ON THE DESIGN OF A SYNCHRONIZING INVERSE OF A CHAOTIC SYSTEM

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Abstract — **The inverse system approach is an uniform view on hiding in and retrieving information from chaotic signals. A clue to the understanding of the system inversion is the relative degree and in connection with it a state transformation into a normal form, both presented in the paper. We point out the differences between the Pecora-Carrol scheme and the inverse system approach. Experimental results from a circuit realization are given as well as new designed examples, representing as far as we know the first two port realizations. A general structure for system inversion is introduced and applied in a novel circuit example.**

I. INTRODUCTION

Recently, the idea to use chaotic systems for information transmission has attracted much attention. Some of the transmission system examples can be treated from the general viewpoint of the *inverse system* concept. The idea is to control a chaotic system, the transmitter, with an information signal. The output of the transmitter, a chaotic broad band signal where the information is hidden, becomes after transmission the input of the receiver which has to retrieve the information signal. Note that both the transmitter and

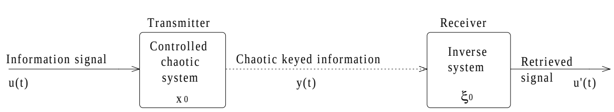

Fig. 1: Inverse system principle

the receiver are nonlinear dynamic systems, the former hiding the information in chaos and the latter extracting the information from chaos.

In practice, the information can only be retrieved, if the inverse system reproduces every input of the original system, at least asymptotically in time, irrespective of the initial conditions of the receiver. In this case, we say that the inverse *synchronises* with the original system.

Morover, it can be shown that an inverse system synchronises with its original if and only if it has unique asymptotic behaviour. It follows that in order to serve our purpose the inverse system has to have unique asymptotic behaviour while the original has to produce a chaotic signal (which is the opposite extreme) and can obviously not synchronise. A more thorough discussion is given in [10].

II. RELATIVE DEGREE

In the following we will show that the inverse system can be of lower dimensionality than the original system. Thus it may be sufficient to investigate a lower dimensional difference system. The number by which the system dimension is decreased by inversion is the *relative degree* of the original system.

A. Analogue Systems

The relative degree, r, defined for bilinear systems, indicates, roughly speaking, the lowest output derivative that is directly influenced by the input. It is defined as follows [1]:

$$\mathbf{L_f L_g^{r-1}} h(\mathbf{x}) \neq \mathbf{0} \text{ and } \mathbf{L_f L_g^{r-2} h(x) = 0} \qquad (1)$$

where $\mathbf{L_a^n} b(x)$ is the n-th Lie derivative. Equs.1 express that the r-th derivative of the output is and the (r-1)th is not influenced by the input.

A clue to the understanding of the system inversion is a state transformation into a normal form according to the relative degree, r , where the output and its first r-1 derivatives are states [1]. This Transformation

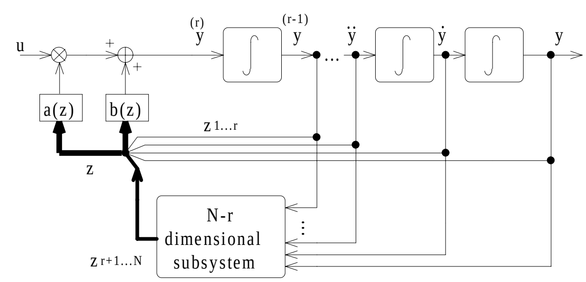

Fig. 2: The structure into which every bilinear system can be transformed

leads again to a bilinear system which is equivalent to the original. The system of Fig. 3 is obviously an inverse of the system of Fig. 2. It shows that r integrators of the original system are converted into differentiators. We conclude:

(i) The inverse of a N-dimensional system with relative degree equal to r is N-r dimensional.

(ii) If the relative degree of an analogue system (represented by state equations) is not zero then its inverse system has a generalised state representation, in which

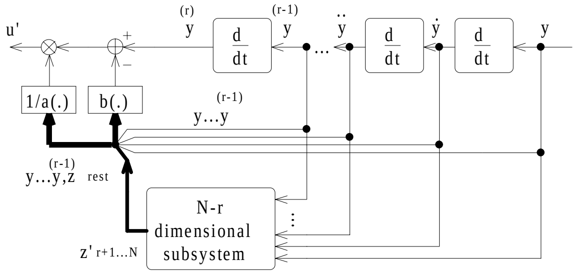

Fig. 3: Inverse of the chain structure, where r former integrators have become differentiators

the state derivatives depend also on derivatives up to the r-1-th order of the input y.

