DIAGRAMMATICS FOR COMODULE MONADS

SEBASTIAN HALBIG AND TONY ZORMAN

Abstract. We extend Willerton's [Wil08] graphical calculus for bimonads to comodule monads, a monadic interpretation of module categories over a monoidal category. As an application, we prove a version of Tannaka–Krein duality for these structures.

Contents

1. Introduction

Given a monad B on a monoidal category $\mathscr{C}$, one might ask to which extent monoidal structures on the category $\mathscr{C}^B$ of B-algebras are controlled by additional structures on the monad itself. This can be seen as an extension of the classical *Tannaka–Krein duality*, and was proved by Moerdijk [Moe02, Theorem 7.1] and McCrudden [McC02, Corollary 3.13].

Theorem 1. *Let B be a monad on a monoidal category $\mathscr{C}$. Bimonad structures on B are in one-to-one correspondence with monoidal structures on $\mathscr{C}^B$, such that the forgetful functor U^B is strict monoidal.*

In [Wil08, Section 2.3], Theorem 1 is proved in a graphical fashion. The aim of this article is to generalise both the statement and its graphical proof to comodule monads, as developed by Aguiar and Chase [AC12]. More precisely, we show the following.

Theorem 2. *Let B be a bimonad on the monoidal category $\mathscr{C}$, and K a monad on a right $\mathscr{C}$-module category $\mathscr{M}$. Coactions of B on K are in bijection with right actions of $\mathscr{C}^B$ on $\mathscr{M}^K$, such that U^K is a strict comodule functor over U^B.*

The article is structured as follows. Section 2 serves to review basic concepts from category theory, as well as their associated string diagrammatic counterparts. Section 3 introduces Willerton's graphical calculus for monoidal categories, and Section 4 extends this construction to module categories and comodule monads, allowing us to prove Theorem 2.

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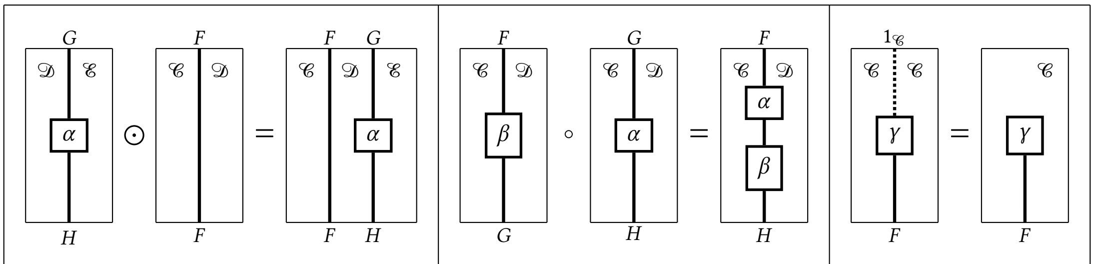
2. 2-Categories and the graphical calculus

We assume the reader's familiarity with basic concepts of the theory of monoidal categories as discussed for example in [ML98; EGNO15; Rie17].

Our study of comodule structures on monads involves—besides the composition of functors and natural transformations—products of categories. To that end we let $\mathbb{X}$ be the monoidal 2-category of (small) categories, functors, and natural transformations, equipped with the monoidal structure induced by the Cartesian product of categories.[1] We use juxtaposition for the horizontal composition of $\mathbb{X}$, or, in case we want to emphasise the direction of composition,

$$- \odot_{\mathcal{A},\mathcal{B},\mathcal{C}} -\colon\ \mathbb{X}(\mathcal{B},\mathcal{C}) \times \mathbb{X}(\mathcal{A},\mathcal{B}) \longrightarrow \mathbb{X}(\mathcal{A},\mathcal{C}), \qquad\qquad \mathcal{A},\mathcal{B},\mathcal{C} \in \mathrm{Ob}(\mathbb{X}).$$

String diagrams will serve as the main tool for doing computations. We closely follow the presentation and conventions in [HZ24]. In the case of a 2-category $\mathbb{X}$, a *string diagram* comprises regions, labelled with categories, strings labelled with functors, and vertices between the strings labelled with natural transformations. If two string diagrams can be transformed into each other, the natural transformations they represent are equal. A more detailed description is given in [JS91; Sel11]. Our convention is to read diagrams from top to bottom and left to right. Horizontal and vertical composition are given by horizontal and vertical gluing of diagrams, respectively. Identity natural transformations are given by unlabelled vertices. The identity functor of a category is represented by the empty edge. If the involved categories are clear from the context, we omit writing them explicitly.

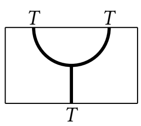


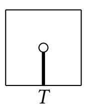
Horizontal composition of natural transformations $\mathrm{id}_F\colon F \Longrightarrow F$ and $\alpha\colon G \Longrightarrow H$ where $F \in \mathbb{X}(\mathcal{C},\mathcal{D})$ and $G, H \in \mathbb{X}(\mathcal{D},\mathcal{C})$.	Vertical composition of two natural transformations $\alpha\colon F \Longrightarrow G$ and $\beta\colon G \Longrightarrow H$, where $F, G, H \in \mathbb{X}(\mathcal{C},\mathcal{D})$.	A natural transformation $\gamma\colon 1_{\mathcal{C}} \Longrightarrow F$ for $1_{\mathcal{C}}, F \in \mathbb{X}(\mathcal{C},\mathcal{C})$.

Recall from e.g. [Bén67] that a *monad* on a category $\mathcal{C} \in \mathbb{X}$ is a functor $T\colon \mathcal{C} \longrightarrow \mathcal{C}$ together with two natural transformations $\mu\colon T^2 \Longrightarrow T$ and $u\colon \mathrm{Id}_{\mathcal{C}} \Longrightarrow T$, called the *multiplication* and *unit* of T, satisfying appropriate *associativity* and *unitality* axioms:

$$\begin{array}{ccc} T^3 & \overset{T\mu}{\Longrightarrow} & T^2 \\ {\scriptstyle \mu T}\Downarrow & & \Downarrow{\scriptstyle \mu} \\ T^2 & \underset{\mu}{\Longrightarrow} & T \end{array} \qquad\qquad \begin{array}{ccccc} T^2 & \overset{uT}{\Longleftarrow} & T & \overset{Tu}{\Longrightarrow} & T \\ & {\scriptstyle \mu}\searrow & \| & \swarrow{\scriptstyle \mu} & \\ & & T & & \end{array}$$

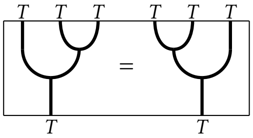
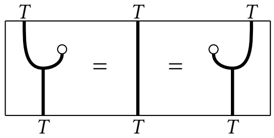
We represent the multiplication and unit of a monad T in terms of string diagrams:

(2.1) 

and 

[1] More generally, our applications of the graphical calculus can be formulated in any (weakly) monoidal 2-category, see [JY21, Chatper 12], which *admits the construction of algebras*, see [Str72, §1].

