Dichotomies uniform on subspaces and formulas for dichotomy spectra

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Abstract

In this note we introduce a notion of dichotomy which generalizes the classical concept of exponential dichotomy and the recent notion of Bohl dichotomy. A key attribute is the discussion of the sets of subspaces of the state space on which the dichotomy estimates are uniform. Two main results are a dichotomy spectral theorem based on our notion of dichotomy which is uniform on subspaces and a formula for the dichotomy spectral intervals which is new for the Bohl dichotomy spectrum as well as for the classical exponential dichotomy spectrum.

1 Introduction

In this paper we develop formulas for the dichotomy spectrum of linear time-varying difference equations on $\mathbb{R}^d$, $d \in \mathbb{N}_{>0}$

$$x(n+1) = A(n)x(n), \qquad n \in \mathbb{T}, \tag{1}$$

with one-sided time $\mathbb{T} = \mathbb{N}$ or two-sided time $\mathbb{T} = \mathbb{Z}$ and invertible coefficients $A\colon \mathbb{T} \to \mathrm{GL}(\mathbb{R}^d)$ such that A and $A^{-1}\colon \mathbb{T} \to \mathrm{GL}(\mathbb{R}^d)$, $n \mapsto A(n)^{-1}$ are bounded.

There is a rich body of literature on hyperbolicity concepts for systems (1), most prominently exponential dichotomy, but also more recently Bohl dichotomy, see Definition 1 and [19] (for the case $\mathbb{T} = \mathbb{N}$) and Remark 2 for a short history and references. In this introduction we observe that a main difference between these known dichotomy notions is the set of subspaces on which the dichotomy estimates for the solutions of (1) are uniform. This will lead us in Definition 4 to a new notion of dichotomy which is uniform on subspaces and which generalizes the concepts of exponential and Bohl dichotomy.

Section 2 contains results on properties of dichotomous systems (1) as a preparation for the main results in Section 4. In Section 3 we recall in Definition 13 Bohl exponents from [19] and investigate their relation to our general notion of dichotomy. Section 4 contains the two main results, a dichotomy spectral theorem based on our general notion of dichotomy which is uniform on subspaces (Theorem 26) and a formula for the dichotomy spectral intervals (Theorem 27).

For an initial value $x_0 \in \mathbb{R}^d$ the solution $x(\cdot, x_0) : \mathbb{T} \to \mathbb{R}^d$ of (1) starting at 0 in x_0 satisfies $x(0, x_0) = x_0$,

$$x(n, x_0) = A(n-1) \cdots A(0) x_0, \qquad n > 0,$$

and if $\mathbb{T} = \mathbb{Z}$

$$x(n, x_0) = A(n)^{-1} \cdots A(-1)^{-1} x_0, \qquad n < 0.$$

Dichotomies are solution estimates on subspaces of the state space $\mathbb{R}^d$ which form a splitting. Denote for $L \subseteq \mathbb{R}^d$ the set of subspaces of L by

$$\mathcal{G}(L) := \{U \subseteq L : U \text{ subspace}\}.$$

For $L_1, L_2 \in \mathcal{G}(\mathbb{R}^d)$ we say that (L_1, L_2) is a splitting of $\mathbb{R}^d$ if $L_1 \oplus L_2 = \mathbb{R}^d$.

Let (L_1, L_2) be a splitting of $\mathbb{R}^d$. We denote by π_{L_1} the projection onto L_1 along L_2. Then $\pi_{L_2} := I - \pi_{L_1}$ is the projection onto L_2 along L_1, where I denotes the identity mapping on $\mathbb{R}^d$. For $V \in \mathcal{G}(\mathbb{R}^d)$ and $i \in \{1, 2\}$

$$\pi_{L_i}[V] := \{\pi_{L_i} v : v \in V\}$$

is the projection of V onto L_i. For the following definition see e.g. [19] and the references therein.

Definition 1 (Exponential and Bohl dichotomy)**.** *Let (L_1, L_2) be a splitting of $\mathbb{R}^d$. System* (1) *has a* Bohl dichotomy *on (L_1, L_2) if there exists an $\alpha > 0$ such that*

$$\begin{aligned} \|x(n, x_0)\| &\le C_1 e^{-\alpha(n-m)} \|x(m, x_0)\|, \qquad && m, n \in \mathbb{T}, n \ge m, x_0 \in L_1, \\ \|x(n, x_0)\| &\ge C_2 e^{\alpha(n-m)} \|x(m, x_0)\|, \qquad && m, n \in \mathbb{T}, n \ge m, x_0 \in L_2, \end{aligned}$$

with constants $C_1, C_2 > 0$ which depend on x_0 in L_1 and L_2, respectively.

In case C_1 and C_2 can be chosen independent of x_0, system (1) *has an* exponential dichotomy.

Remark 2 (History of dichotomy notion)**.**

(a) Foundations. Massera and Schaeffer [29, 30] coined the name exponential dichotomy *and formulated the canonical definition of exponentially dichotomous linear differential equations. The foundational formulation of the theory*

of exponentially dichotomous systems was completed through the monographs of Massera and Schaeffer [28] and Daleckii and Krein [21], which summarized earlier results on exponentially dichotomous systems and outlined paths for the development of this theory by presenting numerous open questions.

For difference equations, this concept first appeared in the work of Coffman and Schaeffer [17], where the notion of uniform exponential dichotomy for infinite-dimensional, one-sided discrete-time systems was introduced, highlighting differences between continuous and discrete-time systems due to the potential non-invertibility of the transition operator. The first definitions of dichotomy for two-sided sequences were provided in [22] and [25, Definition 7.6.4]. Subsequently, many authors have studied the concept of uniform exponential dichotomies for discrete systems with both invertible and non-invertible coefficients [13, 14, 14, 1, 24].

(b) Concepts of uniformity. The term uniform *usually refers to the fact that the dichotomy estimates are uniform with respect to both time and the initial condition. While the role of uniformity with respect to time has been extensively analyzed in the literature (see e.g. [9, 10, 11, 40] and references therein), research of the significance of uniformity with respect to the initial condition has only recently been initiated (see e.g. [12, 23, 8, 19, 20]). These studies examine a type of dichotomy, referred to as* Bohl dichotomy *in [19], where the convergence with respect to time is uniform, but uniformity with respect to the initial condition is restricted to each one-dimensional subspace.*

In the following remark we emphasize the subspaces on which the dichotomy estimates in Definition 1 hold uniformly. Denote for $k \in \mathbb{N}$ and $L \subseteq \mathbb{R}^d$ the set of k-dimensional subspaces of L by

$$\mathcal{G}_k(L) := \{U \in \mathcal{G}(L) : \dim U = k\}.$$

Remark 3 (Uniformity subspaces of Bohl and exponential dichotomy)**.** *If* (1) *has a Bohl dichotomy on* (L_1, L_2) *then, using the homogeneity of the solution* $x(\cdot, cx_0) = cx(\cdot, x_0)$, $c \in \mathbb{R}$, *the dichotomy estimates hold uniformly on all one-dimensional subspaces of* L_1 *and* L_2*, respectively. More precisely, for all* $U \in \mathcal{G}_1(L_1)$*, there exists* $C(U) > 0$ *with*

$$\|x(n, x_0)\| \leq C(U)\mathrm{e}^{-\alpha(n-m)}\|x(m, x_0)\|, \qquad m, n \in \mathbb{T},\ n \geq m,\ x_0 \in U,$$

and for all $U \in \mathcal{G}_1(L_2)$*, there exists* $C(U) > 0$ *with*

$$\|x(n, x_0)\| \geq C(U)\mathrm{e}^{\alpha(n-m)}\|x(m, x_0)\|, \qquad m, n \in \mathbb{T},\ n \geq m,\ x_0 \in U.$$

If (1) *has an exponential dichotomy on* (L_1, L_2) *then the dichotomy estimates hold uniformly on all subspaces of* L_1 *and* L_2*, respectively.*

We introduce a new dichotomy notion based on dichotomy estimates which hold uniformly on a set of subspaces which covers the splitting.

