Pivotality, Twisted Centres, and the Anti-Double of a Hopf Monad

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Based on joint work with
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Let $(\mathscr{C}, \otimes, 1)$ be a **rigid** monoidal category: for all $x \in \mathscr{C}$ there exist left and right dual objects ${}^\star x, x^\star \in \mathscr{C}$, with appropriate evaluation and coevaluation morphisms.

Objects of the **centre** $\mathbb{Z}(\mathscr{C})$ of $\mathscr{C}$ are pairs of an $x \in \mathscr{C}$ and a **half braiding**

$$\sigma_{x,-}\colon\ x \otimes - \xRightarrow{\sim} - \otimes x.$$

A strong monoidal endofunctor $T\colon \mathscr{C} \longrightarrow \mathscr{C}$ is **centralisable** if the following coend exists:

$$Z_T := \int^{x \in \mathscr{C}} {}^\star Tx \otimes - \otimes x.$$

By work of Day and Street, the **central monad** $\mathfrak{D} := Z_{\mathrm{Id}_{\mathscr{C}}}$ has the property $\mathbb{Z}(\mathscr{C}) \simeq \mathscr{C}^{\mathfrak{D}}$.

In fact, $\mathfrak{D}$ has more structure: it is a **bimonad**, a monoid in the category $\mathrm{OpLax}(\mathscr{C}, \mathscr{C})$ of oplax monoidal endofunctors, and furthermore also a **Hopf monad**—its category of algebras is rigid monoidal.

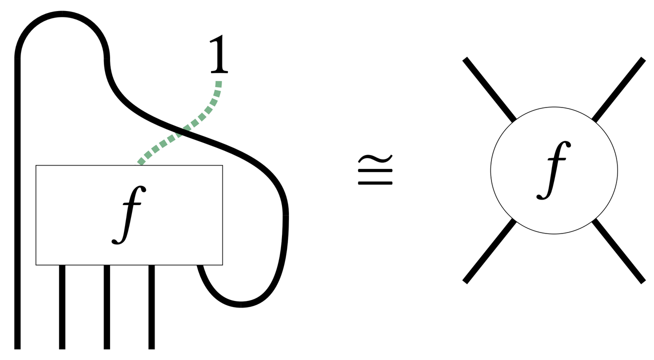
Intertwined with $\mathfrak{D}$ is the **anti-central monad** $\mathfrak{A} := Z_{\star\star(-)}$. Unravelling the structure of $\mathfrak{A}$ over $\mathfrak{D}$ involves the study of twisted centres.

The **left twisted centre** $\mathbb{Z}({}_T\mathscr{C})$ of $\mathscr{C}$ by T comprises half braidings of the form

$$\sigma_{x,-}\colon\ x \otimes - \xRightarrow{\sim} T(-) \otimes x.$$

The centre **acts** on the left twisted centre from the right: $\mathbb{Z}({}_T\mathscr{C}) \curvearrowleft \mathbb{Z}(\mathscr{C})$.

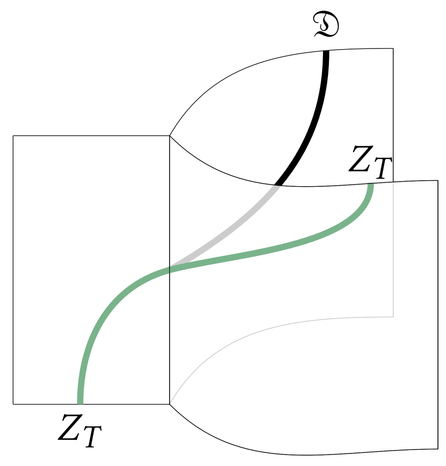
An **invertible** object in $\mathbb{Z}({}_{(-)^{\star\star}}\mathscr{C})$ induces a cyclic action on hom spaces:



The monadic interpretation not only yields $\mathscr{C}^{Z_T} \simeq \mathbb{Z}({}_T\mathscr{C})$, but also reflects the right action: Z_T is a **comodule monad** over $\mathfrak{D}$. This means that there exists a compatible coaction

$$\delta\colon\ Z_T(- \triangleleft =) \Longrightarrow Z_T(-) \triangleleft \mathfrak{D}(=).$$

Comodule monads admit a graphical interpretation, generalising a monoidal string diagrammatic calculus introduced by Willerton:



Theorem (Halbig–Z)
Let $F\colon \mathscr{C} \rightleftarrows \mathfrak{D} : U$ be a strong monoidal adjunction and $G\colon \mathscr{M} \rightleftarrows \mathscr{N} : V$ an adjunction, where $\mathscr{M} \curvearrowleft \mathscr{C}$ and $\mathscr{N} \curvearrowleft \mathfrak{D}$.
Then strong comodule structures on V are in bijective correspondence with lifts of $G \dashv V$ to a comodule adjunction.

Pivotality of a category can be decided by an isomorphism between its central and anti-central monad.

Paper, Poster, References

These results about comodule adjunctions can be applied in various contexts in order to obtain monadic reconstruction theorems.

Corollary (Halbig–Z)
Let B be a bimonad on $\mathscr{C}$ and K a monad on a right $\mathscr{C}$-module category $\mathscr{M}$. Comodule monad structures of K on B are in bijection with right actions of $\mathscr{C}^B$ on $\mathscr{M}^K$ such that U^K is a strict comodule functor over U^B.

Theorem (Stroiński–Z)
If $\mathscr{C}$ is a *nice* abelian category then all *nice* abelian $\mathscr{C}$-module categories $\mathscr{M}$ are of the form $\mathscr{C}^\perp$, for a comonad $\perp$ on $\mathscr{C}$.