B. *Discrete-time Systems*

Translated to discrete-time systems the relative degree gives the number of time steps the current input value is delayed until it directly influences the output. However, discrete-time systems with non zero relative degree cannot be directly inverted, since there is no practical realisation of an inverse of a memory element. Therefore it is reasonable (having inversion in mind) to consider only zero relative degree discrete-time systems.

C. *Equivalent Approach*

Since the relative degree is the *minimal number* of integrations the input signal undergoes until it influences the output it can be recognised from the block diagram of the original system. As an example we consider the block diagram of [2] which represents the Chua's circuit with an input u realized by a current source in parallel to the capacitotor C1 (Fig. 4). Clearly, choosing the capacitor voltage X1 as output, the minimal number of integrators between input and output is one (Fig. 5) but choosing the inductor current X3 as output leads to r = 3. This gives an idea that the relative degree is a feature of the *structure* of a system. That means not the specific functions on the right side of the ODE of the original system determine the relative degree but the structure, i.e. which state is influenced by the input and by which states.

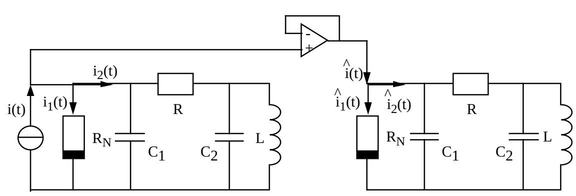

Fig. 4: Example for system inversion: Transmitter and receiver circuit with V_1 transmission at Chua's circuit

D. *Remark*

According to section I. one has to establish unique asymptotic behaviour of a driven N-r dimensional difference system (but may be only for driving signals with certain features) in order to realize synchronization. This is the case with all chaotic synchronization schemes and might be nontrivial since the difference system can be nonlinear.

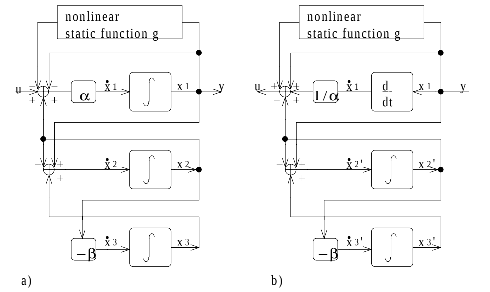

Fig. 5: Example showing that the minimal number of integrators between input and output is converted into differentiators by system inversion, a) original, b) inverse system

The common approach for such problems is the direct method of Ljapunov which is rather restrictive, since it does not admit the temporary increasing of the sytems energy. In [3] the outweighing of expansion by contraction was estimated for a one dimensional difference system in order to establish synchronization. However, in higher dimensional difference systems divergence and convergence may occur in different 'directions' of state space and the estimation of their compensation is nontrivial. Therefore one has to regard the worst case of expansion with respect to a chosen Ljapunov function $V(\mathbf{x})$, i.e. its maximum, for the whole difference state space. The *Matrix Measure induced by* V [4] serves this purpose. It yields an upper bound of the systems energy. If one establishes the *mean value* of the matrix measure to be negative then the trivial solution of the difference system is asymptotically stable and it exists a nonempty basin of attraction, i.e. sychroizatiion can take place.

In case of a quadratic Ljapunov function we suggest an upper bound estimation by means of eigenvalues. However, further consideration and application of this approch is beyond the scope of this paper.

III. Relation of the Inverse System to the Pecora-Carrol Scheme

The experienced reader might be tempted to compare the synchronization of inverse systems with the well known Pecora-Carrol *driving scheme* [5], where a subsystem of the original system is driven by a transmitted state. As an example for driving we present the circuit realization and the block diagram of its receiver of [6] (Figs. 6, 7). The comparison with Fig. 4 and Fig. 5 should make it clear that an inverse system is completely different from a driven subsystem.

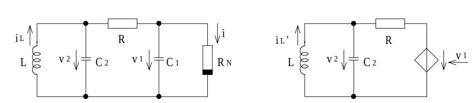

Fig. 6: Example for Driving: Transmitter and receiver circuit for V_1 driving at Chua's circuit

Allthough, in case r¿0 the inverse system represents a driven lower dimensional subsystem too, there are crucial differences between both approaches:

1. Existence of an input to the transmitter in case of the inverse system method while the transmitter of the *driving scheme* of [5] is autonomous.