Definition 4 (Dichotomy uniform on subspaces of a splitting)**.** *Let* (L_1, L_2) *be a splitting of* $\mathbb{R}^d$ *and* $\mathcal{U}_1 \subseteq \mathcal{G}(L_1)$, $\mathcal{U}_2 \subseteq \mathcal{G}(L_2)$ *be covers of* L_1 *and* L_2*, respectively, i.e.*

$$\bigcup \mathcal{U}_1 = L_1 \qquad \text{and} \qquad \bigcup \mathcal{U}_2 = L_2. \tag{2}$$

We say that system (1) *has a* dichotomy on (L_1, L_2) uniformly on the subspaces in $(\mathcal{U}_1, \mathcal{U}_2)$ *if there exists an* $\alpha > 0$ *such that for each* $U_1 \in \mathcal{U}_1$*, there exists* $C(U_1) > 0$ *with*

$$\|x(n, x_0)\| \le C(U_1)\mathrm{e}^{-\alpha(n-m)}\|x(m, x_0)\|, \qquad m, n \in \mathbb{T},\, n \ge m,\, x_0 \in U_1, \tag{3}$$

and for each $U_2 \in \mathcal{U}_2$*, there exists* $C(U_2) > 0$ *with*

$$\|x(n, x_0)\| \ge C(U_2)\mathrm{e}^{\alpha(n-m)}\|x(m, x_0)\|, \qquad m, n \in \mathbb{T},\, n \ge m,\, x_0 \in U_2. \tag{4}$$

We say that system (1) *has a* dichotomy on (L_1, L_2) *if there exist covers* $\mathcal{U}_1 \subseteq \mathcal{G}(L_1)$ *of* L_1 *and* $\mathcal{U}_2 \subseteq \mathcal{G}(L_2)$ *of* L_2 *such that system* (1) *has a dichotomy on* (L_1, L_2) *uniformly on the subspaces in* $(\mathcal{U}_1, \mathcal{U}_2)$*.*

The important and obvious observation that a dichotomy estimate which holds uniformly on a subspace U also holds uniformly on all subspaces of U is formulated in the following theorem.

Theorem 5 (Refining uniformity subspaces of dichotomy)**.** *If system* (1) *has a dichotomy on* (L_1, L_2) *uniformly on the subspaces in* $(\mathcal{U}_1, \mathcal{U}_2)$*, then it also has a dichotomy on* (L_1, L_2) *uniformly on the subspaces in the refinement* $(\mathcal{V}_1, \mathcal{V}_2)$ *with*

$$\mathcal{V}_1 := \bigcup_{U \in \mathcal{U}_1} \mathcal{G}(U) \qquad \text{and} \qquad \mathcal{V}_2 := \bigcup_{U \in \mathcal{U}_2} \mathcal{G}(U).$$

Proof. Let $i \in \{1, 2\}$. The set $\mathcal{V}_i$ consists of all subspaces of the subspaces in $\mathcal{U}_i$ and therefore it also covers L_i. Let $V_i \in \mathcal{V}_i$. Then there exists $U_i \in \mathcal{U}_i$ such that V_i is a subspace of U_i and the dichotomy estimate which holds on U_i also holds on V_i. □

In Remark 3 it was pointed out that for Bohl and exponential dichotomies the estimates are uniform on all one-dimensional subspaces of the corresponding spaces of the splitting. This property also holds for the new dichotomy notion defined in Definition 4.

Remark 6 (Dichotomy is uniform on one-dimensional subspaces)**.** *If* $L \neq \{0\}$ *is a subspace of* $\mathbb{R}^d$ *and* $\mathcal{U} \subseteq \mathcal{G}(L)$ *is a cover of* L *then*

$$\mathcal{G}_1(L) \subseteq \bigcup_{U \in \mathcal{U}} \mathcal{G}(U).$$

As a consequence, if system (1) *has a dichotomy on* (L_1, L_2) *then it also has a Bohl dichotomy on* (L_1, L_2)*.*

The notions of exponential and Bohl dichotomy spectrum (see e.g. [19]) are based on dichotomies of the γ-*shifted* systems for $\gamma \in \mathbb{R}$

$$x(n+1) = e^{-\gamma} A(n) x(n), \qquad n \in \mathbb{T}. \tag{5}$$

Theorem 7 (Exponential and Bohl dichotomy spectrum)**.** *The Bohl dichotomy spectrum of* (1)

$$\Sigma_{\mathrm{BD}}(A) := \{\gamma \in \mathbb{R} : (5) \text{ has no Bohl dichotomy}\},$$

as well as the exponential dichotomy spectrum of (1)

$$\Sigma_{\mathrm{ED}}(A) := \{\gamma \in \mathbb{R} : (5) \text{ has no exponential dichotomy}\},$$

is the nonempty union of at most d compact intervals.

In this paper we generalize Theorem 7 to dichotomies which hold uniformly on subspaces of prescribed dimensions and provide formulas for the endpoints of the spectral intervals (Theorems 26 and 27) which are also new for the exponential and Bohl dichotomy spectrum.

2 Dichotomies

In this section we study dichotomies which are uniform on subspaces of a splitting. The estimates (3) and (4) are the building blocks of Definition 4 which we abbreviate for convenience in the following definition.

Definition 8 (Dichotomy estimates)**.** *Let $\gamma \in \mathbb{R}$ and $U \in \mathcal{G}(\mathbb{R}^d)$.*

(a) We say $\mathrm{D}_1(\gamma, U)$ holds if there exists $C\colon U \to \mathbb{R}_{>0}$, such that

$$\|x(n, x_0)\| \le C(x_0) \mathrm{e}^{\gamma(n-m)} \|x(m, x_0)\|, \qquad m, n \in \mathbb{T},\ n \ge m,\ x_0 \in U.$$

If the mapping C is constant, we say that $\mathrm{D}_1(\gamma, U)$ holds uniformly.

(b) We say $\mathrm{D}_2(\gamma, U)$ holds if there exists $C\colon U \to \mathbb{R}_{>0}$, such that

$$\|x(n, x_0)\| \ge C(x_0) \mathrm{e}^{\gamma(n-m)} \|x(m, x_0)\|, \qquad m, n \in \mathbb{T},\ n \ge m,\ x_0 \in U.$$

If the mapping C is constant, we say that $\mathrm{D}_2(\gamma, U)$ holds uniformly.

Using the notation of Definition 8 system (1) has a dichotomy with splitting (L_1, L_2), if and only if there exists $\alpha > 0$ and a covering $\mathcal{U}_1$ of L_1 and a covering $\mathcal{U}_2$ of L_2, such that $\mathrm{D}_1(-\alpha, U)$ holds uniformly for all $U \in \mathcal{U}_1$ and $\mathrm{D}_2(\alpha, U)$ holds uniformly for all $U \in \mathcal{U}_2$.

Remark 9 (Characterization of dichotomy of γ-shifted system)**.** *For $\gamma \in \mathbb{R}$, the solution of the γ-shifted system* (5) *which starts at* 0 *in $x_0 \in \mathbb{R}^d$ is $\big(\mathrm{e}^{-\gamma n} x(n, x_0)\big)_{n \in \mathbb{T}}$. The γ-shifted system has a dichotomy with splitting (L_1, L_2) if and only if there exists $\alpha > 0$ and a covering $\mathcal{U}_1$ of L_1 and a covering $\mathcal{U}_2$ of L_2, such that $\mathrm{D}_1(\gamma - \alpha, U)$ holds uniformly for all $U \in \mathcal{U}_1$ and $\mathrm{D}_2(\gamma + \alpha, U)$ holds uniformly for all $U \in \mathcal{U}_2$.*

Proposition 10 (Dichotomy estimates are dichotomous)**.** *Let $\alpha > 0$, $\gamma \in \mathbb{R}$ and $U_1, U_2 \in \mathcal{G}(\mathbb{R}^d)$. If $\mathrm{D}_1(\gamma - \alpha, U_1)$ and $\mathrm{D}_2(\gamma + \alpha, U_2)$ hold, then $U_1 \cap U_2 = \{0\}$.*

More generally, suppose that $U \in \mathcal{G}(\mathbb{R}^d)$, $\alpha > 0$ and $\gamma \in \mathbb{R}$. If $\mathrm{D}_2(\gamma + \alpha, U)$ holds and if $\big(\mathrm{e}^{-\gamma n} x(n, x_0)\big)_{n \in \mathbb{N}}$ is bounded for all $x_0 \in U$, then $U = \{0\}$.

