Symmetric Linear Arc Monadic Datalog and Gadget Reductions

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A Datalog program solves a constraint satisfaction problem (CSP) if and only if it derives the goal predicate precisely on the unsatisfiable instances of the CSP. There are three Datalog fragments that are particularly important for finite-domain constraint satisfaction: arc monadic Datalog, linear Datalog, and symmetric linear Datalog, each having good computational properties. We consider the fragment of Datalog where we impose all of these restrictions simultaneously, i.e., we study symmetric linear arc monadic (slam) Datalog. We characterise the CSPs that can be solved by a slam Datalog program as those that have a gadget reduction to a particular Boolean constraint satisfaction problem. We also present exact characterisations in terms of a homomorphism duality (which we call *unfolded caterpillar duality*), and in universal-algebraic terms (using known minor conditions, namely the existence of quasi Maltsev operations and k-absorptive operations of arity nk, for all $n, k \geq 1$). Our characterisations also imply that the question whether a given finite-domain CSP can be expressed by a slam Datalog program is decidable.

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1 Introduction

Datalog is an important concept linking database theory with the theory of constraint satisfaction. It is by far the most intensively studied formalism for polynomial-time tractability in constraint satisfaction. Datalog allows to formulate algorithms that are based on iterating local inferences, aka *constraint propagation* or *establishing local consistency*; this has been made explicit by Feder and Vardi in their groundbreaking work where they also formulate the finite-domain CSP dichotomy conjecture [25]. Following their convention, we say that a Datalog program Π solves a CSP if Π derives the goal predicate on an instance of the CSP if and only if the instance is unsatisfiable.

The class of CSPs that can be solved by Datalog is closed under so-called 'gadget reductions' (a result due to Larose and Zádori [33]). In such a reduction, the variables in an instance of a constraint satisfaction problem are replaced by tuples of variables of some fixed finite length, and the constraints are replaced by *gadgets* (implemented by conjunctive queries; a formal definition can be found in Section 2.3); many of the well-known reductions between computational problems can be phrased as gadget reductions. Datalog is sufficiently powerful to simulate such gadget reductions; this has

been formalised by Atserias, Bulatov, and Dawar in [2] and the connection has been sharpened recently by Dalmau and Opršal [21].

Feder and Vardi showed that Datalog cannot solve systems of linear equations over finite fields, even though such systems can be solved in polynomial time [25]. They suggest that the ability to simulate systems of linear equations should essentially be the only reason for a CSP to not be in Datalog. This conjecture was formalised by Larose and Zádori [33]: they observed that if systems of linear equations admit a gadget reduction to a CSP, then the CSP is not in Datalog, and they conjectured that otherwise the CSP can be solved by Datalog. This conjecture was proved by Barto and Kozik in 2009 [3], long before the resolution of the finite-domain CSP dichotomy conjecture by Bulatov [13] and by Zhuk [37, 38].

Datalog programs can be evaluated in polynomial time; but even a running time in $O(n^3)$ on a sequential computer can be prohibitively expensive in practise. This is one of the reasons why syntactic *fragments* of Datalog have been studied, which often come with better computational properties.

1.1 Arc Monadic Datalog

In monadic Datalog, we restrict the arity of the inferred predicates of the Datalog program to one (i.e., all the *IDBs* are monadic). In *arc Datalog* we restrict each rule to a single input relation symbol (i.e., the body contains a single *EDB*; for formal definitions, see Section 2.4).

An important Datalog fragment is *arc monadic Datalog*, which is still powerful enough to express the famous *arc consistency procedure* in constraint satisfaction. The arc consistency procedure has already been studied by Feder and Vardi [25], and has many favorable properties: it can be evaluated in linear time and linear space. It is used as an important pre-processing step in the algorithms for both of the mentioned CSP dichotomy proofs, and it is also used in many practical implementations of algorithms in constraint satisfaction. The arc consistency procedure is still extremely powerful, and can for instance solve the P-complete HornSat Problem.

Feder and Vardi characterised the power of the arc consistency procedure in terms of *tree duality* (see Section 2.6), a natural combinatorial property which has been studied intensively in the graph homomorphism literature in the 90s (see, e.g., [26, 27]). Their characterisation has several remarkable consequences: one is that also the class of CSPs that can be solved by an arc monadic Datalog program is closed under gadget reductions. Another one is a *collapse result* for Datalog when it comes to finite-domain CSPs, namely that monadic Datalog collapses to arc monadic Datalog: in fact, if a finite-domain CSP can be solved by a monadic Datalog program, then it can already be solved by the arc consistency procedure (i.e., by a program in arc monadic Datalog). This statement is false without the restriction to finite-domain CSPs; in fact, there are infinite-domain CSPs that can be solved by a monadic Datalog program, but not by a program in arc monadic Datalog (Bodirsky and Dalmau [10]).

1.2 Linear Datalog

Besides arc monadic Datalog, there are other natural fragments of Datalog. The most notable one is *linear Datalog*, introduced by Dalmau [19]. Linear Datalog programs can be evaluated in non-deterministic logarithmic space (NL), and hence cannot express P-hard problems (unless P=NL). Dalmau asked whether the converse is true as well, i.e., whether every finite-domain CSP which is in NL can be solved by a linear Datalog program [19]. This is widely treated as a conjecture, to which we refer as the *linear Datalog conjecture*; it is one of the biggest open problems in finite-domain constraint satisfaction.

There are some sufficient conditions for solvability by linear Datalog (see Bulatov, Kozik, and Willard [5] and Carvalho, Dalmau, and Krokhin [16]) and some necessary conditions (Larose and Tesson [32]) but the results still leave a large gap. For examples of CSPs of orientations of trees that fall into this gap, see Bodirsky, Bulín, Starke, and Wernthaler [8]. Again, linear Datalog is closed under gadget reductions [35]. And indeed, if HornSat has a gadget-reduction in a finite-domain CSP, then the finite-domain CSP cannot be solved by a linear Datalog program [1].

1.3 Symmetric Linear Datalog

A further restriction is symmetric linear Datalog, introduced by Egri, Larose, and Tesson [23]. symmetric linear Datalog programs can be evaluated in deterministic logspace (L). Egri, Larose, and Tesson conjecture that every finite-domain CSP which is in L can be be solved by a symmetric linear Datalog program [23]; we refer to this conjecture as the symmetric linear Datalog conjecture. Also symmetric linear Datalog is closed under gadget reductions [35]. Since directed reachability is not in symmetric linear Datalog [24], it follows that every CSP that admits a gadget reduction from directed reachability cannot be solved by a symmetric linear Datalog program. Egri, Larose, and Tesson also suggest that this might be the only additional condition for containment in symmetric linear Datalog, besides the known necessary conditions to be in linear Datalog. Kazda [30] confirms the symmetric linear Datalog conjecture, i.e., he shows that if a finite-domain CSP is in linear Datalog and does not admit a gadget reduction from a CSP that corresponds to the directed reachability problem, then it can be solved by a symmetric linear Datalog program (generalizing an earlier result of Dalmau and Larose [20]).

1.4 Our Contributions

In this paper, we study the Datalog fragment that can be obtained by combining all the previously considered restrictions, namely symmetric linear arc monadic (slam) Datalog. Before stating our result we illustrate this fragment with some examples. For $n \ge 1$, let \mathfrak{P}_n be the directed path with n vertices an n-1 edges. An example of a slam Datalog

program which solves $CSP(\mathfrak{P}_2)$ is

$$A(x) := E(x, y) \qquad \qquad \text{goal} := E(x, y), A(y)$$

(in this case, the program is even recursion-free). An example of a slam Datalog program which solves $CSP(\mathfrak{P}_3)$, this time with recursion and IDBs A and B, is

$$\begin{array}{ll} A(x) := E(x,y) & B(x) := A(y), E(x,y) \\ A(y) := B(x), E(x,y) & \text{goal} := B(y), E(x,y). \end{array}$$

It follows from our results that the class of CSPs that can be solved by slam Datalog programs is closed under gadget reductions, despite the many restrictions that we imposed.

We provide a *full description* of the power of a Datalog fragment in terms of gadget reductions: we show that a CSP can be solved by a slam Datalog program if and only if it has a gadget reduction to $\text{CSP}(\mathfrak{P}_2)$.¹ The particular role of the structure \mathfrak{P}_2 is explained by the fact that it is a representative of the unique class of CSPs which is non-trivial and *weakest* with respect to gadget reductions – a formalisation of this can be found in Section 2.3. This shows that slam Datalog is the smallest non-trivial fragment of Datalog that is closed under gadget reductions.

Our main result (Theorem 3.1) establishes a tight connection between the power of slam Datalog and various central themes in structural combinatorics and universal algebra. Specifically, the power of slam Datalog can be characterised by a new combinatorial duality which we call unfolded caterpillar duality (restricting the concept of caterpillar duality of Carvalho, Dalmau, and Krokhin [17]), and by the existence of a quasi Maltsev polymorphism (a central concept in universal algebra) in combination with kn-ary k-absorbing polymorphisms for every $k, n \geq 1$ (introduced in [17] as well). Our result also implies that the following meta-problem can be decided algorithmically: given a finite structure \mathfrak{B} , can CSP(\mathfrak{B}) be solved by a slam Datalog program?

1.5 Related Results

Solvability of finite-domain CSPs by (unrestricted) Datalog was first studied by Feder and Vardi; they proved that $CSP(\mathfrak{B})$ can be solved by Datalog if and only if \mathfrak{B} has bounded treewidth duality, and they showed that CSPs for systems of linear equations over finite Abelian groups cannot be solved by Datalog. Larose and Zadori [33] showed that solvability by Datalog is preserved by gadget reductions and they asked whether having a gadget reduction from CSPs for systems of linear equations is not only a sufficient, but also a necessary condition for not being solvable by Datalog. This questions was

¹The statement even holds for infinite-domain CSPs, since being solved by an arc monadic Datalog program implies the existence of a finite template [10] and admitting a gadget reduction to a finite-domain CSP implies the existence of a finite template as well [21].

answered positive by Barto and Kozik [4]. Kozik, Krokhin, and Willard [31] gave a characterisation of Datalog in terms of minor conditions.

Linear (but not necessarily symmetric) monadic arc Datalog has been studied by Carvalho, Dalmau, and Krokhin [17]; our proof builds on their result. In their survey on Datalog fragments and dualities in constraint satisfaction [14] Bulatov, Krokhin, and Larose write "it would be interesting to find (...) an appropriate notion of duality for symmetric (Linear) Datalog (...)". We do find such a notion for symmetric linear arc monadic Datalog, namely unfolded caterpillar duality (Theorem 3.1).

2 Preliminaries

We write [n] for the set $\{1, \ldots, n\}$ and [m, n] for the set $\{m, m+1, \ldots, n\}$. We say that a tuple $a \in A^k$, for $k \in \mathbb{N}$, is *injective* if a is injective when viewed as a function from [k] to A.

2.1 Structures and Graphs

We assume familiarity with the concepts of relational structures and first-order formulas from mathematical logic, as introduced for instance in [28]. The arity of a relation symbol R is denoted by $\operatorname{ar}(R)$. If \mathfrak{A} is a τ -structure, then we sometimes use the same symbol for $R \in \tau$ and the respective relation $R^{\mathfrak{A}}$ of \mathfrak{A} . We write $\mathfrak{A}[S]$ for the substructure of \mathfrak{A} induced on S.

