NONLOCAL GAGLIARDO-NIRENBERG-SOBOLEV TYPE INEQUALITY

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Abstract. We establish Gagliardo-Nirenberg-Sobolev type inequalities on nonlocal Sobolev spaces driven by p-Lévy integrable functions, by imposing some appropriate growth conditions on the associated critical function. This gives rise to the embedding of the local and the nonlocal Sobolev spaces into Orlicz type spaces. The Gagliardo-Nirenberg-Sobolev type inequalities, as in the classical context, turn out to have some reciprocity with Poincaré and Poincaré-Sobolev type inequalities. The classical fractional Sobolev inequality is also derived as a direct consequence.

1. Introduction

Classical Sobolev inequalities are ubiquitous within the area of partial differential equations and calculus of variations, and have been investigated by a numerous number of authors. They play crucial roles in existence theory and regularity theory. The Gagliardo-Nirenberg-Sobolev inequality, amongst many others, is certainly the most significant and influential Sobolev inequality. It is the aim of this work to establish analogous of such an inequality on nonlocal Sobolev spaces generated by p-Lévy integrable kernels, that can be seen as a genius generalization of fractional Sobolev-Slobodeckij spaces. Note that our exposition also aims to be as self-contained as possible. We define a nonlocal Sobolev space as follows.

Let $\nu:\mathbb{R}^d\setminus\{0\}\to[0,\infty]$, $d\geq 1$, be the density of a symmetric p-Lévy measure with $1\leq p<\infty$ that is ν is symmetric, i.e., $\nu(h)=\nu(-h)$ for $h\in\mathbb{R}^d\setminus\{0\}$ and ν is p-Lévy integrable, i.e.,

$$\int_{\mathbb{R}^d}(1\wedge|h|^p)\nu(h)\,\mathrm{d}h<\infty. \tag{1.1}$$

Hereafter, we write $|h|=(h_1^2+h_2^2+\cdots+h_d^2)^{1/2}$ and $a\wedge b=\min(a,b)$ for $a,b\in\mathbb{R}$. The nonlocal Sobolev space associated with ν is defined as $W^p_\nu(\mathbb{R}^d)=\big\{u\in L^p(\mathbb{R}^d):\ |u|_{W^p_\nu(\mathbb{R}^d)}<\infty\big\}$ where

$$|u|_{W^p_\nu(\mathbb{R}^d)}=\Big(\iint\limits_{\mathbb{R}^d\,\mathbb{R}^d}|u(x)-u(y)|^p\nu(x-y)\,\mathrm{d}y\,\mathrm{d}x\Big)^{1/p}. \tag{1.2}$$

The space $W^p_\nu(\mathbb{R}^d)$ amounts to a Banach space under the norm

$$\|u\|_{W^p_\nu(\mathbb{R}^d)}=\big(\|u\|^p_{L^p(\mathbb{R}^d)}+|u|^p_{W^p_\nu(\mathbb{R}^d)}\big)^{1/p}.$$

It is noteworthy mentioning that the terminology *nonlocal Sobolev space* to designate the space $W^p_\nu(\mathbb{R}^d)$ is justified as the latter appears as the natural energy space associated with a nonlocal operator, which is a (non)linear p-Lévy integrodifferential operator generated by ν, of the form

$$Lu(x):=\mathrm{p.\,v.}\int_{\mathbb{R}^d}|u(x)-u(y)|^{p-2}(u(x)-u(y))\nu(x-y)\,\mathrm{d}y,\qquad (x\in\mathbb{R}^d).$$

Recent studies regarding function spaces of type $W^p_\nu(\mathbb{R}^d)$ and the analysis of related integrodifferential equations on domains can be found in [FG20]. The class of p-Lévy integrable kernels includes not only integrable functions, but also functions with heavy singularities at the origin A prototypical example is obtained by taking $\nu(h)=s(1-s)|h|^{-d-sp}$, with $s\in(0,1)$. The resulting space is thus the well-known fractional Sobolev-Slobodeckij space $W^{s,p}(\mathbb{R}^d)$ of order s. Just like the

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latter, the nonlocal Sobolev space $W^p_\nu(\mathbb{R}^d)$ also appears as refinement space between $L^p(\mathbb{R}^d)$ and the classical Sobolev space $W^{1,p}(\mathbb{R}^d)$, the space of functions in $L^p(\mathbb{R}^d)$ whose first order distributional derivatives also lie in $L^p(\mathbb{R}^d)$. Furthermore, note that as $s \to 1^-$, [BBM01, FG21] $W^{s,p}(\mathbb{R}^d)$ reduces to the classical Sobolev space $W^{1,p}(\mathbb{R}^d)$. Rigorously speaking, if $|u|_{W^{s,p}(\mathbb{R}^d)}$ is given by (1.2) for $\nu(h) = s(1-s)|h|^{-d-sp}$, ∇u denotes the distributional gradient of u and $|u|_{W^{1,p}(\mathbb{R}^d)} = \|\nabla u\|_{L^p(\mathbb{R}^d)}$ denotes the L^p-norm of $|\nabla u|$, then asymptotically we have

$$\lim_{s\to 1^-} |u|^p_{W^{s,p}(\mathbb{R}^d)} = \frac{|\mathbb{S}^{d-1}|}{p} K_{d,p}|u|^p_{W^{1,p}(\mathbb{R}^d)}. \tag{1.3}$$

Here, $K_{d,p}$ is a universal constant given for any unit vector $e \in \mathbb{S}^{d-1}$ by

$$K_{d,p} = \fint_{\mathbb{S}^{d-1}} |w \cdot e|^p \sigma_{d-1}(w).$$

A side motivation to studying the class of nonlocal Sobolev spaces under consideration is that, the asymptotic convergence in (1.3) remains true for the lager family $(\nu_\varepsilon)_{\varepsilon>0}$ of radial p-Lévy integrable kernels $\nu_\varepsilon : \mathbb{R}^d \setminus \{0\} \to [0,\infty]$ satisfying

$$\int_{\mathbb{R}^d} (1 \wedge |h|^p)\nu_\varepsilon(h)\,\mathrm{d}h = 1 \quad \text{and for all } \delta > 0, \quad \lim_{\varepsilon\to 0^+} \int_{|h|>\delta} (1 \wedge |h|^p)\nu_\varepsilon(h)\,\mathrm{d}h = 0.$$

In fact, as a result, see [FG21, BBM01, FKV20] one finds that

$$\lim_{\varepsilon\to 0^+} \iint_{\mathbb{R}^d\,\mathbb{R}^d} |u(x) - u(y)|^p \nu_\varepsilon(x-y)\,\mathrm{d}y\,\mathrm{d}x = K_{d,p}|u|^p_{W^{1,p}(\mathbb{R}^d)}.$$

