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A semi-algebraic view on quadratic constraints for polynomial systems[☆]

Torbjørn Cunis[a,*], Harald Pfifer[b]

[a] Institute of Flight Mechanics and Control, University of Stuttgart, 70569 Stuttgart, Germany
[b] Chair of Flight Mechanics and Control, TU Dresden, 01307 Dresden, Germany

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ABSTRACT

We show that quadratic constraints admit a semi-algebraic interpretation of dynamic systems. This allows us to improve the analysis of polynomial systems under nonlinear feedback laws by use of the general S-procedure. Extending results to integral quadratic constraints, with the aid of LaSalle's invariance theorem, we obtain a general stability proof for a larger class of multipliers. Numerical results show that the resulting hierarchy of sum-of-squares problems yields much better stability estimates for an exemplary unstable system with nonlinear stabilizing feedback than local approximations.

1. Introduction

We are concerned with the local analysis of polynomial differential equations subject to a bounded, memoryless nonlinear operator that vanishes at the origin. Such problems often arise in the study of dynamic feedback systems, for example, suboptimal model-predictive control (Leung, Liao-McPherson, & Kolmanovsky, 2021) or feedback by neural network controllers (Fazlyab, Morari, & Pappas, 2020; Hashemi, Ruths, & Fazlyab, 2021; Yin, Seiler, & Arcak, 2021), where the nonlinear operators in place would be a convex projection or a hyperbolic tangent, respectively. Among the most basic applications are systems under saturated feedback (Fang, Lin, & Rotea, 2008; Hindi & Boyd, 1998; Ji, Sun, & Liu, 2008). One particular question of interest is the region of attraction of such closed-loop systems, that is, the set of all initial conditions that lead to converging system responses. While the true region of attraction is usually hard to determine (Genesio, Tartaglia, & Vicino, 1985), we are content with computing invariant subsets satisfying a dissipation inequality, typically a sublevel set of a Lyapunov function (La Salle, 1960). Notable approaches include sum-of-squares methods for polynomial nonlinearities (Topcu, Packard, & Seiler, 2008). Sum-of-squares techniques were recently applied to polynomial systems and nonlinear operators satisfying integral quadratic constraints (IQCs) (Iannelli, Seiler, & Marcos, 2019). Most IQCs in literature bound the nonlinear operator globally, i.e., for all possible inputs. This is in stark contrast to the inherently local problem of finding a region of attraction. Hence, conventional IQC approaches are conservative for local analysis. In Summers and Packard (2010), this problem is circumvented by the notion of local IQCs which have been successfully applied to the analysis of closed loop systems under saturation (Knoblach, Pfifer, & Seiler, 2015) or control allocation schemes (Pusch, Ossmann, & Pfifer, 2022). Still, the method is computationally cumbersome relying on a heuristic iteration and does not scale well for multi-input, multi-output nonlinear operators.

Our main contribution is to propose a new parametrization of quadratic constraints as dissipativity conditions with polynomial multipliers, which can be solved as sum-of-squares program, using semi-algebraic sets. This result allows us to compute Lyapunov functions over intersected quadratic constraints to certify local stability of open-loop unstable systems. In extension, we prove a relaxation of the dissipativity constraint in Iannelli et al. (2019) that allows for a larger class of nonnegative multipliers such as (but not limited to) sum-of-squares polynomials; here, our stability result is directly based on La Salle's invariance principle.

Notation Let $\mathbb{R}$ be the space of real numbers, $\mathbb{R}_{\geq 0}$ the set of nonnegative reals, $\mathbb{R}^n$ the Euclidean vector space of dimension n, and $\mathbb{R}^{m \times m}$ the (vector) space of m-by-m matrices. Let $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$ denote the Euclidean vector norm and dot product, respectively, and $\mathcal{L}_{2e}^m$ the set of finite-time square-integrable trajectories $v : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$, that is, $v \in \mathcal{L}_{2e}$ if and only if $\int_0^T \|v(t)\|^2 \mathrm{d}t < \infty$ for any $T \geq 0$. For a symmetric matrix $M \in \mathbb{R}^{m \times m}$ and vector

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* Corresponding author.
E-mail addresses: tcunis@ifr.uni-stuttgart.de (T. Cunis), harald.pfifer@tu-dresden.de (H. Pfifer).