Proof. Let $x_0 \in U_1 \cap U_2$. From $\mathrm{D}_1(\gamma - \alpha, U_1)$ (with $m = 0$ and noting that $\mathbb{N} \subseteq \mathbb{T}$), we conclude that there exists $C_1 > 0$ with

$$\|x(n, x_0)\| \le C_1 \mathrm{e}^{\gamma - \alpha n} \|x_0\|, \qquad n \in \mathbb{N}.$$

In particular, $\big(\mathrm{e}^{-\gamma n} x(n, x_0)\big)_{n \in \mathbb{N}}$ is bounded. The estimate $\mathrm{D}_2(\alpha, U_2)$ yields a $C_2 > 0$ with

$$\|x_0\| \le C_2 \mathrm{e}^{-(\gamma + \alpha) n} \|x(n, x_0)\|, \qquad n \in \mathbb{N}.$$

Letting $n \to \infty$, using the fact that $\alpha > 0$ and the boundedness of $\big(\mathrm{e}^{-\gamma n} x(n, x_0)\big)_{n \in \mathbb{N}}$, we conclude $x_0 = 0$. □

Proposition 11 (Monotonicity of dichotomy subspaces)**.** *Let $\gamma, \widetilde{\gamma} \in \mathbb{R}$ with $\gamma \le \widetilde{\gamma}$. Suppose that the γ-shifted system* (5) *has a dichotomy with splitting (L_1, L_2) and that the $\widetilde{\gamma}$-shifted system has a dichotomy with splitting $(\widetilde{L}_1, \widetilde{L}_2)$. Then $L_1 \subseteq \widetilde{L}_1$. If $\mathbb{T} = \mathbb{Z}$, then $L_2 \supseteq \widetilde{L}_2$.*

Proof. We prove $L_1 \subseteq \widetilde{L}_1$. Let $x_1 \in L_1$. There are $\widetilde{x}_1 \in \widetilde{L}_1$, $\widetilde{x}_2 \in \widetilde{L}_2$ with $x_1 = \widetilde{x}_1 + \widetilde{x}_2$. We show $\widetilde{x}_2 = 0$ by applying Proposition 10 to the $\widetilde{\gamma}$-shifted system, so that $x_1 = \widetilde{x}_1$, i.e. $L_1 \subseteq \widetilde{L}_1$. Indeed, there are $\alpha_1, \alpha_2 > 0$, such that $D_1(\gamma - \alpha_1, \operatorname{span}\{x_1\})$ and $D_1(\widetilde{\gamma} - \alpha_2, \operatorname{span}\{\widetilde{x}_1\})$ and $\mathrm{D}_2(\widetilde{\gamma} + \alpha_2, \operatorname{span}\{\widetilde{x}_2\})$ hold by Remark 6. In particular, for $\alpha := \min(\alpha_1, \alpha_2)$ there is $C > 0$, such that for $n \in \mathbb{N}$,

$$\begin{aligned}
\|\mathrm{e}^{-\widetilde{\gamma} n} x(n, \widetilde{x}_2)\| &= \mathrm{e}^{-\widetilde{\gamma} n} \|x(n, x_1 - \widetilde{x}_1)\| \\
&\le \mathrm{e}^{-\widetilde{\gamma} n} \big(C \mathrm{e}^{(\gamma - \alpha)(n - 0)} \|x(0, x_1)\| + C \mathrm{e}^{(\widetilde{\gamma} - \alpha)(n - 0)} \|x(0, \widetilde{x}_1)\| \big) \\
&= C \big(\mathrm{e}^{(\gamma - \widetilde{\gamma} - \alpha) n} \|x_1\| + \mathrm{e}^{-\alpha n} \|\widetilde{x}_1\| \big).
\end{aligned}$$

Since $\gamma - \widetilde{\gamma} \le 0$, we see that $\big(\mathrm{e}^{-\widetilde{\gamma} n} x(n, \widetilde{x}_2)\big)_{n \in \mathbb{N}}$ is bounded, so that Proposition 10 yields $\operatorname{span}\{\widetilde{x}_2\} = \{0\}$, i.e. $\widetilde{x}_2 = 0$. That $L_2 \supseteq \widetilde{L}_2$ if $\mathbb{T} = \mathbb{Z}$, can be shown similarly. □

Proposition 12 (Uniqueness and dynamic characterization of dichotomy subspaces)**.** *Let $\gamma \in \mathbb{R}$. Suppose that the γ-shifted system* (5) *has a dichotomy with splitting* (L_1, L_2)*. Then*

$$L_1 = \{x_0 \in \mathbb{R}^d : \lim_{n\to\infty} \mathrm{e}^{-\gamma n} x(n, x_0) = 0\},$$

and if $\mathbb{T} = \mathbb{Z}$, then

$$L_2 = \big\{x_0 \in \mathbb{R}^d : \lim_{n\to-\infty} \mathrm{e}^{-\gamma n} x(n, x_0) = 0\big\}.$$

Moreover, if for $\widetilde{\gamma} \in \mathbb{R}$, the $\widetilde{\gamma}$-shifted system has a dichotomy with splitting $(\widetilde{L}_1, \widetilde{L}_2)$ and $\dim L_1 = \dim \widetilde{L}_1$, then $L_1 = \widetilde{L}_1$ and if $\mathbb{T} = \mathbb{Z}$, then $L_2 = \widetilde{L}_2$.

Proof. We prove the representation of L_1 only and set $M := \{x_0 \in \mathbb{R}^d : \lim_{n\to\infty} \mathrm{e}^{-\gamma n} x(n, x_0) = 0\}$. $L_1 \subseteq M$, since there is $\alpha > 0$, such that $D_1(\gamma - \alpha, \operatorname{span}\{x_0\})$ holds for all $x_0 \in L_1$. To show $M \subseteq L_1$, let $x_0 \in M$ and let $x_1 \in L_1$, $x_2 \in L_2$ with $x_0 = x_1 + x_2$. Similar to the proof of $\widetilde{x}_2 = 0$ in the proof of Proposition 11, it follows that $x_2 = 0$ and thus $x_0 = x_1 \in L_1$. Now suppose $\widetilde{\gamma} \in \mathbb{R}$ and assume w.l.o.g. $\widetilde{\gamma} \le \gamma$. From Proposition 11, we obtain that $\widetilde{L}_1 \subseteq L_1$ and since $\dim \widetilde{L}_1 = \dim L_1$, also $\widetilde{L}_1 = L_1$. Similarly one can show that $\widetilde{L}_2 = L_2$ if $\mathbb{T} = \mathbb{Z}$. ☐

3 Bohl exponents and dichotomy estimates

A useful tool for studying properties of dichotomies is the concept of Bohl exponents which were introduced for individual solutions in [16] and later extended to subspace exponents in [15] and [19]. In this section we recall the upper and lower Bohl exponents from [19] and investigate their relation to the dichotomy estimates in Definition 8. We set $\sup \emptyset := -\infty$ and $\inf \emptyset := \infty$.

Definition 13 (Upper and lower Bohl exponent)**.** *For $U \in \mathcal{G}(\mathbb{R}^d)$ we define the* upper Bohl exponent

$$\overline{\beta}(U) := \inf_{N\in\mathbb{N}} \sup \left\{ \frac{1}{n-m} \ln \frac{\|x(n, x_0)\|}{\|x(m, x_0)\|} : m, n \in \mathbb{T},\, n - m > N,\, x_0 \in U \setminus \{0\} \right\}$$

and the lower Bohl exponent

$$\underline{\beta}(U) := \sup_{N\in\mathbb{N}} \inf \left\{ \frac{1}{n-m} \ln \frac{\|x(n, x_0)\|}{\|x(m, x_0)\|} : m, n \in \mathbb{T},\, n - m > N,\, x_0 \in U \setminus \{0\} \right\}.$$

Remark 14 (Basic properties of Bohl exponents)**.**

(a) $\overline{\beta}(\{0\}) = -\infty$ *and* $\underline{\beta}(\{0\}) = \infty$.

(b) If $U \neq \{0\}$ then $-\ln \|A^{-1}\|_\infty \le \underline{\beta}(U) \le \overline{\beta}(U) \le \ln \|A\|_\infty$. In particular,

$$\underline{\beta}(U), \overline{\beta}(U) \in \mathbb{R}, \qquad U \in \mathcal{G}(\mathbb{R}^d) \setminus \{0\}.$$

(c) For $U, V \in \mathcal{G}(\mathbb{R}^d)$

$$U \subseteq V \quad \Rightarrow \quad \underline{\beta}(U) \geq \underline{\beta}(V) \text{ and } \overline{\beta}(U) \leq \overline{\beta}(V).$$

The next two propositions state the relation between the upper and lower Bohl exponents and the dichotomy estimates $D_1(\gamma, U)$ and $D_2(\gamma, U)$, respectively.