Their associativity and unitality then equate to

(2.2)  and 

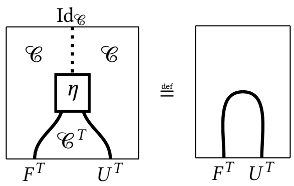
Let us focus on the special case of $\mathbb{X} = \mathbb{C}\mathsf{at}$, though as discussed in [Str72, §1], all of the following constructions generalise to general $\mathbb{X}$ admitting the construction of algebras. There is a 2-category $\mathbb{M}\mathsf{on}$ of monads on $\mathbb{C}\mathsf{at}$. Let

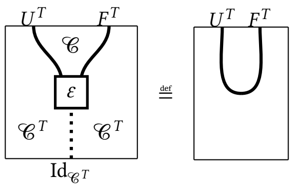
$$\text{Inc}\colon \mathbb{C}\mathsf{at} \longrightarrow \mathbb{M}\mathsf{on}, \qquad \mathscr{C} \longmapsto (1_{\mathscr{C}}, \mathrm{id}_{1_{\mathscr{C}}}, \mathrm{id}_{1_{\mathscr{C}}}) \tag{2.3}$$

be the inclusion 2-functor, which maps any category to its identity monad. The above functor has a right adjoint $\text{Alg}\colon \mathbb{M}\mathsf{on} \longrightarrow \mathbb{C}\mathsf{at}$, which maps any monad $(T, \mu, u)\colon \mathscr{C} \longrightarrow \mathscr{C}$ to its *Eilenberg–Moore category* $\mathscr{C}^T$. Using the previous 2-adjunction, one proves that to every monad $(T, \mu, u)\colon \mathscr{C} \longrightarrow \mathscr{C}$ one can associate adjoint functors

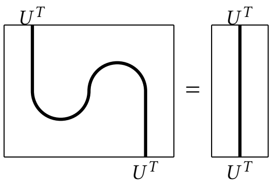
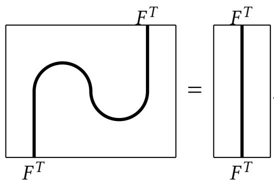
$$F^T\colon \mathscr{C} \longrightarrow \mathscr{C}^T \qquad \text{and} \qquad U^T\colon \mathscr{C}^T \longrightarrow \mathscr{C}, \tag{2.4}$$

whose unit and counit we denote by $\eta\colon 1_{\mathscr{C}} \Longrightarrow U^T F^T$ and $\varepsilon\colon F^T U^T \Longrightarrow 1_{\mathscr{C}^T}$, such that $T = U^T F^T$, $\mu = F^T \varepsilon U^T$, and $u = \eta$. We call the adjunction $F_T\colon \mathscr{C} \rightleftarrows \mathscr{C}^T :U_T$ the Eilenberg–Moore adjunction of (T, μ, u), and depict η and ε by



and


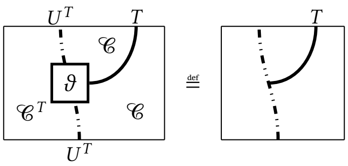
Graphically, the defining equations of adjunctions translate to the snake equations

 and .

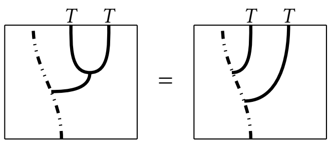
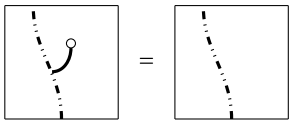
Remark 2.3. As noted in [Wil08], the Eilenberg–Moore category of a monad T can be incorporated into the graphical calculus. Defining the natural transformation

$$\vartheta \stackrel{\text{def}}{=} TU^T = U^T F^T U^T \xrightarrow{U^T \varepsilon} U^T$$

we may write

(2.5) 

We think of $\vartheta\colon TU^T \longrightarrow U^T$ as an action of T on U^T due to the identities

 and 

Any adjunction $F\colon \mathscr{C} \rightleftarrows \mathscr{D} :U$ defines a monad $(UF, F\varepsilon U, \eta)$. A question one might ask is how much the functors F and U "differ" from the free and forgetful functors $F^T\colon \mathscr{C} \longrightarrow \mathscr{C}^T$ and $U^T\colon \mathscr{C}^T \longrightarrow \mathscr{C}$ of T, respectively.

Lemma 2.4 ([Str72, Theorem 3]). *Let T be the monad of the adjunction $F\colon \mathscr{C} \rightleftarrows \mathscr{D} : U$. There exists a unique functor $\Sigma\colon \mathscr{D} \longrightarrow \mathscr{C}^T$ satisfying $\Sigma F = F^T$ and $U^T \Sigma = U$.*

Given an adjunction $F\colon \mathscr{C} \rightleftarrows \mathscr{D} : U$, we call the unique functor $\Sigma\colon \mathscr{D} \longrightarrow \mathscr{C}^T$ discussed in the previous lemma the *comparison functor* and refer to the adjunction $F \dashv U$ as *monadic* if Σ is an equivalence.

3. Bimonads

Bimonads are a vast generalisation of bialgebras. They naturally arise in the study of (rigid) monoidal categories and topological quantum field theories, see amongst others [KL01; Moe02; BV07; BLV11; TV17]. We note that the term "bimonad" is also used by Mesablishvili and Wisbauer for a similar—though distinct—concept; see [MW11]. For this reason, what we call a bimonad is also sometimes called an opmonoidal (or comonoidal) monad in the literature.

Due to the lack of a braiding on the endofunctors $\operatorname{End}(\mathscr{C})$ over $\mathscr{C}$, the naive notion of bialgebra does not generalise to the monadic setting and needs to be adjusted. One possible way of overcoming this problem was introduced and studied by Moerdijk under the name "Hopf monad"[2] in [Moe02]; the idea being that the opmonoidal structure of a functor replaces the bialgebra's comultiplication and counit. Following the conventions of [BV07], we refer to such structures as bimonads.

Definition 3.1. An *opmonoidal functor* between (strict) monoidal categories $(\mathscr{C}, \otimes, 1)$ and $(\mathscr{D}, \otimes, 1)$ is a functor $F\colon \mathscr{C} \longrightarrow \mathscr{D}$ together with two natural transformations[3]

$$F_2\colon F(- \otimes =) \Longrightarrow F- \otimes F= \qquad \text{and} \qquad F_0\colon F1 \longrightarrow 1$$

called the *comultiplication* and *counit*, satisfying *coassociativity* and *counitality*:

$$(3.1)\qquad \begin{array}{ccc} F(- \otimes = \otimes \equiv) & \xrightarrow{F_{2;-,=\otimes\equiv}} & F- \otimes F(= \otimes \equiv) \\ {\scriptstyle F_{2;-\otimes=,\equiv}}\downarrow & & \downarrow{\scriptstyle F- \otimes F_{2;=,\equiv}} \\ F(- \otimes =) \otimes F\equiv & \xrightarrow[F_{2;-,=} \otimes F\equiv]{} & F- \otimes F= \otimes F\equiv \end{array} \qquad \begin{array}{ccccc} F1 \otimes F- & \xleftarrow{F_{2;1,-}} & F- & \xrightarrow{F_{2;-,1}} & F- \otimes F1 \\ & {\scriptstyle F_0 \otimes F-}\searrow & \| & \swarrow{\scriptstyle F- \otimes F_0} & \\ & & F- & & \end{array}$$

If F_2 and F_0 are isomorphisms or identities, we call F *strong monoidal* or *strict monoidal*, respectively.