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$y \in \mathbb{R}^m$, define the quadratic form $\sigma_M(y) = \langle y, My \rangle$. The graph of $\Delta : \mathbb{R}^m \to \mathbb{R}^m$ is the set graph $\Delta = \{(v, w) \in \mathbb{R}^{2m} \mid v = \Delta(w)\}$.

2. Preliminaries

Generally speaking, we consider nonlinear systems of the form

$$\dot{x}(t) = f(x(t), w(t)) \tag{1a}$$

$$v(t) = h(x(t)) \tag{1b}$$

$$w(t) = \Delta(v(t)), \tag{1c}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}^m$ are Lipschitz continuous satisfying $f(0, 0) = 0$, $h(0) = 0$; the nonlinear operator $\Delta : \mathbb{R}^m \to \mathbb{R}^m$ is continuous, bounded,[1] and $\Delta(0) = 0$. A solution of (1) on $x_0 \in \mathbb{R}^n$ is an absolutely continuous trajectory $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ that satisfies (1) for almost all $t \geq 0$ and $x(0) = x_0$.

Definition 1. A set of initial conditions $\Omega \subset \mathbb{R}^n$ is *invariant* with respect to (1) if any solution $x(\cdot)$ on $x_0 \in \Omega$ satisfies $x(t) \in \Omega$ for all $t \geq 0$.

2.1. Problem statement

The region of attraction $\mathcal{R}_\Delta$ is the largest invariant set $R \subset \mathbb{R}^n$ such that all solutions $x(\cdot)$ of (1) with $x(0) \in R$ converge to the origin as $t \to \infty$. Inner estimates of $\mathcal{R}_\Delta$ can be characterized by Lyapunov-type arguments (Iannelli et al., 2019; La Salle, 1960; Topcu et al., 2008). The region of attraction contains some open neighbourhood of the origin if (1) is locally asymptotically stable around the origin, which will be our standing assumption.

Problem 1. Compute a set $R \subseteq \mathcal{R}_\Delta$ which is invariant with respect to (1).

In order to approach this problem, we use (integral) quadratic constraints to describe a class of nonlinear systems, including (1), for which stability can be assessed by polynomial methods.

2.2. Quadratic constraints

A quadratic constraint describes a subset of the input–output space of the nonlinear operator Δ that contains its graph. This subset is a *semi-algebraic* set, that is, it contains all pairs $(v, w) \in \mathbb{R}^{2m}$ that satisfy a polynomial (namely, quadratic) inequality. A nonlinear operator is hence overapproximated by the class of nonlinear operators that satisfy the same quadratic constraint (in the sense to be defined next) and since this class is defined by a polynomial (quadratic) inequality, stability can be assessed as polynomial (matrix) optimization problems.

Definition 2. The operator Δ satisfies the *quadratic constraint* given by a symmetric matrix $M \in \mathbb{R}^{2m \times 2m}$ if for all $(v, w) \in$ graph Δ

$$\sigma_M((v, w)) \geq 0, \tag{2}$$

where $\sigma_M : y \mapsto \langle y, My \rangle$ is a quadratic form on $\mathbb{R}^{2m}$.

A consequence of this definition is that, if Δ satisfies the quadratic constraint given by M and $v \in \mathcal{L}^m_{2e}$, then (2) is satisfied by $(v(t), \Delta \circ v(t))$ for all $t \geq 0$. The choice of M (or σ_M, respectively) directly influences the conservatism of any analysis result. A list of possible choices for M for different nonlinear operators can be found in Megretski and Rantzer (1997, Section VI). Unfortunately, how to optimally pick σ_M is still an open question in the analysis with (integral) quadratic constraints. A common, practical solution is a linear combination of $\sigma_{M_1}, \ldots, \sigma_{M_l}$ with nonnegative constant factors, see e.g. Iannelli et al. (2019) and Pfifer and Seiler (2016). Note that various more complex parametrizations have been studied in literature for specific operators, e.g., Veenman and Scherer (2014). The following result is a special case of Iannelli et al. (2019, Theorem 1) for quadratic constraints based on the aforementioned linear combination. We denote by $\tilde{w}$ the input of the open-loop dynamics defined by (1a) and (1b).

