifestly scale invariant effective potential above.

(b) The same is done at the 2-loop level, i.e. the 2-loop counterterms are determined from the 2-loop self-energy for $h$ and $\mathfrak{D}$ with external momentum $p$.

(c) At the 1-loop level, it is shown that the scale invariant 1-loop counterterms determined above also completely renormalise 1-, 2-, 3- and 4-point Green functions with non-vanishing external momenta calculated using the Lagrangian in mass eigenstates $\{H, S\}$ and with the minimalisation condition (2.47) being used, with potential (2.50).

(d) So far, the potential of the form (2.41) was considered. However, as mentioned in section 2.3, one actually needs to consider the non-Renormalisable potential (2.52) due to quantum scale symmetry. Hence, coupling constants of the form $\lambda_{4+2n}$ for higher dimensional operators have been set to zero at tree-level so far. Now, the Lagrangian is considered with potential (2.52) and non-zero $\lambda_6$ at tree-level, i.e. with $\lambda_6 \neq 0$ but still $\lambda_{4+2n} \equiv 0, \forall\, n \geq 2$ at tree-level. In this theory, the scale invariant effective potential is determined at the 1-loop level, and thus the corresponding scale invariant 1-loop counterterms. Then it is checked that the same scale invariant 1-loop counterterms are obtained from $N$-point Green functions with non-vanishing external momenta computed using the expanded Lagrangian in the broken phase of the theory in flavour eigenstates $\{h, \mathfrak{D}\}$, as well as in mass eigenstates $\{H, S\}$ and the minimalisation conditions (2.47) being used.

(a) The renormalised 1-loop self energies $\Sigma^{(1\mathrm{L})}_{h,\,\mathrm{ren}}$ and $\Sigma^{(1\mathrm{L})}_{\mathfrak{D},\,\mathrm{ren}}$ for $h$ and $\mathfrak{D}$, respectively, computed in a theory with Lagrangian (2.40) and expanded potential (2.42) in flavour eigenstates $\{h, \mathfrak{D}\}$, are given in (3.21) if the replacement $\phi \to h$, $\sigma \to \mathfrak{D}$, $\phi_0 \to v$ and $\sigma_0 \to w$ is made. It can be seen that the scale invariant 1-loop counterterms (3.19), obtained from the manifestly scale invariant effective potential, are sufficient to completely renormalise these self energies, i.e. cancel all divergences of these self energies.

Hence, working in the broken phase of the theory with expanded Lagrangian one obtains the same scale invariant counterterms from Green functions with non-vanishing external momenta as from a manifestly scale invariant approach using $V_{\text{eff}}$.

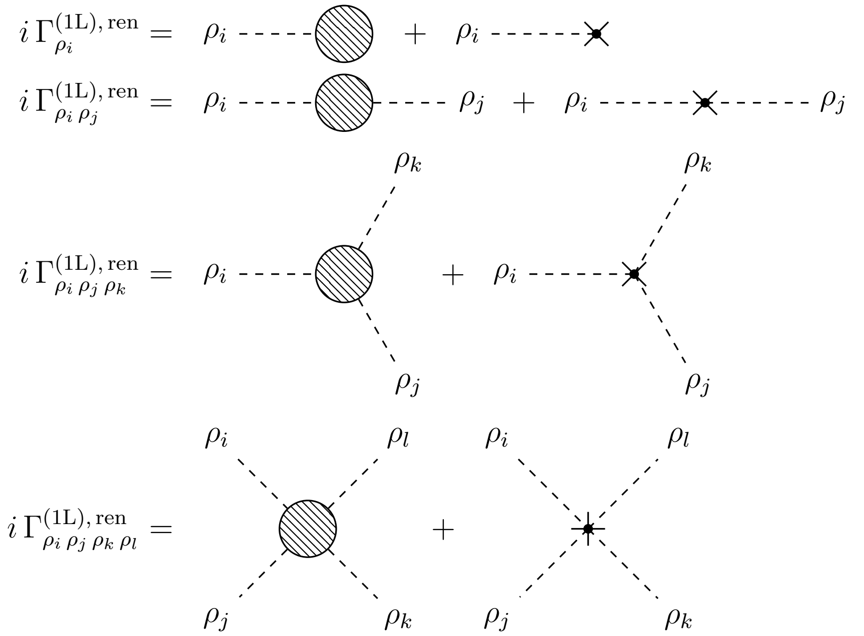
(b) As shown above, counterterms at the 1-loop level do *not* obtain corrections due to QSI, and thus are equivalent to the counterterms of the DReg - regularised theory. However, counterterms at the 2-loop level do obtain new corrections due to new *divergent* quantum corrections introduced by evanescent interactions, i.e. as a result of QSI. For this reason, the validity of working in the broken phase of the theory in the context of QSI should also be checked at the 2-loop level, where the counterterms are distinguishable between SIDReg - and DReg - regularised theories. The results for the renormalised 2-loop self energies $\Sigma^{(2L)}_{h,\,\text{ren}}$ and $\Sigma^{(2L)}_{\mathfrak{D},\,\text{ren}}$ for $h$ and $\mathfrak{D}$, respectively, computed in a theory with Lagrangian (2.40) and expanded potential (2.42) in flavour eigenstates $\{h, \mathfrak{D}\}$, are given in (3.58) to (3.66), again, if the replacement $\phi \to h$, $\sigma \to \mathfrak{D}$, $\phi_0 \to v$ and $\sigma_0 \to w$ is made. Similar to the 1-loop case, it can be seen that the scale invariant 2-loop counterterms (3.47), obtained from the manifestly scale invariant effective potential, are sufficient to completely cancel all $p$ - independent divergencies of these self energies, where $p$ is the external momentum. Therefore, working in the broken phase of the theory with expanded Lagrangian one obtains even at the 2-loop level, where counterterms obtain new corrections due to QSI, the same scale invariant counterterms from Green functions with non-vanishing external momenta as from a manifestly scale invariant approach using $V_{\text{eff}}$.

(c) Now, the Lagrangian (2.40) in mass eigenstates $\{H, S\}$ and with the minimalisation condition (2.47) being used, i.e. with potential (2.50), is considered at the 1-loop level. In this case, i.e. with the minimalisation condition (2.47) being used, the 1-loop counterterms (3.19) are then provided by

$$\begin{aligned}
\delta Z^{(1)}_{V_\phi} \xrightarrow[(2.47)]{} \delta Z^{(1)}_{V_\phi}\Big|_{\text{min}} &= \frac{1}{16\,\pi^2}\,\frac{3}{2}\,\lambda_\phi\,\left(1+\frac{v^4}{9\,w^4}\right)\frac{1}{\epsilon}\\
\delta Z^{(1)}_{V_m} \xrightarrow[(2.47)]{} \delta Z^{(1)}_{V_m}\Big|_{\text{min}} &= \frac{1}{16\,\pi^2}\,\frac{\lambda_\phi}{2}\,\left(1-\frac{4}{3}\frac{v^2}{w^2}+\frac{v^4}{w^4}\right)\frac{1}{\epsilon}\\
\delta Z^{(1)}_{V_\sigma} \xrightarrow[(2.47)]{} \delta Z^{(1)}_{V_\sigma}\Big|_{\text{min}} &= \frac{1}{16\,\pi^2}\,\frac{\lambda_\phi}{6}\,\left(1+9\,\frac{v^4}{w^4}\right)\frac{1}{\epsilon}
\end{aligned} \tag{3.99}$$

Note that these counterterms are still dimensionless because they only contain dimensionless ratios of the VEVs $\{v, w\}$ of the form $\chi_0 = v/w$, and thus are still scale invariant. It has been checked that these counterterms (3.99) completely renormalise all 1-, 2-, 3- and 4-point Green functions, as illustrated in (3.100), with non-vanishing external momenta, computed using the Lagrangian in mass eigenstates $\{H, S\}$ and with the minimalisation condition (2.47) being used, i.e. with potential (2.50). These calculations have been conducted using Mathematica and appropriate packages. All Feynman diagrams have been generated using FeynArts [18], the FeynArts model files have been generated using FeynRules [1, 4], and the generated Feynman diagrams and their amplitudes have been computed using FeynCalc [27, 36, 37] and Package-X [29], which has been connected with FeynCalc using FeynHelpers [35]. The renormalised 1PI $N$-point

Green functions, for $N \in \{1,2,3,4\}$, are



(3.100)

where the blobs represent all 1PI 1-loop diagrams with the corresponding number of external legs and $\{\rho_i\}_{i=1}^2 = \{H, S\}$. Indeed it turns out that the counterterms (3.99) completely renormalise all these 1PI $N$-point Green functions above. Hence, working in the broken phase of the theory with expanded Lagrangian, transformed to mass eigenstates and with the minimalisation condition (2.47) being used, one still obtains the corresponding scale invariant counterterms from Green functions with non-vanishing external momenta, of course only with the minimalisation condition (2.47) being applied.

(d) Considering the QSI 2 Scalar Model with non-zero $\lambda_6$ at tree-level, i.e. considering the non-Renormalisable potential (2.52) with $\lambda_6 \neq 0$ but still $\lambda_{4+2n} \equiv 0, \forall\, n \geq 2$ at tree-level, the 1-loop contribution to the effective potential is still given by (3.10), however, with $M^2_{\rho_k}$ and $c^{(1)}_{\rho_k}$, $\{\rho_k\}_{k=1}^2 = \{H, S\}$, being modified accordingly. Hence, the divergent part of the 1-loop contribution to the effective potential in this scenario is provided by

$$
\begin{aligned}
V_{1L}\big|_{\text{div}} &= -\,\frac{\mu^{2\epsilon}(\sigma)}{64\,\pi^2}\,\sum_{k=1}^{2}\,\Big(\hat{M}^2_{\rho_k}(\phi,\sigma)\Big)^2\,\frac{1}{\epsilon} \\
&= -\,\frac{\mu^{2\epsilon}(\sigma)}{64\,\pi^2}\,\left[\Big(\hat{M}^2_H(\phi,\sigma)\Big)^2 + \Big(\hat{M}^2_S(\phi,\sigma)\Big)^2\right]\,\frac{1}{\epsilon} \\
&= -\,\frac{\mu^{2\epsilon}(\sigma)}{64\,\pi^2}\,\bigg[\frac{1}{4}\,\left(\lambda_\phi^2 + \lambda_m^2 + 20\,\lambda_\phi\,\lambda_6\right)\,\phi^4 + \frac{1}{2}\,\lambda_m\,\left(\lambda_\phi + 4\,\lambda_m + \lambda_\sigma\right)\,\phi^2\,\sigma^2 \\
&\qquad + \frac{1}{4}\,\left(\lambda_m^2 + \lambda_\sigma^2\right)\,\sigma^4 + \lambda_6\,\left(5\,\lambda_\phi - 8\,\lambda_m + \lambda_\sigma\right)\,\frac{\phi^6}{\sigma^2} \\
&\qquad + \lambda_6\,\left(\lambda_m + 25\,\lambda_6\right)\,\frac{\phi^8}{\sigma^4} + 8\,\lambda_6^2\,\frac{\phi^{10}}{\sigma^6} + \lambda_6^2\,\frac{\phi^{12}}{\sigma^8}\bigg]\,\frac{1}{\epsilon}
\end{aligned}
\qquad (3.101)
$$

Subsequently, the 1-loop counterterm potential, necessary for the 1-loop Renormalisation, then reads

$$\widetilde{V}_{\text{tree,ct1}} = \mu^{2\epsilon}(\sigma) \Bigg( \delta Z^{(1)}_{V_\phi} \frac{\lambda_\phi}{4!} \phi^4 + \delta Z^{(1)}_{V_m} \frac{\lambda_m}{4} \phi^2 \sigma^2 + \delta Z^{(1)}_{V_\sigma} \frac{\lambda_\sigma}{4!} \sigma^4 \\ + \delta Z^{(1)}_{V_6} \frac{\lambda_6}{6} \frac{\phi^6}{\sigma^2} + \frac{\delta\lambda^{(1)}_8}{8} \frac{\phi^8}{\sigma^4} + \frac{\delta\lambda^{(1)}_{10}}{10} \frac{\phi^{10}}{\sigma^6} + \frac{\delta\lambda^{(1)}_{12}}{12} \frac{\phi^{12}}{\sigma^8} \Bigg) \tag{3.102}$$

with 1-loop counterterms in the MS-scheme explicitly given by

$$\begin{aligned} &\delta Z^{(1)}_\phi = 0 && \delta Z^{(1)}_\sigma = 0 \\ &\delta Z^{(1)}_{V_\phi} = \frac{1}{16\pi^2} \frac{3}{2} \frac{\lambda_\phi^2 + \lambda_m^2 + 20\,\lambda_m \lambda_6}{\lambda_\phi} \frac{1}{\epsilon} && \delta\lambda^{(1)}_8 = \frac{1}{16\pi^2} 2\,\lambda_6 \Big(\lambda_m + 25\,\lambda_6\Big) \frac{1}{\epsilon} \\ &\delta Z^{(1)}_{V_m} = \frac{1}{16\pi^2} \frac{1}{2} \Big(\lambda_\phi + 4\,\lambda_m + \lambda_\sigma\Big) \frac{1}{\epsilon} && \delta\lambda^{(1)}_{10} = \frac{1}{16\pi^2} 20\,\lambda_6^2 \frac{1}{\epsilon} \\ &\delta Z^{(1)}_{V_\sigma} = \frac{1}{16\pi^2} \frac{3}{2} \frac{\lambda_\sigma^2 + \lambda_m^2}{\lambda_\sigma} \frac{1}{\epsilon} && \delta\lambda^{(1)}_{12} = \frac{1}{16\pi^2} 3\,\lambda_6^2 \frac{1}{\epsilon} \\ &\delta Z^{(1)}_{V_6} = \frac{1}{16\pi^2} \frac{3}{2} \Big(5\,\lambda_\phi - 8\,\lambda_m + \lambda_\sigma\Big) \frac{1}{\epsilon} \end{aligned} \tag{3.103}$$

It can be seen that in this scenario, i.e. for $\lambda_6 \neq 0$ at tree-level, higher dimensional non-polynomial operators of the form (2.51), for $p \leq 4$, are necessary as counterterms already at the 1-loop level. The 1-loop self energies $\Sigma^{(1L)}_{h,\,\text{ren}}$ and $\Sigma^{(1L)}_{\mathfrak{D},\,\text{ren}}$ for $h$ and $\mathfrak{D}$, respectively, are calculated and renormalised analogously to (a) after expanding the Lagrangian with $\lambda_6 \neq 0$ about the fields VEVs $\{v, w\}$ and w.r.t. $\epsilon$. In principle one can still use the results in (3.21), however, with $\hat{M}^2_{\rho_k}$, $\widetilde{\mathcal{V}}_{ijk\cdots}$, $\widetilde{A}_{ij}$ and $\widetilde{B}_{ij}$ modified accordingly because the same Feynman diagrams contribute. For the 1-loop self energy of $h$ one obtains

$$\begin{aligned} -i\,\Sigma^{(1L)}_{h,\text{ren}} &= \frac{1}{2} \widetilde{\mathcal{V}}_{1ik} \widetilde{\mathcal{V}}_{1jl} \int \frac{d^D q}{(2\pi)^D} \big(\widetilde{D}^{-1}_q\big)_{ij} \big(\widetilde{D}^{-1}_{q-p}\big)_{lk} + \frac{1}{2} \widetilde{\mathcal{V}}_{11ij} \int \frac{d^D q}{(2\pi)^D} \big(\widetilde{D}^{-1}_q\big)_{ji} \\ &\quad + i\,p^2\,\delta Z^{(1)}_\phi - i\,\mu_0^{2\epsilon} \Bigg( \frac{\lambda_\phi}{2} \delta Z^{(1)}_{V_\phi} v^2 + \frac{\lambda_m}{2} \delta Z^{(1)}_{V_m} w^2 + 5\,\lambda_6\,\delta Z^{(1)}_{V_6} \frac{v^4}{w^2} \\ &\qquad + 7\,\delta\lambda^{(1)}_8 \frac{v^6}{w^4} + 9\,\delta\lambda^{(1)}_{10} \frac{v^8}{w^6} + 11\,\delta\lambda^{(1)}_{12} \frac{v^{10}}{w^8} \Bigg) \\ &= \frac{i\,\mu_0^{2\epsilon}}{32\pi^2} \Bigg[ \frac{3}{2} \big(\lambda_\phi^2 + \lambda_m^2 + 20\,\lambda_m \lambda_6\big) v^2 + \frac{1}{2} \lambda_m \left(\lambda_\phi + 4\,\lambda_m + \lambda_\sigma\right) w^2 \\ &\qquad + 15\,\lambda_6 \left(5\,\lambda_\phi - 8\,\lambda_m + \lambda_\sigma\right) \frac{v^4}{w^2} + 28\,\lambda_6 \left(\lambda_m + 25\,\lambda_6\right) \frac{v^6}{w^4} \\ &\qquad + 360\,\lambda_6^2 \frac{v^8}{w^6} + 66\,\lambda_6^2 \frac{v^{10}}{w^8} \Bigg] \frac{1}{\epsilon} + \mathcal{O}\left(\epsilon^0\right) \\ &\quad + i\,p^2\,\delta Z^{(1)}_\phi - i\,\mu_0^{2\epsilon} \Bigg( \frac{\lambda_\phi}{2} \delta Z^{(1)}_{V_\phi} v^2 + \frac{\lambda_m}{2} \delta Z^{(1)}_{V_m} w^2 + 5\,\lambda_6\,\delta Z^{(1)}_{V_6} \frac{v^4}{w^2} \\ &\qquad + 7\,\delta\lambda^{(1)}_8 \frac{v^6}{w^4} + 9\,\delta\lambda^{(1)}_{10} \frac{v^8}{w^6} + 11\,\delta\lambda^{(1)}_{12} \frac{v^{10}}{w^8} \Bigg) \end{aligned} \tag{3.104}$$

Whereas for the 1-loop self energy of $\mathfrak{D}$ one obtains

$$
\begin{aligned}
-i\,\Sigma^{(1L)}_{\mathfrak{D},\mathrm{ren}} &= \frac{1}{2}\,\widetilde{\mathcal{V}}_{2ik}\,\widetilde{\mathcal{V}}_{2jl}\int\frac{d^Dq}{(2\,\pi)^D}\left(\widetilde{D}^{-1}_q\right)_{ij}\left(\widetilde{D}^{-1}_{q-p}\right)_{lk} + \frac{1}{2}\,\widetilde{\mathcal{V}}_{22ij}\int\frac{d^Dq}{(2\,\pi)^D}\left(\widetilde{D}^{-1}_q\right)_{ji} \\
&\quad + i\,p^2\,\delta Z^{(1)}_\sigma - i\,\mu_0^{2\epsilon}\left(\frac{\lambda_\sigma}{2}\,\delta Z^{(1)}_{V_\sigma}\,w^2 + \frac{\lambda_m}{2}\,\delta Z^{(1)}_{V_m}\,v^2 + \lambda_6\,\delta Z^{(1)}_{V_6}\,\frac{v^6}{w^4}\right. \\
&\qquad\qquad \left. + \frac{5}{2}\,\delta\lambda^{(1)}_8\,\frac{v^8}{w^6} + \frac{21}{5}\,\delta\lambda^{(1)}_{10}\,\frac{v^{10}}{w^8} + 6\,\delta\lambda^{(1)}_{12}\,\frac{v^{12}}{w^{10}}\right) \\
&= \frac{i\,\mu_0^{2\epsilon}}{32\,\pi^2}\left[\frac{3}{2}\left(\lambda_\sigma^2+\lambda_m^2\right)w^2 + \frac{1}{2}\,\lambda_m\left(\lambda_\phi + 4\,\lambda_m + \lambda_\sigma\right)v^2\right. \\
&\qquad + 3\,\lambda_6\left(5\,\lambda_\phi - 8\,\lambda_m + \lambda_\sigma\right)\frac{v^6}{w^4} + 10\,\lambda_6\left(\lambda_m + 25\,\lambda_6\right)\frac{v^8}{w^6} \\
&\qquad \left. + 168\,\lambda_6^2\,\frac{v^{10}}{w^8} + 36\,\lambda_6^2\,\frac{v^{12}}{w^{10}}\right]\frac{1}{\epsilon} + \mathcal{O}\left(\epsilon^0\right) \\
&\quad + i\,p^2\,\delta Z^{(1)}_\sigma - i\,\mu_0^{2\epsilon}\left(\frac{\lambda_\sigma}{2}\,\delta Z^{(1)}_{V_\sigma}\,w^2 + \frac{\lambda_m}{2}\,\delta Z^{(1)}_{V_m}\,v^2 + \lambda_6\,\delta Z^{(1)}_{V_6}\,\frac{v^6}{w^4}\right. \\
&\qquad\qquad \left. + \frac{5}{2}\,\delta\lambda^{(1)}_8\,\frac{v^8}{w^6} + \frac{21}{5}\,\delta\lambda^{(1)}_{10}\,\frac{v^{10}}{w^8} + 6\,\delta\lambda^{(1)}_{12}\,\frac{v^{12}}{w^{10}}\right)
\end{aligned}
\tag{3.105}
$$

Again, one finds that the scale invariant 1-loop counterterms (3.103), obtained from the manifestly scale invariant effective potential, exactly cancel all divergences of these 1-loop self energies. Hence, the same scale invariant counterterms are obtained from 2-point Green functions with non-zero external momentum $p$, computed using the Lagrangian in the broken phase of the theory, as proposed above.

Analogous to (c) the Lagrangian is furthermore considered in mass eigenstates $\{H, S\}$ and with the minimalisation condition being used. The minimalisation condition for the tree-level potential is also modified for $\lambda_6 \neq 0$ at tree-level and consequently given by

$$
\begin{aligned}
\lambda_m &= -\frac{1}{3}\,\lambda_\phi\,\frac{v^2}{w^2} - 2\,\lambda_6\,\frac{v^4}{w^4} \\
\lambda_\sigma &= \lambda_\phi\,\frac{v^4}{w^4} + 8\,\lambda_6\,\frac{v^6}{w^6}
\end{aligned}
\tag{3.106}
$$

Using these conditions, the 1-loop counterterms (3.103) may then be written as

$$
\begin{aligned}
\delta Z^{(1)}_\phi\Big|_{\min} &= 0, \qquad \delta Z^{(1)}_\sigma\Big|_{\min} = 0 \\
\delta Z^{(1)}_{V_\phi}\Big|_{\min} &= \frac{1}{16\,\pi^2}\,\frac{3}{2}\,\lambda_\phi\left[1 - \frac{20}{3}\,\frac{\lambda_6}{\lambda_\phi}\,\chi_0^2 + \frac{\chi_0^4}{9} - 40\,\frac{\lambda_6^2}{\lambda_\phi^2}\,\chi_0^4 + \frac{4}{3}\,\frac{\lambda_6}{\lambda_\phi}\,\chi_0^6 + 4\,\frac{\lambda_6^2}{\lambda_\phi^2}\,\chi_0^8\right]\frac{1}{\epsilon} \\
\delta Z^{(1)}_{V_m}\Big|_{\min} &= \frac{1}{16\,\pi^2}\,\frac{\lambda_\phi}{2}\left[1 - \frac{4}{3}\,\chi_0^2 + \chi_0^4 - 8\,\frac{\lambda_6}{\lambda_\phi}\,\chi_0^4 + 8\,\frac{\lambda_6}{\lambda_\phi}\,\chi_0^6\right]\frac{1}{\epsilon} \\
\delta Z^{(1)}_{V_\sigma}\Big|_{\min} &= \frac{1}{16\,\pi^2}\,\frac{\lambda_\phi}{6}\left[\frac{1 + 12\,\frac{\lambda_6}{\lambda_\phi}\,\chi_0^2 + 36\,\frac{\lambda_6^2}{\lambda_\phi^2}\,\chi_0^4}{1 + 8\,\frac{\lambda_6}{\lambda_\phi}\,\chi_0^2} + 9\,\chi_0^4 + 72\,\frac{\lambda_6}{\lambda_\phi}\,\chi_0^6\right]\frac{1}{\epsilon}
\end{aligned}
\tag{3.107}
$$

$$
\begin{aligned}
\delta Z^{(1)}_{V_6}\Big|_{\min} &= \frac{1}{16\,\pi^2}\,\frac{15}{2}\,\lambda_\phi\left[1+\frac{8}{15}\,\chi_0^2+\frac{1}{5}\left(1+16\,\frac{\lambda_6}{\lambda_\phi}\right)\chi_0^4+\frac{8}{5}\,\frac{\lambda_6}{\lambda_\phi}\,\chi_0^6\right]\frac{1}{\epsilon}\\
\delta\lambda^{(1)}_{8}\Big|_{\min} &= \frac{1}{16\,\pi^2}\,50\,\lambda_6^2\left[1-\frac{1}{75}\,\frac{\lambda_\phi}{\lambda_6}\,\chi_0^2-\frac{2}{25}\,\chi_0^4\right]\frac{1}{\epsilon}\\
\delta\lambda^{(1)}_{10}\Big|_{\min} &= \frac{1}{16\,\pi^2}\,20\,\lambda_6^2\,\frac{1}{\epsilon}\\
\delta\lambda^{(1)}_{12}\Big|_{\min} &= \frac{1}{16\,\pi^2}\,3\,\lambda_6^2\,\frac{1}{\epsilon}
\end{aligned}
\tag{3.108}
$$

with $\chi_0 = v/w$. Again, the 1PI $N$-point Green functions for $N \in \{1, 2, 3, 4\}$ and with non-vanishing external momenta illustrated in (3.100) have been computed using the Mathematica packages mentioned in (c) and it turned out that the counterterms (3.107) and (3.108) completely renormalise all these 1PI $N$-point Green functions, as expected. Therefore, once again the expected scale invariant counterterms, with the minimalisation condition (3.106) being applied, are obtained even if one works in the broken phase of the theory with expanded Lagrangian, transformed to mass eigenstates $\{H, S\}$ and with the minimalisation condition (3.106) being used.

**Conclusion.**
It has explicitly been shown that working in the broken phase of the theory with expanded Lagrangian is valid even in the context of spontaneously broken *quantum scale symmetry*. The reason for this has already been discussed in the last remark of section 2.1. In this section it has exemplarily been shown for different scenarios and even up to the 2-loop level that indeed the same scale invariant counterterms are obtained. This is an important consistency check because in QSI theories, i.e. theories regularised with SIDReg, the Dilaton appears to an anomalous power in the Lagrangian, and thus it is necessary to expand the Lagrangian in order to derive Feynman rules and perform perturbative calculations, i.e. calculate $N$-point Green functions with non-vanishing external momentum at a certain loop order.

# 4. QSI Gauge Theories

In this chapter the concept of *quantum scale invariance* realised via SIDReg is discussed in the context of gauge theories. QSI has shortly been introduced to gauge theories in [2, 28], however, not in full detail. For this reason, a consistent formulation of a *quantum scale invariant* gauge theory is provided in more detail in the first section of this chapter. This is done by the example of a generic $SU(N)$ gauge theory. In the second and third section of this chapter a quantum scale invariant QED, i.e. a QSI $U(1)$ gauge theory, without and with a toy model Higgs sector, respectively, are introduced to illustrate the concept of QSI gauge theories as well as to prepare and provide all necessary information for the next chapter about muon production.

## 4.1. Consistent Formulation of a QSI Gauge Theory

Analytically continuing the Lagrangian of a gauge theory to $D = 4 - 2\epsilon$ dimensions in a quantum scale invariant way using SIDReg, i.e. $g \to \mu^{\epsilon}(\sigma)\, g$, where $g$ is a generic gauge coupling, breaks gauge invariance [28]. In the following, two different approaches are discussed to avoid the SIDReg-induced breaking of gauge invariance, and thus obtain a quantum scale invariant gauge theory in $D = 4 - 2\epsilon$ dimensions. These are

(a) rescaling the gauge fields by absorbing the gauge coupling into the corresponding gauge field, i.e. $G^a_\mu \to \hat{G}^a_\mu = g\, G^a_\mu$, and *then* afterwards analytically continue the theory to $D = 4 - 2\epsilon$ dimensions in a quantum scale invariant way using SIDReg,

(b) analytically continue the "usual" Lagrangian of the gauge theory, i.e. without rescaling the gauge fields, to $D = 4 - 2\epsilon$ dimensions using SIDReg and also take corrections to the corresponding field strength tensor $F^a_{\mu\nu}$ as well as the corresponding gauge transformations into account.

The Lagrangian of a generic scale invariant $SU(N)$ gauge theory

$$\mathcal{L}_{\text{Gauge}} = \mathcal{L}_{\text{Gauge},cl} + \mathcal{L}_{\text{GF}} + \mathcal{L}_{\text{Ghost}} \tag{4.1}$$

with classical Lagrangian

$$\mathcal{L}_{\text{Gauge},cl} = -\frac{1}{4}\, F^a_{\mu\nu}\, F^{a,\mu\nu} + i\, \overline{\psi}_i \left(\delta_{ij}\, \partial\!\!\!/ - i\, g\, G\!\!\!/^{a}\, T^a_{ij}\right) \psi_j + \frac{1}{2}\, (\partial_\mu \sigma)\, (\partial^\mu \sigma) \tag{4.2}$$

where the Yukawa couplings as well as $\sigma$ self interactions have been neglected, or equivalently, have been set identically to zero at tree-level as they do not contribute to the following analysis of gauge invariance, gauge fixing Lagrangian

$$\mathcal{L}_{\text{GF}} = -B^a\,\partial^\mu G^a_\mu + \frac{\xi}{2}\,B^a\,B^a \tag{4.3}$$

where $B^a$ are Nakanishi-Lautrup fields, and ghost Lagrangian

$$\mathcal{L}_{\text{Ghost}} = \partial^\mu \overline{c}^a\,D^{ac}_\mu c^c \tag{4.4}$$

with the ghosts and anti-ghosts $c^a$ and $\overline{c}^a$, respectively, is considered in this section. Starting with the first approach

**(a) Rescaling the Gauge Fields:**

First, the gauge fields in 4 spacetime dimensions are rescaled as mentioned above, i.e.

$$G^a_\mu \longrightarrow \hat{G}^a_\mu = g\,G^a_\mu \tag{4.5}$$

Since the gauge parameter $\beta^a$ also needs to be rescaled, cf. appendix E.2, the ghost $c^a$ is analogously rescaled, as $\beta^a(x) = \theta\,c^a(x)$. Further, it is convenient to rescale the anti-ghost $\overline{c}^a$ as well, in order to obtain a ghost term similar to (4.4). Thus,

$$c^a \longrightarrow \hat{c}^a = g\,c^a, \qquad \overline{c}^a \longrightarrow \hat{\overline{c}}^a = \frac{1}{g}\,\overline{c}^a \tag{4.6}$$

This leads to

$$\begin{aligned}\mathcal{L}_{\text{Gauge}} = &-\frac{1}{4\,g^2}\,\hat{F}^a_{\mu\nu}\,\hat{F}^{a,\mu\nu} + i\,\overline{\psi}\,\hat{\not{D}}\,\psi + \frac{1}{2}\,(\partial_\mu\sigma)\,(\partial^\mu\sigma)\\ &-\frac{1}{g}\,B^a\,\partial^\mu\hat{G}^a_\mu + \frac{\xi}{2}\,B^a\,B^a + \partial^\mu\hat{\overline{c}}^a\,\hat{D}^{ac}_\mu\hat{c}^c\end{aligned} \tag{4.7}$$

where

$$\begin{aligned}\hat{F}^a_{\mu\nu} &= \partial_\mu\hat{G}^a_\nu - \partial_\nu\hat{G}^a_\mu + f^{abc}\,\hat{G}^b_\mu\,\hat{G}^c_\nu\\ \hat{D}_\mu\,\psi &= \left(\partial_\mu - i\,\hat{G}^a_\mu\,T^a\right)\psi\\ \hat{D}^{ac}_\mu\hat{c}^c &= \left(\delta^{ac}\,\partial_\mu + f^{abc}\,\hat{G}^b_\mu\right)\hat{c}^c\end{aligned} \tag{4.8}$$

and gauge transformations provided in appendix E. Note that physical observables does *not* change under field redefinitions, and consequently, rescaling the gauge fields and the ghosts as in (4.5) and (4.6), respectively, does not change physics. Now, after having rescaled the fields, the Lagrangian (4.7) can analytically be continued to $D = 4 - 2\epsilon$ dimensions using SIDReg, and thus is then given by

$$\begin{aligned}\mathcal{L}^{(D)}_{\text{Gauge}} = &-\frac{1}{4\,g^2}\,\mu^{-2\epsilon}(\sigma)\,\hat{F}^a_{\mu\nu}\,\hat{F}^{a,\mu\nu} + i\,\overline{\psi}\,\hat{\not{D}}\,\psi + \frac{1}{2}\,(\partial_\mu\sigma)\,(\partial^\mu\sigma)\\ &-\frac{1}{g}\,\mu^{-\epsilon}(\sigma)\,B^a\,\partial^\mu\hat{G}^a_\mu + \frac{\xi}{2}\,B^a\,B^a + \partial^\mu\hat{\overline{c}}^a\,\hat{D}^{ac}_\mu\hat{c}^c\\ = &-\frac{1}{4\,g^2}\,\mu^{-2\epsilon}(\sigma)\,\hat{F}^a_{\mu\nu}\,\hat{F}^{a,\mu\nu} + i\,\overline{\psi}\,\hat{\not{D}}\,\psi + \frac{1}{2}\,(\partial_\mu\sigma)\,(\partial^\mu\sigma)\\ &-\frac{1}{2\,\xi\,g^2}\,\mu^{-2\epsilon}(\sigma)\left(\partial^\mu\hat{G}^a_\mu\right)^2 + \partial^\mu\hat{\overline{c}}^a\,\hat{D}^{ac}_\mu\hat{c}^c\end{aligned} \tag{4.9}$$

where in the last line of (4.9) the equation of motion for the Nakanishi-Lautrup field $B^a$, i.e. the gauge condition,

$$B^a = \frac{1}{\xi\, g}\, \mu^{-\epsilon}(\sigma)\, \partial^\mu \hat{G}^a_\mu \tag{4.10}$$

has been used. From dimensional analysis follow the mass dimensions of the quantities in (4.9) in $D = 4 - 2\epsilon$ spacetime dimensions

$$\begin{aligned} &[\hat{G}^a_\mu] = 1, && [\psi] = \frac{3}{2} - \epsilon, && [\sigma] = 1 - \epsilon \\ &[\hat{c}^a] = 0, && [\hat{\bar{c}}^a] = 2 - 2\epsilon, && [B^a] = 2 - \epsilon \\ &[g] = [\xi] = 0 \end{aligned} \tag{4.11}$$

where it can be seen that the rescaled gauge field $\hat{G}^a_\mu$, defined in (4.5), always has mass dimension 1, in contrast to the "usual" gauge field $G^a_\mu$ which has $[G^a_\mu] = 1 - \epsilon$. Now, it remains to show whether the $D$-dimensional Lagrangian (4.9) indeed is BRST invariant, and thus the approach was successful in formulating a *quantum scale invariant* gauge theory consistently. The BRST transformations are given by

$$\begin{aligned} \psi_i &\longmapsto \psi_i + \delta\psi_i & \hat{c}^a &\longmapsto \hat{c}^a + \delta\hat{c}^a \\ \overline{\psi}_i &\longmapsto \overline{\psi}_i + \delta\overline{\psi}_i & \hat{\bar{c}}^a &\longmapsto \hat{\bar{c}}^a + \delta\hat{\bar{c}}^a \\ \hat{G}^a_\mu &\longmapsto \hat{G}^a_\mu + \delta\hat{G}^a_\mu & \sigma &\longmapsto \sigma \\ B^a &\longmapsto B^a + \delta B^a \end{aligned} \tag{4.12}$$

where $\sigma$ transforms trivially, and with

$$\begin{aligned} \delta\psi_i &= \theta\mathcal{Q}\psi_i = i\,\theta\,\hat{c}^a\, T^a_{ij}\,\psi_j \\ \delta\overline{\psi}_i &= \theta\mathcal{Q}\overline{\psi}_i = -i\,\theta\,\hat{c}^a\,\overline{\psi}_j\, T^a_{ji} \\ \delta\hat{G}^a_\mu &= \theta\mathcal{Q}\hat{G}^a_\mu = \theta\hat{D}^{ac}_\mu\hat{c}^c = \theta\,\partial_\mu\hat{c}^a + \theta\, f^{abc}\,\hat{G}^b_\mu\,\hat{c}^c \\ \delta\hat{c}^a &= \theta\mathcal{Q}\hat{c}^a = -\frac{1}{2}\,\theta\, f^{abc}\,\hat{c}^b\,\hat{c}^c \\ \delta\hat{\bar{c}}^a &= \theta\mathcal{Q}\hat{\bar{c}}^a = -\frac{\theta}{g}\,\mu^{-\epsilon}(\sigma)\, B^a = -\frac{\theta}{\xi\, g^2}\,\mu^{-2\epsilon}(\sigma)\,\partial^\mu\hat{G}^a_\mu \\ \delta B^a &= \theta\mathcal{Q}B^a = 0 \end{aligned} \tag{4.13}$$

The first three BRST transformations in (4.13) are given by the gauge transformations (E.12) using $\hat{\beta}^a(x) = \theta\,\hat{c}^a(x)$ for any Grassmann number $\theta$. All BRST transformations in (4.13), in particular the last three, are given in [30, 32] for 4 dimensions. It is straightforward to show that the BRST-operator $\mathcal{Q}$ that generates the BRST transformations in (4.13) is nilpotent as it should be, i.e. $\mathcal{Q}^2 = 0$, by showing that it is nilpotent when acting on the fields $\Psi = (\psi, \overline{\psi}, \hat{G}_\mu, c, \overline{c}, B)$ as well as when acting on any operator $\hat{\mathcal{O}} = \hat{\mathcal{O}}\,(\Psi)$ constructed from those fields [40, 44]. From dimensional analysis and (4.11) it follows that

$$[\hat{\beta}^a] = 0 \quad\Rightarrow\quad [\theta\,\hat{c}^a] = 0 \quad\Rightarrow\quad [\theta] = 0 \tag{4.14}$$

Now it needs to be shown that (4.9) is invariant under (4.13).

**Proposition 4.1.**
*The D-dimensional Lagrangian (4.9) is BRST invariant, i.e. invariant under the BRST transformations (4.13).*

*Proof.*
First, note that the gauge fixing and ghost Lagrangian can be rewritten as a $\mathcal{Q}$-exact term

$$\begin{aligned}\mathcal{L}^{(D)}_{\text{GF+Ghost}} &= \mathcal{L}^{(D)}_{\text{GF}} + \mathcal{L}^{(D)}_{\text{Ghost}} \\ &= \mathcal{Q}\left[-g\,\mu^{\epsilon}(\sigma)\,\hat{\bar{c}}^{a}\left(\frac{\xi}{2}\,B^a - \frac{1}{g}\,\mu^{-\epsilon}(\sigma)\,\partial^\mu \hat{G}^a_\mu\right)\right] \\ &= \frac{\xi}{2}\,B^a\,B^a - \frac{1}{g}\,\mu^{-\epsilon}(\sigma)\,B^a\,\partial^\mu\hat{G}^a_\mu - \hat{\bar{c}}^{a}\,\partial^\mu \hat{D}^{ac}_\mu \hat{c}^c \\ &= -\frac{1}{g}\,\mu^{-\epsilon}(\sigma)\,B^a\,\partial^\mu\hat{G}^a_\mu + \frac{\xi}{2}\,B^a\,B^a + \partial^\mu\hat{\bar{c}}^{a}\,\hat{D}^{ac}_\mu\hat{c}^c\end{aligned} \tag{4.15}$$

where in the third line (4.13) as well as the fact that $\mathcal{Q}$ is a fermionic operator have been used and in the last line the ghost term has been integrated by parts. Now the full Lagrangian (4.9) needs to be considered, and thus

$$\begin{aligned}\delta\mathcal{L}^{(D)}_{\text{Gauge}} &= \delta\mathcal{L}^{(D)}_{\text{Gauge},cl} + \delta\mathcal{L}^{(D)}_{\text{GF}} + \delta\mathcal{L}^{(D)}_{\text{Ghost}} \\ &= \theta\mathcal{Q}\left[-\frac{1}{4\,g^2}\,\mu^{-2\epsilon}(\sigma)\,\hat{F}^a_{\mu\nu}\,\hat{F}^{a,\mu\nu} + i\,\overline{\psi}\,\hat{\not{D}}\,\psi + \frac{1}{2}\,(\partial_\mu\sigma)\,(\partial^\mu\sigma)\right] \\ &\quad + \theta\mathcal{Q}^2\left[-g\,\mu^{\epsilon}(\sigma)\,\hat{\bar{c}}^{a}\left(\frac{\xi}{2}\,B^a - \frac{1}{g}\,\mu^{-\epsilon}(\sigma)\,\partial^\mu\hat{G}^a_\mu\right)\right] \\ &= -\frac{1}{2\,g^2}\,\mu^{-2\epsilon}(\sigma)\,\hat{F}^{a,\mu\nu}\left[-\theta f^{abc}\hat{c}^b\left(\partial_\mu\hat{G}^c_\nu - \partial_\nu\hat{G}^c_\mu + f^{cde}\,\hat{G}^d_\mu\,\hat{G}^e_\nu\right)\right] \\ &\quad + i\,(-i)\,\theta\hat{c}^a\overline{\psi}_k T^a_{ki}\left(\delta_{ij}\not{\partial} - i\hat{\not{G}}^c T^c_{ij}\right)\psi_j + i\overline{\psi}_i\left(\delta_{ij}\not{\partial} - i\hat{\not{G}}^c T^c_{ij}\right)i\theta\hat{c}^a T^a_{jk}\psi_k \\ &\quad + \overline{\psi}_i\,\theta\left(\not{\partial}\hat{c}^a + f^{abc}\hat{\not{G}}^b\hat{c}^c\right)T^a_{ij}\psi_j \\ &= -\frac{1}{2\,g^2}\,\mu^{-2\epsilon}(\sigma)\,f^{abc}\,\theta\,\hat{c}^a\,\hat{F}^{b,\mu\nu}\,\hat{F}^c_{\mu\nu} \\ &\quad + \theta\hat{c}^a\overline{\psi}_k T^a_{kj}\not{\partial}\psi_j - i\theta\hat{c}^a\overline{\psi}_k T^a_{ki}\hat{\not{G}}^c T^c_{ij}\psi_j - \theta\hat{c}^a\overline{\psi}_j\not{\partial}T^a_{jk}\psi_k - \overline{\psi}_j\theta\left(\not{\partial}\hat{c}^a\right)T^a_{jk}\psi_k \\ &\quad + i\overline{\psi}_i\hat{\not{G}}^c T^c_{ij}\theta\hat{c}^a T^a_{jk}\psi_k + \overline{\psi}_i\theta\left(\not{\partial}\hat{c}^a\right)T^a_{ij}\psi_j + \overline{\psi}_i\theta f^{abc}\hat{\not{G}}^b\hat{c}^c T^a_{ij}\psi_j \\ &= 0\end{aligned} \tag{4.16}$$

where in the second line of (4.16) the gauge fixing and ghost Lagrangian have been written as a $\mathcal{Q}$-exact term and in the third line it has been used that $\mathcal{Q}$ is nilpotent. Further, the explicit BRST transformations (4.13) as well as the Jacobi identity for the structure constants $f^{abc}$ in $\delta\hat{F}^a_{\mu\nu} = \theta\mathcal{Q}\hat{F}^a_{\mu\nu}$ have been used in the third line of (4.16). In the last line of (4.16) it has been used that the structure constants $f^{abc}$ are totally antisymmetric as well as the commutation relation $[T^a, T^b] = i f^{abc}T^c$. Hence, the $D$-dimensional Lagrangian (4.9) is BRST invariant. □

Thus, one might conclude that this approach was successful in formulating a quantum scale invariant gauge theory consistently. The $D$-dimensional Lagrangian for a generic QSI $SU(N)$ gauge theory is provided in (4.9).

**Remark.**
*After* the theory was analytically continued to $D = 4 - 2\epsilon$ dimensions, a second field redefinition can be applied where solely the dimensionless gauge coupling $g$ (and *not* the Renormalisation function) is pulled out of the gauge field to obtain $\overline{G}^a_\mu = \mu^\epsilon(\sigma)\, G^a_\mu$, such that

$$\begin{aligned} \overline{G}^a_\mu &= \frac{1}{g}\,\hat{G}^a_\mu = \mu^\epsilon(\sigma)\, G^a_\mu \\ \hat{G}^a_\mu &= g\,\mu^\epsilon(\sigma)\, G^a_\mu = g\,\overline{G}^a_\mu \end{aligned} \tag{4.17}$$

The mass dimension of the gauge field is then still $[\overline{G}^a_\mu] = 1$ and the statements above remain true. However, the $D$-dimensional Lagrangian in this approach looks more similar to (4.1) w.r.t. the gauge couplings, which turns out to be useful in the presence of mixing between the gauge fields as in the Standard Model. In the following, all gauge theories are formulated in terms of the rescaled gauge fields $\overline{G}^a_\mu$, however, the "overbar" is always dropped for simplicity.

Now, continuing with the second more "direct" approach

**(b) Working with non-rescaled Gauge Fields:**

Analytically continuing the theory to $D$ dimensions, the covariant derivative changes as

$$D_\mu \longrightarrow \widetilde{D}_\mu = \partial_\mu - i\, g\, \mu^\epsilon(\sigma)\, G^a_\mu\, T^a, \tag{4.18}$$

and thus the field strength tensor is then given by

$$\widetilde{F}_{\mu\nu} = \widetilde{F}^a_{\mu\nu}\, T^a = \frac{i}{g\,\mu^\epsilon(\sigma)} \left[\widetilde{D}_\mu, \widetilde{D}_\nu\right] \tag{4.19}$$

Evaluating the commutator leads to the following result for the field strength tensor

$$\widetilde{F}^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g\,\mu^\epsilon(\sigma)\, f^{abc}\, G^b_\mu\, G^c_\nu + \epsilon\,\mu^{-1}(\sigma)\,\frac{\partial\mu}{\partial\sigma}\left(\partial_\mu\sigma\, G^a_\nu - \partial_\nu\sigma\, G^a_\mu\right) \tag{4.20}$$

where it can be seen that the field strength tensor obtains an evanescent correction. The classical Lagrangian in $D = 4 - 2\epsilon$ dimensions may then be written as

$$\mathcal{L}^{(D)}_{\text{Gauge},cl} = -\frac{1}{4}\,\widetilde{F}^a_{\mu\nu}\,\widetilde{F}^{a,\mu\nu} + i\,\overline{\psi}_i\left(\delta_{ij}\,\partial\!\!\!/ - i\, g\,\mu^\epsilon(\sigma)\, G\!\!\!/^a\, T^a_{ij}\right)\psi_j + \frac{1}{2}\,(\partial_\mu\sigma)\,(\partial^\mu\sigma) \tag{4.21}$$

with corresponding gauge transformations given in (E.19). Thus, using $\beta^a(x) = \theta\, c^a(x)$, one obtains the following BRST transformations for non-rescaled fields

$$\begin{aligned}
\delta\psi_i &= \theta\mathcal{Q}\psi_i = i\,\theta\,g\,\mu^\epsilon(\sigma)\,c^a\,T^a_{ij}\,\psi_j \\
\delta\overline{\psi}_i &= \theta\mathcal{Q}\overline{\psi}_i = -i\,\theta\,g\,\mu^\epsilon(\sigma)\,c^a\,\overline{\psi}_j\,T^a_{ji} \\
\delta G^a_\mu &= \theta\mathcal{Q}G^a_\mu = \theta\,\widetilde{D}^{ac}_\mu c^c + \epsilon\,\theta\,\mu^{-1}(\sigma)\,\frac{\partial\mu}{\partial\sigma}\,\partial_\mu\sigma\,c^a \\
&\qquad\qquad = \theta\,\partial_\mu c^a + \theta\,g\,\mu^\epsilon(\sigma)\,f^{abc}\,G^b_\mu\,c^c + \epsilon\,\theta\,\mu^{-1}(\sigma)\,\frac{\partial\mu}{\partial\sigma}\,\partial_\mu\sigma\,c^a \\
\delta c^a &= \theta\mathcal{Q}c^a = -\frac{1}{2}\,\theta\,g\,\mu^\epsilon(\sigma)\,f^{abc}\,c^b\,c^c \\
\delta\overline{c}^a &= \theta\mathcal{Q}\overline{c}^a = -\theta\,B^a \\
\delta B^a &= \theta\mathcal{Q}B^a = 0
\end{aligned} \tag{4.22}$$

where

$$\widetilde{D}^{ac}_\mu = \delta^{ac}\,\partial_\mu + g\,\mu^\epsilon(\sigma)\,f^{abc}\,G^b_\mu \tag{4.23}$$

Due to the evanescent correction to the gauge field gauge transformation, cf. (E.19) and (4.22), the chosen gauge fixing condition is corrected correspondingly, and thus the Ansatz for the gauge fixing and ghost Lagrangian is as follows

$$\begin{aligned}
\mathcal{L}^{(D)}_{\text{GF+Ghost}} &= \mathcal{L}^{(D)}_{\text{GF}} + \mathcal{L}^{(D)}_{\text{Ghost}} = \mathcal{Q}\left[-\overline{c}^a\left(\frac{\xi}{2}\,B^a - \partial^\mu\hat{G}^a_\mu - \epsilon\,\mu^{-1}(\sigma)\,\frac{\partial\mu}{\partial\sigma}\,\partial_\mu\sigma\,G^a_\mu\right)\right] \\
&= \frac{\xi}{2}\,B^a\,B^a - B^a\,\partial^\mu G^a_\mu + \partial^\mu\overline{c}^a\,\widetilde{D}^{ac}_\mu c^c \\
&\quad - \epsilon\,\mu^{-1}(\sigma)\,\frac{\partial\mu}{\partial\sigma}\,\partial^\mu\sigma\Big[B^a\,G^a_\mu + \overline{c}^a\,\widetilde{D}^{ac}_\mu c^c - \partial_\mu\overline{c}^a\,c^a\Big] \\
&\quad - \epsilon^2\,\mu^{-2}(\sigma)\left(\frac{\partial\mu}{\partial\sigma}\right)^2\partial^\mu\sigma\,\partial_\mu\sigma\,\overline{c}^a\,c^a
\end{aligned} \tag{4.24}$$

It can be seen that the gauge fixing and the ghost Lagrangian also obtain evanescent corrections, analogously to the kinetic term of the gauge field, cf. (4.20). Consequently, the $D$-dimensional Lagrangian of the considered $SU(N)$ gauge theory is given by

$$\mathcal{L}^{(D)}_{\text{Gauge}} = \mathcal{L}^{(D)}_{\text{Gauge},cl} + \mathcal{L}^{(D)}_{\text{GF}} + \mathcal{L}^{(D)}_{\text{Ghost}} \tag{4.25}$$

with the corresponding Lagrangians provided in (4.21) and (4.24). Now, it remains to show whether (4.25) is BRST invariant under (4.22) and is equivalent to the Lagrangian of approach (a), i.e. (4.9).

