# Forbidden Tournaments and the Orientation Completion Problem* 

Manuel Bodirsky ${ }^{\dagger 1}$ and Santiago Guzmán-Pro ${ }^{\ddagger 1}$<br>${ }^{1}$ Institut für Algebra, TU Dresden


#### Abstract

For a fixed finite set of finite tournaments $\mathcal{F}$, the $\mathcal{F}$-free orientation problem asks whether a given finite undirected graph $G$ has an $\mathcal{F}$-free orientation, i.e., whether the edges of $G$ can be oriented so that the resulting digraph does not embed any of the tournaments from $\mathcal{F}$. We prove that for every $\mathcal{F}$, this problem is in P or NP-complete. Our proof reduces the classification task to a complete complexity classification of the orientation completion problem for $\mathcal{F}$, which is the variant of the problem above where the input is a directed graph instead of an undirected graph, introduced by Bang-Jensen, Huang, and Zhu (2017). Our proof uses results from the theory of constraint satisfaction, and a result of Agarwal and Kompatscher (2018) about infinite permutation groups and transformation monoids.


## 1 Introduction

For a fixed finite set of finite oriented graphs $\mathcal{F}$, the $\mathcal{F}$-free orientation problem asks whether a given finite undirected graph $G$ has an $\mathcal{F}$-free orientation, i.e., whether the edges of $G$ can be oriented so that the resulting digraph does not contain any $F \in \mathcal{F}$ as an induced oriented graph. This problem was first studied from a structural perspective: Skrien 41] proposed structural characterisations of graphs that admit an $\mathcal{F}$-free orientation, where $\mathcal{F}$ is a fixed set of oriented paths on 3 -vertices. The most notorious result in this direction that a connected graph $G$ is a proper circular-arc graph if and only if it admits a $\{(\{1,2,3\},\{(1,2),(1,3)\}),(\{1,2,3\},\{(2,1),(3,1)\})\}$-free orientation 41.

From an algorithmic perspective, the $\mathcal{F}$-free orientation problem can be easily reduced to 2 -SAT if $\mathcal{F}$ is a set of oriented paths on 3 vertices [3]. Later, in [31] the authors extended the previous observation, and showed that for several sets of oriented graphs on 3 vertices the $\mathcal{F}$-free orientation problem is in P - leaving only open the symmetric cases

$$
\begin{aligned}
\mathcal{F} & =\{(\{1,2,3\},\{(1,2),(1,3)\}),(\{1,2,3\},\{(1,2),(2,3),(3,1)\})\} \\
\text { and } \mathcal{F} & =\{(\{1,2,3\},\{(2,1),(3,1)\}),(\{1,2,3\},\{(1,2),(2,3),(3,1)\})\} .
\end{aligned}
$$

[^0]On the other hand, the Roy-Gallai-Hasse-Vitaver Theorem [27,3239,42] implies that the $k$-colouring problem can be interpreted as an $\mathcal{F}$-free orientation problem, i.e., there is a finite set of oriented graphs $\mathcal{F}_{k}$ such that a graph $G$ is $k$-colourable if and only if it admits an $\mathcal{F}_{k}$-free orientation. This yields natural instances of NP-hard $\mathcal{F}$-free orientation problems.

Recently, a thorough study of problems that stem from orientation problems was initiated by Bang-Jensen, Huang, and Zhu [4]. For a fixed class of oriented graphs $\mathcal{C}$, the orientation completion problem asks whether a partially oriented graph $G$ can be completed to an oriented graph in $\mathcal{C}$ by orienting its non-oriented edges. In particular, given a finite set of oriented graphs $\mathcal{F}$, the $\mathcal{F}$ free orientation completion problem generalises the $\mathcal{F}$-free orientation problem. In [4, the authors study the complexity of the orientation completion problem for several classes of digraphs such as in-tournaments, local tournaments, and locally transitive tournaments. They show that for each of these classes, the orientation completion problem is in P or NP-complete.

In this work, we prove that for any finite set of finite tournaments $\mathcal{F}$, the $\mathcal{F}$-free orientation problem is in P or NP-complete. Our proof reduces the classification task to the $\mathcal{F}$-free orientation completion problem, for which we then provide a complete complexity classification. This intermediate result is interesting in its own right, addressing the study initiated by Bang-Jensen, Huang, and Zhu 4].

The rest of this work is structured as follows. In Section 2, we introduce the necessary model theoretic background for this work. Similarly, in Section 3, we provide context for the theory of constraint satisfaction. In Section 4, we prove that for each finite set of tournaments $\mathcal{F}$, there is a Boolean structure whose CSP is polynomial-time equivalent to the $\mathcal{F}$-free orientation problem. This yields a dichotomy for the $\mathcal{F}$-free orientation completion problem, and moreover, we propose a complete classification of the complexity of these problems in terms of $\mathcal{F}$ and the $\mathcal{F}$-free tournaments. In Section 6, we build on the previous classification to classify the complexity of the $\mathcal{F}$-free orientation problem. We will do so by using a result of Agarwal and Kompatscher [1] about infinite permutation groups and transformation monoids which we introduce in Section 5 . Finally, in Section 7 we present our classification from Sections 4 and 6 in graph theoretic terms, and propose some applications which we believe are interesting in their own right to graph theorists. Moreover, these examples can help readers that are less familiar with constraint satisfaction techniques to gain intuition before reading the technical proofs of this paper - readers with a model theory or constraint satisfaction background and motivation can skip this section.

In the remaining of this section we first extend the motivation of our work, by mentioning some relations of the $\mathcal{F}$-free orientation problem to previously studied problems in graph theory, finite model theory, constraint satisfaction theory, and infinite model theory. We conclude this section by introducing the formal setting under which we study the orientation and orientation completion problems.

### 1.1 Related Work

Similar to $\mathcal{F}$-free orientation problems, Damaschke [22] considers characterisations of graph classes by means of finitely many forbidden ordered graphs. For instance, if $\left(P_{3}, \leq\right)$ is the linear ordering of $P_{3}=(\{1,2,3\},\{12,13\})$ where $1 \leq 2 \leq 3$, then a graph $G$ is chordal if and only if it admits a $\left(P_{3}, \leq\right)$-free linear ordering [22]. Shortly after, Duffus, Ginn, and Rödl [23] consider the complexity classification of the following ordering problem: given a linearly ordered graph $(G, \leq)$, decide if an input graph $H$ admits a $(G, \leq)$-free linear ordering. They showed that for almost all 2-connected graphs $G$, the $(G, \leq)$-free ordering is NP-complete, and conjectured that this is the case for all

2-connected graphs unless $G$ is a clique. Similar problems have been considered for circular orderings [30, for so-called tree-layouts of graphs [37, and for 2-edge-coloured graphs [19] (for a comprehensive study of such problems from a structural perspective see [29]). Here we initiate a parallel study to the one started by Duffus, Ginn, and Rödl [23] by considering orientation instead of ordering problems.

Both the $\mathcal{F}$-free orientation problem and the $\mathcal{F}$-free orientation completion problem can be viewed as special cases of a significantly larger class of computational problems, namely the class of problems that can be expressed in the logic $M M S N P_{2}$. In the context of digraphs, a computational problem expressed in $\mathrm{MMSNP}_{2}$ asks "given a digraph $D$, is there an edge colouring of $D$ that avoids some fixed finite set of edge-coloured digraphs?". This logic relates to Feder and Vardi's famous logic MMSNP (for monotone monadic strict NP) as Courcelle's logic $\mathrm{MSO}_{2}$ relates to MSO [5]. It has the same expressive power as guarded disjunctive Datalog [8, which is a formalism studied in database theory. It can be shown that every CSP in MMSNP 2 can be expressed as a Constraint Satisfaction Problem (CSP) for a reduct of a finitely bounded homogeneous structure [11, and hence falls into the scope of the so-called tractability conjecture [16]. This conjecture states that such CSPs are in P or NP-complete, and even provides a mathematical condition to describe the boundary between the cases in P and the NP-complete cases. This condition has numerous equivalent characterisations [6, 7, 9, but despite recent progress [35] the tractability conjecture for $\mathrm{MMSNP}_{2}$ is still wide open. In contrast, the P versus NP-complete dichotomy is true for MMSNP [25, 34 (using the complexity dichotomy for finite-domain CSPs [20, 43]), and even the tractability conjecture has been verified in this case [12. The graph orientation problems studied here cannot be expressed in MMSNP, but it is straightforward to formulate them in MMSNP 2 . Our result not only shows a complexity dichotomy for the $\mathcal{F}$-free orientation problems, but verifies the tractability conjecture for this subclass of $\mathrm{MMSNP}_{2}$.

### 1.2 Formal Setting

This work lies in the intersection of graph theory, model theory, and computational complexity. For this reason, we begin by carefully introducing basic notation and nomenclature; in this section, we start with terminology from graph theory. A digraph $D$ consists of a vertex set $V(D)$ and a binary relation $E(D) \subseteq V(D)^{2}$. The elements of $E(D)$ are called edges, and whenever there are vertices $x, y \in V(D)$ such that $(x, y) \in E(D)$ and $(y, x) \in E(V)$, we write $x y \in E(D)$, and we say that $x y$ is a symmetric edge of $D$ - notice that in graph theory an edge might refer to a symmetric edge in this context. Most digraphs considered here will be loopless, i.e., they do not contain edges of the form $(x, x)$ for $x \in V(D)$. Whenever the digraph is clear from the context, we will simply write $V$ for $V(D)$, and $E$ for $E(D)$.

A (loopless) graph $G$ will be viewed as a (loopless) digraph where $E(G)$ is a symmetric relation, and an oriented graph $O$ is a digraph where $E(O)$ is an anti-symmetric relation (i.e., none of the edges in $E(O)$ is symmetric). It will be convenient to denote by $U$ the relation $E \cup E^{-1}$, i.e, given a digraph $D$ we have $(x, y) \in U$ if and only if $(x, y) \in E(D)$ or $(y, x) \in E(D)$. The underlying graph of a digraph $D$ is the graph $u(D)$ with vertex set $V(D)$ and edge set $U(D)=U$. The relation $U$ will be very important and highly used in this work so, when $D$ is clear from context we will write $U$ for $U(D)$. Given a digraph $D$, we will write $(V, U)$ for the underlying graph of $U$, and $(D, U)$ for the structure that contains both the oriented edges and the edge relation of the underlying graph.

An orientation of a graph $G$ is an oriented graph $G^{\prime}$ such that $u\left(G^{\prime}\right)=G$. A tournament is an orientation of the complete graph $K_{n}$ with $n$ vertices, for some $n \geq 1$, and a graph is semicomplete
if its underlying graph is complete. We denote by $T_{n}$ the transitive tournament on $n$ vertices, and by $\vec{C}_{n}$ the directed cycle on $n$ vertices.

Given digraphs $D$ and $H$ we say that $D$ is a subdigraph of $H$ if $V(D) \subseteq V(H)$ and $E(D) \subseteq$ $E(H)$. We say that $D$ is spanning subdigraph of $H$ is $V(D)=V(H)$. In particular, if $G^{\prime}$ is an orientation of a graph $G$, then $G^{\prime}$ is a spanning subdigraph of $G$. A homomorphism $\varphi: D \rightarrow H$ is a function $\varphi: V(D) \rightarrow V(H)$ such that if $(x, y) \in E(D)$, then $(\varphi(x), \varphi(y)) \in E(H)$ - this notion naturally generalises to homomorphisms of relational structures, which we introduce later. If such a homomorphism exists, we write $D \rightarrow H$, otherwise we write $D \nrightarrow H$. An embedding is an injective homomorphism $\varphi: D \rightarrow H$ such that $(x, y) \in E(D)$ if and only if $(\varphi(x), \varphi(y)) \in E(H)$. Notice that if $H$ is an oriented graph and $T$ is a tournament, then every homomorphism $\varphi: T \rightarrow H$ is an embedding. Finally, given a set of digraphs $\mathcal{F}$, we say that a digraph $D$ is $\mathcal{F}$-free if there is no embedding $\varphi: F \rightarrow D$ for any $F \in \mathcal{F}$. We now formalize the orientation and orientation completion problems in the setting described above.

## $\mathcal{F}$-FREE ORIENTATION PROBLEM

- Input: a finite graph (symmetric digraph) $G$;
- Question: is there an $\mathcal{F}$-free orientation $G^{\prime}$ of $G$ ?


## $\mathcal{F}$-FREE ORIENTATION COMPLETION PROBLEM

- Input: a finite digraph $D$;
- Question: is there a spanning subdigraph $D^{\prime}$ of $D$ such that $D^{\prime}$ is an $\mathcal{F}$-free oriented graph?

Clearly, for every fixed set of oriented graphs $\mathcal{F}$ both the $\mathcal{F}$-free orientation and the $\mathcal{F}$-free orientation completion problem are in the complexity class NP. As previously mentioned, this work studies the $\mathcal{F}$-free orientation (completion) problem when $\mathcal{F}$ is a set of tournaments. Given a tournament $T$, we will often write $T$-free orientation (completion) problem instead of $\{T\}$-free orientation problem. For instance, since every graph admits an acyclic orientation, the $\overrightarrow{C_{3}}$-free orientation problem is trivial and polynomial-time tractable. On the contrary, the $\overrightarrow{C_{3}}$-free orientation completion problem is NP-complete (Corollary 55). Note that if $\mathcal{F}$ contains a tournament with only one vertex, then there is no $\mathcal{F}$-free oriented graph with a non-empty set of vertices, and hence both of the computational problems above are trivial. So we tacitly assume from now on that all vertices in $\mathcal{F}$ have at least two vertices.

## 2 Model Theory Set Up

Relational structures generalise digraphs and allow multiple relations with relations of arbitrary finite arity. They are a natural tool in our study of graph problems. To define them formally, we need the concept of a relational signature, which is a set $\tau$ of relation symbols $R, S, \ldots$, each equipped with an arity $k \in \mathbb{N}$. A $\tau$-structure $\mathfrak{A}$ consists of a set $A$ (the domain) and for each $R \in \tau$ of arity $k$ a relation $R^{\mathfrak{A}} \subseteq A^{k}$. Clearly, a digraph $D$ (and hence also a graph) can be viewed as a structure with domain $V(D)$ and the signature $\{E\}$ where $E$ is a binary relation symbol that denotes the edge relation $E(D)$ of the graph.

A substructure of $\tau$-structure $\mathfrak{A}$ is a $\tau$-structure $\mathfrak{B}$ such that for every $R \in \tau$ of arity $k$ we have $R^{\mathfrak{B}}=R^{\mathfrak{A}} \cap B^{k}$; however, note that a substructure of a digraph when viewed as a structure
corresponds to induced subgraphs in graph theory, rather than subgraphs as introduced earlier. The union of two $\tau$-structures $\mathfrak{A}$ and $\mathfrak{B}$ is the $\tau$-structure $\mathfrak{C}$ with domain $C:=A \cup B$ and the relation $R^{\mathfrak{C}}:=R^{\mathfrak{A}} \cup R^{\mathfrak{B}}$ for every $R \in \tau$. If $\mathfrak{A}$ is a $\tau$-structure and $\mathfrak{A}^{\prime}$ is a $\sigma$-structure with the same domain, for $\sigma \subseteq \tau$, and $R^{\mathfrak{A}}=R^{\mathfrak{A}^{\prime}}$ for every $R \in \sigma$, then $\mathfrak{A}^{\prime}$ is called a reduct of $\mathfrak{A}$, and $\mathfrak{A}$ is called a expansion of $\mathfrak{A}^{\prime}$.

A homomorphism between two $\tau$-structures $\mathfrak{A}$ and $\mathfrak{B}$ is a function $h: A \rightarrow B$ such that for every $R \in \tau$ of arity $k$ we have $a=\left(a_{1}, \ldots, a_{k}\right) \in R^{\mathfrak{A}} \Rightarrow f(a):=\left(f\left(a_{1}\right), \ldots, f\left(a_{k}\right)\right) \in R^{\mathfrak{B}}$. The constraint satisfaction problem of a $\tau$-structure $\mathfrak{B}$ is the class of all finite $\tau$-structures $\mathfrak{A}$ with a homomorphism to $\mathfrak{B}$; for fixed $\mathfrak{B}$ of finite relational signature $\tau$, this class can be viewed as a computational problem, where the input consists of an arbitrary finite $\tau$-structure $\mathfrak{A}$, and the question is to decide whether $\mathfrak{A} \in \operatorname{CSP}(\mathfrak{B})$. For example, $\operatorname{CSP}\left(K_{3}\right)$ can be viewed as the famous graph 3-colouring problem. Two structures are called homomorphically equivalent if there are homomorphism between the structures in both ways. Clearly, homomorphically equivalent structures have the same CSP.

An endomorphism of $\mathfrak{A}$ is a homomorphism from $\mathfrak{A}$ to $\mathfrak{A}$. The set of all endomorphisms of a structure $\mathfrak{A}$ forms a transformation monoid. It is well-known that a transformation monoid $M$ is an endomorphism monoid of a relational structure if and only if it is locally closed, i.e., if $f: A \rightarrow A$ is such that for every $n \in \mathbb{N}$ and $a \in A^{n}$ there exists $g \in M$ such that $f(a)=g(a)$, then $g \in M$; these are the closed sets of the topology of pointwise convergence, which is the product topology on $A^{A}$ where the topoogy on $A$ is taken to be discrete.

A homomorphism is called strong if the implication $\Rightarrow$ in the definition of homomorphisms is replaced by an equivalence $\Leftrightarrow$. An embedding is an injective strong homomorphism. A structure $\mathfrak{A}$ is called a core if all endomorphisms of $\mathfrak{A}$ are embeddings. An isomorphism is a bijective embedding. An automorphism of a structure $\mathfrak{A}$ is an isomorphism of $\mathfrak{A}$ with itself. Note that if $\mathfrak{A}$ is a finite core structure, then all endomorphisms of $\mathfrak{A}$ are automorphisms; this statement is false for general infinite structures $\mathfrak{A}$.

A structure is called homogeneous if every isomorphism between finite substructrues can be extended to an automorphism. It is a well-known fact that a permutation group is the automorphism group of a relational structure $\mathfrak{A}$ if and only if it is closed with respect to the restriction of the topology on $A^{A}$ above to the set of all permutations, which we denote by $\operatorname{Sym}(A)$. If $P$ is a set of permutations, we write $\langle P\rangle$ for the smallest permutation group that contains $P$ and is closed in $\operatorname{Sym}(A)$.

In our study of graph orientation problems, it will be convenient to pass from classes of finite structures to a single countably infinite structure, using Fraïssé theory. We do not need the full power of the theory, but exclusively work with the following fact.

Theorem 1 (see, e.g., [33]). Let $\tau$ be a finite relational signature. Let $\mathcal{C}$ be a class of finite $\tau$ structures which is closed under substructures, isomorphisms, and unions. Then there exists a countably infinite homogeneous $\tau$-structure $\mathfrak{A}$ such that $\mathcal{C}$ equals the class of finite $\tau$-structures that have an embedding into $\mathfrak{A}$. The structure $\mathfrak{A}$ is unique up to isomorphism, and called the Fraïssé-limit of $\mathcal{C}$.

Example 2. Let $\mathcal{C}$ be the class of all finite graphs. Then $\mathcal{C}$ satisfies the assumptions of Theorem 1, and the Fraïssé-limit of $\mathcal{C}$ is called the Rado graph, which we denote by $\mathfrak{R}$. For $n \geq 2$, the Henson graph $\mathfrak{H}_{n}$ is the Fraïssé-limit of the class $\mathcal{C}$ of all finite graphs that do not embed $K_{n}$.

A class $\mathcal{C}$ of finite $\tau$-structures is called finitely bounded if there exists a finite set $\mathcal{F}$ of finite $\tau$-structures such that $\mathfrak{A} \in \mathcal{C}$ if and only if there is no structure in $\mathcal{F}$ which embeds into $\mathfrak{A}$; then
then refer to the elements of $\mathcal{F}$ as the bounds for $\mathcal{C}$. We say that a structure $\mathfrak{B}$ is finitely bounded if the class of all finite structures $\mathfrak{A}$ that embeds into $\mathfrak{B}$ is finitely bounded.

Example 3. The Rado graph $\mathfrak{R}$ and the Henson graphs are finitely bounded. For the Rado graph, the bounds contain the one-vertex loop graph, and the two-vertex graph with a single edge (which forces the edge relation of the Fraïsé-limit to be loopless and symmetric).

We now turn to definitions that are specific for our study of graph orientation problems. Let $\mathcal{F}$ be a finite set of finite tournaments. Then the class of all finite $\mathcal{F}$-free oriented graphs satisfies the conditions from Theorem $\mathbb{\square}$ (here we use our general assumption that all tournaments in $\mathcal{F}$ have at least two vertices), and hence has a countably infinite homogeneous Fraissé-limit $D_{\mathcal{F}}=(V ; E)$. Note that $D_{\mathcal{F}}$ is finitely bounded: as bounds we take the structures in $\mathcal{F}$ and additionally the loop digraph and the two-element digraph which contains a symmetric edge. Clearly, $\operatorname{CSP}\left(D_{\mathcal{F}}\right)$ is in P since it suffices to check whether given given directed graph contains one of the tournaments from $\mathcal{F}$ as a subgraph, which can be tested in polynomial time (where the degree of the polynomial is bounded by the maximal number of elements of the structures in $\mathcal{F}$ ).