Theorem 1. *Let Δ satisfy the quadratic constraints given by $M_1, \ldots, M_l$; if there exist a smooth positive definite[2] function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, nonnegative scalars $\lambda_1, \ldots, \lambda_l \in \mathbb{R}_{\geq 0}$, and a positive definite function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that*

$$\langle \nabla V(x), f(x, \tilde{w}) \rangle + \sum_{i=1}^{l} \lambda_i \sigma_{M_i}((h(x), \tilde{w})) \leq -\gamma(\|x\|) \tag{3}$$

for all $(x, \tilde{w}) \in \Omega \times \mathbb{R}^m$, where $\Omega = \{x \in \mathbb{R}^n \mid V(x) \leq 1\}$ is bounded,[3] then Ω is an invariant subset of $\mathcal{R}_\Delta$. ◁

This result has two notable disadvantages: First, it unnecessarily requires that Δ satisfies the quadratic constraints for all $v \in \mathbb{R}^m$, even though the condition for the Lyapunov function is only assessed on the compact domain Ω; and second, while suitable for open-loop locally stable systems, it cannot account for unstable systems. A solution to overcome the first problem is the notion of local QCs as given in the following definition.

Definition 3. The operator Δ satisfies the local quadratic constraint given by a symmetric matrix M on a subset $\Upsilon \subset \mathbb{R}^m$ if (2) holds for all $(v, w) \in$ graph $\Delta \cap (\Upsilon \times \mathbb{R}^m)$.

Corollary 1. *Let Δ satisfy the local quadratic constraints given by $M_1, \ldots, M_l$ on $\Upsilon \subset \mathbb{R}^m$ and let Ω be as in Theorem 1; if the conditions of Theorem 1 are satisfied and $h(\Omega) = \{h(x) \mid x \in \Omega\} \subseteq \Upsilon$, then Ω is an invariant subset of $\mathcal{R}_\Delta$.* ◁

Although local quadratic constraints avoid the issues of classical quadratic constraints, they require a good a priori choice of the set Υ. In that case, an iterative process can be employed to improve the local constraint (see, e.g., Knoblach et al. (2015, Alg. 1)). For a higher input/output dimension m, the selection becomes tedious if not intractable. Instead, our approach will build upon an implicit local relaxation through the semi-algebraic nature of quadratic constraints, thus avoiding iterations.

Remark 1. Relaxations of (3) for linear systems have also been proposed in Pfifer and Seiler (2016) and Veenman and Scherer (2014); whether these are applicable in the nonlinear case is, to the authors' knowledge, an open question.

3. Semi-algebraic stability analysis

We improve the local stability analysis of polynomial systems under nonlinear operators subject to quadratic constraints. A quadratic constraint defines a semi-algebraic set of points (v, w) satisfying (2) and hence provides an overapproximation of the nonlinear operator. Through the generalized S-procedure for intersections of semi-algebraic sets we can obtain a hierarchy of polynomial inequalities that assert stability.

[1] A function $\psi : \mathbb{R}^m \to \mathbb{R}^m$ is *bounded* if there exists a constant $c \geq 0$ such that $\|\psi(x)\| \leq c\|x\|$ for all $x \in \mathbb{R}^m$.

[2] A function $\psi : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is *positive definite* if $\psi(0) = 0$ and $\psi(x) > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$.

[3] Boundedness of Ω holds, e.g., if the function V is radially unbounded.

Proposition 1. *Let Δ satisfy the quadratic constraints given by $M_1, \dots, M_l$; if there exist a smooth positive definite function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, a positive definite function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, and nonnegative functions $s_1, \dots, s_l : \mathbb{R}^{n+m} \to \mathbb{R}_{\geq 0}$ such that*

$$\langle \nabla V(x), f(x, \tilde{w})\rangle + \sum_{i=1}^{l} s_i(x, \tilde{w})\sigma_{M_i}((h(x), \tilde{w})) \leq -\gamma(\|x\|) \tag{4}$$

for all $(x, \tilde{w}) \in \Omega \times \mathbb{R}^m$, where $\Omega = \{x \in \mathbb{R}^n \mid V(x) \leq 1\}$ is compact, then Ω is an invariant subset of $\mathcal{R}_\Delta$.