**Proposition 4.2.**
*The $D$-dimensional Lagrangian (4.25) is BRST invariant, i.e. invariant under the BRST transformations (4.22).*

*Proof.*
The gauge fixing and the ghost Lagrangian can be written as a $\mathcal{Q}$-exact term, as shown in (4.24), and thus the BRST transformation of $\mathcal{L}^{(D)}_{\text{GF}}$ and $\mathcal{L}^{(D)}_{\text{Ghost}}$ vanishes trivially due to nilpotence of the BRST operator $\mathcal{Q}$. Hence,

$$\begin{aligned} \delta\mathcal{L}^{(D)}_{\text{Gauge}} &= \delta\mathcal{L}^{(D)}_{\text{Gauge},cl} \\ &= -\frac{1}{2}\,\widetilde{F}^{a,\mu\nu}\,\delta\widetilde{F}^{a}_{\mu\nu} + i\,\delta\overline{\psi}_i\left(\delta_{ij}\,\partial\!\!\!/ - i\,g\,\mu^{\epsilon}(\sigma)\,G\!\!\!/^{a}\,T^{a}_{ij}\right)\psi_j \\ &\quad + i\,\overline{\psi}_i\left(\delta_{ij}\,\partial\!\!\!/ - i\,g\,\mu^{\epsilon}(\sigma)\,G\!\!\!/^{a}\,T^{a}_{ij}\right)\delta\psi_j + g\,\mu^{\epsilon}(\sigma)\,\overline{\psi}_i\,\delta G\!\!\!/^{a}\,T^{a}_{ij}\,\psi_j \\ &= -\frac{1}{2}\,\theta\,g\,\mu^{\epsilon}(\sigma)\,f^{abc}\,\widetilde{F}^{a,\mu\nu}\,\widetilde{F}^{b}_{\mu\nu}\,c^{c} \\ &= 0 \end{aligned} \tag{4.26}$$

where in the first line of (4.26) it has been used that the BRST transformation of the gauge fixing and ghost Lagrangian vanishes, as explained above, and in the second line that the Dilaton transforms trivially under BRST. In the third line of (4.26), the explicit BRST transformations (4.22) as well as the Jacobi identity for the structure constants $f^{abc}$ in $\delta\widetilde{F}^{a}_{\mu\nu} = \theta\mathcal{Q}\widetilde{F}^{a}_{\mu\nu}$ and the commutation relation $[T^a, T^b] = i f^{abc} T^c$ have been used. In the last line it has been used that the structure constants $f^{abc}$ are totally antisymmetric. Hence, the $D$-dimensional Lagrangian (4.25) is BRST invariant. □

**Proposition 4.3.**
*The $D$-dimensional Lagrangian (4.9) is equivalent to the $D$-dimensional Lagrangian (4.25), i.e. approach (a) and (b) are equivalent.*

*Proof.*
Starting with Lagrangian (4.9), the gauge coupling $g$ *and* the Renormalisation function $\mu^{\epsilon}(\sigma)$ can be pulled out of the fields $\hat{G}^{a}_{\mu}$, $\hat{c}^{a}$ and $\hat{\overline{c}}^{a}$ in $D = 4 - 2\epsilon$ dimensions, i.e.

$$\hat{G}^{a}_{\mu} = g\,\mu^{\epsilon}(\sigma)\,G^{a}_{\mu}, \qquad \hat{c}^{a} = g\,\mu^{\epsilon}(\sigma)\,c^{a}, \qquad \hat{\overline{c}}^{a} = \frac{1}{g}\,\mu^{-\epsilon}(\sigma)\,\hat{\overline{c}}^{a} \tag{4.27}$$

leading to the Lagrangian (4.25), which can be shown by direct calculation.

Conversely, starting with Lagrangian (4.25), the gauge coupling $g$ *and* the Renormalisation function $\mu^{\epsilon}(\sigma)$ can be absorbed into the fields $G^{a}_{\mu}$, $c^{a}$ and $\overline{c}^{a}$. Then, additional terms coming from commuting derivatives and the Renormalisation function need to be taken into account by subtracting them, leading to the Lagrangian (4.9).

Moreover, note that the same holds true for the BRST transformations in (4.13) and (4.22). □

**Remark.**
It can be seen that Lagrangian (4.9) takes a more convenient form than Lagrangian (4.25) due to the evanescent corrections to the kinetic term of the gauge field as well as to the gauge fixing and ghost Lagrangian, cf. (4.20) and (4.24). For this reason, all gauge theories are formulated in terms of rescaled gauge fields, i.e. in terms of approach (a), in the following. However, they are formulated in terms of $\overline{G}^{a}_{\mu}$, defined in (4.17), as mentioned in the remark above.

## 4.2. Quantum Scale Invariant QED

In order to formulate a quantum scale invariant QED the Dilaton $\sigma$ necessarily needs to be included into the spectrum of the theory as discussed in section 2.1. Therefore, a minimal QSI QED contains the fields $\{\psi_f, A_\mu, \sigma\}$. Moreover, the theory is formulated in terms of the rescaled gauge fields $\overline{A}_\mu$ as introduced in (4.17), however, the "overbar" is dropped for simplicity and convenience. Hence, the quantum scale invariant QED, with $Q = -1$, is given by

$$\begin{aligned} \mathcal{L}_{\text{QED}}^{\text{QSI}} = &-\frac{1}{4}\,\mu^{-2\epsilon}(\sigma)\,F_{\mu\nu}\,F^{\mu\nu} + i\,\overline{\psi}_f\left(\not{\partial} + i\,e\,\not{A}\right)\psi_f + \frac{1}{2}\,(\partial_\mu\sigma)\,(\partial^\mu\sigma) \\ &- y_f\,\mu^{\epsilon}(\sigma)\,\sigma\,\overline{\psi}_f\,\psi_f - \frac{\lambda}{4!}\,\mu^{2\epsilon}(\sigma)\,\sigma^4 - \frac{1}{2\,\xi}\,\mu^{-2\epsilon}(\sigma)\,(\partial^\mu A_\mu)^2 \end{aligned} \tag{4.28}$$

where the ghost term has been neglected as the Faddeev-Popov ghosts in the case of abelian gauge theories completely decouple from the rest of the theory. In the following, $\lambda$ is identically set to zero at tree-level because the theory should be as close as possible to pure QED in the further analysis. Moreover, the Dilaton $\sigma = \mathfrak{D} + w$ then is massless (at tree-level) which is closer to the Standard Model case where the Dilaton is the massless Goldstone boson of (quantum) scale symmetry. Hence, the Lagrangian that is considered in the following, i.e. with $\lambda \equiv 0$ at tree-level, reads as

$$\begin{aligned} \mathcal{L}_{\text{QED}}^{\text{QSI}} &= -\frac{1}{4}\,\mu^{-2\epsilon}(\sigma)\,F_{\mu\nu}\,F^{\mu\nu} + i\,\overline{\psi}_f\left(\not{\partial} + i\,e\,\not{A}\right)\psi_f + \frac{1}{2}\,(\partial_\mu\sigma)\,(\partial^\mu\sigma) \\ &\quad - y_f\,\mu^{\epsilon}(\sigma)\,\sigma\,\overline{\psi}_f\,\psi_f - \frac{1}{2\,\xi}\,\mu^{-2\epsilon}(\sigma)\,(\partial^\mu A_\mu)^2 \\ &= -\frac{1}{4}\,\mu_0^{-2\epsilon}\,F_{\mu\nu}\,F^{\mu\nu} + i\,\overline{\psi}_f\left(\not{\partial} + i\,e\,\not{A}\right)\psi_f + \frac{1}{2}\,(\partial_\mu\mathfrak{D})\,(\partial^\mu\mathfrak{D}) \\ &\quad - \mu_0^{\epsilon}\,y_f\,w\,\overline{\psi}_f\,\psi_f - \mu_0^{\epsilon}\left(1 + \epsilon + \epsilon^2 + \mathcal{O}\left(\epsilon^3\right)\right)y_f\,\mathfrak{D}\,\overline{\psi}_f\,\psi_f \\ &\quad - \frac{1}{2\,\xi}\,\mu_0^{-2\epsilon}\,(\partial^\mu A_\mu)^2 - \mu_0^{\epsilon}\,\frac{\epsilon\,(1+2\epsilon) + \mathcal{O}\left(\epsilon^3\right)}{2\,w}\,y_f\,\mathfrak{D}^2\,\overline{\psi}_f\,\psi_f \\ &\quad + \mu_0^{-2\epsilon}\left(\epsilon\,(1+\epsilon) + \mathcal{O}\left(\epsilon^3\right)\right)\frac{\mathfrak{D}}{w}\left[\frac{1}{2}F_{\mu\nu}\,F^{\mu\nu} + \frac{1}{\xi}\,(\partial^\mu A_\mu)^2\right] + \cdots \end{aligned} \tag{4.29}$$

where the Lagrangian has been expanded about the Dilaton VEV $w$ and w.r.t. $\epsilon$ in the second line of (4.29), and the ellipsis denotes infinitely many terms of higher orders in the fields. The fermion masses are given by $\widetilde{m}_f = m_f = \mu_0^{\epsilon}\,y_f\,w$. The Renormalisation transformations are provided by

$$\begin{aligned} A &\longrightarrow A_0 = \sqrt{Z_A}\,A \\ \psi_f &\longrightarrow \psi_{f,0} = \sqrt{Z_{\psi_f}}\,\psi_f \\ \sigma &\longrightarrow \sigma_0 = \sqrt{Z_\sigma}\,\sigma \\ e &\longrightarrow e_B = \mu^{\epsilon}(\sigma)\,e_0 = \mu^{\epsilon}(\sigma)\,Z_e\,e \\ y_f &\longrightarrow y_{f,B} = \mu^{\epsilon}(\sigma)\,y_{f,0} = \mu^{\epsilon}(\sigma)\,Z_{y_f}\,y_f \\ \lambda &\longrightarrow \lambda_B = \mu^{2\epsilon}(\sigma)\,\lambda_0 = \mu^{2\epsilon}(\sigma)\,Z_\lambda\,\lambda \end{aligned} \tag{4.30}$$

Hence, the 1-loop counterterm Lagrangian is given by

$$\begin{aligned}\mathcal{L}^{\text{QSI}}_{\text{QED,ct1}} = &- \frac{1}{4}\,\delta Z_A\,\mu^{-2\epsilon}(\sigma)\,F_{\mu\nu}\,F^{\mu\nu} + i\,\delta Z_{\psi_f}\,\overline{\psi}_f\,\partial\!\!\!/\,\psi_f + \frac{1}{2}\,\delta Z_\sigma\,(\partial_\mu\sigma)\,(\partial^\mu\sigma)\\ &- \left(\delta Z_{\psi_f} + \delta Z_{y_f} + \frac{1}{2}\,\delta Z_\sigma\right) y_f\,\mu^\epsilon(\sigma)\,\sigma\,\overline{\psi}_f\,\psi_f\\ &- \left(\delta Z_{\psi_f} + \delta Z_e + \frac{1}{2}\,\delta Z_A\right) e\,\overline{\psi}_f\,A\!\!\!/\,\psi_f - \mu^{2\epsilon}(\sigma)\,\frac{\delta\lambda}{4!}\,\sigma^4\end{aligned} \tag{4.31}$$

where the counterterm $\delta\lambda$ needs to be included although $\lambda \equiv 0$ at tree-level, and the counterterm superscripts, indicating the loop-order, have been suppressed because this theory is considered solely at the 1-loop level. Further, it has been used that the Renormalisation of the gauge fixing term does not need to be considered as $Z_\xi = Z_A$ due to Ward identities [8, 35]. Expanding the 1-loop counterterm Lagrangian (4.31) about the Dilaton VEV $w$ and w.r.t. $\epsilon$ then gives

$$\begin{aligned}\mathcal{L}^{\text{QSI}}_{\text{QED,ct1}} = &- \mu_0^{-2\epsilon}\,\frac{1}{4}\,\delta Z_A\,F_{\mu\nu}\,F^{\mu\nu} + i\,\delta Z_{\psi_f}\,\overline{\psi}_f\,\partial\!\!\!/\,\psi_f + \frac{1}{2}\,\delta Z_\sigma\,(\partial_\mu\mathfrak{D})\,(\partial^\mu\mathfrak{D})\\ &- \mu_0^\epsilon\left(\delta Z_{\psi_f} + \delta Z_{y_f} + \frac{1}{2}\,\delta Z_\sigma\right) y_f\,w\,\overline{\psi}_f\,\psi_f\\ &- \mu_0^\epsilon\left(1 + \epsilon + \mathcal{O}\left(\epsilon^2\right)\right)\left(\delta Z_{\psi_f} + \delta Z_{y_f} + \frac{1}{2}\,\delta Z_\sigma\right) y_f\,\mathfrak{D}\,\overline{\psi}_f\,\psi_f\\ &- \left(\delta Z_{\psi_f} + \delta Z_e + \frac{1}{2}\,\delta Z_A\right) e\,\overline{\psi}_f\,A\!\!\!/\,\psi_f\\ &- \mu_0^{2\epsilon}\,\frac{\delta\lambda}{4!}\,w^4 - \mu_0^{2\epsilon}\left(1 + \frac{\epsilon}{2} + \mathcal{O}\left(\epsilon^2\right)\right)\frac{\delta\lambda}{6}\,w^3\mathfrak{D}\\ &- \mu_0^{2\epsilon}\,\frac{1}{2}\left(1 + \frac{7}{6}\,\epsilon + \mathcal{O}\left(\epsilon^2\right)\right)\frac{\delta\lambda}{2}\,w^2\,\mathfrak{D}^2\\ &- \mu_0^{2\epsilon}\left(1 + \frac{13}{6}\,\epsilon + \mathcal{O}\left(\epsilon^2\right)\right)\frac{\delta\lambda}{6}\,w\,\mathfrak{D}^3\\ &- \mu_0^{2\epsilon}\left(1 + \frac{25}{6}\,\epsilon + \mathcal{O}\left(\epsilon^2\right)\right)\frac{\delta\lambda}{4!}\,\mathfrak{D}^4 + \cdots\end{aligned} \tag{4.32}$$

where the ellipsis again denotes infinitely many terms of higher orders in the fields.

Now, the 1-loop counterterms of this theory will be provided in the MS-scheme and determined in Feynman gauge $\xi = 1$ for two different cases, i.e. for the most general case $y_f \neq 0$ and $\lambda \neq 0$ at tree-level

$$\begin{aligned}\delta Z_{\psi_f} &= -\frac{1}{16\,\pi^2}\left(e^2 + \frac{y_f}{2}\right)\frac{1}{\epsilon} & \delta Z_{y_f} &= \frac{1}{16\,\pi^2}\left[\frac{3}{2}\,y_f^2 + \sum_l\,y_l^2 - 3\,e^2\right]\frac{1}{\epsilon}\\ \delta Z_A &= -\frac{1}{16\,\pi^2}\,\frac{4\,N_f\,e^2}{3}\,\frac{1}{\epsilon} & \delta Z_e &= -\frac{1}{2}\,\delta Z_A = \frac{1}{16\,\pi^2}\,\frac{2\,N_f\,e^2}{3}\,\frac{1}{\epsilon}\\ \delta Z_\sigma &= -\frac{1}{16\,\pi^2}\,2\sum_l\,y_l^2\,\frac{1}{\epsilon} & \delta Z_\lambda &= \frac{1}{16\,\pi^2}\left[\frac{3}{2}\,\lambda + 4\sum_l\left(y_l^2 - 6\,\frac{y_l^4}{\lambda}\right)\right]\frac{1}{\epsilon}\end{aligned} \tag{4.33}$$

where $\delta\lambda = \lambda\,(\delta Z_\lambda + 2\,\delta Z_\sigma)$ and $N_f$ is the number of fermions in the theory, and for the case $y_f \neq 0$ and $\lambda \equiv 0$ at tree-level, as discussed above,

$$\begin{aligned}
\delta Z_{\psi_f} &= -\frac{1}{16\,\pi^2}\left(e^2 + \frac{y_f}{2}\right)\frac{1}{\epsilon} & \delta Z_{y_f} &= \frac{1}{16\,\pi^2}\left[\frac{3}{2}\,y_f^2 + \sum_l y_l^2 - 3\,e^2\right]\frac{1}{\epsilon} \\
\delta Z_A &= -\frac{1}{16\,\pi^2}\,\frac{4\,N_f\,e^2}{3}\,\frac{1}{\epsilon} & \delta Z_e &= -\frac{1}{2}\,\delta Z_A = \frac{1}{16\,\pi^2}\,\frac{2\,N_f\,e^2}{3}\,\frac{1}{\epsilon} \\
\delta Z_\sigma &= -\frac{1}{16\,\pi^2}\,2\sum_l y_l^2\,\frac{1}{\epsilon} & \delta\lambda &= -\frac{1}{16\,\pi^2}\,24\sum_l y_l^4\,\frac{1}{\epsilon}
\end{aligned} \tag{4.34}$$

Note that it was assumed that all fermions are leptons, e.g. $f, l \in \{e^-, \mu^-, \tau^-\}$ for 3 lepton flavours. In the case of quarks one also needs to take the corresponding colour factor $N_{c,f}$ into account. The 1-loop counterterms for the case where $y_f \equiv 0$ and $\lambda \equiv 0$ at tree-level, which will also be discussed in chapter 5, can be obtained from (4.34) by $y_f \to 0$ and $\delta y_f = y_f\,(\delta Z_{\psi_f} + \delta Z_{y_f} + \delta Z_\sigma/2) = 0$.

## 4.3. QSI QED with Toy Model Higgs Sector

In this section a quantum scale invariant QED with a toy model Higgs sector is discussed. The scalar potential in this theory contains the 2 Scalar Model as subset and admits dynamical SSB of (quantum) scale symmetry with the Dilaton as associated Goldstone boson. Consequently, the model in this section is closer to a more realistic quantum theory of (quantum) scale invariant electromagnetism than the model in the previous section. The field content of this theory is given by fermions $\psi_f$, the photon $A_\mu$, a Higgs-like boson $\phi = h + v$, the Dilaton $\sigma = \mathfrak{D} + w$ and an additional scalar field $G$. In order to construct such a model with a massless photon the additional symmetry

$$\begin{aligned}
\psi_L &\longmapsto e^{i\,\alpha}\,\psi_L & \Phi &\longmapsto e^{i\,\alpha}\,\Phi \\
\psi_R &\longmapsto \psi_R & \sigma &\longmapsto \sigma \\
A &\longmapsto A, & &
\end{aligned} \tag{4.35}$$

which distinguishes between left- and right-handed fermions, is imposed on the theory beside the other symmetries, such as $U(1)$ gauge symmetry and quantum scale symmetry. Note that $\Phi$ is a complex scalar given by

$$\Phi = \frac{\phi + i\,G}{\sqrt{2}} \tag{4.36}$$

It will turn out that $G$ is the Goldstone boson associated with the additional symmetry (4.35). Again, the theory is formulated in terms of the rescaled gauge fields $\overline{A}_\mu$, with mass dimension 1, as introduced in (4.17) with suppressed "overbar". The Lagrangian of this model is provided by

$$\mathcal{L}^{\text{QSI}}_{\text{QED+Higgs}} = \mathcal{L}^{\text{QSI}}_{\text{Fermion}} + \mathcal{L}^{\text{QSI}}_{\text{Gauge}} + \mathcal{L}^{\text{QSI}}_{\text{Higgs}} + \mathcal{L}^{\text{QSI}}_{\text{Yukawa}} + \mathcal{L}^{\text{QSI}}_{\text{GF}} \tag{4.37}$$

with fermion Lagrangian

$$\begin{aligned}\mathcal{L}^{\text{QSI}}_{\text{Fermion}} &= i\,\overline{\psi}_{f,L}\,\not{D}\,\psi_{f,L} + i\,\overline{\psi}_{f,R}\,\not{D}\,\psi_{f,R} \\ &= i\,\overline{\psi}_f\,\not{D}\,\psi_f = i\,\overline{\psi}_f\left(\not{\partial} + i\,e\,\not{A}\right)\psi_f\end{aligned} \tag{4.38}$$

where $Q = -1$ has been used, gauge Lagrangian

$$\mathcal{L}^{\text{QSI}}_{\text{Gauge}} = -\frac{1}{4}\,\mu^{-2\epsilon}(\sigma)\,F_{\mu\nu}\,F^{\mu\nu} \tag{4.39}$$

Higgs sector

$$\mathcal{L}^{\text{QSI}}_{\text{Higgs}} = \left(\partial^\mu\Phi\right)^\dagger\left(\partial_\mu\Phi\right) + \frac{1}{2}\left(\partial^\mu\sigma\right)\left(\partial_\mu\sigma\right) - \mu^{2\epsilon}(\sigma)\,V_{\text{QED,H}}\left(\Phi,\sigma\right) \tag{4.40}$$

Yukawa sector

$$\begin{aligned}\mathcal{L}^{\text{QSI}}_{\text{Yukawa}} &= -\mu^{\epsilon}(\sigma)\,y_f\,\overline{\psi}_{f,L}\,\Phi\,\psi_{f,R} - \mu^{\epsilon}(\sigma)\,y_f\,\overline{\psi}_{f,R}\,\Phi^*\,\psi_{f,L} \\ &= -\mu^{\epsilon}(\sigma)\,\frac{y_f}{\sqrt{2}}\,\phi\,\overline{\psi}_f\,\psi_f - i\,\mu^{\epsilon}(\sigma)\,\frac{y_f}{\sqrt{2}}\,G\,\overline{\psi}_f\,\gamma_5\,\psi_f\end{aligned} \tag{4.41}$$

and the gauge fixing Lagrangian

$$\mathcal{L}^{\text{QSI}}_{\text{GF}} = -\frac{1}{2\,\xi}\,\mu^{-2\epsilon}(\sigma)\left(\partial^\mu A_\mu\right)^2 \tag{4.42}$$

The scalar potential in (4.40) is explicitly given by

$$\begin{aligned}V_{\text{QED,H}}\left(\Phi,\sigma\right) &= \frac{\lambda_\phi}{3!}\left(\Phi^\dagger\Phi\right)^2 + \frac{\lambda_m}{2}\left(\Phi^\dagger\Phi\right)\sigma^2 + \frac{\lambda_\sigma}{4!}\,\sigma^4 \\ &= \frac{\lambda_\phi}{4!}\,\phi^4 + \frac{\lambda_m}{4}\,\phi^2\,\sigma^2 + \frac{\lambda_\sigma}{4!}\,\sigma^4 \\ &\quad + \frac{\lambda_\phi}{12}\,\phi^2\,G^2 + \frac{\lambda_m}{4}\,\sigma^2\,G^2 + \frac{\lambda_\phi}{4!}\,G^4 \\ &\equiv V_{\text{2SM}}\left(\phi,\sigma\right) + V_{\text{QED},G}\left(\phi,\sigma,G\right)\end{aligned} \tag{4.43}$$

$\Phi$ transforms trivially under $U(1)$ gauge transformations which is necessary to have Yukawa terms of the form (4.41) due to gauge invariance, and thus there is no covariant derivative acting on $\Phi$ in (4.40) which ensures massless photons. Furthermore, note that the additional symmetry (4.35) forbids Yukawa terms with $\sigma$.

Expanding the Lagrangian (4.37) about the fields VEVs $\{v, w\}$ and w.r.t. $\epsilon$, one obtains

$$\mathcal{L}^{\text{QSI}}_{\text{Fermion}} = i\,\overline{\psi}_f\left(\not{\partial} + i\,e\,\not{A}\right)\psi_f \tag{4.44}$$

for the Fermion Lagrangian

$$\begin{aligned}\mathcal{L}^{\text{QSI}}_{\text{Gauge}} = &- \mu_0^{-2\epsilon}\,\frac{1}{4}\,F_{\mu\nu}\,F^{\mu\nu} + \mu_0^{-2\epsilon}\,\frac{1}{2}\,\epsilon\left(1+\epsilon\right)\frac{\mathfrak{D}}{w}\,F_{\mu\nu}\,F^{\mu\nu} \\ &- \mu_0^{-2\epsilon}\,\frac{1}{4}\,\epsilon\left(1+3\,\epsilon\right)\frac{\mathfrak{D}^2}{w^2}\,F_{\mu\nu}\,F^{\mu\nu} + \mu_0^{-2\epsilon}\,\frac{1}{6}\,\epsilon\left(1+4\,\epsilon\right)\frac{\mathfrak{D}^3}{w^3}\,F_{\mu\nu}\,F^{\mu\nu} \\ &- \mu_0^{-2\epsilon}\,\frac{1}{8}\,\epsilon\left(1+\frac{14}{3}\,\epsilon\right)\frac{\mathfrak{D}^4}{w^4}\,F_{\mu\nu}\,F^{\mu\nu} + \mathcal{O}\left(\left(\mathfrak{D}/w\right)^5, \epsilon^3\right)\end{aligned} \tag{4.45}$$

for the gauge Lagrangian

$$
\begin{aligned}
\mathcal{L}^{\mathrm{QSI}}_{\mathrm{Yukawa}} = &- \mu_0^{\epsilon} \, \frac{y_f}{\sqrt{2}} \, v \, \overline{\psi}_f \, \psi_f - \mu_0^{\epsilon} \, \frac{y_f}{\sqrt{2}} \, h \, \overline{\psi}_f \, \psi_f - i \, \mu_0^{\epsilon} \, \frac{y_f}{\sqrt{2}} \, G \, \overline{\psi}_f \, \gamma_5 \, \psi_f \\
&- \mu_0^{\epsilon} \, \epsilon \, (1+\epsilon) \, \frac{y_f}{\sqrt{2}} \, \frac{v}{w} \, \mathfrak{D} \, \overline{\psi}_f \, \psi_f - \mu_0^{\epsilon} \, \epsilon \, (1+\epsilon) \, \frac{y_f}{\sqrt{2}} \, \frac{1}{w} \, h \, \mathfrak{D} \, \overline{\psi}_f \, \psi_f \\
&+ \mu_0^{\epsilon} \, \epsilon \, \frac{y_f}{2\sqrt{2}} \, \frac{v}{w^2} \, \mathfrak{D}^2 \, \overline{\psi}_f \, \psi_f - i \, \mu_0^{\epsilon} \, \epsilon \, (1+\epsilon) \, \frac{y_f}{\sqrt{2}} \, \frac{1}{w} \, \mathfrak{D} \, G \, \overline{\psi}_f \, \gamma_5 \, \psi_f + \cdots
\end{aligned} \tag{4.46}
$$

for the Yukawa Lagrangian, where the ellipsis denotes infinitely many terms of higher orders in the fields as well as in $\epsilon$. The fermion masses in this model are provided by $\widetilde{m}_f = m_f = \mu_0^{\epsilon} \, y_f \, v/\sqrt{2}$. Further, for the gauge fixing Lagrangian one obtains

$$
\begin{aligned}
\mathcal{L}^{\mathrm{QSI}}_{\mathrm{GF}} = &- \mu_0^{-2\epsilon} \, \frac{1}{2\,\xi} \, (\partial^{\mu} A_{\mu})^2 + \mu_0^{-2\epsilon} \, \epsilon \, (1+\epsilon) \, \frac{1}{\xi} \, \frac{\mathfrak{D}}{w} \, (\partial^{\mu} A_{\mu})^2 \\
&- \mu_0^{-2\epsilon} \, \epsilon \, (1+3\,\epsilon) \, \frac{1}{2\,\xi} \, \frac{\mathfrak{D}^2}{w^2} \, (\partial^{\mu} A_{\mu})^2 + \mu_0^{-2\epsilon} \, \epsilon \, (1+4\,\epsilon) \, \frac{1}{3\,\xi} \, \frac{\mathfrak{D}^3}{w^3} \, (\partial^{\mu} A_{\mu})^2 \\
&- \mu_0^{-2\epsilon} \, \epsilon \left(1 + \frac{14}{3}\,\epsilon\right) \frac{1}{4\,\xi} \, \frac{\mathfrak{D}^4}{w^4} \, (\partial^{\mu} A_{\mu})^2 + \mathcal{O}\left((\mathfrak{D}/w)^5 \, , \epsilon^3\right)
\end{aligned} \tag{4.47}
$$

Moreover, for the $D$-dimensional scalar potential $\widetilde{V}_{\mathrm{QED,H}}\,(\Phi,\sigma) = \mu^{2\epsilon}(\sigma)\, V_{\mathrm{QED,H}}\,(\Phi,\sigma)$ one obtains (C.1) for $\widetilde{V}_{\mathrm{2SM}}\,(\phi,\sigma) = \mu^{2\epsilon}(\sigma)\, V_{\mathrm{2SM}}\,(\phi,\sigma)$ as well as

$$
\begin{aligned}
\widetilde{V}_{\mathrm{QED},G}\,(\Phi,\sigma) &= \mu^{2\epsilon}(\sigma) \left( \frac{\lambda_{\phi}}{12} \, \phi^2 \, G^2 + \frac{\lambda_m}{4} \, \sigma^2 \, G^2 + \frac{\lambda_{\phi}}{4!} \, G^4 \right) \\
&= \frac{1}{2} \, \widetilde{M}_G^2 \, G^2 + \frac{1}{2} \, \widetilde{c}_{133} \, h \, G^2 + \frac{1}{2} \, \widetilde{c}_{233} \, \mathfrak{D} \, G^2 + \frac{1}{4} \, \widetilde{c}_{1133} \, h^2 \, G^2 \\
&\quad + \frac{1}{2} \, \widetilde{c}_{1233} \, h \, \mathfrak{D} \, G^2 + \frac{1}{4} \, \widetilde{c}_{2233} \, \mathfrak{D}^2 \, G^2 + \frac{1}{4!} \, \widetilde{c}_{3333} \, G^4 + \cdots
\end{aligned} \tag{4.48}
$$

where the ellipsis denotes infinitely many terms of higher orders in scalar fields and with

$$
\begin{aligned}
\widetilde{M}_G^2 &= M_G^2 = \mu_0^{2\epsilon} \, \frac{\lambda_{\phi}}{6} \, v^2 \, \left( 1 + 3 \, \frac{\lambda_m}{\lambda_{\phi}} \, \frac{w^2}{v^2} \right) \\
\widetilde{c}_{133} &= \mu_0^{2\epsilon} \, \frac{\lambda_{\phi}}{3} \, v \\
\widetilde{c}_{233} &= \mu_0^{2\epsilon} \left[ \lambda_m \, w + \epsilon \, (1+\epsilon) \left( \lambda_m \, w + \frac{\lambda_{\phi}}{3} \, \frac{v^2}{w} \right) + \mathcal{O}\left(\epsilon^3\right) \right] \\
\widetilde{c}_{1133} &= \mu_0^{2\epsilon} \, \frac{\lambda_{\phi}}{3} \\
\widetilde{c}_{1233} &= \mu_0^{2\epsilon} \left[ \frac{2}{3} \, \lambda_{\phi} \, \frac{v}{w} \, \epsilon \, (1+\epsilon) + \mathcal{O}\left(\epsilon^3\right) \right] \\
\widetilde{c}_{2233} &= \mu_0^{2\epsilon} \left[ \lambda_m + \epsilon \left( 3\,\lambda_m - \frac{\lambda_{\phi}}{3} \, \frac{v^2}{w^2} \right) + \epsilon^2 \left( 5\,\lambda_m + \frac{\lambda_{\phi}}{3} \, \frac{v^2}{w^2} \right) + \mathcal{O}\left(\epsilon^3\right) \right] \\
\widetilde{c}_{3333} &= \mu_0^{2\epsilon} \, \lambda_{\phi}
\end{aligned} \tag{4.49}
$$

Note that the minimalisation condition of the full scalar potential $\widetilde{V}_{\text{QED,H}}(\Phi, \sigma)$ is the same as for the 2 Scalar Model, i.e. (2.47). Furthermore, transforming the Higgs-like boson $h$ and the Dilaton $\mathfrak{D}$ in the Lagrangian (4.37) to mass eigenstates $\{H, S\}$ can be done analogously to the 2 Scalar Model discussed in section 2.2, with mass eigenstates given in (2.32) and mixing angle provided in (2.35), i.e.

$$\begin{aligned} H &= c_\beta\, h - s_\beta\, \mathfrak{D} &\Leftrightarrow\quad h &= c_\beta\, H + s_\beta\, S \\ S &= s_\beta\, h + c_\beta\, \mathfrak{D} &\Leftrightarrow\quad \mathfrak{D} &= -s_\beta\, H + c_\beta\, S \end{aligned}$$

The $\widetilde{V}_{\text{2SM}}(H, S)$ part of the scalar potential in mass eigenstates and with the minimalisation condition being used is given in (C.39).

The Renormalisation transformations for this theory are provided by

$$\begin{aligned} A &\longrightarrow A_0 = \sqrt{Z_A}\, A \\ \psi_f &\longrightarrow \psi_{f,0} = \sqrt{Z_{\psi_f}}\, \psi_f \\ \phi &\longrightarrow \phi_0 = \sqrt{Z_\phi}\, \phi \\ \sigma &\longrightarrow \sigma_0 = \sqrt{Z_\sigma}\, \sigma \\ G &\longrightarrow G_0 = \sqrt{Z_G}\, G \\ e &\longrightarrow e_B = \mu^\epsilon(\sigma)\, e_0 = \mu^\epsilon(\sigma)\, Z_e\, e \\ y_f &\longrightarrow y_{f,B} = \mu^\epsilon(\sigma)\, y_{f,0} = \mu^\epsilon(\sigma)\, Z_{y_f}\, y_f \\ \lambda_k &\longrightarrow \lambda_{k,B} = \mu^{2\epsilon}(\sigma)\, \lambda_{k,0} = \mu^{2\epsilon}(\sigma)\, Z_{\lambda_k}\, \lambda_k \end{aligned} \tag{4.50}$$

Where it has been used that in this model, at least at the 1-loop level, it is sufficient to transform left- and right-handed fermions with the same Renormalisation coefficient $Z_{\psi_f}$. Thus, the 1-loop counterterm Lagrangian is given by

$$\mathcal{L}^{\text{QSI}}_{\text{Fermion,ct1}} = i\, \delta Z_{\psi_f}\, \overline{\psi}_f\, \partial\!\!\!/\, \psi_f - \left( \delta Z_{\psi_f} + \delta Z_e + \frac{1}{2}\, \delta Z_A \right) e\, \overline{\psi}_f\, A\!\!\!/\, \psi_f \tag{4.51}$$

for the fermionic part

$$\mathcal{L}^{\text{QSI}}_{\text{Gauge,ct1}} = -\frac{1}{4}\, \mu^{-2\epsilon}(\sigma)\, \delta Z_A\, F_{\mu\nu}\, F^{\mu\nu} \tag{4.52}$$

for the gauge part

$$\begin{aligned} \mathcal{L}^{\text{QSI}}_{\text{Yukawa,ct1}} = &- \mu^\epsilon(\sigma) \left( \delta Z_{\psi_f} + \delta Z_{y_f} + \frac{1}{2}\, \delta Z_\phi \right) \frac{y_f}{\sqrt{2}}\, \phi\, \overline{\psi}_f\, \psi_f \\ &- i\, \mu^\epsilon(\sigma) \left( \delta Z_{\psi_f} + \delta Z_{y_f} + \frac{1}{2}\, \delta Z_G \right) \frac{y_f}{\sqrt{2}}\, G\, \overline{\psi}_f\, \gamma_5\, \psi_f \end{aligned} \tag{4.53}$$

for the Yukawa sector, and

$$
\begin{aligned}
\mathcal{L}^{\text{QSI}}_{\text{Higgs,ct1}} &= \frac{1}{2}\,\delta Z_\phi\,(\partial^\mu\phi)\,(\partial_\mu\phi) + \frac{1}{2}\,\delta Z_\sigma\,(\partial^\mu\sigma)\,(\partial_\mu\sigma) + \frac{1}{2}\,\delta Z_G\,(\partial^\mu G)\,(\partial_\mu G) \\
&\quad - \mu^{2\epsilon}(\sigma)\,\left(\delta Z_{V_\phi}\,\frac{\lambda_\phi}{4!}\,\phi^4 + \delta Z_{V_m}\,\frac{\lambda_m}{4}\,\phi^2\,\sigma^2 + \delta Z_{V_\sigma}\,\frac{\lambda_\sigma}{4!}\,\sigma^4\right) \\
&\quad - \mu^{2\epsilon}(\sigma)\,\left(\delta Z_{V_{G\phi}}\,\frac{\lambda_\phi}{12}\,\phi^2\,G^2 + \delta Z_{V_{Gm}}\,\frac{\lambda_m}{4}\,\sigma^2\,G^2 + \delta Z_{V_G}\,\frac{\lambda_\phi}{4!}\,G^4\right)
\end{aligned}
\tag{4.54}
$$

for the Higgs sector, where

$$
\begin{aligned}
\delta Z_{V_{G\phi}} &\coloneqq \delta Z_{\lambda_\phi} + \delta Z_\phi + \delta Z_G \\
\delta Z_{V_{Gm}} &\coloneqq \delta Z_{\lambda_m} + \delta Z_\sigma + \delta Z_G \\
\delta Z_{V_G} &\coloneqq \delta Z_{\lambda_\phi} + 2\,\delta Z_G
\end{aligned}
\tag{4.55}
$$

and $\{\delta Z_{V_\phi}, \delta Z_{V_m}, \delta Z_{V_\sigma}\}$ as in section 2.3. Again, the counterterm superscripts, indicating the loop-order, have been suppressed because this theory is considered solely at the 1-loop level, and it has been used that the Renormalisation of the gauge fixing term does not need to be considered as $Z_\xi = Z_A$ due to Ward identities [8, 35]. The 1-loop counterterms of this theory in the MS-scheme and determined in Feynman gauge $\xi = 1$ are then given by

$$
\begin{aligned}
\delta Z_{\psi_f} &= -\frac{1}{16\,\pi^2}\left(e^2 + \frac{y_f}{2}\right)\frac{1}{\epsilon} \\
\delta Z_A &= -2\,\delta Z_e = -\frac{1}{16\,\pi^2}\,\frac{4\,N_f\,e^2}{3}\,\frac{1}{\epsilon} \\
\delta Z_\phi &= -\frac{1}{16\,\pi^2}\,\sum_l\,y_l^2\,\frac{1}{\epsilon} \\
\delta Z_\sigma &= 0 \\
\delta Z_G &= -\frac{1}{16\,\pi^2}\,\sum_l\,y_l^2\,\frac{1}{\epsilon} \\
\delta Z_{y_f} &= \frac{1}{16\,\pi^2}\,\frac{1}{2}\left[y_f^2 + \sum_l\,y_l^2 - 6\,e^2\right]\frac{1}{\epsilon} \\
\delta Z_{V_\phi} &= \delta Z_{V_{G\phi}} = \delta Z_{V_G} = \frac{1}{16\,\pi^2}\,\frac{1}{6\,\lambda_\phi}\left[10\,\lambda_\phi^2 + 9\,\lambda_m^2 - 36\,\sum_l\,y_l^4\right]\frac{1}{\epsilon} \\
\delta Z_{V_m} &= \delta Z_{V_{Gm}} = \frac{1}{16\,\pi^2}\,\frac{1}{6}\,[4\,\lambda_\phi + 12\,\lambda_m + 3\,\lambda_\sigma]\,\frac{1}{\epsilon} \\
\delta Z_{V_\sigma} &= \frac{1}{16\,\pi^2}\,\frac{3}{2\,\lambda_\sigma}\left[2\,\lambda_m^2 + \lambda_\sigma^2\right]\frac{1}{\epsilon}
\end{aligned}
\tag{4.56}
$$

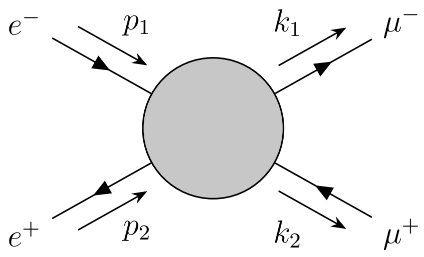
where again $N_f$ is the number of fermions in the theory and it was assumed that all fermions are leptons, e.g. $f, l \in \{e^-, \mu^-, \tau^-\}$ for 3 lepton flavours, as in the previous section.

# 5. Muon Production

In this chapter, muon production is analysed in a *quantum scale invariant* QED. In particular, the scattering process $e^- \, e^+ \longrightarrow \mu^- \, \mu^+$ is considered at the 1-loop level in the framework of the theory discussed in section 4.2. Such scattering processes have not yet been discussed in the framework of a theory with *spontaneously broken quantum scale invariance* regularised using SIDReg. The first section illustrates the general structure of the 1-loop scattering amplitude and its square of the absolute value in order to provide a clear overview of the results discussed the following sections. In the second section, explicit results for the above scattering process in the QSI QED (4.29) are presented and discussed. New finite quantum corrections are expected to emerge at the 1-loop level due to evanescent interactions, i.e. as a result of QSI, as explained in chapter 2. The last section is about IR-divergences occurring in such scattering processes and their implications on QSI theories regularised with SIDReg which contain evanescent interactions.

## 5.1. The Scattering Amplitude at 1-Loop

The amplitude of the scattering process $e^- \, e^+ \longrightarrow \mu^- \, \mu^+$ is given by




$$= i \, \mathcal{M} \left( e^- \, e^+ \longrightarrow \mu^- \, \mu^+ \right) \tag{5.1}$$

where the above diagram with the grey blob represents all *connected* Feynman diagrams with 4 external fermion legs, in particular with external electron and positron in the initial state as well as external muon and anti-muon in the final state. Thus, it contains all possible tree-level, counterterm and loop diagrams contributing to the above scattering process. At the 1-loop level the scattering amplitude may be written as

$$\mathcal{M} = \mathcal{M}_{\text{tree}} + \mathcal{M}_{\text{1L,ren}} + \mathcal{O}\left(\hbar^2\right) = \mathcal{M}_{\text{tree}} + \mathcal{M}_{\text{Res}}^{(1)} + \mathcal{M}_{\text{1L}} + \mathcal{M}_{\text{ct1}} + \mathcal{O}\left(\hbar^2\right) \tag{5.2}$$

with $\mathcal{M}_{\text{1L,ren}} = \mathcal{M}_{\text{Res}}^{(1)} + \mathcal{M}_{\text{1L}} + \mathcal{M}_{\text{ct1}}$, where $\mathcal{M}_{\text{tree}}$ is the tree-level amplitude, $\mathcal{M}_{\text{1L}}$ is the contribution from the (amputated) 1-loop diagrams, $\mathcal{M}_{\text{ct1}}$ is the 1-loop counterterm amplitude and $\mathcal{M}_{\text{Res}}^{(1)}$ is the 1-loop contribution from the Residue of the external particles.

According to the LSZ-formalism, S-matrix elements, i.e. scattering amplitudes, may be calculated by considering only amputated (or truncated) Feynman diagrams and multiplying by a factor of $\sqrt{R}$ for every external particle, where $R$ is the Residue of the propagator of the corresponding external particle, as well as setting the external momenta on the mass shell [8, 38, 40]. The Residue of an external fermion is given by

$$R^{-1} = 1 - \left.\frac{d\Sigma_{\text{ren}}\left(\not{p}\right)}{d\not{p}}\right|_{\not{p}=m_{\text{pol}}} \tag{5.3}$$

where $-\,i\,\Sigma_{\text{ren}}\left(\not{p}\right)$ is the renormalised fermion self energy and $m_{\text{pol}}$ is the pole mass which is provided by

$$m_{\text{pol}} = m + \Sigma_{\text{ren}}\left(\not{p} = m_{\text{pol}}\right) = m + \Sigma_{\text{ren}}\left(\not{p} = m\right) + \mathcal{O}\left(\hbar^2\right), \tag{5.4}$$

and thus

$$R^{-1} = 1 - \left.\frac{d\Sigma_{\text{ren}}\left(\not{p}\right)}{d\not{p}}\right|_{\not{p}=m} + \mathcal{O}\left(\hbar^2\right) \tag{5.5}$$

Now, considering

$$\begin{aligned} R &= 1 + \delta R^{(1)} + \mathcal{O}\left(\hbar^2\right) \\ \Leftrightarrow R^{-1} &= 1 - \delta R^{(1)} + \mathcal{O}\left(\hbar^2\right) \end{aligned} \tag{5.6}$$

and comparing this with (5.5), one obtains at the 1-loop level

$$\delta R^{(1)} = \left.\frac{d\Sigma_{\text{ren}}\left(\not{p}\right)}{d\not{p}}\right|_{\not{p}=m} \tag{5.7}$$

Hence, for the scattering process (5.1) the 1-loop contribution from the Residue of the external particles is provided by

$$\mathcal{M}^{(1)}_{\text{Res}} = \left(\delta R^{(1)}_e + \delta R^{(1)}_\mu\right)\mathcal{M}_{\text{tree}} \tag{5.8}$$

where $\delta R^{(1)}_e$ and $\delta R^{(1)}_\mu$ are the 1-loop contributions to the Residue of the propagator of the electron and the muon, respectively.

**Remark.**
The Residue $R$ depends on the Renormalisation scheme that is used, e.g.

- In the On-shell scheme: $R = 1 \;\Rightarrow\; \delta R^{(1)} = 0$
- In the MS- or $\overline{\text{MS}}$-scheme: $\delta R^{(1)} \neq 0$

The following results will be given in the $\overline{\text{MS}}$-scheme, and thus the Residue contribution $\mathcal{M}^{(1)}_{\text{Res}}$ needs to be taken into account.

In order to compute the cross section, the square of the absolute value of the scattering amplitude (5.2) which is given by

$$\begin{aligned} |\mathcal{M}|^2 &= \mathcal{M}^\dagger_{\text{tree}}\mathcal{M}_{\text{tree}} + \mathcal{M}^\dagger_{\text{tree}}\mathcal{M}_{\text{1L,ren}} + \mathcal{M}^\dagger_{\text{1L,ren}}\mathcal{M}_{\text{tree}} + \mathcal{O}\left(\hbar^2\right) \\ &= \mathcal{M}^\dagger_{\text{tree}}\mathcal{M}_{\text{tree}} + 2\,\text{Re}\left(\mathcal{M}^\dagger_{\text{tree}}\mathcal{M}^{(1)}_{\text{Res}} + \mathcal{M}^\dagger_{\text{tree}}\mathcal{M}_{\text{1L}} + \mathcal{M}^\dagger_{\text{tree}}\mathcal{M}_{\text{ct1}}\right) + \mathcal{O}\left(\hbar^2\right) \end{aligned} \tag{5.9}$$

is needed.

Moreover, it is averaged over the spins of initial particles and summed over the spins of final particles. Therefore, in the scattering process (5.1) one needs to consider

$$\left\langle |\mathcal{M}|^2 \right\rangle = \frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2 = \left\langle |\mathcal{M}|^2 \right\rangle_{\text{tree}} + \left\langle |\mathcal{M}|^2 \right\rangle_{1\text{L}} + \mathcal{O}\left(\hbar^2\right) \tag{5.10}$$

where

$$\begin{aligned} \left\langle |\mathcal{M}|^2 \right\rangle_{\text{tree}} &:= \frac{1}{4} \sum_{\text{spins}} \mathcal{M}^\dagger_{\text{tree}} \mathcal{M}_{\text{tree}} = \left\langle \mathcal{M}^\dagger_{\text{tree}} \mathcal{M}_{\text{tree}} \right\rangle \\ \left\langle |\mathcal{M}|^2 \right\rangle_{1\text{L}} &:= \frac{1}{4} \sum_{\text{spins}} 2 \text{ Re} \left( \mathcal{M}^\dagger_{\text{tree}} \mathcal{M}^{(1)}_{\text{Res}} + \mathcal{M}^\dagger_{\text{tree}} \mathcal{M}_{1\text{L}} + \mathcal{M}^\dagger_{\text{tree}} \mathcal{M}_{\text{ct1}} \right) \\ &= 2 \text{ Re} \left( \frac{1}{4} \sum_{\text{spins}} \mathcal{M}^\dagger_{\text{tree}} \mathcal{M}_{1\text{L,ren}} \right) = 2 \text{ Re} \left( \left\langle \mathcal{M}^\dagger_{\text{tree}} \mathcal{M}_{1\text{L,ren}} \right\rangle \right) \end{aligned} \tag{5.11}$$

Due to $\mathcal{M}_{\text{ct1}}$, the expression (5.10) is UV-finite, however, it is known that such squared scattering amplitudes are IR-divergent and that only the corresponding cross sections, i.e. physical observables, are IR-finite [8, 32]. For more details w.r.t. this, the reader is referred to [8, 32]. Moreover, new quantum corrections due to evanescent interactions, i.e. as a result of QSI, are again expected to emerge at the quantum level. Hence, the 1-loop contribution to (5.10) admits the following general structure

$$\begin{aligned} \left\langle \mathcal{M}^\dagger_{\text{tree}} \mathcal{M}_{1\text{L,ren}} \right\rangle = &\left\langle \mathcal{M}^\dagger_{\text{tree}} \mathcal{M}_{1\text{L,ren}} \right\rangle_{1/\epsilon^2_{\text{IR}}} + \left\langle \mathcal{M}^\dagger_{\text{tree}} \mathcal{M}_{1\text{L,ren}} \right\rangle_{1/\epsilon_{\text{IR}}} \\ &+ \Delta_{\text{IR}} \left\langle \mathcal{M}^\dagger_{\text{tree}} \mathcal{M}_{1\text{L,ren}} \right\rangle_{1/\epsilon_{\text{IR}}} + \left\langle \mathcal{M}^\dagger_{\text{tree}} \mathcal{M}_{1\text{L,ren}} \right\rangle_{\text{fin}} \\ &+ \Delta_{\text{UV}} \left\langle \mathcal{M}^\dagger_{\text{tree}} \mathcal{M}_{1\text{L,ren}} \right\rangle_{\text{fin}} + \Delta_{\text{IR}} \left\langle \mathcal{M}^\dagger_{\text{tree}} \mathcal{M}_{1\text{L,ren}} \right\rangle_{\text{fin}} + \mathcal{O}\left(\epsilon\right) \end{aligned} \tag{5.12}$$

where $\Delta_{\text{UV}}$ and $\Delta_{\text{IR}}$ denote new quantum corrections arising from evanescent interactions cancelling UV-divergences and IR-divergences, respectively. All results in section 5.2 will be given in the form (5.12). Note that IR-divergences are regularised dimensionally and *not* with a regulator mass due to fact that all theories in this thesis are regularised using SIDReg in order to obtain a theory with spontaneously broken quantum scale invariance.