A (directed) graph is a relational structure with a single binary relation E. For instance the clique with n vertices is the graph \mathfrak{K}_n with domain [n] and edges $E^{\mathfrak{K}_n} := \{(a,b) \mid a \neq b\}$. Let \mathfrak{G} be a graph. An (undirected) path from a to b in \mathfrak{G} is a tuple $P = (a_1, \ldots, a_n)$ such that a_1, \ldots, a_n are pairwise distinct, $a_1 = a$, $a_n = b$, and for all $i \in [n-1]$ there is an edge between a_i and a_{i+1} (from a_i to a_{i+1} or from a_{i+1} to a_i). If $i \in [2, n-1]$, then we say that P passes through a_i . A graph \mathfrak{G} is called connected if for any two elements a, b there exists a path from a to b in \mathfrak{G} . A cycle is a path from a to b of length n at least three such that there is an edge between a and b. A graph is called acyclic if it does not contain any cycle. A graph is called a tree if it is connected and acyclic. A graph has girth k if the length of the shortest cycle is k.

2.2 Homomorphisms and CSPs

Let τ be a relational signature and let \mathfrak{A} and \mathfrak{B} be τ -structures. Then a homomorphism from \mathfrak{A} to \mathfrak{B} is a map $h: A \to B$ such that for $R \in \tau$, say of arity k, we have $(h(a_1), \ldots, h(a_k)) \in R^{\mathfrak{B}}$ whenever $(a_1, \ldots, a_k) \in R^{\mathfrak{A}}$. An embedding of \mathfrak{A} into \mathfrak{B} is an injective map $e: A \to B$ such that $(e(a_1), \ldots, e(a_k)) \in R^{\mathfrak{B}}$ if and only if $(a_1, \ldots, a_k) \in R^{\mathfrak{A}}$. We write $\mathfrak{A} \to \mathfrak{B}$ if there exists a homomorphism from \mathfrak{A} to \mathfrak{B} and $\mathfrak{A} \not\to \mathfrak{B}$ if there exists no homomorphism from \mathfrak{A} to \mathfrak{B} .

If τ is a finite relational signature and \mathfrak{B} is a τ -structure, then $\mathrm{CSP}(\mathfrak{B})$ denotes the class of all finite τ -structures \mathfrak{A} such that $\mathfrak{A} \to \mathfrak{B}$. It can be viewed as a computational problem. For example, $\mathrm{CSP}(\mathfrak{K}_n)$ consists of the set of all finite *n*-colourable graphs, and can therefore be viewed as the *n*-colorability problem. Clearly, for finite structures \mathfrak{B} , this problem is always in NP.

A τ -structure \mathfrak{B} is homomorphically equivalent to a τ -structure \mathfrak{C} if there are homomorphisms from \mathfrak{B} to \mathfrak{C} and vice versa. Clearly, homomorphically equivalent structures have the same CSP. A relational τ -structure \mathfrak{C} is called a *core* if all endomorphisms of \mathfrak{C} are embeddings. It is well-known and easy to see that every finite structure \mathfrak{B} is homomorphically equivalent to a core \mathfrak{C} , and that all core structures \mathfrak{C} that are homomorphically equivalent to \mathfrak{B} are isomorphic; therefore, we refer to \mathfrak{C} as *the* core of \mathfrak{B} .

2.3 Primitive Positive Constructions

A τ -formula ϕ is called a *conjunctive query* (in constraint satisfaction and model theory such formulas are called *primitive positive*, or short pp) if it is built from atomic formulas (including atomic formulas of the form x = y) using only conjunction and existential quantification. If \mathfrak{B} is a τ -structure, and ϕ is a conjunctive query over the signature τ , then the relation $R = \{(t_1, \ldots, t_k) \mid \mathfrak{B} \models \phi(t_1, \ldots, t_k)\}$ is called the *relation defined by* ϕ .

Definition 2.1. The *canonical database* of a conjunctive query ϕ over the signature τ is the τ -structure \mathfrak{B} that can be constructed as follows: Let ϕ' be obtained from ϕ be renaming all existentially quantified variables such that no two quantified variables have the same name. Let ϕ'' be obtained from ϕ' by removing all conjuncts of the form x = y in ϕ' and by identifying variables x and y if there is a conjunct x = y in ϕ' . Then \mathfrak{B} is the τ -structure whose domain is the set of variables of ϕ'' such that for every $R \in \tau$ we have

 $R^{\mathfrak{B}} = \{ (v_1, \dots, v_k) \mid R(v_1, \dots, v_k) \text{ is a conjunct of } \phi'' \}.$

The canonical conjunctive query of a structure \mathfrak{B} with signature τ is the τ -formula with variables B given by

$$\bigwedge_{R\in\tau}\bigwedge_{t\in R^{\mathfrak{B}}}R(t_1,\ldots,t_{\operatorname{ar}(R)}).$$

Observe that the canonical database of the canonical conjunctive query of a structure \mathfrak{B} equals \mathfrak{B} . The following concepts have been introduced by Barto, Opršal, and Pinsker [6].

Definition 2.2. A (*d-th*) *pp-power* of a τ -structure \mathfrak{B} is a structure \mathfrak{C} with domain B^d such that every relation of \mathfrak{C} of arity k is definable by a conjunctive query in \mathfrak{B} as a relation of arity dk. A structure is called *primitive positive (pp) constructible of* \mathfrak{B} if it is homomorphically equivalent to a pp-power of \mathfrak{B} .

Primitive positive constructions turned out to the *the* essential tool for classifying the complexity of finite-domain CSPs, because if \mathfrak{C} has a pp-construction in \mathfrak{B} , then there is a so-called *gadget reduction* from $\mathrm{CSP}(\mathfrak{B})$ to $\mathrm{CSP}(\mathfrak{A})$; in fact, the converse is true as well, see Dalmau and Opršal [21].

Definition 2.3. Let \mathcal{B} be a class of finite τ -structures and let \mathcal{C} be a class of finite ρ structures. Then a (*d*-dimensional) gadget reduction from \mathcal{C} to \mathcal{B} consists of conjunctive query ϕ_R of arity dk over the signature τ for every $R \in \rho$ of arity k. This defines the following map r from finite ρ -structures \mathfrak{C} to finite τ -structures:

- replace each element c of \mathfrak{C} by a d-tuple $((c, 1), \ldots, (c, d));$
- for every $R \in \rho$ of arity k and every tuple $(t_1, \ldots, t_k) \in R^{\mathfrak{C}}$, introduce a new element for every existentially quantified variable in ϕ_R and define relations for the relation symbols from τ such that the substructure induced by the new elements and $\{(t_1, 1), \ldots, (t_1, d), \ldots, (t_k, 1), \ldots, (t_k, d)\}$ induce a copy of the canonical database of ϕ_R in the natural way. Let \mathfrak{C}' be the resulting structure.
- let S be the smallest equivalence relation that contains all ordered pairs of elements ((c, i), (d, j)) such that there exists $R \in \rho$ and $(t_1, \ldots, t_k) \in R^{\mathfrak{C}}$ with $t_p = c$, $t_q = d$ such that $\phi_R(x_{1,1}, \ldots, x_{k,d})$ contains the conjunct $x_{p,i} = x_{q,j}$. Then $r(\mathfrak{C}) := \mathfrak{C}'/S$.

For instance, the solution to the Feder-Vardi conjecture mentioned in the introduction states that $CSP(\mathfrak{B})$, for a finite structure \mathfrak{B} , is NP-hard if and only if \mathfrak{K}_3 has a pp-construction in \mathfrak{B} (unless P=NP); by what we have stated above, this is true if and only if the 3-coloring problem has a gadget reduction in $CSP(\mathfrak{B})$.

Given the fundamental importance of conjunctive queries and homomorphisms in database theory, we believe that pp-constructions and gadget reductions are an interesting concept for database theory as well.

2.4 Datalog

Let τ and ρ be finite relational signatures such that $\tau \subseteq \rho$. A *Datalog program* is a finite set of rules of the form

$$\phi_0 := \phi_1, \ldots, \phi_n$$

where each ϕ_i is an atomic τ -formula. The formula ϕ_0 is called the *head* of the rule, and the sequence ϕ_1, \ldots, ϕ_n is called the *body* of the rule. The symbols in τ are called *EDBs* (*extensional database predicates*) and the other symbols from ρ are called *IDBs* (*intensional database predicates*). In the rule heads, only IDBs are allowed. There is one special IDB of arity 0, which is called the *goal predicate*. IDBs might also appear in the rule bodies. We view the set of rules as a recursive specification of the IDBs in terms of the EDBs – for a detailed introduction, see, e.g., [10]. A Datalog program is called

• *linear* if in each rule, at most one IDB appears in the body.

- arc if each rule involves at most one EDB.
- symmetric if it is linear and for every rule $\phi_0 := \phi_1, \phi_2, \ldots, \phi_n$ where ϕ_0 and ϕ_1 are build from IDBs, the Datalog program also contains the reversed rule $\phi_1 := \phi_0, \phi_2, \ldots, \phi_n$.

If \mathfrak{B} is a τ -structure, then we say that $CSP(\mathfrak{B})$ is *solved* by a Datalog program Π with EDBs τ if and only if the goal predicate is derived by Π on a finite τ -structure \mathfrak{A} if and only if there is *no* homomorphism from \mathfrak{A} to \mathfrak{B} .

We say that a Datalog program has width (ℓ, k) if all IDBs have arity at most ℓ , and if every rule has at most k variables. For given (ℓ, k) and a structure \mathfrak{B} , there exists a Datalog program II of width (ℓ, k) with the remarkable property that if some Datalog program solves $\operatorname{CSP}(\mathfrak{B})$, then II solves $\operatorname{CSP}(\mathfrak{B})$. This Datalog program is referred to as the canonical Datalog program for $\operatorname{CSP}(\mathfrak{B})$ of width (ℓ, k) , and is constructed as follows [25]: For every relation R over B of arity at most ℓ , we introduce a new IDB. The empty relation of arity 0 plays the role of the goal predicate. Then II contains all rules $\phi := -\phi_1, \ldots, \phi_n$ with at most k variables such that the formula $\forall \bar{x}(\phi_1 \land \cdots \land \phi_n \Rightarrow \phi)$ holds in the expansion of \mathfrak{B} by all IDBs. If the canonical Datalog program for \mathfrak{B} derives the goal predicate on a finite structure \mathfrak{A} , then there is no homomorphism from \mathfrak{A} to \mathfrak{B} (see, e.g., [10]).

If k is the maximal arity of the EDBs, we may restrict the canonical Datalog program of width (1, k) to those rules with only unary IDBs and at most one EDB; in this case, we obtain the canonical arc monadic Datalog program, which is also known as the *arc consistency procedure*. Analogously, we may define the canonical *linear*, and the canonical *symmetric* Datalog program. We may also combine these restrictions, and in particular obtain a definition the *canonical slam Datalog program*, i.e., the canonical symmetric linear arc monadic Datalog program, which has not yet been studied in the literature before.

The following lemma can be shown analogously to the well-known fact for unrestricted canonical Datalog programs of width (ℓ, k) (see, e.g., [10]).

Lemma 2.4. Let \mathfrak{B} be a finite structure with a finite relational signature, and let Π be the canonical slam Datalog program for \mathfrak{B} . If \mathfrak{A} is a finite structure with a homomorphism to \mathfrak{B} , then Π does not derive false on \mathfrak{A} .

2.5 The Incidence Graph

Several results from graph theory concerning acyclicity and high girth can be generalised to general structures. To formulate these generalisations, we need the concept of an *incidence graph* of a relational structure \mathfrak{A} .