Proposition 15 (Upper Bohl exponent and dichotomy estimate $\mathrm{D}_1(\gamma, U)$)**.** *Let $\mathcal{V} \subseteq \mathcal{G}(\mathbb{R}^d)$ be non-empty and for each $V \in \mathcal{V}$ let $\mathcal{U}(V) \subseteq \mathcal{G}(\mathbb{R}^d)$ be non-empty. Let $\gamma \in \mathbb{R}$.*

- *(a) If there is $V \in \mathcal{V}$, such that for all $U \in \mathcal{U}(V)$ the estimate $\mathrm{D}_1(\gamma, U)$ holds uniformly, then $\gamma \geq \inf_{V \in \mathcal{V}} \sup_{U \in \mathcal{U}(V)} \overline{\beta}(U)$.*
- *(b) If $\gamma > \inf_{V \in \mathcal{V}} \sup_{U \in \mathcal{U}(V)} \overline{\beta}(U)$, then there is $V \in \mathcal{V}$, such that for all $U \in \mathcal{U}(V)$ the estimate $\mathrm{D}_1(\gamma, U)$ holds uniformly.*

Proof. We prove (a). Let $V \in \mathcal{V}$, such that $\mathrm{D}_1(\gamma, U)$ holds for all $U \in \mathcal{U}(V)$. First consider $U \in \mathcal{U}(V) \setminus \{0\}$. We obtain by $\mathrm{D}_1(\gamma, U)$ a constant $C > 0$, such that

$$\frac{1}{n-m} \ln \frac{\|x(n, x_0)\|}{\|x(m, x_0)\|} \leq \frac{1}{n-m} C + \gamma, \qquad m, n \in \mathbb{T},\ n > m, x_0 \in U \setminus \{0\}.$$

Hence for all $N \in \mathbb{N}$, we obtain

$$\frac{1}{n-m} \ln \frac{\|x(n, x_0)\|}{\|x(m, x_0)\|} \leq \frac{C}{N} + \gamma, \qquad m, n \in \mathbb{T},\ n - m > N, x_0 \in U \setminus \{0\}.$$

Thus $\overline{\beta}(U) \leq \gamma$. Also $\overline{\beta}(\{0\}) = -\infty < \gamma$ and we obtain $\sup_{U \in \mathcal{U}(V)} \overline{\beta}(U) \leq \gamma$.

We prove (b). From $\gamma > \inf_{V \in \mathcal{V}} \sup_{U \in \mathcal{U}(V)} \overline{\beta}(U)$ we conclude that there exists $V \in \mathcal{V}$, such that for all $U \in \mathcal{U}(V)$ we have

$$\gamma > \overline{\beta}(U) = \inf_{N \in \mathbb{N}} \sup \left\{ \frac{1}{n-m} \ln \frac{\|x(n, x_0)\|}{\|x(m, x_0)\|} : m, n \in \mathbb{T},\ n - m > N,\ x_0 \in U \setminus \{0\} \right\}.$$

Note that $\mathrm{D}_1(\gamma, \{0\})$ always holds uniformly and for $U \in \mathcal{U}(V) \setminus \{0\}$, there is $N \in \mathbb{N}$ with

$$\gamma > \frac{1}{n-m} \ln \frac{\|x(n, x_0)\|}{\|x(m, x_0)\|}, \qquad m, n \in \mathbb{T},\ n - m > N,\ x_0 \in U \setminus \{0\}.$$

Rearranging yields $\|x(n, x_0)\| \leq \mathrm{e}^{\gamma(n-m)} \|x(m, x_0)\|$ for $m, n \in \mathbb{T}$, $n - m > N$, $x_0 \in U$. To conclude, we have to show that there is $C > 0$, such that

$$\|x(n, x_0)\| \leq C \mathrm{e}^{\gamma(n-m)} \|x(m, x_0)\|, \qquad m, n \in \mathbb{T},\ n - m \leq N,\ x_0 \in U.$$

Indeed, for $x_0 \in U \setminus \{0\}$ this follows from

$$\mathrm{e}^{-\gamma(n-m)} \frac{\|x(n, x_0)\|}{\|x(m, x_0)\|} = \mathrm{e}^{-\gamma(n-m)} \frac{\|A(n-1) \cdots A(m) A(m-1) \cdots A(0) x_0\|}{\|A(m-1) \cdots A(0) x_0\|}$$

$$\leq \mathrm{e}^{-\gamma(n-m)} \|A(n-1)\cdots A(m)\|$$

and by noting that $\mathrm{e}^{-\gamma(n-m)}\|A(n-1)\cdots A(m)\|$ is bounded on $\{(m,n) \in \mathbb{T}^2 : 0 \leq n-m \leq N\}$, since A is bounded. □

Proposition 16 (Lower Bohl exponent and $\mathrm{D}_2(\gamma, U)$)**.** *Let $\mathcal{V} \subseteq \mathcal{G}(\mathbb{R}^d)$ be non-empty and for every $V \in \mathcal{V}$ let $\mathcal{U}(V) \subseteq \mathcal{G}(\mathbb{R}^d)$ be non-empty. Let $\gamma \in \mathbb{R}$.*

(a) If there is $V \in \mathcal{V}$, such that for all $U \in \mathcal{U}(V)$ the estimate $\mathrm{D}_2(\gamma, U)$ holds uniformly, then $\gamma \leq \sup_{V \in \mathcal{V}} \inf_{U \in \mathcal{U}(V)} \underline{\beta}(U)$.

(b) If $\gamma < \sup_{V \in \mathcal{V}} \inf_{U \in \mathcal{U}(V)} \underline{\beta}(U)$, then there is $V \in \mathcal{V}$, such that for all $U \in \mathcal{U}(V)$ the estimate $\mathrm{D}_2(\gamma, U)$ holds uniformly.

Proof. The proof is along the lines of Proposition 15. □

4 The spectral theorem

In this section we generalize Theorem 7 and provide formulas for the spectrum. For each $\gamma \in \mathbb{R}$ which is contained in the Bohl dichotomy resolvent set of (1)

$$\rho_{\mathrm{BD}}(A) := \mathbb{R} \setminus \Sigma_{\mathrm{BD}}(A) = \{\gamma \in \mathbb{R} : (5) \text{ has a Bohl dichotomy}\}$$

or the exponential dichotomy resolvent set

$$\rho_{\mathrm{ED}}(A) := \mathbb{R} \setminus \Sigma_{\mathrm{ED}}(A) = \{\gamma \in \mathbb{R} : (5) \text{ has an exponential dichotomy}\}$$

the γ-shifted system (5) admits a dichotomy on a splitting (L_1, L_2). The subspace L_1 and, in particular, its dimension $k := \dim L_1$ is unique for a fixed γ in the resolvent set (Proposition 12).

Covers $\mathcal{U}_1 \subseteq \mathcal{G}(L_1)$, $\mathcal{U}_2 \subseteq \mathcal{G}(L_2)$ of L_1 and L_2, respectively, on which the dichotomy estimates hold uniformly (cf. Definition 4) are

$$\mathcal{U}_1 = \mathcal{G}_1(L_1), \quad \mathcal{U}_2 = \mathcal{G}_1(L_2) \qquad \text{if } \gamma \in \rho_{\mathrm{BD}}(A)$$

and

$$\mathcal{U}_1 = \mathcal{G}_k(L_1) = \{L_1\}, \quad \mathcal{U}_2 = \mathcal{G}_{d-k}(L_2) = \{L_2\} \qquad \text{if } \gamma \in \rho_{\mathrm{ED}}(A).$$

In both cases

$$\mathcal{U}_1 = \mathcal{G}_{j_1}(L_1), \quad \mathcal{U}_2 = \mathcal{G}_{j_2}(L_2) \tag{6}$$

with $(j_1, j_2) \in \{(1,1), (k, d-k)\}$. We extend this idea and answer in the following proposition for $k \in \{0, \dots, d\}$ the question which $(j_1, j_2) \in \mathbb{N} \times \mathbb{N}$ have the property that for each splitting (L_1, L_2) with $\dim L_1 = k$ the spaces $\mathcal{U}_1$ and $\mathcal{U}_2$ in (6) are covers of L_1 and L_2, respectively (cf. Definition 4).

Proposition 17 (Admissible uniformity dimensions for dichotomy)**.** *For* $k \in \{0, \dots, d\}$ *and* $j_1, j_2 \in \{0, \dots, d\}$*, the following two statements are equivalent:*

(i) For each splitting (L_1, L_2) *with* $\dim L_1 = k$*, the spaces* L_1 *and* L_2 *are covered by their* j_1*- and* j_2*-dimensional subspaces, respectively, i.e.*

$$\bigcup_{U \in \mathcal{G}_{j_1}(L_1)} U = L_1 \qquad \textit{and} \qquad \bigcup_{U \in \mathcal{G}_{j_2}(L_2)} U = L_2.$$

(ii) If $k = 0$ *then* $j_1 = 0$*,* $j_2 \in \{1, \dots, d\}$*. If* $k \in \{1, \dots, d-1\}$ *then* $j_1 \in \{1, \dots, k\}$*,* $j_2 \in \{1, \dots, d-k\}$*. If* $k = d$ *then* $j_1 \in \{1, \dots, d\}$*,* $j_2 = 0$*.*

If (i) and (ii) hold we say that (j_1, j_2) *is* k-admissible.

Proof. This follows from $\mathcal{G}_j(L) = \emptyset$ for $L \in \mathcal{G}(\mathbb{R}^d)$ if $\dim L < j$. □

The following remark is a consequence of the refinement Theorem 5 applied to dichotomies which are uniform on all subspaces of prescribed dimensions (j_1, j_2).

