# THE ORIGIN OF THE ENERGY SPLIT IN PHASE-FIELD FRACTURE AND EIGENFRACTURE

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*Summary* Splits (decompositions) of the mechanical energy potential for regularised fracture models, like phase-field fracture and eigenfracture, are frequently motivated by the statement that crack nucleation and crack growth have to be prevented at compression loads. The intention of the talk is to show that this energy decomposition is an essential part of such fracture models, and to investigate the underlying mathematical statement and its physical interpretation. Finally, a general framework for the derivation of physically based energy splits is presented, which is not limited to certain bulk material models. Illustrative examples with crack face friction, inelasticity and multi-physics are used to present the versatility of the approach.

## THE FREE DISCONTINUITY PROBLEM

Griffith's energetic description of brittle fracture processes can be formulated as variational problem

$$E(\boldsymbol{u},\Gamma) = \int\limits_{\mathcal{B}\setminus\Gamma} \psi(D\boldsymbol{u})\,\mathrm{d}V + G_\mathrm{c}\int\limits_{\Gamma} \mathrm{d}A \rightarrow \mathrm{Min}, \tag{1}$$

where $\mathcal{B}$ is the domain of the body, $\Gamma \subseteq \mathcal{B}$ the domain of the crack, $G_\mathrm{c}$ the fracture toughness and $\boldsymbol{u}$ the displacement field with the weak derivative $D\boldsymbol{u}$. This variational problem belongs to the free discontinuity problems, in which the unknown field $\boldsymbol{u}$ can exhibit discontinuities at the unknown (free) domain $\Gamma$ of the crack.

Regularised formulations for the domain $\Gamma$ have been proposed as an alternative to the variable integration domains with discrete bounds at the discontinuities. Well known regularised formulations for the free discontinuity problem of brittle fracture are for instance

phase-field for fracture
$$E(\boldsymbol{u},p) = \int\limits_{\mathcal{B}} [\psi^- + g(p)\,\psi^+]\,\mathrm{d}V + G_\mathrm{c}\int\limits_{\mathcal{B}} \gamma_l(p)\,\mathrm{d}V \rightarrow \mathrm{Min} \tag{2}$$

and eigenfracture
$$E(\boldsymbol{u},p) = \int\limits_{\mathcal{B}} \psi(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^\mathrm{eig})\,\mathrm{d}V + G_\mathrm{c}\frac{\mathcal{B}_{\boldsymbol{\varepsilon}^\mathrm{eig}\neq 0}}{2\,l} \rightarrow \mathrm{Min}, \tag{3}$$

where $l$ is the regularisation parameter and $\mathcal{B}_{\boldsymbol{\varepsilon}^\mathrm{eig}\neq 0}$ the domain of the crack neighbourhood, compare Fig. 1.

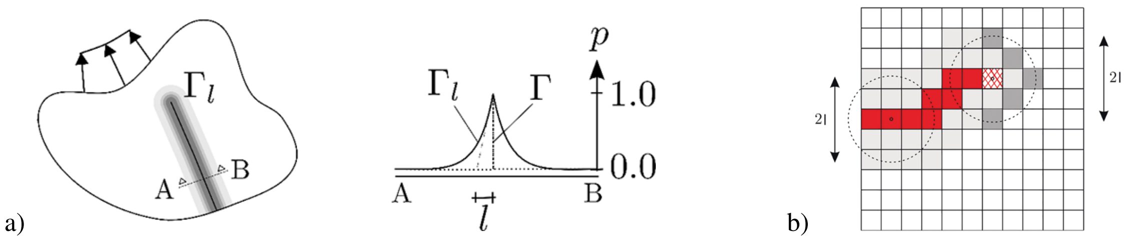


Figure 1. a) Profile of the phase-field field through a crack and b) the crack neighbourhood definition in eigenfracture.

In both formulations, the deformation energy potential is expressed by an integration on the entire domain $\mathcal{B}$, which also includes the crack $\Gamma$. On the one side, the discretisation does not need to be adjusted nor to follow the cracks, which allows to apply common schemes of computational mechanics, like the Finite Element Method. On the other side, the deformation energy potential is not exclusively evaluated at points of solid material without cracks, compare Fig. 2a). Hence, it is further evaluated at locations, where the crack is present in Eqs. (2) and (3). At those *cracked* material points, the derivative of (discontinuous) displacement field does not represent the deformation of the solid material nearby the crack, because it also includes a portion caused by the relative displacements between the crack surfaces, see Fig. 2a).

The discontinuous functions of the free discontinuity problem can be described in the space of Special functions of Bounded Variation (SBV). The properties of this space are intensively studied, e.g. by De Giorgi at al. [1] and Ambrosio et al. [2]. It is shown that the weak derivative $D\boldsymbol{u}$ at the discontinuity additively decomposes into an absolutely continuous part $D^\mathrm{a}\boldsymbol{u}$ and a jump part $D^\mathrm{j}\boldsymbol{u}$ as $D\boldsymbol{u} = D^\mathrm{a}\boldsymbol{u} + D^\mathrm{j}\boldsymbol{u}$. The physical interpretation of this