Proof. Suppose that V, γ, and $s_1, \dots, s_l$ satisfy the assumptions above; take any $x \in \Omega$, let $w = \Delta(h(x))$, and define $\dot{V}(x) = \langle \nabla V(x), f(x, w)\rangle$. Then $\sigma_{M_i}((h(x), w)) \geq 0$ as Δ satisfies the quadratic constraint given by M_i and $s_i(x, w) \geq 0$ by nonnegativity for all $i \in \{1, \dots, l\}$; hence,

$$\dot{V}(x) \leq -\gamma(\|x\|)$$

for all $x \in \Omega$. Since γ is positive definite, $\{0\}$ is the largest invariant subset of Ω such that $\dot{V}$ vanishes. Thus any solution of (1) on $x_0 \in \Omega$ converges to $\{0\}$ by virtue of La Salle (1960, Theorem 2) and Ω is invariant, the desired result. □

For the remainder of this section, we limit ourselves to the scalar operators ($m = 1$) typically encountered in the feedback from suboptimization or neural networks; assuming that Δ is monotone nondecreasing and slope-restricted.

Definition 4. The operator $\Delta : \mathbb{R} \to \mathbb{R}$ is monotone nondecreasing and slope-restricted if $\Delta(0) = 0$ and $\langle \Delta(u) - \Delta(v), c(u - v) - \Delta(u) + \Delta(v)\rangle \geq 0$, where $c \geq 0$, for all $u, v \in \mathbb{R}$.

Operators such as the saturation function, projection onto an interval $\{0\} \subseteq I \subset \mathbb{R}$, and the activation functions ReLU and tanh are monotone nondecreasing and slope-restricted. By definition, the graph of such an operator Δ lies in the sector $[0, c]$, that is,

$$\langle \Delta(v), cv - \Delta(v)\rangle \geq 0 \tag{5}$$

for all $v \in \mathbb{R}$. This equation can equivalently be written in a quadratic form $\sigma_{M_c}((v, w))$ with

$$M_c = \begin{bmatrix} 0 & c/2 \\ c/2 & -1 \end{bmatrix}$$

where $c \geq 0$ is the upper sector bound, implying that Δ satisfies the quadratic constraint given by M_c.

Using some $\hat{v} > 0$, we define

$$v_1(\cdot) = h(x(\cdot)) - \hat{v}, \qquad v_2(\cdot) = h(x(\cdot)) + \hat{v}, \tag{6a}$$

$$w_1(\cdot) = \Delta_1(v_1(\cdot)) + \delta_1, \quad w_2(\cdot) = \Delta_2(v_2(\cdot)) + \delta_2, \tag{6b}$$

$$\Delta_1 : v \mapsto \Delta(v + \hat{v}) - \delta_1, \ \Delta_2 : v \mapsto \Delta(v - \hat{v}) - \delta_2, \tag{6c}$$

where $\delta_1 = \Delta(+\hat{v})$ and $\delta_2 = \Delta(-\hat{v})$ are constants. Clearly, we have that $w_1(\cdot) \equiv w_2(\cdot) \equiv w(\cdot)$.

Lemma 1. *Let Δ be a monotone nondecreasing and slope-restricted operator; the graphs of Δ_1, Δ_2 defined in (6) lie in the sector $[0, c]$.*

Proof. It follows from (6) and the definition of $\delta_{1,2}$ that $\Delta_1(0) = \Delta_2(0) = 0$. Furthermore,

$$\langle \Delta_{1,2}(u) - \Delta_{1,2}(v), c(u - v) - \Delta_{1,2}(u) + \Delta_{1,2}(v)\rangle \equiv$$
$$\langle \Delta(\bar{u}_{1,2}) - \Delta(\bar{v}_{1,2}), c(\bar{u}_{1,2} - \bar{v}) - \Delta(\bar{u}_{1,2}) + \Delta(\bar{v}_{1,2})\rangle,$$

where $\bar{u}_{1,2} = u \pm \hat{v}, \bar{v}_{1,2} = v \pm \hat{v} \in \mathbb{R}$; hence, $\Delta_{1,2}$ satisfy Definition 4 and the sector bound (5) is satisfied. □

In other words, the augmented operator $(\Delta_1, \Delta_2) : \mathbb{R}^2 \to \mathbb{R}^2$ satisfies multiple quadratic constraints based on M_c. Then Proposition 1 can be used to establish asymptotic stability of the augmented system

$$\dot{x}(t) = f(x(t), w_1(t), w_2(t))$$
$$v_1(t) = h(x(t)) - \hat{v}, \quad v_2(t) = h(x(t)) + \hat{v}$$
$$w_1(t) = \Delta_1(v_1(t)), \quad w_2(t) = \Delta_2(v_2(t))$$

using that $w_1(t) \equiv w_2(t)$ for all $t \geq 0$.