Remark 18 (Dichotomy with uniformity dimensions)**.** *Let* $(j_1, j_2) \in \mathbb{N} \times \mathbb{N}$*. If system* (1) *has a dichotomy on* (L_1, L_2) *uniformly on all subspaces of dimensions* (j_1, j_2)*, i.e. with*

$$\mathcal{U}_1 = G_{j_1}(L_1) \qquad \textit{and} \qquad \mathcal{U}_2 = G_{j_2}(L_2),$$

then we say that system (1) *has a* dichotomy with uniformity dimensions (j_1, j_2) *and* (j_1, j_2) *is* k*-admissible with* $k = \dim L_1$*.*

In this case system (1) *has also a dichotomy with uniformity dimensions* (ℓ_1, ℓ_2) *for each* k*-admissible* (ℓ_1, ℓ_2) *which satisfies* $\ell_1 \leq j_1$ *and* $\ell_2 \leq j_2$*.*

We are now ready to define a notion of spectrum based on dichotomies with k-admissible uniformity dimensions (j_{1k}, j_{2k}) for the γ-shifted system (5) where k is the dimension of the dynamically characterized set (Proposition 12)

$$\{x_0 \in \mathbb{R}^d : \lim_{n \to \infty} \mathrm{e}^{-\gamma n} x(n, x_0) = 0\}.$$

The exponential and Bohl dichotomy spectrum are special cases for specific choices of uniformity dimensions (Remark 20).

Definition 19 (Dichotomy resolvent and spectrum)**.** *For each* $k \in \{0, \dots, d\}$ *let* (j_{1k}, j_{2k}) *be* k*-admissible. The* dichotomy resolvent of (1) with uniformity dimensions $J \coloneqq \big((j_{10}, j_{20}), \dots, (j_{1d}, j_{2d})\big) \in (\mathbb{N} \times \mathbb{N})^{d+1}$ *is defined as*

$$\begin{aligned} \rho_J(A) \coloneqq \{\gamma \in \mathbb{R} \,|\, & \textit{the } \gamma\textit{-shifted system has a dichotomy on a splitting } (L_1, L_2), \\ & \textit{uniform on subspaces with dimension } (j_{1k}, j_{2k}), \textit{ where } k = \dim L_1\} \end{aligned}$$

and the dichotomy spectrum of system (1) with admissible uniformity dimensions J is

$$\Sigma_J(A) \coloneqq \mathbb{R} \setminus \rho_J(A).$$

Remark 20 (Special cases of Bohl and exponential dichotomy spectrum)**.**

(a) $\Sigma_{\mathrm{BD}}(A) = \Sigma_J(A)$ *for* $J = \big((0,1),(1,1),\dots,(1,1),(1,0)\big)$.

(b) $\Sigma_{\mathrm{ED}}(A) = \Sigma_J(A)$ *for* $J = \big((0,d),(1,d-1),\dots,(d,0)\big)$.

Remark 21 (Resolvent set characterized by dichotomy estimates)**.** *For admissible uniformity dimensions J the following two statements are equivalent:*

(i) $\gamma \in \rho_J(A)$,

(ii) there exists a splitting (L_1, L_2) *of* $\mathbb{R}^d$ *and* $\alpha > 0$*, such that with* $k := \dim L_1$ *the dichotomy estimate* $D_1(\gamma - \alpha, U)$ *holds uniformly for all* $U \in \mathcal{G}_{j_{1k}}(L_1)$ *and* $D_2(\gamma + \alpha, U)$ *holds uniformly for all* $U \in \mathcal{G}_{j_{2k}}(L_2)$*.*

We now define exponents, called *limiting Bohl exponents*, which turn out to be the boundary points of the dichotomy spectrum. Such descriptions were previously known for the uniform exponential dichotomy only in the one-dimensional case [35] and in the multidimensional case only for the largest and smallest elements of the spectrum [19, Remark 23].

Definition 22 (Limiting Bohl exponents)**.** *For* $j, k \in \mathbb{N}$ *define*

$$\overline{\beta}_{k,j} := \inf_{L \in \mathcal{G}_k(\mathbb{R}^d)} \sup_{U \in \mathcal{G}_j(L)} \overline{\beta}(U),$$

$$\underline{\beta}_{k,j} := \sup_{L \in \mathcal{G}_{d-k}(\mathbb{R}^d)} \inf_{U \in \mathcal{G}_j(L)} \underline{\beta}(U).$$

Remark 23. *Let* $k \in \{0,\dots,d\}$ *and* (j_1, j_2) *be* k*-admissible. From Remark 14 it follows that for* $k = 0$*, resp.* $k = d$*,*

$$\overline{\beta}_{k,j_1} = \overline{\beta}_{0,0} = -\infty \qquad \textit{resp.} \qquad \underline{\beta}_{k,j_2} = \underline{\beta}_{d,0} = \infty.$$

From Remark 14 it also follows that

$$\overline{\beta}_{k,j_1} \in \big[-\ln\|A^{-1}\|_\infty, \ln\|A\|_\infty\big], \qquad k \in \{1,\dots,d\},$$
$$\underline{\beta}_{k,j_2} \in \big[-\ln\|A^{-1}\|_\infty, \ln\|A\|_\infty\big], \qquad k \in \{0,\dots,d-1\}.$$

Lemma 24 (Characterization of dichotomy)**.** *Let* $k \in \{0,\dots,d\}$ *and* (j_1, j_2) *be* k*-admissible. Then the following two statements are equivalent:*

(i) $\gamma \in \big(\overline{\beta}_{k,j_1}, \underline{\beta}_{k,j_2}\big)$,

(ii) there exists a splitting (L_1, L_2) *of* $\mathbb{R}^d$ *with* $\dim L_1 = k$*, such that the* γ*-shifted system* (5) *has a dichotomy on* (L_1, L_2) *with uniformity dimensions* (j_1, j_2)*.*

Proof. (i) $\Rightarrow$ (ii). Let $\gamma \in \big(\overline{\beta}_{k,j_1}, \underline{\beta}_{k,j_2}\big)$. There exists $\alpha > 0$ with $\gamma - \alpha > \overline{\beta}_{k,j_1}$ and $\gamma + \alpha < \underline{\beta}_{k,j_2}$. Applying Proposition 15(b) (where $\mathcal{V}$ is set to $\mathcal{G}_k(\mathbb{R}^d)$ and $\mathcal{U}(V)$ is set to $\mathcal{G}_{j_1}(V)$ for $V \in \mathcal{V}$), we obtain the existence of a space $L_1 \in \mathcal{G}_k(\mathbb{R}^d)$, such that $D_1(\gamma - \alpha, U)$ holds uniformly for all $U \in \mathcal{G}_{j_1}(L_1)$. Applying

Proposition 16(b) (where $\mathcal{V}$ is set to $\mathcal{G}_{d-k}(\mathbb{R}^d)$ and $\mathcal{U}(V)$ is set to $\mathcal{G}_{j_2}(V)$ for $V \in \mathcal{V}$), we obtain the existence $L_2 \in \mathcal{G}_{d-k}(\mathbb{R}^d)$, such that $D_2(\gamma+\alpha, U)$ holds uniformly for all $U \in \mathcal{G}_{j_2}(L_2)$. From Proposition 10, we conclude $L_1 \cap L_2 = \{0\}$, so that (L_1, L_2) is a splitting of $\mathbb{R}^d$ with $\dim L_1 = k$. With Remark 21 (ii) follows.

(ii) $\Rightarrow$ (i). Suppose that (L_1, L_2) is a splitting of $\mathbb{R}^d$ with $\dim L_1 = k$, such that the γ-shifted system (5) has a dichotomy on (L_1, L_2) with uniformity dimensions (j_1, j_2). By Remark 21, there is $\alpha > 0$, such that $D_1(\gamma-\alpha, U)$ holds uniformly for all $U \in \mathcal{G}_{j_1}(L_1)$ and $D_2(\gamma+\alpha, U)$ holds uniformly for all $U \in \mathcal{G}_{j_2}(L_2)$ for some $\alpha > 0$. Propositions 15(a) and 16(a) prove that $\gamma \in \big(\overline{\beta}_{k,j_{1k}}, \underline{\beta}_{k,j_{2k}}\big)$. □

Lemma 25 (Dichotomy resolvent)**.** *Let $J := \big((j_{10}, j_{20}), \dots, (j_{1d}, j_{2d})\big) \in (\mathbb{N} \times \mathbb{N})^{d+1}$ be admissible uniformity dimensions (in the sense of Definition 19). Then*

$$\begin{aligned}\rho_J(A) = \big(-\infty, \underline{\beta}_{0,j_{20}}\big) \cup \\ \big(\overline{\beta}_{1,j_{11}}, \underline{\beta}_{1,j_{21}}\big) \cup \dots \cup \big(\overline{\beta}_{(d-1),j_{1(d-1)}}, \underline{\beta}_{(d-1),j_{2(d-1)}}\big) \cup \\ \big(\overline{\beta}_{d,j_{1d}}, \infty\big)\end{aligned}$$

and the union is ordered with

$$-\ln \|A^{-1}\|_\infty \le \underline{\beta}_{k,j_{2k}} \le \overline{\beta}_{k+1,j_{1(k+1)}} \le \ln \|A\|_\infty, \qquad k \in \{0, \dots, d-1\}.$$

