EXOTIC FUSION SYSTEM ON A SUBGROUP OF THE MONSTER

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Abstract. We prove that an exotic fusion system described by Grazian on a subgroup of the Monster group is block-exotic, thus proving that exotic and block-exotic fusion systems are the same for all p-groups with sectional rank 3, where $p \geq 5$.

1. Introduction

Consider a prime number p and a finite p-group P. We define a *fusion system* as a category where the objects are the subgroups of P, referred to as a category on P, and the morphisms are injective group homomorphisms between these subgroups, subject to specific conditions. If these morphisms adhere to two additional axioms, we term the fusion system *saturated*. For simplicity, we shall henceforth use the term "fusion system" to mean "saturated fusion system."

Every finite group G gives rise to a fusion system $\mathcal{F}_P(G)$ on a Sylow p-subgroup P of G, where the morphisms are defined as conjugation maps induced by a fixed element in G, where they are well-defined. A fusion system constructed in this manner is termed *realizable*, while one that cannot be constructed as such is termed *exotic*. Let k be an algebraically closed field with characteristic p, and let b be a block of kG. In this context, we can also define a fusion system on a defect group P of b by defining the morphisms as well-defined conjugation maps induced by an element in G. This fusion system is denoted by $\mathcal{F}_{(P,e_P)}(G,b)$, where (P,e_P) is a maximal b-Brauer pair. Not every fusion system $\mathcal{F}$ can be realized in this way; if it can, we call $\mathcal{F}$ *block-realizable*; otherwise, it is termed *block-exotic*. The following fact follows from Brauer's Third Main Theorem (see [8, Theorem 3.6]): If G is a finite group and b is the principal p-block of kG, i.e., the block corresponding to the trivial character, with maximal b-Brauer pair (P,e_P), then $P \in \mathrm{Syl}_p(G)$ and $\mathcal{F}_{(P,e_P)}(G,b) = \mathcal{F}_P(G)$. Hence, any realizable fusion system is block-realizable. However, the converse remains an open problem and has been conjectured for some time, see [1, Part IV,7.1] and [3, 9.4]:

Conjecture 1.1. *If $\mathcal{F}$ is an exotic fusion system, then $\mathcal{F}$ is block-exotic.*

This conjecture is hard to tackle, since so far it has been mainly proved for one exotic family of fusion systems at a time, see [7], [3], [9], [10] or [12]. Since block fusion systems "misbehave" with regards to normal subgroups, a descend to normal subgroups and thus a reduction is not easily done and requires more general structures, see [9] or [11] for an explanation.

In this paper, we prove Conjecture 1.1 for all fusion systems on p-groups with sectional rank 3, where $p \geq 5$. A group has sectional rank r when any of its subgroups has at most rank r. In [6], fusion systems on p-groups of sectional rank 3 are studied and if we further assume that $p \geq 5$, Grazian proves in Theorem C that the only options for such a fusion system are either a unique

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exotic system on a group of order 7^5, or the fusion system of $\mathrm{Sp}_4(p)$ on its Sylow p-subgroup. The group of order 7^5 hosting the exotic fusion system is a maximal subgroup of a Sylow 7-subgroup of the Monster group. Here we prove that the fusion system is block-exotic too, giving rise to our main theorem:

Theorem 1.2. *Let $p \geq 5$ be a prime, P a p-group of sectional rank 3 and $\mathcal{F}$ be a fusion system on P with $\mathcal{O}_p(\mathcal{F}) = 1$. Then $\mathcal{F}$ is exotic if and only if it is block-exotic.*

See [1] for details on (block) fusion systems or Section 2 of [10] for a more compact overview of the terms needed. In the only remaining section we prove our main theorem.

2. Proof of Main Theorem

Fix P to be a maximal subgroup of a Sylow 7-subgroup of the monster. We prove that the exotic fusion system on P described by Grazian in [6] is block-exotic too. We first reduce the problem to quasisimple groups and state the reduction theorem we apply. Recall that a fusion system is called reduction simple, when it has no non-trivial proper strongly $\mathcal{F}$-closed subgroups, i.e. subgroups Q such that $\varphi(R) \leq Q$ for all $\varphi \in \mathrm{Mor}(\mathcal{F})$ defined on $R \leq Q$.