4. Extensions to integral quadratic constraints

In order to satisfy a quadratic constraint, the inequality (2) must hold point-wise in time for any pair of input/output signals (v, w) of the nonlinear operator Δ. Quadratic constraints can thus be used to describe nonlinear operators without internal dynamic or memory. Consequently, time-varying operators such as delays can be described by integral quadratic constraints (Megretski & Rantzer, 1997). IQCs were initially specified in the frequency domain but time-domain formulations suitable for Lyapunov-like analysis were derived later (see, e.g., Seiler, 2015). Here, the internal dynamics are represented by a filter ψ with state $x_\psi : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n_\psi}$, output $y_\psi : \mathbb{R}_{\geq 0} \to \mathbb{R}^{2m}$, linear dynamics $f_\psi : \mathbb{R}^{n_\psi} \times \mathbb{R}^{2m} \to \mathbb{R}^{n_\psi}$, and linear output function $h_\psi : \mathbb{R}^{n_\psi} \times \mathbb{R}^{2m} \to \mathbb{R}^{2m}$. We are limiting this study to integral quadratic constraints with finite horizon, so-called *hard* IQCs.

Definition 5. The operator Δ satisfies the hard *integral quadratic constraint* $\Pi = (M, \psi)$ if, for all $N \geq 0$ and $v \in \mathcal{L}_{2e}$,

$$\int_0^N \sigma_M(y_\psi(t))\, \mathrm{d}t \geq 0, \tag{7}$$

subject to

$$\dot{x}_\psi(t) = f_\psi(x_\psi(t), (v(t), w(t))), \quad x_\psi(0) = 0,$$
$$y_\psi(t) = h_\psi(x_\psi(t), (v(t), w(t)))$$

and $(v(t), w(t)) \in \operatorname{graph} \Delta$ for all $t \in [0, N]$.

In this section we consider that Δ satisfies a set of hard IQCs $\Pi_1 = (M_i, \psi_i), \dots, \Pi_l = (M_l, \psi_l)$. We denote the combined dynamics of (1a), (1b), and $\psi_1, \dots, \psi_l$ under input $\tilde{w}$ as

$$\dot{z}(t) = \bar{f}(z(t), \tilde{w}) \tag{8a}$$

$$y_\psi(t) = \bar{h}(z(t), \tilde{w}) \tag{8b}$$

for almost all $t \geq 0$, where $z = (x, x_{\psi_1}, \dots, x_{\psi_l}) \in \mathbb{R}^{n_z}$ is the extended state, $y_\psi = (y_{\psi_1}, \dots, y_{\psi_l}) \in \mathbb{R}^{2lm}$ is the stacked output of $\psi_1, \dots, \psi_l$, and $v(t) \equiv h(x(t))$. The results of Iannelli et al. (2019, Theorem 1) extend Theorem 1 to the integral quadratic constraints $\Pi_1, \dots, \Pi_l$ if there exist a smooth positive definite function $V : \mathbb{R}^{n_z} \to \mathbb{R}_{\geq 0}$, constants $\lambda_1, \dots, \lambda_l \geq 0$, and a positive definite function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that

$$\nabla V(z)^{\mathsf{T}} \bar{f}(z, \tilde{w}) + \sum_{i=1}^{l} \lambda_i \sigma_{M_i}(\bar{h}_i(z, \tilde{w})) \leq -\gamma(\|z\|) \tag{9}$$

for all $(z, \tilde{w}) \in \bar{\Omega} \times \mathbb{R}^m$, where $\bar{\Omega} = \{z | V(z) \leq 1\}$ is bounded and $\Pi_i = (M_i, \psi_i)$ for all $i \in \{1, \dots, l\}$. It should be noted that under (9), the set $\bar{\Omega}$ is invariant with respect to (8) and $\{x \mid V(x, 0) \leq 1\}$ is a subset of $\mathcal{R}_\Delta$, but the latter is not invariant with respect to (1).