Proof. From Lemma 24 we obtain

$$\rho_J(A) = \bigcup_{k=0}^{d} \big(\overline{\beta}_{k,j_{1k}}, \underline{\beta}_{k,j_{2k}}\big).$$

The equality follows from Remark 23. We show for $k \in \{0, \dots, d-1\}$ that $\underline{\beta}_{k,j_{2k}} \le \overline{\beta}_{k+1,j_{1(k+1)}}$, i.e.

$$\sup_{L \in \mathcal{G}_{d-k}(\mathbb{R}^d)} \inf_{U \in \mathcal{G}_{j_{2k}}(L)} \underline{\beta}(U) \le \inf_{L \in \mathcal{G}_{k+1}(\mathbb{R}^d)} \sup_{U \in \mathcal{G}_{j_{1(k+1)}}(L)} \overline{\beta}(U).$$

To this end, let $L \in \mathcal{G}_{d-k}(\mathbb{R}^d)$, $\widetilde{L} \in \mathcal{G}_{k+1}(\mathbb{R}^d)$. Since $\dim L = d-k$, $\dim \widetilde{L} = k+1$, there is $x_0 \in L \cap \widetilde{L}$ with $x_0 \neq 0$. Since $j_{2k}, j_{1(k+1)} > 0$ by admissibility, there is $V \in \mathcal{G}_{j_{2k}}(L)$, $\widetilde{V} \in \mathcal{G}_{j_{1(k+1)}}(\widetilde{L})$ with $x_0 \in V \cap \widetilde{V}$. Thus by Remark 14,

$$\inf_{U \in \mathcal{G}_{j_{2k}}(L)} \underline{\beta}(U) \le \underline{\beta}(V) \le \underline{\beta}(\operatorname{span}\{x_0\}) \le \overline{\beta}(\operatorname{span}\{x_0\}) \le \underline{\beta}(\widetilde{V}) \le \sup_{U \in \mathcal{G}_{j_{1(k+1)}}(\widetilde{L})} \underline{\beta}(U).$$

□

We are now in a position to formulate our first main result on the dichotomy spectrum which generalizes Theorem 7 and provides a spectral flag or spectral filtration of dynamically characterized subspaces which are also related to the formulas for the spectrum given in our second main result Theorem 27 below.

Theorem 26 (Spectral theorem)**.** *Let $J \in (\mathbb{N} \times \mathbb{N})^{d+1}$ be admissible. The dichotomy spectrum $\Sigma_J(A)$ is the nonempty union of at most d compact intervals*

$$\Sigma_J(A) = [a_1, b_1] \cup \cdots \cup [a_\ell, b_\ell],$$

where $a_1 \le b_1 < a_2 \le b_2 < \cdots < a_\ell \le b_\ell$ and $\ell \in \{1, \ldots, d\}$. Setting $b_0 := -\infty$ and $a_{\ell+1} := \infty$, there exists a spectral filtration

$$\{0\} = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_\ell = \mathbb{R}^d,$$

of subspaces $M_k \subseteq \mathbb{R}^d$, $k \in \{0, \ldots, \ell\}$ with

$$M_k = \big\{x_0 \in \mathbb{R}^d \mid \lim_{n \to \infty} \mathrm{e}^{-\gamma n} x(n, x_0) = 0\big\}, \qquad \gamma \in (b_k, a_{k+1}).$$

For two-sided time $\mathbb{T} = \mathbb{Z}$, there exists a spectral decomposition

$$\mathbb{R}^d = W_1 \oplus \cdots \oplus W_\ell$$

into subspaces $W_k \subseteq \mathbb{R}^d$, with

$$W_k = \big\{x_0 \in \mathbb{R}^d \mid \lim_{n \to -\infty} \mathrm{e}^{-\gamma_1 n} x(n, x_0) = \lim_{n \to \infty} \mathrm{e}^{-\gamma_2 n} x(n, x_0) = 0\big\},$$

where $\gamma_1 \in (a_{k-1}, b_k)$, $\gamma_2 \in (b_k, a_{k+1})$ and $k \in \{1, \ldots, \ell\}$.

Proof. We define $k_0 := 0$ and iteratively if k_i is defined, we define $k_{i+1} := \inf\{n \in \{k_i + 1, \ldots, d\} : (\overline{\beta}_{n,j_{1n}}, \underline{\beta}_{n,j_{2n}}) \neq \emptyset\}$, $i \in \mathbb{N}$. We define $\ell \in \{1, \ldots, d\}$ as the maximal number, such that $k_\ell \neq \infty$. Then $k_\ell = d$ and we set

$$a_i := \underline{\beta}_{k_{i-1}, j_{2k_{i-1}}}, \qquad i \in \{1, \ldots, \ell + 1\} \quad \text{and} \quad b_i := \overline{\beta}_{k_i, j_{1k_i}}, \quad i \in \{0, \ldots, \ell\}.$$

By Lemma 25, we obtain $a_1 \le b_1 < a_2 \le b_2 < \cdots < a_\ell \le b_\ell$ and

$$\Sigma_J(A) = [a_1, b_1] \cup \cdots \cup [a_\ell, b_\ell].$$

From Proposition 12 and Lemma 24, we obtain that the space M_k, $k \in \{0, \ldots, \ell\}$, is well-defined. Now suppose $\mathbb{T} = \mathbb{Z}$. Then by Proposition 12 and Lemma 24 the space

$$N_k := \big\{x_0 \in \mathbb{R}^d : \lim_{n \to -\infty} \mathrm{e}^{-\gamma n} x(n, x_0) = 0\big\}, \qquad \gamma \in (b_k, a_{k+1}),$$

is well-defined for $k \in \{0, \ldots, \ell\}$, so that $W_k = M_k \cap N_{k-1}$ for $k \in \{1, \ldots, \ell\}$. We now show the spectral decomposition. To show that the sum is direct, let $\ell, j \in \{1, \ldots, d\}$ with $\ell < j$ and let $x_0 \in (M_\ell \cap N_{\ell-1}) \cap (M_j \cap N_{j-1})$. Since $M_\ell \subseteq M_{j-1}$, we have $x_0 \in M_{j-1} \cap N_{j-1} = \{0\}$ by Proposition 10, proving that $x_0 = 0$. We show inductively that

(*a*) $\qquad M_1 = M_1 \cap N_0.$

$$(b) \qquad \text{If } k > 1 \text{ and } M_{k-1} = \bigoplus_{j=1}^{k-1} M_j \cap N_{j-1} \text{ for } k \in \{1, \dots, d\},$$

$$\text{then } M_k = \bigoplus_{j=1}^{k} M_j \cap N_{j-1}.$$

Part (a) follows, since $N_0 = \mathbb{R}^d$. To show part (b), suppose that $k > 1$ and $M_{k-1} = \bigoplus_{j=1}^{k-1} M_j \cap N_{j-1}$. That $\bigoplus_{j=1}^{k} M_j \cap N_{j-1} \subseteq M_k$, follows from $M_j \subseteq M_k$ for $j \in \{1, \dots, k\}$. Now suppose that $x_0 \in M_k$. Let $x_0 = x_1 + x_2$, with $x_1 \in M_{k-1}$ and $x_2 \in N_{k-1}$. Since $x_0 - x_1 \in M_k$, we have $x_2 = x_0 - x_1 \in M_k \cap N_{k-1}$. By assumption also $x_1 \in M_{k-1} = \bigoplus_{j=1}^{k-1} M_j \cap N_{j-1}$, so that $x_0 = x_1 + x_2 \in \bigoplus_{j=1}^{k} M_j \cap N_{j-1}$, indeed. The theorem now follows, since $M_\ell = \mathbb{R}^d$. □

The proof of Theorem 26 yields explicit formulas for the boundary points of the spectral intervals.