Definition 3.2. An *opmonoidal natural transformation* between opmonoidal functors $F, G\colon \mathscr{C} \longrightarrow \mathscr{D}$ is a natural transformation $\rho\colon F \Longrightarrow G$ such that the diagrams

$$\begin{array}{ccc} F(- \otimes =) & \xrightarrow{\rho_{-\otimes=}} & G(- \otimes =) \\ {\scriptstyle F_{2,-,=}}\downarrow & & \downarrow{\scriptstyle G_{2,-,=}} \\ F- \otimes F= & \xrightarrow[\rho_- \otimes \rho_=]{} & G- \otimes G= \end{array} \qquad \begin{array}{ccc} F1 & \xrightarrow{\rho_1} & G1 \\ & {\scriptstyle F_0}\searrow & \downarrow{\scriptstyle G_0} \\ & & 1 \end{array}$$

commute.

If ρ is additionally a natural isomorphism, we call it an *opmonoidal natural isomorphism.*

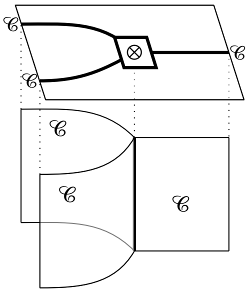
Willerton introduced a graphical calculus for opmonoidal functors in [Wil08]. The key idea is to incorporate the Cartesian product of categories, or, more generally, the tensor product of a monoidal 2-category into the diagrammatic calculus.

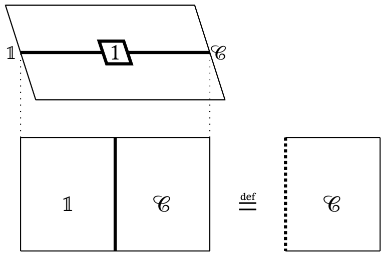
[2] As remarked in [Moe02], the concept of Hopf monads is strictly dual to that of monoidal comonads, which are studied for example in [Boa95].

[3] We view the monoidal unit of $\mathscr{C}$ as a functor $1\colon \mathbb{1} \longrightarrow \mathscr{C}$, where $\mathbb{1}$ is the terminal category.

As before, we consider strings and vertices between them. These are labelled with functors and natural transformations, respectively. The strings and vertices are embedded into bounded rectangles, which we will call sheets. Each (connected) region of a sheet is decorated with a category. The same mechanics as for string diagrams apply: horizontal and vertical gluing represents composition of functors and natural transformations. On top of these operations, we add stacking sheets behind each other to depict the monoidal product in $\mathbb{C}\mathrm{at}$. Our convention is to read diagrams from front to back, left to right, and top to bottom.

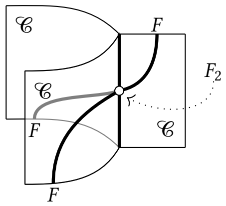
Two of the most vital building blocks in this new graphical language are the tensor product and unit of a monoidal category $(\mathscr{C}, \otimes, 1)$:

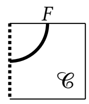




On the left, there are two sheets equating to two copies of $\mathscr{C}$ joined by a line: the tensor product of $\mathscr{C}$. On the right, we have the unit of $\mathscr{C}$ considered as a functor $1\colon \mathbb{1} \longrightarrow \mathscr{C}$, where $\mathbb{1}$ is the terminal category. We represent $\mathbb{1}$ by the empty sheet, and the unit of $\mathscr{C}$ by a dashed line.

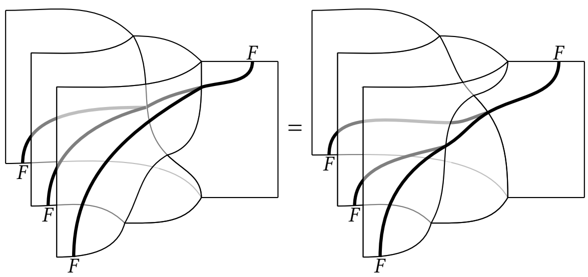
Example 3.3. Consider an opmonoidal functor $(F, F_2, F_0)\colon \mathscr{C} \longrightarrow \mathscr{D}$. Graphically, one may depict F_2 and F_0 like the following:



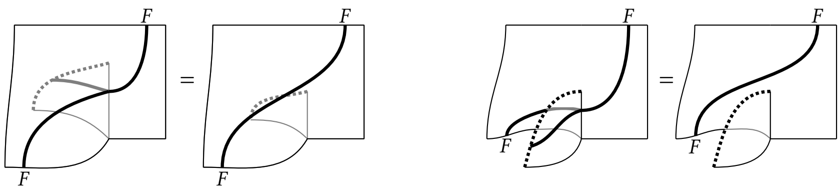


The coassociativity and counitality of Equation (3.1) become:

(3.2)

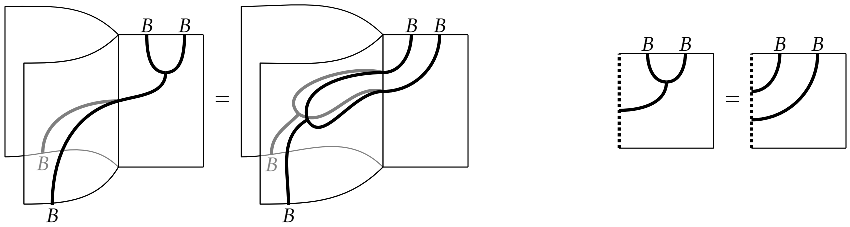


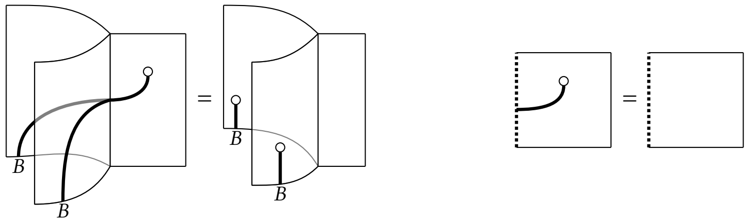
(3.3)



Definition 3.4. A *bimonad* on a monoidal category $\mathscr{C}$ comprises an opmonoidal endofunctor $(B, B_2, B_0)\colon \mathscr{C} \longrightarrow \mathscr{C}$ together with opmonoidal natural transformations $\mu\colon B^2 \Longrightarrow B$ and $\eta\colon \mathrm{Id}_{\mathscr{C}} \Longrightarrow B$ implementing a monad structure on B.