Additionally, the Mandelstam variables are defined as usual, i.e.

$$\begin{aligned} s &= (p_1 + p_2)^2 = (k_1 + k_2)^2 \\ t &= (p_1 - k_1)^2 = (p_2 - k_2)^2 \\ u &= (p_1 - k_2)^2 = (p_2 - k_1)^2 \end{aligned} \tag{5.13}$$

## 5.2. Muon Production in QSI QED

The scattering process (5.1) is discussed in a QSI QED, as introduced in section 4.2, with Lagrangian (4.29) and for 3 fermion flavours, i.e. $f \in \{e^-, \mu^-, \tau^-\}$ and $N_f = 3$. In particular, the scattering process is considered for following two different scenarios

(i) QSI QED with Lagrangian (4.29), however, with $\lambda \equiv 0$ *and* $y_f \equiv 0$ at tree-level, and not just $\lambda \equiv 0$. Thus, this theory is closest to "usual" massless QED because the only new terms in the Lagrangian are the kinetic term of the Dilaton as well as evanescent interactions introduced by the Renormalisation function $\mu(\sigma)$. Moreover, this theory is purely massless even in the broken phase of the theory (at least at tree-level) because the Yukawa terms are set to zero.

(ii) QSI QED with Lagrangian (4.29) where $\lambda \equiv 0$ but $y_f \neq 0$ at tree-level. However, the limit of vanishing fermion masses, i.e. the massless limit, is considered. Note that both, the Yukawa couplings $y_f$ and the Dilaton VEV $w$ remain *non-zero*, but nonetheless the fermion masses $m_f = \mu_0^\epsilon \, y_f \, w$ are set to zero, which appears a bit strange but is motivated by a very high process-energy in the scattering process. In particular, in the case where the invariant mass $s$ (or equivalently the total energy in the C.o.M. frame $\sqrt{s}$) is much greater than the fermion masses, i.e. $s \gg m_f^2$, fermion masses are negligible. The reason for this is that in the purely massless case the IR-divergence structure is more interesting than in the massive case because in the massless case there is not only a simple pole but also a pole of second order in $\epsilon_{\text{IR}}$. Therefore, not only new finite but also a new divergent quantum correction can arise from evanescent interactions if a term $\sim \epsilon$ "meets" a second order pole in $\epsilon_{\text{IR}}$. Practically, this scenario is realised by setting to zero the fermion masses $m_f$ in the free Lagrangian, i.e. in the propagators, but keeping the Yukawa couplings $y_f$ and the Dilaton VEV $w$ non-zero in the interaction Lagrangian.

The results presented in this section have been calculated in Feynman gauge $\xi = 1$ and are provided in the $\overline{\text{MS}}$-scheme. All counterterms in section 4.2, given in the MS-scheme, can still be used after applying (3.23). Further, note that again all Feynman diagrams in this section have been generated using FeynArts [18], the FeynArts model files have been generated using FeynRules [1, 4], and the generated Feynman diagrams and their amplitudes have been computed using FeynCalc [27, 36, 37] and Package-X [29], which has been connected with FeynCalc using FeynHelpers [35].

### 5.2.1. Muon Production in massless QSI QED with $y_f \equiv 0$ and $\lambda \equiv 0$

Starting with the first scenario (i), the tree-level result (in 4 dimensions) is given by

$$\left\langle |\mathcal{M}|^2 \right\rangle_{\text{tree}} = 2 \, e^4 \, \frac{t^2 + u^2}{s^2} \tag{5.14}$$

and the 1-loop results (in $D = 4 - 2\epsilon$ dimensions), provided in the form (5.12) and in the $\overline{\text{MS}}$-scheme, are given as follows:
For the IR-divergences, the second order pole in $\epsilon_{\text{IR}}$ is found to be

$$\left\langle \mathcal{M}^\dagger_{\text{tree}} \mathcal{M}_{\text{1L,ren}} \right\rangle_{1/\epsilon_{\text{IR}}^2} = - \, \mu_0^{4\epsilon} \, \frac{e^6}{2\,\pi^2} \, \frac{t^2 + u^2}{s^2} \, \frac{1}{\epsilon_{\text{IR}}^2} \tag{5.15}$$

whereas the simple pole in $\epsilon_{\mathrm{IR}}$ reads as

$$\left\langle \mathcal{M}^{\dagger}_{\mathrm{tree}} \mathcal{M}_{\mathrm{1L,ren}} \right\rangle_{1/\epsilon_{\mathrm{IR}}} = -\,\mu_0^{4\epsilon}\, \frac{e^6}{4\,\pi^2} \left\{ \frac{t^2 - 4\,t\,u + u^2}{s^2} - 2\,\frac{t^2+u^2}{s^2} \left[ \log\left(-\frac{s}{\mu_0^2}\right) + \log\left(-\frac{t}{\mu_0^2}\right) - \log\left(-\frac{u}{\mu_0^2}\right) \right] \right\} \frac{1}{\epsilon_{\mathrm{IR}}} \tag{5.16}$$

and the new divergent quantum correction to the simple pole in $\epsilon_{\mathrm{IR}}$ is given by

$$\Delta_{\mathrm{IR}} \left\langle \mathcal{M}^{\dagger}_{\mathrm{tree}} \mathcal{M}_{\mathrm{1L,ren}} \right\rangle_{1/\epsilon_{\mathrm{IR}}} = 0 \tag{5.17}$$

The finite 1-loop result for the considered scattering process is provided by

$$\begin{aligned} \left\langle \mathcal{M}^{\dagger}_{\mathrm{tree}} \mathcal{M}_{\mathrm{1L,ren}} \right\rangle_{\mathrm{fin}} = \mu_0^{4\epsilon}\, \frac{e^6}{2\,\pi^2} \Bigg\{ & \frac{(7\,\pi^2 - 50)\,t^2 + 36\,t\,u - (5\,\pi^2+50)\,u^2}{12\,s^2} \\ & + \frac{2\,t^2 - 2\,t\,u + u^2}{s^2} \log\left(-\frac{s}{\mu_0^2}\right) + \frac{2\,t+u}{2\,s} \log\left(-\frac{t}{\mu_0^2}\right) \\ & - \frac{t+2\,u}{2\,s} \log\left(-\frac{u}{\mu_0^2}\right) - \frac{u^2}{s^2} \log^2\left(-\frac{s}{\mu_0^2}\right) \\ & + \frac{t^2-u^2}{4\,s^2} \log^2\left(-\frac{t}{\mu_0^2}\right) + \frac{t^2-u^2}{4\,s^2} \log^2\left(-\frac{u}{\mu_0^2}\right) \\ & - \frac{3\,t^2+u^2}{2\,s^2} \log\left(-\frac{s}{\mu_0^2}\right) \log\left(-\frac{t}{\mu_0^2}\right) \\ & + \frac{t^2+3\,u^2}{2\,s^2} \log\left(-\frac{s}{\mu_0^2}\right) \log\left(-\frac{u}{\mu_0^2}\right) \Bigg\} \end{aligned} \tag{5.18}$$

whereas the new finite quantum correction emerging from UV-divergences is found to be

$$\Delta_{\mathrm{UV}} \left\langle \mathcal{M}^{\dagger}_{\mathrm{tree}} \mathcal{M}_{\mathrm{1L,ren}} \right\rangle_{\mathrm{fin}} = 0 \tag{5.19}$$

and the new finite quantum correction arising from IR-divergences is given by

$$\Delta_{\mathrm{IR}} \left\langle \mathcal{M}^{\dagger}_{\mathrm{tree}} \mathcal{M}_{\mathrm{1L,ren}} \right\rangle_{\mathrm{fin}} = 0 \tag{5.20}$$

**Remark.**

(i) It can be seen that there are *no* new quantum corrections due to QSI at the 1-loop level in the case of vanishing Yukawa and scalar couplings, i.e. $y_f \equiv 0$ and $\lambda \equiv 0$ at tree-level.

(ii) The result for $\left\langle \mathcal{M}^{\dagger}_{\mathrm{tree}} \mathcal{M}_{\mathrm{1L,ren}} \right\rangle$ given above in (5.15) to (5.20) agrees with the 1-loop result provided in [3]. In [3] regular massless QED and here, in the present case, QSI QED with vanishing Yukawa and scalar couplings, i.e. $y_f \equiv 0$ and $\lambda \equiv 0$ at tree-level, has been considered. These two theories only differ in the kinetic term of the Dilaton and the Dilaton-dependent Renormalisation function which are necessary in the QSI theory. Due to vanishing Yukawa couplings the Dilaton does not couple to the fermions directly, but only to the photon via the kinetic term of the photon, as can be seen in (4.29). However, such purely evanescent terms will ultimately not contribute to muon production at the 1-loop level because there are only 2 new (and purely evanescent) diagrams contributing to this scattering process, one with a Dilaton loop starting and ending at the same vertex at the photon propagator and one with a half Dilaton - half photon loop in the photon propagator. The first of these new diagrams is proportional to $(\epsilon + 3\epsilon^2 + \mathcal{O}(\epsilon^3))\,A_0(0) = 0$ and the second one is proportional to $(\epsilon^2 + 2\epsilon^3 + \mathcal{O}(\epsilon^4))\,B_0(s,0,0) \sim (\epsilon^2 + 2\epsilon^3 + \mathcal{O}(\epsilon^4))\,1/\epsilon_{\mathrm{UV}} \to 0$, as $\epsilon \to 0$, and thus non of the new diagrams give rise to a new quantum correction. Hence, new quantum corrections due to QSI are *not* expected to arise at the 1-loop level in the present case. For this reason, it is expected that both theories provide the same result for muon production at the 1-loop level, and thus the agreement with the results in [3] serves as a consistency check for the Mathematica algorithm used to determine the amplitudes in this chapter.

(iii) The theory (4.28) with $y_f \equiv 0$ and $\lambda \equiv 0$ at tree-level represents a minimal QSI QED, and thus is indeed closest to a regular massless QED.

### 5.2.2. Muon Production in massless QSI QED with $y_f \neq 0$ and $\lambda \equiv 0$

Continuing with the second scenario (ii), the tree-level result (in 4 dimensions) is given by

$$\left\langle |\mathcal{M}|^2 \right\rangle_{\mathrm{tree}} = 2\,e^4\,\frac{t^2+u^2}{s^2} + y_e^2\,y_\mu^2 \tag{5.21}$$

and the 1-loop results (in $D = 4 - 2\epsilon$ dimensions), provided in the form (5.12) and in the $\overline{\mathrm{MS}}$-scheme, are given as follows:
For the IR-divergences, the second order pole in $\epsilon_{\mathrm{IR}}$ is found to be

$$\left\langle \mathcal{M}^\dagger_{\mathrm{tree}} \mathcal{M}_{\mathrm{1L,ren}} \right\rangle_{1/\epsilon^2_{\mathrm{IR}}} = -\,\mu_0^{4\epsilon}\,\frac{e^2}{4\,\pi^2}\left(2\,e^4\,\frac{t^2+u^2}{s^2} + y_e^2\,y_\mu^2\right)\frac{1}{\epsilon^2_{\mathrm{IR}}} \tag{5.22}$$

whereas the simple pole in $\epsilon_{\mathrm{IR}}$ reads as

$$\begin{aligned} \left\langle \mathcal{M}^\dagger_{\mathrm{tree}} \mathcal{M}_{\mathrm{1L,ren}} \right\rangle_{1/\epsilon_{\mathrm{IR}}} = -\,\frac{\mu_0^{4\epsilon}}{32\,\pi^2}\Bigg\{ & 8\,e^6\,\frac{t^2 - 4\,t\,u + u^2}{s^2} - 2\,e^4\left(y_e^2 + y_\mu^2\right)\frac{t^2+u^2}{s^2} \\ & + 12\,e^2\,y_e^2\,y_\mu^2 - y_e^4\,y_\mu^2 - y_e^2\,y_\mu^4 \\ & - 8\left(2\,e^6\,\frac{t^2+u^2}{s^2} + e^2\,y_e^2\,y_\mu^2\right)\left[\log\left(-\frac{s}{\mu_0^2}\right)\right. \\ & \left.\left. + \log\left(-\frac{t}{\mu_0^2}\right) - \log\left(-\frac{u}{\mu_0^2}\right)\right]\right\}\frac{1}{\epsilon_{\mathrm{IR}}} \end{aligned} \tag{5.23}$$

and the new divergent quantum correction to the simple pole in $\epsilon_{\mathrm{IR}}$ is given by

$$\Delta_{\mathrm{IR}} \left\langle \mathcal{M}^\dagger_{\mathrm{tree}} \mathcal{M}_{\mathrm{1L,ren}} \right\rangle_{1/\epsilon_{\mathrm{IR}}} = -\frac{\mu_0^{4\epsilon}}{\pi^2}\, e^2\, y_e^2\, y_\mu^2\, \frac{1}{\epsilon_{\mathrm{IR}}} \tag{5.24}$$

The finite 1-loop result for the considered scattering process is provided by

$$\begin{aligned}
\left\langle \mathcal{M}^\dagger_{\mathrm{tree}} \mathcal{M}_{\mathrm{1L,ren}} \right\rangle_{\mathrm{fin}} = \frac{\mu_0^{4\epsilon}}{24\,\pi^2} \Bigg[ & e^6\, \frac{(7\,\pi^2 - 50)\, t^2 + 36\, t\, u - (5\,\pi^2 + 50)\, u^2}{s^2} \\
& - 3\, e^4 \left(y_e^2 + y_\mu^2\right) \frac{t\, u}{s^2} - \frac{12 - \pi^2}{2}\, e^2\, y_e^2\, y_\mu^2 \\
& - 3\, y_e^2\, y_\mu^2 \left(3\, y_e^2 + 3\, y_\mu^2 + 2\, y_\tau^2\right) \Bigg] \\
+ \frac{\mu_0^{4\epsilon}}{16\,\pi^2} \Bigg[ & 8\, e^6\, \frac{2\, t^2 - 2\, t\, u + u^2}{s^2} - e^4 \left(y_e^2 + y_\mu^2\right) \frac{t^2 + u^2}{s^2} \\
+ 2\, e^2\, y_e^2\, y_\mu^2\, & \frac{t^2 - u^2}{s^2} + y_e^2\, y_\mu^2 \left(3\, y_e^2 + 3\, y_\mu^2 + 2\, y_\tau^2\right) \Bigg] \log\left(-\frac{s}{\mu_0^2}\right) \\
+ \frac{\mu_0^{4\epsilon}}{16\,\pi^2} \Bigg[ & 4\, e^6\, \frac{2\, t + u}{s} - e^2\, y_e^2\, y_\mu^2\, \frac{t + 3\, u}{s} \Bigg] \log\left(-\frac{t}{\mu_0^2}\right) \\
- \frac{\mu_0^{4\epsilon}}{16\,\pi^2} \Bigg[ & 4\, e^6\, \frac{t + 2\, u}{s} - e^2\, y_e^2\, y_\mu^2\, \frac{3\, t + u}{s} \Bigg] \log\left(-\frac{u}{\mu_0^2}\right) \\
- \frac{\mu_0^{4\epsilon}}{8\,\pi^2} \Bigg[ & 4\, e^6\, \frac{u^2}{s^2} + e^2\, y_e^2\, y_\mu^2 \Bigg] \log^2\left(-\frac{s}{\mu_0^2}\right) \\
+ \frac{\mu_0^{4\epsilon}}{32\,\pi^2} \Bigg[ & 4\, e^6\, \frac{t^2 - u^2}{s^2} - e^2\, y_e^2\, y_\mu^2 \Bigg] \log^2\left(-\frac{t}{\mu_0^2}\right) \\
+ \frac{\mu_0^{4\epsilon}}{32\,\pi^2} \Bigg[ & 4\, e^6\, \frac{t^2 - u^2}{s^2} + e^2\, y_e^2\, y_\mu^2 \Bigg] \log^2\left(-\frac{u}{\mu_0^2}\right) \\
- \frac{\mu_0^{4\epsilon}}{16\,\pi^2} \Bigg[ & 4\, e^6\, \frac{3\, t^2 + u^2}{s^2} + 3\, e^2\, y_e^2\, y_\mu^2 \Bigg] \log\left(-\frac{s}{\mu_0^2}\right) \log\left(-\frac{t}{\mu_0^2}\right) \\
+ \frac{\mu_0^{4\epsilon}}{16\,\pi^2} \Bigg[ & 4\, e^6\, \frac{t^2 + 3\, u^2}{s^2} + 3\, e^2\, y_e^2\, y_\mu^2 \Bigg] \log\left(-\frac{s}{\mu_0^2}\right) \log\left(-\frac{u}{\mu_0^2}\right)
\end{aligned} \tag{5.25}$$

whereas the new finite quantum correction emerging from UV-divergences is found to be

$$\Delta_{\mathrm{UV}} \left\langle \mathcal{M}^\dagger_{\mathrm{tree}} \mathcal{M}_{\mathrm{1L,ren}} \right\rangle_{\mathrm{fin}} = -\frac{\mu_0^{4\epsilon}}{16\,\pi^2}\, y_e^2\, y_\mu^2 \left(7\, y_e^2 + 7\, y_\mu^2 + 4\, y_\tau^2\right) \tag{5.26}$$

and the new finite quantum correction arising from IR-divergences is given by

$$\begin{aligned}
\Delta_{\mathrm{IR}} \left\langle \mathcal{M}^{\dagger}_{\mathrm{tree}} \mathcal{M}_{\mathrm{1L,ren}} \right\rangle_{\mathrm{fin}} = \frac{\mu_0^{4\epsilon}}{16\,\pi^2} \Bigg[ & 2\,e^4 \left(y_e^2 + y_\mu^2\right) \frac{t^2+u^2}{s^2} \\
& - 24\,e^2\,y_e^2\,y_\mu^2 + 3\,y_e^2\,y_\mu^2 \left(y_e^2 + y_\mu^2\right) \Bigg] \\
+ \frac{\mu_0^{4\epsilon}}{\pi^2}\,e^2\,y_e^2\,y_\mu^2 & \left[ \log\left(-\frac{s}{\mu_0^2}\right) + \log\left(-\frac{t}{\mu_0^2}\right) - \log\left(-\frac{u}{\mu_0^2}\right) \right] \\
- \mu_0^{4\epsilon}\,\frac{5}{2\,\pi^2}\,e^2\,y_e^2\,y_\mu^2 &
\end{aligned} \tag{5.27}$$

**Remark.**

(i) In the case of non-vanishing Yukawa couplings, i.e. $y_f \neq 0$ and $\lambda \equiv 0$ at tree-level, there are not only new finite but also new divergent quantum corrections due to evanescent interactions, i.e. as a result of QSI, which can be seen in (5.24), (5.26) and (5.27).

(ii) The new divergent quantum correction (5.24) arises from the second order pole in $\epsilon_{\mathrm{IR}}$, i.e. from $1/\epsilon_{\mathrm{IR}}^2$, and an evanescent term $\sim \epsilon$ cancelling just one power of $\epsilon_{\mathrm{IR}}$. This new quantum correction is particularly interesting as it changes the IR-divergence structure of the scattering process by introducing a new IR-divergence that ultimately needs to be cancelled at the level of cross sections. This will be discussed in more detail in the last section of this chapter. The emergence of this new divergent quantum correction in the QSI theory and its implications for the IR-finiteness of cross sections was the main motivation for considering the massless limit despite non-vanishing Yukawa couplings, as discussed at the beginning of this section, as there is no $1/\epsilon_{\mathrm{IR}}^2$ divergence in the massive theory, and thus a new IR-divergent quantum correction could not have emerged.

(iii) In (5.27), new finite quantum corrections arising from IR-divergences are presented. The first 3 lines of (5.27) are corrections that have emerged from the simple IR-pole, i.e. from $1/\epsilon_{\mathrm{IR}}$, and the last line in (5.27) has emerged from the second order IR-pole, i.e. from $1/\epsilon_{\mathrm{IR}}^2$. It can be seen that IR-divergences, appearing in scattering processes or decays, can also lead to new (finite) quantum corrections.

(iv) The new finite quantum correction arising from the $1/\epsilon_{\mathrm{UV}}$ divergence is given in (5.26). Such new quantum corrections emerging from UV-divergences are conceptually not new in QSI theories.

(v) All new quantum corrections, given in (5.24), (5.26) and (5.27), are suppressed by very small Yukawa couplings. In particular, in the present theory one finds

$$y_f = \frac{m_f}{w} \sim \frac{m_f}{M_{Pl}} \sim \begin{cases} 10^{-22}, & f = e \\ 10^{-19}, & f = \mu \\ 10^{-18}, & f = \tau \end{cases} \tag{5.28}$$

assuming $w = \langle\sigma\rangle \sim M_{Pl} \sim 10^{18}\,\text{GeV}$. Hence, all new quantum corrections due to QSI are (indirectly) suppressed by the VEV of the Dilaton.

(vi) The scattering amplitude has also been calculated in the massive theory, i.e. the case where the masses have not been neglected. In this case there is indeed *no* second order IR-pole, and thus there is no new *divergent* quantum correction due to QSI, such as (5.24). For this reason, the massive case is less interesting for the next section, and therefore has not been discussed in detail in this thesis.

(vii) The same scattering process has also been considered in the QSI QED with toy model Higgs sector presented in section 4.3. There the scalar sector is similar to that of the quantum scale invariant Standard Model provided in chapter 6 and the Yukawa couplings are given by $y_f = \sqrt{2}\,m_f/v$. Thus, the Yukawa couplings have the same values as in the "usual" Standard Model. However, it turned out that again all new quantum corrections due to QSI are suppressed by the VEV of the Dilaton. In particular, they are suppressed by positive powers of $\chi_0 = v/w$.

## 5.3. IR-Divergences

As mentioned in section 5.1 and explicitly shown in section 5.2, scattering amplitudes are UV-finite after Renormalisation but still IR-divergent. These IR-divergences have to cancel at the level of cross sections, i.e. physical observables must be UV- *and* IR-finite [8, 32]. In particular, in a scattering process beyond tree-level, as considered above, one also needs to consider real emission graphs, i.e. final (and sometimes initial) state radiation, as discussed in [32]. According to the Bloch-Nordsieck and the Kinoshita-Lee-Nauenberg theorem, all IR-divergences in the virtual and real corrections to the cross section cancel each other at a given order of the perturbation theory when they are summed over [8, 32], and thus one obtains an IR-finite result for the cross section.

In subsection 5.2.2, it has been shown that new IR-divergent quantum corrections can emerge due to QSI, and thus change the IR-divergence structure of the scattering amplitude. Therefore, the question arises whether such new IR-divergences also cancel to ultimately give rise to finite cross sections. For this reason, a conjecture about new finite and divergent quantum corrections arising from evanescent interactions cancelling IR-divergences is provided in this section and afterwards exemplarily proven for muon production at the 1-loop level in a massless QSI QED with $y_f \neq 0$ and $\lambda \equiv 0$ at tree-level.

**Conjecture 5.1** (IR-Divergences in the context of Quantum Scale Symmetry)**.**
*All new divergent and finite quantum corrections arising from IR-divergences and evanescent interactions, introduced by the dynamical Renormalisation function in a quantum scale invariant theory, cancel together with the regular IR-divergences when summing over virtual and real corrections to a cross section or decay width at a given order of the perturbation theory.*

**Remark.**

(i) Conjecture 5.1 ensures that physical observables, such as cross sections, remain finite in a quantum scale invariant theory even if the IR-divergence structure is changed by new *divergent* quantum corrections emerging from IR-divergences and evanescent interactions.

(ii) Moreover, also new *finite* quantum corrections arising from evanescent interactions cancelling IR-divergences are ultimately cancelled at the level of cross sections. Hence, only such new *finite* quantum corrections that emerge from UV-divergences will finally contribute to physical observables, such as cross sections, as new corrections introduced by the Renormalisation function in a theory with *spontaneously broken quantum scale symmetry.*

(iii) The idea behind Conjecture 5.1 and the reason for considering it to be true is the following: Contributions from real emission graphs have the same IR-divergences but with opposite sign as the virtual contributions from the corresponding loop diagrams in DReg-regularised theories at every order of the perturbation theory leading to the cancellation of all IR-divergences at the level of cross sections, as discussed above. In SIDReg-regularised theories, i.e. theories with spontaneously broken QSI, new divergent and finite quantum corrections can arise from these IR-divergences and evanescent interactions, as mentioned in Conjecture 5.1. If the evanescent interactions in real emission graphs are the same as in the corresponding loop diagrams, all of these new divergent and finite quantum corrections cancel after summing over virtual and real contributions, as conjectured, because the IR-divergences have opposite signs. There is no reason to think that evanescent interactions in real emission graphs differ from those in the corresponding loop diagrams as the same Feynman rules are used to derive the diagrams and the IR-divergences are the same (with opposite sign), which means that it is reasonable to think that the same interaction terms with the same evanescent corrections contribute to the loop diagrams and the real emission graphs.

In the following, Conjecture 5.1 will exemplarily be proven. It is important to note that this does *not* represent a rigorous prove that holds in a generic theory to all orders of the perturbation theory, however, is sufficient to illustrate the idea of Conjecture 5.1 and show that it is true for the considered case.

First, Conjecture 5.1 is trivially satisfied for massless QSI QED with $\lambda \equiv 0$ *and* $y_f \equiv 0$ at tree-level, i.e. scenario (i) in section 5.2. The reason for this is that in this case there are *no* new quantum corrections at all, as shown in subsection 5.2.1.

Now, it will be shown that Conjecture 5.1 is also satisfied for massless QSI QED with $y_f \neq 0$ and $\lambda \equiv 0$ at tree-level, i.e. scenario (ii) in section 5.2. However, for simplicity only the 1-loop muon vertex corrections contributing to the $e^- e^+ \longrightarrow \mu^- \mu^+$ scattering process are considered, as done in [32] for regular massless QED, and *not* the full scattering amplitude. Therefore, only final state real emission needs to be considered to cancel IR-divergences at the level of cross sections. Since considering only the 1-loop muon vertex corrections provides a scattering amplitude that admits the same structure as the full scattering amplitude, i.e. with non-vanishing second order and simple pole in $\epsilon_{\text{IR}}$ as well as new *divergent* and finite quantum corrections, it is sufficient to restrict the following analysis to the 1-loop muon vertex corrections in order to show the validity of Conjecture 5.1 in the present case. In particular, all Feynman diagrams contributing to the $e^- e^+ \longrightarrow \mu^- \mu^+$ scattering process at tree-level are provided in (F.1), whereas all considered 1-loop diagrams, i.e. diagrams with 1-loop muon vertex corrections, are illustrated in (F.2) and (F.3) in section F.1 of the appendix. In addition to these diagrams, one also needs to consider the muon Residue contribution, i.e. $\mathcal{M}^{(1)}_{\text{Res},\mu} = \delta R^{(1)}_{\mu} \mathcal{M}_{\text{tree}}$, as well as the 1-loop counterterm diagrams containing the $\gamma\mu\mu$ - and the $\mathfrak{D}\mu\mu$ - counterterm for the UV-Renormalisation. In order to cancel the IR-divergences at the level of cross sections, the tree-level cross sections for the scattering processes $e^- e^+ \longrightarrow \mu^- \mu^+ \gamma$ and $e^- e^+ \longrightarrow \mu^- \mu^+ \mathfrak{D}$, i.e. final state real emission graphs, need to be considered, as mentioned above. The tree-level Feynman diagrams for the scattering processes $e^- e^+ \longrightarrow \mu^- \mu^+ \gamma$ are illustrated in (F.4) and (F.5), whereas the tree-level Feynman diagrams for the scattering process $e^- e^+ \longrightarrow \mu^- \mu^+ \mathfrak{D}$ are shown in (F.6) and (F.7) in section F.1 of the appendix. Further, note that in the following Feynman gauge $\xi = 1$ is chosen.

**Remark.** In the following, during the derivation and in intermediate steps, factors of $\mu_0^{n\epsilon}$, for some $n \in \mathbb{Z}$, are suppressed for readability but are implicitly still present and will explicitly be shown only in the final results for the cross sections.

In order to calculate the cross sections for the above described scattering processes $e^- e^+ \longrightarrow \chi^* \longrightarrow X$, where only virtual and real corrections to the muon are considered, it is useful to note that these cross sections factorise into $e^- e^+ \longrightarrow \chi^*$ and $\chi^* \longrightarrow X$, where $\chi \in \{\gamma, \mathfrak{D}\}$ and $X \in \{\mu^- \mu^+, \mu^- \mu^+ \gamma, \mu^- \mu^+ \mathfrak{D}\}$. This has already been shown for regular massless QED in [32], however, it needs to be shown for the present case with non-vanishing Yukawa couplings, i.e. with 2 mediator particles $\{\gamma, \mathfrak{D}\}$. The reason for this is that in principle there are interference terms between photon and Dilaton mediated diagrams. Hence, it is useful to consider the following Lemma first.

**Lemma 5.1.**
*The spin-averaged interference between the electron current coupling to a photon $E_e^\mu := \overline{v}(p_2)\,\gamma^\mu\, u(p_1)$ and the electron current coupling to a Dilaton $E_y := \overline{v}(p_2)\, u(p_1)$ vanishes.*

*Proof.*

$$\begin{aligned} \left\langle E_e^\mu\, E_y^\dagger \right\rangle &= \frac{1}{4} \sum_{\text{spins}} E_e^\mu\, E_y^\dagger = \frac{1}{4} \sum_{\text{spins}} \overline{v}(p_2)\,\gamma^\mu\, u(p_1)\,\overline{u}(p_1)\, v(p_2) = \frac{1}{4}\,\text{Tr}\left[\not{p}_2\,\gamma^\mu\,\not{p}_1\right] \\ &= 0 \end{aligned} \tag{5.29}$$

and analogously for $\left\langle E^\dagger_{e,\mu}\, E_y \right\rangle = 0$. □

Now, consider the factorisation of the cross sections.

**Proposition 5.1** (Factorisation of the Cross Section).
*Considering a quantum scale invariant QED with $y_f \neq 0$ but $\lambda \equiv 0$ at tree-level with Lagrangian (4.29). The cross section for the scattering process $e^- e^+ \longrightarrow \mu^- \mu^+$, at tree-level and for 1-loop muon vertex corrections, factorises as described above, and thus may be written as*

$$\begin{aligned}\sigma_k\left(e\,e \to \mu\,\mu\right) &\\ &= \sigma_k\left(e\,e \to \gamma^* \to \mu\,\mu\right) + \sigma_k\left(e\,e \to \mathfrak{D}^* \to \mu\,\mu\right)\\ &= \frac{e^2}{2\,Q^3}\,\frac{D-2}{D-1}\,\Gamma_k\left(\gamma^* \to \mu\,\mu\right) + \left(1+\epsilon+\epsilon^2+\mathcal{O}\left(\epsilon^3\right)\right)^2 \frac{y_e^2}{2\,Q^3}\,\Gamma_k\left(\mathfrak{D}^* \to \mu\,\mu\right)\end{aligned} \tag{5.30}$$

*where $k \in \{\text{tree}, 1\text{L}\mu\}$, $Q = \sqrt{s}$ is the C.o.M. energy and $\Gamma_k$ are the corresponding decay widths. Further, the cross section for the scattering process $e^- e^+ \longrightarrow \mu^- \mu^+ \chi$, where $\chi \in \{\gamma, \mathfrak{D}\}$, at tree-level factorises analogously, and thus may be written as*

$$\begin{aligned}\sigma_{\text{tree}}\left(e\,e \to \mu\,\mu\,\chi\right) &= \sigma_{\text{tree}}\left(e\,e \to \gamma^* \to \mu\,\mu\,\chi\right) + \sigma_{\text{tree}}\left(e\,e \to \mathfrak{D}^* \to \mu\,\mu\,\chi\right)\\ &= \frac{e^2}{2\,Q^3}\,\frac{D-2}{D-1}\,\Gamma_{\text{tree}}\left(\gamma^* \to \mu\,\mu\,\chi\right)\\ &\quad + \left(1+\epsilon+\epsilon^2+\mathcal{O}\left(\epsilon^3\right)\right)^2 \frac{y_e^2}{2\,Q^3}\,\Gamma_{\text{tree}}\left(\mathfrak{D}^* \to \mu\,\mu\,\chi\right)\end{aligned} \tag{5.31}$$

*Proof.*
Starting with the scattering process $e^- e^+ \longrightarrow \mu^- \mu^+ \chi$ and equation (5.31). The tree-level scattering amplitude for $e^- e^+ \longrightarrow \mu^- \mu^+ \gamma$, with contributing Feynman diagrams (F.4) and (F.5), is given by

$$\begin{aligned}i\,\mathcal{M}_{\text{tree}}\left(e^-\,e^+ \to \mu^-\,\mu^+\,\gamma\right) &= i\,\frac{e^2}{Q^2}\,\overline{v}(p_2)\,\gamma_\mu\,u(p_1)\,\overline{u}(k_1)\,S_e^{\mu\nu}\,v(k_2)\,\varepsilon_\nu^*\\ &\quad + i\left(1+\epsilon+\epsilon^2+\mathcal{O}\left(\epsilon^3\right)\right)\frac{e\,y_e}{Q^2}\,\overline{v}(p_2)\,u(p_1)\,\overline{u}(k_1)\,S_y^{\nu}\,v(k_2)\,\varepsilon_\nu^*\end{aligned} \tag{5.32}$$

with

$$\begin{aligned}S_e^{\mu\nu} :=& -e\left[\gamma^\nu\,\frac{\not{k}_1+\not{k}_F}{\left(k_1+k_F\right)^2}\,\gamma^\mu - \gamma^\mu\,\frac{\not{k}_2+\not{k}_F}{\left(k_2+k_F\right)^2}\,\gamma^\nu\right]\\ &+ 2\left(\epsilon+2\,\epsilon^2+\mathcal{O}\left(\epsilon^3\right)\right)\frac{y_\mu}{e}\,\frac{\eta^{\mu\nu}\,k_F\cdot q + k_F^\nu\,q^\mu - k_F^\mu\,q^\nu}{w\,\left(k_1+k_2\right)^2}\\ S_y^{\nu} :=& -2\left(\epsilon+\epsilon^2+\mathcal{O}\left(\epsilon^3\right)\right)\gamma_\rho\,\frac{\eta^{\nu\rho}\,k_F\cdot\left(k_1+k_2\right) + k_F^\nu\left(k_1+k_2\right)^\rho - k_F^\rho\left(k_1+k_2\right)^\nu}{w\,\left(k_1+k_2\right)^2}\\ &+ \left(1+\epsilon+\epsilon^2+\mathcal{O}\left(\epsilon^3\right)\right)y_\mu\left[\gamma^\nu\,\frac{\not{k}_1+\not{k}_F}{\left(k_1+k_F\right)^2} - \frac{\not{k}_2+\not{k}_F}{\left(k_2+k_F\right)^2}\,\gamma^\nu\right]\end{aligned} \tag{5.33}$$

where the specific kinematic of such processes is provided in (F.8) to (F.12) in section F.1 of the appendix. Hence, the tree-level cross section for $e^- e^+ \longrightarrow \mu^- \mu^+ \gamma$ with scattering amplitude (5.32) is provided by

$$\begin{aligned}\sigma_{\text{tree}}\left(e^- e^+ \to \mu^- \mu^+ \gamma\right) &= \frac{1}{2\,Q^2}\int d\Phi_3\,\left\langle\left|\mathcal{M}_{\text{tree}}\left(e^-\,e^+ \to \mu^-\,\mu^+\,\gamma\right)\right|^2\right\rangle \\ &= \frac{e^4}{2\,Q^6}\,L_e^{\mu\nu}\,X_{e,\mu\nu} + \left(1+\epsilon+\epsilon^2+\mathcal{O}\left(\epsilon^3\right)\right)^2\,\frac{e^2\,y_e^2}{2\,Q^6}\,L_y\,X_y\end{aligned} \tag{5.34}$$

where interference terms between photon and Dilaton mediated contributions vanish due to Lemma 5.1 and with

$$\begin{aligned}L_e^{\mu\nu} &:= \frac{1}{4}\sum_{\text{spins}}\overline{v}(p_2)\,\gamma^\mu\,u(p_1)\,\overline{u}(p_1)\,\gamma^\nu\,v(p_2) = \frac{1}{4}\,\text{Tr}\left[\not{p}_2\,\gamma^\mu\,\not{p}_1\,\gamma^\nu\right] \\ &= p_1^\mu\,p_2^\nu + p_1^\nu\,p_2^\mu - \frac{1}{2}\,Q^2\,\eta^{\mu\nu} \\ X_e^{\mu\nu} &:= \int d\Phi_3\sum_{r,s,t}\overline{u}_r(k_1)\,S_e^{\mu\alpha}\,v_s(k_2)\,\overline{v}_s(k_2)\,\overline{S}_e^{\nu\beta}\,u_r(k_1)\,\varepsilon^*_{t,\alpha}\,\varepsilon_{t,\beta} \\ &= -\int d\Phi_3\,\text{Tr}\left[\not{k}_1\,S_e^{\mu\alpha}\,\not{k}_2\,\overline{S}^\nu_{e,\alpha}\right]\end{aligned} \tag{5.35}$$

where $\overline{S}_{e,\mu\nu} := \gamma^0\,\overline{S}^\dagger_{e,\mu\nu}\,\gamma^0$, and analogously for any other matrix, as well as

$$\begin{aligned}L_y &:= \frac{1}{4}\sum_{\text{spins}}\overline{v}(p_2)\,u(p_1)\,\overline{u}(p_1)\,v(p_2) = \frac{1}{4}\,\text{Tr}\left[\not{p}_2\,\not{p}_1\right] = p_1\cdot p_2 = \frac{Q^2}{2} \\ X_y &:= \int d\Phi_3\sum_{r,s,t}\overline{u}_r(k_1)\,S_y^\mu\,v_s(k_2)\,\overline{v}_s(k_2)\,\overline{S}_y^\nu\,u_r(k_1)\,\varepsilon^*_{t,\mu}\,\varepsilon_{t,\nu} \\ &= -\int d\Phi_3\,\text{Tr}\left[\not{k}_1\,S_y^\mu\,\not{k}_2\,\overline{S}_{y,\mu}\right]\end{aligned} \tag{5.36}$$

Note further that $q_\mu\,X_e^{\mu\nu} = 0$ due to the Ward identity of the mediating photon and that $X_e^{\mu\nu}$ is a Lorentz-covariant function *only* of $q^\mu$ as it is integrated over the other momenta [32], and thus must have the form

$$X_e^{\mu\nu} = \left(q^\mu\,q^\nu - q^2\,\eta^{\mu\nu}\right)\,X_e(q^2) \tag{5.37}$$

Using (5.37) and the explicit form of $L_e^{\mu\nu}$ in (5.35), one obtains

$$L_e^{\mu\nu}\,X_{e,\mu\nu} = -\frac{1}{2}\,\frac{D-2}{D-1}\,Q^2\,\eta^{\mu\nu}\,X_{e,\mu\nu} \tag{5.38}$$

Now, considering the decay $\gamma^* \longrightarrow \mu\,\mu\,\gamma$ and its explicit amplitude

$$i\,\mathcal{M}_{\text{tree}}\left(\gamma^* \to \mu\,\mu\,\gamma\right) = -i\,e\,\overline{u}(k_1)\,S_e^{\mu\nu}\,v(k_2)\,\varepsilon^*_\nu\,\varepsilon_\mu \tag{5.39}$$

one finds

$$\Gamma_{\text{tree}}\left(\gamma^* \to \mu\,\mu\,\gamma\right) = \frac{1}{2\,Q}\int d\Phi_3\sum_{r,s,t_1,t_2}\left|\mathcal{M}_{\text{tree}}\left(\gamma^* \to \mu\,\mu\,\gamma\right)\right|^2 = -\frac{e^2}{2\,Q}\,\eta^{\mu\nu}\,X_{e,\mu\nu} \tag{5.40}$$

and analogously for $\mathfrak{D}^* \longrightarrow \mu\,\mu\,\gamma$

$$i\,\mathcal{M}_{\text{tree}}\left(\mathfrak{D}^* \to \mu\,\mu\,\gamma\right) = i\,e\,\overline{u}(k_1)\,S_y^\mu\,v(k_2)\,\varepsilon_\mu^* \tag{5.41}$$

$$\Gamma_{\text{tree}}\left(\mathfrak{D}^* \to \mu\,\mu\,\gamma\right) = \frac{1}{2\,Q}\int d\Phi_3 \sum_{r,s,t}\Big|\mathcal{M}_{\text{tree}}\left(\mathfrak{D}^* \to \mu\,\mu\,\gamma\right)\Big|^2 = \frac{e^2}{2\,Q}\,X_y \tag{5.42}$$

Hence, due to Lemma 5.1 the cross section for $e^-\,e^+ \longrightarrow \mu^-\,\mu^+\,\gamma$ can be partitioned into one photon and one Dilaton mediated cross section, as shown in (5.34), and then due to the results (5.35) to (5.42) these two cross sections can be factorised and expressed in terms of the corresponding decay widths, i.e.

$$\begin{aligned}\sigma_{\text{tree}}\left(e\,e \to \mu\,\mu\,\gamma\right) &= \sigma_{\text{tree}}\left(e\,e\, \to \gamma^* \to \mu\,\mu\,\gamma\right) + \sigma_{\text{tree}}\left(e\,e \to \mathfrak{D}^* \to \mu\,\mu\,\gamma\right)\\ &= \frac{e^2}{2\,Q^3}\,\frac{D-2}{D-1}\,\Gamma_{\text{tree}}\left(\gamma^* \to \mu\,\mu\,\gamma\right)\\ &\quad + \left(1+\epsilon+\epsilon^2+\mathcal{O}\left(\epsilon^3\right)\right)^2\frac{y_e^2}{2\,Q^3}\,\Gamma_{\text{tree}}\left(\mathfrak{D}^* \to \mu\,\mu\,\gamma\right)\end{aligned} \tag{5.43}$$

which is exactly (5.31) for $\chi = \gamma$. Analogously, following the same steps as above one finds that (5.31) also holds exactly for $\chi = \mathfrak{D}$, i.e. the scattering process $e^-e^+ \to \mu^-\mu^+\mathfrak{D}$ with contributing Feynman diagrams (F.6) and (F.7).

Continuing with the scattering process $e^-\,e^+ \longrightarrow \mu^-\,\mu^+$ and equation (5.30). At tree-level the diagrams (F.1) contribute to this scattering process, and at the 1-loop level only 1-loop muon vertex corrections are considered and provided in (F.2) for photon mediated as well as in (F.3) for Dilaton mediated contributions. In both cases the electron currents are exactly the same as before, and thus Lemma 5.1 can still be used yielding

$$\begin{aligned}\sigma_{\text{tree}}\left(e^-e^+ \to \mu^-\mu^+\right) &= \frac{1}{2\,Q^2}\int d\Phi_2\,\left\langle\left|\mathcal{M}_{\text{tree}}\left(e^-\,e^+ \to \mu^-\,\mu^+\right)\right|^2\right\rangle\\ &= \frac{e^4}{2\,Q^6}\,L_e^{\mu\nu}\,Y_{e,\mu\nu} + \left(1+\epsilon+\epsilon^2+\mathcal{O}\left(\epsilon^3\right)\right)^2\frac{e^2\,y_e^2}{2\,Q^6}\,L_y\,Y_y\end{aligned} \tag{5.44}$$

$$\begin{aligned}\sigma_{1\text{L},\mu}\left(e^-e^+ \to \mu^-\mu^+\right) &= \frac{1}{2\,Q^2}\int d\Phi_2\,\left\langle\left|\mathcal{M}\left(e^-\,e^+ \to \mu^-\,\mu^+\right)\right|^2\right\rangle_{1\text{L},\mu}\\ &= \frac{e^4}{2\,Q^6}\,L_e^{\mu\nu}\,Z_{e,\mu\nu} + \left(1+\epsilon+\epsilon^2+\mathcal{O}\left(\epsilon^3\right)\right)^2\frac{e^2\,y_e^2}{2\,Q^6}\,L_y\,Z_y\end{aligned} \tag{5.45}$$

where $\left\langle|\mathcal{M}|^2\right\rangle_{1\text{L},\mu}$ as in (5.11) containing only 1-loop muon vertex corrections. Hence, the cross section can again be partitioned into one photon and one Dilaton mediated cross section. Now, following the same steps as above, one finds

$$\begin{aligned}\Gamma_{\text{tree}}\left(\gamma^* \to \mu\,\mu\right) &= -\frac{e^2}{2\,Q}\,\eta^{\mu\nu}\,Y_{e,\mu\nu}, &\qquad \Gamma_{\text{tree}}\left(\mathfrak{D}^* \to \mu\,\mu\right) &= \frac{e^2}{2\,Q}\,Y_y\\ \Gamma_{1\text{L},\mu}\left(\gamma^* \to \mu\,\mu\right) &= -\frac{e^2}{2\,Q}\,\eta^{\mu\nu}\,Z_{e,\mu\nu}, &\qquad \Gamma_{1\text{L},\mu}\left(\mathfrak{D}^* \to \mu\,\mu\right) &= \frac{e^2}{2\,Q}\,Z_y\end{aligned} \tag{5.46}$$

Noting that (5.37), and thus (5.38) analogously hold for $Y_e^{\mu\nu}$ and $Z_e^{\mu\nu}$, one ultimately arrives at (5.30) for $\sigma_{\text{tree}}\left(e^- e^+ \to \mu^- \mu^+\right)$ and $\sigma_{1\text{L},\mu}\left(e^- e^+ \to \mu^- \mu^+\right)$. □

**Remark.**
Calculating the cross sections via the associated decay widths simplifies the evaluation of the phase space integrals. In the case of $e^- e^+ \longrightarrow \mu^- \mu^+$ it is useful to note that there is no angular dependence in the spin-summed $\gamma^* \longrightarrow \mu\,\mu$ as well as the spin-summed $\mathfrak{D}^* \longrightarrow \mu\,\mu$ [32], and thus the 2 body phase space integral can trivially be evaluated as in (F.13). Hence, for the considered 1-loop contributions

$$\begin{aligned}\Gamma_{1\text{L},\mu}\left(\gamma^* \to \mu\mu\right) &= \frac{1}{2\,Q}\int d\Phi_2\; 2\,\text{Re}\left(\sum_{r,s,t} \mathcal{M}^\dagger_{\text{tree}} \mathcal{M}_{1\text{L}\mu,\text{ren}}\left(\gamma^* \to \mu\mu\right)\right)\\ &= \frac{1}{2\,Q}\left(\frac{4\,\pi}{Q^2}\right)^{\frac{4-D}{2}} \frac{2^{-D}}{\sqrt{\pi}\,\Gamma\left(\frac{D-1}{2}\right)}\, 2\,\text{Re}\left(\sum_{r,s,t} \mathcal{M}^\dagger_{\text{tree}} \mathcal{M}_{1\text{L}\mu,\text{ren}}\left(\gamma^* \to \mu\mu\right)\right)\end{aligned} \tag{5.47}$$

and similarly for $\Gamma_{1\text{L},\mu}\left(\mathfrak{D}^* \to \mu\mu\right)$ as well as the tree-level contributions.