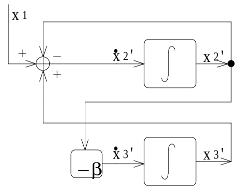

Fig. 7: Block diagram of the driven subsystem of the receiver in Fig. 6

2. Driving only servs the purpose of reconstructing the non transmitted states. The inverse system does this *and* additionally retrieves the transmitter input signal.

3. In terms of circuit realization driving is to replace the memory element belonging to the transmitted state (e.g. a capacitor) in the receiver by a controlled (e.g. voltage-) source, whereas in the inverse system the controlled source has to impose the state of a memory element, but without replacing it.

IV. CIRCUIT EXAMPLES

A. One Port Realizations

All examples published so far realize inversion by treating current and voltage of a 1-port alternatively as input and output. One of the two possible situations is depicted in Fig. 8(a). Even though all exam-

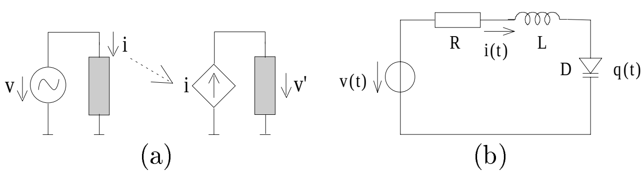

Fig. 8: (a) Inverse system realisation with a 1-port: the $V \rightarrow I \rightarrow V$ method (b) RLDiode one port excited by a voltage signal e(t) (i(t): inductor current, q(t): charge of the diode capacity)

ples we know about are 1-port realisations they have different relative degrees, actually r=0 od r=1. This can be verified by drawing the block diagrams.

The RLDiode circuit is such an example (Fig. 8(b)) [7] well known to produce a chaotic current when excited by a periodic voltage signal. Using an information bearing voltage signal e(t) as input and the inductor current as output leads to a r=1 system the inverse of which has unique asymptotic behaviour. This is easy to show with a Ljapunov function for the one dimensional difference system.

Next we present some experimental resuts, when the input is an AM signal, which should convince that:

1. The principle works with a certain robustness against parameter mismatch between transmitter and receiver.

2. The information can be fairly good hidden in the transmitted broad band signal.

B. Two Port Realizations

It is possible to overcome the restriction to one port realizations of inverse systems by use of an ideal operational amplifier (op.-amp.) i.e. a nullator-norator pair. One of the four possible situations is depicted in Fig. 11. Next we give an two port example which

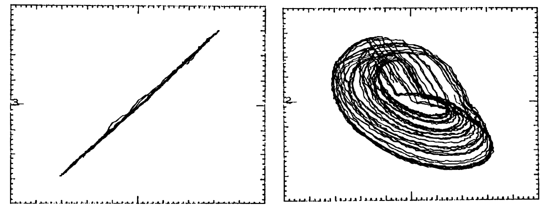

Fig. 9: left: a fairly good transfer characteristic: retrieved signal versus input siganl; right: transmitted signal versus input signal

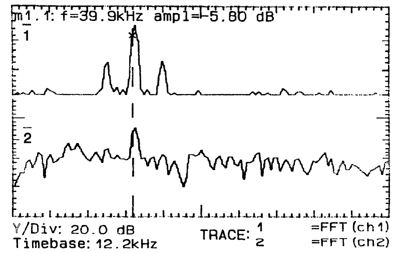

Fig. 10: Frequency spectra showing a fairly good hidden information in the tansmitted signal: ch1-input AM-signal, ch2-transmitted chaotic signal (representing the current)

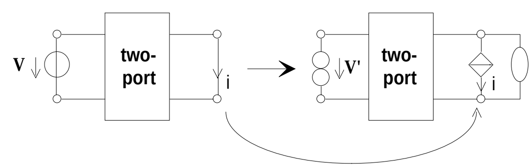

Fig. 11: Two port realization of system inversion, in case of voltage input and current output at another branch, by means of an ideal op.-amp. i.e. a nullator-norator pair

is designed such that its relative degree is equal to the original system dimension i.e. r=N. Following section II. this leads to a zero dimensional inverse system which simply realizes a static function of the transmitted signal and its derivativs.

As discussed in section II.C. choosing the inductor current X3 as output in the Chuas circuit of Fig. 4 leads to r=3=N. The circuit realization of the appropriate inverse system is depicted in Fig. 12. In practical realizations the nonideal characteristics of the op.-amp. have to be taken into account, which could provide an additional state in the inverse system. A paper concernig the correct functioning of nonideal op.-amps. is in preparation.