Graphically, what it means for μ and η to be opmonoidal natural transformations is:

(3.4)


(3.5)


Let $F\colon \mathscr{C} \rightleftarrows \mathscr{D} : U$ be an opmonoidal adjunction between $\mathscr{C}$ and $\mathscr{D}$. The monad of the adjunction $UF\colon \mathscr{C} \longrightarrow \mathscr{C}$ is a bimonad; its comultiplication is defined as the composition

$$UF(- \otimes =) \xrightarrow{UF_{2;-,=}} U(F- \otimes F=) \xrightarrow{U_{2;F-,F=}} UF- \otimes UF=,$$

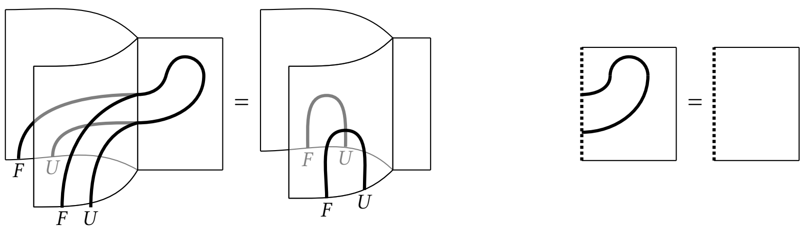
and its counit is given by

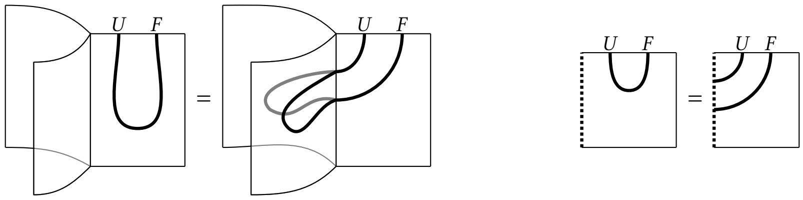
$$UF1 \xrightarrow{UF_0} U1 \xrightarrow{U_0} 1.$$

Adjunctions between monoidal categories are a broad topic with many facets, see for example [AM10, Chapter 3]. For our purposes, we can restrict ourselves to the following situation.

Definition 3.5. We call an adjunction $F\colon \mathscr{C} \rightleftarrows \mathscr{D} : U$ between monoidal categories $\mathscr{C}$ and $\mathscr{D}$ *opmonoidal* if F and U are opmonoidal functors, and the unit and counit of the adjunction are opmonoidal natural transformations. If F and U are moreover strong monoidal, we call $F\colon \mathscr{C} \rightleftarrows \mathscr{D} : U$ a *(strong) monoidal adjunction.*

Suppose $F\colon \mathscr{C} \rightleftarrows \mathscr{D} : U$ to be an opmonoidal adjunction. In our graphical language, the conditions that the unit η and counit ε need to be opmonoidal natural transformations take the following form.

(3.6)


(3.7)


The next result is a slightly simplified version of [TV17, Lemma 7.10].

Lemma 3.6. *Let $F\colon \mathscr{C} \rightleftarrows \mathscr{D} :U$ be a pair of adjoint functors between two monoidal categories. The adjunction $F \dashv U$ is monoidal if and only if U is a strong monoidal functor. That is, the coherence morphisms of U are invertible.*

Let $B\colon \mathscr{C} \longrightarrow \mathscr{C}$ be the bimonad arising from the monoidal adjunction $F\colon \mathscr{C} \rightleftarrows \mathscr{D} :U$. One can show that $\mathscr{C}^B$ carries a monoidal structure such that the forgetful functor $U^B\colon \mathscr{C}^B \longrightarrow \mathscr{C}$ is strict monoidal, see [Moe02; Zaw12; Bö19]. By the above lemma, the adjunction $F^B \dashv U^B$ is monoidal. This raises the question whether the comparison functor—mediating between the two adjunctions—is compatible with this additional structure. Due to [Kel74], see also [BV07, Theorem 2.6], we have the following result.

Lemma 3.7. *Let $F\colon \mathscr{C} \rightleftarrows \mathscr{D} :U$ be a monoidal adjunction and write $B\colon \mathscr{C} \longrightarrow \mathscr{C}$ for its induced bimonad. The comparison functor $\Sigma\colon \mathscr{D} \longrightarrow \mathscr{C}^B$ is strong monoidal and $U^B\Sigma = U$ as well as $\Sigma F = F^B$ as strong and opmonoidal functors, respectively.*

For a more general version see [Bö19, Proposition 3.2].

4. Comodule monads

Monads with a "coaction" over a bimonad were defined and studied by Aguiar and Chase in [AC12]. This concept is needed to obtain an adequate monadic interpretation of twisted centres. We briefly summarise the aspects of the aforementioned article that are needed for our investigation.[4]

Definition 4.1. Let $\mathscr{C}$ be a monoidal category. A *right $\mathscr{C}$-module category* comprises a category $\mathscr{M}$ together with an *action* functor $\triangleleft\colon \mathscr{M} \times \mathscr{C} \longrightarrow \mathscr{M}$, satisfying appropriate *associativity* and *unitality* conditions, see for example [EGNO15, Definition 7.1.1].

Every monoidal category $\mathscr{C}$ is a right $\mathscr{C}$-module category, where $\otimes \overset{\text{def}}{=} \triangleleft$.

Example 4.2. Let $\mathscr{C}$ be a monoidal category, and suppose that $F\colon \mathscr{C} \longrightarrow \mathscr{C}$ is a strong monoidal functor. The action $\triangleleft\colon \mathscr{C} \times \mathscr{C} \xrightarrow{\mathscr{C} \times F} \mathscr{C} \times \mathscr{C} \xrightarrow{\otimes} \mathscr{C}$ endows $\mathscr{C}$ with the structure of a right module category over itself.

To keep our notation concise, for the rest of this article we fix two monoidal categories $(\mathscr{C}, \otimes, 1)$ and $(\mathscr{D}, \otimes, 1)$ and over each a right module category $(\mathscr{M}, \triangleleft)$ and $(\mathscr{N}, \blacktriangleleft)$.

Definition 4.3. Let $F\colon \mathscr{C} \longrightarrow \mathscr{D}$ be an opmonoidal functor. A *(right) comodule functor over F* is a pair (G, δ) comprising a functor $G\colon \mathscr{M} \longrightarrow \mathscr{N}$ together with a natural transformation $\delta\colon G(- \triangleleft =) \Longrightarrow G- \blacktriangleleft F=$, called the *coaction* of G, which is *coassociative* and *counital.*

A comodule functor is called *strong* if its coaction is an isomorphism.

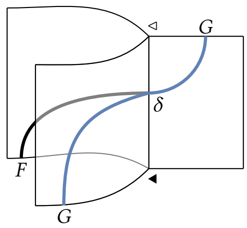
Example 4.4. A *right $\mathscr{C}$-module functor* in the sense of [EGNO15, Definition 7.2.1] comprises a functor $F\colon \mathscr{M} \longrightarrow \mathscr{N}$ between two $\mathscr{C}$-module categories, and a natural transformation $\delta\colon F(- \otimes =) \Longrightarrow F- \otimes =$ whose compatibility with the action equate to F being a comodule functor over $\mathrm{Id}_{\mathscr{C}}\colon \mathscr{C} \longrightarrow \mathscr{C}$.