The most important inequality in the theory Sobolev spaces is the Gagliardo-Nirenberg-Sobolev and reads as follows; for $0 \le s \le 1$ and $1 \le p < \infty$, if the critical Sobolev exponent p^*_s also called the Sobolev conjugate of p satisfies

$$\frac{1}{p^*_s} := \frac{1}{p} - \frac{s}{d} > 0,$$

then there exists a constant $C_s = C(d,s,p) > 0$ such that

$$\Big(\int_{\mathbb{R}^d} |u(x)|^{p^*_s}\,\mathrm{d}x\Big)^{1/p^*_s} \le C_s|u|_{W^{s,p}(\mathbb{R}^d)} \qquad \text{for all } u \in L^{p^*_s}(\mathbb{R}^d). \tag{1.4}$$

The fractional Gagliardo-Nirenberg-Sobolev inequality (1.4) is trivial for $s = 0$, with the convention that $W^{0,p}(\mathbb{R}^d) = L^p(\mathbb{R}^d)$ and the proof for $s = 1$ can be found in any classical book on Sobolev, e.g., [AF03, Bre10, Eva10]. For $s \in (0,1)$, the straightforward proof that we present in Theorem 3.10, for the convenience of the reader, is apparently due to Haim Brezis [Pon16, Proposition 15.5] from a personal communication. It is important to highlight that earlier proofs of the inequality (1.4) exist in the literature as well. For instance a proof using basic analysis tools is well incorporated in [NPV12, Section 6] which, originally springs from [SV11]. See also [BBM02, MS02] where the fractional inequality is established with a robust constant, i.e., with a constant C_s that stays asymptotically equivalent to $s(1-s)$ as $s \to 1^-$. The best constant of the Sobolev inequality, when $s = 1$, is exhibited in [Tal76]. For the special case $p = 2$ and $s \in (0,1)$, the fractional Sobolev inequality is established with best constant in [CT04].

In view of the aforementioned classical (fractional) Gagliardo-Nirenberg-Sobolev inequality (1.4), it is a legitimate right to seek for the analog inequality for the nonlocal Sobolev space $W^p_\nu(\mathbb{R}^d)$. Accordingly, we need to enforce adequate assumptions regarding the p-Lévy kernel ν. First and foremost, for $r > 0$, consider $\eta(r) = (\frac{r}{c_d})^{1/d}$ with $c_d = |B(0,1)|$ and define $w : [0,\infty] \to [0,\infty]$ by

$$w(r) = \Big(|B(0,\eta(r))| \int_{B^c(0,\eta(r))} \nu(h)\,\mathrm{d}h\Big)^{1/p} \quad \text{equally} \quad \frac{w^p(r)}{r} = \int_{B^c(0,\eta(r))} \nu(h)\,\mathrm{d}h. \tag{1.5}$$

For the sake of the reader convenience, we momentarily consider the following standing assumptions on ν, that we improve later on. In what follows, the notation v^{-1} stands for the reciprocal inverse of a bijective function v and should not be confused the fractional inverse $1/v$.

Assumption A **:** The function ν satisfies the p-Lévy integrability condition (1.1), is radial and is almost decreasing, i.e., there is $0 < \kappa \leq 1$ such that

$$\kappa\nu(|x|) \leq \nu(|y|) \quad \text{for all } |x| \geq |y|. \tag{A}$$

Assumption B **:** The mapping $t \mapsto 1/w(1/t)$ is invertible from $[0,\infty]$ to $[0,\infty]$, whose inverse ϕ is a Young function (see below for more details) and will be called *the critical Young function associated with* ν. To be more precise, ϕ is defined by

$$\phi(t) = \Big(\frac{1}{w(1/t)}\Big)^{-1} \quad \text{equivalenty} \quad w(t) = \frac{1}{\phi^{-1}(1/t)}. \tag{B}$$

Assumption C **:** The function $\phi_p : [0,\infty] \to [0,\infty]$ with $\phi_p(t) = \phi(t^{1/p})$ is convex, hence a Young function, and satisfies the growth condition: there is $\theta > 0$, such that

$$\phi_p\big(\theta^p \frac{s}{t}\big) \leq \frac{\phi_p(s)}{\phi_p(t)} \qquad \text{for all } s \leq t. \tag{C}$$

It worth highlighting that ϕ only depends on ν, p and d but, to alleviate the notations, we keep this implicit.

Let us now provide basics notions on Young functions and associated Orlicz spaces. A thorough and extensive study of Orlicz spaces are carried out in the seminal textbooks [RR91,RR02]. See also the traditional references [AF03,HH19,KR61] and the monographs [RGMP16,DHHR11], where the latter offers a treatise on generalized Orlicz spaces, also known as Musielak-Orlicz spaces, including Lebesgue and Sobolev spaces with variable exponents. Recall that a function $\phi : [0,\infty] \to [0,\infty]$ is convex if, $\phi(s + \tau(t-s)) \leq \phi(s) + \tau(\phi(t) - \phi(s))$ for all $s,t \geq 0$ and $\tau \in [0,1]$.

Young function: A convex function $\phi : [0,\infty] \to [0,\infty]$ such that $\phi(0) = 0$ is termed a Young function. Consequently, as a Young function, ϕ is nondecreasing, the mapping $t \mapsto \frac{\phi(t)}{t}$ is nondecreasing on $(0,\infty)$, and, either ϕ is identically zero or $\phi(\infty) = \infty$. Moreover, it is well known that ϕ is continuous on its effective domain, i.e., on the set of elements in $t \in [0,\infty)$ where $\phi(t) < \infty$. A more advanced calculus, e.g., Jensen's theorem [RR91, Theorem 1.3.1], yields the existence of another nondecreasing and right continuous function $b : [0,\infty) \to [0,\infty)$ called the density of ϕ, such that

$$\phi(t) = \int_0^t b(s)\,\mathrm{d}s.$$

This implies that ϕ has left and right derivatives that coincide except possibly on a countable set. To avoid unnecessary pathologies, it is custom to also to assume that ϕ is neither identically zero nor identically infinite on $(0,\infty)$.