The *incidence graph* of a structure \mathfrak{A} with the relational signature τ is the bipartite graph where one color class is A, and the other consists of all pairs of the form (t, R) such that $t \in R^{\mathfrak{A}}$ and $R \in \tau$. We put an edge between a and (t, R) if $t_i = a$ for some i.

The girth of an (undirected) graph \mathfrak{G} is the length of the shortest cycle in \mathfrak{G} . We say that a relational structure is a generalised tree if its incidence graph is a tree. A *leaf* of a generalised tree \mathfrak{T} is an element of T which has degree one in the incidence graph of \mathfrak{T} .

A structure \mathfrak{B} is called *injective* if all tuples that are in some relation in \mathfrak{B} are injective (i.e., have no repeated entries). A structure is called an *(injective) tree* if it is injective and its incidence graph is a tree.

Theorem 2.5 (Sparse incomparability lemma for structures [25]). Let τ be a finite relational signature. Let \mathfrak{A} and \mathfrak{B} be τ -structure with finite domains such that $\mathfrak{A} \not\to \mathfrak{B}$. Then for every $m \in \mathbb{N}$ there exists an injective finite structure \mathfrak{A}' whose incidence graph has girth at least m, such that $\mathfrak{A}' \to \mathfrak{A}$ and $\mathfrak{A}' \not\to \mathfrak{B}$.

2.6 Dualities

For a τ -structure \mathfrak{B} and a class of τ -structures \mathcal{F} the pair $(\mathcal{F}, \mathfrak{B})$ is called a *duality pair* if a finite structure \mathfrak{A} has a homomorphism to \mathfrak{B} if and only if no structure $\mathfrak{F} \in \mathcal{F}$ has a homomorphism to \mathfrak{B} . Several forms of duality pairs will be relevant here, depending on the class of structures \mathcal{F} .

A τ -structure \mathfrak{B} has *finite duality* if there exists a finite set of τ -structures \mathcal{F} such that $(\mathcal{F}, \mathfrak{B})$ is a duality pair. The property of having finite duality is among the very few notions studied in the context of constraint satisfaction which is *not* preserved under gadget reductions, as illustrated in the following example.

Example 2.6. Let $\tau = \{E\}$ be the signature that consists of a single binary relation symbol E whose elements we call *edges*. Let \mathfrak{P}_n be the structure with the domain $\{1, 2, \ldots, n\}$ and edges $\{(1, 2), (2, 3), \ldots, (n - 1, n)\}$. Let \mathfrak{Z}_n be the structure with the domain $\{1, 2, \ldots, 2n + 3\}$ and edges

$$\{(1,2), (2,3), (4,3), (4,5), \dots, (2n-2, 2n-1), (n-1,n)\}.$$

Then

- \mathfrak{P}_2 has finite duality, witnessed by the duality pair ({ \mathfrak{P}_3 }, \mathfrak{P}_2),
- \mathfrak{P}_2 pp-constructs \mathfrak{P}_3 [11], so $\mathrm{CSP}(\mathfrak{P}_3)$ has a gadget reduction to $\mathrm{CSP}(\mathfrak{P}_2)$, but
- \mathfrak{P}_3 does not have finite duality: this is witnessed by the fact that

$$(\{\mathfrak{Z}_n \mid n \in \mathbb{N}\}, \mathfrak{P}_3)$$

is a duality pair [26], and that there is no homomorphism from \mathfrak{Z}_n to \mathfrak{Z}_m for n < m.

$$((6), P)$$

$$((4, 5), E)$$

$$((3, 6), E)$$

$$((3, 6), E)$$

$$((3, 6), E)$$

$$((3, 6), E)$$

$$((3), P)$$

$$((2, 3, 4), R)$$

$$((1, 2, 3), R)$$

$$((3), P)$$

$$((3), P)$$

$$((2, 3, 4), R)$$

$$((1, 2), E)$$

$$((1, 2), E)$$

$$((1, 2), E)$$

Figure 1: An example of the incidence graph of a caterpillar (left) and of a structure that is **not** a caterpillar (right).

Example 2.7. Let $\rho = \{E, Z\}$ be the signature that consists of a binary relation symbol E and a unary relation symbol Z. Let \mathfrak{B}_2 be the structure with domain $\{0, 1\}$ where

$$E^{\mathfrak{B}_2} \coloneqq \{(1,1), (0,1), (1,0)\}$$
$$Z^{\mathfrak{B}_2} \coloneqq \{0\}$$

Let \mathfrak{P}'_2 be the ρ -expansion of \mathfrak{P}_2 where $Z^{\mathfrak{P}'_2} \coloneqq \{0,1\}$. Then $(\{\mathfrak{P}'_2\},\mathfrak{B}_2)$ is a duality pair.

A more robust form of duality is *tree duality*, which plays a central role in constraint satisfaction, and is studied in the graph homomorphism literature in the 90s. A structure \mathfrak{B} has *tree duality* if there exists a (not necessarily finite) set of trees \mathcal{F} such that $(\mathcal{F}, \mathfrak{B})$ is a duality pair. The following is well known; see Theorem 7.4 in [15]. The equivalence of 1. and 3. is from [25]; also see [7].

Theorem 2.8. Let \mathfrak{B} be a finite τ -structure. Then the following are equivalent:

- 1. B has tree duality;
- 2. \mathfrak{B} has a pp-construction in $(\{0,1\};\{0\},\{1\},\{0,1\}^3 \setminus \{(1,1,0)\});$
- 3. \mathfrak{B} can be solved by arc consistency.

There are finite structures with tree duality that have a P-complete CSP, such as the structure $(\{0,1\};\{0\},\{1\},\{0,1\}^3 \setminus \{(1,1,0)\})$, which is essentially the Boolean HornSAT problem. In the following, we therefore introduce more restrictive forms of dualities.

Definition 2.9. A relational structure \mathfrak{A} is called a *generalised caterpillar* if

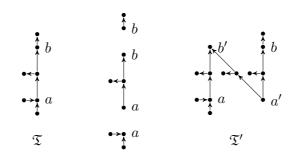


Figure 2: Example of an (a, b)-unfolding \mathfrak{T}' of a tree \mathfrak{T} .

- it is a generalised tree,
- its incidence graph \mathfrak{G} contains a path $P = (a_1, \ldots, a_n)$ such that every vertex in $G \setminus \{a_1, \ldots, a_n\}$ of the form (t, R) is connected (in \mathfrak{G}) to a vertex in P.

See Figure 1 (left) for an example. A relational structure \mathfrak{A} is called an *(injective)* caterpillar if it is injective and a generalised caterpillar (this definition of a caterpillar is equivalent to the one given in [17]). A structure \mathfrak{B} has caterpillar duality if there exists a set of caterpillars \mathcal{F} such that $(\mathcal{F}, \mathfrak{B})$ is a duality pair. The structures \mathfrak{B} with caterpillar duality have been characterised by [17] (Theorem 2.18) in terms of linear arc Datalog.

To capture the power of symmetric linear arc Datalog, we present a more restrictive form of duality. Let \mathfrak{T} be an (injective) tree and let a and b be distinct elements of \mathfrak{T} . Write the canonical query of \mathfrak{T} as $\phi_a \wedge \phi_{a,b} \wedge \phi_b$ where ϕ_a contains all conjuncts of the form $R(\bar{u})$ such that in the incidence graph, all paths from the vertex (\bar{u}, R) to the vertex b pass though a. Similarly, we define ϕ_b , switching the roles of a and b. Note that ϕ_a and ϕ_b do not share any conjuncts. All the remaining conjuncts of the canonical query form ψ . Let ϕ_1 be obtained from ϕ_a (ϕ_b) by existentially quantifying all variables except for a (b), and let ψ be obtained from $\phi_{a,b}$ by existentially quantifying all variables except for a and b. We introduce the following concept; see Figure 2 for an example.

Definition 2.10. Let \mathfrak{T} be an (injective) tree and let a and b be distinct elements of \mathfrak{T} that are not leaves. The (a, b)-unfolding of \mathfrak{T} is the canonical database of the formula

$$\phi_1(a) \wedge \psi(a, b') \wedge \psi(a', b') \wedge \psi(a', b) \wedge \phi_2(b)$$

where ϕ_1 , ϕ_2 , and ψ are as defined above. A *unfolding* of \mathfrak{T} is a structure \mathfrak{T}' that is obtained by a sequence $\mathfrak{T} = \mathfrak{T}_1, \mathfrak{T}_2, \ldots, \mathfrak{T}_n = \mathfrak{T}'$ such that \mathfrak{T}_i is an (a_i, b_i) -unfolding of \mathfrak{T}_{i-1} , for all $i \in \{2, \ldots, n\}$.

Note that an unfolding of a tree \mathfrak{T} is again a tree and has a homomorphism to \mathfrak{T} . It can also be shown that the unfolding of a caterpillar is a caterpillar as well (Lemma 3.7). We say that a structure \mathfrak{B} has *unfolded caterpillar duality* if there exists a set \mathcal{F} of caterpillars such that $(\mathcal{F}, \mathfrak{B})$ is a duality pair, and \mathcal{F} contains every unfolding of a caterpillar in \mathcal{F} . Clearly, unfolded caterpillar duality implies caterpillar duality. Unfolded generalised caterpillar duality is defined analogously.

2.7 Minor conditions

If \mathfrak{B} is a structure and $k \geq 1$, then a polymorphism of \mathfrak{B} of arity k is a homomorphism from \mathfrak{B}^k to \mathfrak{B} . The set of all polymorphisms of \mathfrak{B} is denoted by $\operatorname{Pol}(\mathfrak{B})$. An *(operation)* clone is a set of operations which contains the projections and is closed under composition. Note that $\operatorname{Pol}(\mathfrak{B})$ is a clone. An operation $f: B^n \to B$ is called *idempotent* if $f(x, \ldots, x) = x$ for all $x \in B$. A clone is called *idempotent* if all of its operations are idempotent.

If \mathfrak{A} and \mathfrak{B} are a structures and $k \geq 1$, then a polymorphism of $(\mathfrak{A}, \mathfrak{B})$ of arity k is a homomorphism from \mathfrak{A}^k to \mathfrak{B} . The set of all polymorphisms of $(\mathfrak{A}, \mathfrak{B})$ is denoted by $\operatorname{Pol}(\mathfrak{A}, \mathfrak{B})$. Let $f: A^n \to B$ be a function and let $\sigma: [n] \to [m]$, then the map

$$f_{\sigma} \colon A^m \to B$$

 $(a_1, \dots, a_m) \mapsto f(a_{\sigma(1)}, \dots, a_{\sigma(n)})$

is called a *minor* of f. A *minion* is a set or functions from A^n to B that is closed under taking minors. Note that $Pol(\mathfrak{A}, \mathfrak{B})$ is a minion. Let \mathscr{M} and \mathscr{N} be minions. A map $\xi \colon \mathscr{M} \to \mathscr{N}$ is called a *minion homomorphism* if ξ preserves arity and for every $f \in \mathscr{M}$ of arity n and every map $\sigma \colon [n] \to [m]$ we have

$$\xi(f_{\sigma}) = (\xi(f))_{\sigma}.$$

Let τ be a function signature, i.e., a set of function symbols, each equipped with an arity. A minor condition is a finite set Σ of minor identities, i.e., expressions of the form

$$f(x_1,\ldots,x_n)\approx g(y_1,\ldots,y_m)$$

where f is an n-ary function symbol from τ , g is an m-ary function symbol from τ , and $x_1, \ldots, x_n, y_1, \ldots, y_m$ are (not necessarily distinct) variables. If \mathscr{M} is a minion, then a map $\xi \colon \tau \to \mathscr{M}$ satisfies a minor condition Σ if for every minor identity $f(x_1, \ldots, x_n) \approx g(y_1, \ldots, y_m) \in \Sigma$ and for every assignment $s \colon \{x_1, \ldots, x_n, y_1, \ldots, y_n\} \to B$ we have

$$\xi(f)(s_1(x_1), \dots, s(x_n)) = \xi(g)(s(y_1), \dots, s(y_m)).$$

We say that a minion \mathscr{M} satisfies Σ if there exists a map $\xi \colon \tau \to \mathscr{M}$ that satisfies Σ .² If Σ and Σ' are minor conditions, then we say that Σ implies Σ' if every clone that satisfies Σ also satisfies Σ' . We present some concrete minor conditions that are relevant in the following.