The infinite structures $D_{\mathcal{F}}$ turn out to be closely related to $\mathcal{F}$-free orientation (completion) problems via the following expansions and reducts.

- Let $H_{\mathcal{F}}=(V ; U)$ be the underlying graph of $D_{\mathcal{F}}$, that is, $U=E \cup E^{-1}$. Then a finite undirected graph $G$ has an $\mathcal{F}$-free orientation if and only if it has a homomorphism to $H_{\mathcal{F}}$. Conversely, a finite directed graph has a homomorphism to $H_{\mathcal{F}}$ if and only if the underlying graph has an $\mathcal{F}$-free orientation. Hence, the $\mathcal{F}$-free orientation problem and $\operatorname{CSP}\left(H_{\mathcal{F}}\right)$ are essentially the same problem.
- Similarly as in the previous item, the $\mathcal{F}$-free orientation completion problem may be viewed as $\operatorname{CSP}\left(D_{\mathcal{F}}, U\right)$.

If all tournaments in $\mathcal{F}$ contain a directed cycle, then every finite graph $G$ has an $\mathcal{F}$-free orientation, since we may orient the edges along an arbitrary linear order of the vertices. Hence, the $\mathcal{F}$-free orientation problem is trivial and in P. Note that in this case, $H_{\mathcal{F}}$ is the Rado graph.

Otherwise, there exists a smallest $n=n_{\mathcal{F}} \in \mathbb{N}$ such that $\mathcal{F}$ contains the transitive tournament $T_{n}$ with $n$ vertices. In this case, there exists a largest $k=k_{\mathcal{F}} \in \mathbb{N}$ such that $K_{k}$ has an $\mathcal{F}$-free orientation [24]. Finally, $m_{\mathcal{F}}$ denotes the size of the smallest largest tournament in $\mathcal{F}$. Now we present some easy observations.

Observation 4. It follows from the definition of $n_{\mathcal{F}}$ that if $k \leq n_{\mathcal{F}}-1$, then $T_{k}$ is $\mathcal{F}$-free. Hence, the inequality $k_{\mathcal{F}} \geq n_{\mathcal{F}}-1$ holds. Since every tournament in $\mathcal{F}$ has at least two vertices, we have $n_{\mathcal{F}} \geq 2$ and $k_{\mathcal{F}} \geq 1$.

Observation 5. A notable special case is the situation that the inequality from the previous observation is actually an equality.

- If $n_{\mathcal{F}}=2$ then no graph with a non-empty edge set has an $\mathcal{F}$-free orientation, $k_{\mathcal{F}}=1$, and the $\mathcal{F}$-free orientation problem is trivial and in $P$.
- Suppose that there is no $\mathcal{F}$-free tournament with $n_{\mathcal{F}}$ vertices. Then the $\mathcal{F}$-free orientation problem is equivalent to finding an $n_{\mathcal{F}}$-element clique in a given finite graph, and hence in $P$. This generalises the previous two cases. Note that in this case we have $k_{\mathcal{F}}=n_{\mathcal{F}}-1$.

We will later observe (Lemma 34) that the two cases above correspond to when $H_{\mathcal{F}}$ is the Rado graph, or $H_{\mathcal{F}}$ is a Henson graph.

Lemma 6. Let $\mathcal{F}$ be a finite set of finite tournaments. If $n_{\mathcal{F}} \geq 3$, then the structure $H_{\mathcal{F}}$ is a core.
Proof. First consider the case that $k:=k_{\mathcal{F}}$ as defined above is even. The relation $x \neq y$ can be defined primitively positively by the formula

$$
\exists u_{1}, \ldots, u_{k}\left(U\left(x, u_{1}\right) \wedge \cdots \wedge U\left(x, u_{k / 2}\right) \wedge U\left(u_{k / 2+1}, y\right) \wedge \cdots \wedge U\left(u_{k}, y\right)\right)
$$

Note that if $x=y$ then $x, u_{1}, \ldots, u_{k}$ would induce a clique which does not have an $\mathcal{F}$-free orientation. Similarly, one may show that the complement of the edge relation has a primitive positive definition by removing and edge from $K_{k+1}$. This shows that every endomorphism of $H_{\mathcal{F}}$ is an embedding. The case that $k$ is odd can be handled similarly.

## 3 Constraint Satisfaction Preliminaries

In 2017, Bulatov [20] and Zhuk 43] proved the Feder-Vardi conjecture, i.e., they proved that finitedomain CSPs exhibit a complexity dichotomy. To state their result in its strongest form, we need to introduce the concept of a polymorphism of a $\tau$-structure $\mathfrak{B}$, which is a mapping $f: B^{k} \rightarrow B$, for some $k \in \mathbb{N}$, such that for all $R \in \tau$ and $a_{1}, \ldots, a_{k} \in R^{\mathfrak{B}}$ we have $f\left(a_{1}, \ldots, a_{k}\right) \in R^{\mathfrak{B}}$. The result of Bulatov and Zhuk can be phrased as follows.

Theorem 7 ( 20,43 ). Let $\mathfrak{B}$ be a finite structure. If $\mathfrak{B}$ has a polymorphism which is a weak near unanimity operation, i.e., which satisfies for all $x, y \in B$ that

$$
f(x, \ldots, x, y)=f(x, \ldots, y, x)=\cdots=f(y, x, \ldots, x)
$$

then $\operatorname{CSP}(\mathfrak{B})$ is in $P$.
Before Bulatov and Zhuk proved this theorem, it was already known that finite structures without a weak near unanimity polymorphism can simulate the three-coloring problem in a very specific way: using primitive positive interpretations. Actually, this hardness condition works for arbitrary (not only finite domain) structures, and we recall it in detail in the following.

A primitive positive formula is a formula $\phi\left(y_{1}, \ldots, y_{k}\right)$ of the form

$$
\exists x_{1}, \ldots, x_{n}\left(\psi_{1} \wedge \cdots \wedge \psi_{m}\right)
$$

where $\psi_{1}, \ldots, \psi_{m}$ are atomic formulas with variables from $x_{1}, \ldots, x_{n}$ and the free variables $y_{1}, \ldots, y_{k}$. Note that the equality symbol $=$, the symbol $\top$ for true, and the symbol $\perp$ for false are permitted in atomic formulas. A relation $R \subseteq B^{k}$ is called primitively positively definable in a $\tau$ structure $\mathfrak{B}$ if there exists a primitive positive formula $\phi\left(y_{1}, \ldots, y_{k}\right)$ over the signature $\tau$ such that $R=\left\{\left(b_{1}, \ldots, b_{k}\right) \mid \mathfrak{B} \models \phi\left(b_{1}, \ldots, b_{k}\right)\right\}$. Suppose that $\phi$ is a primitive positive formula build with relation symbols from the signature $\tau$ and without the equality symbol. The canonical database $\mathfrak{A}$ of $\phi$ is the $\tau$-structure whose elements are the (free and existentially quantified) variables of $\phi$, and where $a \in R^{\mathfrak{A}}$, for a relation symbol $R \in \tau$ of arity $k$, if $\phi$ contains the conjunct $R(a)$. We stress that the following two lemmata also hold for structures $\mathfrak{B}$ with an infinite domain.

Lemma 8 (see, e.g., 9,21). Suppose that $R$ is a relation with a primitive positive definition in $\mathfrak{B}$. Then there is a polynomial-time reduction from $\operatorname{CSP}(\mathfrak{B}, R)$ to $\operatorname{CSP}(\mathfrak{B})$.

It turns out that primitive positive definability over a structure $\mathfrak{B}$ with a finite domain can be characterised using the polymorphisms of $\mathfrak{B}$ (we mention that the result below also holds for countably infinite $\omega$-categorical structures $\mathfrak{B}$ [14]; however, this fact is not needed in our proofs).

Theorem 9 (see, e.g., [18,28]). Let $\mathfrak{B}$ be a relational structure with a finite domain. A relation $R$ has a primitive positive definition on $\mathfrak{B}$ if and only if $R$ is preserved by all polymorphisms of $\mathfrak{B}$.

If $\mathfrak{B}$ is a $\tau$-structure, and $\mathfrak{A}$ is a $\sigma$-structure, then a primitive positive interpretation of $\mathfrak{A}$ in $\mathfrak{B}$ is a partial map $I$ from $B^{d}$ to $A$ such that for each atomic formula $\phi\left(y_{1}, \ldots, y_{k}\right)$ over the signature $\sigma$ there exists a primitive positive formula $\phi_{I}$ that defines $\left\{\left(b_{1}, \ldots, b_{k}\right) \in B^{k d} \mid \mathfrak{A} \models \phi\left(I\left(b_{1}\right), \ldots, I\left(b_{k}\right)\right)\right\}$. We refer to $d$ as the dimension of $I$.

Lemma 10 (see, e.g., 9]). Suppose that $\mathfrak{A}$ has a primitive positive interpretation in $\mathfrak{B}$. Then there is a polynomial-time reduction from $\operatorname{CSP}(\mathfrak{A})$ to $\operatorname{CSP}(\mathfrak{B})$.

It is well-known and easy to prove that primitive positive interpretations can be composed (see, e.g., 9]).

Corollary 11. Suppose that $K_{3}$ has a primitive positive interpretation in $\mathfrak{B}$. Then $\operatorname{CSP}(\mathfrak{B})$ is NP-hard.

The special case of the result of Bulatov and of Zhuk for Boolean structures, i.e., structures with a two-element domain, is of particular importance in the present paper. This case has been known since much longer, and it is referred to as Schaefer's theorem. In this special case, one can spell out concrete descriptions of the weak near unanimity polymorphisms that imply polynomial-time tractability of the CSP.

Theorem 12 (Schaefer's theorem). Let $\mathfrak{B}$ be a structure with a domain of size two. Then either $\mathfrak{B}$ interprets $K_{3}$ primitively positively, or $\mathfrak{B}$ has one of the following weak near unanimity polymorphisms

- the binary minimum or maximum operation, i.e., an operation $f$ satisfying $f(x, y)=f(y, x)$ and $f(x, x)=x$ for all $x, y \in B$,
- the ternary majority operation, i.e., the (unique!) operation $f$ satisfying $f(x, x, y)=f(x, y, x)=$ $f(y, x, x)=x$ for all $x, y \in B$,
- the ternary minority operation, i.e., the (unique!) operation $f$ satisfying $f(x, x, y)=f(x, y, x)=$ $f(y, x, x)=y$ for all $x, y \in B$,
- a constant operation, i.e., an operation satisfying $f(x)=f(y)$ for all $x, y \in B$.

In all of these cases, $\operatorname{CSP}(\mathfrak{B})$ is in $P$.
A full complexity classification for Boolean CSPs up to logspace reductions can found in [2]; also see [17]. The following is a well-known fact from linear algebra and will be useful later.

Lemma 13. A Boolean relation $R \subseteq\{0,1\}^{n}$ is preserved by the Boolean minority operation if and only if $R$ is the solution space of a system of linear equalities over the two-element field $\mathbb{F}_{2}$.

The infinite-domain tractability conjecture of Bodirsky and Pinsker from 2011 first appeared in [16]; the formulation below is equivalent to it by results from [6].

Theorem 14. Let $\mathfrak{B}$ be a reduct of a finitely bounded countable homogeneous structure. If $\mathfrak{B}$ does not interpret primitively positively a graph which is homomorphically equivalent to $K_{3}$, then $\operatorname{CSP}(\mathfrak{B})$ is in $P$.

We will verify a strong form of this conjecture for the $\mathcal{F}$-free orientation problem (Theorem 50 . we only require that $\mathfrak{B}=H_{\mathcal{F}}$ does not interpret $K_{3}$ primitively positively). A known obstruction for $\mathfrak{B}$ to admit a primitive positive interpretation of $K_{3}$ is the existence of a so-called pseudo weak near uanimity polymorphism, which is a polymorphism of $\mathfrak{B}$ of arity $k \geq 2$ such that there are endomorphisms $e_{1}, \ldots, e_{k}$ satisfying for all $x, y \in B$ that

$$
e_{1}(f(x, \ldots, x, y))=e_{2}(f(x, \ldots, y, x))=\cdots=e_{k}(f(y, x, \ldots, x))
$$

We will verify in Theorem 50 that if $H_{\mathcal{F}}$ does not admit a primitive positive interpretation of $K_{3}$, then it has a pseudo weak near unanimity polymorphism.

Example 15. Recall from Section 2 that if all tournaments in $\mathcal{F}$ contain a directed cycle, then $H_{\mathcal{F}}$ is isomorphic to the Rado graph $\mathfrak{R}$. Using the homogeneity of $\mathfrak{R}$, it is easy to construct an embedding $f: \mathfrak{R}^{3} \rightarrow \mathfrak{R}$. One may find embeddings $e_{1}, e_{2}, e_{3}$ of $\mathfrak{R}$ into itself such that $e_{1}(f(x, x, y))=$ $e_{2}(f(x, y, x))=e_{3}(f(y, x, x))$, so $\mathfrak{\Re}$ has a ternary pseudo near unanimity polymorphism. The same construction works for the Henson graphs.

## 4 The Orientation Completion Problem

We divide this section into two parts. In the first one, we show that for each finite set of finite tournaments $\mathcal{F}$ there is a Boolean structure whose CSP is equivalent to the $\mathcal{F}$-free orientation completion problem. This naturally yields a complexity classification of the $\mathcal{F}$-free orientation completion problem in terms of Schaefer's cases and the constructed Boolean structure. In the second part, we observe that these cases reduce to only two possibilities: either the Boolean structure primitively positively interprets $K_{3}$, or it is preserved by the minority polymorphism. In particular, this means that either the $\mathcal{F}$-free orientation completion problem reduces (in log-space) to linear equations over $\mathbb{Z}_{2}$, or otherwise, the $\mathcal{F}$-free orientation completion problem is NP-complete. We provide several examples in Section 7

### 4.1 Equivalence to Boolean CSPs

If $T$ is a tournament with vertex set $\{1, \ldots, n\}, n \geq 2$, we define $b_{T} \in\{0,1\}\binom{n}{2}$ as follows. The entries of $b_{T}$ will be indexed by 2-element subsets $\{i, j\}$ of $\{1, \ldots, n\}$ written as $\left(b_{T}\right)_{i j}$. For all $\{i, j\} \subseteq\{1, \ldots, n\}$ with $i<j$ we have that $\left(b_{T}\right)_{i j}=1$ if and only if $E(i, j)$. This coding clearly yields a bijection between $\{0,1\}^{\binom{n}{2}}$ and labeled tournaments with vertex set $\{1, \ldots, n\}$. We illustrate this coding in Fig. 1 .

Let $\mathfrak{B}_{\mathcal{F}}$ be the structure with domain $\{0,1\}$ whose signature contains for every $n \in\left\{2, \ldots, m_{\mathcal{F}}\right\}$ the relation symbol $P_{n}$ of arity $\binom{n}{2}$ which denotes in $\mathfrak{B}_{\mathcal{F}}$ the relation consisting of all $\left.b_{T} \in\{0,1\} \begin{array}{c}n \\ 2\end{array}\right)$ such that the tournament $T$ is $\mathcal{F}$-free. The structure $\left(\mathfrak{B}_{\mathcal{F}}, \mathbf{0}, \mathbf{1}\right)$ is the expansion of $\mathfrak{B}_{\mathcal{F}}$ by the two unary singleton relations $\mathbf{0}:=\{0\}$ and $\mathbf{1}:=\{1\}$.

Example 16. For the sake of clarity, we provide an explicit description of $\mathfrak{B}_{\mathcal{F}}$ if $\mathcal{F}$ is a set of tournaments on 3 vertices. Firstly, it is easy to see that the ternary relation $P_{3}^{\mathfrak{B}_{\mathcal{F}}}$ is empty if


Figure 1: The eight labeled tournaments on 3 vertices. The labels correspond to the associated tuple $\left(x_{1,2}, x_{1,3}, x_{2,3}\right)$ where $x_{i, j}=1$ if $(i, j) \in E(G)$, and $x_{i, j}=0$ if $(j, i) \in E(G)$ for $1 \leq i<j \leq 3$.
$\mathcal{F}=\left\{T_{3}, \overrightarrow{C_{3}}\right\}$, i.e., there is no $\mathcal{F}$-free orientation of $K_{3}$ in this case. If $\mathcal{F}=\varnothing$, then $P_{3}^{\mathfrak{B} \varnothing}=\{0,1\}^{3}$, i.e., any orientation of $K_{3}$ is $\mathcal{F}$-free. If $\mathcal{F}=\left\{T_{3}\right\}$ the relation $P_{3}^{\mathfrak{B}_{\mathcal{F}}}$ is the set $\{(1,0,1),(0,1,0)\}$, since the $T_{3}$-free orientations of $K_{3}$ correspond to both cyclic orientations of $K_{3}$. Finally, if $\mathcal{F}=\overrightarrow{C_{3}}$, then $P_{3}^{\mathfrak{B}_{\mathcal{F}}}=\{0,1\}^{3} \backslash\{(1,0,1),(0,1,0)\}$.

A reduction similar to the reduction in the next theorem has been described in [13].
Theorem 17. The following statements hold for any finite set of finite tournaments $\mathcal{F}$.

1. There is a polynomial-time reduction from the $\mathcal{F}$-free orientation problem to $\operatorname{CSP}\left(\mathfrak{B}_{\mathcal{F}}\right)$.
2. There is a polynomial-time reduction from the $\mathcal{F}$-free orientation completion problem to $\operatorname{CSP}\left(\mathfrak{B}_{\mathcal{F}}, \mathbf{0}, \mathbf{1}\right)$.

Proof. Let $D$ be a given input digraph of the $\mathcal{F}$-free orientation completion problem, and fix an enumeration $\left(v_{1}, \ldots, v_{n}\right)$ of the vertex set $V(D)$. Create a variable $x_{i, j}$ for each $i, j \in\{1, \ldots, n\}$ with $i<j$. Now suppose that $v_{i_{1}}, \ldots, v_{i_{\ell}}$, for $i_{1}<\cdots<i_{\ell}$ and $\ell \leq m_{\mathcal{F}}$, induce a semicomplete digraph. If $\ell=k$ then we add the constraint $P_{k}\left(x_{i_{1}, i_{2}}, x_{i_{1}, i_{3}}, \ldots, x_{i_{k-1}, i_{k}}\right)$. Finally, for each pair of vertices $i<j$ we add the constraint $\mathbf{1}\left(x_{i, j}\right)$ if $(i, j) \in E(D)$ and $(j, i) \notin E(D)$; otherwise, if $(i, j) \notin E(D)$ and $(j, i) \in E(D)$, we add the constraint $\mathbf{0}\left(x_{i, j}\right)$. Clearly, the resulting instance of $\operatorname{CSP}\left(\mathfrak{B}_{\mathcal{F}}, \mathbf{0}, \mathbf{1}\right)$ has a solution if and only if $D$ can be completed to an $\mathcal{F}$-free oriented graph. With similar arguments but omitting the unary constraints $\mathbf{0}(x)$ and $\mathbf{1}(x)$, we obtain a polynomial-time reduction from the $\mathcal{F}$-free orientation problem to $\operatorname{CSP}\left(\mathfrak{B}_{\mathcal{F}}\right)$.

In the rest of this section, we show that there is a polynomial-time reduction from $\operatorname{CSP}\left(\mathfrak{B}_{\mathcal{F}}\right)$ to the $\mathcal{F}$-free orientation completion problem, and use this reduction to classify the complexity of the $\mathcal{F}$-free orientation completion problem.

Given a digraph $D=(V, E)$ and an edge $(x, y) \in E$, we write $D-(x, y)$ to denote the digraph $(V, E \backslash\{(x, y)\})$. Consider a set of tournaments $\mathcal{F}$ and a symmetric edge $x y$ of $D$. We say that $x y$ is free in $D$ (with respect to $\mathcal{F}$ ) if $D-(x, y)$ and $D-(y, x)$ can be completed to an $\mathcal{F}$-free oriented graph. We say that a pair $(x, y)$ forces a pair $(u, v)$ in $D$ (with respect to $\mathcal{F}$ ) if $x y$ and $u v$ are free symmetric edges in $D$, and every $\mathcal{F}$-free orientation completion of $D-(y, x)$ contains $(u, v)$ as an oriented edge. In other words, if $x y$ and $u v$ are free edges in $D$, we say that $(x, y)$ forces $(u, v)$ if any orientation completion $D^{\prime}$ of $D$ such that $(x, y),(v, u) \in E\left(D^{\prime}\right)$ contains some tournament $T$ of $\mathcal{F}$. For instance, in the digraph $D_{1}$ (see Fig. (2) the pair $(x, y)$ forces the pair $(u, v)$ with respect to
$\left\{\overrightarrow{C_{3}}\right\}$. Recall that if $\mathcal{F}$ contains a transitive tournament, we denote by $n_{\mathcal{F}}$ the minimum number of vertices of a transitive tournament in $\mathcal{F}$.

Lemma 18. Let $\mathcal{F}$ be a non-empty finite set of tournaments, and $m$ the minimum number of vertices in a tournament of $\mathcal{F}$. In this case, the following statements are equivalent.