Using $s = Q^2$, the tree-level cross section, in 4 dimensions, is provided by

$$\sigma_{\text{tree}}\left(ee \to \mu\mu\right) = \frac{e^4}{12\,\pi\,s} + \frac{y_e^2\,y_\mu^2}{16\,\pi\,s} \tag{5.48}$$

For the considered 1-loop contributions to the cross section as well as for the $2 \longrightarrow 3$ cross sections, the result will be given in a similar form as in (5.12), i.e.

$$\sigma_k = \sigma_k^{1/\epsilon_{\text{IR}}^2} + \sigma_k^{1/\epsilon_{\text{IR}}} + \Delta_{\text{IR}}\,\sigma_k^{1/\epsilon_{\text{IR}}} + \sigma_k^{\text{fin}} + \Delta_{\text{UV}}\,\sigma_k^{\text{fin}} + \Delta_{\text{IR}}\,\sigma_k^{\text{fin}} + \mathcal{O}\left(\epsilon\right) \tag{5.49}$$

Thus, the 1-loop cross section for $e^- e^+ \longrightarrow \mu^- \mu^+$, containing only 1-loop muon vertex corrections, in the $\overline{\text{MS}}$-scheme, i.e. having used (3.23), may then be given as

$$\sigma_{1\text{L},\mu}^{1/\epsilon_{\text{IR}}^2}\left(ee \to \mu\mu\right) = -\frac{\mu_0^{2\epsilon}}{192\,\pi^3\,s}\left(4\,e^6 + 3\,e^2\,y_e^2\,y_\mu^2\right)\frac{1}{\epsilon_{\text{IR}}^2} \tag{5.50}$$

$$\begin{aligned}\sigma_{1\text{L},\mu}^{1/\epsilon_{\text{IR}}}\left(ee \to \mu\mu\right) = -\,\frac{\mu_0^{2\epsilon}}{2304\,\pi^3\,s}\Bigg[&104\,e^6 - 12\,e^4\,y_\mu^2 + 126\,e^2\,y_e^2\,y_\mu^2 - 9\,y_e^2\,y_\mu^4\\ &- 24\left(4\,e^6 + 3\,e^2\,y_e^2\,y_\mu^2\right)\log\left(\frac{s}{\mu_0^2}\right)\Bigg]\frac{1}{\epsilon_{\text{IR}}}\end{aligned} \tag{5.51}$$

$$\Delta_{\text{IR}}\,\sigma_{1\text{L},\mu}^{1/\epsilon_{\text{IR}}}\left(ee \to \mu\mu\right) = -\,\frac{\mu_0^{2\epsilon}}{16\,\pi^3\,s}\,e^2\,y_e^2\,y_\mu^2\,\frac{1}{\epsilon_{\text{IR}}} \tag{5.52}$$

$$\begin{aligned}\sigma_{1\text{L},\mu}^{\text{fin}}\left(ee \to \mu\mu\right) = &\frac{\mu_0^{2\epsilon}}{3456\,\pi^3\,s}\Bigg[\left(60\,\pi^2 - 464\right)e^6 + 30\,e^4\,y_\mu^2\\ &\qquad + 9\left(5\,\pi^2 - 48\right)e^2\,y_e^2\,y_\mu^2 - 27\,y_e^2\,y_\mu^4\Bigg]\\ &+ \frac{\mu_0^{2\epsilon}}{2304\,\pi^3\,s}\left[208\,e^6 - 24\,e^4\,y_\mu^2 + 198\,e^2\,y_e^2\,y_\mu^2 + 9\,y_e^2\,y_\mu^4\right]\log\left(\frac{s}{\mu_0^2}\right)\\ &- \frac{\mu_0^{2\epsilon}}{96\,\pi^3\,s}\left[4\,e^6 + 3\,e^2\,y_e^2\,y_\mu^2\right]\log^2\left(\frac{s}{\mu_0^2}\right)\end{aligned} \tag{5.53}$$

$$\Delta_{\mathrm{UV}}\,\sigma^{\mathrm{fin}}_{1\mathrm{L},\mu}\,(ee \to \mu\mu) = -\,\frac{3\,\mu_0^{2\epsilon}}{128\,\pi^3\,s}\,y_e^2\,y_\mu^4 \tag{5.54}$$

$$\begin{aligned}\Delta_{\mathrm{IR}}\,\sigma^{\mathrm{fin}}_{1\mathrm{L},\mu}\,(ee \to \mu\mu) = {}& \frac{\mu_0^{2\epsilon}}{384\,\pi^3\,s}\left[4\,e^4\,y_\mu^2 - 84\,e^2\,y_e^2\,y_\mu^2 + 9\,y_e^2\,y_\mu^4\right] \\ &+ \frac{\mu_0^{2\epsilon}}{8\,\pi^3\,s}\,e^2\,y_e^2\,y_\mu^2\,\log\left(\frac{s}{\mu_0^2}\right) - \frac{5\,\mu_0^{2\epsilon}}{32\,\pi^3\,s}\,e^2\,y_e^2\,y_\mu^2\end{aligned} \tag{5.55}$$

Since the 3 body phase space integral cannot be evaluated as trivial as the 2 body phase space integral above, the spin-summed squared amplitudes for the decays $\gamma^* \to \mu\,\mu\,\chi$ and $\mathfrak{D}^* \to \mu\,\mu\,\chi$, where $\chi \in \{\gamma, \mathfrak{D}\}$, are considered first. Using the definitions of energy fractions $x_i$, $i \in \{1, 2, F\}$, in the C.o.M. frame in (F.10), one finds

$$\begin{aligned}\sum_{r,s,t_1,t_2} \left|\mathcal{M}_{\mathrm{tree}}\,(\gamma^* \to \mu\mu\gamma)\,\right|^2 = {}& 4\,e^4\,(D-2)\,\frac{x_1^2 + x_2^2 + \frac{D-4}{2}\,x_F^2}{(1-x_1)\,(1-x_2)} \\ &+ 2\left(\epsilon + 2\epsilon^2 + \mathcal{O}(\epsilon^3)\right)^2 y_\mu^2\,\frac{Q^2}{w^2}\,(D-2)\,\frac{x_F^2}{1-x_F}\end{aligned} \tag{5.56}$$

$$\begin{aligned}&\sum_{r,s,t} \left|\mathcal{M}_{\mathrm{tree}}\,(\mathfrak{D}^* \to \mu\mu\gamma)\,\right|^2 \\ &= 2\left(1 + \epsilon + \epsilon^2 + \mathcal{O}(\epsilon^3)\right)^2 e^2\,y_\mu^2 \frac{(D-2)\,x_F^2 + 4\,(1-x_F)}{(1-x_1)\,(1-x_2)} \\ &+ 2\left(\epsilon + \epsilon^2 + \mathcal{O}(\epsilon^3)\right)^2 e^2\,\frac{Q^2}{w^2}\,\frac{(D-2)\,x_F^2 + 4\,x_2^2 - 4\,x_2\,(2-x_F) + 4\,(1-x_F)}{1-x_F}\end{aligned} \tag{5.57}$$

$$\begin{aligned}&\sum_{r,s,t} \left|\mathcal{M}_{\mathrm{tree}}\,(\gamma^* \to \mu\mu\mathfrak{D})\,\right|^2 \\ &= 2\left(1 + \epsilon + \epsilon^2 + \mathcal{O}(\epsilon^3)\right)^2 e^2\,y_\mu^2 \frac{(D-2)\,x_F^2}{(1-x_1)\,(1-x_2)} \\ &+ 2\left(\epsilon + \epsilon^2 + \mathcal{O}(\epsilon^3)\right)^2 e^2\,\frac{Q^2}{w^2}\,\frac{D\,(2-x_F)^2 + 4x_2^2 - 4x_2\,(2-x_F) - 2\,(2 - 2x_F + x_F^2)}{1-x_F}\end{aligned} \tag{5.58}$$

$$\begin{aligned}\sum_{r,s} \left|\mathcal{M}_{\mathrm{tree}}\,(\mathfrak{D}^* \to \mu\mu\mathfrak{D})\,\right|^2 = {}& 2\left(1 + 2\epsilon + 3\epsilon^2 + \mathcal{O}(\epsilon^3)\right)^2 y_\mu^4 \frac{(x_1-x_2)^2}{(1-x_1)\,(1-x_2)} \\ &+ 2\left(\epsilon + 2\epsilon^2 + \mathcal{O}(\epsilon^3)\right)^2 y_\mu^2\,\frac{Q^2}{w^2}\,(1-x_F)\end{aligned} \tag{5.59}$$

All of the above expressions *either* are already in the form of one of the integrands provided in (F.16) to (F.24) (without the factor (F.15) coming from the 3 body phase space integral) *or* can be rewritten as a combination of these integrands using the relation

(F.12), e.g.

$$\begin{aligned}\frac{(x_1-x_2)^2}{(1-x_1)\,(1-x_2)} &= \frac{2}{D-2}\,\frac{x_1^2+x_2^2+\frac{D-4}{2}\,x_F^2}{(1-x_1)\,(1-x_2)} - 4\,\frac{D-4}{D-2}\,\frac{1}{1-x_2} \\ &\quad + 2\,\frac{D-4}{D-2}\,\frac{x_2\,(2-x_1)}{(1-x_1)\,(1-x_2)} - 2\,\frac{x_1\,x_2}{(1-x_1)\,(1-x_2)} \\ \frac{(D-2)\,x_F^2}{(1-x_1)\,(1-x_2)} &= 2\,\frac{x_1^2+x_2^2+\frac{D-4}{2}\,x_F^2}{(1-x_1)\,(1-x_2)} + 8\,\frac{1}{1-x_2} - 4\,\frac{x_2\,(2-x_1)}{(1-x_1)\,(1-x_2)}\end{aligned} \tag{5.60}$$

and analogously for the other expressions. Using the integrals in (F.16) to (F.24), the 3 body phase space integral (F.14) over the spin-summed squared amplitudes (5.56) to (5.59) can explicitly be evaluated to obtain the corresponding decay widths and using (5.31) in Proposition 5.1 one ultimately obtains the tree-level cross sections for the scattering processes $e^-\,e^+ \longrightarrow \mu^-\,\mu^+\,\chi$, where $\chi \in \{\gamma, \mathfrak{D}\}$. Thus, the tree-level cross section for $e^-\,e^+ \longrightarrow \mu^-\,\mu^+\,\gamma$, written in the form (5.49), then reads as follows

$$\sigma_{\text{tree}}^{1/\epsilon_{\text{IR}}^2}\,(ee \to \mu\mu\gamma) = \frac{\mu_0^{2\epsilon}}{192\,\pi^3\,s}\Big(4\,e^6 + 3\,e^2\,y_e^2\,y_\mu^2\Big)\frac{1}{\epsilon_{\text{IR}}^2} \tag{5.61}$$

$$\begin{aligned}\sigma_{\text{tree}}^{1/\epsilon_{\text{IR}}}\,(ee \to \mu\mu\gamma) = \frac{\mu_0^{2\epsilon}}{1152\,\pi^3\,s}\Bigg[&52\,e^6 + 63\,e^2\,y_e^2\,y_\mu^2 \\ &- 12\,\Big(4\,e^6 + 3\,e^2\,y_e^2\,y_\mu^2\Big)\log\left(\frac{s}{\mu_0^2}\right)\Bigg]\frac{1}{\epsilon_{\text{IR}}}\end{aligned} \tag{5.62}$$

$$\Delta_{\text{IR}}\,\sigma_{\text{tree}}^{1/\epsilon_{\text{IR}}}\,(ee \to \mu\mu\gamma) = \frac{\mu_0^{2\epsilon}}{16\,\pi^3\,s}\,e^2\,y_e^2\,y_\mu^2\,\frac{1}{\epsilon_{\text{IR}}} \tag{5.63}$$

$$\begin{aligned}\sigma_{\text{tree}}^{\text{fin}}\,(ee \to \mu\mu\gamma) = &-\frac{\mu_0^{2\epsilon}}{6912\,\pi^3\,s}\Big[4\,\Big(30\,\pi^2 - 259\Big)\,e^6 + 9\,\Big(10\,\pi^2 - 147\Big)\,e^2\,y_e^2\,y_\mu^2\Big] \\ &- \frac{\mu_0^{2\epsilon}}{576\,\pi^3\,s}\Big[52\,e^6 + 63\,e^2\,y_e^2\,y_\mu^2\Big]\log\left(\frac{s}{\mu_0^2}\right) \\ &+ \frac{\mu_0^{2\epsilon}}{96\,\pi^3\,s}\Big[4\,e^6 + 3\,e^2\,y_e^2\,y_\mu^2\Big]\log^2\left(\frac{s}{\mu_0^2}\right)\end{aligned} \tag{5.64}$$

$$\Delta_{\text{UV}}\,\sigma_{\text{tree}}^{\text{fin}}\,(ee \to \mu\mu\gamma) = 0 \tag{5.65}$$

$$\begin{aligned}\Delta_{\text{IR}}\,\sigma_{\text{tree}}^{\text{fin}}\,(ee \to \mu\mu\gamma) = &\frac{7\,\mu_0^{2\epsilon}}{32\,\pi^3\,s}\,e^2\,y_e^2\,y_\mu^2 - \frac{\mu_0^{2\epsilon}}{8\,\pi^3\,s}\,e^2\,y_e^2\,y_\mu^2\,\log\left(\frac{s}{\mu_0^2}\right) \\ &+ \frac{5\,\mu_0^{2\epsilon}}{32\,\pi^3\,s}\,e^2\,y_e^2\,y_\mu^2\end{aligned} \tag{5.66}$$

Similarly, the tree-level cross section for $e^-\,e^+ \longrightarrow \mu^-\,\mu^+\,\mathfrak{D}$ is found to be

$$\sigma_{\text{tree}}^{1/\epsilon_{\text{IR}}^2}\,(ee \to \mu\mu\mathfrak{D}) = 0 \tag{5.67}$$

$$\sigma_{\text{tree}}^{1/\epsilon_{\text{IR}}}\,(ee \to \mu\mu\mathfrak{D}) = -\frac{\mu_0^{2\epsilon}}{768\,\pi^3\,s}\Big(4\,e^4\,y_\mu^2 + 3\,y_e^2\,y_\mu^4\Big)\frac{1}{\epsilon_{\text{IR}}} \tag{5.68}$$

$$\Delta_{\mathrm{IR}}\, \sigma_{\mathrm{tree}}^{1/\epsilon_{\mathrm{IR}}} \left(ee \to \mu\mu\mathfrak{D}\right) = 0 \tag{5.69}$$

$$\begin{aligned}\sigma_{\mathrm{tree}}^{\mathrm{fin}} \left(ee \to \mu\mu\mathfrak{D}\right) = &-\frac{\mu_0^{2\epsilon}}{4608\,\pi^3\, s}\Big(76\, e^4\, y_\mu^2 + 117\, y_e^2\, y_\mu^4\Big) \\ &+\frac{\mu_0^{2\epsilon}}{384\,\pi^3\, s}\left[4\, e^4\, y_\mu^2 + 3\, y_e^2\, y_\mu^4\right] \log\left(\frac{s}{\mu_0^2}\right)\end{aligned} \tag{5.70}$$

$$\Delta_{\mathrm{UV}}\, \sigma_{\mathrm{tree}}^{\mathrm{fin}} \left(ee \to \mu\mu\mathfrak{D}\right) = 0 \tag{5.71}$$

$$\Delta_{\mathrm{IR}}\, \sigma_{\mathrm{tree}}^{\mathrm{fin}} \left(ee \to \mu\mu\mathfrak{D}\right) = -\frac{\mu_0^{2\epsilon}}{384\,\pi^3\, s}\Big(4\, e^4\, y_\mu^2 + 9\, y_e^2\, y_\mu^4\Big) \tag{5.72}$$

Finally, the total cross section is given by

$$\begin{aligned}\sigma_{\mathrm{total},\mu} \left(ee \to \mu\mu\right) &= \sigma_{\mathrm{tree}} \left(ee \to \mu\mu\right) + \sigma_{\mathrm{1L},\mu} \left(ee \to \mu\mu\right) \\ &\quad + \sigma_{\mathrm{tree}} \left(ee \to \mu\mu\gamma\right) + \sigma_{\mathrm{tree}} \left(ee \to \mu\mu\mathfrak{D}\right) + \mathcal{O}\left(\alpha_i^4\right) \\ &= \sigma_{\mathrm{tree}} \left(ee \to \mu\mu\right) + \sigma_{\mathrm{total,1L}\mu} \left(ee \to \mu\mu\right) + \mathcal{O}\left(\alpha_i^4\right)\end{aligned} \tag{5.73}$$

where $\alpha_i$ are the finestructure constants for $e$ and $y_i$. The full tree-level cross section is to be found in (5.48), whereas the considered 1-loop contribution in the $\overline{\mathrm{MS}}$-scheme and in 4 dimensions is given by

$$\sigma_{\mathrm{total,1L}\mu} \left(ee \to \mu\mu\right) = \sigma_{\mathrm{total,1L}\mu}^{\mathrm{Reg}} \left(ee \to \mu\mu\right) + \Delta_{\mathrm{UV}}\, \sigma_{\mathrm{total,1L}\mu} \left(ee \to \mu\mu\right) \tag{5.74}$$

with the regular 1-loop contribution

$$\begin{aligned}\sigma_{\mathrm{total,1L}\mu}^{\mathrm{Reg}} \left(ee \to \mu\mu\right) = &\frac{1}{512\,\pi^3\, s}\Big(8\, e^6 - 4\, e^4\, y_\mu^2 + 34\, e^2\, y_e^2\, y_\mu^2 - 17\, y_e^2\, y_\mu^4\Big) \\ &- \frac{1}{256\,\pi^3\, s}\left[6\, e^2\, y_e^2\, y_\mu^2 - 3\, y_e^2\, y_\mu^4\right] \log\left(\frac{s}{\mu_0^2}\right)\end{aligned} \tag{5.75}$$

and the new finite quantum correction that emerged from UV-divergences

$$\Delta_{\mathrm{UV}}\, \sigma_{\mathrm{total,1L}\mu} \left(ee \to \mu\mu\right) = -\frac{3}{128\,\pi^3\, s}\, y_e^2\, y_\mu^4 \tag{5.76}$$

Thus, the considered 1-loop contribution to cross section (5.74), containing virtual and real corrections, is UV- *and* IR-finite, as proposed.

**Remark.**

(i) As proposed above, the 1-loop cross section for $e^-\, e^+ \longrightarrow \mu^-\, \mu^+$, containing only 1-loop muon vertex corrections, contains new finite *and* divergent quantum corrections arising from IR-divergences as shown in (5.55) and (5.52), respectively.

(ii) The last term in (5.55) and (5.66) emerged from the second order IR-pole, i.e. from $1/\epsilon_{\mathrm{IR}}^2$, whereas the other terms in (5.55), (5.66) and (5.72) emerge from the simple IR-pole, i.e. from $1/\epsilon_{\mathrm{IR}}$.

(iii) The result of the total 1-loop contribution to the cross section, i.e. the order $\alpha_i^3$ contributions containing virtual and real corrections, is partitioned into a regular contribution (5.75) that also would have been obtained in the DReg-regularised theory and a new finite contribution (5.76) due to evanescent interactions in the QSI theory cancelling UV-divergences, i.e. that is only obtained in the SIDReg-regularised theory. Moreover, the results in (5.75) and (5.76) are given in 4 dimensions as they are finite.

(iv) In (5.74) to (5.76) it can be seen that the considered 1-loop contribution to the cross section of $e^- e^+ \longrightarrow \mu^- \mu^+$ scattering indeed is IR-finite after summing over the contributing final state real emission contributions. Thus, *all* IR-divergences, even the new IR-divergences, cf. (5.52), emerging as a result of evanescent interactions due to QSI, cancel after summing over the virtual and real corrections to the cross section. Moreover, *all* new *finite* quantum corrections to the cross section arising from evanescent interactions cancelling IR-divergences, cf. (5.55), cancel as well. Therefore, *only* new finite quantum corrections emerging from UV-divergences, cf. (5.54), remain as new finite contributions to the cross section as shown in (5.76). Thus, Conjecture 5.1 is true for the present case of QSI QED with non-vanishing Yukawa couplings.

(v) The remaining new finite quantum correction to the cross section given in (5.76) is suppressed by Yukawa couplings to the power of 6, and thus is suppressed by the VEV of the Dilaton to the power of 6 as $y_f = m_f/w$ in the present theory, cf. (5.28).

(vi) Taking the limit of vanishing Yukawa couplings, i.e. $y_i \to 0$, one obtains

$$\sigma_{\text{total},\mu}\left(ee \to \mu\mu\right)\Big|_{y_i \to 0} = \frac{e^4}{12\,\pi\,s} + \frac{e^6}{64\,\pi^3\,s} + \mathcal{O}\left(\alpha^4\right), \tag{5.77}$$

and thus exactly reproduces the result provided in [32], as expected.

# 6. QSI Standard Model

In this chapter a complete *quantum scale invariant* Standard Model is provided. In [13] only the Higgs potential of a QSI Standard Model has been discussed. The purpose of this chapter, however, is to discuss the complete Lagrangian of a QSI Standard Model in full detail, which has not yet been done in the literature so far.

The Lagrangian is formulated in the way that the Renormalisation function $\mu(\sigma)$, but *not* the dimensionless gauge coupling, is absorbed into the gauge fields, such that the gauge fields always have mass dimension 1, as discussed in section 4.1. Hence, for a generic gauge field $A^a_\mu$, one obtains

$$\begin{aligned} &\underline{D=4}: && A^a_\mu \longrightarrow \hat{A}^a_\mu = g\, A^a_\mu = g\, \overline{A}^a_\mu && \Leftrightarrow \quad \overline{A}^a_\mu = A^a_\mu \\ &\underline{D=4-2\epsilon}: && A^a_\mu \longrightarrow \hat{A}^a_\mu = g\, \mu^\epsilon(\sigma)\, A^a_\mu = g\, \overline{A}^a_\mu && \Leftrightarrow \quad \overline{A}^a_\mu = \mu^\epsilon(\sigma)\, A^a_\mu \end{aligned} \tag{6.1}$$

with mass dimensions $[A^a_\mu] = 1-\epsilon$, $[\hat{A}^a_\mu] = 1$ and $[\overline{A}^a_\mu] = 1$. The QSI Standard Model Lagrangian is formulated in terms of $\overline{A}^a_\mu$. However, the "overbar" is dropped in the following for readability, i.e. $\overline{A}^a_\mu \to A^a_\mu$. Note that working with $\overline{A}^a_\mu$ instead of $\hat{A}^a_\mu$ has the advantage that the Lagrangian takes the usual form w.r.t. to the gauge couplings while it is still BRST invariant, as discussed in chapter 4.

The Lagrangian of the complete QSI Standard Model consists in principle of 4 parts, the gauge sector, the fermion sector, the Higgs sector and the Yukawa sector. In addition to that, a gauge-fixing and a ghost sector need to be considered in the quantised theory. Thus, the Lagrangian is given by

$$\begin{aligned} \mathcal{L}_{\text{QSISM},cl} &= \mathcal{L}^{\text{QSI}}_{\text{Gauge}} + \mathcal{L}^{\text{QSI}}_{\text{Fermion}} + \mathcal{L}^{\text{QSI}}_{\text{Higgs}} + \mathcal{L}^{\text{QSI}}_{\text{Yukawa}} \\ \mathcal{L}_{\text{QSISM}} &= \mathcal{L}_{\text{QSISM},cl} + \mathcal{L}^{\text{QSI}}_{\text{GF}} + \mathcal{L}^{\text{QSI}}_{\text{Ghost}} \end{aligned} \tag{6.2}$$

## 6.1. Gauge Field Lagrangian

The gauge field Lagrangian for the QSI Standard Model with rescaled gauge fields, i.e. the Renormalisation function $\mu(\sigma)$ being absorbed into the gauge fields, may be written in terms of the corresponding field strength tensors as

$$\mathcal{L}^{\text{QSI}}_{\text{Gauge}} = -\frac{1}{4}\,\mu^{-2\epsilon}(\sigma)\, G^a_{\mu\nu}\, G^{a,\mu\nu} - \frac{1}{4}\,\mu^{-2\epsilon}(\sigma)\, W^a_{\mu\nu}\, W^{a,\mu\nu} - \frac{1}{4}\,\mu^{-2\epsilon}(\sigma)\, B_{\mu\nu}\, B^{\mu\nu} \tag{6.3}$$

These field strength tensors of the respective gauge groups of the Standard Model are given by

$\underline{SU(3)_c:}$

$$G^a_{\mu\nu} = \partial_\mu\, G^a_\nu - \partial_\nu\, G^a_\mu + g_s\, f^{abc}\, G^b_\mu\, G^c_\nu \tag{6.4}$$

$\underline{SU(2)_L:}$

$$W^a_{\mu\nu} = \partial_\mu\, W^a_\nu - \partial_\nu\, W^a_\mu + g_W\, \varepsilon^{abc}\, W^b_\mu\, W^c_\nu \tag{6.5}$$

$\underline{U(1)_Y:}$

$$B_{\mu\nu} = \partial_\mu\, B_\nu - \partial_\nu\, B_\mu \tag{6.6}$$

Hence, the gauge field Lagrangian (6.3) of the QSI Standard Model is then provided by

$$\begin{aligned}
\mathcal{L}^{\text{QSI}}_{\text{Gauge}} = &-\frac{1}{4}\,\mu^{-2\epsilon}(\sigma)\,\left(\partial_\mu\, G^a_\nu - \partial_\nu\, G^a_\mu\right)^2 - \frac{1}{4}\,\mu^{-2\epsilon}(\sigma)\,\left(\partial_\mu\, W^a_\nu - \partial_\nu\, W^a_\mu\right)^2 \\
&-\frac{1}{4}\,\mu^{-2\epsilon}(\sigma)\,\left(\partial_\mu\, B_\nu - \partial_\nu\, B_\mu\right)^2 \\
&- g_s\,\mu^{-2\epsilon}(\sigma)\, f^{abc}\,\partial_\mu\, G^a_\nu\, G^{b,\mu}\, G^{c,\nu} - g_W\,\mu^{-2\epsilon}(\sigma)\,\varepsilon^{abc}\,\partial_\mu\, W^a_\nu\, W^{b,\mu}\, W^{c,\nu} \\
&-\frac{1}{4}\, g_s^2\,\mu^{-2\epsilon}(\sigma)\, f^{abc}\, f^{ade}\, G^b_\mu\, G^c_\nu\, G^{d,\mu}\, G^{e,\nu} \\
&-\frac{1}{4}\, g_W^2\,\mu^{-2\epsilon}(\sigma)\,\varepsilon^{abc}\,\varepsilon^{ade}\, W^b_\mu\, W^c_\nu\, W^{d,\mu}\, W^{e,\nu}
\end{aligned} \tag{6.7}$$

Transforming the gauge fields to mass eigenstates

$$\begin{aligned}
W^+_\mu &= \frac{1}{\sqrt{2}}\,\left(W^1_\mu - i\, W^2_\mu\right), & W^1_\mu &= \frac{1}{\sqrt{2}}\,\left(W^+_\mu + W^-_\mu\right) \\
W^-_\mu &= \frac{1}{\sqrt{2}}\,\left(W^1_\mu + i\, W^2_\mu\right), & W^2_\mu &= \frac{i}{\sqrt{2}}\,\left(W^+_\mu - W^-_\mu\right) \\
Z_\mu &= c_w\, W^3_\mu - s_w\, B_\mu, & W^3_\mu &= c_w\, Z_\mu + s_w\, A_\mu \\
A_\mu &= s_w\, W^3_\mu + c_w\, B_\mu, & B_\mu &= -s_w\, Z_\mu + c_w\, A_\mu
\end{aligned} \tag{6.8}$$

where

$$\begin{aligned}
s_w &\equiv \sin\left(\vartheta_w\right) = \frac{g_Y}{\sqrt{g_Y^2 + g_W^2}} \\
c_w &\equiv \cos\left(\vartheta_w\right) = \frac{g_W}{\sqrt{g_Y^2 + g_W^2}} \\
t_w &\equiv \tan\left(\vartheta_w\right) = \frac{g_Y}{g_W}
\end{aligned} \tag{6.9}$$

and

$$e = s_w\, g_W = c_w\, g_Y = \frac{g_Y\, g_W}{\sqrt{g_Y^2 + g_W^2}} \tag{6.10}$$

the gauge field Lagrangian may finally be written as

$$
\begin{aligned}
\mathcal{L}^{\mathrm{QSI}}_{\mathrm{Gauge}} = &-\frac{1}{4}\,\mu^{-2\epsilon}(\sigma)\left(\partial_\mu G^a_\nu - \partial_\nu G^a_\mu\right)^2 - \frac{1}{4}\,\mu^{-2\epsilon}(\sigma)\,\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)^2 \\
&-\frac{1}{4}\,\mu^{-2\epsilon}(\sigma)\,\left(\partial_\mu Z_\nu - \partial_\nu Z_\mu\right)^2 \\
&-\frac{1}{2}\,\mu^{-2\epsilon}(\sigma)\,\left(\partial_\mu W^+_\nu - \partial_\nu W^+_\mu\right)\left(\partial_\mu W^-_\nu - \partial_\nu W^-_\mu\right) \\
&-\, g_s\,\mu^{-2\epsilon}(\sigma)\, f^{abc}\partial_\mu G^a_\nu\, G^{b,\mu}\, G^{c,\nu} \\
&- i\, e\,\mu^{-2\epsilon}(\sigma)\,\Big(\partial_\mu A_\nu\, W^{+,\nu}\, W^{-,\mu} - \partial_\mu A_\nu\, W^{+,\mu}\, W^{-,\nu} \\
&\qquad\qquad + \partial_\mu W^+_\nu\, A^\mu\, W^{-,\nu} - \partial_\mu W^+_\nu\, A^\nu\, W^{-,\mu} \\
&\qquad\qquad + \partial_\mu W^-_\nu\, A^\nu\, W^{+,\mu} - \partial_\mu W^-_\nu\, A^\mu\, W^{+,\nu}\Big) \\
&- i\, g_W\, c_w\,\mu^{-2\epsilon}(\sigma)\,\Big(\partial_\mu Z_\nu\, W^{+,\nu}\, W^{-,\mu} - \partial_\mu Z_\nu\, W^{+,\mu}\, W^{-,\nu} \\
&\qquad\qquad + \partial_\mu W^+_\nu\, Z^\mu\, W^{-,\nu} - \partial_\mu W^+_\nu\, Z^\nu\, W^{-,\mu} \\
&\qquad\qquad + \partial_\mu W^-_\nu\, Z^\nu\, W^{+,\mu} - \partial_\mu W^-_\nu\, Z^\mu\, W^{+,\nu}\Big) \\
&-\frac{1}{4}\, g_s^2\,\mu^{-2\epsilon}(\sigma)\, f^{abc}\, f^{ade}\, G^b_\mu\, G^c_\nu\, G^{d,\mu}\, G^{e,\nu} \\
&+\frac{1}{2}\, g_W^2\,\mu^{-2\epsilon}(\sigma)\,\Big(W^+_\mu\, W^{+,\mu}\, W^-_\nu\, W^{-,\nu} - W^+_\mu\, W^{-,\mu}\, W^+_\nu\, W^{-,\nu}\Big) \\
&- e^2\,\mu^{-2\epsilon}(\sigma)\,\Big(A_\mu\, A^\mu\, W^+_\nu\, W^{-,\nu} - A_\mu\, A_\nu\, W^{+,\mu}\, W^{-,\nu}\Big) \\
&- g_W^2\, c_w^2\,\mu^{-2\epsilon}(\sigma)\,\Big(Z_\mu\, Z^\mu\, W^+_\nu\, W^{-,\nu} - Z_\mu\, Z_\nu\, W^{+,\mu}\, W^{-,\nu}\Big) \\
&+ e\, g_W\, c_w\,\mu^{-2\epsilon}(\sigma)\,\Big(A_\mu\, Z^\nu\, W^{+,\mu}\, W^{-,\nu} + A_\mu\, Z_\nu\, W^{+,\nu}\, W^{-,\mu} \\
&\qquad\qquad - 2\, A_\mu\, Z^\mu\, W^+_\nu\, W^{-,\nu}\Big)
\end{aligned}
\tag{6.11}
$$

## 6.2. Fermion Field Lagrangian

The fermion field Lagrangian for the QSI Standard Model is given by

$$
\mathcal{L}^{\mathrm{QSI}}_{\mathrm{Fermion}} = \sum_{\psi_L} i\,\overline{\psi_L}\,\not{D}\,\psi_L + \sum_{\psi_R} i\,\overline{\psi_R}\,\not{D}\,\psi_R \tag{6.12}
$$

where

$$
\begin{aligned}
&\psi_L \in \{L_{L,j}\}_{j=1}^3 \cup \{Q_{L,j}\}_{j=1}^3 \\
&\psi_R \in \{\nu_{R,j}\}_{j=1}^3 \cup \{l_{R,j}\}_{j=1}^3 \cup \{u_{R,j}\}_{j=1}^3 \cup \{d_{R,j}\}_{j=1}^3
\end{aligned}
\tag{6.13}
$$

The covariant derivative acts as follows on the different fermion fields

$$
\begin{aligned}
D_\mu \psi_L &= \begin{cases} \left(\partial_\mu - i g_Y Y B_\mu - i g_W W_\mu^a T_W^a\right)\psi_L, & \psi_L \in \{L_{L,j}\}_{j=1}^3 \\ \left(\partial_\mu - i g_Y Y B_\mu - i g_W W_\mu^a T_W^a - i g_s G_\mu^a T_s^a\right)\psi_L, & \psi_L \in \{Q_{L,j}\}_{j=1}^3 \end{cases} \\
D_\mu \psi_R &= \begin{cases} \left(\partial_\mu - i g_Y Y B_\mu\right)\psi_R, & \psi_R \in \{\nu_{R,j}\}_{j=1}^3 \cup \{l_{R,j}\}_{j=1}^3 \\ \left(\partial_\mu - i g_Y Y B_\mu - i g_s G_\mu^a T_s^a\right)\psi_R, & \psi_R \in \{u_{R,j}\}_{j=1}^3 \cup \{d_{R,j}\}_{j=1}^3 \end{cases}
\end{aligned}
\tag{6.14}
$$

again with rescaled gauge fields of mass dimension 1 where the Renormalisation function $\mu^\epsilon(\sigma)$ is absorbed into the gauge field.
Finally, this leads to the following fermion field Lagrangian

$$
\begin{aligned}
\mathcal{L}_{\text{Fermion}}^{\text{QSI}} =& \sum_f i\,\overline{\psi}_f\,\partial\!\!\!/\,\psi_f \\
&+ \sum_f Q_f\, e\, A_\mu\, \overline{\psi}_f\, \gamma^\mu\, \psi_f \\
&+ \sum_q g_s\, G_\mu^a\, \overline{\psi}_{q,i}\, \gamma^\mu\, T_{s,ij}^a\, \psi_{q,j} \\
&+ \sum_f \frac{g_W}{c_w}\, Z_\mu\, \overline{\psi}_f\, \gamma^\mu \left(g_V^f - g_A^f\,\gamma_5\right)\psi_f \\
&+ \sum_k \frac{1}{\sqrt{2}}\, g_W \Big[ W_\mu^+\, \overline{\nu}_k\, \gamma^\mu\, \frac{1}{2}\left(1-\gamma_5\right) l_k + W_\mu^-\, \overline{l}_k\, \gamma^\mu\, \frac{1}{2}\left(1-\gamma_5\right)\nu_k \\
&\qquad + W_\mu^+\, \overline{u}_k\, \gamma^\mu\, \frac{1}{2}\left(1-\gamma_5\right) d_k + W_\mu^-\, \overline{d}_k\, \gamma^\mu\, \frac{1}{2}\left(1-\gamma_5\right) u_k \Big]
\end{aligned}
\tag{6.15}
$$

where

$$
\begin{aligned}
f &\in \left\{e^-, \mu^-, \tau^-, \nu_e, \nu_\mu, \nu_\tau, d, s, b, u, c, t\right\}, \\
q &\in \{d, s, b, u, c, t\}, \\
k &\in \{1,2,3\} \text{ generation index}, \\
i, j &\in \{1,2,3\} \text{ colour indices}
\end{aligned}
$$

and

$$
g_V^f = \frac{1}{2}\, I_{W,f}^3 - Q_f\, s_w^2, \qquad g_A^f = \frac{1}{2}\, I_{W,f}^3
$$

$$
Q_f = \begin{cases} 0, & f \in \{\nu_e, \nu_\mu, \nu_\tau\} \\ -1, & f \in \{e^-, \mu^-, \tau^-\} \\ +\frac{2}{3}, & f \in \{u, c, t\} \\ -\frac{1}{3}, & f \in \{d, s, b\} \end{cases}
$$

$$
I_{W,f}^3 = \begin{cases} -\frac{1}{2}, & f \in \{e^-, \mu^-, \tau^-, d, s, b\} \\ +\frac{1}{2}, & f \in \{\nu_e, \nu_\mu, \nu_\tau, u, c, t\} \end{cases}
$$

Note the following relation between the hypercharge $Y$, the electric charge $Q$ and $T_W^3$

$$
Y = Q - T_W^3 \tag{6.16}
$$

## 6.3. The Higgs Lagrangian

The Higgs Lagrangian for the QSI Standard Model reads as follows

$$\mathcal{L}_{\text{Higgs}}^{\text{QSI}} = (D_\mu \Phi)^\dagger (D^\mu \Phi) + \frac{1}{2} (\partial_\mu \sigma)(\partial^\mu \sigma) - \mu^{2\epsilon}(\sigma)\, V_{\text{H}}(\Phi, \sigma) \tag{6.17}$$

where

$$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} = \begin{pmatrix} G^+ \\ \frac{\phi + i\, G^0}{\sqrt{2}} \end{pmatrix} \tag{6.18}$$

is the Higgs doublet, with hypercharge $Y_\Phi = 1/2$, the Higgs-like boson $\phi = h + v$ and the Dilaton $\sigma = \mathfrak{D} + w$. The covariant derivative acts on the Higgs doublet as

$$D_\mu \Phi = \left(\partial_\mu - i g_Y Y B_\mu - i g_W W_\mu^a\, T_W^a\right) \Phi \tag{6.19}$$

again with rescaled gauge fields of mass dimension 1 where the Renormalisation function $\mu^\epsilon(\sigma)$ is absorbed into the gauge field. The Higgs potential $V_{\text{H}}$ is explicitly given by

$$\begin{aligned} V_{\text{H}}(\Phi, \sigma) &= \frac{\lambda_\phi}{3!} \left(\Phi^\dagger \Phi\right)^2 + \frac{\lambda_m}{2} \left(\Phi^\dagger \Phi\right) \sigma^2 + \frac{\lambda_\sigma}{4!} \sigma^4 \\ &= \frac{\lambda_\phi}{4!} \phi^4 + \frac{\lambda_m}{4} \phi^2 \sigma^2 + \frac{\lambda_\sigma}{4!} \sigma^4 \\ &\quad + \frac{\lambda_\phi}{3!} \phi^2 G^+ G^- + \frac{\lambda_\phi}{12} \phi^2 \left(G^0\right)^2 + \frac{\lambda_m}{2} \sigma^2 G^+ G^- + \frac{\lambda_m}{4} \sigma^2 \left(G^0\right)^2 \\ &\quad + \frac{\lambda_\phi}{3!} \left(G^+ G^-\right)^2 + \frac{\lambda_\phi}{3!} G^+ G^- \left(G^0\right)^2 + \frac{\lambda_\phi}{4!} \left(G^0\right)^4 \\ &\equiv V_{\text{2SM}}(\phi, \sigma) + V_G\left(\phi, \sigma, G^0, G^\pm\right) \end{aligned} \tag{6.20}$$

where

$$V_{\text{2SM}}(\phi, \sigma) := \frac{\lambda_\phi}{4!} \phi^4 + \frac{\lambda_m}{4} \phi^2 \sigma^2 + \frac{\lambda_\sigma}{4!} \sigma^4 \tag{6.21}$$

is the potential of the 2 Scalar Model which has extensively been discussed in chapter 2 and appendix C, and

$$\begin{aligned} V_G\left(\phi, \sigma, G^0, G^\pm\right) := &\frac{\lambda_\phi}{3!} \phi^2 G^+ G^- + \frac{\lambda_\phi}{12} \phi^2 \left(G^0\right)^2 + \frac{\lambda_m}{2} \sigma^2 G^+ G^- + \frac{\lambda_m}{4} \sigma^2 \left(G^0\right)^2 \\ &+ \frac{\lambda_\phi}{3!} \left(G^+ G^-\right)^2 + \frac{\lambda_\phi}{3!} G^+ G^- \left(G^0\right)^2 + \frac{\lambda_\phi}{4!} \left(G^0\right)^4 \end{aligned} \tag{6.22}$$

is the part of the Higgs potential that contains the Goldstone bosons. Thus, as discussed in section 2.2, the 2 Scalar Model is of great interest for a quantum scale invariant Standard Model as it is a subset of the Higgs Lagrangian.
Expanding the Lagrangian about the fields VEVs $\{v, w\}$ and w.r.t. $\epsilon$, one obtains (C.1) for $\widetilde{V}_{\text{2SM}}(\phi, \sigma) = \mu^{2\epsilon}(\sigma)\, V_{\text{2SM}}(\phi, \sigma)$ as well as

$$
\begin{aligned}
\widetilde{V}_G\left(\phi,\sigma,G^0,G^\pm\right) &= \mu^{2\epsilon}(\sigma)\,V_G\left(\phi,\sigma,G^0,G^\pm\right)\\
&= \frac{1}{2}\,\widetilde{M}_G^2\,\left(G^0\right)^2 + \widetilde{M}_G^2\,G^+\,G^- + \frac{1}{2}\,\widetilde{\mathcal{V}}_{133}\,h\,\left(G^0\right)^2 + \widetilde{\mathcal{V}}_{145}\,h\,G^+\,G^-\\
&\quad + \frac{1}{2}\,\widetilde{\mathcal{V}}_{233}\,\mathfrak{D}\,\left(G^0\right)^2 + \widetilde{\mathcal{V}}_{245}\,\mathfrak{D}\,G^+\,G^- + \frac{1}{4}\,\widetilde{\mathcal{V}}_{1133}\,h^2\,\left(G^0\right)^2\\
&\quad + \frac{1}{2}\,\widetilde{\mathcal{V}}_{1145}\,h^2\,G^+\,G^- + \frac{1}{2}\,\widetilde{\mathcal{V}}_{1233}\,h\,\mathfrak{D}\,\left(G^0\right)^2 + \widetilde{\mathcal{V}}_{1245}\,h\,\mathfrak{D}\,G^+\,G^-\\
&\quad + \frac{1}{4}\,\widetilde{\mathcal{V}}_{2233}\,\mathfrak{D}^2\,\left(G^0\right)^2 + \frac{1}{2}\,\widetilde{\mathcal{V}}_{2245}\,\mathfrak{D}^2\,G^+\,G^- + \frac{1}{4!}\,\widetilde{\mathcal{V}}_{3333}\,\left(G^0\right)^4\\
&\quad + \frac{1}{2}\,\widetilde{\mathcal{V}}_{3345}\,\left(G^0\right)^2\,G^+\,G^- + \frac{1}{4}\,\widetilde{\mathcal{V}}_{4455}\,\left(G^+\,G^-\right)^2 + \cdots
\end{aligned}
\tag{6.23}
$$

where the ellipsis denotes infinitely many terms of higher orders in scalar fields and with

$$
\begin{aligned}
\widetilde{M}_G^2 &= M_G^2 = \mu_0^{2\epsilon}\,\frac{\lambda_\phi}{6}\,v^2\,\left(1 + 3\,\frac{\lambda_m}{\lambda_\phi}\,\frac{w^2}{v^2}\right)\\
\widetilde{\mathcal{V}}_{133} &= \widetilde{\mathcal{V}}_{145} = \mu_0^{2\epsilon}\,\frac{\lambda_\phi}{3}\,v\\
\widetilde{\mathcal{V}}_{233} &= \widetilde{\mathcal{V}}_{245} = \mu_0^{2\epsilon}\,\left[\lambda_m\,w + \epsilon\,(1+\epsilon)\,\left(\lambda_m\,w + \frac{\lambda_\phi}{3}\,\frac{v^2}{w}\right) + \mathcal{O}\left(\epsilon^3\right)\right]\\
\widetilde{\mathcal{V}}_{1133} &= \widetilde{\mathcal{V}}_{1145} = \widetilde{\mathcal{V}}_{3345} = \mu_0^{2\epsilon}\,\frac{\lambda_\phi}{3}\\
\widetilde{\mathcal{V}}_{1233} &= \widetilde{\mathcal{V}}_{1245} = \mu_0^{2\epsilon}\,\left[\frac{2}{3}\,\lambda_\phi\,\frac{v}{w}\,\epsilon\,(1+\epsilon) + \mathcal{O}\left(\epsilon^3\right)\right]\\
\widetilde{\mathcal{V}}_{2233} &= \widetilde{\mathcal{V}}_{2245} = \mu_0^{2\epsilon}\,\left[\lambda_m + \epsilon\,\left(3\,\lambda_m - \frac{\lambda_\phi}{3}\,\frac{v^2}{w^2}\right) + \epsilon^2\,\left(5\,\lambda_m + \frac{\lambda_\phi}{3}\,\frac{v^2}{w^2}\right) + \mathcal{O}\left(\epsilon^3\right)\right]\\
\widetilde{\mathcal{V}}_{3333} &= \mu_0^{2\epsilon}\,\lambda_\phi\\
\widetilde{\mathcal{V}}_{4455} &= \mu_0^{2\epsilon}\,\frac{2}{3}\,\lambda_\phi
\end{aligned}
\tag{6.24}
$$

Obviously, the Goldstone mass $\widetilde{M}_G^2$ in (6.24) vanishes if the minimalisation condition (2.47) of the Higgs potential is used, as expected. Note that the minimalisation condition of the complete Higgs potential indeed is the same as for the 2 Scalar Model in (2.47). Further, the Goldstone mass does not obtain $\epsilon$ - corrections, and thus $\widetilde{M}_G^2 = M_G^2$. Hence, the complete Higgs potential in $D = 4 - 2\epsilon$ dimensions may be written as

$$
\begin{aligned}
\widetilde{V}_{\mathrm{H}}\left(\Phi,\sigma\right) &= \mu^{2\epsilon}(\sigma)\,V_{\mathrm{H}}\left(\Phi,\sigma\right)\\
&= \widetilde{V}_{\mathrm{2SM}}\left(\phi,\sigma\right) + \widetilde{V}_G\left(\phi,\sigma,G^0,G^\pm\right)\\
&\text{"}= \text{(C.1) + (6.23)"}
\end{aligned}
\tag{6.25}
$$

Using (6.19), the kinetic term of the Higgs Lagrangian is given by

$$
\begin{aligned}
\mathcal{L}^{\mathrm{QSI}}_{\mathrm{Higgs,kin}} &= \left(D_\mu \Phi\right)^\dagger \left(D^\mu \Phi\right) + \frac{1}{2}\left(\partial_\mu \sigma\right)\left(\partial^\mu \sigma\right) \\
&= \frac{1}{2}\left(\partial_\mu h\right)\left(\partial^\mu h\right) + \frac{1}{2}\left(\partial_\mu \mathfrak{D}\right)\left(\partial^\mu \mathfrak{D}\right) + \frac{1}{2}\left(\partial_\mu G^0\right)\left(\partial^\mu G^0\right) \\
&\quad + \left(\partial_\mu G^+\right)\left(\partial^\mu G^-\right) \\
&\quad + \frac{1}{8}\frac{g_W^2}{c_w^2}\, v^2\, Z_\mu\, Z^\mu + \frac{1}{4}\, g_W^2\, v^2\, W^+_\mu\, W^{-,\mu} \\
&\quad + \frac{i}{2}\, g_W\, v\, W^-_\mu\, \partial^\mu G^+ - \frac{i}{2}\, g_W\, v\, W^+_\mu\, \partial^\mu G^- + \frac{1}{2}\frac{g_W}{c_w}\, v\, Z_\mu\, \partial^\mu G^0 \\
&\quad + \frac{1}{4}\frac{g_W^2}{c_w^2}\, v\, h\, Z_\mu\, Z^\mu + \frac{1}{2}\, g_W^2\, v\, h\, W^+_\mu\, W^{-,\mu} + \frac{1}{2}\, e\, g_W\, v\, G^+\, W^-_\mu\, A^\mu \\
&\quad + \frac{1}{2}\, e\, g_W\, v\, G^-\, W^+_\mu\, A^\mu - \frac{1}{2}\, e\, g_W\, \frac{s_w}{c_w}\, v\, G^+\, W^-_\mu\, Z^\mu \\
&\quad - \frac{1}{2}\, e\, g_W\, \frac{s_w}{c_w}\, v\, G^-\, W^+_\mu\, Z^\mu + \frac{i}{2}\, g_W\, h\, \partial^\mu G^+\, W^-_\mu - \frac{i}{2}\, g_W\, h\, \partial^\mu G^-\, W^+_\mu \\
&\quad - \frac{i}{2}\, g_W\, \partial^\mu h\, G^+\, W^-_\mu + \frac{i}{2}\, g_W\, \partial^\mu h\, G^-\, W^+_\mu + \frac{1}{2}\frac{g_W}{c_w}\, h\, \partial^\mu G^0\, Z_\mu \\
&\quad - \frac{1}{2}\frac{g_W}{c_w}\, \partial^\mu h\, G^0\, Z_\mu - \frac{1}{2}\, g_W\, \partial^\mu G^0\, G^+\, W^-_\mu - \frac{1}{2}\, g_W\, \partial^\mu G^0\, G^-\, W^+_\mu \\
&\quad + \frac{1}{2}\, g_W\, G^0\, \partial^\mu G^+\, W^-_\mu + \frac{1}{2}\, g_W\, G^0\, \partial^\mu G^-\, W^+_\mu - \frac{i}{2}\, g_W\, \frac{c_{2w}}{c_w}\, G^+\, \partial^\mu G^-\, Z_\mu \\
&\quad + \frac{i}{2}\, g_W\, \frac{c_{2w}}{c_w}\, G^-\, \partial^\mu G^+\, Z_\mu - i\, e\, G^+\, \partial^\mu G^-\, A_\mu + i\, e\, G^-\, \partial^\mu G^+\, A_\mu \\
&\quad + \frac{1}{4}\, g_W^2\, h^2\, W^+_\mu\, W^{-,\mu} + \frac{1}{8}\frac{g_W^2}{c_w^2}\, h^2\, Z_\mu\, Z^\mu - \frac{1}{2}\, e\, g_W\, t_w\, h\, G^+\, W^-_\mu\, Z^\mu \\
&\quad - \frac{1}{2} e\, g_W\, t_w\, h\, G^-\, W^+_\mu\, Z^\mu + \frac{1}{2} e\, g_W\, h\, G^+\, W^-_\mu\, A^\mu + \frac{1}{2} e\, g_W\, h\, G^-\, W^+_\mu\, A^\mu \\
&\quad + \frac{1}{2}\, g_W^2\, G^+\, G^-\, W^+_\mu\, W^{-,\mu} + \frac{1}{4}\, g_W^2\, \left(G^0\right)^2\, W^+_\mu\, W^{-,\mu} \\
&\quad + \frac{i}{2}\, e\, g_W\, t_w\, G^0\, G^+\, W^-_\mu\, Z^\mu - \frac{i}{2}\, e\, g_W\, t_w\, G^0\, G^-\, W^+_\mu\, Z^\mu \\
&\quad - \frac{i}{2}\, e\, g_W\, G^0\, G^+\, W^-_\mu\, A^\mu + \frac{i}{2}\, e\, g_W\, G^0\, G^-\, W^+_\mu\, A^\mu + \frac{1}{8}\frac{g_W^2}{c_w^2}\left(G^0\right)^2 Z_\mu\, Z^\mu \\
&\quad + \frac{1}{4} g_W^2\, \frac{c_{2w}^2}{c_w^2}\, G^+\, G^-\, Z_\mu\, Z^\mu + e\, g_W\, \frac{c_{2w}}{c_w}\, G^+\, G^-\, Z_\mu\, A^\mu + e^2\, G^+\, G^-\, A_\mu\, A^\mu
\end{aligned}
\tag{6.26}
$$

**Remark.**

(i) The mass terms of the gauge fields are given in the third line after the second equality in (6.26). Due to the fact that the Renormalisation function is absorbed into the gauge fields there is no factor of $\mu_0^{2\epsilon}$ in these mass terms. However, as discussed in chapter 2, the factor of $\mu_0^{2\epsilon}$ is part of the squared mass definition, such that the mass has mass dimension 1 even in $D = 4 - 2\epsilon$ dimensions. This can be solved by multiplying with $1 = \mu_0^{-2\epsilon}\,\mu_0^{2\epsilon}$, i.e.

$$
\begin{aligned}
&\frac{1}{8}\,\frac{g_W^2}{c_w^2}\,v^2\,Z_\mu\,Z^\mu + \frac{1}{4}\,g_W^2\,v^2\,W_\mu^+\,W^{-,\mu} \\
&\qquad = \mu_0^{-2\epsilon}\,\mu_0^{2\epsilon}\,\frac{1}{8}\,\frac{g_W^2}{c_w^2}\,v^2\,Z_\mu\,Z^\mu + \mu_0^{-2\epsilon}\,\mu_0^{2\epsilon}\,\frac{1}{4}\,g_W^2\,v^2\,W_\mu^+\,W^{-,\mu} \\
&\qquad = \mu_0^{-2\epsilon}\,\frac{1}{2}\,\widetilde{M}_Z^2\,Z_\mu\,Z^\mu + \mu_0^{-2\epsilon}\,\widetilde{M}_W^2\,W_\mu^+\,W^{-,\mu}
\end{aligned}
\tag{6.27}
$$

Hence, the squared masses of the gauge fields are

$$
\widetilde{M}_Z^2 = M_Z^2 = \mu_0^{2\epsilon}\,\frac{1}{4}\,\frac{g_W^2}{c_w^2}\,v^2, \qquad \widetilde{M}_W^2 = M_W^2 = \mu_0^{2\epsilon}\,\frac{1}{4}\,g_W^2\,v^2 \tag{6.28}
$$

where both masses have mass dimension $[\widetilde{M}_Z] = [\widetilde{M}_W] = 1$, as intended, and do not obtain $\epsilon$ - corrections. The other gauge fields are massless.

Further, note that the prefactor $\mu_0^{-2\epsilon}$ in front of the mass terms (6.27) is necessary in order to be consistent with the kinetic terms of the gauge fields in (6.11) which also obtain a prefactor of $\mu_0^{-2\epsilon}$ after expanding the Lagrangian in (6.11) about the VEV of the Dilaton and w.r.t. $\epsilon$. Thus, the propagators of the gauge fields obtain a global prefactor of $\mu_0^{-2\epsilon}$, which is due to the fact that the Renormalisation function has been absorbed into the gauge fields. Consequently, the whole approach is consistent w.r.t. the gauge fields, as well as their masses and dimensionality.

(ii) The fourth line after the second equality in (6.26) contains the usual bilinear (or kinetic) mixing terms between the gauge fields and the associated Goldstone bosons. Analogous to the usual Standard Model, these terms can be eliminated (at tree-level) using appropriate gauge fixing conditions in the gauge fixing Lagrangian as shown in section 6.5.