As a bimonad B on $\mathscr{C}^H$ is just a monoid in $\mathrm{OpLax}(\mathscr{C}^H, \mathscr{C}^H)$, it can act from the right on another oplax monoidal functor L on $\mathscr{C}^H$. This yields an **oplax monoidal right action** $\alpha\colon LB \Longrightarrow L$, which induces an action of twisted centres $\mathbb{Z}({}_L\mathscr{C}^H) \curvearrowleft \mathbb{Z}({}_B\mathscr{C}^H)$.

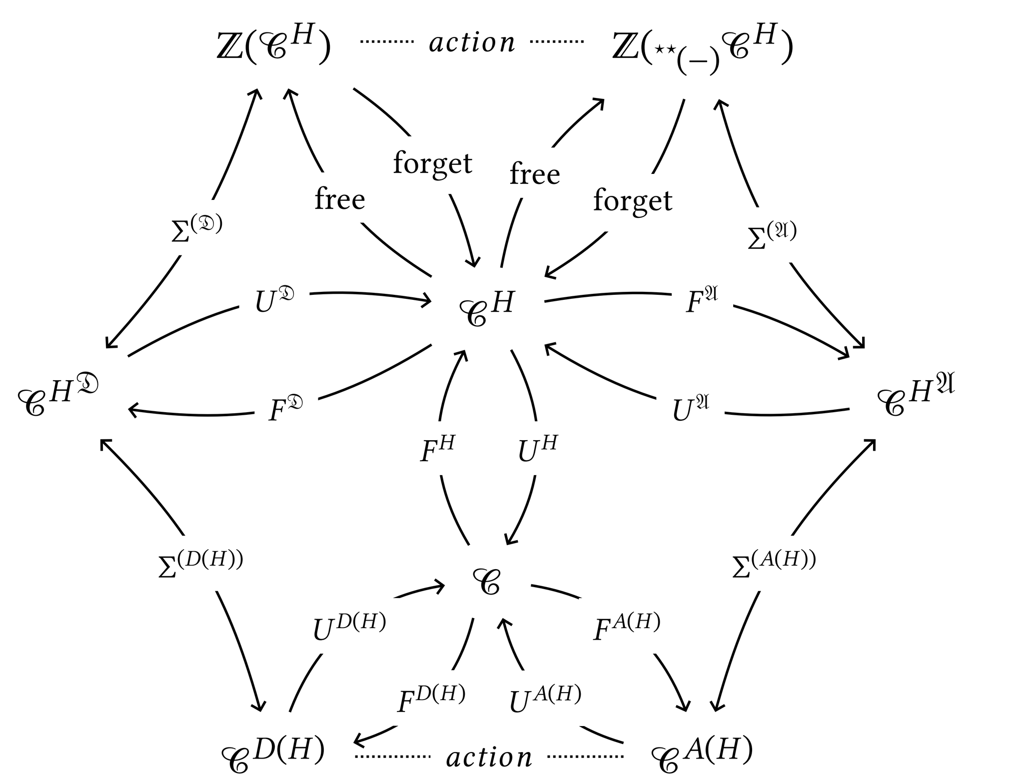
In particular, this action transcends to one of the **cross product** $B \rtimes H$ on $L \rtimes H$, where e.g.,

$$B \rtimes H := U^H B F^H\colon\ \mathscr{C} \longrightarrow \mathscr{C}.$$

Following Bruguières and Virelizier, who proved an analogous result in the bimonadic case, one obtains a version of Beck's theorem of **distributive laws**.

Z_L lifts $Z_{L \rtimes H}$ as a comodule monad, and Z_B lifts $Z_{B \rtimes H}$ as a bimonad. In particular, there exists a comodule distributive law (Ω, Λ), such that $Z_L \rtimes H = Z_{L \rtimes H} \circ_\Omega H$ as well as $Z_B \rtimes H = Z_{B \rtimes H} \circ_\Lambda H$.

The case $B := \mathrm{Id}_{\mathscr{C}^H}$, $L := {}^{\star\star}(-)\colon \mathscr{C}^H \longrightarrow \mathscr{C}^H$ is of particular importance, and we call $D(H) := Z_{\mathrm{Id}_{\mathscr{C}^H}} \rtimes H$ the **double** of H and $A(H) := Z_{\star\star(-)} \rtimes H$ the **anti-double** of H. We are left to untangle a web of adjunctions:



This leads to a result analogous to a theorem by Hajac and Sommerhäuser.

Theorem (Halbig–Z)
The following statements are equivalent:

1. The monoidal unit $1 \in \mathscr{C}$ lifts to $\mathscr{C}^{A(H)}$.
2. $D(H) \cong A(H)$ as comodule monads.
3. $D(H) \cong A(H)$ as monads.

If $\mathscr{C}$ is pivotal, the above statements are also equivalent to H admitting a so-called **pair in involution**.

Pairs in involution for Hopf monads generalise the classical case: they consist of a group-like and a character, such that the square of the antipode mediates between their adjoint actions.

Setting $H := \mathrm{Id}_{\mathscr{C}}$ in the above theorem, and using that $D(\mathrm{Id}_{\mathscr{C}}) \cong \mathfrak{D}$ and $A(\mathrm{Id}_{\mathscr{C}}) \cong \mathfrak{A}$, one obtains an insight about pivotal structures.

Corollary (Halbig–Z)
If $\mathscr{C}$ admits $\mathfrak{D}$ and $\mathfrak{A}$, then $\mathscr{C}$ is pivotal if and only if $\mathfrak{D} \cong \mathfrak{A}$ as monads.