1. An oriented graph $D$ is $\mathcal{F}$-free if and only if it contains no tournament on $m$ vertices.
2. For each digraph $D$ there is an $\mathcal{F}$-free orientation completion of $D$ if and only if every orientation completion of $D$ is $\mathcal{F}$-free.
3. For every digraph $D$, if a pair $(x, y)$ forces a pair $(u, v)$ in $D$ with respect to $\mathcal{F}$, then $x=u$ and $y=v$.
4. For every semicomplete digraph $D$ on $m$ vertices, if a pair $(x, y)$ forces a pair $(u, v)$ in $D$ with respect to $\mathcal{F}$, then $x=u$ and $y=v$.

Proof. Suppose that the first item holds. It immediately follows that a digraph $D$ admits an orientation completion if and only if $D$ contains no semicomplete digraph on $m$ vertices. In this case, any orientation completion of $D$ is $\mathcal{F}$-free. Thus, the first item implies the second one.

Assume the second statement to be true. Since there is a tournament on $m$ vertices in $\mathcal{F}$, it must be the case that no orientation of $K_{m}$ is $\mathcal{F}$-free. By the choice of $m$ it must be the case that $\mathcal{F}$ contains all tournaments on $m$ vertices up to isomorphism, and that $\mathcal{F}$ contains no tournament on less than $m$ vertices. Thus, an oriented graph is $\mathcal{F}$-free if and only if it contains no tournament on $m$ vertices.

Directly from the definition of " $(x, y)$ forces $(u, v)$ " one can notice that the negation of the third statement implies the negation of the second one. Equivalently, the second item implies the third one, and trivially, the third item implies the fourth one. Finally, we argue that the fourth item implies the first one by contraposition. So, assuming the first item is not true, we know that there must be at least one $\mathcal{F}$-free tournament on $m$ vertices. Let $T$ be a tournament in $\mathcal{F}$ with $m$ vertices, and $\left(x_{1}, y_{1}\right), \ldots,\left(x_{n}, y_{n}\right)$ be the edges of $T$. Consider the semicomplete digraph $T^{i}$ recursively defined as $T^{i}:=\left(V(T), E\left(T^{i-1}\right) \cup\left\{\left(y_{i}, x_{i}\right)\right\}\right)$, where $T^{0}:=T$. In particular, notice that $T^{n}$ is the complete graph on $m$ vertices, so $T^{n}$ admits an $\mathcal{F}$-free orientation. Moreover, by the symmetries of complete graphs, for any edge $(x, y) \in T^{n}$ there is an $\mathcal{F}$-free orientation completion of $T^{n}-(x, y)$. Let $l$ be minimal such that $T^{l}$ can be completed to an $\mathcal{F}$-free tournament. Clearly, $x_{l} y_{l}$ is a symmetric edge in $T^{l}$, and since $l \leq n-1$, there is non-symmetric edge in $T^{l}$, namely, $\left(x_{l+1}, y_{l+1}\right) \in E\left(T^{l}\right)$ and $\left(y_{l+1}, x_{l+1}\right) \notin E\left(T^{l}\right)$. From these observations, and by the choice of $l$, it follows that $x_{l} y_{l}$ and $x_{l+1} y_{l+1}$ are (different) symmetric edges in $T^{l+1}$, and ( $x_{l+1}, y_{l+1}$ ) forces $\left(y_{l}, x_{l}\right)$. The claim follows.

It is evident that for every set of tournaments $\mathcal{F}$, every digraph $D$, and every symmetric edge $x y \in E(D)$, the edge $(x, y)$ forces itself. In the proof of the following lemma, we will implicitly use the following observations several times:

1. If $(x, y)$ forces $(u, v)$, then $(v, u)$ forces $(y, x)$.
2. If $(x, y)$ forces $(u, v)$ and $(u, v)$ forces $(a, b)$, then $(x, y)$ forces $(a, b)$.
3. If $\varphi: D \rightarrow D^{\prime}$ is a homomorphism, and $(x, y)$ forces $(u, v)$ in $D$, then $(\varphi(x), \varphi(y))$ forces $(\varphi(u), \varphi(v))$ in $D^{\prime}$.

The following lemma shows that given a digraph $D$ with a pair $(x, y)$ that forces a pair $(u, v)$, we can construct a digraph $D^{\prime}$ with a pair $\left(x^{\prime}, y^{\prime}\right)$ that forces a pair $\left(u^{\prime}, v^{\prime}\right)$, and the latter also forces the former. Moreover, $D^{\prime}$ can be chosen in such a was that the vertices $x^{\prime}, y^{\prime}$ are "far apart" from the pair $u^{\prime}, v^{\prime}$. To this end, we consider the following notion of distance. Given a pair of vertices $x, y$ in a connected digraph $D$, we denote by $d(x, y)$ the distance between $x$ and $x$ in the underlying graph $u(D)$. That is, the number of undirected edges in a shortest path between $x$ and $y$ in $u(D)$. The previously mentioned construction is described in the proof of the following lemma, and illustrated in Fig. 2.

Lemma 19. The following statements are equivalent for a finite set of finite tournaments $\mathcal{F}$.

1. There is a digraph $D$ with two pairs of vertices $(x, y)$ and $(u, v)$ such that $(x, y)$ forces $(u, v)$, and $(x, y) \neq(u, v)$.
2. There is a digraph $D$ with two pairs of vertices $(x, y)$ and $(u, v)$ such that $(x, y)$ forces $(u, v)$, and $|\{x, y, u, v\}|=4$.
3. For every positive integer $k$, there is a digraph $D$ with two pairs of vertices $(x, y)$ and $(u, v)$ such that $(x, y)$ and $(u, v)$ force each other, and $d(a, b) \geq k$ for $a \in\{x, y\}$ and $b \in\{u, v\}$.

Proof. It is evident that the third statement implies the first one. Now, we prove that the first item implies the second one. Suppose that $(x, y) \neq(u, v)$. If $|\{x, y, u, v\}|=4$, then there is nothing left to prove. So suppose that $|\{x, y, u, v\}|=3$. Notice that up to symmetry, there are two cases to consider: when $y=u$, and when $y=v$. We consider the latter one. Consider two copies of $D, D_{1}$ and $D_{2}$, where $\left(x_{i}, y_{i}\right)$ forces $\left(u_{i}, y_{i}\right)$ in $D_{i}$ for each $i \in\{1,2\}$. In this case, let $D^{\prime}$ be the digraph obtained from the disjoint union of $D_{1}$ with $D_{2}$ after identifying $y_{1}$ with $u_{2}$ and $u_{1}$ with $y_{2}$. Using the enumerated observations preceding this lemma, we conclude that $\left(x_{1}, y_{1}\right)$ forces $\left(y_{2}, x_{2}\right)$ in $D^{\prime}$. The case when $y=u$ follows with a similar construction.

Finally, we show that the second statement implies the third one. Let $D, x, y, u$, and $v$ be as in the second statement, and $k \geq 2$. Consider $k$ copies $D_{1}, \ldots, D_{k}$ of $D$ where $\left(x_{i}, y_{i}\right)$ forces $\left(u_{i}, v_{i}\right)$ in $D_{i}$ for each $i \in\{1, \ldots, k\}$. It is not hard to notice that by considering the disjoint union $D_{1}+\cdots+D_{k}$, and identifying $u_{i}$ with $x_{i+1}$, and $v_{i}$ with $y_{i}$ for $i \in[k-1]$, we obtain a digraph $D^{\prime}$ where $\left(x_{1}, x_{2}\right)$ forces $\left(u_{k}, v_{k}\right)$, and $d(a, b) \geq k$ for $a \in\left\{x_{1}, y_{1}\right\}$ and $b \in\left\{u_{k}, v_{k}\right\}$. Finally, consider $D^{\prime}$ together with a disjoint copy $D^{\prime \prime}$ of itself, and the following identifications $u_{k}^{\prime} \sim x_{1}^{\prime \prime}, u_{k}^{\prime \prime} \sim x_{1}^{\prime}$, $v_{k}^{\prime} \sim y_{1}^{\prime \prime}$, and $v_{k}^{\prime \prime} \sim y_{1}^{\prime}$. It is not hard to see that we obtain a digraph $\left(D^{\prime}+D^{\prime \prime}\right) / \sim$ with two pairs of vertices $\left(x_{1}^{\prime}, y_{1}^{\prime}\right)$ and $\left(u_{1}^{\prime}, v_{1}^{\prime}\right)$ that force each other and $d(a, b) \geq k$ for $a \in\left\{x_{1}^{\prime}, y_{1}^{\prime}\right\}$ and $b \in\left\{u_{1}^{\prime}, v_{1}^{\prime}\right\}$. The lemma is now proved.

Recall that the $\mathcal{F}$-free orientation completion problem and $\operatorname{CSP}\left(D_{\mathcal{F}}, U\right)$ are trivially polynomialtime equivalent. We will see that there is a primitive positive interpretation of $\left(\mathfrak{B}_{\mathcal{F}}, \mathbf{0}, \mathbf{1}\right)$ in $\left(D_{\mathcal{F}}, U\right)$. This will yield a polynomial-time reduction from $\operatorname{CSP}\left(\mathfrak{B}_{\mathcal{F}}, \mathbf{0}, \mathbf{1}\right)$ to $\operatorname{CSP}\left(D_{\mathcal{F}}, U\right)$, and thus, also to the $\mathcal{F}$-free orientation completion problem. To do so, we first consider the 4 -ary relation $S_{4}$ defined by

$$
\begin{equation*}
S_{4}\left(x_{1}, x_{2}, x_{3}, x_{4}\right) \Leftrightarrow\left(\left(E\left(x_{1}, x_{2}\right) \wedge E\left(x_{3}, x_{4}\right)\right) \vee\left(E\left(x_{2}, x_{1}\right) \wedge E\left(x_{4}, x_{3}\right)\right)\right) \tag{1}
\end{equation*}
$$

Intuitively, $S_{4}$ encodes that the first and last pair of vertices are adjacent, and both edges have the same direction.


Figure 2: Three digraphs $D_{1}, D_{2}$, and $D_{3}$ where in each case, the pair $(x, y)$ forces the pair $(u, v)$ with respect to $\left\{\overrightarrow{C_{3}}\right\}$. In $D_{1}$, the cardinality of $\{x, y, u, v\}$ is 3 . In $D_{2},|\{x, y, u, v\}|=4$, and it is obtained from $D_{1}$ as in the proof of Lemma 19. Finally, in $D_{3}, d(a, b) \geq 2$ for $a \in\{x, y\}$ and $b \in\{u, v\}$, and $D_{3}$ is obtained from $D_{2}$ as in the proof of Lemma 19 for $k=2$.

Lemma 20. For every non-empty finite set of tournaments $\mathcal{F}$, there is a primitive positive definition of $S_{4}$ in $\left(D_{\mathcal{F}}, U\right)$.

Proof. By Lemma 18, and by the third part of Lemma 19, there is a digraph $D$ with two pairs of vertices $(x, y)$ and $(u, v)$ such that $(x, y)$ and $(u, v)$ force each other, and $d(a, b) \geq 4$ for $a \in\{x, y\}$ and $b \in\{u, v\}$. Interpret $D$ as an $\{E, U\}$-structure $D_{U}$, where $(x, y) \in E\left(D_{U}\right)$ if and only if $(x, y) \in E(D)$ and $(y, x) \notin E(D)$, and $(x, y) \in U\left(D_{U}\right)$ if and only if $(x, y) \in E(D)$ and $(y, x) \in E(D)$. In other words, the interpretation of $U$ in $D^{\prime}$ corresponds to the symmetric symmetric edges of $D$, and the interpretation of $E$ corresponds to the anti-symmetric edges of $D$. Let $\phi\left(x_{1}, \ldots, x_{n}\right)$ be the canonical conjunctive query of $D_{U}$, where $x_{1}$ corresponds to $x, x_{2}$ to $y, x_{3}$ to $u$, and $x_{4}$ to $v$. With this setting, if $\phi_{S}\left(x_{1}, x_{2}, x_{3}, x_{4}\right)$ is the primitive positive-formula $\exists x_{5}, \ldots, x_{n} \phi\left(x_{1}, \ldots, x_{n}\right)$, then $\phi_{S}$ implies $S_{4}$. Now, we briefly argue that $S_{4}$ also implies $\phi_{S}$. Let $y_{1}, \ldots, y_{4}$ be four vertices of $D$ such that $S_{4}\left(y_{1}, y_{2}, y_{3}, y_{4}\right)$, and without loss of generality assume that $\left(y_{1}, y_{2}\right),\left(y_{3}, y_{4}\right) \in E\left(D_{\mathcal{F}}\right)$. Suppose that $\left|\left\{y_{1}, y_{2}, y_{3}, y_{4}\right\}\right|=4$ and let $D^{\prime}$ be a $\mathcal{F}$-free orientation of $D$ where $\left(x_{1}, x_{2}\right),\left(x_{3}, x_{4}\right) \in E\left(D^{\prime}\right)$. For each $i, j \in\{1,2,3,4\}$, if there is an edge $\left(y_{i}, y_{j}\right)$ in $D_{\mathcal{F}}$, then add an edge $\left(x_{i}, x_{j}\right)$ to $D^{\prime}$ obtaining an oriented graph $D^{\prime \prime}$. Notice that since $d\left(x_{i}, x_{j}\right) \geq 4$ for $i \in\{1,2\}$ and $j \in\{3,4\}$, any triangle of $D^{\prime \prime}$ is a triangle of $D^{\prime}$ and so, $D^{\prime \prime}$ is an $\mathcal{F}$-free oriented graph. Thus, there is partial automorphism that maps $y_{i} \mapsto x_{i}$ for $i \in\{1,2,3,4\}$, and by homogeneity of $D_{\mathcal{F}}$ this can be extended to an automorphism $f: D_{\mathcal{F}} \rightarrow D_{\mathcal{F}}$. Since every edge of $D^{\prime}$ is an edge of $D^{\prime \prime}$ and $\phi_{S}$ is a primitive positive formula, it is the case that $\phi_{S}$ is true of $x_{1}, x_{2}, x_{3}, x_{4}$ in $D_{\mathcal{F}}$. Therefore, since primitive positive formulas are preserved by automorphisms, and $f^{-1}\left(x_{i}\right)=y_{i}$ for $i \in\{1,2,3,4\}$, we conclude that $D_{\mathcal{F}} \models \phi_{S}\left(y_{1}, y_{2}, y_{3}, y_{4}\right)$. Finally, the cases when $\left|\left\{y_{1}, y_{2}, y_{3}, y_{4}\right\}\right| \in\{2,3\}$ follow similarly, but instead of obtaining $D^{\prime \prime}$ from $D^{\prime}$ by adding new edges, we obtain $D^{\prime \prime}$ from $D^{\prime}$ by identifying $x_{i}$ with $x_{j}$ whenever $y_{i}=y_{j}$. Again, the observation that $D^{\prime \prime}$ is a $\mathcal{F}$-free oriented graph follows from the fact that $d\left(x_{i}, x_{j}\right) \geq 4$ for $i \in\{1,2\}$ and $j \in\{3,4\}$. With the corresponding identifications, we obtain a partial automorphism of $D_{\mathcal{F}}$ that defined by mapping $y_{i} \mapsto x_{i}$ for $i \in\{1,2,3,4\}$. Finally, using the fact that primitive positive formulas are preserved under homomorphisms (and $D^{\prime \prime}$ is a homomorphic image of $\left.D^{\prime}\right)$ and under automorphisms, we conclude that $D_{\mathcal{F}} \models \phi_{S}\left(y_{1}, y_{2}, y_{3}, y_{4}\right)$.

Now, we show that using the relation $S_{4}$ we can primitively positively interpret $\left(\mathfrak{B}_{\mathcal{F}}, \mathbf{0}, \mathbf{1}\right)$ in $\left(D_{\mathcal{F}}, U, S_{4}\right)$. Also, recall that $H_{\mathcal{F}}$ is the underlying graph of $D_{\mathcal{F}}$, i.e., the reduct of $\left(D_{\mathcal{F}}, U\right)$ after forgetting the relation $E$. We also see that we can primitively positively interpret $\mathfrak{B}_{\mathcal{F}}$ in $\left(H_{\mathcal{F}}, S_{4}\right)$.

Lemma 21. For every finite set of finite tournaments $\mathcal{F}$, there is a primitive positive interpretation of $\left(\mathfrak{B}_{\mathcal{F}}, \mathbf{0}, \mathbf{1}\right)$ in $\left(D_{\mathcal{F}}, U, S_{4}\right)$, and a primitive positive interpretation of $\mathfrak{B}_{\mathcal{F}}$ in $\left(H_{\mathcal{F}}, S_{4}\right)$.

Proof. We first consider the primitive positive interpretation of $\left(\mathfrak{B}_{\mathcal{F}}, \mathbf{0}, \mathbf{1}\right)$ in $\left(D_{\mathcal{F}}, U, S_{4}\right)$. The dimension of the interpretation is 2 , and the domain formula is $\top_{I}(x, y):=U(x, y)$. Equality $={ }_{I}\left(x_{1}, y_{1}, x_{2}, y_{2}\right)$ is defined by $S_{4}\left(x_{1}, y_{1}, x_{2}, y_{2}\right)$, and the unary relations $\mathbf{0}_{I}(x, y)$ and $\mathbf{0}_{I}(x, y)$ are defined by $E(y, x)$ and $E(x, y)$, respectively. Finally, for each positive integer $n$, the $2\binom{n}{2}$-ary relation

$$
\delta_{P_{n}}\left(x_{1,2}, y_{1,2}, x_{1,3}, y_{1,3}, \ldots, x_{n-1, n}, y_{n-1, n}\right)
$$

expresses that there are $n$ vertices $k_{1}, \ldots, k_{n}$ such that:

1. $U\left(k_{i}, k_{j}\right)$ for each $i, j \in[n]$, i.e., the vertices $k_{1}, \ldots, k_{n}$ induce a tournament in $D_{\mathcal{F}}$, and
2. for each pair $i<j$ the 4 -tuple $\left(x_{i, j}, y_{i, j}, k_{i}, k_{j}\right)$ belongs to $S_{4}$, i.e., the edges $k_{i} k_{j}$ and $x_{i, j} y_{i, j}$ have the same orientation in $D_{\mathcal{F}}$.

The same interpretation without the defining formulas $\delta_{0}$ and $\delta_{1}$ yield a primitive positive interpretation of $\mathfrak{B}_{\mathcal{F}}$ in $\left(H_{\mathcal{F}}, S_{4}\right)$. Notice that in this case, the first item means that the vertices $k_{1}, \ldots, k_{n}$ induce a clique in $H_{\mathcal{F}}$.

Using these two lemmas, we can easily prove the following statement.
Proposition 22. Let $\mathcal{F}$ be a finite set of finite tournaments. If $\mathcal{F}$ is not the empty set, then there is primitive positive interpretation of $\left(\mathfrak{B}_{F}, \mathbf{0}, \mathbf{1}\right)$ in $\left(D_{\mathcal{F}}, U\right)$.

Proof. By Lemma 20, $S_{4}$ has a primitive positive definition in $\left(D_{\mathcal{F}}, U\right)$, and by Lemma 21, there is primitive positive interpretation of $\left(\mathfrak{B}_{\mathcal{F}}, \mathbf{0}, \mathbf{1}\right)$ in $\left(D_{\mathcal{F}}, U, S_{4}\right)$.

Theorem 17 asserts that the $\mathcal{F}$-free orientation completion problem reduces in polynomial-time to $\operatorname{CSP}\left(\mathfrak{B}_{\mathcal{F}}, \mathbf{0}, \mathbf{1}\right)$. Also, as mentioned in Section 2 the $\mathcal{F}$-free orientation completion problem and $\operatorname{CSP}\left(D_{\mathcal{F}}, U\right)$ are polynomial-time equivalent. Finally, Proposition 22 together with Lemma 10 show that $\operatorname{CSP}\left(B_{\mathcal{F}}, \mathbf{0}, \mathbf{1}\right)$ reduces in polynomial-time to $\operatorname{CSP}\left(D_{\mathcal{F}}, U\right)$. Thus, the following statement is proved by the arguments in this paragraph.

Theorem 23. The following problems are polynomial-time equivalent for each set finite set of finite tournaments $\mathcal{F}$.

1. The $\mathcal{F}$-free orientation completion problem.
2. $\operatorname{CSP}\left(\mathfrak{B}_{\mathcal{F}}, \mathbf{0}, \mathbf{1}\right)$.
3. $\operatorname{CSP}\left(D_{\mathcal{F}}, U\right)$.

If a Boolean structure $\mathfrak{B}$ contains no constant endomorphism, then it is a core, and hence $\operatorname{CSP}(\mathfrak{B})$ and $\operatorname{CSP}(\mathfrak{B}, \mathbf{0}, \mathbf{1})$ are polynomial-time equivalent (see, e.g., 9 ). Notice that for a set of tournaments $\mathcal{F}$ the structure $\mathfrak{B}_{\mathcal{F}}$ contains a constant endomorphism if and only if for each $n \leq m_{\mathcal{F}}$ either $P_{n}$ is empty or $P_{n}$ contains both constant tuples. In particular, if $\mathcal{F}$ contains a transitive
tournament and there is at least one $\mathcal{F}$-free tournament on tournament on $n_{\mathcal{F}}$ vertices, then $P_{n_{\mathcal{F}}}$ is neither empty nor contains a constant tuple, and thus, $\mathfrak{B}_{\mathcal{F}}$ contains no constant endomorphism. With these arguments in mind, the following statement is an immediate implication of Theorem 23,

Corollary 24. Let $\mathcal{F}$ be a finite set of finite tournaments that contains a transitive tournament. If there is at least one $\mathcal{F}$-free tournament with $n_{\mathcal{F}}$-vertices, then the following problems are polynomialtime equivalent.