V. DESIGN OF INVERSE SYSTEMS

If a system has a relative degree r=N, its inverse is zero dimensional. Since in this case one does not have to care about asymptotic uniqueness it seems desirable to choose input and output of a chaotic system so that r=N. However, in this case any added channel noise leads to serious errors in signal recovery because it is differentiated several times. Therefore we propose a zero relative degree structure in the sequel.

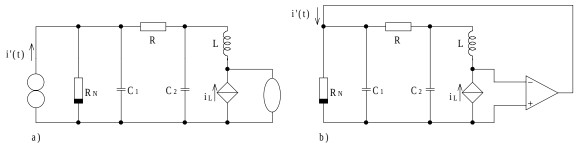

Fig. 12: Two port realization of an inverse system (for r=3=N i.e. a static inverse) by means of a) a nullator-norator pair b) appropriate use of an op.-amp.

A. General Structure

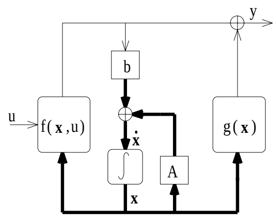

Fig. 13: Genaral structure: original system; bold lines: vector instead of scalar, for discrete-time system the vector integrator has to be replaced by a vector delay

At this point it becomes clear how such a crucial system feature as chaotic motion (in the autonomous case u=0) can be converted into unique asymptotic behaviour by system inversion:

Consider the different roles of $f(\mathbf{x}))$ and $g(\mathbf{x})$. While $f(\mathbf{x}))$ represents the recrsive, i.e. the *feed-back* part, $g(\mathbf{x}))$ serves only as part of the output function, i.e. as *feed-forward* part. Obviously, the (nonlinear) function $f(\mathbf{x})$ is responsible for the chaotic motion of the original system whereas $g(\mathbf{x})$ does not influence it at all. In the inverse system the roles of $f(\mathbf{x})$ and $g(\mathbf{x})$ are exchanged. Therefore by choosing them of different nature one can achieve qualitatively different behaviour in the original and inverse system.

Our goal is an inverse system with unique asymptotic behaviour. Since this is easy to establish for linear systems, we choose $g(\mathbf{x})$ as a linear function of states $g(\mathbf{x}) = \mathbf{c^T} \cdot \mathbf{x}$ (Fig. 14).

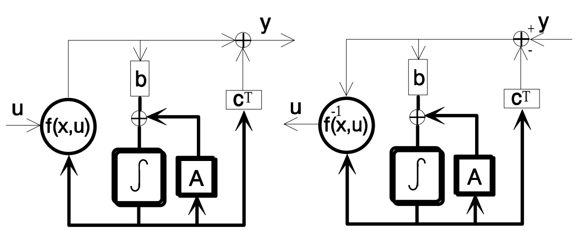

Fig. 14: Genaral structure and its inverse, bold lines: vector instead of scalar, for discrete-time system the vector integrator has to be replaced by a vector delay

Our proposed structure has the following features:

(1) Since it has zero relative degree, it is applicable to discrete-time systems as well.

(2) It is invertible if $f(\mathbf{x}, \mathbf{u})$ is invertible with respect to u.

(3) It contains a unique static nonlinear system which is real valued, i. e. it is similar to a Lur'e system [8].

(4) Its inverse and therefore also the difference system is a *linear* system with only a nonlinear output function.

$$\Delta\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{b} \cdot \mathbf{c^T}) \cdot \mathbf{\Delta x} \qquad (2)$$

(5) Synchronisation (asymptotic uniqueness of the inverse system behaviour) is because of (4) easy to establish.

(6) Provided the pair $(\mathbf{A}, \mathbf{b})$ is controllable it is possible to design the synchronisation speed by setting the poles of the linear inverse system by applying the Ackermann formula in order to choose $\mathbf{c^T}$ [9].

B. Design Example

Here we apply our general structure to Chua's circuit, which is evidently a Lur'e type system. Since its linear part is already passive we choose $\mathbf{c^T} = \mathbf{0}$. The resulting circuit and its inverse is depicted in Fig. 15. It represents again a two port realization. By simulation

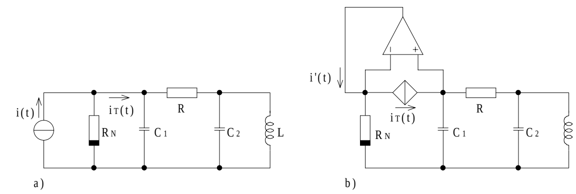

Fig. 15: Design example: genaral structure applied to Chua circuit - a two port realization

we obtained good synchronisation results even under assumption of a nonideal op.-amp., where we used a one-pole modell.

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