Convex conjugate: To a Young function ϕ one associates the convex complementary, also called the convex conjugate, $\widetilde{\phi} : [0,\infty] \to [0,\infty]$, which is simultaneously defined as follows

$$\widetilde{\phi}(t) = \sup\big\{ts - \phi(s) : \ s > 0\big\} = \int_0^t \widetilde{b}(s)\,\mathrm{d}s.$$

Here, $\widetilde{b}(t) = \sup\{s > 0 : \ b(s) < t\}$ is the right inverse of b. Clearly, $\widetilde{\phi}$ is also a Young function, i.e., convex and $\widetilde{\phi}(0) = 0$. Note that, in virtue of the Fenchel-Moreau theorem, the couple $(\phi, \widetilde{\phi})$ is uniquely defined provided that ϕ is lower semi-continuous and additionally ϕ we have

$$\phi(t) = \sup\big\{ts - \widetilde{\phi}(s) : \ s > 0\big\} = \int_0^t b(s)\,\mathrm{d}s.$$

Analogously, the couple $(b, \widetilde{b})$ is uniquely determined and b is also the right inverse of $\widetilde{b}$, i.e., $b(t) = \sup\{s > 0 : \ \widetilde{b}(s) < t\}$. Furthermore, if b is strictly increasing then $\widetilde{b} = b^{-1}$,the inverse of b.

N-function: A Young function $\phi : [0,\infty] \to [0,\infty]$ is called a $N-$function (Nice Young function) if its density $b : [0,\infty) \to [0,\infty)$ is nondecreasing, right continuous and satisfies $0 < b(t) < \infty$ for $t > 0$, $\lim\limits_{t\to 0^+} b(t) = 0$ and $\lim\limits_{t\to\infty} b(t) = \infty$. This is equivalent to say that ϕ is continuous, increasing, convex and in addition the mapping $t \mapsto \frac{\phi(t)}{t}$, $t > 0$ is increasing and satisfies

$$\lim_{t\to 0^+} \frac{\phi(t)}{t} = \lim_{t\to\infty} \frac{t}{\phi(t)} = 0. \tag{1.6}$$

Orlicz space: Next, we write $K^\phi(\mathbb{R}^d)$ and $L^\phi(\mathbb{R}^d)$ respectively to denote the Orlicz class and the Orlicz space with respect to the Young function ϕ defined by

$$K^\phi(\mathbb{R}^d) = \Big\{ u : \mathbb{R}^d \to \mathbb{R} \text{ meas.} : \int_{\mathbb{R}^d} \phi\big(|u(x)|\big)\,\mathrm{d}x < \infty \Big\},$$
$$L^\phi(\mathbb{R}^d) = \Big\{ u : \mathbb{R}^d \to \mathbb{R} \text{ meas.} : \int_{\mathbb{R}^d} \phi\Big(\frac{|u(x)|}{\lambda}\Big)\,\mathrm{d}x < \infty \text{ for some } \lambda > 0 \Big\}.$$

It is worthwhile noticing that the Orlicz class $K^\phi(\mathbb{R}^d)$ is a convex set of functions and that $L^\phi(\mathbb{R}^d)$ is the linear hull of $K^\phi(\mathbb{R}^d)$. In addition, $u \in L^\phi(\mathbb{R}^d)$ if and only if $u \in \lambda K^\phi(\mathbb{R}^d)$ for some $\lambda > 0$. The space $L^\phi(\mathbb{R}^d)$ is a Banach space furnished with the Luxemburg norm $\|\cdot\|_{L^\phi(\mathbb{R}^d)}$ defined as the Minkowski functional or gauge of $K^\phi(\mathbb{R}^d)$ by

$$\|u\|_{L^\phi(\mathbb{R}^d)} = \inf\Big\{ \lambda > 0 \;:\; \int_{\mathbb{R}^d} \phi\Big(\frac{|u(x)|}{\lambda}\Big)\,\mathrm{d}x \le 1 \Big\}. \tag{1.7}$$

Obviously, by Fatou's lemma we have

$$\int_{\mathbb{R}^d} \phi\Big(\frac{|u(x)|}{\|u\|_{L^\phi(\mathbb{R}^d)}}\Big)\,\mathrm{d}x \le 1. \tag{1.8}$$

Beside the Luxemburg norm $\|\cdot\|_{L^\phi(\mathbb{R}^d)}$, an equivalence is the Orlicz norm $|\cdot|_{L^\phi(\mathbb{R}^d)}$, with

$$|u|_{L^\phi(\mathbb{R}^d)} = \sup\Big\{ \int_{\mathbb{R}^d} u(x)v(x)\,\mathrm{d}x \;:\; \int_{\mathbb{R}^d} \widetilde{\phi}(|v(x)|)\,\mathrm{d}x \le 1 \Big\}.$$

Moreover, the following comparison holds for all $u \in L^\phi(\mathbb{R}^d)$,

$$\|u\|_{L^\phi(\mathbb{R}^d)} \le |u|_{L^\phi(\mathbb{R}^d)} \le 2\|u\|_{L^\phi(\mathbb{R}^d)}.$$

For the particular Young function $\phi_L(t) = t^p/p$ with $1 < p < \infty$, the Orlicz space $L^{\phi_L}(\mathbb{R}^d)$ coincides with the well-known Lebesgue space $L^p(\mathbb{R}^d)$. In addition we have $\widetilde{\phi}_L(t) = t^{p'}/p'$ with p' satisfying the relation $pp' = p + p'$. Another example of Young function is $t \mapsto \max(t^{p_1}, t^{p_2})$ with $1 \le p_1, p_2 < \infty$ whose associated Orlicz space is $L^{p_1}(\mathbb{R}^d) \cap L^{p_2}(\mathbb{R}^d)$. Note in passing that $t \mapsto \min(t^{p_1}, t^{p_2})$ is not convex unless $p_1 = p_2$. One can however check that $L^{p_1}(\mathbb{R}^d) + L^{p_2}(\mathbb{R}^d)$ is the Orlicz space associated with the Young function given by

$$t \mapsto \int_0^t \frac{\min(s^{p_1}, s^{p_2})}{s}\,\mathrm{d}s.$$

The computation of the Luxemburg norm is not often straightforward. To illustrate this, let $E \subset \mathbb{R}^d$ be a measurable set with finite Lebesgue measure, i.e., $|E| < \infty$ and consider $\mathbb{1}_E$ be its characteristic function, i.e., $\mathbb{1}_E(x) = 1$ if $x \in E$ and $\mathbb{1}_E(x) = 0$ elsewhere, then

$$\|\mathbb{1}_E\|_{L^\phi(\mathbb{R}^d)} = \frac{1}{\phi^{-1}(1/|E|)}. \tag{1.9}$$

Indeed, for $\lambda > 0$ we have $\int_{\mathbb{R}^d} \phi(\frac{\mathbb{1}_E}{\lambda})(x)\,\mathrm{d}x = |E|\phi(\frac{1}{\lambda})$ which is less than 1 if and only if

$$\frac{1}{\phi^{-1}(1/|E|)} \le \lambda \quad \text{and hence} \quad \frac{1}{\phi^{-1}(1/|E|)} \le \|\mathbb{1}_E\|_{L^\phi(\mathbb{R}^d)}.$$

Therefore, the formula (1.9) holds as it suffices to choose $\lambda = \frac{1}{\phi^{-1}(1/|E|)}$ to have the reverse inequality. In the same spirit, using Jensen's inequality one establishes that

$$|\mathbb{1}_E|_{L^\phi(\mathbb{R}^d)} = |E|\widetilde{\phi}^{-1}(1/|E|).$$

Throughout this note, we only use the Luxemburg norm $\|\cdot\|_{L^\phi(\mathbb{R}^d)}$ defined as in (1.7). Assuming the Orlicz space $L^\phi(\mathbb{R}^d)$ associated with the critical function ϕ is equipped with the norm $\|\cdot\|_{L^\phi(\mathbb{R}^d)}$, our main result (see Theorem 3.7 for a general version) reads as follows.