4.1. An extended stability theorem

While the definition of integral quadratic constraints trivially extends to linear combinations with nonnegative constants, the extension of the semi-algebraic approach to IQCs is difficult. As step towards such an extension, we are going to prove an extended stability theorem which generalizes the result of Iannelli et al. (2019, Theorem 1). Let $\bar{\sigma}_M : \mathbb{R}^{2lm} \to \mathbb{R}^l$ denote the element-wise quadratic form $\bar{\sigma}_M : (h_1, \ldots, h_l) \mapsto (\sigma_{M_1}(h_1), \ldots, \sigma_{M_l}(h_l))$ where $\Pi_i = (M_i, \psi_i)$ for all $i \in \{1, \ldots, l\}$. We introduce the additional state trajectories

$$\mu(t) = \int_0^t \bar{\sigma}_M(\bar{h}(z(\tau), w'(\tau)))\,\mathrm{d}\tau \tag{10}$$

subject to (8) and $w' \in \mathcal{L}_{2e}^m$, where $\mu(\cdot)$ lies in the nonnegative orthant if $w' = \Delta \circ v$. By introducing the integral quadratic form of (7) as an additional state, we turn the dissipativity condition (9) into a standard Lyapunov inclusion constraint. In addition, we obtain an upper bound on the value the integral state μ can reach along converging solutions.

Theorem 2. *Let Δ satisfy the hard IQCs $\Pi_1, \ldots, \Pi_l$ and suppose that $\hat{\Omega} \subset \mathbb{R}^{n_z} \times \mathbb{R}^l$ be compact as well as invariant with respect to (8), (10) for all $w' \in \mathcal{L}_{2e}^m$ such that $y_{\psi_i}(\cdot)$ satisfies (7) for M_i, where $\Pi_i = (M_i, \psi_i)$, for all $i \in \{1, \ldots, l\}$ and $N \geq 0$; if there exist a smooth positive definite function $V : \mathbb{R}^{n_z} \times \mathbb{R}^l \to \mathbb{R}_{\geq 0}$ and a positive definite function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that*

$$\nabla_z V(z, \mu)^{\mathrm{T}} \bar{f}(z, \tilde{w}) + \nabla_\mu V(z, \mu)^{\mathrm{T}} \bar{\sigma}_M(\bar{h}(z, \tilde{w})) \leq -\gamma(\|z\|) \tag{11}$$

for all $(z, \mu, \tilde{w}) \in \hat{\Omega} \times \mathbb{R}^m$, then $\Omega_0 = \{x \mid (x, 0, 0) \in \hat{\Omega}\}$ is a subset of $\mathcal{R}_\Delta$.

Proof. The function $\rho : (z, \mu) \mapsto \gamma(\|z\|)$ is positive definite in z and nonnegative in μ. Hence, any solution $(z, \mu) : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n_z} \times \mathbb{R}^l$ of (8), (10) on $(z(0), \mu(0)) \in \hat{\Omega}$ satisfies

$$\dot{\mu}(t) = (\bar{\sigma}_M \circ \bar{h})(z(t), \Delta(v(t))) \tag{12}$$

and converges to $X = \{(0, \mu) \mid \mu \in \mathbb{R}^l\}$ by La Salle (1960, Theorem 1). Since $\bar{f}$ is independent of μ, this is the desired result. □

If (12) holds for all (z, μ) in some bounded level set $\hat{\Omega}_V$ of V, then $\hat{\Omega}_V$ is invariant with respect to all solutions of (8) and (10) which satisfy the IQC condition (7).

Corollary 2. *Let Δ satisfy the hard IQCs $\Pi_1, \ldots, \Pi_l$ and suppose that Eq. (11) holds for the function V on the compact set $\hat{\Omega}$, both as defined in Theorem 2; if the solution $z : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n_z}$ of (8) for $w \in \mathcal{L}_{2e}^m$ such that $y_\psi(\cdot)$ satisfies Π and $z(0) \in \Omega_0 \times \{0\}$, then*

$$\sup_{t \geq 0} \mu(t) \leq \alpha^{-1}(V(z(0), 0)),$$

where $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a continuous, strictly increasing function satisfying $\alpha(\|(z, \mu)\|) \leq V(z, \mu)$ on $\hat{\Omega}$. ◁

Note that the lower bound $\alpha(\cdot)$ exists since V is a continuous, positive definite function and $\hat{\Omega}$ is compact (Kellett, 2014).

4.2. New classes of multipliers

One might think that the assumptions of Theorem 2 are more difficult to prove given the additional dynamics; however, its result is in fact necessary for for the dissipativity condition (9). Being independent of μ, (11) reduces to (9) when taking

$$\hat{V}(z, \mu) \equiv V(z) + \lambda_i |\mu_i|$$

with V and $\lambda_1, \ldots, \lambda_l$ as in Eq. (9), noting that the restriction of $|\mu_i|$ onto the nonnegative reals is differentiable with $\nabla_{\mu_i \geq 0} |\mu_i| = 1$. We generalize this observation into a characterization of invariance.