²It is convenient and standard practise to notationally drop the distinction between $f \in \tau$ and $\xi(f) \in \mathcal{M}$.

Definition 2.11. An operation $m: B^3 \to B$ is called a *quasi Maltsev operation* if it satisfies the minor condition

$$m(x, x, y) \approx m(y, x, x) \approx m(y, y, y).$$

A Maltsev operation is an idempotent quasi Maltsev operation. A quasi minority operation is a quasi Maltsev operation m that additionally satisfies

$$m(x, y, x) \approx m(x, x, x)$$

and a *minority operation* is an idempotent quasi minority operation.

Definition 2.12. An operation $f: B^m \to B$ is called a *quasi majority operation* if it satisfies the minor condition

$$m(x, x, y) \approx m(x, y, x) \approx m(y, x, x) \approx m(x, x, x).$$

A majority operation is an idempotent quasi majority operation.

The following was shown in in [25, 29]; for the exact formulation, see [9].

Proposition 2.13. Let \mathfrak{B} be a finite relational τ -structure. Then the following are equivalent:

- \mathfrak{B} has a quasi majority polymorphism.
- Every relation with a primitive positive definition in \mathfrak{B} has a definition by a conjunction of primitive positive formulas, each with at most two free variables.

Example 2.14. Generalising Example 2.7, the structure \mathfrak{B}_n has domain $\{0, 1\}$, signature $\{\mathbf{0}, R_n\}$ and the relations $\{0\}$ and $\{0, 1\}^n \setminus \{(0, \ldots, 0)\}$. Define the structure \mathfrak{F}_n with domain $\{1, \ldots, n\}$, signature $\{\mathbf{0}, R_n\}$, $\mathbf{0} \coloneqq \{1, \ldots, n\}$, and $R_n \coloneqq \{(1, \ldots, n)\}$. Note that $(\{\mathfrak{F}_n\}, \mathfrak{B}_n)$ is a duality pair and that \mathfrak{F}_n is a tree, but not a caterpillar. The structure \mathfrak{B}_n has finite duality, but no quasi Maltsev polymorphism and no quasi majority polymorphism. It follows from Theorems 2.18 and 3.1 that $\mathrm{CSP}(\mathfrak{B})$ can be solved by linear arc monadic Datalog but not by slam Datalog.

Definition 2.15. Let $k, n \in \mathbb{N}_{>0}$. An operation $f: B^{kn} \to B$ is called *k*-block symmetric if it satisfies the following condition

$$f(x_{11}, \dots, x_{1k}, \dots, x_{n1}, \dots, x_{nk}) \approx f(y_{11}, \dots, y_{1k}, \dots, y_{n1}, \dots, y_{nk})$$
(1)

whenever $\{S_1, \ldots, S_n\} = \{T_1, \ldots, T_n\}$ where $S_i = \{x_{i1}, \ldots, x_{ik}\}$ and $T_i = \{y_{i1}, \ldots, y_{ik}\}$. If k = 1 or n = 1 then f is called *totally symmetric*.

If f is k-block symmetric and S_1, \ldots, S_n are subsets of B of size at most k, then we also write $f(S_1, \ldots, S_n)$ instead of $f(x_{11}, \ldots, x_{1k}, \ldots, x_{n1}, \ldots, x_{nk})$ where $\{x_{i1}, \ldots, x_{ik}\} = S_i$.

We say that f is k-absorptive if it satisfies

$$f(S_1, S_2, \ldots, S_n) \approx f(S_2, S_2, S_3, \ldots, S_n)$$

whenever $S_2 \subseteq S_1$.

Remark 2.16. Note that every structure with a 2-absorptive polymorphism f of arity 6 also has the quasi majority polymorphism m given by

$$m(x, y, z) \coloneqq f(x, y, z, x, y, z),$$

because $\{x\} \subseteq \{x, z\}$ and hence

$$m(x, x, z) = f(x, x, z, x, x, z) = f(x, x, x, x, x, x) = m(x, x, x)$$

and similarly m(x, z, x) = m(z, x, x) = m(x, x, x).

The list of equivalent statements from Theorem 2.8 can now be extended as follows.

Theorem 2.17 ([22, 25]). Let \mathfrak{B} be a finite τ -structure. Then \mathfrak{B} has tree duality if and only if \mathfrak{B} has totally symmetric polymorphisms of all arities.

We will make crucial use of the following theorem.

Theorem 2.18 (Theorem 16 in [17]). Let \mathfrak{B} be a finite relational τ -structure. Then the following are equivalent.

- 1. \mathfrak{B} has caterpillar duality.
- 2. $\operatorname{CSP}(\mathfrak{B})$ can be solved by a linear arc monadic Datalog program.
- 3. Pol(\mathfrak{B}) contains for every $k, n \geq 1$ an k-absorbing operation of arity kn.
- 4. \mathfrak{B} is homomorphically equivalent to a structure \mathfrak{B}' with binary polymorphisms \sqcup and \sqcap such that (B', \sqcup, \sqcap) is a (distributive) lattice.

2.8 Indicator structures

In this section we revisit a common theme in constraint satisfaction, the concept of an *indicator structure* of a minor condition. To simplify the presentation, we only define the indicator structure for minor conditions with only one function symbol. For our purposes, this is without loss of generality, because for clones over a finite domain, every minor condition is equivalent to such a restricted minor condition. If $f: C^n \to C$ and $g: C^m \to C$ are operations, then the *star product* f * g is defined to be the operation defined as

$$(x_{1,1},\ldots,x_{n,m})\mapsto f(g(x_{1,1},\ldots,x_{1,m}),\ldots,g(x_{n,1},\ldots,x_{n,m})).$$

Lemma 2.19. Let Σ be a minor condition. Then there exists a minor condition Σ' with a single function symbol such that a clone over a finite domain satisfies Σ if and only if it satisfies Σ' .

Proof. First note that for every clone \mathscr{D} on a finite set there exists an *idempotent* clone \mathscr{C} on a finite set which is equivalent to it with respect to minion homomorphisms, i.e., there are minion homomorphism from \mathscr{D} to \mathscr{C} and vice versa. It is well-known and easy to see that if f_1, \ldots, f_n are the function symbols that appear in Σ , and \mathscr{C} satisfies Σ , \mathscr{C} also contains an operation g of arity m such that for every $i \in [n]$ there exists $\alpha_i \colon [m] \to [k]$ such that $g_{\alpha_i} = f_i$ (use that \mathscr{C} is closed under the star product and idempotent). Note that \mathscr{C} satisfies a minor identity $(f_i)_\beta \approx (f_j)_\gamma$ if and only if \mathscr{C} satisfies a minor identity $(g_{\alpha_i})_\beta \approx (g_{\alpha_j})_\gamma$.

If \mathfrak{B} is a relational τ -structure and \sim is an equivalence relation on B, the $\mathfrak{B}/_{\sim}$ is the τ -structure whose domain are the equivalence classes of \sim , and where $R(C_1, \ldots, C_k)$ holds if there exist $a_1 \in C_1, \ldots, a_k \in C_k$ such that $R(a_1, \ldots, a_k)$ holds in \mathfrak{B} .

Definition 2.20. Let \mathfrak{B} be a relational τ -structure and let Σ be a minor condition with a single function symbol f of arity m. Let \sim be the smallest equivalence relation on B^m such that $a \sim b$ if Σ contains $f(x_1, \ldots, x_m) \approx f(y_1, \ldots, y_m)$ such that there is a map $s: \{x_1, \ldots, x_m, y_1, \ldots, y_m\} \to B$ with $a = (s(x_1), \ldots, s(x_m))$ and $b = (s(y_1), \ldots, s(y_m))$. Then the *indicator structure of* Σ with respect to \mathfrak{B} is the τ -structure $\mathfrak{B}^m/_{\sim}$.

The following is straightforward from the definitions.

Lemma 2.21. Let \mathfrak{B} be a structure and Σ be a minor condition with a single function symbol f. Then \mathfrak{B} has a polymorphism satisfying Σ if and only if the indicator structure of Σ with respect to \mathfrak{B} has a homomorphism to \mathfrak{B} .

3 Results

In this section we state and prove our main result (Theorem 3.1), which characterises the power of slam Datalog in many different ways, including descriptions in terms of ppconstructability in \mathfrak{P}_2 , minor conditions, unfolded caterpillar duality, and homomorphic equivalence to a structure with both lattice and quasi Maltsev polymorphisms.

Theorem 3.1. Let \mathfrak{B} be a structure with a finite domain and a finite relational signature τ . Then the following are equivalent.

- 1. $Pol(\mathfrak{B})$ contains a quasi Maltsev operation and k-absorptive operations of arity nk, for all $n, k \geq 1$.
- 2. The canonical slam Datalog program for \mathfrak{B} solves $CSP(\mathfrak{B})$.
- 3. Some slam Datalog program solves $CSP(\mathfrak{B})$.

- 4. \mathfrak{B} has unfolded caterpillar duality.
- 5. If $\operatorname{Pol}(\mathfrak{B})$ does not satisfy a minor condition Σ , then Σ implies $f(x) \approx f(y)$.
- 6. Every minor condition that holds in $Pol(\mathfrak{P}_2)$ also holds in $Pol(\mathfrak{B})$.
- 7. There is a minion homomorphism from $\operatorname{Pol}(\mathfrak{P}_2)$ to $\operatorname{Pol}(\mathfrak{B})$.
- 8. There is a pp-construction of \mathfrak{B} in \mathfrak{P}_2 .
- 9. \mathfrak{B} is homomorphically equivalent to a structure \mathfrak{B}' such that $\operatorname{Pol}(\mathfrak{B}')$ contains a quasi Maltsev operation and operations \sqcup and \sqcap such that (B', \sqcup, \sqcap) forms a (distributive) lattice.

Moreover, if one of these items holds, then there exists a structure \mathfrak{B}' with a binary relational signature such that $\operatorname{Pol}(\mathfrak{B}') = \operatorname{Pol}(\mathfrak{B})$, and all the statements hold for \mathfrak{B}' in place of \mathfrak{B} as well.

We first prove the equivalence of (1)-(6) in cyclic order. We then explain how the equivalence of (6)-(8) follows from general results in the literature, and finally show the equivalence of (1) and (9). The proof of the theorem stretches over the following subsections.

Example 3.2. The structure \mathfrak{T}_n is the transitive tournament with n vertices, i.e., it has the domain [n] and the binary relation <. Note that \mathfrak{T}_2 equals \mathfrak{P}_2 . It is easy to see that $(\{\mathfrak{P}_{n+1}\},\mathfrak{T}_n)$ is a duality pair. Since \mathfrak{P}_{n+1} is a caterpillar Theorem 2.18 implies that $\mathrm{CSP}(\mathfrak{T}_n)$ can be solved by a linear arc monadic Datalog program. However, \mathfrak{T}_n does not have a quasi Maltsev polymorphism for $n \geq 3$, and hence Theorem 3.1 implies that $\mathrm{CSP}(\mathfrak{T}_n)$ cannot be solved by slam Datalog.