Theorem 1.1. *Let* ***Assumption*** *A,* ***Assumption*** *B and* ***Assumption*** *C be in force. For $t \geq 2$, define $\Theta_t = t[2\kappa^2 C_p(t)\phi(\frac{\theta}{t})]^{-1/p}$ with $C_p(t) = \frac{t^p-2}{t^p-1}$. The following inequality holds*

$$\|u\|_{L^\phi(\mathbb{R}^d)} \leq \Theta_t \Big(\iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} |u(x)-u(y)|^p \nu(x-y)\,\mathrm{d}y\,\mathrm{d}x \Big)^{1/p} \quad \text{for all} \quad u \in L^\phi(\mathbb{R}^d). \tag{1.10}$$

Accordingly, the embeddings $W^p_\nu(\mathbb{R}^d) \hookrightarrow L^\phi(\mathbb{R}^d)$ and $W^{1,p}(\mathbb{R}^d) \hookrightarrow L^\phi(\mathbb{R}^d)$ are continuous.

The embedding $W^{1,p}(\mathbb{R}^d) \hookrightarrow L^\phi(\mathbb{R}^d)$ reminisces the so called Trudinger inequality [Mos71, Tru67] which implies that, for a smooth set $\Omega \subset \mathbb{R}^d$ and the Young function $\psi(t) = e^{t^{d/d-1}} - 1$, $p = d > 1$, the embedding $W^{1,p}(\Omega) \hookrightarrow L^\psi(\Omega)$ is continuous; $L^\psi(\Omega)$ is the Orlicz space on Ω associated with ψ. A substantial amount of results appeared in the wake of [Tru67], dealing with embedding of Sobolev spaces (eventually of Orlicz-Sobolev Spaces) into Orlicz spaces. A non-exhaustive list of reference on this and related topics includes [Cia96, Cia04, Cia05, Iul17, PR18, CPS20].

Of course, in accordance to the fractional Sobolev inequality (1.4), we show later in Example 2.5 that taking the fractional kernel $\nu(h) = |h|^{-d-sp}$ with $sp < d$, which fits the requirements of Theorem 1.1, gives $\phi(t) = ct^{p^*_s}$ whose corresponding Orlicz space is $L^{p^*_s}(\mathbb{R}^d)$. See also [ACPS21] for embeddings of Fractional Orlicz-Sobolev spaces into Orlicz type spaces. Let us mention that, in this particular case, the critical exponent p^*_s (or the critical Young function $\phi(t) = ct^{p^*_s}$) can be anticipated using an elementary scaling argument. Unfortunately it is not possible to forecast the critical Young function ϕ associated with a general kernel ν, using a scaling argument. For instance, if $0 < s_1 < s_2 < 1$, the kernel $\nu(h) = \max(|h|^{-d-s_1p}, |h|^{-d-s_2p})$ or $\nu(h) = \min(|h|^{-d-s_1p}, |h|^{-d-s_2p})$, does not permit the use of a scaling argument. Observe however that we obtain the critical Young function ϕ in a more constructive, but still less explicit, manner. The abstract aspect of the kernel ν under consideration forces a more general setting. Therefore, we will see later in Theorem 3.7 that, Theorem 1.1 still holds under weakened assumptions on ν. For instance, it appears that the decaying condition (A) can be completely dropped.

It appears as a natural question to know if it is possible to obtain the analog of the Gagliardo-Nirenberg-Sobolev inequality for functions restricted on an open set Ω different from $\mathbb{R}^d$. The answer to this important question turns out to be strongly related to the so called Poincaré-Sobolev type inequality. Indeed, as a second main result of this note, we formulate some interplay between Gagliardo-Nirenberg-Sobolev type inequalities, Poincaré-Sobolev type inequalities and Poincaré type inequalities. The global idea here, summarizes as follows, under that assumption of Theorem 1.1, if $\Omega \subset \mathbb{R}^d$ is sufficiently smooth then there is a constant also depending on Ω such that,

$$\|u - \fint_\Omega u\,\|_{L^\phi(\Omega)} \leq C \Big(\iint\limits_{\Omega\Omega} |u(x)-u(y)|^p \nu(x-y)\,\mathrm{d}y\,\mathrm{d}x \Big)^{1/p} \quad \text{for all} \quad u \in L^\phi(\Omega).$$

The rest of the paper is structured as follows. In Section 2, we comment in details our standing **Assumption** A, **Assumption** B and **Assumption** C by explaining their needfulness or not, and additionally providing some illustrative examples. Section 3 is dedicated to the proof of the main result Theorem 1.1 and its generalization in Theorem 3.7 with relaxed assumptions. This gives us an opportunity to revisit fractional Gagliardo-Nirenberg-Sobolev inequality with an alternative proof. Finally, Section 4 we establish some reciprocity relations between Gagliardo-Nirenberg-Sobolev type inequalities, Poincaré-Sobolev type inequalities and Poincaré type inequalities.

Through out, $B(x,r) := \{y \in \mathbb{R}^d : \ |y-x| < r\}$ denotes the open ball with radius $r > 0$ and centered at $x \in \mathbb{R}^d$ and its closure is denoted by $\overline{B}(x,r)$. On many estimates, $C > 0$ is a generic constant depending on the local inputs.

2. Miscellaneous

In this section we discuss **Assumption** A, **Assumption** B and **Assumption** C and provide at the end, some examples of kernels. We also collect some useful basics results on Orlicz spaces needed in the sequel. Let us first comment on the aforementioned assumptions and explain their necessity.