Theorem 2.1. *[10, Theorem 3.5] Let P be a non-abelian p-group such that $Z(P)$ is cyclic and let $\mathcal{F}$ be a reduction simple fusion system on P. If $\mathcal{F}$ is block-realizable, then there exists a fusion system $\mathcal{F}_0$ on P and a quasisimple group L with an $\mathcal{F}_0$-block, where $\mathcal{O}_p(\mathcal{F}_0) = 1$.*

Proposition 2.2. *Let $\mathcal{F}$ be the exotic fusion system on P. If $\mathcal{F}$ is block-realizable, it is block-realizable by the block of a finite quasisimple group.*

Proof. Clearly P has cyclic centre. Assume $1 \neq N \leq P$ is strongly $\mathcal{F}$-closed. In particular $N \trianglelefteq P$, which implies $Z(P) \leq N$. Thus, as in the proof of [5, Theorem 4.3.1], we obtain $N = P$, which means that $\mathcal{F}$ is reduction simple. By Theorem 2.1, there exists a fusion system $\mathcal{F}_0$ on P and a quasisimple group L with an $\mathcal{F}_0$-block, where $\mathcal{O}_p(\mathcal{F}_0) = 1$. However, by Theorem C of [6] we deduce $\mathcal{F} = \mathcal{F}_0$. □

The following result is useful for proving block-exoticity.

Proposition 2.3. *[10, Proposition 4.3] Let G be a quasisimple finite group and denote the quotient $G/Z(G)$ by $\overline{G}$. Suppose $\overline{G} = G(q)$ is a finite group of Lie type and let p be a prime number ≥ 7, $(p, q) = 1$. Let D be a p-group such that $Z(D)$ is cyclic of order p and $Z(D) \subseteq [D, D]$. If D is a defect group of a block of G, then there are $n, k \in \mathbb{N}$ and a finite group H with $\mathrm{SL}_n(q^k) \leq H \leq \mathrm{GL}_n(q^k)$ (or $\mathrm{SU}_n(q^k) \leq H \leq \mathrm{GU}_n(q^k)$) such that there is a block c of H with non-abelian defect group D' such that D'/Z is of order $|D/Z(D)|$ for some $Z \leq D' \cap Z(H)$.*

Proposition 2.4. *If G is as in the previous proposition, then G has no blocks with defect groups isomorphic to P.*

Proof. Recall $|P| = 7^5$ and $Z(P) \cong C_7$. We apply the previous proposition with P taking the role of D. Let H, D' be as in its assertion with $p = 7$. Assume first $H \leq \mathrm{GL}_n(q^k)$ and let a be such that $|q^k - 1|_7 = 7^a$. Then, since $\mathrm{SL}_n(q^k) \leq H$, we have $|D'| = |P/Z(P)| \cdot |Z| = 7^4|Z| \leq 7^4|Z(H)| \leq 7^4|Z(\mathrm{SL}_n(q^k))| \leq 7^{4+a}$. Now the block of $k\,\mathrm{GL}_n(q^k)$ covering c has a defect group of order at most 7^{2a+4}. But it is a well-known fact, that (non-abelian) defect groups of $\mathrm{GL}_n(q^k)$ have order at least 7^{7a+1}, see [4, Theorem 3C]. Thus, $7^{7a+1} \leq 7^{2a+4}$, which is a contradiction. The case $H \leq \mathrm{GU}_n(q^k)$ can be shown in the same fashion by considering the 7-part of $q^k + 1$ instead of $q^k - 1$. □

Proof of Theorem 1.2. By [6, Theorem C], the only exotic fusion system on such a group is a fusion system on a maximal subgroup P of a Sylow 7-subgroup of the Monster with $|P| = 7^5$. Thus, by [8, Theorem 3.6], we only need to prove block-exoticity of this system, denote it by $\mathcal{F}$.
By Proposition 2.2, if $\mathcal{F}$ is block-realizable, we may assume that it is block-relizable by the block of a finite quasisimple group G having P as defect group. We use the classification of finite simple groups to exclude all possibilites for G. Clearly, we can assume that G is non-abelian.
Firstly, assume $G/Z(G)$ is an alternating group $\mathfrak{A}_m$. Then P is isomorphic to a Sylow 7-subgroup of some symmetric group $\mathfrak{S}_{7w}$ with $w \leq 6$. Define the cycle $\sigma_i = ((i-1)7+1, \dots, i7)$ and the subgroup $S' = \langle \sigma_1, \dots, \sigma_6 \rangle \leq \mathfrak{A}_m$. Then $S' \in \mathrm{Syl}_7(\mathfrak{A}_m)$. But this group is abelian, which means that $P \notin \mathrm{Syl}_7(\mathfrak{A}_m)$.
Next, assume G is a group of Lie type. First assume the latter group is defined over a field of characteristic 7, then P is a Sylow 7-subgroup of G by [2, Theorem 6.18] and thus $\mathcal{F}$ cannot be exotic. In particular, we can assume G is defined over a field of order coprime to 7. However, by Proposition 2.4, we see that case in cross characteristic is not possible either.
Finally, assume $G/Z(G)$ is one of the sporadic groups. By, [3, Theorem 9.22], the fusion system of a block of a sporadic group can not be exotic. This proves the theorem. □

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