1. The $\mathcal{F}$-free orientation completion problem.
2. $\operatorname{CSP}\left(\mathfrak{B}_{\mathcal{F}}, \mathbf{0}, \mathbf{1}\right)$.
3. $\operatorname{CSP}\left(\mathfrak{B}_{\mathcal{F}}\right)$.
4. $\operatorname{CSP}\left(D_{\mathcal{F}}, U\right)$.

### 4.2 Complexity Classification

Theorem 23 together with Schaefer's theorem yield a classification of the $\mathcal{F}$-free orientation completion problem in terms of the Boolean structure $\left(\mathfrak{B}_{\mathcal{F}}, \mathbf{0}, \mathbf{1}\right)$. In this section, we see that if $\left(\mathfrak{B}_{F}, \mathbf{0}, \mathbf{1}\right)$ does not primitively positively interpret $K_{3}$, then it is preserved by the Boolean minority operation, or by a constant operation.

Lemma 25. Let $\mathcal{F}$ be a finite set of finite tournaments. The following statements are equivalent for each positive integer $n \leq m_{\mathcal{F}}$ such that $T_{n}$ is $\mathcal{F}$-free.

1. Every tournament on $n$ vertices is $\mathcal{F}$-free.
2. $P_{n}=\{0,1\}^{\binom{n}{2}}$.
3. $P_{n}$ is preserved by the minimum operation.
4. $P_{n}$ is preserved by the maximum operation.
5. $P_{n}$ is preserved by the majority operation.
6. $P_{n}$ is preserved by the minority operation.

Proof. The first two items are clearly equivalent, and the second item implies 3-6. Denote by $b_{\mathbf{0}}$ (resp. $b_{\mathbf{1}}$ ) the constant 0 (resp. constant 1) tuples of arity $\binom{n}{2}$. Since $T_{n}$ is $\mathcal{F}$-free, the tuples $b_{\mathbf{0}}$ and $b_{1}$ belong to $P_{n}$. It is not hard to notice that for any pair of tuples $b, b^{\prime} \in P_{n}$ the equalities $\operatorname{majority}\left(b_{\mathbf{1}}, b, b^{\prime}\right)=\max \left(b, b^{\prime}\right)$, and majority $\left(b_{\mathbf{0}}, b, b^{\prime}\right)=\min \left(b, b^{\prime}\right)$ hold. Thus, if $P_{n}$ is preserved by the majority operation, then it is preserved by the minimum and the maximum operations.

To conclude the proof, we show that each of the statements $3,4,6$ imply the first two. Suppose that $P_{n}$ is preserved by the minimum operation. For $i, j \in[n]$ with $i<j$, we denote by $b^{i j}$ the tuple where $\left(b^{i j}\right)_{k l}=1$ if and only if $i=k$ and $j=l$. We show that each $b^{i j}$ belongs to $P_{n}$. Given $i<j$ consider the following permutations of $T_{n}$ where the edge set is defined by the linear ordering of $[n]$ :

$$
\begin{aligned}
& T^{1}:=(n, n-1, \ldots, j+1, j-1, j-2, \ldots, i, j, i-1, i-2, \ldots 1) \\
& \text { and } \quad T^{2}:=(n, n-1, \ldots, j+1, i, j, j-1, \ldots, i+1, i-1, i-2, \ldots, 1)
\end{aligned}
$$

With a simple computation of the minimum operation one can notice that $\min \left(b_{T^{1}}, b_{T^{2}}\right)=b^{i j}$. Since $T_{n}$ is $\mathcal{F}$-free, it follows that $b_{T^{1}}, b_{T^{2}} \in P_{n}$, and so $b^{i j} \in P_{n}$. It is not hard to notice that the tuple $b^{i j}$ encode those tournaments obtained from $T_{n}$ be reversing the orientation of one arc. Similarly, the tuples $c^{i j}$ where $\left(c^{i j}\right)_{k l}=0$ if and only if $i=k$ and $l=j$, encode the same family of tournaments (up to isomorphism). Thus, it is also the case that for each $i, j \in[n]$ with $1 \leq i<j \leq n$, all tuples $c^{i j}$ belong to $P_{n}$. Finally, consider a set of pairs $\left\{\left(i_{1}, j_{1}\right), \ldots,\left(i_{l}, j_{l}\right)\right\} \subseteq[n]^{2}$ with $i_{k}<j_{k}$ for each $k \in[l]$. By composing the minimum operation as $\min \left(c^{i_{1} j_{1}}, \min \left(c^{i_{2} j_{2}}, \ldots\right)\right)$, we obtain a tuple $b$ where $b_{i j}=0$ if and only if $(i, j)=\left(i_{k}, j_{k}\right)$ for some $k \in[l]$. Thus, we conclude that each $\left.b \in\{0,1\} \begin{array}{c}n \\ 2\end{array}\right)$ belongs to $P_{n}$, i.e., $P_{n}=\{0,1\}^{\binom{n}{2}}$, and so, the third item implies the second one. The case when $\mathcal{F}$ is preserved by the maximum operation follows with dual arguments.

Finally, suppose that $P_{n}$ is preserved by the minority operation. For a pair of tuples $b, b^{\prime}$ we denote by $b+b^{\prime}$ the coordinate-wise addition modulo 2 , and notice that minority $\left(b_{\mathbf{0}}, b, b^{\prime}\right)=b+b^{\prime}$. Thus, since $P_{n}$ is preserved by the minority operation, and $b_{0} \in P_{n}$, we conclude that $P_{n}$ is closed under addition modulo 2. Evidently, every tuple $b \in\{0,1\} \begin{gathered}\binom{n}{2}\end{gathered}$ can be expressed as a sum of tuples of the form $b^{i j}$ (introduced in the previous paragraph). Hence, it suffices to show that $b^{i j} \in P_{n}$ for each pair $i<j$. To do so, consider the following permutations of $T_{n}$ where the edge set is define by the linear ordering of $[n]$ :

$$
\begin{aligned}
& T^{1}:=(1, \ldots, i-1, i, i+2, \ldots, j, i+1, j+1, j+2, \ldots, n), \\
\text { and } \quad & T^{2}:=(1, \ldots, i-1, i+1, i+2, \ldots, j, i, j+1, j+2, \ldots, n) .
\end{aligned}
$$

It is not hard to notice that $b^{i j}=b_{T^{1}}+b_{T^{2}}$, and since $T_{n}$ is $\mathcal{F}$-free and $P_{n}$ is closed under addition modulo 2 , we conclude that $b^{i j} \in P_{n}$. So, by the arguments above we conclude that $P_{n}=\{0,1\}^{\binom{n}{2}}$. The equivalence between $1-6$ is now proved.

Recall that $m_{\mathcal{F}}$ denotes the maximum number of vertices of a tournament in $\mathcal{F}$.
Lemma 26. Let $\mathcal{F}$ be a finite set of finite tournaments. The following statements are equivalent for each positive integer $n \leq m_{\mathcal{F}}$.

1. Either all tournaments on $n$ vertices are $\mathcal{F}$-free, or no tournament on $n$ vertices is $\mathcal{F}$-free.
2. Either $P_{n}=\varnothing$ or $P_{n}=\{0,1\}\binom{n}{2}$.
3. $P_{n}$ is preserved by the minimum operation.
4. $P_{n}$ is preserved by the maximum operation.
5. $P_{n}$ is preserved by the majority operation.

Proof. It is evident that the first two items are equivalent, and that each of these implies the rest. We show that each of $3-5$ imply the first two. To do so, we will see that in each case, if $P_{n} \neq \varnothing$, then $P_{n}$ contains the constant tuple $b_{1}$ (where all entries are 1), i.e., $T_{n}$ is $\mathcal{F}$-free. We will thus conclude by Lemma 25 that $P_{n}=\{0,1\}^{\binom{n}{2}}$. To begin with, suppose that $P_{n}$ is preserved by the maximum operation, and that there is some tuple $b_{T} \in P_{n}$ for some $\mathcal{F}$-free tournament $T$ with vertex set $[n]$. If $b_{T}=b_{\mathbf{1}}$, there is nothing left to prove, so suppose $b_{i j}=0$ for some $1 \leq i<j \leq n$. Let $T^{\prime}$ be the permutation of $T$ obtained from transposing $i$ with $j$. Since $\left(b_{T}\right)_{i j}=0$ and $\left(b_{T^{\prime}}\right)_{i j}=1$, it follows that the number of coordinates which equal 1 in $\max \left(b_{T}, b_{T^{\prime}}\right)$ is strictly larger than those which
equal 1 in $b_{T}$. Since $P_{n}$ is preserved by maximum operation, we can iterate this procedure to see that $b_{1} \in P_{n}$. Thus $T_{n}$ is $\mathcal{F}$-free and so, using Lemma 25 we conclude that $P_{n}=\{0,1\}\binom{n}{2}$. The case when $P_{n}$ is preserved by the minimum operation follows from dual arguments.

Finally, suppose that $P_{n}$ is preserved by majority, and that there is some $\mathcal{F}$-free tournament with vertex set $[n]$, i.e., $b_{T} \in P_{n}$. We begin by showing that there is a tuple $b_{T^{\prime}} \in P_{n}$ such that $\left(b_{T^{\prime}}\right)_{1 j}=1$ for all $1<j \leq n$. Let $l$ be the maximum outdegree of $T$, and suppose $l<n-1$ (otherwise, there is nothing left to prove). Consider two permutations $T^{1}$ and $T^{2}$ of $T$ such that for $i \in\{1,2\}$, the vertex 1 is the vertex of largest outdegree of $T^{i}$ and $\{i+1, \ldots, i+l\}$ are its outneighbours. Consider a third labeling $T^{3}$ such that $(1,2),(1, i+1) \in E\left(T^{3}\right)$. Clearly, if $b=$ majority $\left(b_{T^{1}}, b_{T^{2}}, b_{T^{3}}\right)$, then $b_{1 j}=1$ for all $2 \leq j \leq l+1$. Since $P_{n}$ is preserved by the majority operation, we can proceed inductively to find a tuple $b \in P_{n}$ such that $b_{1 j}=1$ for all $1<j \leq n$. With a similar finite inductive argument over $k \in[n]$, we can find a tuple $b \in P_{n}$ such that $b_{i j}=1$ for all $i \leq k$ and $j>i$. Thus, we conclude that $b_{1} \in P_{n}$ and so, $T_{n}$ is $F$-free. Hence, by Lemma 25, we conclude that $P_{n}=\{0,1\}^{\binom{n}{2}}$. The claim follows.

Building on Lemma 26, we prove the following statement.
Lemma 27. Let $\mathcal{F}$ be a finite set of finite tournaments. If $\mathfrak{B}_{\mathcal{F}}$ does not interpret $K_{3}$ primitively positively, then $\mathfrak{B}_{\mathcal{F}}$ is preserved by the Boolean minority operation or a constant operation.

Proof. If $\mathfrak{B}_{\mathcal{F}}$ does not interpret $K_{3}$ primitively positively, then, by Schaefer's theorem, $\mathfrak{B}_{\mathcal{F}}$ is preserved by the minimum, the maximum, the majority, the minority, or the constant operation. If either of the last two cases holds, the claim is proved. Otherwise, suppose that $\mathfrak{B}_{\mathcal{F}}$ is preserved by the minimum, the maximum, or the majority operation. Then, for each $n \leq m_{\mathcal{F}}$ the relation $P_{n}$ is preserved by one of these operations. By Lemma [26, we conclude that for each $n \leq m_{\mathcal{F}}$ either $P_{n}=\varnothing$ or $P_{n}=\{0,1\} \begin{gathered}\binom{n}{2}\end{gathered}$. Hence, each $P_{n}$ is trivially preserved by any constant operation, and so, $\mathfrak{B}_{\mathcal{F}}$ is preserved by a constant operation.

We are now ready to state the proposed classification for the complexity of $\mathcal{F}$-free orientation completion problems.

Theorem 28. Let $\mathcal{F}$ be a finite set of finite tournaments. Then exactly one of the following two cases applies.

- $K_{3}$ has a primitive positive interpretation in $\left(\mathfrak{B}_{\mathcal{F}}, \mathbf{0}, \mathbf{1}\right)$ and in $\left(D_{\mathcal{F}}, U\right)$. In this case, $\operatorname{CSP}\left(D_{\mathcal{F}}, U\right)$ and the $\mathcal{F}$-free orientation completion problem are NP-complete.
- $\left(B_{\mathcal{F}}, \mathbf{0}, \mathbf{1}\right)$ has the the minority operation as polymporphism, and $\left(D_{\mathcal{F}}, U\right)$ has a ternary pseudo near unanimity polymorphism. In this case, $\operatorname{CSP}\left(D_{\mathcal{F}}, U\right)$ and the $\mathcal{F}$-free orientation problem are in $P$.

Proof. The two cases are mutually disjoint: if $\left(B_{\mathcal{F}}, \mathbf{0}, \mathbf{1}\right)$ is has the minority operation as polymorphism, then every structure with a first-order interpretation in it has such a polymorphism as well (see, e.g., Corollary 6.5.16 in [9]), but $K_{3}$ does not have such a polymorphism (see, e.g., Proposition 6.1.43 in [9]).

Now we see that one of the two cases holds, and first suppose that $K_{3}$ has a primitive positive interpretation in $\left(\mathfrak{B}_{\mathcal{F}}, \mathbf{0}, \mathbf{1}\right)$. Since $\left(\mathfrak{B}_{\mathcal{F}}, \mathbf{0}, \mathbf{1}\right)$ has a primitive positive interpretation in $\left(D_{\mathcal{F}}, U\right)$, by composing these interpretations, we obtain a primitive positive interpretation of $K_{3}$ in $\left(\mathfrak{B}_{\mathcal{F}}, \mathbf{0}, \mathbf{1}\right)$.

The NP-hardness of $\operatorname{CSP}\left(D_{\mathcal{F}}, U\right)$ and the $\mathcal{F}$-free orientation completion problem now follows from Lemma 10 .

Otherwise, if $K_{3}$ does not have a primitive positive interpretation in $\left(\mathfrak{B}_{\mathcal{F}}, \mathbf{0}, \mathbf{1}\right)$, then Schaefer's theorem (Theorem (12) implies that $\operatorname{CSP}\left(\mathfrak{B}_{\mathcal{F}}, \mathbf{0}, \mathbf{1}\right)$ is in P , and $\left(\mathfrak{B}_{\mathcal{F}}, \mathbf{0}, \mathbf{1}\right)$ has a polymorphism which is a constant operation, or the minimum, maximum, majority, or minority operation. Lemma 27, and the obvious fact that $\left(\mathfrak{B}_{\mathcal{F}}, \mathbf{0}, \mathbf{1}\right)$ cannot be preserved by constant operations, imply that $\left(\mathfrak{B}_{\mathcal{F}}, \mathbf{0}, \mathbf{1}\right)$ has a minority polymorphism. In this case we construct the ternary pseudo near unanimity polymorphism of $\left(D_{\mathcal{F}}, U\right)$ as follows. Consider the digraph with domain $V^{3}$ and edge set

$$
\left\{\left(\left(u_{1}, u_{2}, u_{3}\right),\left(v_{1}, v_{2}, v_{3}\right)\right)\left|\left(u_{1}, v_{1}\right),\left(u_{2}, v_{2}\right),\left(u_{3}, v_{3}\right) \in U,\left|\left\{i \mid\left(u_{i}, v_{i}\right) \in E\right\}\right| \in\{0,2\}\right\} .\right.
$$

Since $\mathfrak{B}_{\mathcal{F}}$ has a minority polymorphism, every finite subgraph $F$ of this graph has an embedding $h$ to $D_{\mathcal{F}}$, and by the homogeneity of $D_{\mathcal{F}}$ there exist automorphisms $e_{1}, e_{2}$ of $D_{\mathcal{F}}$ such that for all $x, y \in V$ with $(x, x, y),(x, y, x),(y, x, x) \in V(F)$ we have $e_{1}\left(h(x, x, y)=e_{2}(h(x, y, x))=h(y, x, x)\right.$. The existence of a ternary pseudo weak near unanimity polymorphism can be shown as Proposition 6.6 in [16. The polynomial-time tractability of $\operatorname{CSP}\left(D_{\mathcal{F}}, U\right)$ and the $\mathcal{F}$-free orientation completion problem now follows from the polynomial-time tractability of $\operatorname{CSP}\left(\mathfrak{B}_{\mathcal{F}}, \mathbf{0}, \mathbf{1}\right)$ via Theorem 17

## 5 Symmetries

Recall that, given a set of tournaments $\mathcal{F}$, we denote by $D_{\mathcal{F}}$ the countable universal homogeneous $\mathcal{F}$-free digraph, and by $H_{\mathcal{F}}$ its its underlying graph. In order to classify the complexity of the $\mathcal{F}$-free orientation problem, it will be highly useful to understand the symmetries of $H_{\mathcal{F}}$ in terms of the symmetries of $D_{\mathcal{F}}$. Specifically, we use the description of the automorphism group of $H_{\mathcal{F}}$ in terms of the automorphism group of $D_{\mathcal{F}}$ proposed by Agarwal and Kompatscher [1]. To provide examples of these symmetries we consider the four non-isomorphic tournaments on 4 vertices $T_{4}$, $T C_{4}, C_{3}^{-}$, and $C_{3}^{+}$depicted in Fig. 3.


Figure 3: The four non-isomorphic oriented tournaments on 4 vertices
If $D$ is a digraph, then we denote by $-D$ the digraph obtained by flipping the orientation of all arcs of $D$, and we call it the flip of $D$. It is not hard to notice that if $D$ is a $T_{k}$-free digraph, then $-D$ is also $T_{k}$-free. The following example shows that there are finite sets $\mathcal{F}$ of tournaments that are not preserved by flips.

Example 29. Note that $-\left(C_{3}^{-}\right)$is isomorphic to $C_{3}^{+}$. Hence, $C_{3}^{-}$is a $\left\{T_{4}, C_{3}^{+}\right\}$-free oriented graph whose flip is not $\left\{T_{4}, C_{3}^{+}\right\}$-free.

There is a second source of possible symmetries between $\mathcal{F}$-free oriented graphs. Given a vertex $a \in V(D)$, the switch of $D$ with respect to $a$ is the oriented graph obtained by switching the orientation of all arcs incident to $a$. We denote the resulting oriented graph by $s w_{a}(D)$.

Example 30. The digraph $T C_{4}^{+}$is the switch of $T_{4}$ with respect $v$, where $v$ is a any vertex of $T_{4}$ with positive in- and out-degree.

We say that a class of digraphs $\mathcal{C}$ is preserved by switches (resp., flips) if for every $D \in \mathcal{C}$ and each $a \in V(D)$ it is the case that $\mathcal{C}$ contains a digraph isomorphic to $s w_{a}(D)$ (resp., $-D$ ). It is evident that the class of $\mathcal{F}$-free digraphs is preserved by switches (resp., flips) if and only if $\mathcal{F}$ is preserved by switches (resp., flips). For instance, the previous example shows that the class of $T_{4}$-free graphs is preserved by flips but not by switches.

Since $H_{\mathcal{F}}$ is the underlying graph of $D_{\mathcal{F}}$, every automorphism of $D_{\mathcal{F}}$ is an automorphism of $H_{\mathcal{F}}$. Agarwal and Kompatscher [1] gave a description of the possible automorphism groups of $H_{\mathcal{F}}$ in terms of the automorphism group of $D_{\mathcal{F}}$, and a few additional permutations that we introduce in the following paragraphs.

For any set $X$, we denote by $\operatorname{Sym}(X)$ the permutation group of all permutation of $X$. If $-D_{\mathcal{F}} \cong D_{\mathcal{F}}$, we denote by - any isomorphism $-: D_{\mathcal{F}} \rightarrow-D_{\mathcal{F}}$. In other words, $-: V\left(D_{\mathcal{F}}\right) \rightarrow$ $V\left(D_{\mathcal{F}}\right)$ is a bijection such that $(x, y) \in E\left(D_{\mathcal{F}}\right)$ if and only if $(-(y),-(x)) \in E\left(D_{\mathcal{F}}\right)$. Similarly, if $s w_{a}\left(D_{\mathcal{F}}\right) \cong D_{\mathcal{F}}$ for some $a \in V\left(D_{\mathcal{F}}\right)$, we denote by sw any isomorphism sw: $D_{\mathcal{F}} \rightarrow s w_{a}\left(D_{\mathcal{F}}\right)$. That is, sw : $V\left(D_{\mathcal{F}}\right) \rightarrow V\left(D_{\mathcal{F}}\right)$ is a bijection such that $(a, x) \in E\left(D_{\mathcal{F}}\right)$ if and only if $(\operatorname{sw}(x), \operatorname{sw}(a)) \in$ $E\left(D_{\mathcal{F}}\right),(x, a) \in E\left(D_{\mathcal{F}}\right)$ if and only if $(\operatorname{sw}(a), \operatorname{sw}(x)) \in E\left(D_{\mathcal{F}}\right)$, and whenever $x \neq a \neq y$, there is an edge $(x, y) \in E\left(D_{\mathcal{F}}\right)$ if and only if $(\operatorname{sw}(x), \operatorname{sw}(y)) \in E\left(D_{\mathcal{F}}\right)$. The next two lemmas follow from Lemma 2.4 in [1].