Theorem 3. *Let Δ satisfy the hard IQCs $\Pi_1, \ldots, \Pi_l$; if there exist a smooth function $V : \mathbb{R}_{\geq 0} \times \mathbb{R}^{n_z} \to \mathbb{R}_{\geq 0}$, differentiable nonnegative functions $s = (s_1, \ldots, s_l) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}^l$, and a scalar $r > 0$ such that*

$$\begin{aligned}&\nabla_z V(t, z)^{\mathrm{T}} \bar{f}(z, \tilde{w}) + \mu_1 \dot{s}_1(t) + \cdots + \mu_l \dot{s}_l(t) \\ &\quad + \nabla_t V(t, z) + s(t)^{\mathrm{T}} \bar{\sigma}_M(\bar{h}(z, \tilde{w})) \leq 0\end{aligned} \tag{13}$$

for all $t \geq 0$ and $(z, \mu, \tilde{w}) \in \mathbb{R}^{n_z} \times \mathbb{R}^l \times \mathbb{R}^m$ with $V(0, z) \leq r$ and $\mu \in \mathbb{R}_{\geq 0}^l$, then $\Omega_0 = \{x \mid V(0, x, 0) \leq r\}$ is invariant with respect to (1).

Proof. Integration of (13) along a solution $(z, \mu) : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n_z + l}$ of (1) and (12) with $V(0, z(0)) \leq r$ and $\mu(0) = 0$ yields

$$0 \geq V(T, z(T)) - V(0, z_0) + \sum_i s_i(T)\,\mu_i(T).$$

By nonnegativity, the multipliers s_i certify dissipativity of V for (1) since Δ satisfies the hard IQCs $\Pi_1, \ldots, \Pi_l$, that is, $\mu_1(T), \ldots, \mu_l(T) \geq 0$ for all $T \geq 0$. □

5. Numerical examples

We consider the problem of the open-loop unstable linear system

$$A = \begin{bmatrix} 0 & 1 \\ 14.7150 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 30 \end{bmatrix} \tag{14a}$$

with saturated LQR feedback (with $Q = I_2$ and $R = 1$)

$$K = \begin{bmatrix} -1.6043 & -1.0521 \end{bmatrix} \tag{14b}$$

$$\phi : v \mapsto \mathrm{sat}(v), \tag{14c}$$

where $\mathrm{sat} : \mathbb{R} \to \mathbb{R}$ is the saturation function. As $A + BK$ is Hurwitz, the system (14) is locally stable but not globally. The operator $\Delta : v \mapsto v - \mathrm{sat}(v)$ satisfies $\Delta(\hat{v}) = 0$ for all $\hat{v} \in [-1, 1]$. Hence, define $v_{1,2} = Kx \mp 1$ and $\Delta_{1,2} = \Delta(v \pm 1)$; then $\Delta_{1,2}$ satisfy the quadratic constraint given by $M_{c=1}$ virtue of Lemma 1. Let $g_{1,2} = \sigma_{M_c}((v_{1,2}, w)) \geq 0$ be the corresponding polynomial inequalities.

In order to estimate the region of attraction of (14), we postulate the bilinear sum-of-squares optimization problem

$$\max b \tag{15a}$$

$$\text{s.t.} \quad s_g\,(V - g) \succeq \nabla V^{\mathrm{T}} f + \sum_{i \in \{1,2\}} s_i\,g_i + \epsilon x^{\mathrm{T}} x \tag{15b}$$

$$s_b\,(P - b) \succeq V - g \tag{15c}$$

$$V - \epsilon x^{\mathrm{T}} x, s_1, s_2, s_g, s_b \succeq 0, \tag{15d}$$

where V is a polynomial decision variable of fixed degree, b and g are scalar decision variables, the multipliers $s_{1,2}$, s_g, and s_b are sum-of-squares decision variables of fixed degree, $f = (A + BK)x - Bw$ are the closed-loop dynamics, P is a given shape function, and $\epsilon > 0$; the notation "$\succeq$" denotes nonnegativity in the sum-of-squares sense. We choose for P an ellipsoidal shape which is elongated along the levels of Kx. We have solved (15) using 250 iterations of coordinate descent (Chakraborty, Seiler, & Balas, 2011). The resulting level sets $\Omega = \{x \mid V(x) \leq g\}$ and $\mathcal{E} = \{x \mid P(x) \leq b\}$ are shown in Fig. 1. The optimal value is $\hat{b} = 8.728\,03$.