(iii) The Higgs potential in (6.20) to (6.22) is given in the unbroken phase of the theory, i.e. in a manifestly scale invariant form. In (C.1) and (6.23), or equivalently in (6.25), the Higgs potential is given in the broken phase of the theory. The kinetic part of the Higgs Lagrangian is given in the broken phase of the theory in (6.26). In order to obtain the kinetic part of the Higgs Lagrangian in a manifestly scale invariant form, the Lagrangian (6.26) can be used with the replacement $h \to \phi$, $\mathfrak{D} \to \sigma$ and $v \to 0$ being used.

(iv) The Higgs Lagrangian can be transformed to mass eigenstates $\{H, S\}$ analogously to the 2 Scalar Model discussed in section 2.2, with mass eigenstates given in (2.32) and mixing angle provided in (2.35), i.e.

$$
\begin{aligned}
H &= c_\beta\,h - s_\beta\,\mathfrak{D} \quad &\Leftrightarrow \quad h &= c_\beta\,H + s_\beta\,S \\
S &= s_\beta\,h + c_\beta\,\mathfrak{D} \quad &\Leftrightarrow \quad \mathfrak{D} &= -s_\beta\,H + c_\beta\,S
\end{aligned}
$$

## 6.4. The Yukawa Lagrangian

The Yukawa Lagrangian of the QSI Standard Model is provided by

$$\begin{aligned}\mathcal{L}^{\mathrm{QSI}}_{\mathrm{Yukawa}} = -\sum_{i,j}\Big(&\mu^\epsilon(\sigma)\,Y'_{l,ij}\,\overline{L'_{L,i}}\,\Phi\,l'_{R,j} + \mu^\epsilon(\sigma)\,Y'^{*}_{l,ij}\,\overline{l'_{R,j}}\,\Phi^\dagger\,L'_{L,i}\\ &+\mu^\epsilon(\sigma)\,Y'_{d,ij}\,\overline{Q'_{L,i}}\,\Phi\,d'_{R,j} + \mu^\epsilon(\sigma)\,Y'^{*}_{d,ij}\,\overline{d'_{R,j}}\,\Phi^\dagger\,Q'_{L,i}\\ &+\mu^\epsilon(\sigma)\,Y'_{u,ij}\,\overline{Q'_{L,i}}\,\widetilde{\Phi}\,u'_{R,j} + \mu^\epsilon(\sigma)\,Y'^{*}_{u,ij}\,\overline{u'_{R,j}}\,\widetilde{\Phi}^\dagger\,Q'_{L,i}\Big)\end{aligned} \tag{6.29}$$

where flavour eigenstates are marked by a prime, and related to mass eigenstates by the following unitary transformation

$$\begin{aligned} f'_{L,k} &= U^{f\,*}_{L,ik} f_{L,i}\\ f'_{R,k} &= U^{f\,*}_{R,ik} f_{R,i}\end{aligned} \tag{6.30}$$

Further, the Higgs doublet is given in (6.18) and

$$\widetilde{\Phi} = i\,\sigma^2\,\Phi^* = \begin{pmatrix}\frac{\phi - i\,G^0}{\sqrt{2}}\\ -G^-\end{pmatrix} \tag{6.31}$$

Transforming from flavour to mass eigenstates and using the explicit form of the Higgs and fermion doublets leads to

$$\begin{aligned}\mathcal{L}^{\mathrm{QSI}}_{\mathrm{Yukawa}} &= -\sum_f\left(\frac{1}{\sqrt{2}}\,y_f\,\mu^\epsilon(\sigma)\,\phi\,\overline{\psi}_f\,\psi_f - i\,\sqrt{2}\,I^3_{W,f}\,y_f\,\mu^\epsilon(\sigma)\,G^0\,\overline{\psi}_f\,\gamma_5\,\psi_f\right)\\ &\quad-\sum_i y_{l,i}\,\mu^\epsilon(\sigma)\Big(G^+\,\overline{\nu}_i\,P_R\,l_i + G^-\,\overline{l}_i\,P_L\,\nu_i\Big)\\ &\quad-\sum_{i,j}\Big[\mu^\epsilon(\sigma)\,V_{CKM,ij}\,G^+\,\overline{u}_i\Big(y_{d,j}\,P_R - y_{u,i}\,P_L\Big)\,d_j\\ &\qquad+\mu^\epsilon(\sigma)\,V^*_{CKM,ij}\,G^-\,\overline{d}_j\Big(y_{d,j}\,P_L - y_{u,i}\,P_R\Big)\,u_i\Big]\\ &= -\sum_f\Big(\frac{1}{\sqrt{2}}\,y_f\,\mu^\epsilon(\sigma)\,v\,\overline{\psi}_f\,\psi_f + \frac{1}{\sqrt{2}}\,y_f\,\mu^\epsilon(\sigma)\,h\,\overline{\psi}_f\,\psi_f\\ &\qquad- i\,\sqrt{2}\,I^3_{W,f}\,y_f\,\mu^\epsilon(\sigma)\,G^0\,\overline{\psi}_f\,\gamma_5\,\psi_f\Big)\\ &\quad-\sum_i y_{l,i}\,\mu^\epsilon(\sigma)\Big(G^+\,\overline{\nu}_i\,P_R\,l_i + G^-\,\overline{l}_i\,P_L\,\nu_i\Big)\\ &\quad-\sum_{i,j}\Big[\mu^\epsilon(\sigma)\,V_{CKM,ij}\,G^+\,\overline{u}_i\Big(y_{d,j}\,P_R - y_{u,i}\,P_L\Big)\,d_j\\ &\qquad+\mu^\epsilon(\sigma)\,V^*_{CKM,ij}\,G^-\,\overline{d}_j\Big(y_{d,j}\,P_L - y_{u,i}\,P_R\Big)\,u_i\Big]\end{aligned} \tag{6.32}$$

where $V_{CKM} = U_L^u \left(U_L^d\right)^\dagger$ is the CKM matrix and the fermion masses are given by

$$\widetilde{m}_f = m_f = \mu_0^\epsilon \frac{1}{\sqrt{2}} y_f\, v \tag{6.33}$$

with $f \in \{e^-, \mu^-, \tau^-, d, s, b, u, c, t\}$. The fermion masses do not obtain $\epsilon$ - corrections, i.e. $\widetilde{m}_f = m_f$. Note that it has been assumed that neutrinos are massless.

## 6.5. The Gauge Fixing Lagrangian

In order to write down the gauge fixing Lagrangian, appropriate gauge fixing conditions are needed. Recall that there are bilinear (or kinetic) mixing terms between the gauge fields $\{Z, W^\pm\}$ and the Goldstone bosons $\{G^0, G^\pm\}$ in (6.26). These bilinear mixing terms can be removed (at tree-level) by choosing the following gauge fixing conditions in $D = 4 - 2\epsilon$ dimensions

$$\begin{aligned}
\widetilde{G}^a \left[G_\mu^a\right] &= \mu^{-\epsilon}(\sigma)\, \partial^\mu G_\mu^a = 0 \\
\widetilde{G}^A \left[A_\mu\right] &= \mu^{-\epsilon}(\sigma)\, \partial^\mu A_\mu = 0 \\
\widetilde{G}^Z \left[Z_\mu\right] &= \mu^{-\epsilon}(\sigma)\, \partial^\mu Z_\mu - \frac{1}{2}\, \xi_Z\, \frac{g_W}{c_w}\, \mu^\epsilon(\sigma)\, v\, G^0 = 0 \\
\widetilde{G}^\pm \left[W_\mu^\pm\right] &= \mu^{-\epsilon}(\sigma)\, \partial^\mu W_\mu^\pm \mp \frac{i}{2}\, \xi_W\, g_W\, \mu^\epsilon(\sigma)\, v\, G^\pm = 0
\end{aligned} \tag{6.34}$$

The gauge fixing Lagrangian in terms of Nakanishi-Lautrup fields $\{B_G^a, B_A, B_Z, B_\pm\}$ may then be written as

$$\begin{aligned}
\mathcal{L}_{\text{GF}}^{\text{QSI}} = & \frac{\xi_G}{2}\, B_G^a\, B_G^a - \mu^{-\epsilon}(\sigma)\, B_G^a\, \partial^\mu G_\mu^a \\
& + \frac{\xi_A}{2}\, B_A^2 - \mu^{-\epsilon}(\sigma)\, B_A\, \partial^\mu A_\mu \\
& + \frac{\xi_Z}{2}\, B_Z^2 - \mu^{-\epsilon}(\sigma)\, B_Z \left(\partial^\mu Z_\mu - \frac{1}{2}\, \xi_Z\, \frac{g_W}{c_w}\, \mu^{2\epsilon}(\sigma)\, v\, G^0\right) \\
& + \xi_W\, B_+\, B_- - \mu^{-\epsilon}(\sigma)\, B_+ \left(\partial^\mu W_\mu^- + \frac{i}{2}\, \xi_W\, g_W\, \mu^{2\epsilon}(\sigma)\, v\, G^-\right) \\
& \qquad - \mu^{-\epsilon}(\sigma)\, B_- \left(\partial^\mu W_\mu^+ - \frac{i}{2}\, \xi_W\, g_W\, \mu^{2\epsilon}(\sigma)\, v\, G^+\right)
\end{aligned} \tag{6.35}$$

Obviously, the gauge fixing Lagrangian in this form is formulated in the broken phase of the theory. Choosing Landau gauge $\xi_G = \xi_A = \xi_Z = \xi_\pm = 0$, the Lagrangian is given by

$$\begin{aligned}
\mathcal{L}_{\text{GF}}^{\text{QSI}} = & - \mu^{-\epsilon}(\sigma)\, B_G^a\, \partial^\mu G_\mu^a - \mu^{-\epsilon}(\sigma)\, B_A\, \partial^\mu A_\mu - \mu^{-\epsilon}(\sigma)\, B_Z\, \partial^\mu Z_\mu \\
& - \mu^{-\epsilon}(\sigma)\, B_+\, \partial^\mu W_\mu^- - \mu^{-\epsilon}(\sigma)\, B_-\, \partial^\mu W_\mu^+
\end{aligned} \tag{6.36}$$

which is manifestly scale invariant, i.e. in the unbroken phase of the theory.

The equations of motion for the Nakanishi-Lautrup fields, derived from the Lagrangian (6.35), are

$$
\begin{aligned}
B_G^a &= \frac{1}{\xi_G}\,\mu^{-\epsilon}(\sigma)\,\partial^\mu G_\mu^a \\
B_A &= \frac{1}{\xi_A}\,\mu^{-\epsilon}(\sigma)\,\partial^\mu A_\mu \\
B_Z &= \frac{1}{\xi_Z}\,\mu^{-\epsilon}(\sigma)\left(\partial^\mu Z_\mu - \frac{1}{2}\,\xi_Z\,\frac{g_W}{c_w}\,\mu^{2\epsilon}(\sigma)\,v\,G^0\right) \\
B_\pm &= \frac{1}{\xi_W}\,\mu^{-\epsilon}(\sigma)\left(\partial^\mu W_\mu^\pm \mp \frac{i}{2}\,\xi_W\,g_W\,\mu^{2\epsilon}(\sigma)\,v\,G^\pm\right)
\end{aligned}
\tag{6.37}
$$

Using these eqautions of motions (6.37), the gauge fixing Lagrangian may be written as

$$
\begin{aligned}
\mathcal{L}_{\mathrm{GF}}^{\mathrm{QSI}} =& -\frac{1}{2\,\xi_G}\,\mu^{-2\epsilon}(\sigma)\,F_G^a\,F_G^a - \frac{1}{2\,\xi_A}\,\mu^{-2\epsilon}(\sigma)\,F_A^2 \\
& - \frac{1}{2\,\xi_Z}\,\mu^{-2\epsilon}(\sigma)\,F_Z^2 - \frac{1}{\xi_W}\,\mu^{-2\epsilon}(\sigma)\,F_+\,F_- \\
=& -\frac{1}{2\,\xi_G}\,\mu^{-2\epsilon}(\sigma)\left(\partial^\mu G_\mu^a\right)^2 - \frac{1}{2\,\xi_A}\,\mu^{-2\epsilon}(\sigma)\,(\partial^\mu A_\mu)^2 \\
& - \frac{1}{2\,\xi_Z}\,\mu^{-2\epsilon}(\sigma)\,(\partial^\mu Z_\mu)^2 - \frac{1}{\xi_W}\,\mu^{-2\epsilon}(\sigma)\left(\partial^\mu W_\mu^+\right)\left(\partial^\nu W_\nu^-\right) \\
& - \frac{1}{8}\,\xi_Z\,\frac{g_W^2}{c_w^2}\,\mu^{2\epsilon}(\sigma)\,v^2\left(G^0\right)^2 - \frac{1}{4}\,\xi_W\,g_W^2\,\mu^{2\epsilon}(\sigma)\,v^2\,G^+\,G^- \\
& - \frac{i}{2}\,g_W\,v\,W_\mu^-\,\partial^\mu G^+ + \frac{i}{2}\,g_W\,v\,W_\mu^+\,\partial^\mu G^- - \frac{1}{2}\,\frac{g_W}{c_w}\,v\,Z_\mu\,\partial^\mu G^0
\end{aligned}
\tag{6.38}
$$

where the last 3 terms in the last line after the second equality in (6.38) have been integrated by parts such that they exactly cancel the bilinear mixing terms in (6.26), as proposed, and

$$
\begin{aligned}
F_G^a &= \partial^\mu G_\mu^a \\
F_A &= \partial^\mu A_\mu \\
F_Z &= \partial^\mu Z_\mu - \frac{1}{2}\,\xi_Z\,\frac{g_W}{c_w}\,\mu^{2\epsilon}(\sigma)\,v\,G^0 \\
F_\pm &= \partial^\mu W_\mu^\pm \mp \frac{i}{2}\,\xi_W\,g_W\,\mu^{2\epsilon}(\sigma)\,v\,G^\pm
\end{aligned}
\tag{6.39}
$$

## 6.6. The Ghost Lagrangian

The ghost Lagrangian of the QSI Standard Model can be derived as follows, using (6.39),

$$
\mathcal{L}_{\mathrm{Ghost}}^{\mathrm{QSI}} = -\int d^4y\;\overline{\eta}^B(x)\,\frac{\delta F_B(x)}{\delta\omega^C(y)}\,\eta^C(y)
\tag{6.40}
$$

where it is implicitly summed over $B$ and $C$, $\eta^B$ are the ghosts $\{c^a, b_A, b_Z, b_\pm\}$ and $\omega^B$ are the gauge fields $\{\beta^a, \alpha_A, \alpha_Z, \alpha^\pm\}$ in the corresponding gauge transformations.

Hence,

$$\begin{aligned}\mathcal{L}^{\text{QSI}}_{\text{Ghost}} = -\int d^4y \, \Bigg\{ & \sum_{a,c=1}^{8} \overline{c}^a(x)\, \frac{\delta F^a_G(x)}{\delta \beta^c(y)}\, c^c(y) \\ & + \sum_{a=1}^{4} \Bigg( \overline{b}_A(x)\, \frac{\delta F_A(x)}{\delta\alpha^a(y)} + \overline{b}_Z(x)\, \frac{\delta F_Z(x)}{\delta\alpha^a(y)} \\ & + \overline{b}_+(x)\, \frac{\delta F_+(x)}{\delta\alpha^a(y)} + \overline{b}_-(x)\, \frac{\delta F_-(x)}{\delta\alpha^a(y)} \Bigg) b_a(y) \Bigg\}\end{aligned} \tag{6.41}$$

In order to find the variations $\delta F_B$, infinitesimal gauge transformations need to be considered. The gauge transformations in the QSI Standard Model, i.e. for the gauge group $SU(3)_c \times SU(2)_L \times U(1)_Y$, are given by

$$\begin{aligned} U_{SU(3)_c} &= e^{i\, g_s\, \beta^a\, T^a_s} \\ U_{SU(2)_L} &= e^{i\, g_W\, \alpha^a\, T^a_W} \\ U_{U(1)_Y} &= e^{i\, g_Y\, \alpha^4\, Y} \end{aligned} \tag{6.42}$$

which lead to the following infinitesimal gauge transformations for the gauge fields

$$\begin{aligned} \delta G^a_\mu &= \partial_\mu \beta^a + g_s\, f^{abc}\, G^b_\mu\, \beta^c \\ \delta W^a_\mu &= \partial_\mu \alpha^a + g_W\, \varepsilon^{abc}\, W^b_\mu\, \alpha^c \\ \delta B_\mu &= \partial_\mu \alpha^4 \end{aligned} \tag{6.43}$$

Transforming to gauge field mass eigenstates and introducing

$$\begin{aligned} \alpha^\pm &= \frac{1}{\sqrt{2}}\left(\alpha^1 \mp i\,\alpha^2\right), & \alpha^1 &= \frac{1}{\sqrt{2}}\left(\alpha^+ + \alpha^-\right) \\ \alpha_Z &= c_w\,\alpha^3 - s_w\,\alpha^4, & \alpha^2 &= \frac{i}{\sqrt{2}}\left(\alpha^+ - \alpha^-\right) \\ \alpha_A &= s_w\,\alpha^3 + c_w\,\alpha^4, & \alpha^3 &= c_w\,\alpha_Z + s_w\,\alpha_A \\ & & \alpha^4 &= -s_w\,\alpha_Z + c_w\,\alpha_A \end{aligned} \tag{6.44}$$

the following infinitesimal gauge transformations for the gauge fields in mass eigenstates are obtained

$$\begin{aligned} \delta G^a_\mu &= \partial_\mu \beta^a + g_s\, f^{abc}\, G^b_\mu\, \beta^c \\ \delta W^+_\mu &= \partial_\mu \alpha^+ - i\, g_W \Big[ \big(c_w\, Z_\mu + s_w\, A_\mu\big)\, \alpha^+ - \big(c_w\,\alpha_Z + s_w\,\alpha_A\big)\, W^+_\mu \Big] \\ \delta W^-_\mu &= \partial_\mu \alpha^- + i\, g_W \Big[ \big(c_w\, Z_\mu + s_w\, A_\mu\big)\, \alpha^- - \big(c_w\,\alpha_Z + s_w\,\alpha_A\big)\, W^-_\mu \Big] \\ \delta Z_\mu &= \partial_\mu \alpha_Z + i\, g_W\, c_w \Big( \alpha^+\, W^-_\mu - \alpha^-\, W^+_\mu \Big) \\ \delta A_\mu &= \partial_\mu \alpha_A + i\, e \Big( \alpha^+\, W^-_\mu - \alpha^-\, W^+_\mu \Big) \end{aligned} \tag{6.45}$$

Using that the Higgs doublet (6.18) transforms non-trivially under $SU(2)_L \times U(1)_Y$, as well as the relations

$$G^\pm = \phi^\pm, \qquad G^0 = \frac{i}{\sqrt{2}} \Big( \big(\phi^0\big)^* - \phi^0 \Big) \tag{6.46}$$

the infinitesimal gauge transformations of the Goldstone bosons are found to be

$$\begin{aligned} \delta G^\pm &= \pm\frac{i}{2}\, g_W \left(\phi \pm i\, G^0\right) \alpha^\pm \pm i \left( e\, \alpha_A + \frac{g_W}{2}\, \frac{c_{2w}}{c_w}\, \alpha_Z \right) G^\pm \\ \delta G^0 &= \frac{1}{2}\, g_W \left(\alpha^+ G^- + \alpha^- G^+\right) - \frac{1}{2}\, \frac{g_W}{c_w}\, \alpha_Z\, \phi \end{aligned} \tag{6.47}$$

Therefore, the variations $\delta F_B$ are explicitly given by

$$\begin{aligned} \delta F_G^a &= \partial^\mu \delta G_\mu^a \\ \delta F_A &= \partial^\mu \delta A_\mu \\ \delta F_Z &= \partial^\mu \delta Z_\mu - \frac{1}{2}\, \xi_Z\, \frac{g_W}{c_w}\, \mu^{2\epsilon}(\sigma)\, v\, \delta G^0 \\ \delta F_\pm &= \partial^\mu \delta W_\mu^\pm \mp \frac{i}{2}\, \xi_W\, g_W\, \mu^{2\epsilon}(\sigma)\, v\, \delta G^\pm \end{aligned} \tag{6.48}$$

Finally, after performing the variations and integrating over $y$ as well as integrating by parts w.r.t. $x$ in (6.41), the ghost Lagrangian is found to be

$$\begin{aligned} \mathcal{L}_{\text{Ghost}}^{\text{QSI}} = {} & \partial^\mu \overline{c}^a\, D_\mu^{ac} c^c + \partial^\mu \overline{b}_A\, \partial_\mu b_A + \partial^\mu \overline{b}_Z\, \partial_\mu b_Z - \frac{1}{4}\, \xi_Z\, \frac{g_W^2}{c_w^2}\, \mu^{2\epsilon}(\sigma)\, v^2\, \overline{b}_Z\, b_Z \\ & + \partial^\mu \overline{b}_+\, \partial_\mu b_+ - \frac{1}{4}\, \xi_W\, g_W^2\, \mu^{2\epsilon}(\sigma)\, v^2\, \overline{b}_+\, b_+ \\ & + \partial^\mu \overline{b}_-\, \partial_\mu b_- - \frac{1}{4}\, \xi_W\, g_W^2\, \mu^{2\epsilon}(\sigma)\, v^2\, \overline{b}_-\, b_- \\ & - \frac{1}{4}\, \xi_W\, g_W^2\, \mu^{2\epsilon}(\sigma)\, v\, h\, \overline{b}_+\, b_+ - \frac{1}{4}\, \xi_W\, g_W^2\, \mu^{2\epsilon}(\sigma)\, v\, h\, \overline{b}_-\, b_- \\ & - \frac{i}{4}\, \xi_W\, g_W^2\, \mu^{2\epsilon}(\sigma)\, v\, G^0\, \overline{b}_+\, b_+ + \frac{i}{4}\, \xi_W\, g_W^2\, \mu^{2\epsilon}(\sigma)\, v\, G^0\, \overline{b}_-\, b_- \\ & - \frac{1}{4}\, \xi_Z\, \frac{g_W^2}{c_w^2}\, \mu^{2\epsilon}(\sigma)\, v\, h\, \overline{b}_Z\, b_Z \\ & - \frac{1}{4}\, \xi_W\, g_W^2\, \frac{c_{2w}}{c_w}\, \mu^{2\epsilon}(\sigma)\, v\, G^+\, \overline{b}_+\, b_Z - \frac{1}{4}\, \xi_W\, g_W^2\, \frac{c_{2w}}{c_w}\, \mu^{2\epsilon}(\sigma)\, v\, G^-\, \overline{b}_-\, b_Z \\ & - \frac{1}{4}\, \xi_Z\, \frac{g_W^2}{c_w}\, \mu^{2\epsilon}(\sigma)\, v\, G^-\, \overline{b}_Z\, b_+ - \frac{1}{4}\, \xi_Z\, \frac{g_W^2}{c_w}\, \mu^{2\epsilon}(\sigma)\, v\, G^+\, \overline{b}_Z\, b_- \\ & - \frac{1}{4}\, \xi_W\, e\, g_W\, \mu^{2\epsilon}(\sigma)\, v\, G^+\, \overline{b}_+\, b_A - \frac{1}{4}\, \xi_W\, e\, g_W\, \mu^{2\epsilon}(\sigma)\, v\, G^-\, \overline{b}_-\, b_A \\ & - i\, e\, A_\mu\, \partial^\mu \overline{b}_+\, b_+ + i\, e\, A_\mu\, \partial^\mu \overline{b}_-\, b_- - i\, g_W\, c_w\, Z_\mu\, \partial^\mu \overline{b}_+\, b_+ \\ & + i\, g_W\, c_w\, Z_\mu\, \partial^\mu \overline{b}_-\, b_- + i\, g_W\, c_w\, W_\mu^+\, \partial^\mu \overline{b}_+\, b_Z - i\, g_W\, c_w\, W_\mu^-\, \partial^\mu \overline{b}_-\, b_Z \\ & + i\, g_W\, c_w\, W_\mu^-\, \partial^\mu \overline{b}_Z\, b_+ - i\, g_W\, c_w\, W_\mu^+\, \partial^\mu \overline{b}_Z\, b_- + i\, e\, W_\mu^+\, \partial^\mu \overline{b}_+\, b_A \\ & - i\, e\, W_\mu^-\, \partial^\mu \overline{b}_-\, b_A + i\, e\, W_\mu^-\, \partial^\mu \overline{b}_A\, b_+ - i\, e\, W_\mu^+\, \partial^\mu \overline{b}_A\, b_- \end{aligned} \tag{6.49}$$

where

$$D^{ac}_\mu = \delta^{ac}\,\partial_\mu + g_s\,f^{abc}\,G^b_\mu \tag{6.50}$$

Obviously, the ghost Lagrangian in (6.49) is formulated in the broken phase of the theory. Choosing Landau gauge $\xi_G = \xi_A = \xi_Z = \xi_\pm = 0$, the Lagrangian is given by

$$\begin{aligned}
\mathcal{L}^{\text{QSI}}_{\text{Ghost}} = {} & \partial^\mu \overline{c}^a\, D^{ac}_\mu c^c + \partial^\mu \overline{b}_A\, \partial_\mu b_A + \partial^\mu \overline{b}_Z\, \partial_\mu b_Z + \partial^\mu \overline{b}_+\, \partial_\mu b_+ + \partial^\mu \overline{b}_-\, \partial_\mu b_- \\
& - i\,e\,A_\mu\,\partial^\mu \overline{b}_+\, b_+ + i\,e\,A_\mu\,\partial^\mu \overline{b}_-\, b_- - i\,g_W\,c_w\,Z_\mu\,\partial^\mu \overline{b}_+\, b_+ \\
& + i\,g_W\,c_w\,Z_\mu\,\partial^\mu \overline{b}_-\, b_- + i\,g_W\,c_w\,W^+_\mu\,\partial^\mu \overline{b}_+\, b_Z - i\,g_W\,c_w\,W^-_\mu\,\partial^\mu \overline{b}_-\, b_Z \\
& + i\,g_W\,c_w\,W^-_\mu\,\partial^\mu \overline{b}_Z\, b_+ - i\,g_W\,c_w\,W^+_\mu\,\partial^\mu \overline{b}_Z\, b_- + i\,e\,W^+_\mu\,\partial^\mu \overline{b}_+\, b_A \\
& - i\,e\,W^-_\mu\,\partial^\mu \overline{b}_-\, b_A + i\,e\,W^-_\mu\,\partial^\mu \overline{b}_A\, b_+ - i\,e\,W^+_\mu\,\partial^\mu \overline{b}_A\, b_-
\end{aligned} \tag{6.51}$$

which is manifestly scale invariant, i.e. in the unbroken phase of the theory.

# 7. QSI SM Effective Potential

In this chapter the effective potential for the complete quantum scale invariant Standard Model, discussed in chapter 6, is determined at the 1-loop level. It is shown that this effective potential indeed is quantum scale invariant, similar to the 2 Scalar Model in chapter 3. The effective potential in the QSI Standard Model at the 1-loop level has been determined in [13] for a Higgs potential containing $\lambda_6 \neq 0$ at tree-level. In this thesis, the effective potential is determined in a more Feynman diagrammatic approach than in [13] using the full QSI SM Lagrangian (6.2) and for the Higgs potential (6.20), i.e. with $\lambda_{4+2n} \equiv 0, \forall\, n$, especially $\lambda_6 \equiv 0$, at tree-level.

Analogous to chapter 3, the Lagrangian is not expanded about the fields VEVs, and thus the theory is considered in a manifestly scale invariant form, however, a field shift is applied to the Lagrangian. Due to gauge invariance it is sufficient to perform the field shift on only one of the four scalar fields in the Higgs doublet (6.18) [10]. Thus, the field shift is given by

$$
\begin{aligned}
&\Phi \longrightarrow \Phi + \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}\,\phi_0 \end{pmatrix} = \begin{pmatrix} G^+ \\ \frac{\phi+\phi_0+i\,G^0}{\sqrt{2}} \end{pmatrix} \\
&\sigma \longrightarrow \sigma + \sigma_0
\end{aligned}
\tag{7.1}
$$

where $\phi_0$ and $\sigma_0$ are background fields. Furthermore, it is chosen to work in Landau gauge, i.e.

$$
\xi = \xi_G = \xi_A = \xi_Z = \xi_W = 0 \tag{7.2}
$$

due to the following reasons

- the propagators of the gauge fields $\{G^a, A, Z, W^\pm\}$ are transverse [10]
- the associated ghosts are massless and couple only to the gauge fields [10], i.e. the ghosts decouple from the Higgs and Goldstone bosons
- Landau gauge ensures that the Lagrangian is manifestly scale invariant, as discussed in chapter 6.

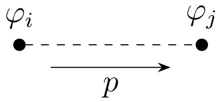
The field dependent masses are given in (C.6) for $\widetilde{M}_{ij}^2$ and in (C.17, C.18) for $\widetilde{M}_H^2$ & $\widetilde{M}_S^2$ if the VEVs $\{v, w\}$ are replaced by the background fields $\{\phi_0, \sigma_0\}$ as well as given by

$$
\begin{aligned}
&\widetilde{M}_G^2 = M_G^2 = \mu^{2\epsilon}(\sigma_0)\,\frac{\lambda_\phi}{6}\,\phi_0^2\left(1 + 3\,\frac{\lambda_m}{\lambda_\phi}\,\frac{\sigma_0^2}{\phi_0^2}\right) \\
&\widetilde{M}_Z^2 = M_Z^2 = \mu^{2\epsilon}(\sigma_0)\,\frac{1}{4}\,\frac{g_W^2}{c_w^2}\,\phi_0^2, \qquad \widetilde{M}_W^2 = M_W^2 = \mu^{2\epsilon}(\sigma_0)\,\frac{1}{4}\,g_W^2\,\phi_0^2 \\
&\widetilde{m}_f = m_f = \mu^{\epsilon}(\sigma_0)\,\frac{1}{\sqrt{2}}\,y_f\,\phi_0
\end{aligned}
\tag{7.3}
$$

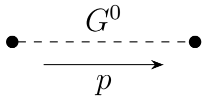
Analogous to chapter 3, the interaction coefficients $\widetilde{\mathcal{V}}_{ijk\cdots}$ are also field dependent and the relations (3.4) and (3.5) as well as

$$
\begin{aligned}
\frac{\partial \widetilde{M}_G^2}{\partial \phi_0} &= \widetilde{\mathcal{V}}_{133} = \widetilde{\mathcal{V}}_{145}, & \frac{\partial \widetilde{M}_G^2}{\partial \sigma_0} &= \widetilde{\mathcal{V}}_{233} = \widetilde{\mathcal{V}}_{245} \\
\frac{\partial \widetilde{M}_Z^2}{\partial \phi_0} &= 2\,\frac{\widetilde{M}_Z^2}{\phi_0}, & \frac{\partial \widetilde{M}_Z^2}{\partial \sigma_0} &= 2\,\epsilon\,(1+\epsilon)\,\frac{\widetilde{M}_Z^2}{\sigma_0} + \mathcal{O}\left(\epsilon^3\right) \\
\frac{\partial \widetilde{M}_W^2}{\partial \phi_0} &= 2\,\frac{\widetilde{M}_W^2}{\phi_0}, & \frac{\partial \widetilde{M}_W^2}{\partial \sigma_0} &= 2\,\epsilon\,(1+\epsilon)\,\frac{\widetilde{M}_W^2}{\sigma_0} + \mathcal{O}\left(\epsilon^3\right) \\
\frac{\partial \widetilde{m}_f}{\partial \phi_0} &= \frac{\widetilde{m}_f}{\phi_0}, & \frac{\partial \widetilde{m}_f}{\partial \sigma_0} &= \epsilon\,(1+\epsilon)\,\frac{\widetilde{m}_f}{\sigma_0} + \mathcal{O}\left(\epsilon^3\right)
\end{aligned}
\tag{7.4}
$$

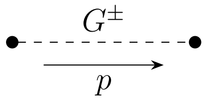
are necessary for the derivation of the effective potential in a diagrammatic approach. Moreover, the necessary Feynman rules, here in a general gauge, are provided by the propagators



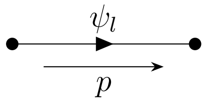

$$= i\left(\widetilde{D}_p^{-1}\right)_{ij}$$



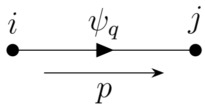

$$= \frac{i}{p^2 - \left(\widetilde{M}_G^2 + \xi_Z\,\widetilde{M}_Z^2\right)}$$



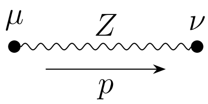

$$= \frac{i}{p^2 - \left(\widetilde{M}_G^2 + \xi_W\,\widetilde{M}_W^2\right)}$$



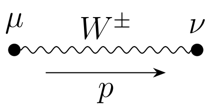

$$= \frac{i\left(\not{p} + \widetilde{m}_l\right)}{p^2 - \widetilde{m}_l^2} \tag{7.5}$$




$$= \frac{i\left(\not{p} + \widetilde{m}_q\right)}{p^2 - \widetilde{m}_q^2}\,\delta^{ij}$$

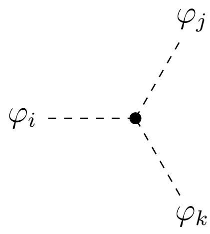



$$= \mu^{2\epsilon}(\sigma_0)\,\frac{-\,i}{p^2 - \widetilde{M}_Z^2}\left(\eta^{\mu\nu} - (1-\xi_Z)\,\frac{p^\mu\,p^\nu}{p^2 - \xi_Z\,\widetilde{M}_Z^2}\right)$$

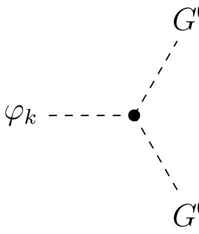



$$= \mu^{2\epsilon}(\sigma_0)\,\frac{-\,i}{p^2 - \widetilde{M}_W^2}\left(\eta^{\mu\nu} - (1-\xi_W)\,\frac{p^\mu\,p^\nu}{p^2 - \xi_W\,\widetilde{M}_W^2}\right)$$

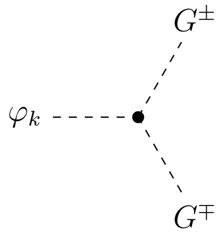
the scalar 3-interactions




$$= \quad -i\,\widetilde{\mathcal{V}}_{ijk}$$

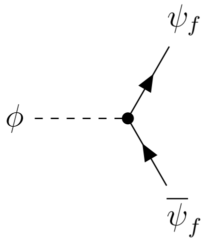



$$= \quad -i\,\widetilde{\mathcal{V}}_{k33},$$

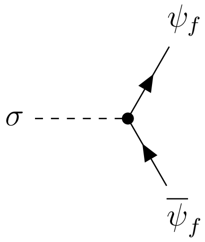



$$= \quad -i\,\widetilde{\mathcal{V}}_{k45} \tag{7.6}$$

where $i, j, k \in \{1, 2\}$, the Yukawa interactions

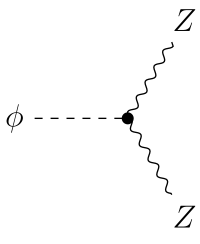



$$= \quad -\mu^{\epsilon}(\sigma_0)\,\frac{i}{\sqrt{2}}\,y_f$$

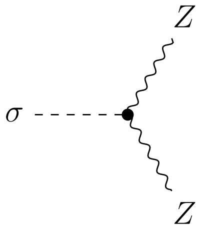



$$= \quad -\mu^{\epsilon}(\sigma_0)\,\epsilon\,(1+\epsilon)\,\frac{i}{\sqrt{2}}\,y_f\,\frac{\phi_0}{\sigma_0} + \mathcal{O}\left(\epsilon^3\right) \tag{7.7}$$

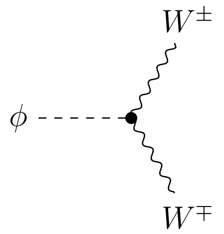
and the scalar - gauge boson interactions



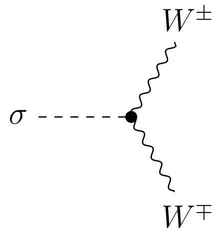

$$= \frac{i}{2}\,\frac{g_W^2}{c_w^2}\,\phi_0\,\eta^{\mu\nu}$$




$$= \mu^{-2\epsilon}(\sigma_0)\frac{i\,2\epsilon\,(1+\epsilon)}{\sigma_0}\left[(p_1 \cdot p_2)\,\eta^{\mu\nu} - \left(1 - \frac{1}{\xi_Z}\right) p_1^{\mu}\,p_2^{\nu}\right] + \mathcal{O}\left(\epsilon^3\right) \tag{7.8}$$

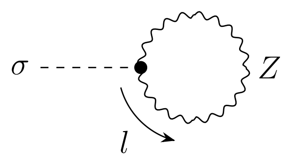



$$= \frac{i}{2}\, g_W^2\, \phi_0\, \eta^{\mu\nu}$$




$$= \mu^{-2\epsilon}(\sigma_0) \frac{i\, 2\epsilon\,(1+\epsilon)}{\sigma_0} \left[ (p_1 \cdot p_2)\, \eta^{\mu\nu} - \left(1 - \frac{1}{\xi_W}\right) p_1^\mu\, p_2^\nu \right] + \mathcal{O}\left(\epsilon^3\right) \tag{7.9}$$

The reason for providing the Feynman rules in a general gauge becomes obvious in (7.8) and (7.9) for the Feynman rules of $\sigma$ with the corresponding gauge fields. It can be seen that these two Feynman rules contain $1/\xi_i$, and thus would diverge for $\xi_i \to 0$, which is due to the prefactor of $\mu^{-2\epsilon}(\sigma)$ in the gauge fixing Lagrangian, as shown in section 6.5. In order to obtain results in Landau gauge (7.2), the limit $\xi_i \to 0$ needs to be taken after evaluating the corresponding Feynman diagrams in a general gauge, which is shown in the following for the tadpole diagram with external $\sigma$-leg and $Z$-loop. Hence, in Landau gauge this tadpole diagram evaluates as




$$\begin{aligned}
&= \lim_{\xi_Z \to 0} \frac{1}{2} \int \frac{d^D l}{(2\,\pi)^D} \Bigg\{ \mu^{-2\epsilon}(\sigma_0) \frac{i\, 2\epsilon\,(1+\epsilon)}{\sigma_0} \left[ l^2\, \eta^{\mu\nu} - \left(1 - \frac{1}{\xi_Z}\right) l^\mu\, l^\nu \right] \\
&\qquad \times \mu^{2\epsilon}(\sigma_0)\, \frac{-\,i}{l^2 - \widetilde{M}_Z^2} \left( \eta_{\mu\nu} - (1-\xi_Z)\, \frac{l_\mu\, l_\nu}{l^2 - \xi_Z\, \widetilde{M}_Z^2} \right) + \mathcal{O}\left(\epsilon^3\right) \Bigg\} \\
&= \lim_{\xi_Z \to 0} \int \frac{d^D l}{(2\,\pi)^D} \Bigg\{ \frac{\epsilon\,(1+\epsilon)}{\sigma_0}\, \frac{1}{l^2 - \widetilde{M}_Z^2} \left[ D\, l^2 + \left(1 - \frac{1}{\xi_Z}\right) \left( \frac{l^4}{l^2 - \xi_Z\, \widetilde{M}_Z^2} - l^2 \right) \right] \\
&\qquad + \mathcal{O}\left(\epsilon^3\right) \Bigg\} \\
&= \int \frac{d^D l}{(2\,\pi)^D}\, \frac{\epsilon\,(1+\epsilon) + \mathcal{O}\left(\epsilon^3\right)}{\sigma_0}\, \frac{D\, l^2 - \widetilde{M}_Z^2}{l^2 - \widetilde{M}_Z^2}
\end{aligned} \tag{7.10}$$

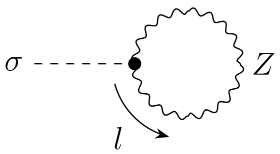
where in the last line the limit $\xi_Z \to 0$ has been taken explicitly. Now, the two parts of the final integral in (7.10) are considered separately, starting with

$$
\begin{aligned}
\int \frac{d^D l}{(2\,\pi)^D} \, \frac{\epsilon\,(1+\epsilon) + \mathcal{O}\,(\epsilon^3)}{\sigma_0} \, \frac{D\, l^2}{l^2 - \widetilde{M}_Z^2} \\
&= \frac{i\, D}{(4\,\pi)^{D/2}} \, \frac{\epsilon\,(1+\epsilon) + \mathcal{O}\,(\epsilon^3)}{\sigma_0} \, \frac{\Gamma\,(1 + D/2)\; \Gamma\,(-D/2)}{\Gamma\,(D/2)} \, \left(\widetilde{M}_Z^2\right)^{2-\epsilon} \\
&= \frac{i}{(4\,\pi)^{D/2}} \, \frac{\Gamma\,(1 + D/2)\; \Gamma\,(-D/2)}{\Gamma\,(D/2)} \, \frac{\partial}{\partial \sigma_0} \left(\widetilde{M}_Z^2\right)^{2-\epsilon} \\
&= \frac{\partial}{\partial \sigma_0} \, \frac{i}{(16\,\pi)^2} \, \mu^{2\epsilon}(\sigma_0) \, \frac{D}{2} \, \Gamma\,(\epsilon - 2) \left(\hat{M}_Z^2\right)^2 \left(\frac{4\,\pi\, \mu^2(\sigma_0)}{\widetilde{M}_Z^2}\right)^{\epsilon} \\
&= -\,\frac{D}{2} \, \frac{\partial}{\partial \sigma_0} \int \frac{d^D l}{(2\,\pi)^D} \, \log\left(l^2 - \widetilde{M}_Z^2\right)
\end{aligned}
\tag{7.11}
$$

where the momentum integral was evaluated explicitly in the first line, the corresponding relation for $\widetilde{M}_Z^2$ in (7.4) has been used in the second line, $\hat{M}_Z^2 = \mu^{-2\epsilon}(\sigma_0)\, \widetilde{M}_Z^2 = g_W^2 / \, (4\, c_w^2)\; \phi_0^2$ has been used in the third line, i.e. $\mu^{2\epsilon}(\sigma_0)$ has been pulled out of $\widetilde{M}_Z^2$, and (D.4) has been used in the last line. The second part of the final integral in (7.10) evaluates as

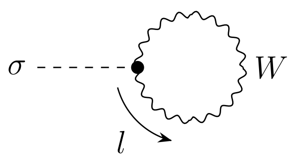
$$
\int \frac{d^D l}{(2\,\pi)^D} \, \frac{\epsilon\,(1+\epsilon) + \mathcal{O}\,(\epsilon^3)}{\sigma_0} \, \frac{-\,\widetilde{M}_Z^2}{l^2 - \widetilde{M}_Z^2} = \frac{1}{2} \, \frac{\partial}{\partial \sigma_0} \int \frac{d^D l}{(2\,\pi)^D} \, \log\left(l^2 - \widetilde{M}_Z^2\right) \tag{7.12}
$$

where the corresponding relation for $\widetilde{M}_Z^2$ in (7.4) has been used. Hence, in Landau gauge the tadpole diagram in (7.10) is given by




$$
= \frac{1 - D}{2} \, \frac{\partial}{\partial \sigma_0} \int \frac{d^D l}{(2\,\pi)^D} \, \log\left(l^2 - \widetilde{M}_Z^2\right) \tag{7.13}
$$

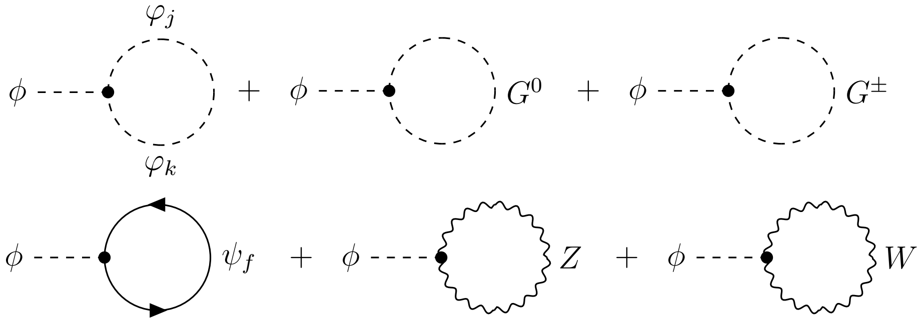
Analogously, the tadpole diagram with external $\sigma$-leg and $W$-loop in Landau gauge is provided by




$$
= (1 - D) \, \frac{\partial}{\partial \sigma_0} \int \frac{d^D l}{(2\,\pi)^D} \, \log\left(l^2 - \widetilde{M}_W^2\right) \tag{7.14}
$$

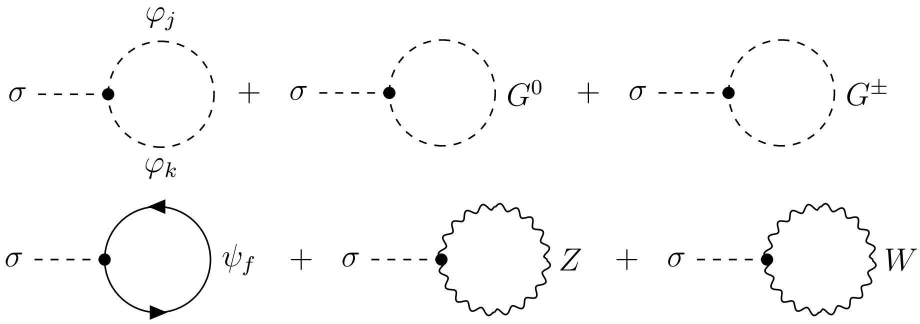
which does *not* have a symmetry factor of 1/2 due to the $W$-loop. The other tadpole diagrams with external $\sigma$-leg, i.e. with scalar and fermion loops, and all contributing tadpole diagrams with external $\phi$-leg can be evaluated more straightforward and analogous to section 3.1. Hence, at the 1-loop level one obtains

$$-i\,\frac{\partial V_{1\mathrm{L}}}{\partial\phi_0}=i\,\frac{\delta\Gamma_{1\mathrm{L}}}{\delta\phi}\bigg|_{\substack{\phi=\phi_0\\ \sigma=\sigma_0}}$$




$$\begin{aligned}
&=\frac{1}{2}\int\frac{d^Dl}{(2\,\pi)^D}\,\widetilde{\mathcal{V}}_{1jk}\left(\widetilde{D}_l^{-1}\right)_{jk}+\frac{1}{2}\int\frac{d^Dl}{(2\,\pi)^D}\,\frac{\widetilde{\mathcal{V}}_{133}}{l^2-\widetilde{M}_G^2}+\int\frac{d^Dl}{(2\,\pi)^D}\,\frac{\widetilde{\mathcal{V}}_{145}}{l^2-\widetilde{M}_G^2}\\
&\quad-\sum_f N_{c,f}\;\mathrm{Tr}\int\frac{d^Dl}{(2\,\pi)^D}\,\frac{\widetilde{m}_f}{\phi_0}\,\frac{\not{l}+\widetilde{m}_f}{l^2-\widetilde{m}_f^2}\\
&\quad+\int\frac{d^Dl}{(2\,\pi)^D}\,\frac{\widetilde{M}_Z^2}{\phi_0}\,\frac{D-1}{l^2-\widetilde{M}_Z^2}+\int\frac{d^Dl}{(2\,\pi)^D}\,\frac{2\,\widetilde{M}_W^2}{\phi_0}\,\frac{D-1}{l^2-\widetilde{M}_W^2}\\
&=-\frac{1}{2}\,\frac{\partial}{\partial\phi_0}\int\frac{d^Dl}{(2\,\pi)^D}\,\left[\log\left(l^2-\widetilde{M}_H^2\right)+\log\left(l^2-\widetilde{M}_S^2\right)+3\,\log\left(l^2-\widetilde{M}_G^2\right)\right]\\
&\quad+\sum_f 2\,N_{c,f}\,\frac{\partial}{\partial\phi_0}\int\frac{d^Dl}{(2\,\pi)^D}\,\log\left(l^2-\widetilde{m}_f^2\right)\\
&\quad+\frac{\partial}{\partial\phi_0}\int\frac{d^Dl}{(2\,\pi)^D}\left[\frac{1-D}{2}\,\log\left(l^2-\widetilde{M}_Z^2\right)+(1-D)\,\log\left(l^2-\widetilde{M}_W^2\right)\right]
\end{aligned}\tag{7.15}$$

$$-i\,\frac{\partial V_{1\mathrm{L}}}{\partial\sigma_0}=i\,\frac{\delta\Gamma_{1\mathrm{L}}}{\delta\sigma}\bigg|_{\substack{\phi=\phi_0\\ \sigma=\sigma_0}}$$




$$\begin{aligned}
&=-\frac{1}{2}\,\frac{\partial}{\partial\sigma_0}\int\frac{d^Dl}{(2\,\pi)^D}\,\left[\log\left(l^2-\widetilde{M}_H^2\right)+\log\left(l^2-\widetilde{M}_S^2\right)+3\,\log\left(l^2-\widetilde{M}_G^2\right)\right]\\
&\quad+\sum_f 2\,N_{c,f}\,\frac{\partial}{\partial\sigma_0}\int\frac{d^Dl}{(2\,\pi)^D}\,\log\left(l^2-\widetilde{m}_f^2\right)\\
&\quad+\frac{\partial}{\partial\sigma_0}\int\frac{d^Dl}{(2\,\pi)^D}\left[\frac{1-D}{2}\,\log\left(l^2-\widetilde{M}_Z^2\right)+(1-D)\,\log\left(l^2-\widetilde{M}_W^2\right)\right]
\end{aligned}$$

where in (7.15) relations (3.4), (3.5) as well as (7.4), and in particular for $\partial V_{1L}/\partial\sigma_0$ (7.10) to (7.14) have been used. Further, it is summed implicitly over $j, k \in \{1, 2\}$ as well as explicitly summed over $f \in \{e^-, \mu^-, \tau^-, d, s, b, u, c, t\}$ and the colour factor $N_{c,f}$ is provided by

$$N_{c,f} = \begin{cases} 1, & f \in \{e^-, \mu^-, \tau^-\} \\ 3, & f \in \{d, s, b, u, c, t\} \end{cases} \tag{7.16}$$

Thus, from (7.15) it can be seen that the 1-loop contribution to the effective potential in the QSI Standard Model is given by

$$\begin{aligned} V_{1L} = -\, i \int \frac{d^D l}{(2\,\pi)^D} \Bigg\{ & \frac{1}{2} \left[ \log\left(l^2 - \widetilde{M}_H^2\right) + \log\left(l^2 - \widetilde{M}_S^2\right) \right] \\ & + \frac{3}{2} \log\left(l^2 - \widetilde{M}_G^2\right) - \sum_f 2\, N_{c,f} \log\left(l^2 - \widetilde{m}_f^2\right) \\ & - \frac{1-D}{2} \log\left(l^2 - \widetilde{M}_Z^2\right) - (1-D) \log\left(l^2 - \widetilde{M}_W^2\right) \Bigg\} \end{aligned} \tag{7.17}$$