3.1 Symmetrizing Linear Arc Monadic Datalog

The following lemma is used for the implication from (1) to (2) in the proof of Theorem 3.1. Note that in the canonical linear arc monadic Datalog program Π we can use the 'strongest possible rules'³ when deriving the goal predicate. However, the canonical slam Datalog program Π_S might need to use weaker rules in order to be able to apply symmetric rules later on in the derivation. See Example 3.5.

Lemma 3.3. Let \mathfrak{B} be a finite structure with relational signature τ such that $\operatorname{Pol}(\mathfrak{B})$ contains a quasi Maltsev operation. Let Π be the canonical linear arc monadic Datalog program for \mathfrak{B} and Π_S be the canonical slam Datalog program for \mathfrak{B} . Then Π can derive the goal predicate on a finite τ -structure \mathfrak{A} if and only if Π_S can derive the goal predicate on \mathfrak{A} .

³These comments are intended to illustrate the challenges in the proof of next lemma; it will not be necessary to formalise what we mean by strongest possible rules.

Proof. Note that every rule of Π_S is also a rule of Π . Hence if Π_S can derive the goal predicate on \mathfrak{A} , then so can Π . Let

$$\vdash_{R_0} P_0(a_0) \vdash_{R_1} \cdots \vdash_{R_n} P_n(a_n) \vdash_{R_{n+1}} G$$

be a derivation of Π on \mathfrak{A} that derives the goal predicate G. For $i \in [n]$ define the primitive positive formula Φ_i as follows. Suppose that the rule R_i is of the form $P_i(y) := \Psi_i(x_1, \ldots, x_k) \wedge P_{i-1}(x)$ for some atomic formula Ψ_i . We may assume that both x and y are among the variables x_1, \ldots, x_k ; otherwise, the canonical database of $P_i(y) \wedge \Psi_i(x_1, \ldots, x_k) \wedge P_{i-1}(x)$ is not connected. Since Π solves a CSP we may assume without loss of generality that such rules R_i were not used in the derivation of the goal predicate.

- If $x \neq y$, then $\Phi_i(x, y)$ is obtained from $\Psi_i(x_1, \ldots, x_k)$ by existentially quantifying all variables except for x and y,
- otherwise, x = y and $\Phi_i(x, x')$ is obtained from $\Psi_i(x_1, \ldots, x_k)$ by existentially quantifying all variables except for x and adding the conjunct x = x'.

Define $\Phi_0(x)$ and $\Phi_{n+1}(x)$ from the rules R_0 and R_{n+1} in a similar fashion. For $i \in [n]$ define the binary relation \rightarrow_i on B that contains all tuples (b, b') such that $\mathfrak{B} \models \Phi_i(b, b')$. We may assume without loss of generality that

$$P_0 = \{b \in B \mid \mathfrak{B} \models \Phi_0(b)\} = \Phi_0^{\mathfrak{B}}$$

and that for all $b' \in P_{i+1}$ there exists a $b \in P_i$ such that $b \to_i b'$. In particular, this implies that P_0 is pp-definable.

Let $Q_0, \ldots, Q_n \subseteq B$ be the smallest sets such that

- $P_i \subseteq Q_i$,
- $b \in Q_i$ and $b \to_i b'$ implies $b' \in Q_{i+1}$, and
- $b' \in Q_{i+1}$ and $b \to_i b'$ implies $b \in Q_i$.

Note that there can be $b \in Q_i$ such that there is no $b' \in B$ with $b \to_i b'$. Analogously, there can be $b' \in Q_{i+1}$ such that there is no $b \in B$ with $b \to_i b'$. Let $\tilde{R}_0, \ldots, \tilde{R}_{n+1}$ be the rules obtained from R_0, \ldots, R_{n+1} by replacing each occurrence of P_i by Q_i for all $i \in [0, n]$. We will now show that

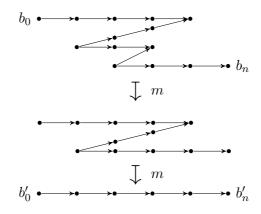
$$\vdash_{\tilde{R}_0} Q_0(a_0) \vdash_{\tilde{R}_1} \cdots \vdash_{\tilde{R}_n} Q_n(a_n) \vdash_{\tilde{R}_{n+1}} G$$

is a derivation of Π_S on \mathfrak{A} . It suffices to show that \tilde{R}_i is a rule of Π_S for all $i \in [0, n+1]$. By definition $P_0 \subseteq Q_0$. Hence $\mathfrak{B} \models \forall x(\Phi_0(x) \Rightarrow P_0(x))$ implies $\mathfrak{B} \models \forall x(\Phi_0(x) \Rightarrow Q_0(x))$. Therefore, \tilde{R}_0 is a rule of Π_S .

Let $i \in [n]$. To show that $\mathfrak{B} \models \forall x((\Phi_i(x, y) \land Q_{i-1}(x)) \Rightarrow Q_i(y))$ let $b \in Q_{i-1}$ and $b' \in B$ be such that $\mathfrak{B} \models \Phi_i(b, b')$. Then $b \to_i b'$ and $b' \in Q_i$ by the definition of Q_i . Analogously, we show that $\mathfrak{B} \models \forall x((\Phi_i(x, y) \land Q_i(x)) \Rightarrow Q_{i-1}(y))$. Therefore, \tilde{R}_i is a rule of Π_S .

To prove that R_{n+1} is a rule in Π_S we will show that for all $b_n \in Q_n$ we have $\mathfrak{B} \not\models \Phi_{n+1}(b_n)$. Let $b_n \in Q_n$ and assume that $\mathfrak{B} \models \Phi_{n+1}(b_n)$. By the definition of Q_0, \ldots, Q_n and the condition that for all $b' \in P_{i+1}$ there exists a $b \in P_i$ such that $b \to_i b'$ we know that there exists a $b_0 \in P_0$ such that b_0 and b_n are connected by the symmetric transitive closure of $\bigcup_{i=1}^n \to_i$.

Let *m* be a quasi Maltsev polymorphism of \mathfrak{B} . Applying *m* repeatedly to the connection of b_0 and b_n as indicated in the following picture:



we can conclude that there exist $b'_0, b'_1, \ldots, b'_n \in B$ such that

$$b'_0 \to_1 b'_1 \to_2 \dots \to_n b'_n$$

and $\mathfrak{B} \models \Phi_{n+1}(b'_n)$. Since P_0 is pp-definable, it is preserved by m and therefore $b'_0 \in P_0$. Hence, $b'_2 \in P_2, \ldots, b'_n \in P_n$. This contradicts that R_{n+1} is a rule of Π (as $\mathfrak{B} \models \Phi_{n+1}(b'_n) \wedge P_n(b'_n)$). Hence, $\mathfrak{B} \not\models \Phi_{n+1}(b_n)$ and \tilde{R}_{n+1} is a rule of Π_S as desired. \square

Remark 3.4. Observe that the canonical database of

$$\Phi_0(x_0) \wedge \Phi_1(x_0, x_1) \wedge \dots \wedge \Phi_n(x_{n-1}, x_n) \wedge \Phi_{n+1}(x_n)$$

is a generalised caterpillar.

Example 3.5. Consider the structure \mathfrak{B} with domain $\{0, 0', 1, a, b, b'\}$, binary relation $E = \{(0, 1), (0', 1), (a, b), (a, b')\}$, and all constants. Let \mathfrak{A} be the structure with domain $\{0, b\}, E = \{(0, b)\}$, and all constants. Clearly, $\mathfrak{A} \not\to \mathfrak{B}$. The canonical linear arc monadic Datalog program Π for \mathfrak{B} can derive the goal predicate using the derivation

$$\vdash_{R_0} \{0\}(0) \vdash_{R_1} \{1\}(b) \vdash_{R_2} G.$$

The canonical slam Datalog program Π_S for \mathfrak{B} can also derive the goal predicate on \mathfrak{A} . It cannot use the rule R_1 , because R_1 is not symmetric. However, it may use a different rule \tilde{R}_1 :

$$\vdash_{\tilde{R}_0} \{0, 0'\}(0) \vdash_{\tilde{R}_1} \{1\}(b) \vdash_{R_2} G$$

Note that R_0 is also a rule of Π_S but in order to apply \tilde{R}_1 the program needs to use the rule \tilde{R}_0 which is weaker than the rule R_0 (in the sense that the derived IDB is a strict superset).

3.2 Proving Unfolded Caterpillar Duality

This section is devoted to the proof of the implication (3) to (4) in Theorem 3.1. We first prove a general result about obstruction sets for finite-domain CSPs that closes a gap in the presentation of the proof of Lemma 21 in [17] and essentially follows from the sparse incomparability lemma (Theorem 2.5); we thank Víctor Dalmau for clarification.

Lemma 3.6. Let \mathfrak{B} be a finite structure and let \mathcal{F} be a class of finite structures such that $(\mathcal{F}, \mathfrak{B})$ is a duality pair. Define

$$\mathcal{F}' \coloneqq \{\mathfrak{F} \in \mathcal{F} \mid \mathfrak{F} \text{ is injective}\}.$$

Then $(\mathcal{F}', \mathfrak{B})$ is a duality pair.

Proof. Let τ be the signature of \mathfrak{B} ; let m be the maximal arity of the relation symbols in τ . Let \mathfrak{A} be a finite τ -structure which does not homomorphically map to \mathfrak{B} . By Theorem 2.5, there exists an injective finite structure \mathfrak{A}' whose incidence graph has girth at least 3, and which homomorphically maps to \mathfrak{A} but not to \mathfrak{B} . There exists a $\mathfrak{F} \in \mathcal{F}$ which homomorphically maps to \mathfrak{A}' . Since \mathfrak{A}' is injective, so is \mathfrak{F} . It follows that $\mathfrak{F} \in \mathcal{F}'$. This implies that $(\mathcal{F}', \mathfrak{B})$ is a duality pair.

Note that Theorem 2.18 implies that if $CSP(\mathfrak{B})$ is solved by a linear arc monadic Datalog program, then \mathfrak{B} has caterpillar duality; the proof given in [17] only shows generalised caterpillar duality. However, Lemma 3.6 implies that \mathfrak{B} in this case also has (injective) caterpillar duality.

In order to prove the implication (3) to (4) in Theorem 3.1, it only remains to prove that \mathfrak{B} also has *unfolded* caterpillar duality (Lemma 3.8). We first prove the following lemma, which has already been mentioned in Section 2.6.

Lemma 3.7. An unfolding of a caterpillar is a caterpillar as well.

Proof. Let \mathfrak{D} be a caterpillar and let \mathfrak{D}' be an (a, b)-unfolding of \mathfrak{D} for two non-leafs $a, b \in D$. Let $P = (a_1, \ldots, a_n)$ be a longest possible path in the incidence graph of \mathfrak{D} which shows that \mathfrak{D} is a caterpillar. Note that since P is longest possible it must pass through all non-leafs of \mathfrak{D} (see Figure 1). In particular, P passes through a and through

b; without loss of generality it can be written as

$$(\bar{u}, a, \bar{v}, b, \bar{w}).$$

Let ϕ_1 , ϕ_2 , and ψ be obtained from \mathfrak{D} as in the definition of the (a, b)-unfolding of \mathfrak{D} , so that \mathfrak{D}' is the canonical databases of $\phi_1(a) \wedge \psi(a, b') \wedge \psi(a', b') \wedge \psi(a', b) \wedge \phi_2(b)$. Note that

- (\bar{u}, a) is a path in the incidence graph of the canonical database of $\phi_1(a)$,
- (a, \overline{v}, b) is a path in the incidence graph of the canonical database of $\psi(a, b)$, and
- (b, \bar{w}) is a path in the incidence graph of the canonical database of $\phi_2(b)$.