Example 4.5 ([AC12, Section 6.1])**.** If B is a bialgebra over a commutative ring $\Bbbk$, then $B \otimes_{\Bbbk} -\colon \Bbbk\text{-Mod} \longrightarrow \Bbbk\text{-Mod}$ becomes an opmonoidal functor. Let A be a right B-comodule algebra, and suppose C is a $\Bbbk$-subalgebra of the B-coinvariants of A. The coaction $\nu\colon A \longrightarrow A \otimes_{\Bbbk} B$ is a map of C-C-bimodules, which turns $A \otimes_C -\colon C\text{-Mod} \longrightarrow C\text{-Mod}$

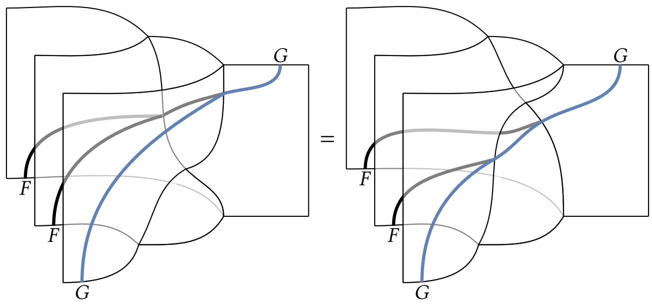
[4] We slightly deviate from [AC12] in that we study right comodule monads as opposed to their left versions.

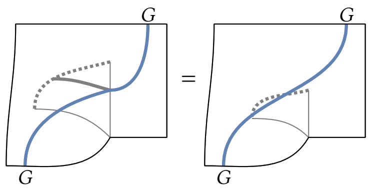
into a right comodule functor over $B \otimes_{\Bbbk} -$. The action of C-Mod on the category of $\Bbbk$-modules is given by tensoring over $\Bbbk$.

The string diagrammatic calculus of Section 3 can be adapted to this setting as follows: in addition to combining sheets using tensor products, we now additionally consider actions to do so as well. In order to keep track of which splitting occurred, functors between the module categories will be coloured blue. As an example, consider the coaction $\delta\colon G(- \triangleleft =) \Longrightarrow G- \blacktriangleleft F=$ from Definition 4.3. It is represented by



The mentioned compatibility of the coaction with the comultiplication and counit of F result in diagrams analogous to (3.2) and (3.3)—except that one of the strings will be in a different colour.

(4.1)


(4.2)


Definition 4.6. Suppose that $(G, \delta^{(G)})$, $(K, \delta^{(K)})\colon \mathcal{M} \longrightarrow \mathcal{N}$ are comodule functors over $B, F\colon \mathscr{C} \longrightarrow \mathscr{D}$. A *comodule natural transformation* from G to K comprises a pair of natural transformations $\phi\colon G \Longrightarrow K$ and $\psi\colon B \Longrightarrow F$, such that

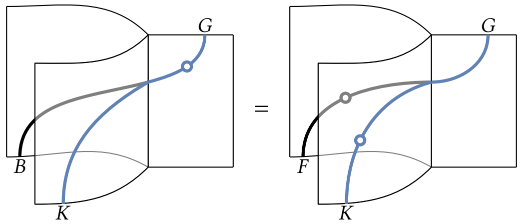
(4.3)
$$\begin{array}{ccc} G(m \triangleleft x) & \xrightarrow{\phi_{m \triangleleft x}} & K(m \triangleleft x) \\ \Big\downarrow{\scriptstyle \delta^{(G)}_{m,x}} & & \Big\downarrow{\scriptstyle \delta^{(K)}_{m,x}} \\ Gm \blacktriangleleft Bx & \xrightarrow[\phi_m \blacktriangleleft \psi_x]{} & Km \blacktriangleleft Fx \end{array}$$

commutes, for all $x \in \mathscr{C}$ and $m \in \mathcal{M}$.

We call (ϕ, ψ) a *morphism of comodule functors* if $B = F$ and $\psi = \mathrm{id}_B$.

Given a comodule natural transformation $(\phi\colon G \Longrightarrow K,\ \psi\colon B \Longrightarrow F)$, the graphical version of Diagram (4.3) is displayed in our next picture, where the blue dot indicates ϕ

and the black dot indicates ψ.



Remark 4.7. Suppose the pair $\phi\colon G \Longrightarrow K$ and $\psi\colon B \Longrightarrow F$ constitute a comodule natural transformation. We can view $\phi\colon G \Longrightarrow K$ as a morphism of comodule functors over F if we equip G with a new coaction. It is given for all $x \in \mathscr{C}$ and $m \in \mathscr{M}$ by

$$G(m \triangleleft x) \xrightarrow{\delta^{(G)}_{m,x}} Gm \blacktriangleleft Bx \xrightarrow{Gm \blacktriangleleft \psi_x} Gm \blacktriangleleft Fx.$$

It follows that by altering the involved coactions suitably, comodule natural transformations and morphisms of comodule functors can be identified with each other.

Let (B, μ, η, B_2, B_0) be a bimonad on $\mathscr{C}$. The unit $\eta\colon \mathrm{Id}_{\mathscr{C}} \longrightarrow B$ implements a coaction on $\mathrm{Id}_{\mathscr{M}}\colon \mathscr{M} \longrightarrow \mathscr{M}$ via

$$\mathrm{id}_m \triangleleft \eta_x\colon \mathrm{Id}_{\mathscr{M}}(m \triangleleft x) \longrightarrow \mathrm{Id}_{\mathscr{M}} m \triangleleft Bx, \qquad \text{for all } x \in \mathscr{C}, m \in \mathscr{M}.$$

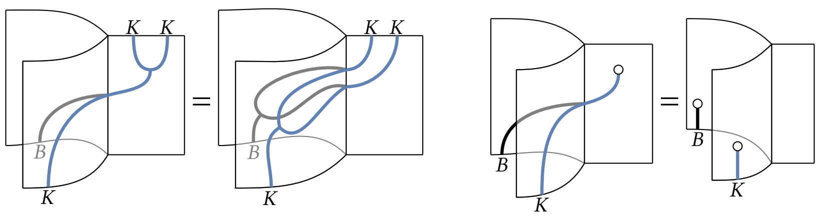
Using the multiplication $\mu\colon B^2 \Longrightarrow B$, we can equip the composition $GK\colon \mathscr{M} \longrightarrow \mathscr{M}$ of two B-comodule functors $G, K\colon \mathscr{M} \longrightarrow \mathscr{M}$ with a comodule structure:

$$GK(- \triangleleft =) \xrightarrow{G\delta^{(K)}} G(K- \triangleleft B=) \xrightarrow{\delta^{(K)}} GK- \triangleleft B^2= \xrightarrow{\mathrm{id} \triangleleft \mu} GK- \triangleleft B=.$$

Due to the associativity and unitality of the multiplication of B, the category $\mathrm{Com}(B, \mathscr{M})$ of comodule endofunctors on $\mathscr{M}$ over B is monoidal. Studying its monoids will be a main focus of the rest of this article.

Definition 4.8. Consider a bimonad B on $\mathscr{C}$. A *comodule monad* over B on $\mathscr{M}$ is a comodule endofunctor $(K, \delta)\colon \mathscr{M} \longrightarrow \mathscr{M}$ together with morphisms of comodule functors $\mu\colon K^2 \Longrightarrow K$ and $\eta\colon \mathrm{Id}_{\mathscr{M}} \Longrightarrow K$ such that (K, μ, η) is a monad.