Lemma 31. The following statements are equivalent for a finite set of finite tournaments $\mathcal{F}$.

1. $\mathcal{F}$ is preserved by flips.
2. The class of $\mathcal{F}$-free digraphs is preserved by flips.
3. $D_{\mathcal{F}}$ is isomorphic to $-D_{\mathcal{F}}$.
4. $-: V\left(D_{\mathcal{F}}\right) \rightarrow V\left(D_{\mathcal{F}}\right)$ exists and $-\in \operatorname{Aut}\left(H_{\mathcal{F}}\right)$.

Proof. The first two items are clearly equivalent. By definition of $-: V\left(D_{\mathcal{F}}\right) \rightarrow V\left(D_{\mathcal{F}}\right)$, the last two items are clearly equivalent. Moreover, it is immediate to notice that if $-: V\left(D_{\mathcal{F}}\right) \rightarrow V\left(D_{\mathcal{F}}\right)$ exists, then - defines an automorphism of $H_{\mathcal{F}}$. The equivalence of the second and third statements follows from Lemma 2.4 in [1].

Lemma 32. The following statements are equivalent for a finite set of finite tournaments $\mathcal{F}$.

1. $\mathcal{F}$ is preserved by switches.
2. The class of $\mathcal{F}$-free digraphs is preserved by switches.
3. $D_{\mathcal{F}}$ is isomorphic to $s w_{a}\left(D_{\mathcal{F}}\right)$ for some $a \in V\left(D_{\mathcal{F}}\right)$.
4. $D_{\mathcal{F}}$ is isomorphic to $s w_{a}\left(D_{\mathcal{F}}\right)$ for each $a \in V\left(D_{\mathcal{F}}\right)$.
5. sw : $V\left(D_{\mathcal{F}}\right) \rightarrow V\left(D_{\mathcal{F}}\right)$ exists and $\mathrm{sw} \in \operatorname{Aut}\left(H_{\mathcal{F}}\right)$.

Proof. Similarly as before, the equivalence between items $2-4$ follow from Lemma 2.4 in [1]. While the first two statements are clearly equivalent, and the last two are equivalent by the definition of sw.

As anticipated, the following statement asserts that $\operatorname{Aut}\left(H_{\mathcal{F}}\right)$ is either $\operatorname{Aut}\left(D_{\mathcal{F}}\right)$ or the smallest closed supergroup of $\operatorname{Aut}\left(D_{\mathcal{F}}\right)$ that contains -, sw, or both. The following statement is an adaptation of the third statement of Theorem 2.2 in [1] to the notation of the present work.

Theorem 33 (Theorem 2.2(iii) in [1]). Let $\mathcal{F}$ be a finite set of finite tournaments. Then, $H_{\mathcal{F}}$ is the Rado graph, $H_{\mathcal{F}}$ is a Henson graph, or $H_{\mathcal{F}}$ is not homogeneous. In the last case, the automorphism group of $H_{\mathcal{F}}$ equals one of the permutation groups from the following list.

1. $\operatorname{Aut}\left(D_{\mathcal{F}}\right)$.
2. $\left\langle\operatorname{Aut}\left(D_{\mathcal{F}}\right) \cup\{-\}\right\rangle$.
3. $\left\langle\operatorname{Aut}\left(D_{\mathcal{F}}\right) \cup\{\mathrm{sw}\}\right\rangle$.
4. $\left\langle\operatorname{Aut}\left(D_{\mathcal{F}}\right) \cup\{-, \mathrm{sw}\}\right\rangle$.

Our approach for proving dichotomy for the $\mathcal{F}$-free orientation problem will follow a case distinction on $\operatorname{Aut}\left(H_{\mathcal{F}}\right)$ according to the cases listed above. For this, it will be convenient to describe for which sets $\mathcal{F}$ the graph $H_{\mathcal{F}}$ is the Rado graph or a Henson graph.

Lemma 34. The following statements hold for a finite set of finite tournaments $\mathcal{F}$.

1. $H_{\mathcal{F}}$ is the Rado graph $\mathfrak{R}$ if and only if $\mathcal{F}$ contains no transitive tournament.
2. $H_{\mathcal{F}}$ is a Henson graph $\mathfrak{H}_{n}$ if and only if

- $\mathcal{F}$ contains a transitive tournament,
- $n_{\mathcal{F}}=n$, and
- there is no $\mathcal{F}$-free tournament with $n$ vertices.

Proof. If $\mathcal{F}$ contains no transitive tournament, then $\mathfrak{R}$ admits an $\mathcal{F}$-free orientation (orient the edges of the Rado graph according to any linear ordering of $V(\Re)$ ), and so it follows that $H_{\mathcal{F}}$ is the Rado graph. If $\mathcal{F}$ contains a transitive tournament, then it follows from [24] that there is a complete graph that does not admit an $\mathcal{F}$-free orientation, and thus $H_{\mathcal{F}}$ is not the Rado graph.

With similar arguments as in the Rado case, one can notice that if $\mathcal{F}$ contains a transitive tournament and there is no $\mathcal{F}$-free, then $H_{\mathcal{F}}$ is the Henson graph $\mathfrak{H}_{n_{\mathcal{F}}}$. Now, we show that the converse statement holds by contraposition. First, if $n \leq n_{\mathcal{F}}$, then $K_{n}$ admits an $\mathcal{F}$-free orientation, so $H_{\mathcal{F}}$ is not $\mathfrak{H}_{n}$. Also, if $\mathcal{F}$ contains no transitive tournament, then there is nothing to prove given the first statement. So, suppose that $\mathcal{F}$ contains a transitive tournament, that there is some $\mathcal{F}$-free tournament on $n_{\mathcal{F}}$ vertices, and $n>n_{\mathcal{F}}$. It suffices to show that there is a graph $G$ with no complete subgraph on $n_{\mathcal{F}}+1$ vertices such that $G$ does not admit an $\mathcal{F}$-free orientation, i.e., $H_{\mathcal{F}}$ does not embed so $H_{\mathcal{F}}$ is not the Henson graph $\mathfrak{H}_{n}$ for any $n>n_{\mathcal{F}}$. It is well-known that for every positive integer $k \geq 3$, there is there is a $K_{k}$-free graph $G$ such that every 2-edge-colouring of $G$ yields a monochromatic complete graph on $k-1$ vertices (see, e.g., [26]). Let $G$ be such a graph for $k=n_{\mathcal{F}}+1$. Consider any orientation $G^{\prime}$ of $G$, and any linear ordering $\leq$ of $V(G)$. Now, colour an edge $x y$ of $G$ with blue if $x \leq y$ and $(x, y) \in E\left(G^{\prime}\right)$ or if $y \leq x$ and $(y, x) \in E\left(G^{\prime}\right)$; otherwise, colour
$x y$ with red. By the choice of $G$, there must $k$ vertices $v_{1}, \ldots, v_{k}$ that induce a monochromatic clique. It is not hard to notice that $v_{1}, \ldots, v_{k}$ must induce a transitive tournament on $G^{\prime}$, so $G^{\prime}$ contains a transitive tournament on $n_{\mathcal{F}}+1$ vertices. Since this holds for any orientation $G^{\prime}$ of $G$, we conclude that $G$ does not admit an $\mathcal{F}$-free orientation. Thus, $G$ embeds in the Henson graph $\mathfrak{H}_{n}$ for each $n \geq n_{\mathcal{F}}+1$, but not in $H_{\mathcal{F}}$. This concludes the proof.

Finally, it will also be convenient to have an alternative description of $\operatorname{Aut}\left(H_{\mathcal{F}}\right)$ if it contains the action sw. Let $P \subseteq V^{3}$ be the ternary relation that contains all triples $(i, j, k) \in V^{3}$ such that $U(i, j), U(j, k), U(i, k)$, and $|\{(i, j),(j, k),(i, k)\} \cap E|$ is even.

Theorem 35. For a finite set of finite tournaments $\mathcal{F}$ the following equalities hold.

1. $\left\langle\operatorname{Aut}\left(D_{\mathcal{F}}\right) \cup\{\mathrm{sw}\}\right\rangle=\operatorname{Aut}(V ; P)$, and
2. $\left\langle\operatorname{Aut}\left(D_{\mathcal{F}}\right) \cup\{-, \mathrm{sw}\}\right\rangle=\langle\operatorname{Aut}(V ; P) \cup\{-\}\rangle$.

Proof. First observe that $P$ is preserved by sw; this implies the inclusions $\subseteq$ in the two statements. Now suppose for contradiction that there exists $\alpha \in \operatorname{Aut}(V ; P) \backslash\left\langle\operatorname{Aut}\left(D_{\mathcal{F}}\right) \cup\{\mathrm{sw}\}\right\rangle$. Then Theorem 33 implies that $\operatorname{Aut}\left(H_{\mathcal{F}}\right)$ contains -, which is a contradiction because - clearly does not preserve $P$.

Theorem 36. For all finite sets $\mathcal{F}$ of finite tournaments, $H_{\mathcal{F}}$ is a model complete core.
Proof. The fact that $H_{\mathcal{F}}$ is core follows from Lemma 6, and the model completeness of $H_{\mathcal{F}}$ is a side product of the proof in [1] (similarly as in [15]).

## 6 The Orientation Problem

We prove the complexity dichotomy for the $\mathcal{F}$-free orientation problem following a similar idea as we proved the dichotomy for the $\mathcal{F}$-free orientation completion problem: we show that for each finite set of finite tournaments $\mathcal{F}$, there is a Boolean structure whose CSP is polynomial-time equivalent to the $\mathcal{F}$-free orientation problem. We will do so by a case distinction over the automorphism group of $H_{\mathcal{F}}$, and repeatedly use the following lemma (see [9]).

Lemma 37. If $\mathfrak{C}$ is an $\omega$-categorical model-complete core, then all orbits of $k$-tuples of $\operatorname{Aut}(\mathfrak{C})$ are primitively positively definable in $\mathfrak{C}$.

Throughout the remaining of this section, let $\mathcal{F}$ be a fixed finite set of finite tournaments.

### 6.1 The Standard Case

We begin by considering the case where $\operatorname{Aut}\left(H_{\mathcal{F}}\right)=\operatorname{Aut}\left(D_{\mathcal{F}}\right)$.
Proposition 38. Let $\mathcal{F}$ be a finite set of finite tournaments with $n_{\mathcal{F}} \geq 3$. If $\operatorname{Aut}\left(H_{\mathcal{F}}\right)=\operatorname{Aut}\left(D_{\mathcal{F}}\right)$, then the relation $E$ has a primitive positive definition in $H_{\mathcal{F}}$.

Proof. Note that $E$ consists of one orbit of pairs in $\operatorname{Aut}\left(H_{\mathcal{F}}\right)$. Since $H_{\mathcal{F}}$ is a model-complete core by Theorem [36, it follows that the relation $E$ has a primitive positive definition in $H_{\mathcal{F}}$ by Lemma 37.

Corollary 39. Let $\mathcal{F}$ be a finite set of finite tournaments with $n_{\mathcal{F}} \geq 3$. If $\operatorname{Aut}\left(H_{\mathcal{F}}\right)=\operatorname{Aut}\left(D_{\mathcal{F}}\right)$, then there exists a primitive positive interpretation of $\mathfrak{B}_{\mathcal{F}}$ in $H_{\mathcal{F}}$, and there exists a polynomial-time reduction from $\operatorname{CSP}\left(\mathfrak{B}_{\mathcal{F}}\right)$ to the $\mathcal{F}$-free orientation problem.

Proof. Since $E$ has a primitive positive definition in $H_{\mathcal{F}}$ by Proposition 38, the statement is an immediate consequence of Lemma 21. The polynomial-time reduction follows from Lemma 10.

### 6.2 The Flipping Case

We write $N$ for the binary relation on $V$ defined as

$$
N:=\left\{(x, y) \in V^{2} \mid \neg U(x, y) \wedge x \neq y\right\}
$$

Let $O$ be the arity four relation on $V$ defined as

$$
O:=\left\{(x, y, u, v) \in V^{2} \mid S_{4}(x, y, u, v) \wedge N(x, u) \wedge N(x, v) \wedge N(y, u) \wedge N(y, v)\right\}
$$

where $S_{4}$ is the relation defined in Eq. (11).
Lemma 40. Let $\mathcal{F}$ be a finite set of finite tournaments with $n_{\mathcal{F}} \geq 3$. If $\operatorname{Aut}\left(H_{\mathcal{F}}\right)=\left\langle\operatorname{Aut}\left(D_{\mathcal{F}}\right) \cup\right.$ $\{-\}\rangle$, then the relation $S_{4}$ has a primitive positive definition in $H_{\mathcal{F}}$.

Proof. Note that the relation $O$ consists of one orbit of pairs in Aut $\left(H_{\mathcal{F}}\right)$. Since $H_{\mathcal{F}}$ is a modelcomplete core by Theorem 36, the relation $O$ has a primitive positive definition in $H_{\mathcal{F}}$ by Lemma 37 , We claim that $S_{4}$ has the following primitive positive definition.

$$
\exists a, b(O(x, y, a, b) \wedge O(a, b, u, v))
$$

If $(x, y, u, v)$ satisfies this formula and $a, b$ are witnesses for the existentially quantified variables, then $(x, y) \in E$ if and only if $(a, b) \in E$, which in turn is the case if and only if $(u, v) \in E$. Hence, the given formula implies $S_{4}(x, y, u, v)$. Conversely, if $(x, y) \in E$ and $(u, v) \in E$, then we may pick $(a, b) \in E$ such that $(p, q) \in N$ for all $p \in\{x, y, u, v\}$ and $q \in\{a, b\}$, and hence $(x, y, u, v)$ satisfies the given formula.

Corollary 41. Let $\mathcal{F}$ be a finite set of finite tournaments with $n_{\mathcal{F}} \geq 3$. If $\operatorname{Aut}\left(H_{\mathcal{F}}\right)=\left\langle\operatorname{Aut}\left(D_{\mathcal{F}}\right) \cup\right.$ $\{-\}\rangle$, then $\mathfrak{B}_{\mathcal{F}}$ has a primitive positive interpretation in $H_{\mathcal{F}}$, and there exists a polynomial-time reduction from $\operatorname{CSP}\left(\mathfrak{B}_{\mathcal{F}}\right)$ to the $\mathcal{F}$-free orientation problem.
Proof. The first statement follows directly from Lemma 40 via Lemma 21. The second statement then follows via Lemma 10.

### 6.3 The Switching Case

So far, we have solely worked with the Boolean structure $\mathfrak{B}_{\mathcal{F}}$. Now, when $\operatorname{Aut}\left(H_{\mathcal{F}}\right)$ contains sw, we consider an auxiliary Boolean structure which we define in the following paragraphs. If $T$ is a tournament with vertex set $\{1, \ldots, n\}$, then $c_{T} \in\{0,1\} \begin{gathered}\binom{k}{3}\end{gathered}$ is defined as follows. The entries of $c_{T}$ will be indexed by 3 -element subsets $\{i, j, k\}$ of $\{1, \ldots, n\}$, written as $\left(c_{T}\right)_{i j k}$. For all $\{i, j, k\} \in\binom{V(T)}{3}$ with $i<j<k$ we have that $\left(c_{T}\right)_{i j k}=0$ if and only if $|\{(i, j),(j, k),(i, k)\} \cap E(T)|$ is even. Equivalently, $c_{T}$ is defined by the following equation over $\mathbb{Z}_{2}$

$$
\begin{equation*}
\left(c_{T}\right)_{i j k}=\left(b_{T}\right)_{i j}+\left(b_{T}\right)_{i k}+\left(b_{T}\right)_{j k} \tag{2}
\end{equation*}
$$

Recall that the tuples $b_{T}$ fully determines the labeled tournament $T$. Note that there can be different tournaments $T$ and $T^{\prime}$ such that $c_{T}=c_{T^{\prime}}$. Nonetheless, we argue that the tuples $c_{T}$ determine whether $T$ is $\mathcal{F}$-free whenever $\mathcal{F}$ is preserved by switch.

Observation 42. Let $\mathcal{F}$ be a finite set of tournaments preserved by switch, and $T, T^{\prime}$ a pair of tournaments. If $c_{T}=c_{T^{\prime}}$, then $T$ is $\mathcal{F}$-free if and only if $T^{\prime}$ is $\mathcal{F}$-free.

Proof. This is a consequence of Lemma 32 and Theorem 35 .
Definition 1. The structure $\mathfrak{C}_{\mathcal{F}}$ has domain $\{0,1\}$ and the signature which contains for every $k \leq m_{\mathcal{F}}$ the relation symbol $Q_{k}$ of arity $\binom{k}{3}$ which denotes in $\mathfrak{C}_{\mathcal{F}}$ the relation consisting of all $c \in\{0,1\}^{\binom{k}{3}}$ such that there exists an $\mathcal{F}$-free tournament $T$ with $c_{T}=c$.

We proceed to observe that $\mathfrak{B}_{\mathcal{F}}$ and $\mathfrak{C}_{\mathcal{F}}$ are mutually pp-definable. To do so, we will use the following lemma which is similar to Lemma 27.

Lemma 43. Let $\mathcal{F}$ be a finite set of finite tournaments with $\mathrm{sw} \in \operatorname{Aut}\left(H_{\mathcal{F}}\right)$. If $\mathfrak{C}_{\mathcal{F}}$ does not interpret $K_{3}$ primitively positively, then $\mathfrak{C}_{\mathcal{F}}$ is preserved by the Boolean minority operation or a constant operation.

Proof. If $\mathfrak{C}_{\mathcal{F}}$ is not NP-hard, it falls into one of the Schaefer's cases [40]: $\mathfrak{C}_{\mathcal{F}}$ is preserved by a constant operation, by min, by max, by majority, or by minority. If the first or last case holds, then there is nothing to be shown. We show that if $\mathfrak{C}_{\mathcal{F}}$ is preserved by min, by max, or by majority, then $\mathfrak{C}_{\mathcal{F}}$ is also preserved by the constant 0 operation. It suffices to show that for every $n \leq m_{\mathcal{F}}$, the relation $Q_{n}$ is either empty or contains the tuple with all entries 0 . For each $t \in Q_{n}$, define a tournament $T(t)$ on $\{1, \ldots, m\}$ by setting $(i, j) \in E(T(t))$ if $\left(t_{i}, t_{j}, t_{m+1}\right) \in P$, and setting $(j, i) \in E(T(t))$ if $\left(t_{i}, t_{j}, t_{m+1}\right) \notin P$. Denote by $\mathcal{T}$ the set of all tournaments on $\{1, \ldots, m\}$ obtained in this way, and notice that $P_{n-1}\left(\mathfrak{B}_{\mathcal{T}}\right)$ is preserved by min, max, or majority, because $Q_{n}$ is preserved by min, max, or majority. Thus, by Lemma [26, either $P_{n-1}\left(\mathfrak{B}_{\mathcal{T}}\right)=\varnothing$ or $P_{n-1}\left(\mathfrak{B}_{\mathcal{T}}\right)=\{0,1\}{ }_{\binom{n-1}{2}}$. If the former case holds, then $Q_{n}$ is empty and the claim follows; if the latter holds, then $P_{n-1}\left(\mathfrak{B}_{\mathcal{T}}\right)$ contains both constant tuples, i.e., $T_{n-1} \in \mathcal{T}$. Let $t \in Q_{n}$ be such that $T(t)=T_{n-1}$, i.e., $t_{i j n}=0$ for all $1 \leq i<j \leq n-1$. Since $t=c_{T}$ for some $\mathcal{F}$-free tournament $T$, we have that $t_{i j k}=t_{i j n}+t_{i k n}+t_{j k n}=0$ for all $1 \leq i \leq j \leq k \leq n-1$. Hence, $t$ is the constant 0 -tuple, and $t \in Q_{n}$. The claim now follows.

Let $R_{4} \subseteq\{0,1\}^{4}$ be the 4 -ary Boolean relation consisting of all the tuples $(i, j, k, l)$ such that $i+j+k+l=0 \bmod 2$.