Theorem 27 (Formula for the dichotomy spectrum)**.** *Let $J \in (\mathbb{N} \times \mathbb{N})^{d+1}$ be admissible. Let $k_i := \dim M_i$ denote the dimension of the spectral flag subspaces M_i, $i \in \{0, \dots, \ell\}$, of Theorem 26. The following formula holds for the dichotomy spectrum $\Sigma_J(A) = [a_1, b_1] \cup \cdots \cup [a_\ell, b_\ell]$ of system* (1)*.*

$$a_i = \underline{\beta}_{k_{i-1}, j_{2k_{i-1}}} \qquad \textit{and} \qquad b_i = \overline{\beta}_{k_i, j_{1k_i}}, \qquad i \in \{1, \dots, \ell\}.$$

Remark 28 (Formulas for the Bohl and exponential dichotomy spectrum)**.** *Using the notation of Theorem 27, we obtain the following formulas for the boundary points of the Bohl dichotomy spectrum and the exponential dichotomy spectrum (cf. Remark 20).*

(a) The boundary points of the Bohl dichotomy spectrum are

$$a_i = \underline{\beta}_{k_{i-1}, 1} \qquad \textit{and} \qquad b_i = \overline{\beta}_{k_i, 1}, \qquad i \in \{1, \dots, \ell\}.$$

(b) The boundary points of the exponential dichotomy spectrum are

$$a_i = \underline{\beta}_{k_{i-1}, d-k_{i-1}} \qquad \textit{and} \qquad b_i = \overline{\beta}_{k_i, d-k_i}, \qquad i \in \{1, \dots, \ell\}.$$

In case system (1) has a dichotomy then the limiting Bohl exponents of Definition 22 can also be expressed in terms of the splitting of the dichotomy.

Corollary 29 (Limiting Bohl exponents in case of dichotomy)**.** *If system* (1) *has a dichotomy on (L_1, L_2) with uniformity dimensions (j_1, j_2) then*

$$\overline{\beta}_{k, j_1} = \sup_{U \in \mathcal{G}_{j_1}(L_1)} \overline{\beta}(U) \tag{7}$$

where $k = \dim L_1$, and if $\mathbb{T} = \mathbb{Z}$, then

$$\underline{\beta}_{k, j_2} = \inf_{U \in \mathcal{G}_{j_2}(L_2)} \overline{\beta}(U). \tag{8}$$

Proof. The first part follows readily from Lemma 24 and we only proof formula (7). Suppose that system (1) has a dichotomy with splitting (L_1, L_2) uniform on subspaces of dimensions j_1 and j_2. Let $\alpha > 0$ with $-\alpha \in (\overline{\beta}_{k,j_1}, 0)$, so that $\mathrm{D}_1(-\alpha, U)$ holds uniformly for all $U \in \mathcal{G}_{j_1}(L_1)$ by Proposition 15(b). Let $k \coloneqq \dim L_1$. If $k = 0$, then $j_1 = 0$ and (7) is satisfied (cf. Remark 14). Let $k > 0$. Suppose that $L_1' \in \mathcal{G}_k(\mathbb{R}^d)$ with

$$\sup_{U' \in \mathcal{G}_{j_1}(L_1')} \overline{\beta}(U') \leq \sup_{U \in \mathcal{G}_{j_1}(L_1)} \overline{\beta}(U). \tag{9}$$

It holds that $\overline{\beta}(U) \leq -\alpha$ for all $U \in \mathcal{G}_{j_1}(L_1)$ (cf. Proposition 15(a) with $\mathcal{V} \coloneqq \{L_1\}$ and $\mathcal{U}(L_1) \coloneqq \mathcal{G}_{j_1}(L_1)$). By the inequality (9), also $\overline{\beta}(U') \leq -\alpha$ for all $U' \in \mathcal{G}_{j_1}(L_1')$. By Proposition 15(b), we conclude that $\mathrm{D}_1(-\alpha/2, U')$ holds for all $U' \in \mathcal{G}_{j_1}(L_1')$, so that

$$\lim_{n\to\infty} x(n, x_0) = 0, \qquad x_0 \in L_1'.$$

This implies that $L_1' \subseteq L_1$ by Proposition 12. But since $\dim L_1 = \dim L_1'$, also $L_1 = L_1'$. We conclude that for all $L_1' \in \mathcal{G}_k(\mathbb{R}^d)$, we have

$$\sup_{U' \in \mathcal{G}_{j_1}(L_1')} \overline{\beta}(U') \geq \sup_{U \in \mathcal{G}_{j_1}(L_1)} \overline{\beta}(U).$$

Taking the infimum over all $L_1' \in \mathcal{G}_k(\mathbb{R}^d)$ yields formula (7). ☐

Remark 30 (Open problem)**.** *It is not clear to the authors, if* (8) *holds for all* L_2*, complementary to* L_1 *in the case* $\mathbb{T} = \mathbb{N}$*.*

5 Maximal subspaces of uniformity

In this section we introduce maximality of the uniformity dimensions of a dichotomous system in Theorem 31 and discuss its dependence on the dichotomy splitting in Theorems 33 and 34. An open problem and a conjecture are formulated in Remark 35.

Theorem 31 (Maximal uniformity dimensions)**.** *If system* (1) *has a dichotomy on* (L_1, L_2) *there exist* $u_1 \in \{0, \dots, \dim L_1\}$ *and* $u_2 \in \{0, \dots, \dim L_2\}$ *such that*

(a) system (1) *has a dichotomy on* (L_1, L_2) *with uniformity dimensions* (u_1, u_2)*,*

(b) if (1) *has a dichotomy on* (L_1, L_2) *with uniformity dimensions* (ℓ_1, ℓ_2) *then*

$$\ell_1 \in \{0, \dots, u_1\} \qquad \text{and} \qquad \ell_2 \in \{0, \dots, u_2\}.$$

(u_1, u_2) *are called* maximal uniformity dimensions*.*

In case system (1) has a dichotomy on (L_1, L_2) then by Proposition 12 the space L_1 is unique and hence the maximal uniformity dimension u_1 does not depend on L_1. If $\mathbb{T} = \mathbb{Z}$, the space L_2 is unique and in this case the maximal uniformity dimension u_2 does not depend on L_2. We provide a partial answer to the question how u_2 does depend on L_2 in Theorem 34 and formulate an open problem in Remark 35. As preparatory results we prove the following Lemma and Theorem 33.

Lemma 32 (Dichotomy estimate follows from uniform estimate at later time)**.** *Let $\gamma \in \mathbb{R}$, $U \in \mathcal{G}(\mathbb{R}^d)$, $C > 0$ and $m_0 \in \mathbb{N}$ with*

$$\|x(n, x_0)\| \geq C e^{\gamma(n-m)} \|x(m, x_0)\|, \qquad m, n \in \mathbb{N},\ n \geq m \geq m_0,\ x_0 \in U. \tag{10}$$

Then $\mathrm{D}_2(\gamma, U)$ holds uniformly.

Proof. If $m_0 = 0$, then the statement of Lemma 32 is clear. Suppose that $m_0 \geq 1$. By induction, it suffices to show that

$$\|x(n, x_0)\| \geq C' e^{\gamma(n-m_0+1)} \|x(m_0 - 1, x_0)\|, \qquad n \in \mathbb{N},\ n \geq m_0 - 1,\ x_0 \in U \tag{11}$$

holds for some $C' > 0$. Let us denote

$$a = \max\left\{\|A\|_\infty, \|A^{-1}\|_\infty\right\}.$$

Considering (10) for n set to $n+1$ and for m set to m_0, we get

$$\|x(n+1, x_0)\| \geq C e^{\gamma(n-m_0+1)} \|x(m_0, x_0)\|, \qquad m, n \in \mathbb{N},\ n \geq m_0 - 1,\ x_0 \in U$$

and consequently

$$\|A^{-1}(m_0)\| \cdot \|x(n+1, x_0)\| \geq C e^{\gamma(n-m_0+1)} \|A^{-1}(m_0)\| \cdot \|x(m_0, x_0)\|.$$

Noting that

$$\begin{aligned} \|A^{-1}(m_0)\| \|x(n+1, x_0)\| &\leq a \|x(n+1, x_0)\| \\ &= a \|A(n) x(n, x_0)\| \\ &\leq a^2 \|x(n, x_0)\| \end{aligned}$$

and

$$\|A^{-1}(m_0)\| \|x(m_0, x_0)\| \geq \|A^{-1}(m_0) x(m_0, x_0)\| = \|x(m_0 - 1, x_0)\|,$$

we obtain

$$a^2 \|x(n, x_0)\| \geq C e^{\gamma(n-m_0+1)} \|x(m_0 - 1, x_0)\|.$$

This proves inequality (10) with $C' = C/a^2$. □

Theorem 33 (Independence of maximal uniformity dimensions of splitting)**.** *Suppose that $T = \mathbb{N}$. Let (L_1, L_2) and (L_1, L_2') be splittings of $\mathbb{R}^d$. Suppose that System* (1) *has a dichotomy on (L_1, L_2) with maximal uniformity dimensions (u_1, u_2) and on (L_1, L_2') and with maximal uniformity dimensions (u_1', u_2'). Then*

$$u_1 = u_1'. \tag{12}$$

If additionally for each $V' \in \mathcal{G}_{u_2}(L_2')$, we have $\dim V'_{L_1} \leq u_1$, then