The conditions for the multiplication and unit of a comodule monad $K\colon \mathscr{M} \longrightarrow \mathscr{M}$ over a bimonad $B\colon \mathscr{C} \longrightarrow \mathscr{C}$ to be morphisms of comodule functors amount to



Notice how these conditions are analogous to those in Diagrams 3.4 and 3.5.

Example 4.9. Consider the poset $(\mathbb{R}, \leq)$. It is a monoidal category with addition as tensor product and $0 \in \mathbb{R}$ as unit. Hasegawa and Lemay, see [HL18], defined a bimonad using the ceiling function. Let

$$H\colon \mathbb{R} \longrightarrow \mathbb{R}, \qquad x \longmapsto \lceil x \rceil \stackrel{\text{def}}{=} \min\{\, n \in \mathbb{Z} \mid n \geq x \,\}.$$

As $\lceil x + y \rceil \leq \lceil x \rceil + \lceil y \rceil$ for all $x, y \in \mathbb{R}$ and $\lceil 0 \rceil = 0$, the functor H is opmonoidal with comultiplication and counit given by the unique arrows

$$H_1\colon H0 = 0 \longrightarrow 0 \qquad \text{and} \qquad H_{2;x,y}\colon H(x+y) \longrightarrow Hx + Hy, \text{ for all } x, y \in \mathbb{R}.$$

The idempotence of the ceiling function implies that the identity natural transformation defines a multiplication $\mu\colon H^2 \longrightarrow H$. Its unit corresponds to $(x \longrightarrow Hx)_{x\in\mathbb{R}}$.

Given a number $a \in [0,1)$ in the half-open interval bounded by 0 and 1, define

$$C_a\colon \mathbb{R} \longrightarrow \mathbb{R}, \qquad x \longmapsto \min\{\, a+n \mid n \in \mathbb{Z}, a+n \geq x \,\}.$$

In case $a = 0$, we have $C_a = H$. Otherwise $C_a(0) = a > 0$ and C_a cannot be opmonoidal. Nonetheless, $C_a(x+y) \leq C_a x + \lceil y \rceil = C_a x + Hy$ holds, and the unique natural arrow

$$\delta^{(a)}_{x,y}\colon C_a(x+y) \longrightarrow C_a x + Hy, \qquad \text{for all } x, y \in \mathbb{R}$$

defines a coaction of H on C. Again, $C_a^2 = C_a$ is idempotent and $x \leq C_a(x)$ for all $x \in \mathbb{R}$. Thus, it is a comodule monad over H.

Example 4.10. Let $(\mathscr{V}, \otimes, 1)$ be a closed symmetric monoidal category. A $\mathscr{V}$-category $\mathscr{C}$ is said to be *copowered* over $\mathscr{V}$, see [Kel05, Section 3.7], if there exists a functor $- \cdot = \colon \mathscr{C} \times \mathscr{V} \longrightarrow \mathscr{C}$, such that for all $c \in \mathscr{C}$ we have

$$c \cdot -\colon \mathscr{V} \rightleftarrows \mathscr{C} \ :\mathscr{C}(c, -).$$

One obtains $c \cdot (v \otimes w) \cong (c \cdot v) \cdot w$ by the Yoneda lemma:

$$\begin{aligned} \mathscr{C}\big(c \cdot (v \otimes w), x\big) &\cong \mathscr{V}\big(v \otimes w, \mathscr{C}(c,x)\big) \cong \mathscr{V}\big(w, \mathscr{V}(v, \mathscr{C}(c,x))\big) \\ &\cong \mathscr{V}\big(w, \mathscr{C}((c \cdot v), x)\big) \cong \mathscr{C}\big((c \cdot v) \cdot w, x\big). \end{aligned}$$

In fact, this turns $\mathscr{C}$ into a $\mathscr{V}$-module category, and therefore the identity monad on $\mathscr{C}$ into a comodule monad over $\mathrm{Id}_{\mathscr{V}}$ with trivial coaction.

Suppose that $B \overset{\text{def}}{=} UF$ is a bimonad on $\mathscr{V}$. The unit $\eta^{(B)}\colon \mathrm{Id}_{\mathscr{V}} \longrightarrow B$ is a morphism of bimonads and we may extend the coaction of $\mathrm{Id}_{\mathscr{C}}$ to

$$\delta\colon - \cdot = \xrightarrow{- \cdot \eta^{(B)}} - \otimes B=.$$

Remark 4.11. Let $B\colon \mathscr{C} \longrightarrow \mathscr{C}$ be a bimonad and $(K, \delta)\colon \mathscr{M} \longrightarrow \mathscr{M}$ a comodule monad over it. The coaction of K allows us to define an action $\widehat{\triangleleft}\colon \mathscr{M}^K \times \mathscr{C}^B \longrightarrow \mathscr{M}^K$. For any two modules $(m, \vartheta_m) \in \mathscr{M}^K$ and $(x, \vartheta_x) \in \mathscr{C}^B$, it is given by

$$(m, \vartheta_m) \,\widehat{\triangleleft}\, (x, \vartheta_x) \overset{\text{def}}{=} \big(m \triangleleft x, (\vartheta_m \triangleleft \vartheta_x) \circ \delta_{m,x}\big).$$

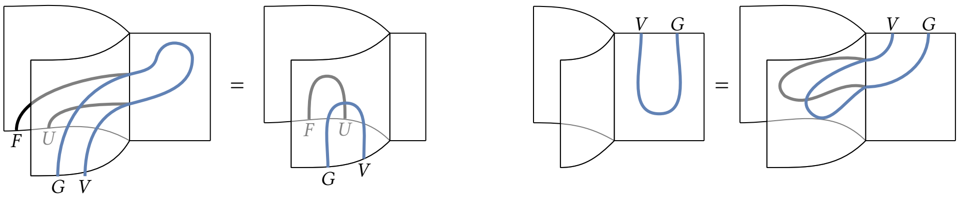
The axioms of the coaction of B on K translate precisely to the compatibility of the action of $\mathscr{C}^B$ on $\mathscr{M}^K$ with the tensor product and unit of $\mathscr{C}^B$.

We have already seen that monads and adjunctions are in close correspondence, and that additional structures on the monads have their counterparts expressed in terms of the units and counits of adjunctions. In the case of comodule monads this is slightly more complicated as we have two adjunctions to consider: one corresponding to the bimonad and one to the comodule monad.