Furthermore, each of these paths witnesses that the corresponding canonical database is a caterpillar. Let (a, \bar{v}_1, b') , (a', \bar{v}_2, b') , (a', \bar{v}_3, b) be the corresponding paths in the incidence graph of the canonical database of $\psi(a, b')$, $\psi(a', b')$, and $\psi(b', b)$, respectively. Let \tilde{v}_2 be \bar{v}_2 in reversed order. Then

$$P' \coloneqq (\bar{u}, a, \bar{v}_1, b', \tilde{v}_2, a', \bar{v}_3, b, \bar{w})$$

is a path in the incidence graph \mathfrak{G}' of \mathfrak{D}' . We claim that P' witnesses that \mathfrak{D}' is a caterpillar. This follows from the observation that if \mathfrak{C}_i , for $i \in \{1, 2\}$, is a caterpillar with witnessing path $P_i = (\bar{u}_i, a_i)$ such that $a_i \in C_i$, then the structure obtained by taking the disjoint union of \mathfrak{C}_1 and \mathfrak{C}_2 and identifying a_1 and a_2 is a caterpillar, witnessed by the path $(\bar{u}_1, a_1, \bar{u}_2)$.

The statement for unfoldings in general follows by induction.

Lemma 3.8. If $CSP(\mathfrak{B})$ is solved by a slam Datalog program Π , then \mathfrak{B} has unfolded caterpillar duality.

Proof. As we have explained above, Theorem 2.18 (in combination with Lemma 3.6) implies that there exists a set of caterpillars \mathcal{F} such that $(\mathcal{F}, \mathfrak{B})$ is a duality pair. Let \mathcal{F}' be the closure of \mathcal{F} by all unfoldings of caterpillars in \mathcal{F} . By Lemma 3.7, we have that \mathcal{F}' is a set of caterpillars as well. It remains to show that $(\mathcal{F}', \mathfrak{B})$ is a duality pair. Since $\mathcal{F} \subseteq \mathcal{F}'$ it suffices to show that no element in \mathcal{F}' maps homomorphically to \mathfrak{B} . Let $\mathfrak{F} \in \mathcal{F}, a, b \in F$, and \mathfrak{F}' be the (a, b)-unfolding of \mathfrak{F} (Definition 2.10). Let ϕ_1, ϕ_2 , and ψ be obtained from \mathfrak{F} as in the definition of the (a, b)-unfolding of \mathfrak{F} so that \mathfrak{F}' is the canonical database of $\phi_1(a) \land \psi(a, b') \land \psi(a', b') \land \psi(a', b) \land \phi_2(b)$. Assume without loss of generality that for every existentially quantified variable v in $\psi(a, b')$ the corresponding variables in $\psi(a', b')$ and $\psi(b', a)$ are v' and v'', respectively. Let $P_0(v_0) \vdash_{R_1} \cdots \vdash_{R_n} P_n(v_n)$ be a derivation of Π on \mathfrak{F} (assuming that Π already derived P_0 on a) with $v_0 = a, v_n = b$, and $v_1, \ldots, v_{n-1} \in F \setminus \{a, b\}$ such that P_{i-1} occurs in the body of R_i for every $i \in [n]$. Then v_1, \ldots, v_n are in the canonical database of $\psi(a, b)$. Hence v_i, v_i' , and v_i'' are elements in \mathfrak{F}' for every $i \in [n]$. For every rule R_i let R_i denote the reversed rule, which is also a rule of Π since Π is symmetric. Note that

$$P_0(a) \vdash_{R_1} \cdots \vdash_{R_n} P_n(b') \vdash_{\tilde{R}_n} P_{n-1}(v'_{n-1})$$
$$\cdots \vdash_{\tilde{R}_1} P_0(a') \vdash_{R_1} P_1(v''_1) \cdots \vdash_{R_n} P_n(b)$$

is a derivation of Π on \mathfrak{F}' . Therefore, any derivation d of Π on \mathfrak{F} deriving the IDB P on an element v can be transformed into a derivation d' of Π on \mathfrak{F}' such that

- d' derives the IDB P on v if v is in the canonical databases of $\phi_1(a)$ or $\phi_2(b)$ and
- d' derives the IDB P on v or on v'' if v is in the canonical database of $\psi(a, b)$ and $v \notin \{a, b\}$.

Since $\mathfrak{F} \in \mathcal{F}$ and Π solves $CSP(\mathfrak{B})$, we have that Π derives the goal predicate on \mathfrak{F} . Therefore, Π derives the goal predicate on \mathfrak{F}' as well. Hence, \mathfrak{F}' does not map homomorphically to \mathfrak{B} . It follows by induction that no unfolding of an element in \mathcal{F} maps homomorphically to \mathfrak{B} . Therefore, $(\mathcal{F}', \mathfrak{B})$ is a duality pair. \Box

3.3 Using Unfolded Caterpillar Duality

This section proves the implication (4) to (5) in Theorem 3.1.

Lemma 3.9. Let \mathfrak{B} be a relational structure with unfolded caterpillar duality. If \mathfrak{B} does not satisfy a minor condition Σ , then Σ implies $f(x) \approx f(y)$.

Proof. Let τ be the signature of \mathfrak{B} . Suppose that $\mathfrak{B} \not\models \Sigma$. By Lemma 2.19, we may assume that Σ only involves a single function symbol f of arity K. Let \mathfrak{I} be the indicator structure of Σ with respect to \mathfrak{B} . Then $\mathfrak{I} \not\rightarrow \mathfrak{B}$ (Lemma 2.21). By the caterpillar duality of \mathfrak{B} , there must be a caterpillar \mathfrak{C} with a homomorphism to \mathfrak{I} that does not have a homomorphism to \mathfrak{B} . In the following we will show that either there exists an unfolding of \mathfrak{C} that has a homomorphism to \mathfrak{B} , which contradicts the unfolded caterpillar duality of \mathfrak{B} , or that Σ implies $f(x) \approx f(y)$, which concludes the proof.

Let P be a path witnessing that \mathfrak{C} is a caterpillar. We may assume that P is of the form

$$(v_0, (s_1, R_1), \ldots, (s_n, R_n), v_n).$$

This assumption is without loss of generality: if P instead starts as follows

$$((s_1, R_1), v_1, \dots)$$

then either R_1 is not unary, and we may prepend to P an entry v_0 of s_1 which is different from v_1 . Or R_1 is unary, in which case (s_1, R_1) is a leaf (in the incidence graph of \mathfrak{C}) and we may simply discard this first vertex of P. In both cases, the new path still witnesses that \mathfrak{C} is a caterpillar. Analogously, we can ensure that P does not end in a vertex of the form (s_n, R_n) . Let m + 1 be the maximal degree of any v_i in the incidence graph of \mathfrak{C} . Define the primitive positive formulas ϕ_1, \ldots, ϕ_n as follows. For every $i \in [n]$ we obtain $\phi_i(v_{i-1}, v_i)$ from $R_i(s_i)$ (seen as an atomic formula) by existentially quantifying all variables except for v_{i-1} and v_i . Furthermore, for every $i \in [0, n]$ and every neighbor (t, R) of v_i in the incidence graph of \mathfrak{C} that is not in P, let $\phi(x)$ be the formula obtained from R(t) by existentially quantifying all variables except for v_i .

Let h be a homomorphism from \mathfrak{C} to \mathfrak{I} . For every $i \in [0, n]$ fix tuples $t_{i,0}, \ldots, t_{i,m+1}$ in $h(v_i) \subseteq B^K$ such that

- for every $i \in [0, n]$ and every formula $\phi(x)$ corresponding to a neighbour (t, R) of v_i in the incidence graph of \mathfrak{C} that is not in P there is a $j \in [m]$ such that $\mathfrak{B} \models \phi(t_{i,j,k})$ for all $k \in [K]$, and
- for every $i \in [n]$ we have that $\mathfrak{B} \models \phi_i(t_{i-1,m+1,k}, t_{i,0,k})$ for all $k \in [K]$.

See Figure 3 for a visualization of the tuples in an example.

Define the binary relations ~ and \rightarrow_a on $[0, n] \times [0, m + 1] \times [K]$ for $a \in [n]$: for $(c, i, k), (d, j, \ell) \in [n] \times [0, m + 1] \times [K]$ we have

- $(c, i, k) \sim (d, j, \ell)$ if and only if c = d, $|i j| \le 1$, and $t_{c,i,k} = t_{d,j,\ell}$ and
- $(c,i,k) \rightarrow_a (d,j,\ell)$ if and only if c+1 = d = a, i = m+1, j = 0, and $\mathfrak{B} \models \phi_d(t_{c,i,k}, t_{d,j,\ell})$.

We write \leftarrow_a for the converse of \rightarrow_a . Define \leftrightarrow^* as the transitive symmetric closure of $\sim \cup \rightarrow_1 \cup \cdots \cup \rightarrow_n$. Note that \leftrightarrow^* is an equivalence relation. Now we consider two cases. The first case is that there are no $k, \ell \in [K]$ such that $(0,0,k) \leftrightarrow^* (n, m+1, \ell)$. Define the map

$$\begin{aligned} \pi\colon [n]\times [0,m+1]\times [K] &\to \{x,y\} \\ (c,i,k) &\mapsto \begin{cases} x & \text{if } (c,i,k) \leftrightarrow^* (0,0,\ell) \text{ for some } \ell \\ y & \text{otherwise.} \end{cases} \end{aligned}$$

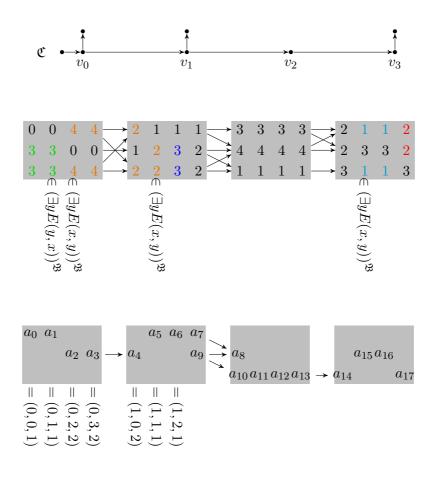
Observe that

- 1. $\pi((0,0,k)) = x$ for all $k \in [K]$ and that $\pi((n,m+1,\ell)) = y$ for all $\ell \in [K]$.
- 2. Let $c \in [n]$ and let $i \in [0, m]$. Then Σ implies $f(t_{c,i}) \approx f(t_{c,i+1})$. Also $t_{c,j_1,k_1} = t_{c,j_2,k_2}$ implies $\pi(c, j_1, k_1) = \pi(c, j_2, k_2)$. Hence Σ implies

$$f(\pi(c, i, 1), \dots, \pi(c, i, K)) \approx f(\pi(c, i + 1, 1), \dots, \pi(c, i + 1, K)).$$

3. Let $c \in [n-1]$, then $\pi(c, m+1, k) = \pi(c+1, 0, k)$ for all $k \in [K]$. Hence,

$$f(\pi(c, m+1, 1), \dots, \pi(c, m+1, K)) = f(\pi(c+1, 0, 1), \dots, \pi(c+1, 0, K)).$$



$$A_0 = \{0, 1, 2, 3\}A_1 = \{4, 5, 6, 7\}A_2 = \{8\}$$

$$A_3 = \{9\} \qquad A_4 = \{10, \dots, 13\}A_5 = \{14, \dots, 17\}$$

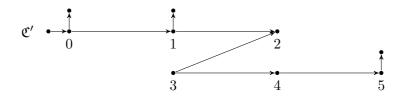


Figure 3: At the top there is a picture of a caterpillar \mathfrak{C} . Below there is a visualisation of the tuples $t_{0,0}, \ldots, t_{3,3}$ introduced in the proof of Lemma 3.9. The first column is $t_{0,0}$, the second $t_{0,1}$, and so on. The condition used to identify tuples is the ternary quasi minority condition. The colors indicate the equivalence classes of the \leftrightarrow^* relation. Below we have the partially labelled sequence a_0, \ldots, a_{17} and the resulting sequence A_0, \ldots, A_5 . At the bottom there is the constructed unfolding of \mathfrak{C} .