Assumption A: Although, the class of radial and almost decreasing p-Lévy integrable kernels is fairly large, we will see later that this assumption can be completely dropped by the mean of the Schwarz symmetrization rearrangement see for instance Theorem 3.6. Rather, having an almost decreasing p-Lévy integrable kernels, allows us to get a quicker constructive approach of the critical Young function ϕ as given in (B). Beside this, the p-Lévy integrability condition, i.e., $\nu \in L^1(\mathbb{R}^d, 1 \wedge |h|^p \,\mathrm{d}h)$, though does not really play any role in the proof Theorem 1.1, can neither be improved nor completely dropped. Indeed, this condition renders the space $W^p_\nu(\mathbb{R}^d)$ more consistent, in a sense that it warrants the space $W^p_\nu(\mathbb{R}^d)$ to contain at least smooth functions of compact support. The p-Lévy integrability draws a borderline for which a space of type $W^p_\nu(\mathbb{R}^d)$ is trivial or not. This is illustrated by the next proposition, see [FG21, Proposition 2.14].

Proposition 2.1. *Let $\nu : \mathbb{R}^d \setminus \{0\} \to [0,\infty]$ be symmetric. The following assertions hold true.*

(i) If $\nu \in L^1(\mathbb{R}^d)$, then $W^p_\nu(\mathbb{R}^d) = L^p(\mathbb{R}^d)$ with equivalence in norm.

(ii) If ν is radial and $\int_{B(0,\delta)} |h|^p \nu(h)\,\mathrm{d}h = \infty$ for some $\delta > 0$, and thus $\nu \notin L^1(\mathbb{R}^d, 1 \wedge |h|^p \,\mathrm{d}h)$, then the only smooth functions contained in $W^p_\nu(\mathbb{R}^d)$ are constants.

(iii) If $\nu \in L^1(\mathbb{R}^d, 1 \wedge |h|^p \,\mathrm{d}h)$ then the embedding $W^{1,p}(\mathbb{R}^d) \hookrightarrow W^p_\nu(\mathbb{R}^d)$ is continuous and hence $C^\infty_c(\mathbb{R}^d) \subset W^p_\nu(\mathbb{R}^d)$. In addition, if ν is radial then for some constants $C_1, C_2 > 0$, we have

$$C_1 \|u\|_{W^{1,p}(\mathbb{R}^d)} \leq \|u\|_{W^p_\nu(\mathbb{R}^d)} \leq C_2 \|u\|_{W^{1,p}(\mathbb{R}^d)} \quad \text{for all } u \in W^{1,p}(\mathbb{R}^d). \tag{2.1}$$

Warning! The equivalence (2.1) does not imply that $W^{1,p}(\mathbb{R}^d) = W^p_\nu(\mathbb{R}^d)$.

Assumption B: The **Assumption** B turns out to be weaker than **Assumption** C. Indeed, the function $t \mapsto \phi_p(t) = \phi(t^{1/p})$ being a Young function and hence convex and nondecreasing implies that $\phi(t) = \phi_p(t^p)$ is also convex and hence a Young function. Furthermore, it is natural to require the function ϕ to be invertible as it rules out pathological functions. Note that, assuming ϕ is a Young function, one views from (B) that $\phi(0) = 0 = w(0)$ and $\phi(\infty) = \infty = w(\infty)$ and hence that $t \mapsto \phi(t)$ is invertible from $[0,\infty]$ to $[0,\infty]$ if and only if $r \mapsto w^p(r)$ is. Therefore, **Assumption** B is somewhat superfluous and is simply an accessory to define the critical Young function ϕ and the associated Orlicz space $L^\phi(\mathbb{R}^d)$.

Assumption C: The **Assumption** C essentially constitutes the most fundamental and quite vital property needed on ϕ in order to establishing our main result. Next, we explain how the **Assumption** C globally infers certain growth conditions on ϕ. First of all, the growth condition (C) is clearly equivalent to say that

$$\phi\big(\theta \frac{s}{t}\big) \leq \frac{\phi(s)}{\phi(t)} \quad \text{for all } s \leq t \quad \text{or equaly} \quad \theta \leq \phi^{-1}\big(\frac{s}{t}\big)\frac{\phi^{-1}(t)}{\phi^{-1}(s)} \quad \text{for all } s \leq t. \tag{2.2}$$

The latter suggests that the growth behavior of ϕ is not far from that of a polynomial growth [Mal85]; see for instance Proposition 2.2 below. This behavior can be expected, since regarding the inequality of interest (1.10) in Theorem 1.1. Since at the first glance, in comparison with the fractional Sobolev space $W^{s,p}(\mathbb{R}^d)$, one can expect that the space $W^p_\nu(\mathbb{R}^d)$ is embedded in another Lebesgue space. Nevertheless, it is worth noticing that fractional kernels of the form $\nu(h) = |h|^{-d-sp}$ with $s \in (0,1)$ are the only radial functions satisfying **Assumption** A, **Assumption** B and **Assumption** C that can produce polynomial critical Young functions of the form $\phi(t) = ct^q$;

see Example 2.5 and Theorem 2.6 below. Observe that letting $s = t\tau$ with $0 \leq \tau \leq 1$ then the condition (C) (see also (2.2)) is also to equivalent to say that ϕ satisfies sup-multiplicative condition

$$\phi(\theta\tau)\phi(t) \leq \phi(t\tau) \qquad \text{for all } t \geq 0 \text{ and } \tau \in [0,1]$$

or that ϕ^{-1} satisfies the sub-multiplicative condition

$$\theta\phi^{-1}(\tau t) \leq \phi^{-1}(\tau)\phi^{-1}(t) \qquad \text{for all } t \geq 0 \text{ and } \tau \in [0,1].$$

Let us now highlight some consequences of the fact $\phi_p(t) = \phi(t^{1/p})$ is a Young function. Before let us observe that if ϕ is exists, i.e., as given in (B) then taking $r = \phi(t)$ for $t > 0$, in virtue of (1.5) and (B) we obtain

$$\frac{\phi(t)}{t^p} = \frac{r}{[\phi^{-1}(r)]^p} = rw^p(1/r) = \int_{|h|^d \geq \frac{1}{c_d\phi(t)}} \nu(h)\,\mathrm{d}h. \tag{2.3}$$

In particular, if $\lim\limits_{t\to 0^+} \phi(t) = 0$ and $\lim\limits_{t\to\infty} \phi(t) = \infty$ then

$$\lim_{t\to\infty} \frac{\phi(t)}{t^p} = \int_{\mathbb{R}^d} \nu(h)\,\mathrm{d}h \quad \text{and} \quad \lim_{t\to 0^+} \frac{\phi(t)}{t^p} = \lim_{r\to\infty} \int_{|h|\geq r} \nu(h)\,\mathrm{d}h. \tag{2.4}$$

The next proposition shows that the convexity of $\phi_p(t) = \phi(t^{1/p})$ induces via the ratio $\frac{\phi(t)}{t^p}$ certain geometry growth comparisons between $\phi(t)$ and t^p at both near the origin and the infinity.