Evaluating the momentum integral in (7.17) using (D.4) and replacing the background fields $\{\phi_0, \sigma_0\}$ by the fields $\{\phi, \sigma\}$, analogous to chapter 3, in order to finally obtain the 1-loop contribution of the effective potential in terms of the fields, one obtains

$$\begin{aligned} V_{1L}(\phi, \sigma) = -\frac{\mu^{2\epsilon}(\sigma)}{16\,\pi^2} \Bigg\{ & \sum_{k=1}^{2} \frac{\hat{M}_{\rho_k}^4(\phi,\sigma)}{4} \left[ \frac{1}{\epsilon} + \frac{3}{2} - \log\left( \frac{M_{\rho_k}^2(\phi,\sigma)}{4\,\pi\,\mu^2(\sigma)}\, e^{\gamma_E} \right) \right] \\ & + \frac{3}{4} \hat{M}_G^4(\phi,\sigma) \left[ \frac{1}{\epsilon} + \frac{3}{2} - \log\left( \frac{M_G^2(\phi,\sigma)}{4\,\pi\,\mu^2(\sigma)}\, e^{\gamma_E} \right) \right] \\ & - \sum_f N_{c,f}\, \hat{m}_f^4(\phi) \left[ \frac{1}{\epsilon} + \frac{3}{2} - \log\left( \frac{m_f^2(\phi,\sigma)}{4\,\pi\,\mu^2(\sigma)}\, e^{\gamma_E} \right) \right] \\ & + \frac{3}{4} \hat{M}_Z^4(\phi) \left[ \frac{1}{\epsilon} + \frac{5}{6} - \log\left( \frac{M_Z^2(\phi,\sigma)}{4\,\pi\,\mu^2(\sigma)}\, e^{\gamma_E} \right) \right] \\ & + \frac{3}{2} \hat{M}_W^4(\phi) \left[ \frac{1}{\epsilon} + \frac{5}{6} - \log\left( \frac{M_W^2(\phi,\sigma)}{4\,\pi\,\mu^2(\sigma)}\, e^{\gamma_E} \right) \right] \\ & + \Delta U_{1L}(\phi,\sigma) \Bigg\} + \mathcal{O}(\epsilon) \end{aligned} \tag{7.18}$$

where $\{\rho_k\}_{k=1}^2 = \{H, S\}$ and

$$\Delta U_{1L}(\phi,\sigma) \coloneqq -\frac{\mu^{2\epsilon}(\sigma)}{32\,\pi^2} \sum_{k=1}^{2} \hat{M}_{\rho_k}^4(\phi,\sigma)\, c_{\rho_k}^{(1)}(\phi,\sigma) \tag{7.19}$$

is exactly the same new finite quantum correction as for the 2 Scalar Model in chapter 3. In order to renormalise the 1-loop contribution to the effective potential in the MS-scheme, the counterterm potential $\widetilde{V}_{\mathrm{tree,ct1}}$ need to exactly cancel the divergent part of (7.18), i.e.

$$\begin{aligned}
V_{1\mathrm{L}}\big|_{\mathrm{div}} &+ \widetilde{V}_{\mathrm{tree,ct1}} \\
&= -\frac{\mu^{2\epsilon}(\sigma)}{16\,\pi^2}\Bigg\{\sum_{k=1}^{2}\frac{\hat{M}^4_{\rho_k}(\phi,\sigma)}{4} + \frac{3}{4}\,\hat{M}^4_G\,(\phi,\sigma) \\
&\qquad - \sum_f N_{c,f}\,\hat{m}^4_f\,(\phi) + \frac{3}{4}\,\hat{M}^4_Z\,(\phi) + \frac{3}{2}\,\hat{M}^4_W\,(\phi)\Bigg\}\frac{1}{\epsilon} \\
&\qquad + \mu^{2\epsilon}(\sigma)\left(\delta Z^{(1)}_{V_\phi}\,\frac{\lambda_\phi}{4!}\,\phi^4 + \delta Z^{(1)}_{V_m}\,\frac{\lambda_m}{4}\,\phi^2\,\sigma^2 + \delta Z^{(1)}_{V_\sigma}\,\frac{\lambda_\sigma}{4!}\,\sigma^4\right) \\
&\stackrel{!}{=} 0
\end{aligned} \tag{7.20}$$

Hence, the 1-loop counterterms in the MS-scheme are given by

$$\begin{aligned}
\delta Z^{(1)}_{V_\phi} &= \frac{1}{16\,\pi^2}\,\frac{3}{2\,\lambda_\phi}\left[\frac{4}{3}\,\lambda^2_\phi + \lambda^2_m - \sum_f 4\,N_{c,f}\,y^4_f + \frac{3}{2}\,g^4_W + \frac{3}{4}\,\frac{g^4_W}{c^4_w}\right]\frac{1}{\epsilon} \\
\delta Z^{(1)}_{V_m} &= \frac{1}{16\,\pi^2}\,\frac{2\,\lambda_\phi + 4\,\lambda_m + \lambda_\sigma}{2}\,\frac{1}{\epsilon} \\
\delta Z^{(1)}_{V_\sigma} &= \frac{1}{16\,\pi^2}\,\frac{3}{2}\,\frac{\lambda^2_\sigma + 4\,\lambda^2_m}{\lambda_\sigma}\,\frac{1}{\epsilon}
\end{aligned} \tag{7.21}$$

Finally, after Renormalisation and then going back to 4 dimensions, i.e. $\epsilon \to 0$, the result for the effective potential in the QSI Standard Model up to the 1-loop level is given by

$$V_{\mathrm{eff}}(\Phi,\sigma) = V_{\mathrm{H}}(\Phi,\sigma) + V_{1\mathrm{L,reg}}(\phi,\sigma) + \Delta U_{1\mathrm{L}}(\phi,\sigma) + \mathcal{O}\left(\hbar^2\right) \tag{7.22}$$

where $V_{\mathrm{H}}$ and $\Delta U_{1\mathrm{L}}\,(\phi,\sigma)$ are to be found in (6.20) and (7.19), respectively, and $V_{1\mathrm{L,reg}}$ in the $\overline{\mathrm{MS}}$-scheme is provided by

$$\begin{aligned}
V^{\overline{\mathrm{MS}}}_{1\mathrm{L,reg}}\,(\phi,\sigma) = \frac{1}{16\,\pi^2}\Bigg\{&\sum_{k=1}^{2}\frac{M^4_{\rho_k}\,(\phi,\sigma)}{4}\left[\log\left(\frac{M^2_{\rho_k}\,(\phi,\sigma)}{\mu^2(\sigma)}\right) - \frac{3}{2}\right] \\
&+ \frac{3}{4}\,M^4_G\,(\phi,\sigma)\left[\log\left(\frac{M^2_G\,(\phi,\sigma)}{\mu^2(\sigma)}\right) - \frac{3}{2}\right] \\
&- \sum_f N_{c,f}\,m^4_f\,(\phi)\left[\log\left(\frac{m^2_f\,(\phi)}{\mu^2(\sigma)}\right) - \frac{3}{2}\right] \\
&+ \frac{3}{4}\,M^4_Z\,(\phi)\left[\log\left(\frac{M^2_Z\,(\phi)}{\mu^2(\sigma)}\right) - \frac{5}{6}\right] \\
&+ \frac{3}{2}\,M^4_W\,(\phi)\left[\log\left(\frac{M^2_W\,(\phi)}{\mu^2(\sigma)}\right) - \frac{5}{6}\right]\Bigg\}
\end{aligned} \tag{7.23}$$

**Remark.**

(i) Again, it has been used that in 4 dimensions, $\hat{M}_i^2$ is identical to $M_i^2$. Furthermore, in (7.18), (7.20) and (7.23) it has been used that in $D = 4 - 2\epsilon$ dimensions the masses $m_f$, $M_Z$ and $M_W$ depend on $\sigma$ only via the factor $\mu^\epsilon(\sigma)$, as can be seen in (7.3), and thus these masses do *not* depend on $\sigma$ in 4 dimensions and if they are labelled with a "hat", as $\hat{M}_i$ are the masses without factors of $\mu^\epsilon(\sigma)$.

(ii) As in the case of the 2 Scalar Model discussed in chapter 3, it can be seen that beside the regular Coleman-Weinberg term $V_{\text{1L,reg}}$, given in (7.23), a new finite quantum correction $\Delta U_{\text{1L}}$, provided in (7.19), is obtained due to evanescent interactions introduced by the Renormalisation function, i.e. as a result of QSI. Again, this new quantum correction contains a higher dimensional non-polynomial operator of the form $\phi^6/\sigma^2$.

(iii) It is important to note that this new finite quantum correction is exactly the same as in the 2 Scalar Model in (3.15). Hence, in the QSI Standard Model there are *no* additional new finite quantum corrections at the 1-loop level compared to the 2 Scalar Model. The reason for this that, in contrast to $\widetilde{M}_H^2$ and $\widetilde{M}_S^2$, the masses $\widetilde{M}_G^2$, $\widetilde{m}_f$, $\widetilde{M}_Z^2$ and $\widetilde{M}_W^2$ do *not* obtain evanescent corrections, as can be seen in (7.3). However, additional new quantum corrections are expected to emerge at the 2-loop level because at the 2-loop level not only the masses, cf. (7.17), but also the coupling constants contribute to the effective potential and due to the fact that there are more coupling constants obtaining evanescent corrections in the QSI Standard Model than in the 2 Scalar Model.

(iv) Again, the 1-loop effective potential (7.22) is a homogeneous function of the fields, and thus satisfies (2.13). No massive parameters are introduced at the quantum level due to the usage of SIDReg with a dynamical Renormalisation function $\mu(\sigma)$ instead of DReg. Hence, the QSI Standard Model indeed is scale invariant at the quantum level (at least at the 1-loop-level), i.e. quantum scale invariant, as intended.

(v) The results in this chapter are in perfect agreement with [13] for $\lambda_6 \equiv 0$. This confirms once again that using a Feynman diagrammatic approach with an expanded Lagrangian provides the same manifestly scale invariant effective potential as working directly with equation (15) in [13]. Moreover, said equation in [13] has been derived from a Feynman diagrammatic approach in this chapter as shown in (7.15) and (7.17).

(vi) The result of the effective potential (7.22), which agrees with the literature [13], also works as a consistency check of the QSI Standard Model provided in chapter 6 formulated in terms of rescaled gauge fields with absorbed gauge couplings.

# 8. Summary and Outlook

The concept of spontaneously broken quantum scale symmetry as well as its realisation via SIDReg and its implications have extensively been discussed. In this context, the 2 Scalar Model has been introduced in detail, similarly to [11, 14]. The QSI 2 Scalar Model is the simplest model with dynamical SSB of quantum scale symmetry, but nonetheless physically relevant due to its connection to the QSI Standard Model Higgs sector. Therefore, it is an excellent model to illustrate the concepts of QSI. The effective potential of the 2 Scalar Model has been determined at the 2-loop level and compared to the results in the literature [14]. This 2-loop effective potential is not only manifestly scale invariant but also obtains new finite and divergent quantum corrections that emerge from evanescent interactions cancelling UV-divergences. These evanescent interactions are introduced to the theory by the Dilaton-dependent Renormalisation function, and thus are a consequence of spontaneously broken QSI. The new divergent quantum corrections are of particular interest, as they do not only change the divergence structure of the theory at the 2-loop level, and consequently change the 2-loop counterterms, but also introduce higher dimensional non-polynomial operators that need to be renormalised implying non-Renormalisability of the theory. Furthermore, the $\beta$ - functions of the QSI 2 Scalar Model have been determined at the 2-loop level and compared with the literature [14] as well. It has been shown that at the 1-loop level they are the same as in the DReg-regularised theory, however, at the 2-loop level one obtains new corrections to the $\beta$ - functions, and thus the 2-loop running of the couplings in the QSI theory is different from the usual DReg-regularised theory. The reason for this are the new divergent quantum corrections, mentioned above, leading to corrections of the 2-loop counterterms. Particularly interesting is that a theory that is regularised using SIDReg is scale invariant even at the quantum level, where scale invariance is only broken spontaneously, but nonetheless admits non-zero $\beta$ - functions, and thus a running of the couplings, implying the vanishing of the $\beta$ - functions is not necessary for QSI.

One might draw the conclusion that there are 3 main consequences of spontaneously broken quantum scale symmetry, realised via SIDReg, that affect physics. First, all scales, including the Renormalisation scale, are generated dynamically via SSB of scale symmetry implying the absence of anomalous breaking of scale symmetry as there is no initial scale in the theory. Second, new finite and divergent quantum corrections arising from evanescent interactions generated by the Dilaton-dependent Renormalisation function. Third, non-Renormalisability due to an infinite amount of those evanescent interactions.

Moreover, it has explicitly been shown that working in the broken phase of the theory with expanded Lagrangian is valid even in the context of spontaneously broken quantum scale symmetry. In particular, it has been shown in the QSI 2 Scalar Model that the same scale invariant counterterms, even at the 2-loop level with new divergent quantum corrections, are obtained from Green-functions with non-vanishing external momenta computed from a Lagrangian in the broken phase of the theory. At the 2-loop level, this has been done using the corresponding 2-loop self energies. This is an important consistency check, which has not been done to this extend in the literature so far, because in QSI theories, i.e. theories regularised with SIDReg, the Dilaton appears to an anomalous power in the Lagrangian, and thus it is necessary to expand the Lagrangian in order to derive Feynman rules and perform perturbative calculations.

The concept of spontaneously broken quantum scale symmetry in the context of gauge theories has been discussed in more detail than in the literature so far. The consistent formulation of a QSI gauge theory has been presented by the example of a generic $SU(N)$ gauge theory using two approaches, one with rescaled and the other one with non-rescaled fields. Furthermore, the physically relevant scattering process $e^- \, e^+ \longrightarrow \mu^- \, \mu^+$ has been considered at the 1-loop level in the context of spontaneously broken quantum scale symmetry, which has not been done before. It has been shown that new finite and divergent quantum corrections can arise due to evanescent interactions not only by cancelling UV- but also by cancelling IR-divergences. New IR-divergences due to QSI are of particular interest, as they also need to be cancelled by real emission corrections at the level of the cross section at every order in the perturbation theory. In this context, it has been conjectured and exemplarily proven that such new quantum corrections arising from IR-divergences cancel together with the regular IR-divergences leaving an UV- and IR-finite result, containing only the regular result and possible new finite quantum corrections emerging from UV-divergences. This is an important result to ensure the finiteness of physical observables in theories with spontaneously broken quantum scale symmetry.

Finally, a complete quantum scale invariant Standard Model has been formulated, which has not been done to this extend in the literature so far. In the framework of this QSI Standard Model the 1-loop effective potential has been determined using a Feynman diagrammatic appraoch and the background field shifted Lagrangian. It has been shown that this 1-loop effective potential is manifestly scale invariant and it has been compared with the literature [13] as a consistency check. At the 1-loop level, the QSI Standard Model effective potential also obtains a new finite quantum correction arising from evanescent interactions cancelling UV-divergences, which is exactly the same as that of the QSI 2 Scalar Model.

Spontaneously broken quantum scale symmetry realised via a manifestly scale invariant Regularisation provides several aspects that are not only conceptually interesting but also from a phenomenological point of view, especially w.r.t. new quantum corrections emerging as a result of evanescent interactions. In future studies, the following aspects and problems in the context of spontaneously broken quantum scale symmetry could be analysed

- Further investigations of Conjecture 5.1, e.g. in other theories, for different processes or in more generality.

- Determining the effective potential of the quantum scale invariant Standard Model at the 2-loop level, which has not been done in the literature so far.

- Implications of spontaneously broken quantum scale symmetry to vacuum stability, inflation and the Higgs mass using the 2-loop effective potential of the QSI Standard Model.

- Further phenomenological analyses, e.g. in the full QSI Standard Model presented in chapter 6. Since the Higgs sector obtains the most modifications compared to the "usual" Standard Model, one could focus these investigations on Higgs physics. Especially, Higgs decays such as $H \longrightarrow \bar{b}\, b$ and $H \longrightarrow \tau^-\, \tau^+$ as well as Dilaton-Higgs scattering are of particular interest and could be used to determine a lower bound on the Dilaton VEV $\langle \sigma \rangle$.

- Implications of spontaneously broken quantum scale symmetry in Cosmology and in QFT in curved spacetime.

- In contrast to this thesis where global quantum scale symmetry has been considered, one could conduct further investigations of local, i.e. gauged, quantum scale symmetry (which has already been done in the literature).

- Although quantum scale invariance has been introduced to provide an alternative to Supersymmetry as BSM physics, one could nonetheless introduce spontaneously broken quantum scale symmetry to a supersymmetric theory. If this theory is also invariant under special conformal transformations one obtains a quantum superconformal field theory. In supersymmetric theories, one needs to use a manifestly scale invariant version of dimensional reduction (DRed), i.e. SIDRed instead of SIDReg.

- In such a supersymmetric theory one could analyse whether it is possible to construct a dynamical Renormalisation function containing a nilpotent Grassmann number. In this case, the Renormalisation function admits a finite power series, and thus only a finite number of new evanescent interactions are introduced to theory.

# A. Conformal Group

The conformal symmetry group of spacetime is an extension of the Poincare group and consists of the Poincare transformations, special conformal transformations and dilatations (i.e. scaling transformations). In $D = 3 + 1$ dimensions, the conformal symmetry has 15 degrees of freedom, namely, 10 for the Poincare group, 4 for special conformal transformations and 1 for dilatations. The 15 generators of the conformal group are given by the 4 generators of translations $\hat{P}_\mu$, the 6 Lorentz generators $\hat{M}_{\mu\nu}$, the dilatation generator $\hat{D}$ and the 4 generators of special conformal transformations $\hat{K}_\mu$. Accordingly, the conformal algebra $\{\hat{P}_\mu, \hat{M}_{\mu\nu}, \hat{D}, \hat{K}_\mu\}$ is an extension of the Poincare algebra by the dilatation and the special conformal generators $\hat{D}$ and $\hat{K}_\mu$, respectively, and is given by the commutation relations

$$\begin{aligned}
[\hat{P}_\mu, \hat{P}_\nu] &= 0 \\
[\hat{P}_\mu, \hat{M}_{\rho\sigma}] &= i\left(\eta_{\mu\rho}\hat{P}_\sigma - \eta_{\mu\sigma}\hat{P}_\rho\right) \\
[\hat{M}_{\mu\nu}, \hat{M}_{\rho\sigma}] &= i\left(\eta_{\nu\rho}\hat{M}_{\mu\sigma} + \eta_{\mu\sigma}\hat{M}_{\nu\rho} - \eta_{\mu\rho}\hat{M}_{\nu\sigma} - \eta_{\nu\sigma}\hat{M}_{\mu\rho}\right) \\
[\hat{D}, \hat{P}_\mu] &= i\hat{P}_\mu \\
[\hat{D}, \hat{M}_{\mu\nu}] &= 0 \\
[\hat{D}, \hat{D}] &= 0 \\
[\hat{D}, \hat{K}_\mu] &= -i\hat{K}_\mu \\
[\hat{K}_\mu, \hat{P}_\nu] &= 2i\left(\eta_{\mu\nu}\hat{D} - \hat{M}_{\mu\nu}\right) \\
[\hat{K}_\mu, \hat{M}_{\rho\sigma}] &= i\left(\eta_{\mu\rho}\hat{K}_\sigma - \eta_{\mu\sigma}\hat{K}_\rho\right) \\
[\hat{K}_\mu, \hat{K}_\nu] &= 0
\end{aligned} \tag{A.1}$$

Note that, without special conformal transformations, the (closed) algebra $\{\hat{P}_\mu, \hat{M}_{\mu\nu}, \hat{D}\}$ is also an extension of the Poincare algebra by only the dilatation generator $\hat{D}$ and given by the commutation relations

$$\begin{aligned}
[\hat{P}_\mu, \hat{P}_\nu] &= 0, \quad [\hat{D}, \hat{M}_{\mu\nu}] = 0, \quad [\hat{D}, \hat{D}] = 0 \\
[\hat{P}_\mu, \hat{M}_{\rho\sigma}] &= i\left(\eta_{\mu\rho}\hat{P}_\sigma - \eta_{\mu\sigma}\hat{P}_\rho\right) \\
[\hat{M}_{\mu\nu}, \hat{M}_{\rho\sigma}] &= i\left(\eta_{\nu\rho}\hat{M}_{\mu\sigma} + \eta_{\mu\sigma}\hat{M}_{\nu\rho} - \eta_{\mu\rho}\hat{M}_{\nu\sigma} - \eta_{\nu\sigma}\hat{M}_{\mu\rho}\right) \\
[\hat{D}, \hat{P}_\mu] &= i\hat{P}_\mu
\end{aligned} \tag{A.2}$$

A particular representation for the generators of the conformal group is given by the following set of differential operators acting on a scalar field $\phi(x)$ with scaling dimension $\Delta_\phi$

$$\begin{aligned}\hat{P}_\mu &= -i\,\partial_\mu \\ \hat{M}_{\mu\nu} &= i\,(x_\mu\,\partial_\nu - x_\nu\,\partial_\mu) \\ \hat{D} &= -i\,(\Delta_\phi + x^\mu\,\partial_\mu) \\ \hat{K}_\mu &= -i\,\big(2\,x_\mu\,x^\nu\,\partial_\nu - x^2\,\partial_\mu + 2\,\Delta_\phi\,x_\mu\big)\end{aligned} \tag{A.3}$$

and, analogously, acting on $N$-point Green functions $G^{(N)}(x_1,\ldots,x_N)$ constructed from scalar fields $\phi_j(x)$ with scaling dimension $\Delta_\phi$

$$\begin{aligned}\hat{P}_\mu &= -i\sum_{j=1}^{N}\frac{\partial}{\partial x_j^\mu} \\ \hat{M}_{\mu\nu} &= i\sum_{j=1}^{N}\left(x_{j,\mu}\,\frac{\partial}{\partial x_j^\nu} - x_{j,\nu}\,\frac{\partial}{\partial x_j^\mu}\right) \\ \hat{D} &= -i\left(N\,\Delta_\phi + \sum_{j=1}^{N} x_j^\mu\,\frac{\partial}{\partial x_j^\mu}\right) \\ \hat{K}_\mu &= -i\sum_{j=1}^{N}\left[\left(2\,x_{j,\mu}\,x_{j,\nu} - \eta_{\mu\nu}\,x_j^2\right)\frac{\partial}{\partial x_{j,\nu}} + 2\,\Delta_\phi\,x_{j,\mu}\right]\end{aligned} \tag{A.4}$$

For more details, the reader is referred to [21, 31]. Note, however, the different sign convention for the generators $\hat{P}_\mu$, $\hat{D}$ and $\hat{K}_\mu$ (compared with [21]) in order to be consistent with [31, 39, 40, 42], which also affects the signs in the commutator relations above.

# B. Renormalisation Function

In this chapter of the appendix, power series of the Renormalisation function

$$\mu(\sigma) = z\,\sigma^{\frac{2}{D-2}} = z\,\sigma^{\frac{1}{1-\epsilon}}, \tag{B.1}$$

expanded w.r.t. $\epsilon$ and $\mathfrak{D}/w$, with three different exponents that usually appear in the Lagrangian are provided.

$$\begin{aligned}\mu^{\epsilon}(\sigma) = \mu_0^{\epsilon} \Bigg( 1 &+ \frac{\mathfrak{D}}{w}\,\epsilon\,(1+\epsilon) - \frac{1}{2}\frac{\mathfrak{D}^2}{w^2}\,\epsilon + \frac{1}{3}\frac{\mathfrak{D}^3}{w^3}\,\epsilon\left(1 - \frac{1}{2}\epsilon\right) \\ &- \frac{1}{4}\frac{\mathfrak{D}^4}{w^4}\,\epsilon\left(1-\frac{5}{6}\epsilon\right) + \mathcal{O}\left(\left(\mathfrak{D}/w\right)^5, \epsilon^3\right)\Bigg)\end{aligned} \tag{B.2a}$$

$$\begin{aligned}\mu^{2\epsilon}(\sigma) = \mu_0^{2\epsilon} \Bigg( 1 &+ 2\,\frac{\mathfrak{D}}{w}\,\epsilon\,(1+\epsilon) - \frac{\mathfrak{D}^2}{w^2}\,\epsilon\,(1-\epsilon) + \frac{2}{3}\frac{\mathfrak{D}^3}{w^3}\,\epsilon\,(1-2\epsilon) \\ &- \frac{1}{2}\frac{\mathfrak{D}^4}{w^4}\,\epsilon\left(1-\frac{8}{3}\epsilon\right) + \mathcal{O}\left(\left(\mathfrak{D}/w\right)^5, \epsilon^3\right)\Bigg)\end{aligned} \tag{B.2b}$$

$$\begin{aligned}\mu^{-2\epsilon}(\sigma) = \mu_0^{-2\epsilon} \Bigg( 1 &- 2\,\frac{\mathfrak{D}}{w}\,\epsilon\,(1+\epsilon) + \frac{\mathfrak{D}^2}{w^2}\,\epsilon\,(1+3\epsilon) - \frac{2}{3}\frac{\mathfrak{D}^3}{w^3}\,\epsilon\,(1+4\epsilon) \\ &+ \frac{1}{2}\frac{\mathfrak{D}^4}{w^4}\,\epsilon\left(1+\frac{14}{3}\epsilon\right) + \mathcal{O}\left(\left(\mathfrak{D}/w\right)^5, \epsilon^3\right)\Bigg)\end{aligned} \tag{B.2c}$$

where $\sigma = \mathfrak{D} + w$ and $\mu_0 = \mu(\langle\sigma\rangle) = z\,\langle\sigma\rangle^{\frac{1}{1-\epsilon}} \equiv z\,w^{\frac{1}{1-\epsilon}}$.

# C. The 2 Scalar Model

In this chapter of the appendix explicit expressions for the coefficients of the 2 Scalar potential and other expressions in the 2 Scalar Model, such as counterterms, are provided.

## C.1. The 2 Scalar Potential

First, the coefficients of the potential of the 2 Scalar Model (2.41)

$$\begin{aligned}\widetilde{V}(h+w,\mathfrak{D}+w) = \widetilde{V}(v,w) + \widetilde{T}_{\varphi_i}\,\varphi_i + \frac{1}{2}\,\widetilde{M}^2_{ij}\,\varphi_i\,\varphi_j + \frac{1}{3!}\,\widetilde{\mathcal{V}}_{ijk}\,\varphi_i\,\varphi_j\,\varphi_k \\ + \frac{1}{4!}\,\widetilde{\mathcal{V}}_{ijkl}\,\varphi_i\,\varphi_j\,\varphi_k\,\varphi_l + \frac{1}{5!}\,\widetilde{\mathcal{V}}_{ijklm}\,\varphi_i\,\varphi_j\,\varphi_k\,\varphi_l\,\varphi_m \\ + \frac{1}{6!}\,\widetilde{\mathcal{V}}_{ijklmn}\,\varphi_i\,\varphi_j\,\varphi_k\,\varphi_l\,\varphi_m\,\varphi_n + \ldots\end{aligned} \tag{C.1}$$

where $\{\varphi_i\}_{i=1}^2 = \{h,\mathfrak{D}\}$, are considered.

$$\widetilde{V}(v,w) = \mu_0^{2\epsilon}\left(\frac{\lambda_\phi}{4!}\,v^4 + \frac{\lambda_m}{4}\,v^2\,w^2 + \frac{\lambda_\sigma}{4!}\,w^4\right) \tag{C.2}$$

The tadpoles are given by

$$\widetilde{T}_{\varphi_i} = \mu_0^{2\epsilon}\,\frac{v^3}{6}\,\left(t_{\varphi_i} + \epsilon\,t^{(1)}_{\varphi_i} + \epsilon^2\,t^{(2)}_{\varphi_i} + \mathcal{O}\left(\epsilon^3\right)\right) \tag{C.3}$$

with

$$\begin{aligned}t_h &= \lambda_\phi + 3\,\lambda_m\,\frac{w^2}{v^2} \\ t^{(1)}_h &= t^{(2)}_h = 0\end{aligned} \tag{C.4a}$$

$$\begin{aligned}t_{\mathfrak{D}} &= 3\,\lambda_m\,\frac{w}{v} + \lambda_\sigma\,\frac{w^3}{v^3} \\ t^{(1)}_{\mathfrak{D}} &= t^{(2)}_{\mathfrak{D}} = \frac{1}{2}\,\lambda_\phi\,\frac{v}{w} + 3\,\lambda_m\,\frac{w}{v} + \frac{1}{2}\,\lambda_\sigma\,\frac{w^3}{v^3}\end{aligned} \tag{C.4b}$$

The matrix of the squared masses is provided by

$$\widetilde{\mathcal{M}}^2_{\phi\sigma} = \begin{pmatrix}\widetilde{M}^2_{11} & \widetilde{M}^2_{12} \\ \widetilde{M}^2_{21} & \widetilde{M}^2_{22}\end{pmatrix}, \quad \text{where} \quad \widetilde{M}^2_{12} = \widetilde{M}^2_{21} \tag{C.5}$$

where

$$\widetilde{M}^2_{ij} = \mu_0^{2\epsilon}\,\frac{v^2}{2}\,\left(u_{ij} + \epsilon\, u^{(1)}_{ij} + \epsilon^2\, u^{(2)}_{ij} + \mathcal{O}\left(\epsilon^3\right)\right) \tag{C.6}$$

with

$$\begin{aligned}
u_{11} &= \lambda_\phi + \lambda_m\,\frac{w^2}{v^2}\\
u^{(1)}_{11} &= u^{(2)}_{11} = 0\\
u_{12} &= 2\,\lambda_m\,\frac{w}{v}\\
u^{(1)}_{12} &= u^{(2)}_{12} = \frac{2}{3}\,\lambda_\phi\,\frac{v}{w} + 2\,\lambda_m\,\frac{w}{v}\\
u_{22} &= \lambda_m + \lambda_\sigma\,\frac{w^2}{v^2}\\
u^{(1)}_{22} &= -\frac{1}{6}\,\lambda_\phi\,\frac{v^2}{w^2} + 3\,\lambda_m + \frac{7}{6}\,\lambda_\sigma\,\frac{w^2}{v^2}\\
u^{(2)}_{22} &= \frac{1}{6}\,\lambda_\phi\,\frac{v^2}{w^2} + 5\,\lambda_m + \frac{3}{2}\,\lambda_\sigma\,\frac{w^2}{v^2}
\end{aligned} \tag{C.7}$$

Note that $u_{ij}, u^{(1)}_{ij}$ and $u^{(2)}_{ij}$ are symmetric. Then, the squared masses without evanescent corrections, c.f. (2.34), are given by

$$M^2_{ij} = \mu_0^{2\epsilon}\,\frac{v^2}{2}\,u_{ij} \tag{C.8}$$

The interaction coefficients are given by the following expressions, for the 3-interactions

$$\widetilde{\mathcal{V}}_{ijk} = \mu_0^{2\epsilon}\,\widehat{\mathcal{V}}_{ijk} = \mu_0^{2\epsilon}\,v\,\left(k_{ijk} + \epsilon\, k^{(1)}_{ijk} + \epsilon^2\, k^{(2)}_{ijk} + \mathcal{O}\left(\epsilon^3\right)\right) \tag{C.9}$$

with

$$\begin{aligned}
k_{111} &= \lambda_\phi & k_{112} &= \lambda_m\,\frac{w}{v}\\
k^{(1)}_{111} &= k^{(2)}_{111} = 0 & k^{(1)}_{112} &= k^{(2)}_{112} = \lambda_\phi\,\frac{v}{w} + \lambda_m\,\frac{w}{v}\\
k_{122} &= \lambda_m & k_{222} &= \lambda_\sigma\,\frac{w}{v}\\
k^{(1)}_{122} &= -\frac{1}{3}\,\lambda_\phi\,\frac{v^2}{w^2} + 3\,\lambda_m & k^{(1)}_{222} &= \frac{1}{6}\,\lambda_\phi\,\frac{v^3}{w^3} + \lambda_m\,\frac{v}{w} + \frac{13}{6}\,\lambda_\sigma\,\frac{w}{v}\\
k^{(2)}_{122} &= \frac{1}{3}\,\lambda_\phi\,\frac{v^2}{w^2} + 5\,\lambda_m & k^{(2)}_{222} &= -\frac{1}{3}\,\lambda_\phi\,\frac{v^3}{w^3} + 4\,\lambda_m\,\frac{v}{w} + \frac{11}{3}\,\lambda_\sigma\,\frac{w}{v}
\end{aligned} \tag{C.10}$$

the 4-interactions

$$\widetilde{\mathcal{V}}_{ijkl} = \mu_0^{2\epsilon}\,\widehat{\mathcal{V}}_{ijkl} = \mu_0^{2\epsilon}\,\left(k_{ijkl} + \epsilon\, k^{(1)}_{ijkl} + \epsilon^2\, k^{(2)}_{ijkl} + \mathcal{O}\left(\epsilon^3\right)\right) \tag{C.11}$$

with

$$\begin{aligned}
k_{1111} &= \lambda_\phi & k_{1112} &= 0 \\
k^{(1)}_{1111} &= k^{(2)}_{1111} = 0 & k^{(1)}_{1112} &= k^{(2)}_{1112} = 2\,\lambda_\phi\,\frac{v}{w} \\
k_{1122} &= \lambda_m & k_{1222} &= 0 \\
k^{(1)}_{1122} &= -\lambda_\phi\,\frac{v^2}{w^2} + 3\,\lambda_m & k^{(1)}_{1222} &= \frac{2}{3}\,\lambda_\phi\,\frac{v^3}{w^3} + 2\,\lambda_m\,\frac{v}{w} \\
k^{(2)}_{1122} &= \lambda_\phi\,\frac{v^2}{w^2} + 5\,\lambda_m & k^{(2)}_{1222} &= -\frac{4}{3}\,\lambda_\phi\,\frac{v^3}{w^3} + 8\,\lambda_m\,\frac{v}{w} \\
k_{2222} &= \lambda_\sigma \\
k^{(1)}_{2222} &= -\frac{1}{2}\,\lambda_\phi\,\frac{v^4}{w^4} - \lambda_m\,\frac{v^2}{w^2} + \frac{25}{6}\,\lambda_\sigma \\
k^{(2)}_{2222} &= \frac{4}{3}\,\lambda_\phi\,\frac{v^4}{w^4} - 2\,\lambda_m\,\frac{v^2}{w^2} + 10\,\lambda_\sigma
\end{aligned} \tag{C.12}$$

the 5-interactions

$$\widetilde{\mathcal{V}}_{ijklm} = \mu_0^{2\epsilon}\,\widehat{\mathcal{V}}_{ijklm} = \mu_0^{2\epsilon}\,\frac{1}{v}\,\left(k_{ijklm} + \epsilon\,k^{(1)}_{ijklm} + \epsilon^2\,k^{(2)}_{ijklm} + \mathcal{O}\left(\epsilon^3\right)\right) \tag{C.13}$$

with

$$\begin{aligned}
k_{11111} &= 0 & k_{11112} &= 0 \\
k^{(1)}_{11111} &= k^{(2)}_{11111} = 0 & k^{(1)}_{11112} &= k^{(2)}_{11112} = 2\,\lambda_\phi\,\frac{v}{w} \\
k_{11122} &= 0 & k_{11222} &= 0 \\
k^{(1)}_{11122} &= -2\,\lambda_\phi\,\frac{v^2}{w^2} & k^{(1)}_{11222} &= 2\,\lambda_\phi\,\frac{v^3}{w^3} + 2\,\lambda_m\,\frac{v}{w} \\
k^{(2)}_{11122} &= 2\,\lambda_\phi\,\frac{v^2}{w^2} & k^{(2)}_{11222} &= -4\,\lambda_\phi\,\frac{v^3}{w^3} + 8\,\lambda_m\,\frac{v}{w} \\
k_{12222} &= 0 & k_{22222} &= 0 \\
k^{(1)}_{12222} &= -2\,\lambda_\phi\,\frac{v^4}{w^4} - 2\,\lambda_m\,\frac{v^2}{w^2} & k^{(1)}_{22222} &= 2\,\lambda_\phi\,\frac{v^5}{w^5} + 2\,\lambda_m\,\frac{v^3}{w^3} + 2\,\lambda_\sigma\,\frac{v}{w} \\
k^{(2)}_{12222} &= \frac{16}{3}\,\lambda_\phi\,\frac{v^4}{w^4} - 4\,\lambda_m\,\frac{v^2}{w^2} & k^{(2)}_{22222} &= -\frac{19}{3}\,\lambda_\phi\,\frac{v^5}{w^5} + 2\,\lambda_m\,\frac{v^3}{w^3} + \frac{31}{3}\,\lambda_\sigma\,\frac{v}{w}
\end{aligned} \tag{C.14}$$

and for the 6-interactions

$$\widetilde{\mathcal{V}}_{ijklmn} = \mu_0^{2\epsilon}\,\widehat{\mathcal{V}}_{ijklmn} = \mu_0^{2\epsilon}\,\frac{1}{v^2}\,\left(k_{ijklmn} + \epsilon\,k^{(1)}_{ijklmn} + \epsilon^2\,k^{(2)}_{ijklmn} + \mathcal{O}\left(\epsilon^3\right)\right) \tag{C.15}$$

with

$$\begin{aligned}
&k_{111111} = 0 && k_{111112} = 0 \\
&k^{(1)}_{111111} = k^{(2)}_{111111} = 0 && k^{(1)}_{111112} = k^{(2)}_{111112} = 0 \\
&k_{111122} = 0 && k_{111222} = 0 \\
&k^{(1)}_{111122} = -2\,\lambda_\phi\,\frac{v^2}{w^2} && k^{(1)}_{111222} = 4\,\lambda_\phi\,\frac{v^3}{w^3} \\
&k^{(2)}_{111122} = 2\,\lambda_\phi\,\frac{v^2}{w^2} && k^{(2)}_{111222} = -8\,\lambda_\phi\,\frac{v^3}{w^3} \\
&k_{112222} = 0 && k_{122222} = 0 \\
&k^{(1)}_{112222} = -6\,\lambda_\phi\,\frac{v^4}{w^4} - 2\,\lambda_m\,\frac{v^2}{w^2} && k^{(1)}_{122222} = 8\,\lambda_\phi\,\frac{v^5}{w^5} + 4\,\lambda_m\,\frac{v^3}{w^3} \\
&k^{(2)}_{112222} = 16\,\lambda_\phi\,\frac{v^4}{w^4} - 4\,\lambda_m\,\frac{v^2}{w^2} && k^{(2)}_{122222} = -\frac{76}{3}\,\lambda_\phi\,\frac{v^5}{w^5} + 4\,\lambda_m\,\frac{v^3}{w^3} \\
&k_{222222} = 0 \\
&k^{(1)}_{222222} = -10\,\lambda_\phi\,\frac{v^6}{w^6} - 6\,\lambda_m\,\frac{v^4}{w^4} - 2\,\lambda_\sigma\,\frac{v^2}{w^2} \\
&k^{(2)}_{222222} = \frac{107}{3}\,\lambda_\phi\,\frac{v^6}{w^6} - 2\,\lambda_m\,\frac{v^4}{w^4} - \frac{19}{3}\,\lambda_\sigma\,\frac{v^2}{w^2}
\end{aligned} \tag{C.16}$$

The coefficients $\widetilde{\mathcal{V}}_{ijk\cdots}$, and thus $k_{ijk\cdots}$ and $k^{(s)}_{ijk\cdots}, \forall\, s$ are symmetric.
The eigenvalues of the matrix of the squared masses (C.5) are generically given by

$$\begin{aligned}
\widetilde{M}^2_H &= \frac{1}{2}\left(\widetilde{M}^2_{11} + \widetilde{M}^2_{22} + \sqrt{\left(\widetilde{M}^2_{11} - \widetilde{M}^2_{22}\right)^2 + 4\left(\widetilde{M}^2_{12}\right)^2}\right) \\
\widetilde{M}^2_S &= \frac{1}{2}\left(\widetilde{M}^2_{11} + \widetilde{M}^2_{22} - \sqrt{\left(\widetilde{M}^2_{11} - \widetilde{M}^2_{22}\right)^2 + 4\left(\widetilde{M}^2_{12}\right)^2}\right)
\end{aligned} \tag{C.17}$$

and may be written as

$$\begin{aligned}
\widetilde{M}^2_H &= M^2_H + \epsilon\, m^{(1)}_H + \epsilon^2\, m^{(2)}_H + \mathcal{O}\left(\epsilon^3\right) \\
&= M^2_H\left(1 + \epsilon\, c^{(1)}_H + \epsilon^2\, c^{(2)}_H + \mathcal{O}\left(\epsilon^3\right)\right) \\
\widetilde{M}^2_S &= M^2_S + \epsilon\, m^{(1)}_S + \epsilon^2\, m^{(2)}_S + \mathcal{O}\left(\epsilon^3\right) \\
&= M^2_S\left(1 + \epsilon\, c^{(1)}_S + \epsilon^2\, c^{(2)}_S + \mathcal{O}\left(\epsilon^3\right)\right)
\end{aligned} \tag{C.18}$$

where

$$\begin{aligned}
M_H^2 &= \mu_0^{2\epsilon}\,\frac{v^2}{4}\,\left(u_{11}+u_{22}+\sqrt{\left(u_{11}-u_{22}\right)^2+4\,u_{12}^2}\right)\\
&= \mu_0^{2\epsilon}\,\frac{1}{4}\,\left(\left(\lambda_\phi+\lambda_m\right)\,v^2+\left(\lambda_m+\lambda_\sigma\right)\,w^2+2\,\Lambda_m\right)\\
M_S^2 &= \mu_0^{2\epsilon}\,\frac{v^2}{4}\,\left(u_{11}+u_{22}-\sqrt{\left(u_{11}-u_{22}\right)^2+4\,u_{12}^2}\right)\\
&= \mu_0^{2\epsilon}\,\frac{1}{4}\,\left(\left(\lambda_\phi+\lambda_m\right)\,v^2+\left(\lambda_m+\lambda_\sigma\right)\,w^2-2\,\Lambda_m\right)
\end{aligned} \tag{C.19}$$

and for the evanescent corrections

$$\begin{aligned}
m_H^{(1)} = M_H^2\,c_H^{(1)} &= \mu_0^{2\epsilon}\,\frac{v^2}{4}\,\left(u_{11}^{(1)}+u_{22}^{(1)}+\frac{\left(u_{11}-u_{22}\right)\left(u_{11}^{(1)}-u_{22}^{(1)}\right)+4\,u_{12}\,u_{12}^{(1)}}{\sqrt{\left(u_{11}-u_{22}\right)^2+4\,u_{12}^2}}\right)\\
&= \mu_0^{2\epsilon}\,\frac{1}{24\,w^2}\,\left(-\lambda_\phi\,v^4+18\,\lambda_m\,v^2\,w^2+7\,\lambda_\sigma\,w^4+\Omega_1\right)\\
m_S^{(1)} = M_S^2\,c_S^{(1)} &= \mu_0^{2\epsilon}\,\frac{v^2}{4}\,\left(u_{11}^{(1)}+u_{22}^{(1)}-\frac{\left(u_{11}-u_{22}\right)\left(u_{11}^{(1)}-u_{22}^{(1)}\right)+4\,u_{12}\,u_{12}^{(1)}}{\sqrt{\left(u_{11}-u_{22}\right)^2+4\,u_{12}^2}}\right)\\
&= \mu_0^{2\epsilon}\,\frac{1}{24\,w^2}\,\left(-\lambda_\phi\,v^4+18\,\lambda_m\,v^2\,w^2+7\,\lambda_\sigma\,w^4-\Omega_1\right)
\end{aligned} \tag{C.20}$$

and

$$\begin{aligned}
m_H^{(2)} = M_H^2\,c_H^{(2)} = \mu_0^{2\epsilon}\,\frac{v^2}{4}\,&\Bigg(u_{11}^{(2)}+u_{22}^{(2)}+\frac{1}{2\,\left[\left(u_{11}-u_{22}\right)^2+4\,u_{12}^2\right]^{3/2}}\\
&\times\Bigg[\left(\left(u_{11}-u_{22}\right)^2+4\,u_{12}^2\right)\Bigg(\left(u_{11}^{(1)}-u_{22}^{(1)}\right)^2+4\,\left(u_{12}^{(1)}\right)^2\\
&\quad+2\,\left(u_{11}-u_{22}\right)\,\left(u_{11}^{(2)}-u_{22}^{(2)}\right)+8\,u_{12}\,u_{12}^{(2)}\Bigg)\\
&\quad-\left(\left(u_{11}-u_{22}\right)\,\left(u_{11}^{(1)}-u_{22}^{(1)}\right)+4\,u_{12}\,u_{12}^{(1)}\right)^2\Bigg]\Bigg)\\
= \mu_0^{2\epsilon}\,\frac{1}{24\,w^2}\,&\left(\lambda_\phi\,v^4+30\,\lambda_m\,v^2\,w^2+9\,\lambda_\sigma\,w^4+\Omega_2\right)
\end{aligned} \tag{C.21}$$

$$
\begin{aligned}
m_S^{(2)} = M_S^2\, c_S^{(2)} = \mu_0^{2\epsilon}\, \frac{v^2}{4} \Bigg( & u_{11}^{(2)} + u_{22}^{(2)} - \frac{1}{2\,\left[(u_{11}-u_{22})^2 + 4\,u_{12}^2\right]^{3/2}} \\
& \times \Bigg[ \left((u_{11}-u_{22})^2 + 4\,u_{12}^2\right) \Bigg( \left(u_{11}^{(1)} - u_{22}^{(1)}\right)^2 + 4\,\left(u_{12}^{(1)}\right)^2 \\
& \quad + 2\,(u_{11}-u_{22})\,\left(u_{11}^{(2)} - u_{22}^{(2)}\right) + 8\,u_{12}\,u_{12}^{(2)} \Bigg) \\
& \quad - \left((u_{11}-u_{22})\,\left(u_{11}^{(1)} - u_{22}^{(1)}\right) + 4\,u_{12}\,u_{12}^{(1)}\right)^2 \Bigg] \Bigg) \\
= \mu_0^{2\epsilon}\, \frac{1}{24\,w^2}\, & \left(\lambda_\phi\, v^4 + 30\,\lambda_m\, v^2\, w^2 + 9\,\lambda_\sigma\, w^4 - \Omega_2\right)
\end{aligned}
\tag{C.22}
$$

where the square of $\Lambda_m$ in (C.19) is defined by

$$
\begin{aligned}
\Lambda_m^2 \coloneqq \frac{1}{4} \Big[ & (\lambda_\phi - \lambda_m)^2\; v^4 + 2\,(\lambda_\phi\,\lambda_m - \lambda_\phi\,\lambda_\sigma \\
& + 7\,\lambda_m^2 + \lambda_m\,\lambda_\sigma)\, v^2\, w^2 + (\lambda_m - \lambda_\sigma)^2\; w^4 \Big]
\end{aligned}
\tag{C.23}
$$

and the parameters $\Omega_1$ and $\Omega_2$ in (C.20) and (C.21, C.22), respectively, are given by

$$
\begin{aligned}
\Omega_1 \coloneqq \frac{1}{2\,\Lambda_m} \Big[ & \lambda_\phi\,(\lambda_\phi - \lambda_m)\; v^6 + \left(15\,\lambda_\phi\,\lambda_m + 18\,\lambda_m^2 - \lambda_\phi\,\lambda_\sigma\right)\; v^4\, w^2 \\
& + \left(78\,\lambda_m^2 - 7\,\lambda_\phi\,\lambda_\sigma + 25\,\lambda_m\,\lambda_\sigma\right)\; v^2\, w^4 + 7\,\lambda_\sigma\,(\lambda_\sigma - \lambda_m)\; w^6 \Big]
\end{aligned}
\tag{C.24}
$$

$$
\begin{aligned}
\Omega_2 \coloneqq \frac{1}{24\,w^2\,\Lambda_m^3} \Big[ & \lambda_\phi \left\{ 13\,\lambda_\phi^3 - 39\,\lambda_\phi^2\,\lambda_m + 27\,\lambda_\phi\,\lambda_m^2 + 3\,\lambda_m^3 \right\} v^{10}\, w^2 \\
& + \Big\{ 5\,\lambda_\phi^2\,\lambda_m\,(25\,\lambda_\phi + 6\,\lambda_\sigma) - 23\,\lambda_\phi^3\,\lambda_\sigma - 423\,\lambda_\phi\,\lambda_m^3 \\
& \quad + 3\,\lambda_\phi\,\lambda_m^2\,(16\,\lambda_\phi + 3\,\lambda_\sigma) + 90\,\lambda_m^4 \Big\}\, v^8\, w^4 \\
& + \Big\{ \lambda_\phi^2\,\lambda_\sigma\,(7\,\lambda_\sigma - 27\,\lambda_\phi) + \lambda_\phi\,\lambda_m\,\lambda_\sigma\,(65\,\lambda_\phi + 9\,\lambda_\sigma) + 2034\,\lambda_m^4 \\
& \quad + \lambda_\phi\,\lambda_m^2\,(553\,\lambda_\phi - 615\,\lambda_\sigma) + 3\,\lambda_m^3\,(151\,\lambda_\phi + 99\,\lambda_\sigma) \Big\}\, v^6\, w^6 \\
& + \Big\{ 3\,\lambda_\phi\,\lambda_m^2\,(27\,\lambda_\phi + \lambda_\sigma) - \lambda_\phi\,\lambda_m\,\lambda_\sigma\,(81\,\lambda_\phi + 361\,\lambda_\sigma) + 3438\,\lambda_m^4 \\
& \quad + \lambda_m^2\,\lambda_\sigma\,(351\,\lambda_\sigma - 521\,\lambda_\phi) + 3\,\lambda_m^3\,(261\,\lambda_\phi + 641\,\lambda_\sigma) \Big\}\, v^4\, w^8 \\
& + \Big\{ 342\,\lambda_m^4 - 609\,\lambda_m^3\,\lambda_\sigma + \lambda_m^2\,\lambda_\sigma\,(292\,\lambda_\sigma - 81\,\lambda_\phi) \\
& \quad - 81\,\lambda_\phi\,\lambda_\sigma^3 + 9\,\lambda_m\,\lambda_\sigma^2\,(18\,\lambda_\phi + 19\,\lambda_\sigma) \Big\}\, v^2\, w^{10} \\
& - 27\,\lambda_\sigma\,(\lambda_m - \lambda_\sigma)^3\; w^{12} \Big]
\end{aligned}
\tag{C.25}
$$