Therefore, Σ implies

$$f(x, \dots, x) = f(\pi(0, 0, 1), \dots, \pi(0, 0, K))$$

$$\approx f(\pi(0, 1, 1), \dots, \pi(0, 1, K))$$

$$\vdots$$

$$\approx f(\pi(n, m + 1, 1), \dots, \pi(n, m + 1, K)) = f(y, \dots, y).$$

Hence, Σ implies $g(x) \approx g(y)$, as desired.

Now we consider the case that there are $k, \ell \in [K]$ such that $(0, 0, k) \leftrightarrow^* (n, m+1, \ell)$. Then there exists a sequence a_0, \ldots, a_N of distinct elements in $[0, n] \times [0, m+1] \times [K]$ such that $a_0 = (0, 0, k)$, $a_N = (n, m+1, \ell)$, and for every $i \in [N]$ we have $a_{i-1} \sim a_i$, $a_{i-1} \rightarrow_j a_i$, or $a_{i-1} \leftarrow_j a_i$ for some $j \in [n]$. We will use this sequence to construct an unfolding of \mathfrak{C} that has a homomorphism to \mathfrak{B} . See Figure 3 for an example of such a sequence and the construction that follows. First group the sequence $0, \ldots, N$ into the sequence A_0, \ldots, A_M such that

- $a_0 \in A_0$ and $a_N \in A_M$,
- for all $i \in [0, M]$ there are $j \in [0, N]$ and $0 \le k \le N j$ such that $A_i = \{j, j+1, \dots, j+k\},\$
- for all $i \in [0, M]$ and $j \in A_i$ with $j + 1 \in A_i$ we have $a_j \sim a_{j+1}$,
- for all $0 \leq i_1 < i_2 \leq M$ and all $j_1 \in A_{i_1}, j_2 \in A_{i_2}$ we have $j_1 < j_2$, and
- for all $i \in [M]$ we have $\max(A_{i-1}) + 1 = \min(A_i)$ and there is a $j \in [n]$ with $a_{\max(A_{i-1})} \rightarrow_j a_{\min(A_i)}$ or $a_{\max(A_{i-1})} \leftarrow_j a_{\min(A_i)}$.

We write $A_i \rightarrow_j A_{i+1}$ if $a_{\max(A_{i-1})} \rightarrow_j a_{\min(A_i)}$ and $A_i \leftarrow_j A_{i+1}$ if $a_{\max(A_{i-1})} \leftarrow_j a_{\min(A_i)}$ Define the primitive positive formula $\Phi(0, \ldots, M)$ by adding conjuncts in the following way: For all *i* and all *j*

- if $A_i \rightarrow_j A_{i+1}$, then add the conjunct $\phi_j(i, i+1)$,
- if $A_i \leftarrow_i A_{i+1}$, then add the conjunct $\phi_i(i+1,i)$,
- if $A_i \rightarrow_j A_{i+1} \rightarrow_{j+1} A_{i+2}$, then for all $\phi(x)$ corresponding to a neighbour (t, R) of v_j in the incidence graph of \mathfrak{C} that is not in P add the conjunct $\phi(i+1)$, and
- if $A_i \leftarrow_{j+1} A_{i+1} \leftarrow_j A_{i+2}$, then for all $\phi(x)$ corresponding to a neighbour (t, R) of v_j in the incidence graph of \mathfrak{C} that is not in P add the conjunct $\phi(i+1)$.

Denote the canonical database of Φ by \mathfrak{C}' . Note that \mathfrak{C}' is a caterpillar. Observe that, by definition, $t_{a_j} = t_{a_{j'}}$ for all $j, j' \in A_i$. Hence, the map $\iota: [0, M] \to B$ that maps i to t_a for some (any) a for which there is an $j \in A_i$ with $a = a_j$ is well defined. The map ι is a satisfying assignment of Φ in \mathfrak{B} :

- Every conjunct of the form $\phi_j(i, i+1)$ is satisfied, since $a = a_{\max(A_i)} \rightarrow_j a_{\min(A_{i+1})} = a'$ implies $\mathfrak{B} \models \phi_j(t_a, t_{a'})$ by definition of \rightarrow_j .
- The argument for conjuncts of the form $\phi_j(i+1,i)$ is analogous.
- The conjuncts of the form $\phi(i)$ are satisfied: Let $(d, j, \ell) = a_{\min(A_i)}$. Then, by construction of the *t*'s, there is j' such that $\mathfrak{B} \models t_{d,j',k}$ for all $k \in [K]$. By definition of Φ we added $\phi(i)$ only if there is some $j' \in A_i$ with $a_{j'} = (d, j', k)$ for some $k \in [K]$.

Therefore, ι can be extended to a homomorphism from \mathfrak{C}' to \mathfrak{B} . The proof that \mathfrak{C}' is an unfolding of \mathfrak{C} is technical and can for example be done by induction on the number of *orientation changes* in the sequence A_0, \ldots, A_M . However, it is easy to see that \mathfrak{C}' must indeed be an unfolding of \mathfrak{C} .

Remark 3.10. The conclusion of Lemma 3.9 can be strengthened as follows: if \mathfrak{B} does not satisfy a minor condition Σ , then Σ implies $f(x) \approx f(y)$ even with respect to the class of all minions, i.e., it then holds that every *minion* that satisfies Σ also satisfies $f(x) \approx f(y)$. However, in this case we cannot use Lemma 2.19, which is a statement for clones rather than minions. Instead, one can then use a more general notion of indicator structure adapt the entire proof to this more general setting. Since we do not need this for proving our main result, Theorem 3.1, we have decided for the weaker result which allows for a less technical proof.

3.4 Proof of the main result

We finally prove Theorem 3.1.

Proof of Theorem 3.1. For the implication $(1) \Rightarrow (2)$ let Π_S be the canonical slam Datalog program for \mathfrak{B} and let Π be the canonical linear arc monadic Datalog program for \mathfrak{B} . Since \mathfrak{B} has k-absorptive operations of arity nk for all $n, k \ge 1$ we can apply Theorem 2.18 to conclude that Π solves $\operatorname{CSP}(\mathfrak{B})$. Furthermore, \mathfrak{B} has a quasi Maltsev polymorphism. Hence, Lemma 3.3 implies that Π and Π_S can derive the goal predicate on the same instances of $\operatorname{CSP}(\mathfrak{B})$. Therefore, Π_S solves $\operatorname{CSP}(\mathfrak{B})$.

The implication $(2) \Rightarrow (3)$ is trivial, the implication $(3) \Rightarrow (4)$ by Lemma 3.8, and the implication $(4) \Rightarrow (5)$ by Lemma 3.9.

For the implication from (5) to (6), suppose that Σ is a minor condition that holds in $\operatorname{Pol}(\mathfrak{P}_2)$. Since all polymorphisms of \mathfrak{P}_2 are idempotent, Σ does not imply $f(x) \approx f(y)$. Hence, the contraposition of (5) implies that $\operatorname{Pol}(\mathfrak{B})$ does not satisfy Σ .

The implication from (6) to (1) is clear since $Pol(\mathfrak{P}_2)$ is preserved by the Boolean minority operation and by the *nk*-ary Boolean operation

$$(x_{11},\ldots,x_{1k},\ldots,x_{n1},\ldots,x_{nk})\mapsto\bigvee_{i\in[n]}\bigwedge_{j\in[k]}x_{ij}$$

which is k-absorptive.

The equivalence between (6), (7), and (8) follows from well-known general results [6]. The equivalence of (1) and (9) follows the equivalence of items (3) and (4) in Theorem 2.18 and from the fact that the existence of a quasi Maltsev polymorphism is preserved by homomorphic equivalence.

For the final statement of the theorem, let \mathfrak{B}' be the structure with the same domain as \mathfrak{B} which contains all binary relations that are primitively positively definable in \mathfrak{B} . First recall from Remark 2.16 that (1) implies that \mathfrak{B} has a quasi majority polymorphism, and hence every relation of \mathfrak{B} is equivalent to a conjunction of binary relations of \mathfrak{B}' (Proposition 2.13), which shows that $\operatorname{Pol}(\mathfrak{B}) = \operatorname{Pol}(\mathfrak{B}')$. In particular, \mathfrak{B}' has k-absorptive polymorphisms of arity nk, for all $n, k \geq 1$, and hence the theorem applies to \mathfrak{B}' in place of \mathfrak{B} as well.

Remark 3.11. Consider the poset of all finite structures ordered by primitive positive constructability. It is well known that the structure $\mathfrak{C}_1 := (\{0\}, \{(0,0)\})$ is a representative of the top element of this poset and that it has exactly one lower cover with representative \mathfrak{B}_2 . We claim that \mathfrak{T}_3 is a representative of a lower cover of \mathfrak{B}_2 in the poset of all finite structures ordered by primitive positive constructability. The structure $\mathfrak{T}_3 := (\{0, 1, 2\}, \{(0, 1), (0, 2), (1, 2)\})$ satisfies all conditions Σ that do not imply the quasi Maltsev condition (see, e.g., [11]). Clearly, \mathfrak{T}_3 does not have a primitive positive construction in \mathfrak{B}_2 , because \mathfrak{T}_3 does not have a quasi Maltsev polymorphism. Since min and max are polymorphisms of \mathfrak{T}_3 , Theorem 2.18 implies that \mathfrak{T}_3 has kn-ary kabsorbing polymorphisms for all $n, k \geq 1$. Let \mathfrak{B} be a structure with a primitive positive construction in \mathfrak{T}_3 which does not admit a primitive positive construction of \mathfrak{T}_3 . Then \mathfrak{B} must have a quasi Maltsev polymorphism and kn-ary k-absorbing polymorphisms for all $n, k \geq 1$. By Theorem 3.1, \mathfrak{B} has a primitive positive construction in \mathfrak{B}_2 , which proves the claim. It is still open what other lower covers \mathfrak{B}_2 has.