Definition 4.12. Consider two adjunctions $F\colon \mathscr{C} \rightleftarrows \mathscr{D} \ :U$ and $G\colon \mathscr{M} \rightleftarrows \mathscr{N} \ :V$ such that $F \dashv U$ is monoidal and G, V are comodule functors over F, U. We call the pair $(G \dashv V, F \dashv U)$ a *comodule adjunction* if the following two identities hold:

$$\begin{array}{ccc} m \triangleleft x & \xrightarrow{\eta^{(G\dashv V)}_{m \triangleleft x}} & VG(m \triangleleft x) \\ {\scriptstyle \eta^{(G\dashv V)}_m \triangleleft \eta^{(F\dashv U)}_x}\downarrow & & \downarrow{\scriptstyle V\delta^{(G)}_{m,x}} \\ VGm \triangleleft UFx & \xleftarrow[\delta^{(V)}_{Gm \blacktriangleleft Fx}]{} & V(Gm \blacktriangleleft Fx) \end{array} \qquad \begin{array}{ccc} GV(n \blacktriangleleft y) & \xrightarrow{G\delta^{(V)}_{n,y}} & G(Vn \triangleleft Uy) \\ {\scriptstyle \varepsilon^{(G\dashv V)}_{n \blacktriangleleft y}}\downarrow & & \downarrow{\scriptstyle \delta^{(G)}_{Vn \triangleleft Uy}} \\ n \blacktriangleleft y & \xleftarrow[\varepsilon^{(G\dashv V)}_n \blacktriangleleft \varepsilon^{(F\dashv U)}_y]{} & GVn \blacktriangleleft FUy \end{array}$$

From their string diagrammatic representations, one observes that the conditions required by Definition 4.12 are analogous to Diagrams 3.6 and 3.7:

(4.4)


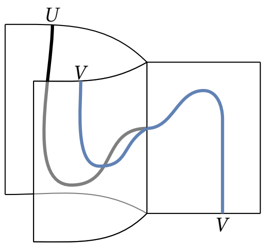
Our main theorem extends [AC12, Proposition 4.1.2] and [TV17, Lemma 7.10]; we prove it analogously to the latter.

Theorem 4.13. *Let $\mathscr{C}$ and $\mathscr{D}$ be monoidal categories, and suppose that $\mathscr{M}$ and $\mathscr{N}$ are right $\mathscr{C}$- and $\mathscr{D}$-module categories, respectively. Suppose that $F\colon \mathscr{C} \rightleftarrows \mathscr{D} : U$ is a monoidal adjunction, and let $G\colon \mathscr{M} \rightleftarrows \mathscr{N} : V$ be any adjunction. Lifts of $G \dashv V$ to a comodule adjunction are in bijection with lifts of $V\colon \mathscr{N} \longrightarrow \mathscr{M}$ to a strong comodule functor.*

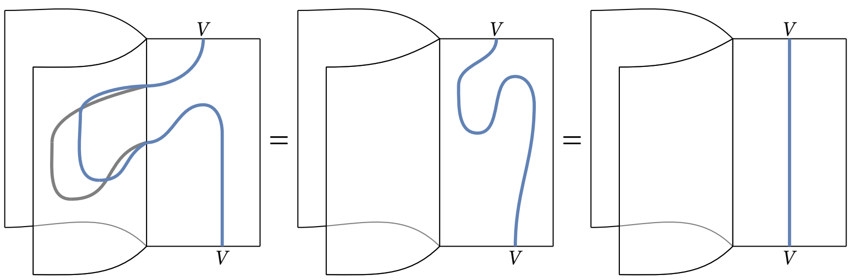
Proof. Let $G \dashv V$ be a comodule adjunction over $F \dashv U$, and write $\delta^{(V)}$ and $\delta^{(G)}$ for the coactions of V and G, respectively. For all $m \in \mathscr{M}$ and $x \in \mathscr{C}$, we define the inverse $\delta^{-(V)}\colon V{-} \triangleleft U{=} \Longrightarrow V({-} \blacktriangleleft {=})$ of $\delta^{(V)}$ via

$$Vn \triangleleft Uy \xrightarrow{\eta^{(G\dashv V)}_{Vn \triangleleft Uy}} VG(Vn \triangleleft Uy) \xrightarrow{V\delta^{(G)}_{Vn,Uy}} V(GVm \blacktriangleleft FUx) \xrightarrow{V\left(\varepsilon^{(G\dashv V)}_m \blacktriangleleft \varepsilon^{(F\dashv U)}_x\right)} V(m \blacktriangleleft x).$$

That is,

(4.5)


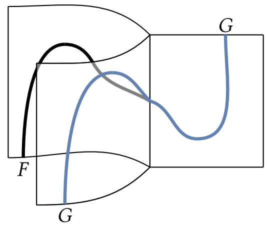
Using that G and V are part of a comodule adjunction, a straightforward computation proves $\delta^{-(V)} \circ \delta^{(V)} = \mathrm{Id}_{V(- \blacktriangleleft =)}$:



A similar strategy can be used to show that $\delta^{(V)} \circ \delta^{-(V)} = \mathrm{id}_{V- \triangleleft U=}$. Thus, $\delta^{(V)}$ is a natural isomorphism and therefore V is a strong comodule functor.

Conversely, suppose that $(V, \delta^{(V)})\colon \mathcal{N} \longrightarrow \mathcal{M}$ is a strong comodule functor over U. Define an arrow $\delta^{(G)}\colon G(- \triangleleft =) \Longrightarrow G- \blacktriangleleft F=$ by

(4.6)

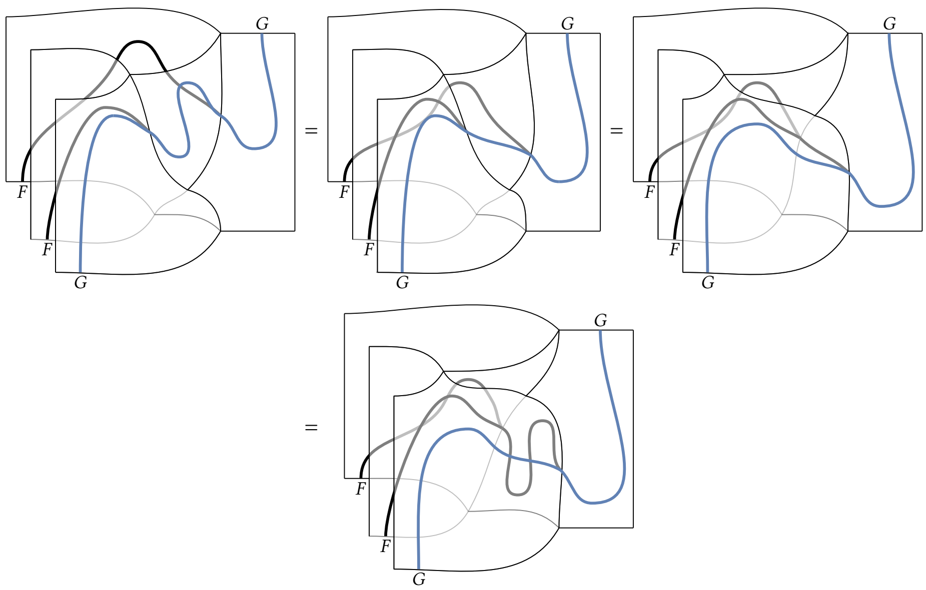


Suppose that η and ε are the unit and counit of the adjunction $F \dashv U$. Due to [TV17, Lemma 7.10], the comultiplication and counit of $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ are for all $x, y \in \mathcal{C}$ given by