Lemma 44. Let $\mathcal{F}$ be a finite set of finite tournaments. If $\mathcal{F}$ is preserved by switch, then each pair of the following structures are mutually primitively positively definable

$$
\mathfrak{B}_{\mathcal{F}},\left(\mathfrak{B}_{\mathcal{F}}, R_{4}\right),\left(\mathfrak{C}_{\mathcal{F}}, R_{4}\right), \mathfrak{C}_{\mathcal{F}}
$$

Proof. We first see that $\mathfrak{B}_{\mathcal{F}}$ and $\left(\mathfrak{B}_{\mathcal{F}}, R_{4}\right)$ are mutually pp-definable, and we only argue the nontrivial direction. By Theorem 9, it suffices to show that $R_{4}$ is preserved by all polymorphisms of $\mathfrak{B}_{\mathcal{F}}$. By Lemma 27, if $\mathfrak{B}_{\mathcal{F}}$ has a polymorphism $f$ which is not a projection, then it is generated by the minority, or by constant operations, i.e., $f$ is obtained by composing projections with the minority or the constant operation (see e.g., [38]). Since $R_{4}$ is preserved by projections, by the minority, and by the constant operation, it follows inductively that $R_{4}$ is preserved by compositions of these
operations. Therefore, $R_{4}$ is preserved by all polymorphisms of $\mathfrak{B}_{\mathcal{F}}$ and thus, it has a pp-definition in $\mathfrak{B}_{\mathcal{F}}$. With similar arguments, and using Lemma 43, one can prove that $\mathfrak{C}_{\mathcal{F}}$ and $\left(\mathfrak{C}_{\mathcal{F}}, R_{4}\right)$ are mutually pp-definable. To conclude the proof we show that $\left(\mathfrak{B}_{\mathcal{F}}, R_{4}\right)$ and $\left(\mathfrak{C}_{\mathcal{F}}, R_{4}\right)$ pp-define one another. It follows immediately from Eq. (22) that $Q_{n}^{\mathfrak{B}_{\mathcal{F}}}$ is primitively positively defined in $\left(\mathfrak{B}_{\mathcal{F}}, R_{4}\right)$ by

$$
\exists b_{12}, \ldots, b_{n-1, n}\left(P_{n}\left(b_{12}, \ldots, b_{n-1, n}\right) \wedge \bigwedge_{1 \leq i<j<k \leq n} R_{4}\left(b_{i j}, b_{i l}, b_{j k}, c_{i j k}\right)\right)
$$

Finally, consider the $\binom{k}{2}$-ary relation $P_{n}^{\prime}\left(b_{12}, \ldots, b_{n-1, n}\right)$ defined by the formula

$$
\exists c_{123}, \ldots, c_{n-2, n-1, n}\left(Q_{n}\left(c_{123}, \ldots, c_{n-2, n-1, n}\right) \wedge \bigwedge_{1 \leq i<j<k \leq n} R_{4}\left(b_{i j}, b_{i k}, b_{j k}, c_{i j k}\right)\right)
$$

Notice that, for a tournament $T$ the formula $b_{T} \in P_{n}^{\prime}$ if and only if there is an $\mathcal{F}$-free tournament $T^{\prime}$ such that $c_{T}=c_{T^{\prime}}$. Since sw $\in \operatorname{Aut}\left(H_{\mathcal{F}}\right)$, by Observation 42 we conclude that $T$ is also $\mathcal{F}$-free, and so $b_{T} \in P_{n}^{\mathfrak{B}_{\mathcal{F}}}$. Conversely, if $b_{T} \in P_{n}^{\mathfrak{B}_{\mathcal{F}}}$, then the tuple $c_{T}$ is a witness to the fact that $b_{T} \in P_{n}^{\prime}$.

The following statement is an immediate implication of this lemma, Lemma 10, and Theorem 17
Proposition 45. Let $\mathcal{F}$ be a finite set of finite tournaments. If $\mathrm{sw} \in \operatorname{Aut}\left(H_{\mathcal{F}}\right)$, then there is a polynomial-time reduction from the $\mathcal{F}$-free orientation problem to $\operatorname{CSP}\left(\mathfrak{C}_{\mathcal{F}}\right)$.

The proof of the following theorem is similar to the proof of Lemma 20 We use the ternary relation $P$ introduced before Theorem 35 to now define

$$
\begin{gathered}
S_{6}:=\left\{(a, b, c, u, v, w) \in V^{6} \mid U(a, b) \wedge U(b, c) \wedge U(a, c) \wedge U(u, v) \wedge U(v, w) \wedge U(u, w)\right. \\
\text { and } P(a, b, c) \Leftrightarrow P(u, v, w)\} .
\end{gathered}
$$

Lemma 46. For every finite set of finite tournaments $\mathcal{F}$, the Boolean structure $\mathfrak{C}_{\mathcal{F}}$ has a primitive positive interpretation in $\left(H_{\mathcal{F}}, S_{6}\right)$.

Proof. Our interpretation $I$ has dimension three, is defined on all triples $(a, b, c)$ that induce a $K_{3}$ in $H_{\mathcal{F}}$, and is given by $I(a, b, c):=0$ if $(a, b, c) \in P$, and $I(a, b, c):=1$ otherwise. The domain formula $\top_{I}(a, b, c)$ is $U(a, b) \wedge U(b, c) \wedge U(a, c)$. The interpreting formula $={ }_{I}(a, b, c, u, v, w)$ for equality is $S_{6}(a, b, c, u, v, w)$. For $n \leq k_{\mathcal{F}}$, the interpreting formula

$$
\left(R_{n}\right)_{I}\left(a_{123}, b_{123}, c_{123}, \ldots, a_{n-2, n-1, n}, b_{n-2, n-1, n}, c_{n-2, n-1, n}\right)
$$

for $R_{n}$ is

$$
\exists x_{1}, \ldots, x_{n} \bigwedge_{\{i, j\} \in\binom{\{1, \ldots, n\}}{2}} U\left(x_{i}, x_{j}\right) \wedge \bigwedge_{\{i, j, k\} \in\binom{\{1, \ldots, n\}}{3}} S_{6}\left(a_{i j k}, b_{i j k}, c_{i j k}, x_{i}, x_{j}, x_{k}\right)
$$

It is straightforward to verify that $S_{6}(a, b, c, u, v, w)$ if and only if $I(a, b, c)=I(u, v, w)$. Now suppose that $\left(H_{\mathcal{F}}, S_{6}\right) \models\left(R_{n}\right)_{I}\left(a_{123}, \ldots, c_{n-2, n-1, n}\right)$, and let $d_{1}, \ldots, d_{n}$ be witnesses for the existentially quantified variables in this formula. Note that $d_{1}, \ldots, d_{n}$ must induce a $K_{n}$ in $H_{\mathcal{F}}$; let $T$ be the tournament induced in $D_{\mathcal{F}}$. Then $c_{T} \in R_{n}$ by the definition of $R_{n}$, which means that $\left(I\left(a_{123}, b_{123}, c_{123}\right), \ldots, I\left(a_{n-2, n-1, n}, b_{n-2, n-1, n}, c_{n-2, n-1, n}\right)\right) \in R_{n}$ by the properties of $S_{6}$. All of the implications in this argument can be reserved, which completes the proof.

To conclude the proof, it now suffices to show that $S_{6}$ has a primitive positive definition in $H_{\mathcal{F}}$ when sw $\in \operatorname{Aut}\left(H_{\mathcal{F}}\right)$. Again, we consider the cases $-\in \operatorname{Aut}\left(H_{\mathcal{F}}\right)$ and $-\notin \operatorname{Aut}\left(H_{\mathcal{F}}\right)$ separately.

Lemma 47. Let $\mathcal{F}$ be a finite set of finite tournaments. If $\operatorname{Aut}\left(H_{\mathcal{F}}\right)=\left\langle\operatorname{Aut}\left(D_{\mathcal{F}}\right) \cup\{-, \mathrm{sw}\}\right\rangle$, then there exists a primitive positive definition of $S_{6}$ in $H_{\mathcal{F}}$.

Proof. Notice that if $n_{\mathcal{F}}=3$, then $T_{3} \in \mathcal{F}$, and since $\overrightarrow{C_{3}}$ can be obtained as a switch of $T_{3}$, the triangle does not admit an $\mathcal{F}$-free orientation. The latter claims holds trivially if $n_{\mathcal{F}} \leq 2$. So, if $n_{\mathcal{F}} \leq 3$, then $S_{6}=\varnothing$, and thus it is primitive positive defined by formula $\perp$. Otherwise, if $n_{\mathcal{F}} \geq 4$, we proceed similarly to Lemma 40, but instead of $O$ using the relation

$$
\begin{aligned}
\left(P\left(x_{1}, x_{2}, x_{3}\right) \Leftrightarrow\left(P\left(y_{1}, y_{2}, y_{3}\right)\right)\right. & \wedge \bigwedge_{i, j \in\{1,2,3\}} N\left(x_{i}, y_{j}\right) \\
& \wedge U\left(x_{1}, x_{2}\right) \wedge U\left(x_{2}, x_{3}\right) \wedge U\left(x_{1}, x_{3}\right) \\
& \wedge U\left(y_{1}, y_{2}\right) \wedge U\left(y_{2}, y_{3}\right) \wedge U\left(y_{1}, y_{3}\right)
\end{aligned}
$$

which consists of just one orbit of 6 -tuples in the group $\operatorname{Aut}\left(H_{\mathcal{F}}\right)$, and hence has a primitive positive definition in the model-complete core structure $H_{\mathcal{F}}$.

Finally, we consider the most technical case, namely, when $\operatorname{Aut}\left(H_{\mathcal{F}}\right)=\left\langle\operatorname{Aut}\left(D_{\mathcal{F}}\right) \cup\{\mathrm{sw}\}\right\rangle$.
Lemma 48. Let $\mathcal{F}$ be a finite set of finite tournaments. If $\operatorname{Aut}\left(H_{\mathcal{F}}\right)=\left\langle\operatorname{Aut}\left(D_{\mathcal{F}}\right) \cup\{\mathrm{sw}\}\right\rangle$, then $S_{6}$ has a primitive positive definition in $H_{\mathcal{F}}$.

Proof. With the same arguments as in the proof of Lemma 47, we see that if if $n_{\mathcal{F}} \leq 3$, then $S_{6}$ is primitive positive defined by formula $\perp$. Suppose that $n_{\mathcal{F}} \geq 4$, and notice that in this case, any orientation of $K_{3}$ is $\mathcal{F}$-free, fact which we use below. Now, since $\operatorname{Aut}\left(H_{\mathcal{F}}\right)=\left\langle\operatorname{Aut}\left(D_{\mathcal{F}}\right) \cup\{\right.$ sw $\left.\}\right\rangle$, it follows from Theorem 35 that the relation $P$ consists of one orbit of triples of $\operatorname{Aut}\left(H_{\mathcal{F}}\right)$. Hence, $P$ has a primitive positive definition in $H_{\mathcal{F}}$, because $H_{\mathcal{F}}$ is a model-complete core and by Lemma 37 The same holds for

$$
Q:=\left\{(a, b, c) \in V^{3} \mid U(a, b) \wedge U(b, c) \wedge U(a, c) \wedge \neg P(a, b, c)\right\}
$$

instead of $P$. Therefore, it suffices to prove that $S_{6}$ is primitive positive definable in $(V ; P, Q)$.
Let $T \in \mathcal{F}$, and $\phi$ be a conjunction of all atomic $\{P, Q\}$-formulas of the form $P(i, j, k)$ for $\{i, j, k\} \in\binom{\{1, \ldots, k\}}{3}$ such that $\left(c_{T}\right)_{i j k}=0$, and of the form $Q(i, j, k)$ such that $\left(c_{T}\right)_{i j k}=1$. Clearly, $\phi$ is unsatisfiable in $(V ; P, Q)$. Let $\psi$ be a maximal satisfiable subset of the conjuncts of $\phi$. Since $T_{3}$ and $\overrightarrow{C_{3}}$ are $\mathcal{F}$-free, each conjunct of $\phi$ is satisfiable and so $\psi$ has at least one conjunct $\chi_{1}$. Let $\psi^{\prime}$ be the remaining conjuncts of $\psi$, i.e., $\psi^{\prime}:=\psi \backslash\left\{\chi_{1}\right\}$. Let $\chi_{2}$ be any conjunct of $\phi$ which is not in $\psi$. Note that $\psi^{\prime}$ implies $\chi_{1} \Rightarrow \neg \chi_{2}$. We may assume that $\chi_{1}$ is of the form $P(i, j, k)$ and $\chi_{2}$ is of the form $Q(i, j, k)$. To see that this is without loss of generality, first suppose that otherwise all conjuncts in $\phi \backslash \psi$ are of the form $P(i, j, k)$. Clearly, we may assume that two of the variables $a, b \in\{1, \ldots, k\}$ of $\chi_{2}$ are smallest among all the variables of $\psi$. Let $\tilde{T}$ be an isomorphic copy obtained from $T$ by exchanging the labels $a$ and $b$; then the construction of $\phi$ above, carried out with $\tilde{T}$ instead of $T$, has the desired property. The case that all conjuncts of $\psi$ are of the form $Q(i, j, k)$ can be treated similarly.

Note that $\chi_{1}$ and of $\chi_{2}$ cannot have the same set of variables; so we may assume without loss of generality that $\chi_{1} \Rightarrow \chi_{2}$ is of the form $P(a, b, c) \Rightarrow P\left(a^{\prime}, b^{\prime}, c^{\prime}\right)$ where $a$ might be the same variable
as $a^{\prime}$, and $b$ might be the same variable as $b^{\prime}$, but $c$ and $c^{\prime}$ are different variables. Let $\psi^{\prime \prime}$ be the primitive positive formula obtained from $\psi^{\prime}$ by existentially quantifying all variables except for $a, b, c, a^{\prime}, b^{\prime}, c^{\prime}$, and let $\eta$ be the formula

$$
\exists a^{\prime}, b^{\prime}, c^{\prime}, u^{\prime}, v^{\prime}, w^{\prime}\left(\psi^{\prime \prime}\left(a, b, c, a^{\prime}, b^{\prime}, c^{\prime}\right) \wedge \psi^{\prime \prime}\left(b^{\prime}, c^{\prime}, a^{\prime}, v^{\prime}, w^{\prime}, u^{\prime}\right) \wedge \psi^{\prime \prime}\left(w^{\prime}, u^{\prime}, v^{\prime}, w, u, v\right)\right)
$$

First note that if $(a, b, c) \in P$, then the first conjunct of $\eta$ implies that $\left(a^{\prime}, b^{\prime}, c^{\prime}\right) \in P$, the second that $\left(u^{\prime}, v^{\prime}, w^{\prime}\right) \in P$, and the third that $(u, v, w) \in P$. Conversely, $\eta(a, b, c, u, v, w)$ implies that if $(u, v, w) \in P$, then $(a, b, c) \in P$.

We claim that the following formula $\nu(a, b, c, u, v, w)$ defines $S_{6}$ :

$$
\exists a^{\prime}, b^{\prime}, c^{\prime}, u^{\prime}, v^{\prime}, w^{\prime}\left(\eta\left(a, b, c, a^{\prime}, b^{\prime}, c^{\prime}\right) \wedge \eta\left(a^{\prime}, b^{\prime}, c^{\prime}, u^{\prime}, v^{\prime}, w^{\prime}\right) \wedge \eta\left(u^{\prime}, v^{\prime}, w^{\prime}, u, v, w\right)\right)
$$

It is easy to see that $\nu(a, b, c, u, v, w)$ implies $S_{6}(a, b, c, u, v, w)$.
Now suppose that $(a, b, c, u, v, w) \in S_{6}$. First consider the case that $(a, b, c) \in P$ and $(u, v, w) \in$ $P$. The case that $(a, b, c) \in Q$ and $(u, v, w) \in Q$ can be treated similarly. Let $G$ be the canonical database of the primitive positive $\{E\}$-formula obtained from replacing the $\{P, Q\}$-atoms of $\nu(a, b, c, u, v, w)$ by their primitive positive definition in $H_{\mathcal{F}}$. Let $D$ be an $\mathcal{F}$-free orientation of $G$. We assume that the tournament $\{a, b, c\}$ induces the same tournament in $D_{\mathcal{F}}$ and in $D$, and that $\{u, v, w\}$ induces the same tournament in $D_{\mathcal{F}}$ and in $D$. This is without loss of generality, because otherwise we may repeatedly compose $e$ with $\operatorname{sw}_{a}$ for $a \in\{a, b, c, u, v, w\}$ to obtain such an orientation.

We now add directed edges to $D$ such that $\{a, b, c, u, v, w\}$ induces the same digraph in $D_{\mathcal{F}}$ and in $D$. Note that the resulting digraph is still $\mathcal{F}$-free, because every tournament that embeds into $D$ must either embed into $\{a, b, c, u, v, w\}$, into $\left\{a, b, c, a^{\prime}, b^{\prime}, c^{\prime}\right\}$, into $\left\{a^{\prime}, b^{\prime}, c^{\prime}, u^{\prime}, v^{\prime}, w^{\prime}\right\}$, or into $\left\{u^{\prime}, v^{\prime}, w^{\prime}, u, v, w\right\}$, because all edges of $D$ are covered by one of these subsets. Hence, $D$ has an embedding $e$ into $D_{\mathcal{F}}$, and by homogeneity we may assume that $e(a)=a, e(b)=b$, and $e(c)=c$, $e(u)=u, e(v)=v$, and $e(w)=w$. This shows that $\nu(a, b, c, u, v, w)$ holds in $H_{\mathcal{F}}$.

Corollary 49. Let $\mathcal{F}$ be a finite set of finite tournaments. If $\operatorname{Aut}\left(H_{\mathcal{F}}\right) \in\left[\left\langle\operatorname{Aut}\left(D_{\mathcal{F}}\right) \cup\{\mathrm{sw}\}\right\rangle\right.$, $\left.\left\langle\operatorname{Aut}\left(D_{\mathcal{F}}\right) \cup\{-, \mathrm{sw}\}\right\rangle\right]$, then there exists a primitive positive interpretation of $\mathfrak{B}_{\mathcal{F}}$ in $H_{\mathcal{F}}$, and there exists a polynomial-time reduction from $\operatorname{CSP}\left(\mathfrak{B}_{\mathcal{F}}\right)$ to the $\mathcal{F}$-free orientation problem.

Proof. By Theorem [33, either $\operatorname{Aut}\left(H_{\mathcal{F}}\right)=\left\langle\operatorname{Aut}\left(D_{\mathcal{F}}\right) \cup\{-, \mathrm{sw}\}\right\rangle$ or $\operatorname{Aut}\left(H_{\mathcal{F}}\right)=\left\langle\operatorname{Aut}\left(D_{\mathcal{F}}\right) \cup\{\mathrm{sw}\}\right\rangle ;$ the primitive positive definition of $S_{6}$ in $H_{\mathcal{F}}$ now follows by Lemmas 47 and 48 respectively. Thus, by Lemma46, $\mathfrak{C}_{\mathcal{F}}$ has a primitive positive interpretation in $H_{\mathcal{F}}$ and so, by Lemma 44 and composing interpretations, we conclude that $\mathfrak{B}_{\mathcal{F}}$ has a primitive positive interpretation in $H_{\mathcal{F}}$. Finally, the polynomial time reduction follows from from Lemma 10

### 6.4 Complexity Classification

Finally, we put all these cases together and obtain the following result.
Theorem 50. Let $\mathcal{F}$ be a finite set of finite tournaments. Then exactly one of the following two cases applies.

- $K_{3}$ has a primitive positive interpretation in $\mathfrak{B}_{\mathcal{F}}$ and in $H_{\mathcal{F}}$. In this case, $\operatorname{CSP}\left(H_{\mathcal{F}}\right)$ and the $\mathcal{F}$-free orientation problem are NP-complete.
- $\mathfrak{B}_{\mathcal{F}}$ has a constant operation or the minority operation as a polymorphism, and $H_{\mathcal{F}}$ has a ternary pseudo near unanimity polymorphism. In this case, $\operatorname{CSP}\left(H_{\mathcal{F}}\right)$ and the $\mathcal{F}$-free orientation problem are in $P$.

Proof. It is well-known that the two cases are mutually disjoint: if $H_{\mathcal{F}}$ has a ternary pseudo near unanimity polymorphism, then every structure with a first-order interpretation in $H_{\mathcal{F}}$ has such a polymorphism as well (see, e.g., Corollary 6.5.16 in [9]), but $K_{3}$ does not have such a polymorphism (see, e.g., Proposition 6.1.43 in [9).

We prove that one of the two cases applies by a case distinction on $H_{\mathcal{F}}$ and on $\operatorname{Aut}\left(H_{\mathcal{F}}\right)$ according to Theorem [33. If $H_{\mathcal{F}}$ is the Rado graph (in particular, if $\mathcal{F}$ is empty), then, by Lemma 34, $\mathcal{F}$ contains no transitive tournament, so for all $n \in\left\{2, \ldots, m_{\mathcal{F}}\right\}$ we have that $P_{n}^{\mathfrak{B}_{\mathcal{F}}}$ contains both constant $\binom{n}{2}$-tuples and so, $\mathfrak{B}_{\mathcal{F}}$ has a constant polymorphism. Moreover, the Rado graph (Example 2) clearly has a pseudo near unanimity polymorphism (Example 15) and the statement is trivial.

If $H_{\mathcal{F}}$ is a Henson graph $\mathfrak{H}_{m}$, then, by Lemma $34 \mathcal{F}$ contains a transitive tournament, there is no $\mathcal{F}$-free tournament on $n_{\mathcal{F}}$ vertices, and $m=n_{\mathcal{F}}$. So, $P_{n}^{\mathfrak{B}_{\mathcal{F}}}=\emptyset$ for all $n \in\left\{n_{\mathcal{F}}, \ldots, m_{\mathcal{F}}\right\}$, and $P_{n}^{\mathfrak{B}_{\mathcal{F}}}$ contains both constant $\binom{n}{2}$-tuples for all other $n<n_{\mathcal{F}}$. Hence, $\mathfrak{B}_{\mathcal{F}}$ again has a constant polymorphism. Moreover, since $H_{\mathcal{F}}$ is the Henson graph $H_{n_{\mathcal{F}}}$ (Example 3), it also has a pseudo near unanimity polymorphism as was mentioned earlier (Example 15). The $\mathcal{F}$-free orientation problem is in P , because it suffices to check whether the give input graph contains $K_{n_{\mathcal{F}}}$.