The propagator for the 2 particles $h$ and $\mathfrak{D}$ with squared-mass matrix (C.5) reads

$$
\widetilde{D}_p = p^2 - \widetilde{\mathcal{M}}_{\phi\sigma}^2, \quad \text{or} \quad \left(\widetilde{D}_p\right)_{ij} = p^2\,\delta_{ij} - \widetilde{M}_{ij}^2
\tag{C.26}
$$

and thus the inverse propagator can be written as

$$\widetilde{D}_p^{-1} = \frac{\widetilde{A}}{p^2 - \widetilde{M}_H^2} + \frac{\widetilde{B}}{p^2 - \widetilde{M}_S^2} \tag{C.27}$$

for symmetric matrices $\widetilde{A}$ and $\widetilde{B}$ that satisfy

$$\widetilde{B}_{ij} = \delta_{ij} - \widetilde{A}_{ij} \quad \Leftrightarrow \quad \widetilde{A}_{ij} = \delta_{ij} - \widetilde{B}_{ij} \tag{C.28}$$

and are given by

$$\begin{aligned} \widetilde{A}_{11} = 1 - \widetilde{B}_{11} = \widetilde{B}_{22} = 1 - \widetilde{A}_{22} &= \frac{\widetilde{M}_H^2 - \widetilde{M}_{22}^2}{\widetilde{M}_H^2 - \widetilde{M}_S^2} \\ \widetilde{A}_{12} = \widetilde{A}_{21} = -\widetilde{B}_{12} = -\widetilde{B}_{21} &= \frac{\widetilde{M}_{12}^2}{\widetilde{M}_H^2 - \widetilde{M}_S^2} \\ \widetilde{A}_{22} = 1 - \widetilde{B}_{22} = \widetilde{B}_{11} = 1 - \widetilde{A}_{11} &= \frac{\widetilde{M}_H^2 - \widetilde{M}_{11}^2}{\widetilde{M}_H^2 - \widetilde{M}_S^2} \end{aligned} \tag{C.29}$$

Again, these matrix elements may be written in terms of an $\epsilon$ - expansion as follows

$$\begin{aligned} \widetilde{A}_{ij} &= A_{ij} + \epsilon\, A_{ij}^{(1)} + \epsilon^2\, A_{ij}^{(2)} + \mathcal{O}\left(\epsilon^3\right) \\ \widetilde{B}_{ij} &= B_{ij} + \epsilon\, B_{ij}^{(1)} + \epsilon^2\, B_{ij}^{(2)} + \mathcal{O}\left(\epsilon^3\right) \end{aligned} \tag{C.30}$$

which, due to (C.28), obey

$$\begin{aligned} A_{ij} = \delta_{ij} - B_{ij} \quad &\Leftrightarrow \quad B_{ij} = \delta_{ij} - A_{ij} \\ B_{ij}^{(1)} &= -A_{ij}^{(1)} \\ B_{ij}^{(2)} &= -A_{ij}^{(2)} \end{aligned} \tag{C.31}$$

Explicit expressions for the coefficients in (C.30) are given by

$$\begin{aligned} A_{11} &= 1 - B_{11} = B_{22} = 1 - A_{22} \\ &= \frac{1}{2} + \frac{u_{11} - u_{22}}{2\sqrt{\left(u_{11} - u_{22}\right)^2 + 4\, u_{12}^2}} \\ &= \frac{1}{2} + \frac{1}{4\,\Lambda_m}\left(\lambda_\phi\, v^2 - \lambda_m\left(v^2 - w^2\right) - \lambda_\sigma\, w^2\right) \end{aligned} \tag{C.32}$$

$$\begin{aligned} A_{12} &= A_{21} = -B_{12} = -B_{21} \\ &= \frac{u_{12}}{\sqrt{\left(u_{11} - u_{22}\right)^2 + 4\, u_{12}^2}} \\ &= \frac{\lambda_m\, v\, w}{\Lambda_m} \end{aligned} \tag{C.33}$$

$$
\begin{aligned}
A^{(1)}_{11} = -B^{(1)}_{11} = -A^{(1)}_{22} = B^{(1)}_{22} &= 2\,u_{12}\,\frac{u_{12}\,\left(u^{(1)}_{11}-u^{(1)}_{22}\right)-u^{(1)}_{12}\,\left(u_{11}-u_{22}\right)}{\left[\left(u_{11}-u_{22}\right)^2+4\,u_{12}^2\right]^{3/2}} \\
&= \frac{\lambda_m\,v^2}{6\,\Lambda_m^3}\left[\lambda_\phi\,\left(3\,\lambda_m-2\,\lambda_\phi\right)\,v^4+2\,\left(\lambda_\phi\,\lambda_\sigma-4\,\lambda_\phi\,\lambda_m-6\,\lambda_m^2\right)\,v^2\,w^2\right. \\
&\qquad \left. -\,\lambda_m\,\left(6\,\lambda_m+\lambda_\sigma\right)\,w^4\right]
\end{aligned} \tag{C.34}
$$

$$
\begin{aligned}
A^{(1)}_{12} = A^{(1)}_{21} = -B^{(1)}_{12} = -B^{(1)}_{21} &= \left(u_{11}-u_{22}\right)\,\frac{u^{(1)}_{12}\,\left(u_{11}-u_{22}\right)-u_{12}\,\left(u^{(1)}_{11}-u^{(1)}_{22}\right)}{\left[\left(u_{11}-u_{22}\right)^2+4\,u_{12}^2\right]^{3/2}} \\
= \frac{v}{24\,w\,\Lambda_m^3}&\left[\lambda_\phi\,\left(2\,\lambda_\phi^2-5\,\lambda_\phi\,\lambda_m+3\,\lambda_m^2\right)\,v^6\right. \\
&+\left\{5\,\lambda_\phi\,\lambda_m\,\left(2\,\lambda_\phi+\lambda_\sigma\right)-4\,\lambda_\phi^2\,\lambda_\sigma+\lambda_\phi\,\lambda_m^2-12\,\lambda_m^3\right\}v^4\,w^2 \\
&+\left\{2\,\lambda_\phi\,\lambda_\sigma^2+\lambda_m^2\,\left(14\,\lambda_\phi-13\,\lambda_\sigma\right)-9\,\lambda_\phi\,\lambda_m\,\lambda_\sigma+6\,\lambda_m^3\right\}v^2\,w^4 \\
&\left.+\,\lambda_m\,\left(6\,\lambda_m^2-5\,\lambda_m\,\lambda_\sigma-\lambda_\sigma^2\right)\,w^6\right]
\end{aligned} \tag{C.35}
$$

and finally

$$
\begin{aligned}
A^{(2)}_{11} = -B^{(2)}_{11} = -A^{(2)}_{22} = B^{(2)}_{22} &= \frac{1}{\mathcal{N}_u^5}\left\{\mathcal{Z}_{01}\left[u^{(1)}_{12}\,\left(\left(u_{11}-u_{22}\right)^2-8\,u_{12}^2\right)\right.\right. \\
&\qquad \left.\left.-\,3\,u_{12}\,\left(u_{11}-u_{22}\right)\,\left(u^{(1)}_{11}-u^{(1)}_{22}\right)\right]+2\,u_{12}\,\mathcal{N}_u^2\,\mathcal{Z}_{02}\right\} \\
= \frac{v^2}{288\,w^2\,\Lambda_m^5}&\left[\lambda_\phi^2\left\{-4\,\lambda_\phi^3+20\,\lambda_\phi^2\,\lambda_m-31\,\lambda_\phi\,\lambda_m^2+15\,\lambda_m^3\right\}v^{10}\right. \\
&+\lambda_\phi\left\{12\,\lambda_\phi^3\,\lambda_\sigma-20\,\lambda_\phi^2\,\lambda_m\,\left(3\,\lambda_\phi+2\,\lambda_\sigma\right)\right. \\
&\qquad \left.+\,\lambda_\phi\,\lambda_m^2\,\left(180\,\lambda_\phi+31\,\lambda_\sigma\right)+5\,\lambda_\phi\,\lambda_m^3-192\,\lambda_m^4\right\}v^8\,w^2 \\
&+2\left\{-6\,\lambda_\phi^3\,\lambda_\sigma^2-\lambda_\phi^2\,\lambda_m^2\,\left(132\,\lambda_\phi+19\,\lambda_\sigma\right)+\lambda_\phi\,\lambda_m^3\,\left(44\,\lambda_\phi-163\,\lambda_\sigma\right)\right. \\
&\qquad \left.+\,2\,\lambda_\phi^2\,\lambda_m\,\lambda_\sigma\,\left(28\,\lambda_\phi+5\,\lambda_\sigma\right)+450\,\lambda_\phi\,\lambda_m^4+144\,\lambda_m^5\right\}v^6\,w^4 \\
&-2\left\{-2\,\lambda_\phi^2\,\lambda_\sigma^3+\lambda_m^4\,\left(270\,\lambda_\phi-96\,\lambda_\sigma\right)+\lambda_\phi\,\lambda_m^3\,\left(236\,\lambda_\phi-463\,\lambda_\sigma\right)\right. \\
&\qquad \left.+\,\lambda_\phi\,\lambda_m^2\,\lambda_\sigma\,\left(71\,\lambda_\sigma-118\,\lambda_\phi\right)+22\,\lambda_\phi^2\,\lambda_m\,\lambda_\sigma^2+1008\,\lambda_m^5\right\}v^4\,w^6 \\
&+\lambda_m\left\{-8\,\lambda_\phi\,\lambda_\sigma^3+12\,\lambda_m^3\,\left(29\,\lambda_\sigma-31\,\lambda_\phi\right)\right. \\
&\qquad \left.+\,\lambda_m^2\,\lambda_\sigma\,\left(184\,\lambda_\phi-117\,\lambda_\sigma\right)+49\,\lambda_\phi\,\lambda_m\,\lambda_\sigma^2-468\,\lambda_m^4\right\}v^2\,w^8 \\
&\left.+\,3\,\lambda_m^2\,\left(-7\,\lambda_\sigma^3+27\,\lambda_m\,\lambda_\sigma^2+16\,\lambda_m^2\,\lambda_\sigma-36\,\lambda_m^3\right)\,w^{10}\right]
\end{aligned} \tag{C.36}
$$

$$
\begin{aligned}
A^{(2)}_{12} &= A^{(2)}_{21} = -B^{(2)}_{12} = -B^{(2)}_{21} \\
&= \frac{1}{\mathcal{N}_u^5} \Bigg\{ \mathcal{Z}_{01} \Bigg[ \left( u^{(1)}_{11} - u^{(1)}_{22} \right) \left[ (u_{11} - u_{22})^2 - 2\, u_{12}^2 \right] \\
&\qquad\qquad + 6\, u_{12}\, u^{(1)}_{12}\, (u_{11} - u_{22}) \Bigg] - (u_{11} - u_{22})\, \mathcal{N}_u^2\, \mathcal{Z}_{02} \Bigg\} \\
&= -\frac{v}{576\, w^3\, \Lambda_m^5} \Bigg[ \lambda_\phi^2\, (2\,\lambda_\phi - 3\,\lambda_m)\, (\lambda_\phi - \lambda_m)^2\, v^{12} \\
&\quad + 2\,\lambda_\phi \Big\{ 30\,\lambda_m^4 + 22\,\lambda_\phi\,\lambda_m^3 - 3\,\lambda_\phi^3\, (2\,\lambda_\phi + \lambda_\sigma) \\
&\qquad + \lambda_\phi^2\,\lambda_m\, (57\,\lambda_\phi + 7\,\lambda_\sigma) - \lambda_\phi\,\lambda_m^2\, (99\,\lambda_\phi + 4\,\lambda_\sigma) \Big\}\, v^{10}\, w^2 \\
&\quad + \Big\{ -75\,\lambda_m^5 - 972\,\lambda_\phi\,\lambda_m^4 + 2\,\lambda_\phi^3\,\lambda_\sigma\, (17\,\lambda_\phi + 3\,\lambda_\sigma) \\
&\qquad + 2\,\lambda_\phi^2\,\lambda_m^2\, (237\,\lambda_\phi + 28\,\lambda_\sigma) + \lambda_\phi\,\lambda_m^3\, (301\,\lambda_\phi + 160\,\lambda_\sigma) \\
&\qquad - \lambda_\phi^2\,\lambda_m \left( 84\,\lambda_\phi^2 + 184\,\lambda_\phi\,\lambda_\sigma + 7\,\lambda_\sigma^2 \right) \Big\}\, v^8\, w^4 \\
&\quad + 2 \Big\{ 1116\,\lambda_m^5 + 408\,\lambda_\phi\,\lambda_m^4 + \lambda_m^3 \left( 379\,\lambda_\phi^2 - 651\,\lambda_\phi\,\lambda_\sigma \right) \\
&\qquad - 42\,\lambda_m^4\,\lambda_\sigma + \lambda_\phi\,\lambda_m^2 \left( 71\,\lambda_\sigma^2 - 5\,\lambda_\phi\,\lambda_\sigma - 108\,\lambda_\phi^2 \right) \\
&\qquad + \lambda_\phi\,\lambda_m\,\lambda_\sigma \left( 75\,\lambda_\phi^2 + 13\,\lambda_\phi\,\lambda_\sigma \right) - 15\,\lambda_\phi^3\,\lambda_\sigma^2 - \lambda_\phi^2\,\lambda_\sigma^3 \Big\}\, v^6\, w^6 \\
&\quad + \Big\{ 324\,\lambda_m^5 + 6\,\lambda_\phi\,\lambda_m^2\,\lambda_\sigma\, (41\,\lambda_\phi - 91\,\lambda_\sigma) \\
&\qquad - \lambda_\phi\,\lambda_m\,\lambda_\sigma^2\, (37\,\lambda_\phi - 44\,\lambda_\sigma) + 12\,\lambda_m^4\, (45\,\lambda_\phi + 92\,\lambda_\sigma) \\
&\qquad + 6\,\lambda_\phi^2\,\lambda_\sigma^3 + \lambda_m^3 \left( 59\,\lambda_\sigma^2 + 540\,\lambda_\phi\,\lambda_\sigma - 264\,\lambda_\phi^2 \right) \Big\}\, v^4\, w^8 \\
&\quad + 2 \Big\{ 72\,\lambda_m^5 - 6\,\lambda_m^4\, (13\,\lambda_\phi - 31\,\lambda_\sigma) - 20\,\lambda_\phi\,\lambda_m\,\lambda_\sigma^3 + \lambda_\phi\,\lambda_\sigma^4 \\
&\qquad + \lambda_m^2\,\lambda_\sigma^2\, (8\,\lambda_\phi + 41\,\lambda_\sigma) + \lambda_m^3\,\lambda_\sigma\, (89\,\lambda_\phi - 103\,\lambda_\sigma) \Big\}\, v^2\, w^{10} \\
&\quad - \lambda_m\, (\lambda_m - \lambda_\sigma)^2 \left( 36\,\lambda_m^2 + 24\,\lambda_m\,\lambda_\sigma - 11\,\lambda_\sigma^2 \right) w^{12} \Bigg]
\end{aligned}
\tag{C.37}
$$

where

$$
\begin{aligned}
\mathcal{N}_u &\coloneqq \sqrt{(u_{11} - u_{22})^2 + 4\, u_{12}^2} \\
\mathcal{Z}_{0k} &\coloneqq u_{12} \left( u^{(k)}_{11} - u^{(k)}_{22} \right) - u^{(k)}_{12}\, (u_{11} - u_{22})
\end{aligned}
\tag{C.38}
$$

In mass eigenstates $H$ and $S$, as well as having used the minimalisation conditions (2.47), the 2 Scalar potential may be written as

$$
\begin{aligned}
\widetilde{V}(H,S) &= \frac{1}{2}\, \widetilde{M}_H^2\, H^2 + \frac{1}{3!}\, \widetilde{\lambda}_{ijk}\, \rho_i\, \rho_j\, \rho_k + \frac{1}{4!}\, \widetilde{\lambda}_{ijkl}\, \rho_i\, \rho_j\, \rho_k\, \rho_l \\
&\quad + \frac{1}{5!}\, \widetilde{\lambda}_{ijklm}\, \rho_i\, \rho_j\, \rho_k\, \rho_l\, \rho_m + \frac{1}{6!}\, \widetilde{\lambda}_{ijklmn}\, \rho_i\, \rho_j\, \rho_k\, \rho_l\, \rho_m\, \rho_n + \ldots
\end{aligned}
\tag{C.39}
$$

where $\{\rho_i\}_{i=1}^2 = \{H, S\}$.
The squared masses are then provided by

$$\begin{aligned} \widetilde{M}_H^2 &\xrightarrow[(2.47)]{} \widetilde{M}_H^2 = M_H^2 = \frac{1}{3}\,\mu_0^{2\epsilon}\,\lambda_\phi\, v^2 \left(1 + \frac{v^2}{w^2}\right) \\ \widetilde{M}_S^2 &\xrightarrow[(2.47)]{} \widetilde{M}_S^2 = M_S^2 = 0 \end{aligned} \tag{C.40}$$

The interaction coefficients are given by the following expressions, for the 3-interactions

$$\widetilde{\lambda}_{ijk} = \mu_0^{2\epsilon}\,\widehat{\lambda}_{ijk} = \mu_0^{2\epsilon}\,\lambda_\phi\, v\,\left(g_{ijk} + \epsilon\, g_{ijk}^{(1)} + \epsilon^2\, g_{ijk}^{(2)} + \mathcal{O}\left(\epsilon^3\right)\right) \tag{C.41}$$

with

$$\begin{aligned} g_{111} &= \frac{\left(w^2 - v^2\right)\sqrt{v^2 + w^2}}{w^3} \\ g_{111}^{(1)} &= g_{111}^{(2)} = -2\,\frac{v^2\sqrt{v^2 + w^2}}{w^3} \\ g_{112} &= g_{112}^{(1)} = g_{112}^{(2)} = \frac{2}{3}\,\frac{v\sqrt{v^2 + w^2}}{w^2} \\ g_{122} &= g_{122}^{(1)} = g_{122}^{(2)} = 0 \\ g_{222} &= g_{222}^{(1)} = g_{222}^{(2)} = 0 \end{aligned} \tag{C.42}$$

the 4-interactions

$$\widetilde{\lambda}_{ijkl} = \mu_0^{2\epsilon}\,\widehat{\lambda}_{ijkl} = \mu_0^{2\epsilon}\,\lambda_\phi\,\left(g_{ijkl} + \epsilon\, g_{ijkl}^{(1)} + \epsilon^2\, g_{ijkl}^{(2)} + \mathcal{O}\left(\epsilon^3\right)\right) \tag{C.43}$$

with

$$\begin{aligned} g_{1111} &= \frac{v^4 - 2\,v^2\,w^2 + w^4}{w^4} & g_{1112} &= \frac{v\,\left(w^2 - v^2\right)}{w^3} \\ g_{1111}^{(1)} &= \frac{4\,v^4 - 8\,v^2\,w^2}{w^4} & g_{1112}^{(1)} &= \frac{v\,\left(2\,w^2 - 4\,v^2\right)}{w^3} \\ g_{1111}^{(2)} &= \frac{12\,v^4 - 8\,v^2\,w^2}{w^4} & g_{1112}^{(2)} &= \frac{v\,\left(2\,w^2 - 8\,v^2\right)}{w^3} \\ g_{1122} &= \frac{2}{3}\,\frac{v^2}{w^2} & g_{1222} &= 0 \\ g_{1122}^{(1)} &= 2\,\frac{v^2}{w^2} & g_{1222}^{(1)} &= 0 \\ g_{1122}^{(2)} &= \frac{10}{3}\,\frac{v^2}{w^2} & g_{1222}^{(2)} &= 0 \\ g_{2222} &= 0 \\ g_{2222}^{(1)} &= 0 \\ g_{2222}^{(2)} &= 0 \end{aligned} \tag{C.44}$$

the 5-interactions

$$\widetilde{\lambda}_{ijklm} = \mu_0^{2\epsilon}\,\widehat{\lambda}_{ijklm} = \mu_0^{2\epsilon}\,\frac{\lambda_\phi}{v}\,\left(g_{ijklm} + \epsilon\, g_{ijklm}^{(1)} + \epsilon^2\, g_{ijklm}^{(2)} + \mathcal{O}\left(\epsilon^3\right)\right) \tag{C.45}$$

with

$$
\begin{aligned}
& g_{11111} = 0 && g_{11112} = 0 \\
& g^{(1)}_{11111} = -\frac{10}{3}\,\frac{v^2\,\left(v^4+3\,w^4\right)}{w^5\,\sqrt{v^2+w^2}} && g^{(1)}_{11112} = 2\,\frac{v\,\left(v^4-2\,v^2\,w^2+w^4\right)}{w^4\,\sqrt{v^2+w^2}} \\
& g^{(2)}_{11111} = -\frac{10}{3}\,\frac{v^2\,\left(v^4-12\,v^2\,w^2+3\,w^4\right)}{w^5\,\sqrt{v^2+w^2}} && g^{(2)}_{11112} = 2\,\frac{v\,\left(5\,v^4-10\,v^2\,w^2+w^4\right)}{w^4\,\sqrt{v^2+w^2}} \\
& g_{11122} = 0 && g_{11222} = 0 \\
& g^{(1)}_{11122} = 2\,\frac{v^2\,\left(w^2-v^2\right)}{w^3\,\sqrt{v^2+w^2}} && g^{(1)}_{11222} = \frac{4}{3}\,\frac{v^3}{w^2\,\sqrt{v^2+w^2}} \\
& g^{(2)}_{11122} = 2\,\frac{v^2\,\left(3\,w^2-5\,v^2\right)}{w^3\,\sqrt{v^2+w^2}} && g^{(2)}_{11222} = \frac{16}{3}\,\frac{v^3}{w^2\,\sqrt{v^2+w^2}} \\
& g_{12222} = 0 && g_{22222} = 0 \\
& g^{(1)}_{12222} = 0 && g^{(1)}_{22222} = 0 \\
& g^{(2)}_{12222} = 0 && g^{(2)}_{22222} = 0
\end{aligned}
\tag{C.46}
$$

and for the 6-interactions

$$
\widetilde{\lambda}_{ijklmn} = \mu_0^{2\epsilon}\,\widehat{\lambda}_{ijklmn} = \mu_0^{2\epsilon}\,\frac{\lambda_\phi}{v^2}\,\left(g_{ijklmn} + \epsilon\,g^{(1)}_{ijklmn} + \epsilon^2\,g^{(2)}_{ijklmn} + \mathcal{O}\left(\epsilon^3\right)\right)
\tag{C.47}
$$

with

$$
\begin{aligned}
& g_{111111} = 0 && g_{111112} = 0 \\
& g^{(1)}_{111111} = -10\,\frac{v^4\,\left(v^4+2\,v^2\,w^2+3\,w^4\right)}{w^6\,\left(v^2+w^2\right)} && g^{(1)}_{111112} = \frac{10}{3}\,\frac{v^3\,\left(v^4+3\,w^4\right)}{w^5\,\left(v^2+w^2\right)} \\
& g^{(2)}_{111111} = 10\,\frac{v^4\,\left(3\,v^4+10\,v^2\,w^2+3\,w^4\right)}{w^6\,\left(v^2+w^2\right)} && g^{(2)}_{111112} = -\frac{10}{3}\,\frac{v^3\,\left(v^4+12\,v^2\,w^2+3\,w^4\right)}{w^5\,\left(v^2+w^2\right)} \\
& g_{111122} = 0 && g_{111222} = 0 \\
& g^{(1)}_{111122} = -2\,\frac{v^2\,\left(v^4-2\,v^2\,w^2+w^4\right)}{w^4\,\left(v^2+w^2\right)} && g^{(1)}_{111222} = -2\,\frac{v^3\,\left(w^2-v^2\right)}{w^3\,\left(v^2+w^2\right)} \\
& g^{(2)}_{111122} = 2\,\frac{v^2\,\left(w^4+6\,v^2\,w^2-3\,v^4\right)}{w^4\,\left(v^2+w^2\right)} && g^{(2)}_{111222} = -2\,\frac{v^3\,\left(w^2-3\,v^2\right)}{w^3\,\left(v^2+w^2\right)} \\
& g_{112222} = 0 && g_{122222} = 0 \\
& g^{(1)}_{112222} = -\frac{4}{3}\,\frac{v^4}{w^2\,\left(v^2+w^2\right)} && g^{(1)}_{122222} = 0 \\
& g^{(2)}_{112222} = -\frac{8}{3}\,\frac{v^4}{w^2\,\left(v^2+w^2\right)} && g^{(2)}_{122222} = 0 \\
& g_{222222} = 0 && \\
& g^{(1)}_{222222} = 0 && \\
& g^{(2)}_{222222} = 0 &&
\end{aligned}
\tag{C.48}
$$

The coefficients $\widetilde{\lambda}_{ijk\cdots}$, and thus $g_{ijk\cdots}$ and $g^{(s)}_{ijk\cdots}, \forall\, s$ are symmetric.

## C.2. Counterterms

Expanding the 1-loop counterterm Lagrangian in (2.59) about the fields VEVs and w.r.t. $\epsilon$ one obtains

$$\begin{aligned}\mathcal{L}_{\text{ct1}} &= \frac{1}{2}\,\delta Z^{(1)}_{\phi}\,\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}\,\delta Z^{(1)}_{\sigma}\,\partial_\mu\sigma\partial^\mu\sigma \\ &\quad - \mu^{2\epsilon}(\sigma)\,\left(\delta Z^{(1)}_{V_\phi}\,\frac{\lambda_\phi}{4!}\,\phi^4 + \delta Z^{(1)}_{V_m}\,\frac{\lambda_m}{4}\,\phi^2\,\sigma^2 + \delta Z^{(1)}_{V_\sigma}\,\frac{\lambda_\sigma}{4!}\,\sigma^4\right) \\ &= \frac{1}{2}\,\delta Z^{(1)}_{\phi}\,\partial_\mu h\,\partial^\mu h + \frac{1}{2}\,\delta Z^{(1)}_{\sigma}\,\partial_\mu\mathfrak{D}\,\partial^\mu\mathfrak{D} - \delta\widetilde{\mathcal{V}}_0 - \delta\widetilde{T}_{\varphi_i}\,\varphi_i - \frac{1}{2}\,\delta\widetilde{\mathcal{V}}_{ij}\,\varphi_i\,\varphi_j \\ &\quad - \frac{1}{3!}\,\delta\widetilde{\mathcal{V}}_{ijk}\,\varphi_i\,\varphi_j\,\varphi_k - \frac{1}{4!}\,\delta\widetilde{\mathcal{V}}_{ijkl}\,\varphi_i\,\varphi_j\,\varphi_k\,\varphi_l + \ldots\end{aligned} \tag{C.49}$$

where $\{\varphi_i\}^2_{i=1} = \{h, \mathfrak{D}\}$. The counterterm coefficients of $\mathcal{L}_{\text{ct1}}$ are, for the constant term

$$\delta\widetilde{\mathcal{V}}_0 = \mu_0^{2\epsilon}\left(\frac{\delta Z^{(1)}_{V_\phi}\,\lambda_\phi}{4!}\,v^4 + \frac{\delta Z^{(1)}_{V_m}\,\lambda_m}{4}\,v^2\,w^2 + \frac{\delta Z^{(1)}_{V_\sigma}\,\lambda_\sigma}{4!}\,w^4\right) \tag{C.50}$$

for the tadpole counterterms

$$\delta\widetilde{T}_{\varphi_i} = \mu_0^{2\epsilon}\,\frac{v^3}{6}\,\left(\delta t_{\varphi_i} + \epsilon\,\delta t^{(1)}_{\varphi_i} + \mathcal{O}\left(\epsilon\right)\right) \tag{C.51}$$

with

$$\begin{aligned}\delta t_h &= \delta Z^{(1)}_{V_\phi}\,\lambda_\phi + 3\,\delta Z^{(1)}_{V_m}\,\lambda_m\,\frac{w^2}{v^2} \\ \delta t^{(1)}_h &= 0\end{aligned} \tag{C.52a}$$

$$\begin{aligned}\delta t_{\mathfrak{D}} &= 3\,\delta Z^{(1)}_{V_m}\,\lambda_m\,\frac{w}{v} + \delta Z^{(1)}_{V_\sigma}\,\lambda_\sigma\,\frac{w^3}{v^3} \\ \delta t^{(1)}_{\mathfrak{D}} &= \frac{1}{2}\,\delta Z^{(1)}_{V_\phi}\,\lambda_\phi\,\frac{v}{w} + 3\,\delta Z^{(1)}_{V_m}\,\lambda_m\,\frac{w}{v} + \frac{1}{2}\,\delta Z^{(1)}_{V_\sigma}\,\lambda_\sigma\,\frac{w^3}{v^3}\end{aligned} \tag{C.52b}$$

for the mass counterterms

$$\delta\widetilde{\mathcal{V}}_{ij} = \mu_0^{2\epsilon}\,\frac{v^2}{2}\,\left(\delta k_{ij} + \epsilon\,\delta k^{(1)}_{ij} + \mathcal{O}\left(\epsilon\right)\right) \tag{C.53}$$

with

$$\begin{aligned}
\delta k_{11} &= \delta Z^{(1)}_{V_\phi}\,\lambda_\phi + \delta Z^{(1)}_{V_m}\,\lambda_m\,\frac{w^2}{v^2} \\
\delta k^{(1)}_{11} &= 0 \\
\delta k_{12} &= 2\,\delta Z^{(1)}_{V_m}\,\lambda_m\,\frac{w}{v} \\
\delta k^{(1)}_{12} &= \frac{2}{3}\,\delta Z^{(1)}_{V_\phi}\,\lambda_\phi\,\frac{v}{w} + 2\,\delta Z^{(1)}_{V_m}\,\lambda_m\,\frac{w}{v} \\
\delta k_{22} &= \delta Z^{(1)}_{V_m}\,\lambda_m + \delta Z^{(1)}_{V_\sigma}\,\lambda_\sigma\,\frac{w^2}{v^2} \\
\delta k^{(1)}_{22} &= -\frac{1}{6}\,\delta Z^{(1)}_{V_\phi}\,\lambda_\phi\,\frac{v^2}{w^2} + 3\,\delta Z^{(1)}_{V_m}\,\lambda_m + \frac{7}{6}\,\delta Z^{(1)}_{V_\sigma}\,\lambda_\sigma\,\frac{w^2}{v^2}
\end{aligned} \tag{C.54}$$

for the 3-interaction counterterms

$$\delta\widetilde{\mathcal{V}}_{ijk} = \mu_0^{2\epsilon}\,v\,\left(\delta k_{ijk} + \epsilon\,\delta k^{(1)}_{ijk} + \mathcal{O}\left(\epsilon\right)\right) \tag{C.55}$$

with

$$\begin{aligned}
\delta k_{111} &= \delta Z^{(1)}_{V_\phi}\,\lambda_\phi \\
\delta k^{(1)}_{111} &= 0 \\
\delta k_{112} &= \delta Z^{(1)}_{V_m}\,\lambda_m\,\frac{w}{v} \\
\delta k^{(1)}_{112} &= \delta Z^{(1)}_{V_\phi}\,\lambda_\phi\,\frac{v}{w} + \delta Z^{(1)}_{V_m}\,\lambda_m\,\frac{w}{v} \\
\delta k_{122} &= \delta Z^{(1)}_{V_m}\,\lambda_m \\
\delta k^{(1)}_{122} &= -\frac{1}{3}\,\delta Z^{(1)}_{V_\phi}\,\lambda_\phi\,\frac{v^2}{w^2} + 3\,\delta Z^{(1)}_{V_m}\,\lambda_m \\
\delta k_{222} &= \delta Z^{(1)}_{V_\sigma}\,\lambda_\sigma\,\frac{w}{v} \\
\delta k^{(1)}_{222} &= \frac{1}{6}\,\delta Z^{(1)}_{V_\phi}\,\lambda_\phi\,\frac{v^3}{w^3} + \delta Z^{(1)}_{V_m}\,\lambda_m\,\frac{v}{w} + \frac{13}{6}\,\delta Z^{(1)}_{V_\sigma}\,\lambda_\sigma\,\frac{w}{v}
\end{aligned} \tag{C.56}$$

for the 4-interaction counterterms

$$\delta\widetilde{\mathcal{V}}_{ijkl} = \mu_0^{2\epsilon}\,\left(\delta k_{ijkl} + \epsilon\,\delta k^{(1)}_{ijkl} + \mathcal{O}\left(\epsilon\right)\right) \tag{C.57}$$

with

$$
\begin{aligned}
\delta k_{1111} &= \delta Z^{(1)}_{V_\phi}\,\lambda_\phi \\
\delta k^{(1)}_{1111} &= 0 \\
\delta k_{1112} &= 0 \\
\delta k^{(1)}_{1112} &= 2\,\delta Z^{(1)}_{V_\phi}\,\lambda_\phi\,\frac{v}{w} \\
\delta k_{1122} &= \delta Z^{(1)}_{V_m}\,\lambda_m \\
\delta k^{(1)}_{1122} &= -\delta Z^{(1)}_{V_\phi}\,\lambda_\phi\,\frac{v^2}{w^2} + 3\,\delta Z^{(1)}_{V_m}\,\lambda_m \\
\delta k_{1222} &= 0 \\
\delta k^{(1)}_{1222} &= \frac{2}{3}\,\delta Z^{(1)}_{V_\phi}\,\lambda_\phi\,\frac{v^3}{w^3} + 2\,\delta Z^{(1)}_{V_m}\,\lambda_m\,\frac{v}{w} \\
\delta k_{2222} &= \delta Z^{(1)}_{V_\sigma}\,\lambda_\sigma \\
\delta k^{(1)}_{2222} &= -\frac{1}{2}\,\delta Z^{(1)}_{V_\phi}\,\lambda_\phi\,\frac{v^4}{w^4} - \delta Z^{(1)}_{V_m}\,\lambda_m\,\frac{v^2}{w^2} + \frac{25}{6}\,\delta Z^{(1)}_{V_\sigma}\,\lambda_\sigma
\end{aligned}
\tag{C.58}
$$

Note that the power series in (C.51), (C.53), (C.55) and (C.57) are indeed only shown up to the order of $\mathcal{O}(\epsilon^0)$, due to the fact that 1-loop counterterms contain $1/\epsilon$ poles. In terms of mass eigenstates $H$ and $S$, and with the minimalisation conditions (2.47) being used, the 1-loop counterterm Lagrangian (2.60) is given by

$$
\begin{aligned}
\mathcal{L}_{\text{ct1}} = {} & \frac{1}{2}\,\delta Z_H\,\partial_\mu H \partial^\mu H + \frac{1}{2}\,\delta Z_S\,\partial_\mu S \partial^\mu S + \delta Z_{HS}\,\partial_\mu H \partial^\mu S \\
& - \mu_0^{2\epsilon}\,\delta V_0 - \mu_0^{2\epsilon}\,(\delta T_H + \epsilon\,\delta Y_1)\,H - \mu_0^{2\epsilon}\,(\delta T_S + \epsilon\,\delta Y_2)\,S \\
& - \frac{1}{2}\,(\delta Z_H + \delta Z_{M_H} + \epsilon\,\delta Y_{11})\,M_H^2\,H^2 - \frac{1}{2}\,\mu_0^{2\epsilon}\left(\delta M_S^2 + \epsilon\,\delta Y_{22}\right) S^2 \\
& - \frac{1}{2}\,\mu_0^{2\epsilon}\left(\delta M_{HS}^2 + \epsilon\,\delta Y_{12}\right) H\,S - \mu_0^{2\epsilon}\,(\delta Z_{111} + \epsilon\,\delta Y_{111})\,\frac{\lambda_\phi}{3!}\,v\,H^3 \\
& - \mu_0^{2\epsilon}\,(\delta Z_{112} + \epsilon\,\delta Y_{112})\,\frac{\lambda_\phi}{2}\,v\,H^2\,S - \mu_0^{2\epsilon}\,(\delta Z_{122} + \epsilon\,\delta Y_{122})\,\frac{\lambda_\phi}{2}\,v\,H\,S^2 \\
& - \mu_0^{2\epsilon}\,(\delta Z_{222} + \epsilon\,\delta Y_{222})\,\frac{\lambda_\phi}{3!}\,v\,S^3 - \mu_0^{2\epsilon}\,(\delta Z_{1111} + \epsilon\,\delta Y_{1111})\,\frac{\lambda_\phi}{4!}\,H^4 \\
& - \mu_0^{2\epsilon}\,(\delta Z_{1112} + \epsilon\,\delta Y_{1112})\,\frac{\lambda_\phi}{3!}\,H^3\,S - \mu_0^{2\epsilon}\,(\delta Z_{1122} + \epsilon\,\delta Y_{1122})\,\frac{\lambda_\phi}{4}\,H^2\,S^2 \\
& - \mu_0^{2\epsilon}\,(\delta Z_{1222} + \epsilon\,\delta Y_{1222})\,\frac{\lambda_\phi}{3!}\,H\,S^3 - \mu_0^{2\epsilon}\,(\delta Z_{2222} + \epsilon\,\delta Y_{2222})\,\frac{\lambda_\phi}{4!}\,S^4 + \cdots
\end{aligned}
\tag{C.59}
$$

The counterterms in (C.59) that are not multiplied by $\epsilon$ are defined by

$$
\begin{aligned}
\delta Z_H &= \frac{w^2}{v^2 + w^2}\,\delta Z_\phi + \frac{v^2}{v^2 + w^2}\,\delta Z_\sigma \\
\delta Z_S &= \frac{w^2}{v^2 + w^2}\,\delta Z_\sigma + \frac{v^2}{v^2 + w^2}\,\delta Z_\phi \\
\delta Z_{HS} &= \frac{v\,w}{v^2 + w^2}\,\delta Z_\phi - \frac{v\,w}{v^2 + w^2}\,\delta Z_\sigma
\end{aligned}
\tag{C.60}
$$

$$
\begin{aligned}
\delta V_0 &= \frac{\lambda_\phi}{4!}\, v^4 \, \left(\delta Z_{V_\phi} + \delta Z_{V_\sigma} - 2\,\delta Z_{V_m}\right) \\
\delta T_H &= \frac{\lambda_\phi}{6}\, v^3\, \frac{v^2\, \left(\delta Z_{V_m} - \delta Z_{V_\sigma}\right) + w^2\, \left(\delta Z_{V_\phi} - \delta Z_{V_m}\right)}{w\sqrt{v^2+w^2}} \\
\delta T_S &= \frac{\lambda_\phi}{6}\, \frac{v^4}{\sqrt{v^2+w^2}}\, \left(\delta Z_{V_\phi} + \delta Z_{V_\sigma} - 2\,\delta Z_{V_m}\right) \\
\delta Z_H + \delta Z_{M_H} &= \frac{v^4\, \left(3\,\delta Z_{V_\sigma} - \delta Z_{V_m}\right) + 4\,v^2\,w^2\,\delta Z_{V_m} + w^4\, \left(3\,\delta Z_{V_\phi} - \delta Z_{V_m}\right)}{2\,\left(v^2+w^2\right)^2} \\
\delta Z_{M_H} &= \frac{1}{2\,\left(v^2+w^2\right)^2}\Big[v^4\,\left(3\,\delta Z_{V_\sigma} - \delta Z_{V_m} - 2\,\delta Z_\sigma\right) + 2\,v^2\,w^2\,(2\,\delta Z_{V_m} \\
&\qquad - \delta Z_\phi - \delta Z_\sigma) + w^4\, \left(3\,\delta Z_{V_\phi} - \delta Z_{V_m} - 2\,\delta Z_\phi\right)\Big] \\
\delta M_S^2 &= \frac{\lambda_\phi}{2}\, \frac{v^4}{v^2+w^2}\, \left(\delta Z_{V_\phi} + \delta Z_{V_\sigma} - 2\,\delta Z_{V_m}\right) \\
\delta M_{HS}^2 &= \lambda_\phi\, \frac{v^3}{w\,\left(v^2+w^2\right)}\, \left(v^2\,\left(\delta Z_{V_m} - \delta Z_{V_\sigma}\right) + w^2\,\left(\delta Z_{V_\phi} - \delta Z_{V_m}\right)\right) \\
\delta Z_{111} &= \frac{w^6\,\delta Z_{V_\phi} + \delta Z_{V_m}\,\left(v^2\,w^4 - v^4\,w^2\right) - v^6\,\delta Z_{V_\sigma}}{w^3\,\left(v^2+w^2\right)^{3/2}} \\
\delta Z_{112} &= \frac{v\,w^4\,\left(3\,\delta Z_{V_\phi} - \delta Z_{V_m}\right) + 4\,v^3\,w^2\,\delta Z_{V_m} + v^5\,\left(3\,\delta Z_{V_\sigma} - \delta Z_{V_m}\right)}{3\,w^2\,\left(v^2+w^2\right)^{3/2}} \\
\delta Z_{122} &= \frac{v^2\,\left(v^2\,\left(\delta Z_{V_m} - \delta Z_{V_\sigma}\right) + w^2\,\left(\delta Z_{V_\phi} - \delta Z_{V_m}\right)\right)}{w\,\left(v^2+w^2\right)^{3/2}} \\
\delta Z_{222} &= \frac{v^3\,\left(\delta Z_{V_\phi} - 2\,\delta Z_{V_m} + \delta Z_{V_\sigma}\right)}{\left(v^2+w^2\right)^{3/2}} \\
\delta Z_{1111} &= \frac{w^8\delta Z_{V_\phi} - 2\,v^4\,w^4\,\delta Z_{V_m} + v^8\,\delta Z_{V_\sigma}}{w^4\,\left(v^2+w^2\right)^2} \\
\delta Z_{1112} &= \frac{v\,w^6\,\delta Z_{V_\phi} + \left(v^3\,w^4 - v^5\,w^2\right)\,\delta Z_{V_m} - v^7\,\delta Z_{V_\sigma}}{w^3\,\left(v^2+w^2\right)^2} \\
\delta Z_{1122} &= \frac{v^2\,\left(w^4\,\left(3\,\delta Z_{V_\phi} - \delta Z_{V_m}\right) + 4\,v^2\,w^2\,\delta Z_{V_m} + v^4\,\left(3\,\delta Z_{V_\sigma} - \delta Z_{V_m}\right)\right)}{3\,w^2\,\left(v^2+w^2\right)^2} \\
\delta Z_{1222} &= \frac{v^3\,\left(w^2\,\left(\delta Z_{V_\phi} - \delta Z_{V_m}\right) + v^2\,\left(\delta Z_{V_m} - \delta Z_{V_\sigma}\right)\right)}{w\,\left(v^2+w^2\right)^2} \\
\delta Z_{2222} &= \frac{v^4\,\left(\delta Z_{V_\phi} - 2\,\delta Z_{V_m} + \delta Z_{V_\sigma}\right)}{\left(v^2+w^2\right)^2}
\end{aligned}
\tag{C.61}
$$

and similar expressions for the $\delta Y_{ij\dots}$.

For the calculation of the 2-loop effective potential, the mass counterterms at the 1-loop level, given by the 2nd derivatives of $\widetilde{V}_{\text{tree,ct1}}$, are needed up the order of $\mathcal{O}(\epsilon)$.

$$\delta\widetilde{\mathcal{V}}_{ij}(\phi_0,\sigma_0) = \left.\frac{\partial^2 \widetilde{V}_{\text{tree,ct1}}}{\partial\varphi_i \partial\varphi_j}\right|_{\substack{\phi=\phi_0 \\ \sigma=\sigma_0}} = \frac{\mu^{2\epsilon}(\sigma_0)}{16\,\pi^2}\left(\delta u_{ij} + \epsilon\,\delta u_{ij}^{(1)} + \epsilon^2\,\delta u_{ij}^{(2)}\right)\frac{1}{\epsilon} \tag{C.62}$$

with

$$\begin{aligned} \delta u_{11} &= \frac{3}{4}\left(\lambda_\phi^2 + \lambda_m^2\right)\phi_0^2 + \frac{\lambda_m}{4}\left(\lambda_\phi + 4\,\lambda_m + \lambda_\sigma\right)\sigma_0^2 \\ \delta u_{11}^{(1)} &= \delta u_{11}^{(2)} = 0 \end{aligned} \tag{C.63}$$

$$\begin{aligned} \delta u_{12} &= \frac{\lambda_m}{2}\left(\lambda_\phi + 4\,\lambda_m + \lambda_\sigma\right)\phi_0\,\sigma_0 \\ \delta u_{12}^{(1)} = \delta u_{12}^{(2)} &= \frac{\lambda_m}{2}\left(\lambda_\phi + 4\,\lambda_m + \lambda_\sigma\right)\phi_0\,\sigma_0 + \frac{1}{2}\left(\lambda_\phi^2 + \lambda_m^2\right)\frac{\phi_0^3}{\sigma_0} \end{aligned} \tag{C.64}$$

and

$$\begin{aligned} \delta u_{22} &= \frac{\lambda_m}{4}\left(\lambda_\phi + 4\,\lambda_m + \lambda_\sigma\right)\phi_0^2 + \frac{3}{4}\left(\lambda_m^2 + \lambda_\sigma^2\right)\sigma_0^2 \\ \delta u_{22}^{(1)} &= \frac{3}{4}\lambda_m\left(\lambda_\phi + 4\,\lambda_m + \lambda_\sigma\right)\phi_0^2 + \frac{7}{8}\left(\lambda_m^2 + \lambda_\sigma^2\right)\sigma_0^2 - \frac{1}{8}\left(\lambda_\phi^2 + \lambda_m^2\right)\frac{\phi_0^4}{\sigma_0^2} \\ \delta u_{22}^{(2)} &= \frac{5}{4}\lambda_m\left(\lambda_\phi + 4\,\lambda_m + \lambda_\sigma\right)\phi_0^2 + \frac{9}{8}\left(\lambda_m^2 + \lambda_\sigma^2\right)\sigma_0^2 + \frac{1}{8}\left(\lambda_\phi^2 + \lambda_m^2\right)\frac{\phi_0^4}{\sigma_0^2} \end{aligned} \tag{C.65}$$

where the explicit expressions (3.19) for the 1-loop counterterms in the MS-scheme have been used. Note that $\delta\widetilde{\mathcal{V}}_{ij}(\phi_0,\sigma_0)$ in (C.62) is the same as in (C.53) this time, however, with the VEVs $\{v, w\}$ being replaced by the field shifts $\{\phi_0, \sigma_0\}$ and in a notation similar to [14].

# D. Loop Functions

In this chapter of the appendix, some loop-functions at the 1-loop and the 2-loop order are provided which are used in this thesis. Particular attention is paid to the specifics of quantum scale invariance introduced by the Renormalisation function $\mu(\sigma)$ in SIDReg. This is especially important w.r.t. the treatment of factors of $\mu_0$, where

- $\mu_0 = \mu(w) = z\, w^{\frac{1}{1-\epsilon}}$, in the usual case
- $\mu_0 = \mu(\sigma_0) = z\, \sigma_0^{\frac{1}{1-\epsilon}}$, in the case of the shifted Lagrangian used for computing $V_{\text{eff}}$

Before the loop functions are discussed, however, some particular $\epsilon$ - expansions are provided.

$$\begin{aligned}
\Gamma(\epsilon) &= \frac{1}{\epsilon} - \gamma_E + \epsilon \left(\frac{1}{2}\gamma_E^2 + \frac{\pi^2}{12}\right) + \mathcal{O}\left(\epsilon^2\right) \\
\Gamma(\epsilon-1) &= -\frac{1}{\epsilon} + \gamma_E - 1 + \epsilon \left(\gamma_E - 1 - \frac{1}{2}\gamma_E^2 - \frac{\pi^2}{12}\right) + \mathcal{O}\left(\epsilon^2\right) \\
\Gamma(\epsilon-2) &= \frac{1}{2}\left[\frac{1}{\epsilon} + \frac{3}{2} - \gamma_E + \epsilon\left(\frac{21}{12} - \frac{3}{2}\gamma_E + \frac{1}{2}\gamma_E^2 + \frac{\pi^2}{12}\right)\right] + \mathcal{O}\left(\epsilon^2\right) \\
X^\epsilon &= 1 + \epsilon\, \log\left(X\right) + \frac{1}{2}\epsilon^2\, \log^2\left(X\right) + \mathcal{O}\left(\epsilon^3\right)
\end{aligned} \tag{D.1}$$

## D.1. 1-Loop-Functions

The well-known $A_0$ - function in a QSI theory, using SIDReg, is given by

$$\begin{aligned}
\frac{i}{16\,\pi^2} A_0\left(\widetilde{M}_\rho^2\right) &= \int \frac{d^D k}{(2\,\pi)^D} \frac{1}{k^2 - \widetilde{M}_\rho^2} = -\frac{i}{(4\,\pi)^{D/2}} \Gamma\left(1 - \frac{D}{2}\right) \left(\widetilde{M}_\rho^2\right)^{\frac{D}{2}-1} \\
&= -\frac{i}{(4\,\pi)^2} \widetilde{M}_\rho^2\, \Gamma\left(\epsilon - 1\right) \left(\frac{4\,\pi}{\widetilde{M}_\rho^2}\right)^\epsilon \\
&= -\frac{i}{(4\,\pi)^2} \left(\mu_0^{-2\epsilon}\, \widetilde{M}_\rho^2\right)\, \Gamma\left(\epsilon - 1\right) \left(\frac{4\,\pi\,\mu_0^2}{\widetilde{M}_\rho^2}\right)^\epsilon \\
&= \frac{i}{(4\,\pi)^2} \hat{M}_\rho^2 \left[\frac{1}{\epsilon} + 1 - \log\left(\frac{M_\rho^2}{4\,\pi\,\mu_0^2} e^{\gamma_E}\right) + c_\rho^{(1)} + \epsilon\, \frac{\widetilde{A}_0^\epsilon\left(M_\rho^2\right)}{M_\rho^2}\right] + \mathcal{O}\left(\epsilon^2\right)
\end{aligned} \tag{D.2}$$

where $\rho \in \{H, S\}$, $\hat{M}_\rho^2 := \mu_0^{-2\epsilon} M_\rho^2$ and

$$\widetilde{A}_0^\epsilon\left(M_\rho^2\right) := \frac{M_\rho^2}{2}\left[\log^2\left(\frac{M_\rho^2}{4\,\pi\,\mu_0^2}\,e^{\gamma_E}\right) - 2\,\log\left(\frac{M_\rho^2}{4\,\pi\,\mu_0^2}\,e^{\gamma_E}\right) + 2 + \frac{\pi^2}{6}\right.$$
$$\left. - 2\,c_\rho^{(1)}\,\log\left(\frac{M_\rho^2}{4\,\pi\,\mu_0^2}\,e^{\gamma_E}\right) + 2\,c_\rho^{(2)}\right] \tag{D.3}$$

Further

$$\begin{aligned}
\int \frac{d^D k}{(2\,\pi)^D}\,\log\left(k^2 - \widetilde{M}_\rho^2\right) &= \frac{i}{(4\,\pi)^{D/2}}\,\frac{2}{D}\,\Gamma\left(1 - \frac{D}{2}\right)\left(\widetilde{M}_\rho^2\right)^{D/2} \\
&= -\frac{i}{(4\,\pi)^{D/2}}\,\Gamma\left(-\frac{D}{2}\right)\left(\widetilde{M}_\rho^2\right)^{D/2} \\
&= -\frac{i}{(4\,\pi)^2}\left(\widetilde{M}_\rho^2\right)^2\,\Gamma\left(\epsilon - 2\right)\left(\frac{4\,\pi}{\widetilde{M}_\rho^2}\right)^\epsilon \\
&= -\frac{i}{(4\,\pi)^2}\,\mu_0^{2\epsilon}\left(\mu_0^{-2\epsilon}\,\widetilde{M}_\rho^2\right)^2\,\Gamma\left(\epsilon - 2\right)\left(\frac{4\,\pi\,\mu_0^2}{\widetilde{M}_\rho^2}\right)^\epsilon \\
&= -\frac{i}{(4\,\pi)^2}\,\mu_0^{2\epsilon}\,\frac{\left(\hat{M}_\rho^2\right)^2}{2}\left[\frac{1}{\epsilon} + \frac{3}{2} - \log\left(\frac{M_\rho^2}{4\,\pi\,\mu_0^2}\,e^{\gamma_E}\right)\right. \\
&\qquad \left. + 2\,c_\rho^{(1)} + \epsilon\,\frac{\widetilde{Q}_0^\epsilon\left(M_\rho^2\right)}{M_\rho^2}\right] + \mathcal{O}\left(\epsilon^2\right)
\end{aligned} \tag{D.4}$$

where $\rho \in \{H, S\}$, $\hat{M}_\rho^2 = \mu_0^{-2\epsilon} M_\rho^2$, as before, and

$$\widetilde{Q}_0^\epsilon\left(M_\rho^2\right) := \frac{M_\rho^2}{2}\left[\log^2\left(\frac{M_\rho^2}{4\,\pi\,\mu_0^2}\,e^{\gamma_E}\right) - 3\,\log\left(\frac{M_\rho^2}{4\,\pi\,\mu_0^2}\,e^{\gamma_E}\right) + \frac{21}{6} + \frac{\pi^2}{6}\right.$$
$$\left. + 4\,c_\rho^{(1)}\left(1 + \frac{1}{2}\,c_\rho^{(1)} - \log\left(\frac{M_\rho^2}{4\,\pi\,\mu_0^2}\,e^{\gamma_E}\right)\right) + 4\,c_\rho^{(2)}\right] \tag{D.5}$$

**Remark.**

(i) $M_\rho^2$ and $\widetilde{M}_\rho^2$, $\rho \in \{H, S\}$, are the squared mass eigenvalues of $\mathcal{M}_{\phi\sigma}^2$ and $\widetilde{\mathcal{M}}_{\phi\sigma}^2$, given in (C.19) and (C.17) & (C.18), respectively, with mass dimension $[M_\rho^2] = 2$ and $[\widetilde{M}_\rho^2] = 2$, even in $D = 4 - 2\epsilon$ dimensions. As discussed in chapter 2 and visible in (C.19) squared masses always have mass dimension 2 in every spacetime dimension, and contain a factor of $\mu_0^{2\epsilon}$ which ensures this. The reason for this is that there are actually no mass terms in scale invariant theories, however, masses arise either after SSB or due to the field shift (3.1), and thus dependent on either the VEVs or the background fields, respectively. Hence, masses consist of VEVs or background fields with mass dimension $1 - \epsilon$ and come with an factor of $\mu_0^\epsilon$ to an appropriate power to ultimately give rise to a mass of mass dimension 1.