Remark 3.12. Yet another condition on finite structures \mathfrak{B} that is equivalent to the conditions in Theorem 3.1 has been found by Vucaj and Zhuk [36]: they prove that there is a minion homomorphism from $\operatorname{Pol}(\mathfrak{P}_2)$ to $\operatorname{Pol}(\mathfrak{B})$ (item 7) if and only if $\operatorname{Pol}(\mathfrak{B})$ contains totally symmetric polymorphisms of all arities and generalised quasi minority polymorphisms of all odd arities $n \geq 3$. A operation $f: B^n \to B$, for odd $n \geq 3$, is called a generalised quasi minority if it satisfies

$$f(x_1, \dots, x_n) \approx f(x_{\pi(1)}, \dots, x_{\pi(n)}) \qquad \text{for every } \pi \in S_n$$

and $f(x, x, x_3, \dots, x_n) \approx f(y, y, x_3, x_4, \dots, x_n).$

However, it is not clear to us whether this characterisation can be used to prove the consequence of our main result from Remark 3.11.

Remark 3.13. A minion \mathscr{M} is called a *core* if every minion homomorphism from \mathscr{M} to \mathscr{M} is injective. We say that \mathscr{N} is a *minion core of* \mathscr{M} if \mathscr{M} and \mathscr{N} are homomorphically

equivalent (i.e., there is a homomorphism from \mathscr{M} to \mathscr{N} and vice versa) and \mathscr{N} is a minion core. Note that if \mathscr{M} is *locally finite*, i.e., if $\mathscr{M}^{(n)}$ is finite for every $n \in \mathbb{N}$, then there exists a minion \mathscr{N} which is a minion core of \mathscr{M} , and \mathscr{N} is unique up to isomorphism. We therefore call it *the* minion core of \mathscr{M} . From personal communication with Libor Barto we learned that the equivalent items of Theorem 3.1 apply if and only if the minion core of Pol(\mathfrak{B}) equals Pol($\mathfrak{C}_2, \mathfrak{B}_2$).

4 Decidability of Meta-Problem

There are many interesting results and open problems about *algorithmic meta-problems* in constraint satisfaction; we refer to [18]. The natural algorithmic meta-problem in the context of our work is the one addressed in the following proposition.

Proposition 4.1. There is an algorithm which decides whether the CSP of a given finite structure \mathfrak{B} can be solved by a slam Datalog program, and if so, computes such a program.

Proof. The following algorithm can be used to test whether \mathfrak{B} has k-absorptive operations of arity nk, for all $n, k \geq 1$. It is well-known that the existence of a quasi Maltsev polymorphism can be decided in non-deterministic polynomial time (see, e.g., [18]). Hence, the statement then follows from Theorem 3.1.

Let m be the maximal arity of the relations of \mathfrak{B} . Let $n_0 := m \binom{|B|}{|B|/2}$ and $k_0 := m|B|$. Note that \mathfrak{B} has k-absorptive polymorphisms of arity nk, for all k, n, if and only if it has k_0 -absorptive polymorphisms of arity n_0k_0 (similarly as the well-known fact that \mathfrak{B} has totally symmetric polymorphisms of all arities if and only if it has totally symmetric polymorphisms of arity m|B|; the term $\binom{|B|}{|B|/2}$ bounds the size of antichains in the set of all subsets of B. Also see [16] for the case of k-absorptive polymorphisms). Let Σ be the minor condition for the existence of k_0 -absorptive operations of arity n_0k_0 . Let \mathfrak{C} be the indicator structure of Σ with respect to \mathfrak{B} as defined in Section 2.8; clearly, this structure can be computed in doubly exponential time. We may then find a nondeterministic algorithm with the same time bound that tests whether there exists a homomorphism from \mathfrak{C} to \mathfrak{B} . The non-determinism for checking whether $\mathfrak{C} \to \mathfrak{B}$ can be eliminated by standard self-reduction techniques (again see, e.g., [18]). For the second part of the statement, note that there are for a given \mathfrak{B} only finitely many potential rules of a slam Datalog program, and one can compute for a given rule whether it is part of the canonical slam Datalog program of $CSP(\mathfrak{B})$.

5 Remarks on Related Results

The following remarks show that the results of Carvalho, Dalmau and Krokhin [16] can be extended in the same spirit as our Theorem 3.1.

Remark 5.1. Let \mathfrak{D}_2 be the structure $(\{0, 1\}; \{0\}, \{1\}, \leq)$, also known as st-Con. Theorem 2.18 of Carvalho, Dalmau and Krokhin can be extended in the same spirit as our Theorem 3.1, by adding the following equivalent items:

- 5. Every minor condition that holds in $Pol(\mathfrak{D}_2)$ also holds in $Pol(\mathfrak{B})$.
- 6. There is a minion homomorphism from $Pol(\mathfrak{D}_2)$ to $Pol(\mathfrak{B})$.
- 7. \mathfrak{B} has a primitive positive construction in \mathfrak{D}_2 .

The equivalence of 5., 6., and 7. follows immediately from the general results in [6].

4. \Rightarrow 7. It is well known that $\operatorname{Pol}(\mathfrak{D}_2)$ is generated by the two binary operations \lor and \land .⁴ Let \mathfrak{B} be a structure that is homomorphically equivalent to a structure \mathfrak{B}' with binary polymorphisms \sqcup and \sqcap such that (B', \sqcup, \sqcap) is a distributive lattice. Note that $(\{0,1\},\lor,\land)$ is a distributive lattice as well. Let ι be the map that maps terms over \lor,\land to terms over \sqcup,\sqcap by replacing \lor and \land by \sqcup and \sqcap , respectively. Define the map $\xi \colon \operatorname{Pol}(\mathfrak{D}_2) \to \operatorname{Pol}(\mathfrak{B}')$ as follows. Since $\operatorname{Pol}(\mathfrak{D}_2)$ is generated by \lor and \land , for every $f \in \operatorname{Pol}(\mathfrak{D}_2)$ there is a $\{\land,\lor\}$ -term t whose term operation is f. Define $\xi(f)$ as the term operation of $\iota(t)$. Note that this term operation is a polymorphism of \mathfrak{B}' . It is clear that ξ is a minion homomorphism (even a clone homomorphism). We still need to show that ξ is well defined. Let t and t' be two $\{\land,\lor\}$ -terms that both have the term operation $f \in \operatorname{Pol}(\mathfrak{D}_2)$. Since $(\{0,1\},\lor,\land)$ is a distributive lattice, there is a set \mathcal{I} of subsets of [n] such that f is the term operation of

$$s \coloneqq \bigwedge_{I \in \mathcal{I}} \bigvee_{i \in I} x_i.$$

Furthermore, t and t' can both be rewritten (using associativity, commutativity, distributivity, and idempotence) into the term s. Therefore, $\iota(t)$ and $\iota(t')$ can also both be rewritten into the term $\iota(s)$. Since (B', \sqcup, \sqcap) is a distribute lattice, the term operations of $\iota(t)$, $\iota(t')$, and $\iota(s)$ are the same. Hence, ξ is well defined.

5. \Rightarrow 4. holds since \mathfrak{D}_2 has for every $n, k \geq 1$ a k-absorbing polymorphism of arity kn.

Remark 5.2. Let $\mathfrak{B}_{\infty}^{\leq}$ be the structure with the domain $\{0, 1\}$ and the signature $\{\mathbf{0}, \leq R_1, R_2, \ldots\}$ where $\mathbf{0} \coloneqq \{0\}, \leq \coloneqq \{(0, 0), (0, 1), (1, 1)\}$, and $R_n \coloneqq \{0, 1\}^n \setminus \{(0, \ldots, 0)\}$ for every $n \geq 1$. It is well known that $\operatorname{Pol}(\mathfrak{B}_{\infty}^{\leq})$ is generated by the operation m given by $(x, y, z) \mapsto x \land (y \lor z)$.⁵ Carvalho, Dalmau and Krokhin also introduce another type

⁴Proof sketch: clearly, \lor and \land preserve the relations of \mathfrak{D}_2 . For the converse inclusion, it suffices to verify that every relation that is preserved by \land and \lor has a primitive positive definition in \mathfrak{D}_2 (see, e.g. [34]). Every Boolean relation preserved by \lor and \land has a definition in CNF which is both Horn and dual Horn, so consists of clauses that can be defined using the relations in \mathfrak{D}_2 . This implies the claim.

⁵Proof sketch: clearly, every relation of $\mathfrak{B}_{\infty}^{\leq}$ is preserved by m. For the converse inclusion, it suffices to verify that every relation that is preserved by m has a primitive positive definition in $\mathfrak{B}_{\infty}^{\leq}$ (see, e.g., [34]). First note that $m(x, y, y) = x \wedge y$, and hence every Boolean relation R preserved by m

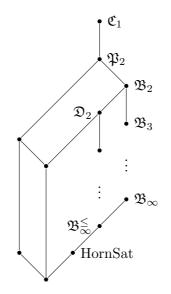


Figure 4: The lattice of 2-element structures with respect to pp-constructability [12]; the structures HornSat, \mathfrak{D}_2 , and \mathfrak{P}_2 were relevant in this text for characterisations of Datalog fragments (for arc monadic, linear arc monadic, and slam Datalog, respectively).

of duality in their paper: jellyfish duality. Their characterization in Theorem 18 in [16] can be extended by the following items:

- 6. Every minor condition that holds in $\operatorname{Pol}(\mathfrak{B}_{\infty}^{\leq})$ also holds in $\operatorname{Pol}(\mathfrak{B})$.
- 7. There is a minion homomorphism from $\operatorname{Pol}(\mathfrak{B}_{\infty}^{\leq})$ to $\operatorname{Pol}(\mathfrak{B})$.
- 8. \mathfrak{B} has a primitive positive construction in $\mathfrak{B}_{\infty}^{\leq}$.

The proof is analogous to the proof in Remark 5.1.

6 Conclusion and Open Problems

We characterised the unique submaximal element in the primitive positive constructability poset on finite structures, linking concepts from homomorphism dualities, Datalog fragments, minor conditions, and minion homomorphisms. It is now tempting to further descend in the pp-constructability poset of finite structures in order to obtain a more systematic understanding. Particularly attractive are other dividing lines in the poset

has a Horn definition; pick such a definition ϕ which is shortest possible. Suppose for contradiction that a Horn clause in ϕ contains a positive literal ψ_1 and two negative literals ψ_2 and ψ_3 . By the minimality assumption there are tuples $t_1, t_2, t_3 \in \mathbb{R}$ such that t_i satisfies ϕ_i and no other literal in that clause. Then $m(t_1, t_2, t_3)$ satisfies none of ψ_1, ψ_2, ψ_3 , a contradiction. It follows that each clause can be defined using the relations in $\mathfrak{B}_{\infty}^{\leq}$ and the statement follows.

that are relevant for the complexity of the constraint satisfaction problem. We propose the following problems for future research.

- Is there a countable set of structures $\mathfrak{C}_1, \mathfrak{C}_2, \ldots$ such that \mathfrak{B} does not have a pp-construction in \mathfrak{P}_2 if and only if one of the structures $\mathfrak{C}_1, \mathfrak{C}_2, \ldots$ has a pp-construction in \mathfrak{B} ? This is true if we restrict to digraphs [11] and if we restrict to 3-element structures [36]. Our result shows that \mathfrak{T}_3 must belong to this set (Remark 3.11).
- Characterise all finite structures that are primitively positively constructible in a finite structure that has finite duality. Are these exactly the finite structures whose polymorphism clones have Hagemann-Mitschke chains of some length and extended k-absorptive polymorphisms of arity kn + 1, for all $n, k \ge 1$, as defined in [17]? Is there a Datalog fragment that corresponds to this class?
- What is the precise computational complexity the Meta-Problem of deciding whether the CSP of a given finite structure \mathfrak{B} can be solved by a slam Datalog program? The algorithm from Proposition 4.1 only provides a deterministic doubly exponential time algorithm.

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