Otherwise, if $H_{\mathcal{F}}$ is not the Rado graph, nor a Henson graph, then Theorem 33 implies that $\operatorname{Aut}\left(H_{\mathcal{F}}\right) \in\left[\operatorname{Aut}\left(D_{\mathcal{F}}\right),\left\langle\operatorname{Aut}\left(D_{\mathcal{F}}\right) \cup\{-, \mathrm{sw}\}\right\rangle\right]$, and Lemma 34 implies that $\mathcal{F}$ contains a transitive tournament, and that there is at least one $\mathcal{F}$-free oriented tournament on $n_{\mathcal{F}}$ vertices. In particular, this implies that $\mathfrak{B}_{\mathcal{F}}$ does not have a constant polymorphism, since $P_{n_{\mathcal{F}}}^{\mathfrak{B}_{\mathcal{F}}}$ is neither empty, nor contains constant tuples. Hence, by Lemma 27, $\mathfrak{B}_{\mathcal{F}}$ either has the minority operation as polymorphism, or it pp-interprets $K_{3}$. If the former holds, then the minority operation is a polymorphism of $\left(\mathfrak{B}_{\mathcal{F}}, \mathbf{0}, \mathbf{1}\right)$ as well, and thus, $\left(D_{\mathcal{F}}, U\right)$ has a pseudo near unanimity polymorphism $f$ by Theorem [28. Clearly, $f$ is also a pseudo near unanimity polymorphism of $H_{\mathcal{F}}$. The fact that the $\mathcal{F}$-free orientation problem and $\operatorname{CSP}\left(H_{\mathcal{F}}\right)$ are in P follows from the polynomial-time tractability of $\operatorname{CSP}\left(\mathfrak{B}_{\mathcal{F}}\right)$ and Theorem [17. Finally, suppose that $\mathfrak{B}_{\mathcal{F}}$ primitively positively interprets $K_{3}$. Since $\operatorname{Aut}\left(H_{\mathcal{F}}\right) \in\left[\operatorname{Aut}\left(D_{\mathcal{F}}\right),\left\langle\operatorname{Aut}\left(D_{\mathcal{F}}\right) \cup\{-, \mathrm{sw}\}\right\rangle\right]$, then, by Theorem 33, it must be the case that either $\operatorname{Aut}\left(H_{\mathcal{F}}\right)=\operatorname{Aut}\left(D_{\mathcal{F}}\right), \operatorname{Aut}\left(H_{\mathcal{F}}\right)=\left\langle\operatorname{Aut}\left(D_{\mathcal{F}}\right) \cup\{-\}\right\rangle$, or $\operatorname{Aut}\left(H_{\mathcal{F}}\right) \in\left[\left\langle\operatorname{Aut}\left(D_{\mathcal{F}}\right) \cup\{\mathrm{sw}\}\right\rangle,\left\langle\operatorname{Aut}\left(D_{\mathcal{F}}\right) \cup\{-, \mathrm{sw}\}\right\rangle\right]$. So, in each case by Corollaries 39, 41] and 49, respectively, there is a primitive positive interpretation of $\mathfrak{B}_{\mathcal{F}}$ in $H_{\mathcal{F}}$. Finally, by the well-known fact that pp-interpretations compose, we conclude that $K_{3}$ has a primitive positive interpretation in $H_{\mathcal{F}}$, and $\operatorname{CSP}\left(H_{\mathcal{F}}\right)$ and the $\mathcal{F}$-free orientation problem are NP-complete by Lemma 10 ,

## 7 Examples and Applications

If the reader is not familiar with constraint satisfaction theory, they might find Theorems 28 and 50 not transparent enough. In order to address a broader audience, we first describe the minority operation in terms of tournaments, and then, using this description, we propose simpler versions of Theorems 28 and 50. Some readers might find these versions more natural, and others might find it redundant; the latter can skip the following subsection.

We conclude this section by providing some examples and applications of Theorems 28 and 50 to certain natural instances of the $\mathcal{F}$-free orientation and orientation completion problems.

### 7.1 Minority and Tournaments

Consider three tournaments $T^{1}, T^{2}$, and $T^{3}$ with vertex set $[n]$. The minority operation maps the triple $\left(T^{1}, T^{2}, T^{3}\right)$ to a tournament with vertex set $[n]$ which we denote by minority $\left(T^{1}, T^{2}, T^{3}\right)$ and the edge set is defined as follows. For $1 \leq i<j \leq n$ there is an edge $(i, j)$ in minority $\left(T^{1}, T^{2}, T^{3}\right)$ if either $(i, j) \in E\left(T^{1}\right) \cap E\left(T^{2}\right) \cap E\left(T^{3}\right)$ or if $(i, j)$ is an edge in exactly one of the tournaments $T^{1}$, $T^{2}$, and $T^{3}$. Intuitively, when deciding whether there is an edge $(i, j)$ in minority $\left(T^{1}, T^{2}, T^{3}\right)$, we first check if the three tournaments agree on $(i, j)$, and otherwise, we take the minority vote. We extend these operation to take as input triples of tournaments $\left(T^{1}, T^{2}, T^{3}\right)$ with different sizes of vertex sets. Suppose that $V\left(T^{1}\right)=\left[n_{1}\right], V\left(T^{2}\right)=\left[n_{2}\right]$, and $V\left(T^{3}\right)=\left[n_{3}\right]$, where $n_{1} \leq n_{2} \leq n_{3}$. We define minority $\left(T^{1}, T^{2}, T^{3}\right):=\operatorname{minority}\left(T^{1}, T^{2}\left[n_{1}\right], T^{3}\left[n_{1}\right]\right)$.


Figure 4: An illustration of the minority operation acting on a triple $\left(T^{1}, T^{2}, T^{3}\right)$ of tournaments isomorphic to $T C_{4}$, which yields a tournament isomorphic $T_{4}$.

We say that a set of tournaments $\mathcal{T}$ is preserved by the minority operation if for every three tournaments $T^{1}, T^{2}, T^{3}$ with vertex set $\left[n_{1}\right],\left[n_{2}\right]$, and $\left[n_{3}\right]$, respectively, whenever each $T^{i}$, with $i \in\{1,2,3\}$, is isomorphic to some tournament in $\mathcal{T}$, then minority $\left(T^{1}, T^{2}, T^{3}\right)$ is isomorphic to some tournament in $\mathcal{T}$. For instance, the illustration of this operation in Fig. 4 shows if a set $\mathcal{T}$ contains $T C_{4}$, but not $T_{4}$, then it is not preserved by the minority operation.

Clearly, the previously defined operation on tournaments is a translation of the minority operation on Boolean relational structures (Theorem [12). We will use these translations to propose a classification of the complexity of the $\mathcal{F}$-free orientation completion problem. Given a non-empty finite set of finite tournaments $\mathcal{F}$, we denote by $\mathcal{F}_{f}$ the set of $\mathcal{F}$-free tournaments on at most $m_{\mathcal{F}}$ vertices.

Lemma 51. Let $\mathcal{F}$ be a finite set of finite tournaments. The Boolean structure $\mathfrak{B}_{\mathcal{F}}$ is preserved by the minority Boolean operation if and only if $\mathcal{F}_{f}$ is preserved by the minority operation.

Proof. By definition of $P_{n}\left(\mathfrak{B}_{\mathcal{F}}\right)$, it follows that for each $n \leq m_{\mathcal{F}}$ the relation $P_{n}$ is preserved by the minority operation if and only if for any three $\mathcal{F}$-free tournaments $T^{1}, T^{2}, T^{3}$ with vertex set $[n]$, it is the case that minority $\left(T^{1}, T^{2}, T^{3}\right)$ is $\mathcal{F}$-free. Thus, it suffices to prove that the latter condition holds if and only if $\mathcal{F}_{f}$ is preserved by the minority operation. One implication is trivial, we prove the remaining one by contraposition. Suppose $\mathcal{F}_{f}$ is not preserved by minority, and let $T^{1}, T^{2}, T^{3} \in \mathcal{F}_{f}$ such that majority $\left(T^{1}, T^{2}, T^{3}\right) \notin \mathcal{F}_{f}$. Without loss of generality, assume that $T^{1}$ has the minimum number of vertices $n$ amongst these three tournaments. Since $\mathcal{F}_{f}$ is closed under taking subtournaments, then $T^{2}[n]$ and $T^{3}[n]$ belong to $\mathcal{F}_{f}$. The claim follows since
$\operatorname{minority}\left(T^{1}, T^{2}[n], T^{3}[n]\right)$ is equal (by definition) to minority $\left(T^{1}, T^{2}, T^{3}\right)$ which does not belong to $\mathcal{F}_{f}$.

Now, we translate Theorems 28 and 50 in terms of the minority operation and tournaments.
Corollary 52. For every finite set of finite tournaments $\mathcal{F}$ one of the following cases holds.

1. $\mathcal{F}_{f}$ is preserved by the minority operation. In this case, the $\mathcal{F}_{f}$-free orientation completions of a digraph $D$ correspond to the solution space of a system of linear equations over $\mathbb{Z}_{2}$ (constructed from $D$ and $\mathcal{F}$ ).
2. Otherwise, $\mathcal{F}$-free orientation completion problem is NP-complete.

In the first case, the $\mathcal{F}$-free orientation completion problem is in $P$.
Proof. It follows directly from Theorem 28 and Lemma 51 ,
Corollary 53. For every finite set of finite tournaments $\mathcal{F}$ one of the following cases holds.

1. $\mathcal{F}$ contains no transitive tournament. In this case, every graph admits an $\mathcal{F}$-free orientation.
2. $\mathcal{F}_{f}$ is preserved by the minority operation. In this case, the $\mathcal{F}_{f}$-free orientations of a graph $G$ correspond to the solution space of a system of linear equations over $\mathbb{Z}_{2}$ (constructed from $G$ and F).
3. Otherwise, the $\mathcal{F}$-free orientation completion problem is NP-complete.

In cases 1 and 2, the $\mathcal{F}$-free orientation problem is in $P$.
Proof. Immediate implication of Theorem 50 and Lemma 51.
We can already illustrate these corollaries using Fig. 3.
Example 54. Notice that by the minority operation depicted in Fig. 3, if $\mathcal{F}$ contains $T_{4}$ and $T C_{4}$ if $\mathcal{F}$-free, then $\mathcal{F}_{f}$ is not preserved by the minority operation. Thus, in any such case, the $\mathcal{F}$-free orientation and the $\mathcal{F}$-free orientation completion problems are NP-complete.

### 7.2 Sets with $T_{3}$ or $\overrightarrow{C_{3}}$

Now, we consider the special cases when $\mathcal{F}$ contains some small tournament. Specifically, when $\mathcal{F}$ contains at least one tournament on at most 3 vertices. The cases when $\mathcal{F}$ contains $T_{1}$ is trivial, and it is also not hard to settle the case when $\mathcal{F}$ contains $T_{2}$.

Recall that given a digraph $D$, we write $U$ to denoted the symmetric closure of $E$. It is not hard to notice that an oriented graph $D$ can be described as a $\mathbb{Z}_{2}$-colouring $c: U \rightarrow \mathbb{Z}_{2}$ such that for every pair $i j \in U$ the equality $c(i, j)+c(j, i)=1$ is satisfied. Equivalently, orientation completions of a digraph $D$ are in one to one correspondence with solutions to the linear equation $x_{i j}+x_{j i}=1$ over $\mathbb{Z}_{2}$, where $i, j$ ranges over adjacent vertices of $G$, and $x_{i j}=1$ for each $(i, j) \in E$ such that $(j, i) \notin E$. Now, notice that every hamiltonian oriented path of $\overrightarrow{C_{3}}$ has an even number of forward edges, but there are hamiltonian oriented paths of $T_{3}$ with different parity. Thus, the $T_{3}$-free
orientation completions of $D$ are in one-to-one correspondence to the solutions of the system with variables $x_{i j}$ for $i j \in U$ and linear equations

$$
\begin{aligned}
x_{i j} & =1 \text { where }(i, j) \in E \text { and }(j, i) \notin E, \\
x_{i j}+x_{j i} & =1 \text { where } i j \in U, \\
x_{i j}+x_{j k} & =0 \text { where } i j, j k, i k \in U .
\end{aligned}
$$

We denote by $\operatorname{Sys}_{3}(D)$ the previously described system of linear equations. For the following statement, we assume that the input digraph of the $\mathcal{F}$-free orientation problem is given with vertex set $[n]$ for $n=|V(D)|$. Clearly, this assumption is done without loss of generality.

Corollary 55. Let $\mathcal{F}$ be a finite set of finite tournaments. If the smallest tournament in $\mathcal{F}$ has exactly three vertices, then one of the following holds.

1. $\mathcal{F}$ contains both $T_{3}$ and $\overrightarrow{C_{3}}$. In this case, a digraph $D$ admits an $\mathcal{F}$-free orientation completion if and only if the $D$ contains no semicomplete digraph on 3 vertices.
2. $\mathcal{F}$ contains a $T_{3}$ but not $\overrightarrow{C_{3}}$. In this case, the $\mathcal{F}$-free orientations of a digraph $D$ correspond to the solution space of $S_{3} s_{3}(D)$.
3. $\mathcal{F}$ contains a $\overrightarrow{C_{3}}$ but not $T_{3}$. In this case, the $\mathcal{F}$-free orientation completion problem is NPcomplete.

In cases 1 and 2, the $\mathcal{F}$-free orientation completion problem is in $P$.
Proof. The first case is immediate to see. To prove the second case, notice that every tournament on 4 vertices contains a transitive triangle and so, an orientation completion $D^{\prime}$ of $D$ is $\mathcal{F}$-free if and only if it is $T_{3}$-free. Thus, the second case follows from the two paragraphs preceding this corollary. Finally, the case when $\overrightarrow{C_{3}} \in \mathcal{F}$ but $T_{3} \notin \mathcal{F}$ follows by noticing that $P_{3}^{\mathfrak{B}_{\mathcal{F}}}$ is not preserved by minority: the tuples $(1,1,0),(0,1,1)$, and $(1,1,1)$ correspond to three permutations of $T_{3}$ (i.e., three $\mathcal{F}$-free tournaments), while minority $((1,1,0),(0,1,1),(1,1,1))=(0,1,0)$ which corresponds to a directed triangle (i.e., a non- $\mathcal{F}$-free tournament). Thus, the $\mathcal{F}$-free orientation completion problem in NP-complete by Theorem 28.

Notice that if $G$ is a graph with vertex set $[n]$, then $\operatorname{Sys}_{3}(G)$ does not contain equations of the form $x_{i j}=1$.

Corollary 56. Let $\mathcal{F}$ be a finite set of finite tournaments. If the smallest tournament in $\mathcal{F}$ has exactly three vertices, then one of the following holds.

1. $\mathcal{F}$ contains both $T_{3}$ and $\overrightarrow{C_{3}}$. In this case, a graph $G$ admits an $\mathcal{F}$-free orientation if and only if the $G$ is $K_{3}$-free.
2. $\mathcal{F}$ contains a $T_{3}$ but not $\overrightarrow{C_{3}}$. In this case, the $\mathcal{F}$-free orientations of a graph $G$ correspond to the solution space of $\operatorname{Sys}_{3}(G)$.
3. $\mathcal{F}$ contains $\overrightarrow{C_{3}}$ but not $T_{3}$. In this case, every graph admits an $\mathcal{F}$-free orientation.

In each of these cases, the $\mathcal{F}$-free orientation completion problem is in $P$.

Proof. Cases 1 and 3 are trivial, and case 2 follows as a particular instance of the second case in Corollary 55.

It is also possible to have an ad-hoc reduction from an NP-complete problem to the $\overrightarrow{C_{3}}$ orientation completion problem. The most natural problem in this scenario is not-all-equal 3-SAT: the input is a 3 -SAT instance; and a solution is where at least one variable per clause is false, and one is true. As a sanity check, one can easily verify that the gadget $\left(D, x_{0}, x_{1}, y_{0}, y_{1}, z_{0}, z_{1}\right)$ in Fig. 5 yields a reduction from not-all-equal 3 -SAT to the $\overrightarrow{C_{3}}$-free orientation. To see this, simply notice that $\left(x_{0}, x_{1}\right)$ and $(a, b)$ force each other, and symmetrically, $\left(y_{0}, y_{1}\right)$ and $(b, c)$ force each other, and $\left(z_{0}, z_{1}\right)$ and $(c, a)$ force each other. Since the triangle $a b c$ must be oriented transitively, it must be the case that at least one of the edge $x_{0} x_{1}, y_{0} y_{1}, z_{0} z_{1}$ is oriented from 0 to 1 , but not the three of them. Moreover, any transitive orientation of $a b c$ extends to a $\overrightarrow{C_{3}}$-free orientation of $D$. With these arguments, one can notice that given a instance $\phi$ of not-all-equal 3-SAT, there is a digraph $D$ such that $D$ admits a $\overrightarrow{C_{3}}$-free orientation completion if and only if $\phi$ is a yes instance to not-all-equal 3-sat (and $D$ can be constructed in polynomial time with respect to $D$ ). Actually, the reader familiar with pp-interpretations, can notice that the gadget $\left(D, x_{0}, x_{1}, y_{0}, y_{1}, z_{0}, z_{1}\right)$ can be translated into a primitive positive interpretation of $\left(\{0,1\},\{0,1\}^{3} \backslash\{(0,0,0),(1,1,1)\}\right)$ in $D_{\mathcal{F}}$ - which is guaranteed to exist by Theorem 28 and the well-known fact that $K_{3}$ pp-interprets any finite structure [9, and that pp-interpretations compose.

$\left(D, x_{0}, x_{1}, y_{0}, y_{1}, z_{0}, z_{1}\right)$
Figure 5: A gadget for reducing not-all-equal 3-Sat to the $\overrightarrow{C_{3}}$-free orientation completion. Equivalently, the interpreting formula for the primitive positive interpretation of $\left(\{0,1\},\{0,1\}^{3} \backslash\right.$ $\{(0,0,0),(1,1,1)\})$ in $\left(D_{\mathcal{F}}, U\right)$ for $\mathcal{F}=\left\{\overrightarrow{C_{3}}\right\}$.

### 7.3 Tournaments on four vertices

In the previous subsection we considered all cases where $\mathcal{F}$ contains at least one tournament on at most 3 vertices. In this subsection, we apply our main results to classify the complexity of the $\mathcal{F}$-free orientation and orientation completion problem if all tournaments in $\mathcal{F}$ have exactly 4 vertices. Similarly as in the previous subsection, we introduce three families of systems of linear equations over $\mathbb{Z}_{2}$, whose solution space will encode $\mathcal{F}$-free orientation completions for certain sets $\mathcal{F}$.

Recall that in Fig. 3 we depicted all four tournaments on 4 vertices, up to isomorphism. Notice that if $T_{4}$ is described by the linear ordering of [4], then the cycle 1342 has 3 forward edges and 1 backward edge, i.e., $x_{12}+x_{23}+x_{34}+x_{41}=1$ according to the coding described in the previous subsection. Similarly, it is not hard to find a hamiltonian oriented cycle $i j k l$ in $T C_{4}$ such that $x_{i j}+x_{j k}+x_{k l}+x_{l i}=1$. On the contrary, it is evident that every hamiltonian cycle $i j k l$ of $C_{3}^{+}$and of $C_{3}^{-}$have an even number of forward edges, i.e., $x_{i j}+x_{j k}+x_{k l}+x_{l i}=0$. Thus, similar as the set up of $\operatorname{Sys}_{3}(D)$, there is a system of linear equations $\operatorname{Sys}_{4}(D)$ whose solution space correspond to the $\left\{T_{4}, T C_{4}\right\}$-free orientation completions of a digraph $D$.

It is even simpler to verify that each vertex in $C_{3}^{+}$has an even number of outneighbours, while for $T_{4}, T C_{4}$, and $C_{3}^{-}$, there is at least one vertex with an odd number of outneighours. Again, this leads to a system of linear equations $\operatorname{Sys}_{+}(D)$ with variables $x_{i j}$ where $i j \in U(D)$, such that the solution space (over $\mathbb{Z}_{2}$ ) of $\operatorname{Sys}_{+}(D)$ corresponds to the $\left\{T_{4}, T C_{4}, C_{3}^{-}\right\}$-free orientation completions of $D$. Moreover, notice that $C_{3}^{-}$is the flip of $C_{3}^{+}$, thus an oriented graph $D^{\prime}$ is $\left\{T_{4}, T C_{4}, C_{3}^{-}\right\}$-free if and only $-D$ is $\left\{T_{4}, T C_{4}, C_{3}^{+}\right\}$-free. This means that if $x$ is a solution to $\operatorname{Sys}_{+}(D)$ and we interpret $x_{i j}=1$ as an edge $(j, i)$ (instead of $(i, j)$ ), we see that the solution space of $\operatorname{Sys}_{+}(D)$ also encondes the $\left\{T_{4}, T C_{4}, C_{3}^{+}\right\}$-free orientation completions of $D$.

Corollary 57. If $\mathcal{F}$ is a non-empty set of tournaments on 4 vertices, then one of the following holds.

1. $\mathcal{F}$ contains all tournaments on 4 vertices, up to isomorphism. In this case, a digraph $D$ admits an $\mathcal{F}$-free orientation completion if and only if it does not contain a semicomplete digraph on 4 vertices.
2. $\mathcal{F}$ contains both $T_{4}$ and $T C_{4}$. In this case, the $\mathcal{F}$-free orientation completions of a digraph $D$ correspond to the solution space of $\operatorname{Sys}_{4}(D)$ or of $\operatorname{Sys}_{+}(D)$.
3. $\mathcal{F}$ contains at most one of $T_{4}$ or $T C_{4}$. In this case, the $\mathcal{F}$-free orientation completion is NP-complete.