(ii) The $A_0$ - function in (D.2) has mass dimension $2 - 2\epsilon$, as can be seen from the corresponding momentum integral expression. Hence, the last line in (D.2) must also have mass dimension $2 - 2\epsilon$.

(iii) The LHS of (D.4) has mass dimension $D = 4 - 2\epsilon$, which can also be seen in equation (3.8) since the effective potential has mass dimension $D = 4 - 2\epsilon$. Thus, the RHS of (D.4) must also have mass dimension $D = 4 - 2\epsilon$.

(iv) In the penultimate step of (D.2) and (D.4) the factor of $\mu_0^{2\epsilon}$ is pulled out of the first squared mass. This is done in order to obtain an appropriate factor of $\mu_0$ in the logarithms, because the arguments in the logarithms have to have mass dimension 0. Then, $\hat{M}_\rho^2 := \mu_0^{-2\epsilon} M_\rho^2$ is defined for convenience, which has mass dimension $[\hat{M}_\rho^2] = 2 - 2\,\epsilon$, and thus is the squared mass without the factor of $\mu_0^{2\epsilon}$.

(v) From the penultimate to the last step in (D.2) and (D.4), the $\epsilon$-expansion for $\widetilde{M}_\rho^2$ in (C.18) has been used and then $\widetilde{M}_\rho^2$, $\Gamma(\epsilon - 1)$ and $\Gamma(\epsilon - 2)$, respectively, as well as $(.)^\epsilon$ have been expanded w.r.t. $\epsilon$.

(vi) Finally, an expression of mass dimension $2-2\epsilon$ and $D = 4-2\epsilon$ in (D.2) and (D.4), respectively, each with dimensionless arguments in the logarithms is obtained, since $[\hat{M}_\rho^2] = 2 - 2\epsilon$, $[M_\rho^2] = 2$ and $[\mu_0^2] = 2$, as expected.

(vii) After Renormalisation, in the limit $\epsilon \to 0$, $\hat{M}_\rho^2$ will be identical to $M_\rho^2$.

Beside the above two loop-integrals, the divergent part of the $B_0$ - function in QSI theories is needed for the computation of 1-loop counterterms in MS or $\overline{\text{MS}}$ scheme.

$$
\begin{aligned}
\frac{i}{16\,\pi^2}\, B_0\left(p, \widetilde{M}_1, \widetilde{M}_2\right) &= \int \frac{d^D k}{(2\,\pi)^D}\, \frac{1}{k^2 - \widetilde{M}_1^2}\, \frac{1}{(k-p)^2 - \widetilde{M}_2^2} \\
&= \frac{i}{16\,\pi^2}\, \mu_0^{-2\epsilon} \int_0^1 dx\ (4\,\pi)^\epsilon\ \Gamma\left(\epsilon\right) \left(\frac{\mu_0^2}{\widetilde{Q}_B(x)}\right)^\epsilon \\
&= \frac{i}{16\,\pi^2}\, \mu_0^{-2\epsilon}\, \frac{1}{\epsilon} + \mathcal{O}\left(\epsilon^0\right)
\end{aligned}
\tag{D.6}
$$

where $\widetilde{M}_i^2$ is a generic squared mass in a QSI theory, such as e.g. (C.17) & (C.18), and

$$
\widetilde{Q}_B(x) := p^2\ (1-x)^2 + \widetilde{M}_1^2\, x - \left(p^2 - \widetilde{M}_2^2\right)\ (1-x) \tag{D.7}
$$

Note that the same technical details, especially w.r.t. factors of $\mu_0$, apply as in the loop-integrals above.

## D.2. 2-Loop-Functions

The 2-loop functions considered in this section are discussed and provided in [5, 6, 7, 10, 15, 22, 23, 24, 25, 26].

$$
\begin{aligned}
J\left(\widetilde{x},\widetilde{y}\right) &= -\left(16\,\pi^2\right)^2 \int \frac{d^D k}{(2\,\pi)^D} \int \frac{d^D l}{(2\,\pi)^D}\,\frac{1}{k^2-\widetilde{x}}\,\frac{1}{l^2-\widetilde{y}} \\
&= \mu_0^{-4\epsilon}\left\{\frac{\widetilde{x}\,\widetilde{y}}{\epsilon^2} - \frac{\widetilde{x}\,\widetilde{y}}{\epsilon}\left(\log\left(\frac{\widetilde{x}}{4\,\pi\,\mu_0^2}\,e^{\gamma_E}\right) + \log\left(\frac{\widetilde{y}}{4\,\pi\,\mu_0^2}\,e^{\gamma_E}\right) - 2\right)\right. \\
&\qquad - \widetilde{x}\,\widetilde{y}\left[2\,\log\left(\frac{\widetilde{x}}{4\,\pi\,\mu_0^2}\,e^{\gamma_E}\right) + 2\,\log\left(\frac{\widetilde{y}}{4\,\pi\,\mu_0^2}\,e^{\gamma_E}\right)\right. \\
&\qquad\qquad \left.\left. - \frac{1}{2}\,\log^2\left(\frac{\widetilde{x}}{4\,\pi\,\mu_0^2}\,e^{\gamma_E}\,\frac{\widetilde{y}}{4\,\pi\,\mu_0^2}\,e^{\gamma_E}\right) - \left(3+\frac{\pi^2}{6}\right)\right]\right\} + \mathcal{O}\left(\epsilon\right) \\
&= \frac{\hat{x}\,\hat{y}}{\epsilon^2} - \frac{\hat{x}\,\hat{y}}{\epsilon}\left(\log\left(\frac{x}{4\,\pi\,\mu_0^2}\,e^{\gamma_E}\right) + \log\left(\frac{y}{4\,\pi\,\mu_0^2}\,e^{\gamma_E}\right) - c_x^{(1)} - c_y^{(1)} - 2\right) \\
&\quad - \hat{x}\,\hat{y}\left[\left(2 + c_x^{(1)} + c_y^{(1)}\right)\,\log\left(\frac{x}{4\,\pi\,\mu_0^2}\,e^{\gamma_E}\right) + \left(2 + c_x^{(1)} + c_y^{(1)}\right)\,\log\left(\frac{y}{4\,\pi\,\mu_0^2}\,e^{\gamma_E}\right)\right. \\
&\qquad - \frac{1}{2}\,\log^2\left(\frac{x}{4\,\pi\,\mu_0^2}\,e^{\gamma_E}\,\frac{y}{4\,\pi\,\mu_0^2}\,e^{\gamma_E}\right) - \left(3+\frac{\pi^2}{6}\right) \\
&\qquad \left. - \left(c_x^{(1)} + c_y^{(1)}\right) - \left(c_x^{(2)} + c_y^{(2)} + c_x^{(1)}\,c_y^{(1)}\right)\right] + \mathcal{O}\left(\epsilon\right)
\end{aligned} \tag{D.8}
$$

where

$$
\widetilde{x}_i = x_i\,\left(1 + \epsilon\,c_{x_i}^{(1)} + \epsilon^2\,c_{x_i}^{(2)} + \mathcal{O}\left(\epsilon^3\right)\right) \tag{D.9}
$$

and $\hat{x}_i \coloneqq \mu_0^{-2\epsilon}\,x_i$, with $\{x_i\}_{i=1}^3 = \{x, y, z\}$.

**Remark.**

(i) The same technical details, especially w.r.t. factors of $\mu_0$, apply as discussed in the previous section for the 1-loop functions.

(ii) The mass dimensions, in $D = 4 - 2\epsilon$ spacetime dimensions, are given by $[\widetilde{x}] = 2$, $[x] = 2$ and $[\hat{x}] = 2 - 2\epsilon$, meaning that the factor of $\mu_0^{2\epsilon}$ is implicitly contained in $\widetilde{x}$ and $x$, whereas $\hat{x}$ does not contain $\mu_0^{2\epsilon}$.

(iii) In 4 dimensions, i.e. in the limit $\epsilon \to 0$, $\hat{x}$ is identical to $x$.

(iv) As a starting point the definition of $J$ in [6] was used. The arguments $\widetilde{x}$ and $\widetilde{y}$ have then been expanded w.r.t. $\epsilon$.

$$
\begin{aligned}
I\left(\widetilde{x},\widetilde{y},\widetilde{z}\right) &= \left(16\,\pi^2\right)^2 \int \frac{d^D k}{(2\,\pi)^D} \int \frac{d^D l}{(2\,\pi)^D} \, \frac{1}{k^2-\widetilde{x}} \, \frac{1}{l^2-\widetilde{y}} \, \frac{1}{(k-l)^2-\widetilde{z}} \\
&= \mu_0^{-4\epsilon} \Bigg\{ -\frac{\widetilde{x}+\widetilde{y}+\widetilde{z}}{2\,\epsilon^2} + \frac{1}{\epsilon}\left[\widetilde{x}\,\overline{\log}\left(\widetilde{x}\right) + \widetilde{y}\,\overline{\log}\left(\widetilde{y}\right) + \widetilde{z}\,\overline{\log}\left(\widetilde{z}\right) - \frac{3}{2}\,\left(\widetilde{x}+\widetilde{y}+\widetilde{z}\right)\right] \\
&\qquad + \frac{1}{2}\left[\widetilde{x}\,\overline{\log}\left(\widetilde{y}\right)\,\overline{\log}\left(\widetilde{z}\right) + \widetilde{y}\,\overline{\log}\left(\widetilde{x}\right)\,\overline{\log}\left(\widetilde{z}\right) + \widetilde{z}\,\overline{\log}\left(\widetilde{x}\right)\,\overline{\log}\left(\widetilde{y}\right)\right] \\
&\qquad - \frac{\widetilde{x}+\widetilde{y}+\widetilde{z}}{2}\left(7+\frac{\pi^2}{6}\right) \\
&\qquad - \frac{1}{2}\,\left[\widetilde{x}\,\overline{\log}\left(\widetilde{x}\right) + \widetilde{y}\,\overline{\log}\left(\widetilde{y}\right) + \widetilde{z}\,\overline{\log}\left(\widetilde{z}\right)\right]\left[\overline{\log}\left(\widetilde{x}\right) + \overline{\log}\left(\widetilde{y}\right) + \overline{\log}\left(\widetilde{z}\right) - 6\right] \\
&\qquad - \frac{\Delta\left(\widetilde{x},\widetilde{y},\widetilde{z}\right)}{2\,\widetilde{z}}\,\Phi\left(\widetilde{x},\widetilde{y},\widetilde{z}\right)\Bigg\} + \mathcal{O}\left(\epsilon\right) \\
&= \mu_0^{-2\epsilon} \Bigg\{ -\frac{\hat{x}+\hat{y}+\hat{z}}{2\,\epsilon^2} + \frac{1}{\epsilon}\Bigg[\hat{x}\,\overline{\log}\left(x\right) + \hat{y}\,\overline{\log}\left(y\right) + \hat{z}\,\overline{\log}\left(z\right) - \frac{3}{2}\,\left(\hat{x}+\hat{y}+\hat{z}\right) \\
&\qquad - \frac{1}{2}\,\left(\hat{x}\,c_x^{(1)} + \hat{y}\,c_y^{(1)} + \hat{z}\,c_z^{(1)}\right)\Bigg] - \frac{\hat{x}+\hat{y}+\hat{z}}{2}\left(7+\frac{\pi^2}{6}\right) \\
&\qquad + \frac{1}{2}\left[\hat{x}\,\overline{\log}\left(y\right)\,\overline{\log}\left(z\right) + \hat{y}\,\overline{\log}\left(x\right)\,\overline{\log}\left(z\right) + \hat{z}\,\overline{\log}\left(x\right)\,\overline{\log}\left(y\right)\right] \\
&\qquad - \frac{1}{2}\,\left[\hat{x}\,\overline{\log}\left(x\right) + \hat{y}\,\overline{\log}\left(y\right) + \hat{z}\,\overline{\log}\left(z\right)\right]\left[\overline{\log}\left(x\right) + \overline{\log}\left(y\right) + \overline{\log}\left(z\right) - 6\right] \\
&\qquad + \hat{x}\,c_x^{(1)}\,\overline{\log}\left(x\right) + \hat{y}\,c_y^{(1)}\,\overline{\log}\left(y\right) + \hat{z}\,c_z^{(1)}\,\overline{\log}\left(z\right) \\
&\qquad - \frac{1}{2}\,\hat{x}\,\left(c_x^{(1)} + c_x^{(2)}\right) - \frac{1}{2}\,\hat{y}\,\left(c_y^{(1)} + c_y^{(2)}\right) - \frac{1}{2}\,\hat{z}\,\left(c_z^{(1)} + c_z^{(2)}\right) \\
&\qquad - \frac{\Delta\left(\hat{x},\hat{y},\hat{z}\right)}{2\,\hat{z}}\,\Phi\left(x,y,z\right)\Bigg\} + \mathcal{O}\left(\epsilon\right) \\
&= \mu_0^{-2\epsilon} \Bigg\{ -\frac{\hat{x}+\hat{y}+\hat{z}}{2\,\epsilon^2} + \frac{1}{\epsilon}\Bigg[\hat{x}\,\overline{\log}\left(x\right) + \hat{y}\,\overline{\log}\left(y\right) + \hat{z}\,\overline{\log}\left(z\right) - \frac{3}{2}\,\left(\hat{x}+\hat{y}+\hat{z}\right) \\
&\qquad - \frac{1}{2}\,\left(\hat{x}\,c_x^{(1)} + \hat{y}\,c_y^{(1)} + \hat{z}\,c_z^{(1)}\right)\Bigg] - \frac{1}{2}\left[\hat{x}\,\overline{\log}^2\left(x\right) + \hat{y}\,\overline{\log}^2\left(y\right) + \hat{z}\,\overline{\log}^2\left(z\right)\right] \\
&\qquad + \hat{x}\,\left(1+c_x^{(1)}\right)\,\overline{\log}\left(x\right) + \hat{y}\,\left(1+c_y^{(1)}\right)\,\overline{\log}\left(y\right) + \hat{z}\,\left(1+c_z^{(1)}\right)\,\overline{\log}\left(z\right) \\
&\qquad - \left(1+\frac{\pi^2}{12}+\frac{c_x^{(1)}}{2}+\frac{c_x^{(2)}}{2}\right)\,\hat{x} - \left(1+\frac{\pi^2}{12}+\frac{c_y^{(1)}}{2}+\frac{c_y^{(2)}}{2}\right)\,\hat{y} \\
&\qquad - \left(1+\frac{\pi^2}{12}+\frac{c_z^{(1)}}{2}+\frac{c_z^{(2)}}{2}\right)\,\hat{z}\Bigg\} + I_{\mathrm{SM}}\left(x,y,z\right) + \mathcal{O}\left(\epsilon\right)
\end{aligned}
\tag{D.10}
$$

where

$$\overline{\log}\,(x) \coloneqq \log\left(\frac{x}{4\,\pi\,\mu_0^2}\,e^{\gamma_E}\right) \tag{D.11}$$

and, defined according to [7, 15, 23],

$$\begin{aligned} I_{\mathrm{SM}}\,(x,y,z) \coloneqq \mu_0^{-4\epsilon}\Bigg\{ &\frac{1}{2}\,(x-y-z)\,\overline{\log}\,(y)\,\overline{\log}\,(z) + \frac{1}{2}\,(y-x-z)\,\overline{\log}\,(x)\,\overline{\log}\,(z) \\ &+ \frac{1}{2}\,(z-x-y)\,\overline{\log}\,(x)\,\overline{\log}\,(y) + 2\,x\,\overline{\log}\,(x) + 2\,y\,\overline{\log}\,(y) \\ &+ 2\,z\,\overline{\log}\,(z) - \frac{5}{2}\,(x+y+z) - \frac{1}{2}\,\xi(x,y,z)\Bigg\} \end{aligned} \tag{D.12}$$

with, for $x, y \leq z$,

$$\begin{aligned} \xi(x,y,z) &\coloneqq \frac{\Delta\,(x,y,z)}{z}\,\Phi\,(x,y,z) \\ &= R\left[2\,\log\left(\frac{z+x-y-R}{2\,z}\right)\,\log\left(\frac{z+y-x-R}{2\,z}\right) - \log\left(\frac{x}{z}\right)\,\log\left(\frac{y}{z}\right)\right. \\ &\quad \left. - 2\,\mathrm{Li}_2\left(\frac{z+x-y-R}{2\,z}\right) - 2\,\mathrm{Li}_2\left(\frac{z+y-x-R}{2\,z}\right) + \frac{\pi^2}{3}\right] \end{aligned} \tag{D.13}$$

$$R \coloneqq \sqrt{x^2+y^2+z^2-2\,x\,y-2\,x\,z-2\,y\,z} \tag{D.14}$$

Further, this can be expressed in terms of $\Delta$ and $\Phi$, as done in [6], with

$$\Delta\,(x,y,z) \coloneqq x^2+y^2+z^2-2\,(x\,y+x\,z+y\,z) \tag{D.15}$$

$$\begin{aligned} \Phi\,(x,y,z) \coloneqq \frac{1}{\lambda}\Big[&2\,\log\,(x_+)\,\log\,(x_-) - \log\,(u)\,\log\,(v) \\ &- 2\,\left(\mathrm{Li}_2\,(x_+) + \mathrm{Li}_2\,(x_-)\right) + \frac{\pi^2}{3}\Big] \end{aligned} \tag{D.16}$$

with

$$\begin{aligned} &\mathrm{Li}_2\,(s) \coloneqq -\int_0^s dt\,\frac{\log\,(1-t)}{t}, \qquad \text{the dilogarithm} \\ &u \coloneqq \frac{x}{z}, \qquad v \coloneqq \frac{y}{z} \\ &\lambda \coloneqq \sqrt{(1-u-v)^2 - 4\,u\,v} = \frac{1}{z}\,R \\ &x_\pm \coloneqq \frac{1}{2}\,(1 \pm (u-v) - \lambda) \end{aligned} \tag{D.17}$$

Note that the definition of $\Phi\,(x,y,z)$ is valid for

$$u = \frac{x}{z} \leq 1 \quad \text{and} \quad v = \frac{y}{z} \leq 1$$

The other branches of $\Phi(x,y,z)$ can be obtained using the symmetry properties [6]

$$\Phi(x,y,z) = \Phi(y,x,z)\,, \qquad x\,\Phi(x,y,z) = z\,\Phi(z,y,x)$$

The derivatives of $\Phi(x,y,z)$ can be obtained via the following recursive relation, given in [6],

$$\begin{aligned}\Delta(x,y,z)\,\frac{\partial\Phi(x,y,z)}{\partial x} &= (y+z-x)\,\Phi(x,y,z) \\ &+ \frac{z}{x}\left[(y-z)\,\log\left(\frac{z}{y}\right) + x\left(\log\left(\frac{x}{y}\right) + \log\left(\frac{x}{z}\right)\right)\right]\end{aligned} \tag{D.18}$$

Derivatives of $\Phi$ w.r.t. $y$ and $z$ can be obtained by using the recursive relation and the symmetry properties above.

# E. Gauge Theories

In this chapter of the appendix gauge transformations for a generic $SU(N)$ gauge theory in 4 and $D = 4-2\epsilon$ spacetime dimensions as well as for rescaled and non-rescaled gauge fields with mass dimensions $[\hat{G}^a_\mu] = 1$ and $[G^a_\mu] = 1-\epsilon$, respectively, are displayed.

## E.1. Gauge Transformations in 4 Dimensions

(a) For rescaled gauge fields $G^a_\mu \to \hat{G}^a_\mu = g\, G^a_\mu$ one obtains the following relation for the gauge transformed gauge field $\hat{G}'^a_\mu$ [32]

$$\begin{aligned} \left(\partial_\mu - i\,\hat{G}'^a_\mu\,T^a\right) U\,\Psi &= U\left(\partial_\mu - i\,\hat{G}^a_\mu\,T^a\right)\Psi \\ \Longrightarrow\ \hat{G}'^a_\mu\,T^a &= U\,\hat{G}^a_\mu\,T^a\,U^{-1} - i\,(\partial_\mu U)\,U^{-1} \\ &= \hat{G}^a_\mu\,T^a + \partial_\mu\hat{\beta}^a\,T^a + f^{abc}\,\hat{G}^b_\mu\,\hat{\beta}^c\,T^a \end{aligned} \tag{E.1}$$

for

$$U = e^{i\,\hat{\beta}^a\,T^a} \tag{E.2}$$

and with mass dimensions (in 4 spacetime dimensions)

$$[\hat{G}^a_\mu] = 1, \qquad [\hat{\beta}^a] = 0, \qquad [g] = 0 \tag{E.3}$$

Thus, the infinitesimal gauge transformations are given by

$$\begin{aligned} \psi_i &\longmapsto \psi_i + i\,\hat{\beta}^a\,T^a_{ij}\,\psi_j \\ \overline{\psi}_i &\longmapsto \overline{\psi}_i - i\,\hat{\beta}^a\,\overline{\psi}_j\,T^a_{ji} \\ \hat{G}^a_\mu &\longmapsto \hat{G}^a_\mu + \partial_\mu\hat{\beta}^a + f^{abc}\,\hat{G}^b_\mu\,\hat{\beta}^c \end{aligned} \tag{E.4}$$

(b) For non-rescaled gauge fields $G^a_\mu$ one obtains the following relation for the gauge transformed gauge field $G'^a_\mu$ [32]

$$\begin{aligned} \left(\partial_\mu - i\,g\,G'^a_\mu\,T^a\right) U\,\Psi &= U\left(\partial_\mu - i\,g\,G^a_\mu\,T^a\right)\Psi \\ \Longrightarrow\ G'^a_\mu\,T^a &= U\,G^a_\mu\,T^a\,U^{-1} - \frac{i}{g}\,(\partial_\mu U)\,U^{-1} \\ &= G^a_\mu\,T^a + \partial_\mu\beta^a\,T^a + g\,f^{abc}\,G^b_\mu\,\beta^c\,T^a \end{aligned} \tag{E.5}$$

for

$$U = e^{i\, g\, \beta^a\, T^a} \tag{E.6}$$

and with mass dimensions (in 4 spacetime dimensions)

$$[G^a_\mu] = 1, \qquad [\beta^a] = 0, \qquad [g] = 0 \tag{E.7}$$

Thus, the infinitesimal gauge transformations are given by

$$\begin{aligned} \psi_i &\longmapsto \psi_i + i\, g\, \beta^a\, T^a_{ij}\, \psi_j \\ \overline{\psi}_i &\longmapsto \overline{\psi}_i - i\, g\, \beta^a\, \overline{\psi}_j\, T^a_{ji} \\ G^a_\mu &\longmapsto G^a_\mu + \partial_\mu \beta^a + g\, f^{abc}\, G^b_\mu\, \beta^c \end{aligned} \tag{E.8}$$

## E.2. Gauge Transformations in $D = 4 - 2\epsilon$ Dimensions

(a) In the case where the gauge fields are rescaled and *then* the theory is analytically extended to $D = 4-2\epsilon$ dimensions, i.e. $G^a_\mu \to \hat{G}^a_\mu = g\, \mu^\epsilon(\sigma)\, G^a_\mu$, one obtains the following relation for the gauge transformed gauge field $\hat{G}'^a_\mu$

$$\begin{aligned} &\left(\partial_\mu - i\, \hat{G}'^a_\mu\, T^a\right) U\, \Psi = U \left(\partial_\mu - i\, \hat{G}^a_\mu\, T^a\right) \Psi \\ \Longrightarrow\ &\hat{G}'^a_\mu\, T^a = U\, \hat{G}^a_\mu\, T^a\, U^{-1} - i\, (\partial_\mu U)\, U^{-1} \\ &\qquad = \hat{G}^a_\mu\, T^a + \partial_\mu \hat{\beta}^a\, T^a + f^{abc}\, \hat{G}^b_\mu\, \hat{\beta}^c\, T^a \end{aligned} \tag{E.9}$$

for

$$U = e^{i\, \hat{\beta}^a\, T^a} \tag{E.10}$$

and with mass dimensions (in $D = 4 - 2\epsilon$ spacetime dimensions)

$$[\hat{G}^a_\mu] = 1, \qquad [\hat{\beta}^a] = 0, \qquad [g] = 0 \tag{E.11}$$

Thus, the infinitesimal gauge transformations are given by

$$\begin{aligned} \psi_i &\longmapsto \psi_i + i\, \hat{\beta}^a\, T^a_{ij}\, \psi_j \\ \overline{\psi}_i &\longmapsto \overline{\psi}_i - i\, \hat{\beta}^a\, \overline{\psi}_j\, T^a_{ji} \\ \hat{G}^a_\mu &\longmapsto \hat{G}^a_\mu + \partial_\mu \hat{\beta}^a + f^{abc}\, \hat{G}^b_\mu\, \hat{\beta}^c \end{aligned} \tag{E.12}$$

It can be seen that these gauge transformations are equivalent to those in 4 dimensions.

Note that, *after* the theory was analytically continued to $D = 4 - 2\epsilon$ dimensions, solely the dimensionless gauge coupling $g$ (and *not* the Renormalisation function) can be pulled out of the gauge field to obtain $\overline{G}^a_\mu = \mu^\epsilon(\sigma)\, G^a_\mu$, such that

$$\begin{aligned} \overline{G}^a_\mu &= \frac{1}{g}\, \hat{G}^a_\mu = \mu^\epsilon(\sigma)\, G^a_\mu \\ \hat{G}^a_\mu &= g\, \mu^\epsilon(\sigma)\, G^a_\mu = g\, \overline{G}^a_\mu \end{aligned} \tag{E.13}$$

and analogous for $\beta^a$. In this case the infinitesimal gauge transformations are then provided by

$$\begin{aligned} \psi_i &\longmapsto \psi_i + i\, g\, \overline{\beta}^a\, T^a_{ij}\, \psi_j \\ \overline{\psi}_i &\longmapsto \overline{\psi}_i - i\, g\, \overline{\beta}^a\, \overline{\psi}_j\, T^a_{ji} \\ \overline{G}^a_\mu &\longmapsto \overline{G}^a_\mu + \partial_\mu \overline{\beta}^a + g\, f^{abc}\, \overline{G}^b_\mu\, \overline{\beta}^c \end{aligned} \tag{E.14}$$

which look exactly like the "usual" gauge transformations in 4 dimensions (cf. case (b) in section E.1). The mass dimensions are still the same as in (E.11), i.e.

$$[\overline{G}^a_\mu] = 1, \qquad [\overline{\beta}^a] = 0, \qquad [g] = 0 \tag{E.15}$$

(b) For non-rescaled gauge fields $G^a_\mu$ one obtains the following relation for the gauge transformed gauge field $G'^a_\mu$

$$\begin{aligned} &\left(\partial_\mu - i\, g\, \mu^\epsilon(\sigma)\, G'^a_\mu\, T^a\right) U\, \Psi = U \left(\partial_\mu - i\, g\, \mu^\epsilon(\sigma)\, G^a_\mu\, T^a\right) \Psi \\ \Longrightarrow\ & G'^a_\mu\, T^a = U\, G^a_\mu\, T^a\, U^{-1} - \frac{i}{g}\, \mu^{-\epsilon}(\sigma)\, (\partial_\mu U)\, U^{-1} \\ &\qquad = G^a_\mu\, T^a + \partial_\mu \beta^a\, T^a + g\, \mu^\epsilon(\sigma)\, f^{abc}\, G^b_\mu\, \beta^c\, T^a + \epsilon\, \mu^{-1}(\sigma)\, \frac{\partial \mu}{\partial \sigma}\, \partial_\mu \sigma\, \beta^a\, T^a \end{aligned} \tag{E.16}$$

for

$$U = e^{i\, g\, \mu^\epsilon(\sigma)\, \beta^a\, T^a} \tag{E.17}$$

and with mass dimensions (in $D = 4 - 2\epsilon$ spacetime dimensions)

$$[G^a_\mu] = 1 - \epsilon, \qquad [\beta^a] = -\epsilon, \qquad [g] = 0 \tag{E.18}$$

Thus, the infinitesimal gauge transformations are given by

$$\begin{aligned} \psi_i &\longmapsto \psi_i + i\, g\, \mu^\epsilon(\sigma)\, \beta^a\, T^a_{ij}\, \psi_j \\ \overline{\psi}_i &\longmapsto \overline{\psi}_i - i\, g\, \mu^\epsilon(\sigma)\, \beta^a\, \overline{\psi}_j\, T^a_{ji} \\ G^a_\mu &\longmapsto G^a_\mu + \partial_\mu \beta^a + g\, \mu^\epsilon(\sigma)\, f^{abc}\, G^b_\mu\, \beta^c + \epsilon\, \mu^{-1}(\sigma)\, \frac{\partial \mu}{\partial \sigma}\, \partial_\mu \sigma\, \beta^a \end{aligned} \tag{E.19}$$

It can be seen that the gauge transformation of the gauge field obtains a non-trivial evanescent correction due to the Renormalisation function.

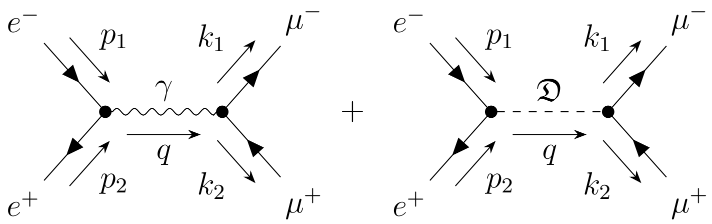
# F. Muon Production

The Feynman diagrams of the considered process in section 5.3, i.e. muon production at the 1-loop level, considering only the 1-loop muon vertex corrections, in massless QSI QED with $y_f \neq 0$ and $\lambda \equiv 0$ and Lagrangian (4.29), are provided in the first section of this chapter of the appendix. Moreover, the 3 body phase space integral and some solutions for specific kinematics which are necessary for the real emission graphs in the above mentioned process are given in the second section.

## F.1. Feynman Diagrams

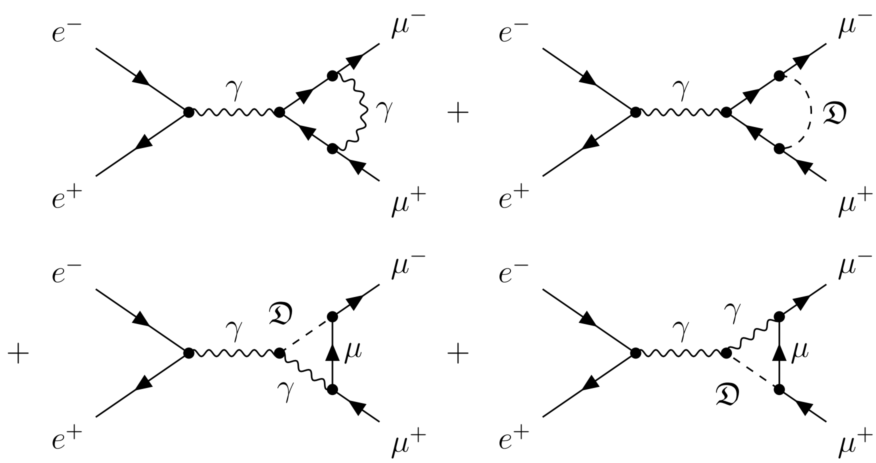
In the considered theory (4.29), there are 2 Feynman diagrams contributing to the scattering process $e^- \, e^+ \longrightarrow \mu^- \, \mu^+$ at tree-level, which are

$$i\,\mathcal{M}_{\text{tree}} \;=$$



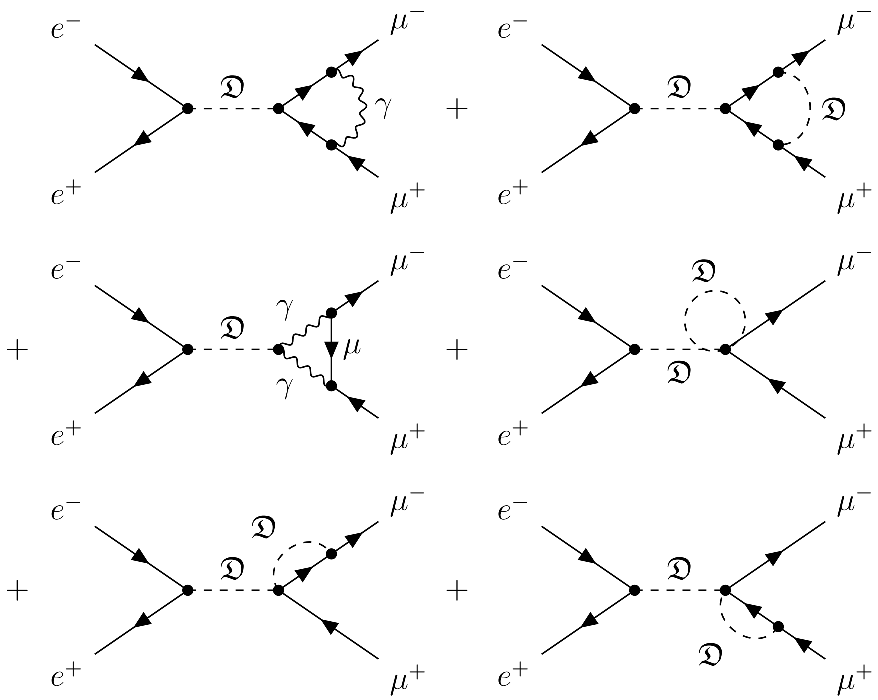

(F.1)

At the 1-loop level, there are 10 Feynman diagrams containing a 1-loop muon vertex correction, 4 of them with a photon mediator as illustrated in (F.2)



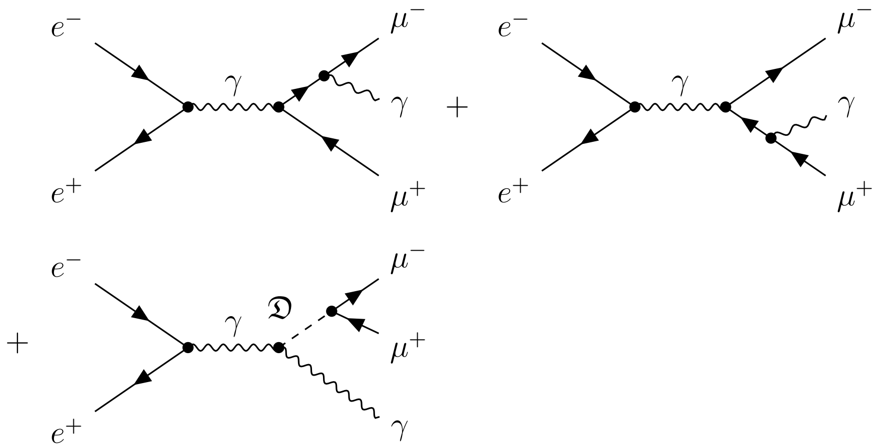

(F.2)

and the other 6 with a Dilaton mediator as shown below in (F.3).



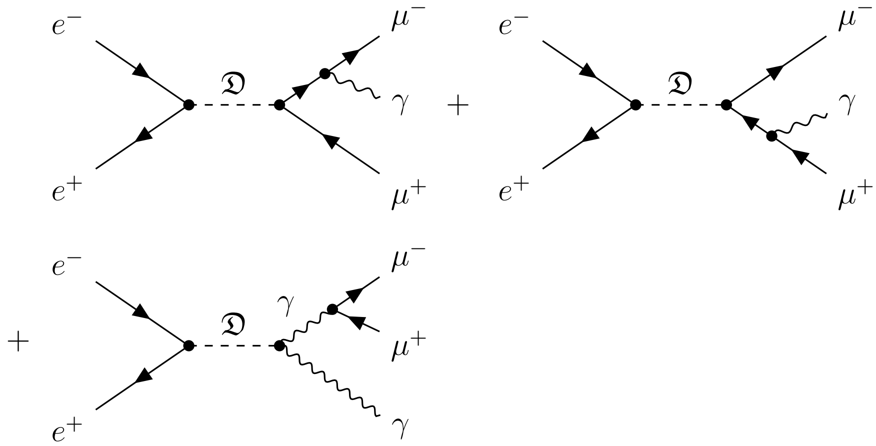

(F.3)

Moreover, for the scattering process $e^- e^+ \longrightarrow \mu^- \mu^+ \gamma$, i.e. final state real photon emission graphs, there are 6 tree-level Feynman diagrams, where 3 of them are photon mediated as provided in (F.4)



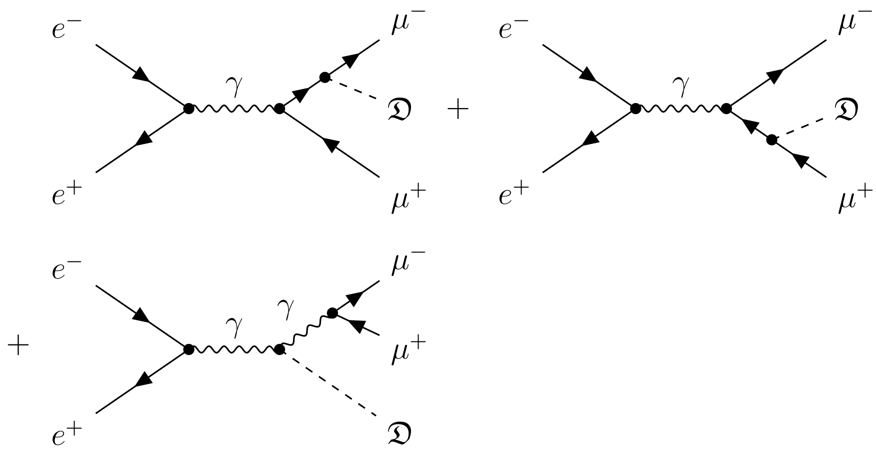

(F.4)

and the other 3 of them are Dilaton mediated as illustrated in (F.5).



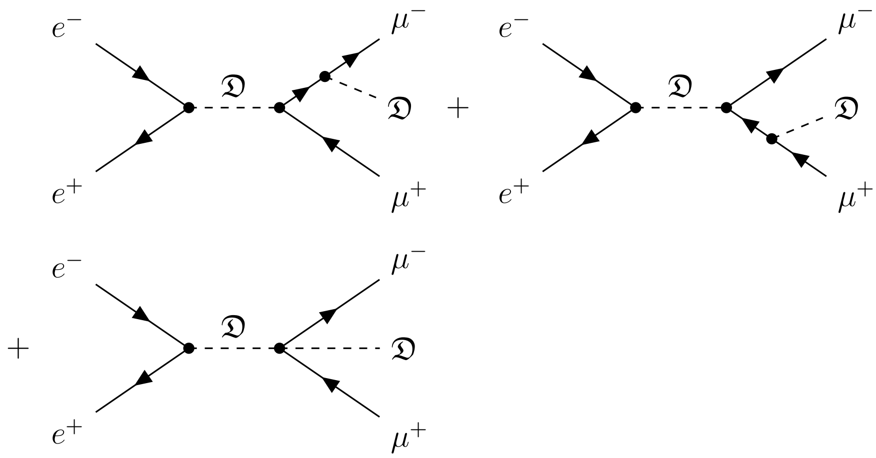

(F.5)

Additionally, for the scattering process $e^- \, e^+ \longrightarrow \mu^- \, \mu^+ \, \mathfrak{D}$, i.e. final state real Dilaton emission graphs, there are 6 tree-level Feynman diagrams, where 3 of them are photon mediated as given in (F.6)



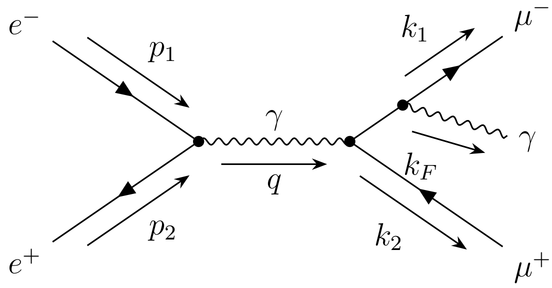

(F.6)

and the other 3 of them are Dilaton mediated, provided below in (F.7).




(F.7)

The kinematic of the $2 \longrightarrow 3$ scattering processes is given by $p_1$, $p_2$, $k_1$ and $k_2$ for the electron, positron, muon and antimuon, respectively, as well as $k_F$ for either the final state photon or the final state Dilaton, depending on the considered process. This labelling of the momenta is exemplarily illustrated for one of the $e^- e^+ \longrightarrow \mu^- \mu^+ \gamma$ diagrams in (F.8).




(F.8)

In the present case of massless QSI QED with $y_f \neq 0$ and $\lambda \equiv 0$ the squared momenta evaluate as follows

$$\begin{aligned} p_1^2 = p_2^2 = k_1^2 = k_2^2 = k_F^2 = 0 \\ q^2 = (p_1 + p_2)^2 = (k_1 + k_2 + k_F)^2 =: Q^2 \end{aligned} \tag{F.9}$$

where $Q = \sqrt{s}$ is the center-of-mass energy. Defining the energy fractions $x_i$, $i \in \{1, 2, F\}$, in the C.o.M. frame analogous to [32] as

$$x_i := \frac{2\, k_i \cdot q}{Q^2} \tag{F.10}$$

the kinematic can further be specified by

$$\begin{aligned} (k_1 + k_2)^2 = 2\, k_1\, k_2 = Q^2\, (1 - x_F) \\ (k_1 + k_F)^2 = 2\, k_1\, k_F = Q^2\, (1 - x_2) \\ (k_2 + k_F)^2 = 2\, k_2\, k_F = Q^2\, (1 - x_1) \end{aligned} \tag{F.11}$$

with relation

$$x_1 + x_2 + x_F = 2 \tag{F.12}$$

## F.2. 3 Body Phase Space Integral

First, consider the 2 body phase space integral in $D = 4 - 2\epsilon$ dimensions over 1, as it is also needed in section 5.3, which is given by [32]

$$\int d\Phi_2 = \left(\frac{4\,\pi}{Q^2}\right)^{\frac{4-D}{2}} \frac{2^{-D}}{\sqrt{\pi}\,\Gamma\left(\frac{D-1}{2}\right)} \tag{F.13}$$

Now, consider the 3 body phase space integral in $D = 4 - 2\epsilon$ dimensions over some function $f = f(k_1, k_2, k_F)$ which may be written as [32]

$$\begin{aligned}\int d\Phi_3\, f\,(k_1, k_2, k_F) = \left(\frac{4\,\pi}{Q^2}\right)^{4-D} & \frac{Q^2}{128\,\pi^3\,\Gamma\,(D-2)} \\ \times \int_0^1 dx_1 \int_{1-x_1}^1 & dx_2\, \frac{f(x_1, x_2, x_F)}{\big[\,(1-x_1)\,(1-x_2)\,(1-x_F)\,\big]^{\frac{4-D}{2}}}\Bigg|_{x_F = 2 - x_1 - x_2}\end{aligned} \tag{F.14}$$

in terms of the energy fractions in the C.o.M. frame $x_i$, $i \in \{1, 2, F\}$, defined in (F.10).

In the following, the 3 body phase space integral is explicitly evaluated for some specific integrands which are necessary for the $2 \longrightarrow 3$ scattering processes considered in section 5.3. Note that the integrals over $x_1$ and $x_2$ have been performed exactly and then afterwards the result has been expanded w.r.t. $\epsilon$. This is important as the expansion in $\epsilon$ does *not* commute with the phase space integration in general. Before the results are presented it is convenient to define

$$\Delta_\Phi\,(x_1, x_2, x_F) \coloneqq \frac{1}{\big[\,(1-x_1)\,(1-x_2)\,(1-x_F)\,\big]^{\frac{4-D}{2}}} \tag{F.15}$$

Now, explicit results are provided for the following integrals:

$$\begin{aligned}I_1 &\coloneqq \int_0^1 dx_1 \int_{1-x_1}^1 dx_2\, \frac{x_1^2 + x_2^2 + \frac{D-4}{2}\,x_F^2}{(1-x_1)\,(1-x_2)}\,\Delta_\Phi\,(x_1, x_2, x_F)\Bigg|_{x_F = 2 - x_1 - x_2} \\ &= \frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{19}{2} - \pi^2 + \mathcal{O}\,(\epsilon)\end{aligned} \tag{F.16}$$

$$I_2 \coloneqq \int_0^1 dx_1 \int_{1-x_1}^1 dx_2\, \frac{x_F^2}{1-x_F}\,\Delta_\Phi\,(x_1, x_2, x_F)\Bigg|_{x_F = 2 - x_1 - x_2} = -\frac{1}{\epsilon} - \frac{23}{6} + \mathcal{O}\,(\epsilon) \tag{F.17}$$

$$I_3 \coloneqq \int_0^1 dx_1 \int_{1-x_1}^1 dx_2\, \Delta_\Phi\,(x_1, x_2, x_F)\Bigg|_{x_F = 2 - x_1 - x_2} = \frac{1}{2} + \mathcal{O}\,(\epsilon) \tag{F.18}$$

$$\begin{aligned}I_4 &\coloneqq \int_0^1 dx_1 \int_{1-x_1}^1 dx_2\, \frac{x_1^2 + x_2^2 + \frac{D-4}{2}\,x_F^2}{1-x_F}\,\Delta_\Phi\,(x_1, x_2, x_F)\Bigg|_{x_F = 2 - x_1 - x_2} \\ &= -\frac{2}{3\,\epsilon} - \frac{2}{3} + \mathcal{O}\,(\epsilon)\end{aligned} \tag{F.19}$$

$$\begin{aligned}I_5 &\coloneqq \int_0^1 dx_1 \int_{1-x_1}^1 dx_2\, \frac{x_1\,x_2}{1-x_F}\,\Delta_\Phi\,(x_1, x_2, x_F)\Bigg|_{x_F = 2 - x_1 - x_2} \\ &= -\frac{1}{6\,\epsilon} - \frac{1}{12} + \mathcal{O}\,(\epsilon)\end{aligned} \tag{F.20}$$

$$I_6 \coloneqq \int_0^1 dx_1 \int_{1-x_1}^1 dx_2\, \frac{1}{1-x_2}\,\Delta_\Phi\,(x_1, x_2, x_F)\Bigg|_{x_F = 2 - x_1 - x_2} = -\frac{1}{\epsilon} - 3 + \mathcal{O}\,(\epsilon) \tag{F.21}$$

$$I_7 \coloneqq \int_0^1 dx_1 \int_{1-x_1}^1 dx_2 \, \frac{x_2\,(2-x_1)}{(1-x_1)\,(1-x_2)} \, \Delta_\Phi\,(x_1, x_2, x_F)\bigg|_{x_F=2-x_1-x_2} \tag{F.22}$$
$$= \frac{1}{\epsilon^2} - \frac{\pi^2}{2} - \frac{1}{2} + \mathcal{O}\,(\epsilon)$$

$$I_8 \coloneqq \int_0^1 dx_1 \int_{1-x_1}^1 dx_2 \, \frac{x_1\,x_2}{(1-x_1)\,(1-x_2)} \, \Delta_\Phi\,(x_1, x_2, x_F)\bigg|_{x_F=2-x_1-x_2} \tag{F.23}$$
$$= \frac{1}{\epsilon^2} + \frac{2}{\epsilon} + \frac{13}{2} - \frac{\pi^2}{2} + \mathcal{O}\,(\epsilon)$$

$$I_9 \coloneqq \int_0^1 dx_1 \int_{1-x_1}^1 dx_2 \, (1-x_F)\, \Delta_\Phi\,(x_1, x_2, x_F)\bigg|_{x_F=2-x_1-x_2} = \frac{1}{6} + \mathcal{O}\,(\epsilon) \tag{F.24}$$

Furthermore, note the following relation

$$I_2 = \frac{1}{D-2}\Big(2\,I_4 - 8\,I_3 + 4\,I_5\Big) \tag{F.25}$$

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## Erklärung

Hiermit erkläre ich, dass ich diese Arbeit im Rahmen der Betreuung am Institut für Kern- und Teilchenphysik ohne unzulässige Hilfe Dritter verfasst und alle Quellen als solche gekennzeichnet habe.

Matthias Weißwange
Dresden, 18. Oktober 2021