$$u_2 \leq u_2'. \tag{13}$$

Proof. The equality (12) is clear. Let $\alpha > 0$ be such that $\mathrm{D}_1(-\alpha, U)$ holds uniformly for all $U \in \mathcal{G}_{j_1}(L_1)$ and $\mathrm{D}_2(\alpha, U)$ holds uniformly for all $U \in \mathcal{G}_{j_2}(L_2)$ for all $j_1 \in \{0, ..., u_1\}$ and $j_2 \in \{0, ..., u_2\}$. Let $V' \in \mathcal{G}_{u_2}(L_2')$. To prove the inequality (13), it suffices to show that the estimate $\mathrm{D}_2(\alpha, V')$ holds uniformly. Since $\dim \pi_{L_1}[V'] \le u_1$ and $\dim \pi_{L_2}[V'] \le \dim V' = u_2$, there exist constants $C_1, C_2 > 0$ such that

$$\|x(n, x_0)\| \le C_1 \mathrm{e}^{-\alpha(n-m)} \|x(m, x_0)\|, \qquad m, n \in \mathbb{N},\ n \ge m,\ x_0 \in \pi_{L_1}[V'] \tag{14}$$

and

$$\|x(n, x_0)\| \ge C_2 \mathrm{e}^{\alpha(n-m)} \|x(m, x_0)\|, \qquad m, n \in \mathbb{N},\ n \ge m,\ x_0 \in \pi_{L_2}[V']. \tag{15}$$

Let $K > 0$ be such that

$$C_1 \le K \qquad \text{and} \qquad C_2 \ge K^{-1}.$$

Then (14) and (15) imply that

$$\|x(n, x_0)\| \le K \mathrm{e}^{-\alpha(n-m)} \|x(m, x_0)\|, \qquad m, n \in \mathbb{N},\ n \ge m,\ x_0 \in \pi_{L_1}[V'] \tag{16}$$

and

$$\|x(n, x_0)\| \ge K^{-1} \mathrm{e}^{\alpha(n-m)} \|x(m, x_0)\|, \qquad m, n \in \mathbb{N},\ n \ge m,\ x_0 \in \pi_{L_2}[V']. \tag{17}$$

Consider $v' \in V'$ with $\|v'\| = 1$ and note that $v' \ne 0$ implies $\pi_{L_2} v' \ne 0$. We have

$$\|x(n, v')\| \ge \|x(n, \pi_{L_2} v'\| - \|x(n, \pi_{L_1} v')\|, \qquad n \in \mathbb{N}$$

and by applying (16) and (17), we get

$$\begin{aligned}
\|x(n, v')\| &\ge K^{-1} \mathrm{e}^{\alpha(n-m)} \|x(m, \pi_{L_2} v')\| - K \mathrm{e}^{-\alpha(n-m)} \|x(m, \pi_{L_1} v')\| \\
&\ge K^{-1} \mathrm{e}^{\alpha(n-m)} \|x(m, \pi_{L_2} v')\| - K \mathrm{e}^{\alpha(n-m)} \|x(m, \pi_{L_1} v')\| \\
&= \mathrm{e}^{\alpha(n-m)} (K^{-1} \|x(m, \pi_{L_2} v')\| - K \|x(m, \pi_{L_1} v')\|), \\
&\mathrm{e}^{\alpha(n-m)} (K^{-1} - K \frac{\|x(m, \pi_{L_1} v')\|}{\|x(m, \pi_{L_2} v')\|}) \|x(m, \pi_{L_2} v')\|, \qquad m, n \in \mathbb{N},\ n \ge m.
\end{aligned}$$

Taking in (14) $x_0 = \pi_{L_1} v'$ and $m = 0$, and in (15) $x_0 = \pi_{L_2} v'$ and $m = 0$, we get

$$\frac{\|x(k, \pi_{L_1} v')\|}{\|x(k, \pi_{L_2} v')\|} \le \frac{K^2}{\mathrm{e}^{2\alpha k}} \frac{\|\pi_{L_1} v'\|}{\|\pi_{L_2} v'\|}, \qquad k \in \mathbb{N} \tag{18}$$

and therefore

$$\|x(n, v')\| \ge \mathrm{e}^{\alpha(n-m)} \left(K^{-1} - \frac{K^3}{\mathrm{e}^{2\alpha m}} \frac{\|\pi_{L_1} v'\|}{\|\pi_{L_2} v'\|} \right) \|x(m, \pi_{L_2} v')\|, \qquad m, n \in \mathbb{N},\ n \ge m. \tag{19}$$

The function $v' \mapsto \frac{\|\pi_{L_1} v'\|}{\|\pi_{L_2} v'\|}$ defined on the compact set $\{v' \in V : \|v'\| = 1\}$ is continuous and therefore attains its maximum value. Hence

$$C' := \max\left\{\frac{\|\pi_{L_1} v'\|}{\|\pi_{L_2} v'\|} : v' \in V,\, \|v'\| = 1\right\} \in [0, \infty),$$

is well-defined. The inequality (19) then implies that

$$\|x(n, v')\| \geq \mathrm{e}^{\alpha(n-m)}(K^{-1} - \frac{K^3}{\mathrm{e}^{2\alpha m}} C')\|x(m, \pi_{L_2} v')\|, \qquad m, n \in \mathbb{N},\, n \geq m.$$

Consider $m_0 \in \mathbb{N}$ such that

$$K^{-1} - \frac{K^3}{\mathrm{e}^{2\alpha m_0}} C' > 0,$$

for all $m \geq m_0$. Define

$$C'' := K^{-1} - \frac{K^3}{\mathrm{e}^{2\alpha m_0}} C'.$$

We get

$$\|x(n, v')\| \geq \mathrm{e}^{\alpha(n-m)} C'' \|x(m, \pi_{L_2} v')\|, \qquad m, n \in \mathbb{N},\, n \geq m \geq m_0, \tag{20}$$

since

$$K^{-1} - \frac{K^3}{\mathrm{e}^{2\alpha m}} C' \geq C''$$

for $m \geq m_0$. Note that by (18), we have

$$\begin{aligned} \|x(k, v')\| &\leq \|x(k, \pi_{L_2} v'\| + \|x(k, \pi_{L_1} v')\| \\ &\leq \|x(k, \pi_{L_2} v'\| + \frac{K^2}{\mathrm{e}^{2\alpha k}} C' \|x(k, \pi_{L_2} v')\| \\ &= (1 + \frac{K^2}{\mathrm{e}^{2\alpha k}} C')\|x(k, \pi_{L_2} v')\| \\ &\leq (1 + K^2 C')\|x(k, \pi_{L_2} v')\|, \qquad k \in \mathbb{N} \end{aligned}$$

and therefore

$$\|x(m, y(v'))\| \geq (1 + K^2 C')^{-1} \|x(m, v')\|, \qquad m \in \mathbb{N}. \tag{21}$$

From the inequalities (20) and (21), we get

$$\|x(n, v')\| \geq \mathrm{e}^{\alpha(n-m)} C'' (1 + K^2 C')^{-1} \|x(m, v')\|, \qquad m, n \in \mathbb{N},\, n \geq m \geq m_0.$$

The statement now follows from Lemma 32. □

Theorem 34 (Maximal uniformity dimensions for one-sided time)**.** *Suppose that $T = \mathbb{N}$. Let (L_1, L_2) be a splitting of $\mathbb{R}^d$. Suppose that System* (1) *has a dichotomy on (L_1, L_2) with maximal uniformity dimensions (u_1, u_2). Denote by $(u_1, u_2^\perp)$ the maximal uniformity dimensions of system* (1) *with respect to the splitting $(L_1, L_1^\perp)$. Then it holds that $u_2 \leq u_2^\perp$.*

Proof. This follows from Theorem 33, noting that for each $V' \in \mathcal{G}_{u_2}(L_1^\perp)$, we have $\dim V_{L_1}[V'] = 0 \leq u_1$ $\square$

In the following remark we list those cases in which Theorem 33 yields a unique maximal uniformity dimension of a dichotomy, i.e. independent of the splitting, and we formulate a conjecture on the general dependence of the maximal uniformity dimensions on the subspace L_2 which is complementary to L_1.

Remark 35 (Dependence of maximal uniformity dimensions on splitting for one-sided time)**.** *Suppose that system* (1) *with* $\mathbb{T} = \mathbb{N}$ *has a dichotomy on a splitting* (L_1, L_2).

(a) The maximal uniformity dimensions do not depend on L_2 *in case* $\dim L_1 = 1$ *or* $\dim L_1 = d - 1$.

(b) Conjecture: There exists $A\colon \mathbb{N} \to \mathrm{GL}(\mathbb{R}^d)$ *such that the maximal uniformity dimension* u_2 *depends on the choice of* L_2.

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