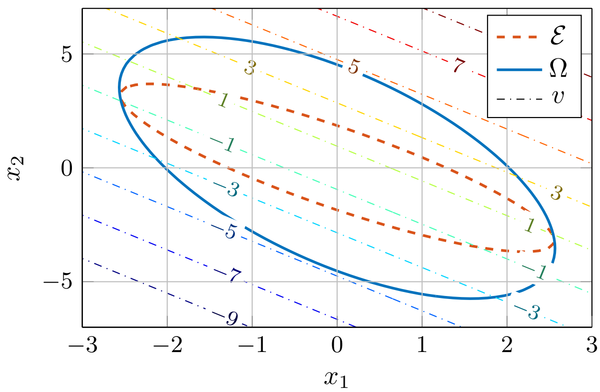

Fig. 1. Region of attraction estimate Ω and inscribing ellipsoid $\mathcal{E}$ for (14) by polynomial combination of constraints $g_{1,2}$, along with feedback $v = Kx$.

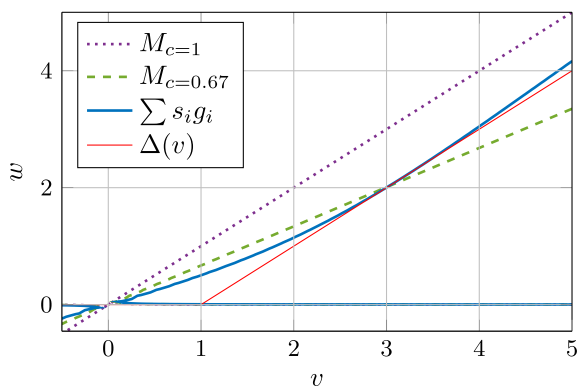

Fig. 2. Comparison of quadratic constraint given by $M_{c=1}$, local quadratic constraint by $M_{c=0.67}$, and polynomial combination of constraints $g_{1,2}$ for the nonlinear operator $\Delta(\cdot)$ in (14). Due to symmetry only the first quadrant is shown.

For comparison, we estimate the region of attraction by a local sector constraint. We have found experimentally that the closed-loop of (14) can be shown to be stable by Corollary 1 if Δ satisfies the local quadratic constraint given by M_c with $c \le 0.67$. This local sector corresponds to a maximum feedback value $v_{\max} = 3.0303$, into which the region of attraction estimate Ω must be embedded. We have computed a Lyapunov function V and level set g such that Ω satisfies this constraint while maximizing the inscribing ellipsoid $\mathcal{E}$ with $\hat{b} = 2.990\,72$.

We also applied the method of Fang et al. (2008), based upon an exact ellipsoidal characterization of invariant sets for linear systems with a single saturated input (Hu & Lin, 2002), which yields $\hat{b} = 10.774\,73$. While the approach of intersected quadratic constraints is applicable to polynomial multi-input systems with higher-order Lyapunov functions, we have obtained a result that is very close to the best (quadratic) solution. Moreover, our approach returned an estimate which is much larger than the estimate using a local sector constraint. This is achieved by considering the globally unstable dynamics directly rather than a conservative relaxation. Here, the sum term $\sum s_i g_i$ in (15), evaluated for the optimal solution, effectively serves as a higher-order polynomial approximation of the graph of Δ than the quadratic constraint as depicted in Fig. 2: Both the polynomial approximation and the local quadratic constraint exclude the open-loop case, which is included in the 'global' quadratic constraint given by $M_{c=1}$; yet, the local quadratic constraint is only valid for inputs $v \le 2/c \approx 3$ whereas the polynomial approximation is valid for all $v \in \mathbb{R}$.

6. Conclusion

Quadratic constraints for the assessment of stability under nonlinear or unknown operators cannot be directly applied to open-loop unstable systems. As multiple quadratic constraints define a semialgebraic set, we have derived a dissipativity condition with polynomial multipliers from the generalized S-procedure. Moreover, we have extended this result to integral quadratic constraints by virtue of LaSalle's invariance principle.

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