In cases 1 and 2, the $\mathcal{F}$-free orientation completion problem is in $P$.
Proof. The first case is immediate, and the second one is argued in the paragraph preceding this statement. To prove item 3 , first notice that the case when $T_{4} \in \mathcal{F}$ and $T C_{4} \notin \mathcal{F}$, follows from Example 54 Otherwise, suppose that $T_{4} \notin \mathcal{F}$, and so, $T_{4}$ is $\mathcal{F}$-free. Since $\mathcal{F}$ contains at least one tournament on 4 vertices, and $T_{4}$ is $\mathcal{F}$-free, it follows from Lemma 25, that $P_{n}$ is not preserved by the Boolean minority operation. The hardness of the $\mathcal{F}$-free orientation completion problem now follows from Theorem 28 .

Corollary 58. If $\mathcal{F}$ is a set of tournaments on 4 vertices, then one of the following holds.

1. $\mathcal{F}$ contains all tournaments on 4 vertices, up to isomorphism. In this case, a graph $G$ admits an $\mathcal{F}$-free orientation if and only if $G$ is $K_{4}$-free.
2. $\mathcal{F}$ contains both $T_{4}$ and $T C_{4}$. In this case, the $\mathcal{F}$-free orientations of a graph $G$ correspond to the solution space of $\operatorname{Sys}_{4}(G)$ or of $S y s_{+}(G)$.
3. $\mathcal{F}$ does not contain $T_{4}$. In this case, every graph admits an $\mathcal{F}$-free orientation.
4. $\mathcal{F}$ contains $T_{4}$ but not $T C_{4}$. In this case, the $\mathcal{F}$-free orientation problem is NP-complete.

In cases 1-3, the $\mathcal{F}$-free orientation completion problem is in $P$.
Proof. Cases 1 and 2 follow as particular instances of cases 1 and 2 from Corollary 57. Case 3 is trivial, and the fourth one follows from Example 54.

### 7.4 Transitive Tournaments

We conclude our series of examples by considering the cases where $\mathcal{F}$ does not contain a transitive tournament, and where $\mathcal{F}$ only contains transitive tournaments.

Proposition 59. Let $\mathcal{F}$ be a non-empty finite set of finite tournaments. If $\mathcal{F}$ does not contain any transitive tournament, then the following statements hold.

1. The $\mathcal{F}$-free orientation completion problem is NP-complete. Moreover, this problem remains $N P$-hard even when the input is restricted to digraphs with no semicomplete subdigraph with $m_{\mathcal{F}}+1$ vertices.
2. The $\mathcal{F}$-free orientation problem is trivial and in $P$.

Proof. We only prove the first statement as the second one is evident. Let $\mathcal{F}^{\prime}$ be the set obtained from $\mathcal{F}$ by adding all tournament on $m_{\mathcal{F}}+1$ vertices. It is not hard to notice that the $\mathcal{F}$-free orientation completion problem restricted to digraphs no semicomplete digraph on $m_{\mathcal{F}}+1$ vertices is polynomial time equivalent to the $\mathcal{F}^{\prime}$-free orientation completion problem: on the one hand, if $D$ is a digraph with no semicomplete graph on $m_{\mathcal{F}}+1$ vertices, then an orientation completion $D^{\prime}$ of $D$ is $\mathcal{F}$-free if and only if $D^{\prime}$ is $\mathcal{F}^{\prime}$-free; on the other one, if $D$ is an input to the $\mathcal{F}^{\prime}$-free orientation completion, then one can first verify (in polynomial-time) if $D$ contains a semicomplete digraph on $m_{\mathcal{F}}+1$-vertices (if it does, reject), and the solve the $\mathcal{F}$-free orientation completion problem. Finally, the fact that the $\mathcal{F}^{\prime}$-free orientation completion problem is NP-complete, follows from the observation that $T_{m_{\mathcal{F}}}$ is $\mathcal{F}^{\prime}$-free, but since $\mathcal{F}$ is non-empty, there must be at lest one tournament $T$ on $m_{\mathcal{F}}$ vertices that is no $\mathcal{F}$-free. Thus, $T$ is not $\mathcal{F}^{\prime}$-free but $T_{m_{\mathcal{F}}}$ is $\mathcal{F}^{\prime}$-free, hence, by Lemma 26 we conclude that $P_{m_{\mathcal{F}}^{\prime}}$ cannot be preserved by the minority operation. The hardness of the $\mathcal{F}^{\prime}$-free orientation completion problem now follows from Theorem 28 , and so, the $\mathcal{F}$-free orientation completion problem is NP-complete even when the input is restricted so digraphs with no semicomplete digraph on $m_{\mathcal{F}}+1$ vertices.

We highlight that Proposition 59 provides several instances of sets $\mathcal{F}$ such that the $\mathcal{F}$-free orientation problem is in $P$, while the $\mathcal{F}$-free orientation completion problem is NP-complete. It turns out that in "almost" any other case, the orientation and orientation completion problems are equivalent.

Theorem 60. For a finite set of non-empty finite tournament $\mathcal{F}$ one of the following statements holds.

1. $\mathcal{F}$ contains a transitive tournament and at least one tournament with $n_{\mathcal{F}}$ vertices is $\mathcal{F}$-free. In this case, the $\mathcal{F}$-free orientation and the $\mathcal{F}$-free orientation completion problems are polynomial time equivalent.
2. $\mathcal{F}$ contains a transitive tournament and all tournaments in $\mathcal{F}$ have at least $n_{\mathcal{F}}$ vertices. In this case, the $\mathcal{F}$-free orientation and the $\mathcal{F}$-free orientation completion problems are polynomial time equivalent.
3. Otherwise, the $\mathcal{F}$-free orientation completion problem is $N P$-complete, and the $\mathcal{F}$-free orientation problem is in $P$.

Proof. In case 1, it follows from Corollary 24, that the $\mathcal{F}$-free orientation completion problem an $\operatorname{CSP}\left(\mathfrak{B}_{\mathcal{F}}\right)$ are polynomial time equivalent. And thus, since $\operatorname{CSP}\left(\mathfrak{B}_{\mathcal{F}}\right)$ and the $\mathcal{F}$-free orientation completion problem are polynomial-time equivalent, the claim follows. To prove the second statement, notice that if $\mathcal{F}$ does not contain all tournaments on $n_{\mathcal{F}}$ vertices, then we are in case 1 . Now, if $\mathcal{F}$ contains all tournaments on $n_{\mathcal{F}}$ vertices, then the $\mathcal{F}$-free orientation corresponds to finding $n_{\mathcal{F}}$ complete graphs in an input graph $G$, and the $\mathcal{F}$-free orientation completion problem corresponds to finding semicomplete graphs on $n_{\mathcal{F}}$ vertices in the input digraph $D$. Thus, both problems are in P , and we conclude that the second statement holds.

Now we prove the third statement. If $\mathcal{F}$ does not contain a transitive tournament, the claim follows from Proposition 59, Otherwise, $\mathcal{F}$ contains a transitive tournament, there is no $\mathcal{F}$-free tournament on $n_{\mathcal{F}}$ vertices, and $\mathcal{F}$ contains some tournament on less than $n_{\mathcal{F}}$ vertices. On the one hand, this means that the $\mathcal{F}$-free orientation problem reduces to determining if the input graph is $K_{n_{\mathcal{F}}}$-free, and thus it is in P . On the other one, if $\mathcal{F}^{\prime}$ is obtained from $\mathcal{F}$ by removing all tournament on $n_{\mathcal{F}}$ vertices, then the $\mathcal{F}^{\prime}$-free orientation completion problem in NP-complete by Proposition 59 , Moreover, it reaming NP-complete when restricted to input digraphs with no semicomplete digraph on $m_{F^{\prime}}+1$ vertices. This problem is clearly equivalent to the restriction of the $\mathcal{F}$-free orientation completion problem to digraphs with no semicomplete digraph on $m_{F^{\prime}}+1$ vertices. Thus, the (general) $\mathcal{F}$-free orientation completion problem must be NP-complete as well.

On the opposite side of the cases consider in Proposition 59, is the case when $\mathcal{F}$ only contains transitive tournaments. In this case, by Theorem 60, the $\mathcal{F}$-free orientation completion and the $\mathcal{F}$ free orientation problems are polynomial-time equivalent. Also, notice that these cases boil down to the case when $\mathcal{F}=\left\{T_{k}\right\}$ for some integer $k$ (larger forbidden transitive tournaments are redundant).

Theorem 61. The following statements hold for a positive integer $k \geq 4$.

1. The $T_{k}$-free orientation completion problem is $N P$-complete. This problem remains NPcomplete when the input is restricted to digraphs with no semicomplete subdigraph with $k+1$ vertices.
2. The $T_{k}$-free orientation problem is NP-complete. This problem remains NP-complete when the input is restricted to $K_{k+1}$-free graphs.

Proof. Clearly, it suffices to prove the second part of statements 1 and 2. Let $\mathcal{F}_{k}$ be the set of tournaments that contains $T_{k}$ and all tournaments on $k+1$ vertices. With similar arguments as in the proof of Proposition 59, one can notice that the $\mathcal{F}_{k}$-free orientation (resp. completion) problem is
polynomial-time equivalent to the $T_{k}$-free orientation (resp. completion) problem when the input is restricted to $K_{k+1}$-free graphs (resp. digraphs with no subdigraph with $k+1$ vertices). Moreover, by the second statement of Theorem 60, the $\mathcal{F}_{k}$-free orientation and the $\mathcal{F}_{k}$-free orientation completion problems are polynomial-time equivalent. Therefore, to prove the whole theorem, if suffices to prove the $\mathcal{F}_{k}$-free orientation completion problem is NP-complete for $k \geq 4$. To do so, notice that if $D$ is a digraph and $D^{\prime}$ is obtained from $D$ by adding a universal sink, then $D$ can be completed to a $\mathcal{F}_{k}$-free oriented graph if and only if $D^{\prime}$ can be completed to a $\mathcal{F}_{k+1}$-free oriented graph. Thus, the $\mathcal{F}_{k}$-free orientation completion problem reduces in polynomial-time to the $\mathcal{F}_{k+1}$-free orientation completion problem. Hence, we conclude the proof by showing that the $\mathcal{F}_{4}$-free orientation completion problem is NP-complete. Clearly, $T_{4} \in \mathcal{F}_{4}$ and $T C_{4}$ is $\mathcal{F}_{4}$-free so, it follows from Example 54 that the $\mathcal{F}_{4}$-free orientation completion problem is NP-complete. Both statements now follow.

## 8 Conclusion and Outlook

From a structural perspective, a family of graph obstructions to the class of graphs that admit a $T_{3}$-free orientation was described in 31. In light of the hardness of the $T_{4}$-free orientation problem (Corollary 58), it might be hard to extend such a description for the class of graphs that admit a $T_{4}$-free orientation, but it could be interesting to understand the structure of graphs that admit a $\left\{T_{4}, T C_{4}\right\}$-free orientation, and of those that admit a $\left\{T_{4}, T C_{4}, C_{3}^{+}\right\}$-free orientation.

From a computational complexity point of view, a first natural extension of this work would be to classify the complexity of the $\mathcal{F}$-free orientation (completion) problem if $\mathcal{F}$ is any finite set of oriented graphs. In general, this might not be equivalent to a (possibly infinite) CSP, but if $\mathcal{F}$ consists of connected oriented graphs, and it is closed under homomorphisms, then the class of graphs that admit an $\mathcal{F}$-free orientation corresponds to the CSP of some (possibly infinite) graph $G$ (see e.g., 9 ). Such a restriction on the forbidden set $\mathcal{F}$, is also a particular instance of the larger class of problems that can be expressed in the logic $\mathrm{MMSNP}_{2}$. Some of the techniques we used to classify the computational complexity of the $\mathcal{F}$-free orientation problem might be useful to prove a complexity dichotomy $\mathrm{MMSNP}_{2}$. One would have to overcome the following obstacles.

- For general problems expressible in $\mathrm{MMSNP}_{2}$, the finite structures we work with instead of $\mathfrak{B}_{\mathcal{F}}$ and $\mathfrak{C}_{\mathcal{F}}$ will in general have more than two elements, which means that we cannot use lemmata that explicitly rely on the Schaefer's cases, such as Lemma 26,
- For the $\mathcal{F}$-free orientation problem, the concept of force that we introduced in Section 4 is particularly pleasant, since flipping arguments corresponds to Boolean complementation. The combinatorics of forcing will be more involved in the general case.
- For the structures needed to formulate problems $\mathrm{MMSNP}_{2}$ as CSPs, there is no known generalisation of the result of Kompatscher and Agarval [1], which was crucial in our proof.

However, our hope is that a combination of ideas from the present paper with more recent results in the theory of constraint satisfaction, e.g., from [10, 12, 35, 36] can eventually lead to a proof of a complexity dichotomy for all of $\mathrm{MMSNP}_{2}$.

## References

[1] L. Agarwal and M. Kompatscher. $2^{\aleph_{0}}$ pairwise nonisomorphic maximal-closed subgroups of $\operatorname{Sym}(\mathbb{N})$ via the classification of the reducts of the Henson digraphs. Journal of Symbolic Logic, 83(2):395-415, 2018.
[2] E. Allender, M. Bauland, N. Immerman, H. Schnoor, and H. Vollmer. The complexity of satisfiability problems: Refining Schaefer's theorem. Journal of Computer and System Sciences, 75(4):245-254, 2009.
[3] J. Bang-Jensen, J. Huang, and E. Prisner. In-tournament digraphs. Journal of Combinatorial Theory Series B, 59(2):267-287, 1993.
[4] J. Bang-Jensen, J. Huang, and X. Zhu. Completing orientations of partially oriented graphs. Journal of Graph Theory, 87(3):285-304, 2017.
[5] A. Barsukov. On dichotomy above Feder and Vardi's logic, 2022. PhD thesis, École Doctorale Science pour l'Engénieur, Clermont-Ferrand.
[6] L. Barto, M. Kompatscher, M. Olšák, T. V. Pham, and M. Pinsker. Equations in oligomorphic clones and the constraint satisfaction problem for $\omega$-categorical structures. Journal of Mathematical Logic, 19(2):\#1950010, 2019.
[7] L. Barto and M. Pinsker. The algebraic dichotomy conjecture for infinite domain constraint satisfaction problems. In Proceedings of the 31th Annual IEEE Symposium on Logic in Computer Science - LICS'16, pages 615-622, 2016. Preprint arXiv:1602.04353.
[8] M. Bienvenu, B. ten Cate, C. Lutz, and F. Wolter. Ontology-based data access: A study through disjunctive Datalog, CSP, and MMSNP. ACM Transactions of Database Systems, 39(4):33, 2014.
[9] M. Bodirsky. Complexity of Infinite-Domain Constraint Satisfaction. Lecture Notes in Logic (52). Cambridge University Press, Cambridge, United Kingdom; New York, NY, 2021.
[10] M. Bodirsky and B. Bodor. Canonical polymorphisms of Ramsey structures and the unique interpolation property. In Proceedings of the Symposium on Logic in Computer Science (LICS), 2021.
[11] M. Bodirsky, S. Knäuer, and F. Starke. ASNP: A tame fragment of existential second-order logic. In M. Anselmo, G. D. Vedova, F. Manea, and A. Pauly, editors, Beyond the Horizon of Computability - 16th Conference on Computability in Europe, CiE 2020, Fisciano, Italy, June 29 - July 3, 2020, Proceedings, volume 12098 of Lecture Notes in Computer Science, pages 149-162. Springer, 2020.
[12] M. Bodirsky, F. R. Madelaine, and A. Mottet. A proof of the algebraic tractability conjecture for monotone monadic SNP. SIAM J. Comput., 50(4):1359-1409, 2021.
[13] M. Bodirsky and A. Mottet. A dichotomy for first-order reducts of unary structures. Logical Methods in Computer Science, 14(2), 2018.
[14] M. Bodirsky and J. Nešetřil. Constraint satisfaction with countable homogeneous templates. Journal of Logic and Computation, 16(3):359-373, 2006.
[15] M. Bodirsky and M. Pinsker. Minimal functions on the random graph. Israel Journal of Mathematics, 200(1):251-296, 2014.
[16] M. Bodirsky, M. Pinsker, and A. Pongrácz. Projective clone homomorphisms. Journal of Symbolic Logic, 86(1):148-161, 2021.
[17] M. Bodirsky and A. Vucaj. Two-element structures modulo primitive positive constructability. Algebra Universalis, 81(20), 2020. Preprint available at ArXiv:1905.12333.
[18] V. G. Bodnarčuk, L. A. Kalužnin, V. N. Kotov, and B. A. Romov. Galois theory for Post algebras, part I and II. Cybernetics, 5:243-539, 1969.
[19] J. Bok, S. Guzmán-Pro, C. Hernández-Cruz, and N. Jedličková. On the expressive power of 2-edge-colourings. In preparation.
[20] A. A. Bulatov. A dichotomy theorem for nonuniform CSPs. In 58th IEEE Annual Symposium on Foundations of Computer Science, FOCS 2017, Berkeley, CA, USA, October 15-17, pages 319-330, 2017.
[21] A. A. Bulatov, A. A. Krokhin, and P. G. Jeavons. Classifying the complexity of constraints using finite algebras. SIAM Journal on Computing, 34:720-742, 2005.
[22] P. Damaschke. Forbidden ordered subgraphs. In Topics in Combinatorics and Graph Theory, pages 219-229. R. Bodendiek and R. Hennm Eds, 1990.
[23] D. Duffus, M. Ginn, and V. Rödl. On the computational complexity of ordered subgraph recognition. Random Struct. Alg., 7:223-268, 1995.
[24] P. Erdős and L. Moser. On the representation of directed graphs as unions of orderings. A Magyar Tudományos Akadémia, Matematikai Kutató Intézetének Közleményei, 9:125-132, 1964.
[25] T. Feder and M. Y. Vardi. The computational structure of monotone monadic SNP and constraint satisfaction: a study through Datalog and group theory. SIAM Journal on Computing, 28:57-104, 1999.
[26] J. Folkman. Graphs with monochromatic complete subgraphs in every edge coloring. SIAM Journal on Applied Mathematics, 18(3):19-24, 1970.
[27] T. Gallai. On directed paths and circuits. In Theory of Graphs, pages 115-118. Academic Press, New York, 1968.
[28] D. Geiger. Closed systems of functions and predicates. Pacific Journal of Mathematics, 27:95100, 1968.
[29] S. Guzmán-Pro. Local expressions of hereditary classes. PhD thesis, Facultad de Ciencias, UNAM, 2023.
[30] S. Guzmán-Pro, P. Hell, and C. Hernández-Cruz. Describing hereditary properties by forbidden circular orderings. Applied Mathematics and Computation, 438(1):127555, 2023.
[31] S. Guzmán-Pro and C. Hernández-Cruz. Orientations without forbidden patterns on three vertices. arXiv:2003.05606, 2020.
[32] M. Hasse. Zur algebraischen Begründung der Graphentheorie. I, Math. Nachr., 28:275-290, 1964/1965.
[33] W. Hodges. A shorter model theory. Cambridge University Press, Cambridge, 1997.
[34] G. Kun. Constraints, MMSNP, and expander relational structures. Combinatorica, 33(3):335347, 2013.
[35] A. Mottet and M. Pinsker. Smooth approximations and CSPs over finitely bounded homogeneous structures. 2020. Preprint arXiv:2011.03978.
[36] A. Mottet and M. Pinsker. Cores over Ramsey structures. Journal of Symbolic Logic, 86(1):352361, 2021.
[37] C. Paul and E. Protopapas. Tree-layout based graph classes: proper chordal graphs. arXiv.2211.07550, 2022.
[38] E. L. Post. The two-valued iterative systems of mathematical logic, volume 5. Princeton University Press, Princeton, 1941.
[39] B. Roy. Nombre chromatique et plus longs chemins d'un graphe. Rev. Fr. Inform. Rech. Oper., 1:129-132, 1967.
[40] T. J. Schaefer. The complexity of satisfiability problems. In Proceedings of the Symposium on Theory of Computing (STOC), pages 216-226, 1978.
[41] D. J. Skrien. A relationship between triangulated graphs, comparability graphs, proper interval graphs, proper circular-arc graphs, and nested interval graphs. Journal of Graph Theory, $6(3): 309-316,1982$.
[42] L. M. Vitaver. Determination of minimal colouring of vertices of a graph by means of boolean powers of the incidence matrix. Dokl. Akad. Nauk SSSR (in Russian), 147:758-759, 1962.
[43] D. N. Zhuk. A proof of CSP dichotomy conjecture. In 58th IEEE Annual Symposium on Foundations of Computer Science, FOCS 2017, Berkeley, CA, USA, October 15-17, pages 331-342, 2017. https://arxiv.org/abs/1704.01914.


[^0]:    *Both authors have been funded by the European Research Council (Project POCOCOP, ERC Synergy Grant 101071674). Views and opinions expressed are however those of the authors only and do not necessarily reflect those of the European Union or the European Research Council Executive Agency. Neither the European Union nor the granting authority can be held responsible for them.
    ${ }^{\dagger}$ manuel.bodirsky@tu-dresden.de
    ${ }^{\ddagger}$ santiago.guzman_pro@tu-dresden.de