Proposition 2.2. *Assuming $t \mapsto \phi_p(t) = \phi(t^{1/p})$ is an invertible Young function, the following assertions are true.*

(i) *ϕ is also a Young function.*
(ii) *The mappings $t \mapsto \phi(t)$ and $t \mapsto \frac{\phi(t)}{t^p}$ are increasing.*
(iii) *If $1 < p < \infty$ then ϕ is an N-function.*
(iv) *Let $\delta_0 = \frac{\phi(t_0)}{t_0^p}$ for fixed $t_0 > 0$ then we have*

$$\phi(t) \leq \delta_0 t^p \quad \text{if } 0 \leq t \leq t_0 \qquad \text{and} \qquad \phi(t) \geq \delta_0 t^p \quad \text{if } t \geq t_0.$$

(v) *Let $\delta_0' = \frac{w^p(r_0)}{r_0}$ for fixed $r_0 > 0$ then we have*

$$\int_{B^c(0,\eta(r))} \nu(h)\,\mathrm{d}h = \frac{w^p(r)}{r} \geq \delta_0' \quad \text{if } 0 \leq r \leq r_0 \quad \text{and} \quad \int_{B^c(0,\eta(r))} \nu(h)\,\mathrm{d}h = \frac{w^p(r)}{r} \leq \delta_0' \quad \text{if } r \geq r_0.$$

(vi) *If $\nu \in L^1(\mathbb{R}^d, 1 \wedge |h|^p\,\mathrm{d}h)$ and $\nu \notin L^1(\mathbb{R}^d)$ then ϕ_p is an N-function, equally we have*

$$\lim_{t\to 0^+} \frac{\phi(t)}{t^p} = \lim_{t\to\infty} \frac{t^p}{\phi(t)} = 0.$$

(vii) *If $\nu \in L^1(\mathbb{R}^d, 1 \wedge |h|^p\,\mathrm{d}h)$ is radial, then $r \mapsto \frac{d}{dr}\Big(\frac{w^p(r)}{r}\Big)$ solve the ordinary differential equation*

$$\nu(\eta(r)) = -\frac{d}{dr}\Big(\frac{w^p(r)}{r}\Big) \quad \text{and} \quad \lim_{r\to\infty} \frac{w^p(r)}{r} = 0. \tag{2.5}$$

Proof. As an invertible Young function $t \mapsto \phi_p(t)$ is increasing and hence the assertion *(i)* is clear since $\phi(t) = \phi_p(t^p)$ and $t \mapsto t^p$ is also convex. In view of proving *(ii)*, observe that ϕ is increasing as it is an invertible Young function. Since ϕ_p is convex and invertible with $\phi_p(0) = 0$, we get

$$\phi_p(s^p) = \phi_p\Big(\frac{s^p}{t^p}t^p\Big) < \frac{s^p}{t^p}\phi_p(t^p) \quad \text{that is} \quad \frac{\phi(s)}{s^p} < \frac{\phi(t)}{t^p}, \quad \text{for all } s < t.$$

The assertion *(iii)* follows from *(i)* and *(ii)*. The assertions *(iv)* and *(v)* are consequences of *(ii)*. Whereas, *(vi)* clearly follows from (2.4). Differentiating the relation (1.5) gives *(vii)* since by change of variables we get

$$\frac{w^p(r)}{r} = \int_{B^c(0,\eta(r))} \nu(h)\,\mathrm{d}h = dc_d \int_{\eta(r)}^{\infty} \nu(\tau)\tau^{d-1}\,\mathrm{d}\tau = \int_r^{\infty} \nu(\eta(\tau'))\,\mathrm{d}\tau'.$$

□

Next, we need the following result owed to [RR91, Theorem 5.1.3].

Theorem 2.3. *Let $D \subset \mathbb{R}^d$ be measurable and let ϕ_i, $i = 1,2$ be a pair of Young functions. If $|D| < \infty$ (resp. $|D| = \infty$) and $\phi_1(t) \le \phi_2(ct)$ for all $t \ge t_0$ for some $c > 0$ and $t_0 > 0$ (resp. $t_0 = 0$) then the embedding $L^{\phi_2}(D) \hookrightarrow L^{\phi_1}(D)$ is continuous. The converse holds true as well.*

Proof. The case $|D| = \infty$ and $t_0 = 0$ is straightforward and one has $\|u\|_{L^{\phi_1}(D)} \le c\|u\|_{L^{\phi_2}(D)}$. Now, assume $|D| < \infty$, for $u \in L^{\phi_2}(D)$, consider $A = \{x \in D : |u(x)| \le ct_0\|u\|_{L^{\phi_2}(D)}\}$ and put $T = \phi_1(t_0)|D| + 1$. Since $\phi_1(\frac{t}{T}) \le \frac{1}{T}\phi_1(t)$ for $t > 0$, recalling (1.8), one gets

$$\begin{aligned}\int_D \phi_1\Big(\frac{|u(x)|}{Tc\|u\|_{L^{\phi_2}(D)}}\Big)\,\mathrm{d}x &= \int_A \phi_1\Big(\frac{|u(x)|}{Tc\|u\|_{L^{\phi_2}(D)}}\Big)\,\mathrm{d}x + \int_{D\setminus A} \phi_1\Big(\frac{|u(x)|}{Tc\|u\|_{L^{\phi_2}(D)}}\Big)\,\mathrm{d}x\\ &\le \frac{1}{T}\Big(\phi_1(t_0)|A| + \int_{D\setminus A} \phi_2\Big(\frac{|u(x)|}{\|u\|_{L^{\phi_2}(D)}}\Big)\,\mathrm{d}x\Big)\\ &\le \frac{1}{T}\big(\phi_1(t_0)|D| + 1\big) = 1.\end{aligned}$$

Accordingly, this implies that

$$\|u\|_{L^{\phi_1}(D)} \le cT\|u\|_{L^{\phi_2}(D)}. \tag{2.6}$$

Conversely, assume there is no constant $c > 0$ such that $\phi_1(t) \le \phi_2(ct)$ for all $t > t_0 > 0$. Then one can construct an increasing sequence $0 < t_0 < \cdots < t_k < t_{k+1} \cdots$ such that $\phi_1(t_k) > \phi_2(2^k k^2 t_k)$. In particular, $\phi_2(t_k) > 0$ and the convexity implies $\phi_1(t_k) > 2^k\phi_2(k^2 t_k)$. Fix $D_0 \subset D$ such that $0 < |D_0| < \infty$ and let $D_k \subset D_0$ be disjoint measurable sets such that $|D_k| > 0$ and

$$|D_k| = \frac{\phi_2(t_1)|D_0|}{2^k\phi_2(k^2 t_k)}, \quad \text{and hence} \quad \sum_{k=1}^{\infty} |D_k| < |D_0|.$$