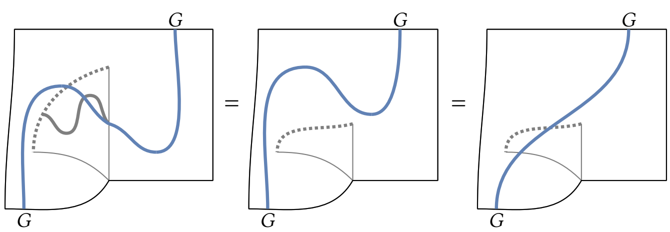
$$F_{2,x,y} \overset{\text{def}}{=} F(x \otimes y) \xrightarrow{F(\eta_x \otimes \eta_y)} F(UFx \otimes UFy) \xrightarrow{FU_{-2,Fx,Fy}} FU(Fx \otimes Fy) \xrightarrow{\varepsilon_{Fx \otimes Fx}} Fx \otimes Fy,$$

$$F_0 \overset{\text{def}}{=} F1 \xrightarrow{FU_{-0}} FU1 \xrightarrow{\epsilon_1} 1.$$

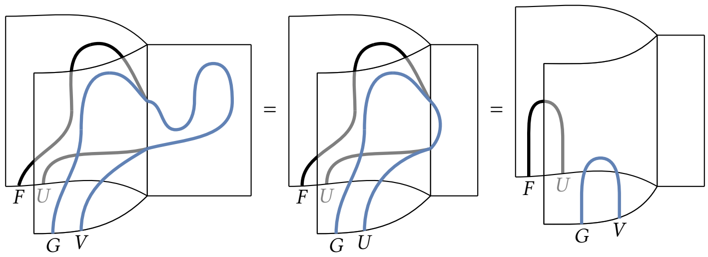
Note that, graphically, F_2 looks just like Diagram (4.6), with black strings taking the place of blue ones. The following shows that $\delta^{(G)}\colon G(- \triangleleft =) \Longrightarrow G- \blacktriangleleft F=$ is a coaction; i.e., satisfies the coassociativity and counitality conditions given in Diagrams 4.1 and 4.2:



and

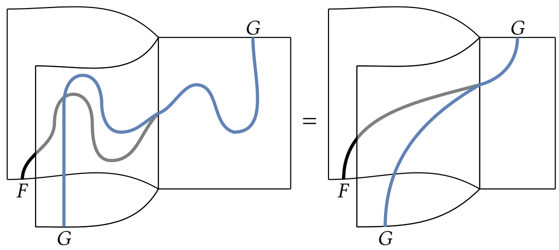


A straightforward computation proves that the unit of the adjunction $G \dashv V$ satisfies the axioms displayed in Diagram (4.4):



An analogous computation for the counit shows that $G \dashv V$ is a comodule adjunction.

To see that these constructions are inverse to each other, first suppose that we have a comodule adjunction $(G, \delta^{(G)}) \dashv (V, \delta^{(V)})$. By utilising $\delta^{-(V)}$ as given in Diagram (4.5), we obtain another coaction $\lambda^{(G)}$ on G, see Diagram (4.6). A direct computation shows that $\delta^{(G)} = \lambda^{(G)}$:



The converse direction is clear since the map which associates to any strong comodule structure on V a comodule adjunction $G \dashv V$ preserves the coaction of V. □

The philosophy that monads and adjunctions are two sides of the same coin extends to the comodule setting. Suppose that we have a monoidal adjunction $F\colon \mathscr{C} \rightleftarrows \mathscr{D} :U$ and over it a comodule adjunction $G\colon \mathscr{M} \rightleftarrows \mathscr{N} :V$. As stated in [AC12, Proposition 4.3.1], the bimonad $B \overset{\text{def}}{=} UF$ admits a coaction on the monad $K \overset{\text{def}}{=} VG$. For any $m \in \mathscr{M}$ and $x \in \mathscr{C}$ it is given by

$$VG(m \triangleleft x) \xrightarrow{V\delta^{(G)}_{m,x}} V(Gm \blacktriangleleft Fx) \xrightarrow{\delta^{(V)}_{Gm,Fx}} Km \triangleleft Bx.$$

Using the previous result, we clarify the structure of comparison functors associated to comodule adjunctions. Its proof is analogous to [BV07, Theorem 2.6].

Proposition 4.14. *Consider a comodule adjunction $G\colon \mathscr{M} \rightleftarrows \mathscr{N} :V$ over a monoidal adjunction $F\colon \mathscr{C} \rightleftarrows \mathscr{D} :U$, and denote the associated comodule monad and bimonad by $K \overset{\text{def}}{=} VG\colon \mathscr{M} \longrightarrow \mathscr{M}$ and $B \overset{\text{def}}{=} UF\colon \mathscr{C} \longrightarrow \mathscr{C}$, respectively. The comparison functor $\Sigma^K\colon \mathscr{N} \longrightarrow \mathscr{M}^K$ is a strong comodule functor over $\Sigma^B\colon \mathscr{D} \longrightarrow \mathscr{C}^B$. We furthermore have that*

$$U^K \Sigma^K = V \qquad \text{and} \qquad \Sigma^K G = F^K$$

as comodule functors.

Proof. For any $n \in \mathscr{N}$ we have $\Sigma^K n = (Vn, V\varepsilon_n)$ and a direct computation shows that the coaction of V lifts to a coaction of Σ^B on Σ^K. That is,

$$U^K \delta^{(\Sigma^K)}_{n,y} = \delta^{(V)}_{n,y}, \qquad \text{for all } n \in \mathscr{N} \text{ and } y \in \mathscr{D}.$$

Since U^K is conservative and $\delta^{(V)}$ is an isomorphism by Theorem 4.13, Σ^K is a strong comodule functor.

The U^B coaction on U^K is given by the identity natural transformation and we get $U^K\Sigma^K = V$ as comodule functors. Lastly, we compute for any $x \in \mathscr{C}$ and $m \in \mathscr{M}$,

$$\delta^{(\Sigma^K G)}_{m,x} = \delta^{(U^K\Sigma^K G)}_{m,x} = \delta^{(VG)}_{m,x} = \delta^{(K)}_{m,x} = \delta^{(U^K F^K)}_{m,x} = \delta^{(F^K)}_{m,x}.$$

□

Theorem 4.13 yields furthermore a description of a comodule monad's coaction in terms of its Eilenberg–Moore adjunction; i.e., Theorem 2, our desired analogue of Theorem 1.

Proof of Theorem 2. Suppose $\mathscr{C}^B$ acts from the right on $\mathscr{M}^K$ such that U^K is a strict comodule functor. Due to Theorem 4.13, $K = U^K F^K$ is a comodule monad via the coaction

$$\delta^{(K)} \overset{\text{def}}{=} \delta^{(U^K)} \circ U^K \delta^{(F^K)} = U^K \delta^{(F^K)}. \tag{4.7}$$

Conversely, if K is a comodule monad, $\mathscr{M}^K$ becomes a suitable right module over $\mathscr{C}^B$ with the action as given in Remark 4.11.

Since the coaction on K and the action of $\mathscr{C}^B$ on $\mathscr{M}^K$ determine the coactions of F^K uniquely, the above constructions are inverse to each other by Theorem 4.13. □

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