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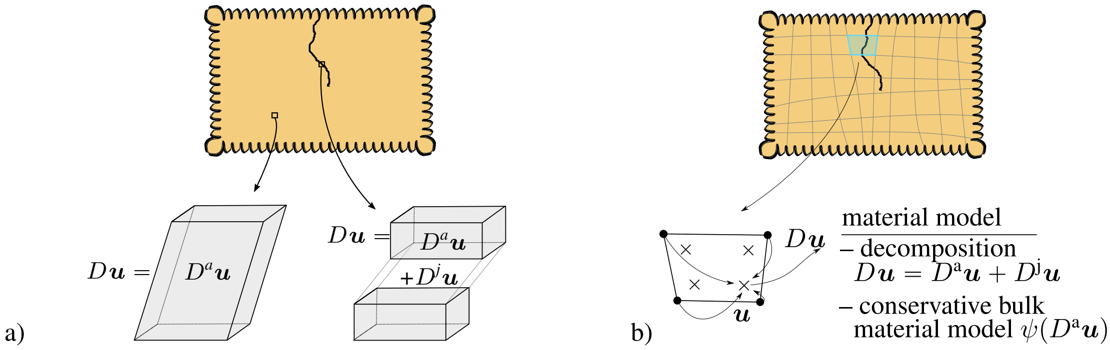


Figure 2. a) Decomposition of the weak derivative $D\boldsymbol{u}$ and b) the crack neighbourhood definition in eigenfracture.

decomposition in the context of brittle fracture relates the absolutely continuous part $D^{\mathrm{a}}\boldsymbol{u}$ to the solid deformation and the jump part $D^{\mathrm{j}}\boldsymbol{u}$ to the contribution of the crack, compare Fig. 2a). Then, the free discontinuity problem of Eq. (1) can be rewritten as

$$\begin{aligned} E(\boldsymbol{u},\Gamma) &= \int_{\mathcal{B}} [\psi(D\boldsymbol{u}) - \psi(D^{\mathrm{j}}\boldsymbol{u})]\ \mathrm{d}V + G_{\mathrm{c}} \int_{\Gamma} \mathrm{d}A \\ &= \int_{\mathcal{B}} \psi(D^{\mathrm{a}}\boldsymbol{u})\ \mathrm{d}V + G_{\mathrm{c}} \int_{\Gamma} \mathrm{d}A \rightarrow \mathrm{Min.} \end{aligned} \tag{4}$$

The comparison of this formulation to the regularised formulations in Eqs. (2) and (3) shows that the used energy decompositions are realisations of the weak derivative decomposition.

## PHYSICAL SPLITS MODELS

Having the mathematical basis and physical interpretation of the splits identified, artificial split approaches can be overcome and replaced by physical models. The combination of a discrete crack model for a small crack portion and the established concept of computational homogenisation yields the variational framework of *Representative Crack Elements* (RCE), which allows to derive the energy decomposition through the determination of $D^{\mathrm{a}}\boldsymbol{u}$ as $\boldsymbol{H}|_{\boldsymbol{x}} = \bar{\boldsymbol{H}} + \frac{1}{l^1}[\![\bar{\boldsymbol{u}}]\!] \otimes \boldsymbol{N}_1$ for mechanical problems (compare Fig. 3).

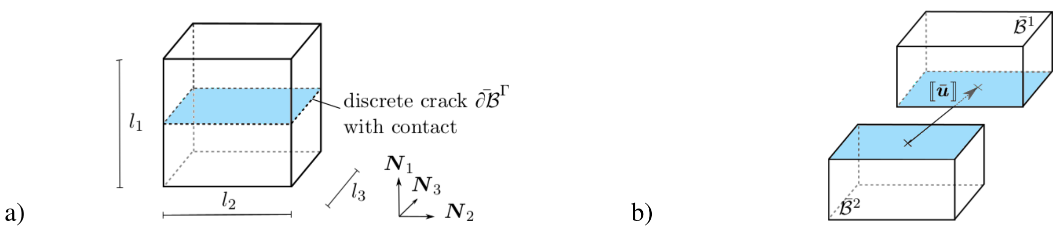


Figure 3. a) Realisation for a representative crack element and b) its deformed state.

Finally, the RCE can be treated as an ordinary problem of solid mechanics with contact at the crack surface. Conventional material models for the bulk can be applied to the RCE. Moreover, the solution of the problem is cheap, because the entire problem can be formulated w.r.t. the unknown jump of the displacement field $[\![\bar{\boldsymbol{u}}]\!]$. The RCE framework has been successfully applied for instance to anisotropic elasticity [3, 4, 5], crack face friction and dissipative materials [6], thermo-mechanics [7] and electro-mechanics [8].

## References


[1] E. De Giorgi, M. Carriero, and A. Leaci, Archive for Rational Mechanics and Analysis **108**, 195–218 (1989).
[2] L. Ambrosio, N. Fusco, and D. Pallara, Functions of Bounded Variation and Free Discontinuity Problems (Clarendon Press, 2000).
[3] J. Storm, D. Supriatna, and M. Kaliske, International Journal for Numerical Methods in Engineering **121**, 779–805 (2020).
[4] J. Storm, A. Qinami, and M. Kaliske, Mechanics Research Communications **116**, 103747 (2021).
[5] C. Steinke, J. Storm, and M. Kaliske, International Journal of Fracture **237**, 15–46 (2022).
[6] J. Storm, B. Yin, and M. Kaliske, International Journal for Numerical Methods in Engineering (submitted).
[7] J. Storm, B. Yin, and M. Kaliske, Computational Mechanics **69**, 1165–1176 (2022).
[8] M. Kaliske, J. Storm, A. Kanan, and W. Klausler, Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences **380**, 20210329 (2022).