To conclude, we show that the function $u = \sum_{k=1}^{\infty} kt_k \mathbb{1}_{D_k}$ (which is supported in D_0) belongs in $L^{\phi_2}(\mathbb{R}^d)$ but not in $L^{\phi_1}(\mathbb{R}^d)$. Indeed, for any integer $n \ge 1$ we have

$$\begin{aligned}\int_D \phi_2(nu(x))\,\mathrm{d}x &= \sum_{k=1}^{\infty} \phi_2(nkt_k)|D_k| \le \sum_{k=1}^{n} \phi_2(nkt_k)|D_k| + \sum_{k\ge n+1} \phi_2(k^2 t_k)|D_k|\\ &= \sum_{k=1}^{n} \phi_2(nkt_k)|D_k| + \phi_2(t_1)|D_0| \sum_{k\ge n+1} \frac{1}{2^k} < \infty.\end{aligned}$$

This shows that $u \in L^{\phi_1}(\mathbb{R}^d)$. However, for any $\varepsilon > 0$, recalling that $\phi_1(t_k) > 2^k\phi_2(k^2 t_k)$ we have

$$\begin{aligned}\int_D \phi_1(\varepsilon u(x))\,\mathrm{d}x &\ge \sum_{k\ge \frac{1}{\varepsilon}} \phi_1(\varepsilon kt_k)|D_k| \quad \ge \sum_{k\ge\frac{1}{\varepsilon}} \phi_1(t_k)|D_k| \quad (\text{since } \ \varepsilon k \ge 1)\\ &\ge \sum_{k\ge\frac{1}{\varepsilon}} 2^k\phi_2(k^2 t_k)|D_k| = \phi_2(t_1)|D_0| \sum_{k\ge\frac{1}{\varepsilon}} 1 = \infty.\end{aligned}$$

This implies $u \notin L^{\phi_1}(\mathbb{R}^d)$ and hence that $L^{\phi_2}(\mathbb{R}^d) \not\subset L^{\phi_1}(\mathbb{R}^d)$. The proof is complete. □

Corollary 2.4. *Assume $t \mapsto \phi_p(t) = \phi(t^{1/p})$ is an invertible Young function, then the embedding $L^{\phi}(\mathbb{R}^d) \hookrightarrow L^p_{\mathrm{loc}}(\mathbb{R}^d)$ is continuous.*

Proof. According to Proposition 2.2, $\phi(t) \ge \delta_0 t^p$ for all $t \ge t_0$ with $t_0 > 0$ fixed. The claim follows since for a compact set $D \subset \mathbb{R}^d$, Theorem 2.3 implies $\|u\|_{L^p(D)} \le C\|u\|_{L^{\phi}(D)} \le C\|u\|_{L^{\phi}(\mathbb{R}^d)}$. □

Let us provide examples for which the main inequality (1.10) holds. First of all, we deal with the classical fractional Sobolev space $W^{s,p}(\mathbb{R}^d)$.

Example 2.5. For $s \in (0,1)$, consider the kernel $\nu(h) = |h|^{-d-sp}$, $h \neq 0$ so that $W^p_\nu(\mathbb{R}^d) = W^{s,p}(\mathbb{R}^d)$. A painless computation through polar coordinates in (1.5) and (B) yields that

$$w(r) = \gamma_s^{1/p}\, r^{1/p^*_s} \qquad \text{and} \qquad \phi(t) = \gamma_s^{p^*_s/p}\, t^{p^*_s}, \tag{2.7}$$

where, recalling $c_d = |B(0,1)|$, we set

$$\frac{1}{p^*_s} = \frac{1}{p} - \frac{s}{d} \qquad \text{and} \qquad \gamma_s = \frac{dc_d^{1+\frac{sp}{d}}}{sp}.$$

Observe that $1/p^*_s > 0$ if and if $p^*_s \geq p \geq 1$ and, hence if and only if $\phi_p(t) = \phi(t^{1/p}) = \gamma_s^{p^*_s/p} t^{p^*_s/p}$ is convex. Moreover, for all $s, t > 0$ we have

$$\phi_p\big(\theta^p \frac{s}{t}\big) = \frac{\phi_p(s)}{\phi_p(t)} \quad \text{with} \quad \theta = \frac{1}{\gamma_s^{1/p}}.$$

Whence, **Assumption** A, **Assumption** B and **Assumption** C are fulfilled provided that $1/p^*_s > 0$.

In connection with Example 2.5, the next result shows that, for ν radial, the Orlicz space $L^\phi(\mathbb{R}^d)$ is a Lebesgue space, i.e., $\phi(t) = ct^q$ if and only if the space $W^p_\nu(\mathbb{R}^d)$ is a fractional Sobolev space.

Theorem 2.6. *Let $\nu : \mathbb{R}^d \setminus \{0\} \to [0, \infty]$ be radial. Assume that ν, associated with $\phi(t) = ct^q$ for some $q, c > 0$, satisfies **Assumption** A, **Assumption** B and **Assumption** C. Then necessarily $\frac{1}{p} - \frac{1}{d} < \frac{1}{q} < \frac{1}{p}$ and there exists $s \in (0,1)$, in fact, $s = \frac{d}{p} - \frac{d}{q}$, such that $\nu(h) = C_{p,q,d}|h|^{-d-sp}$, for some constant $C_{p,q,d} > 0$ depending on c, p, q and d.*

Proof. First of all, in virtue of **Assumption** C, observe that $\phi_p(t) = ct^{q/p}$ with $c > 0$ is convex if and only if $q \geq p \geq 1$. The relation (B) implies that $w(r) = c^{1/q} r^{1/q}$ and hence (1.5) amounts to

$$c^{p/q} r^{p/q-1} = \int_{B^c(0,\eta(r))} \nu(h)\, \mathrm{d}h = dc_d \int_{\eta(r)}^\infty \nu(\tau)\tau^{d-1}\, \mathrm{d}\tau = \int_r^\infty \nu(\eta(\tau'))\, \mathrm{d}\tau'.$$

Differentiating both sides and letting $\rho = \eta(r) = \big(\frac{r}{c_d}\big)^{1/d}$ yields

$$\nu(\rho) = c^{p/q}(1 - \frac{p}{q}) c_d^{p/q-2}\, \rho^{-d+dp/q-d} = C_{p,q,d}\, \rho^{-d-sp} \qquad \text{with } s = \frac{d}{p} - \frac{d}{q} \in [0, d].$$

In short, $\nu(h) = C_{p,q,d}\, |h|^{-d-sp}$. Finally observe that, by **Assumption** A, $\nu \in L^1(\mathbb{R}^d, 1 \wedge |h|^p\, \mathrm{d}h)$ if and only if $s \in (0,1)$ that is $\frac{1}{p} - \frac{1}{d} < \frac{1}{q}$ and $q > p$. This ends the proof. □

Remark 2.7. Theorem 2.6 implies that we always have $L^\phi(\mathbb{R}^d) \not\subset L^p(\mathbb{R}^d)$. Furthermore, there is no radial kernel ν for which one has $\phi(t) = ct^p$.

Example 2.8. Assume $\nu \in L^1(\mathbb{R}^d)$ is radial and has full support so that ϕ exists. As explained previously, in this situation $W^p_\nu(\mathbb{R}^d) = L^p(\mathbb{R}^d)$ with equivalence in norm. Moreover, the relation (2.3) implies $\phi(t) \leq \|\nu\|_{L^1(\mathbb{R}^d)} t^p$ for all $t > 0$. Whence, according to Theorem 2.3 we get the continuous embedding $L^p(\mathbb{R}^d) \hookrightarrow L^\phi(\mathbb{R}^d)$ and we have

$$\|u\|_{L^\phi(\mathbb{R}^d)} \leq \|\nu\|^{1/p}_{L^1(\mathbb{R}^d)} \|u\|_{L^p(\mathbb{R}^d)} \quad \text{for all } u \in L^\phi(\mathbb{R}^d).$$

Together with Corollary 2.4, we get the continuous embeddings $L^p(\mathbb{R}^d) \hookrightarrow L^\phi(\mathbb{R}^d) \hookrightarrow L^p_{\text{loc}}(\mathbb{R}^d)$.

For a concrete example, let us define the family of Young functions $\phi^a(t) = \ln(a + e^{t^p}) - \ln(a+1)$ here $a > 0$ is a fixed parameter. Note that $\phi(t) \leq t^p$, for all $t > 0$. Clearly $\phi^a_p(t) = \phi^a(t^{1/p}) = \ln(a + e^t) - \ln(a+1)$ also a Young function. Moreover, each ϕ^a satisfies (C) with $\theta = 1$. Last one defines the kernel $\nu^a \in L^1(\mathbb{R}^d)$ associated with ϕ^a through the relation

$$\nu^a(\eta(r)) = -\frac{d}{dr}\Big(\frac{1}{r\xi^a(r)}\Big) \quad \text{with } \xi^a(r) := [\phi^a_p]^{-1}(1/r) = \ln((a+1)e^{1/r} - a).$$

We leave all computational details to the interested reader; who should also checks that, each ν^a fulfills **Assumption** A, **Assumption** B and **Assumption** C.

Before giving additional examples, let us recall without proof the following basic result characterizing the intersection and the sum of Orlicz spaces.

Theorem 2.9. *Let ϕ_i, $i=1,2$ be a pair of Young functions. For the Young function $\phi(t) = \max(\phi_1(t), \phi_2(t))$ we have $L^{\phi_1}(\mathbb{R}^d)\cap L^{\phi_2}(\mathbb{R}^d) = L^{\phi}(\mathbb{R}^d)$ and*

$$\frac{1}{2}\|u\|_{L^{\phi}(\mathbb{R}^d)} \leq \max(\|u\|_{L^{\phi_1}(\mathbb{R}^d)}, \|u\|_{L^{\phi_2}(\mathbb{R}^d)}) \leq \|u\|_{L^{\phi}(\mathbb{R}^d)}.$$

In addition, for any Young function ψ such that $\psi(t)\leq \phi(ct)$ for some $c>0$, then we have the continuous embedding $L^{\phi_1}(\mathbb{R}^d)\cap L^{\phi_2}(\mathbb{R}^d)\hookrightarrow L^{\psi}(\mathbb{R}^d)$.
Analogously, for the function $\phi(t)=\min(\phi_1(t),\phi_2(t))$ we have $L^{\phi_1}(\mathbb{R}^d)+L^{\phi_2}(\mathbb{R}^d) = L^{\phi}(\mathbb{R}^d)$. Note that here ϕ is not necessarily convex and thus, is identified with its greatest convex minorant $\phi_{\min}$ defined by

$$\phi_{\min}(t) = \int_0^t \frac{\min(\phi_1(s),\phi_2(s))}{s}\,\mathrm{d}s.$$

So that we have, $\phi_{\min}(t)\leq\phi(t)\leq\phi_{\min}(2t)$ and hence $\frac{1}{2}\|u\|_{L^{\phi}(\mathbb{R}^d)} \leq \|u\|_{L^{\phi_{\min}}(\mathbb{R}^d)} \leq \|u\|_{L^{\phi}(\mathbb{R}^d)}$. Moreover, if $\|\cdot\|_{L^{\phi_1}(\mathbb{R}^d)+L^{\phi_2}(\mathbb{R}^d)}$ denotes the natural norm on $L^{\phi_1}(\mathbb{R}^d)+L^{\phi_2}(\mathbb{R}^d)$ then

$$\frac{1}{4}\|u\|_{L^{\phi}(\mathbb{R}^d)} \leq \|u\|_{L^{\phi_1}(\mathbb{R}^d)+L^{\phi_2}(\mathbb{R}^d)} \leq 2\|u\|_{L^{\phi}(\mathbb{R}^d)}.$$

Recall that $\|u\|_{L^{\phi_1}(\mathbb{R}^d)+L^{\phi_2}(\mathbb{R}^d)} = \inf\big\{\|u_1\|_{L^{\phi_1}(\mathbb{R}^d)}+\|u_2\|_{L^{\phi_2}(\mathbb{R}^d)}:\ u=u_1+u_2,\, u_i\in L^{\phi_1}(\mathbb{R}^d)\big\}$.

In particular case of Lebesgue spaces, if $1\leq p_1\leq p_2$ and $q\in[p_1,p_2]$ then $t^q\leq \max(t^{p_1},t^{p_2})$ for all $t>0$ and hence $L^{p_1}(\mathbb{R}^d)\cap L^{p_2}(\mathbb{R}^d)\hookrightarrow L^q(\mathbb{R}^d)$ as $L^{p_1}(\mathbb{R}^d)\cap L^{p_2}(\mathbb{R}^d)$ is the Orlicz space associated with the Young function $t\mapsto \max(t^{p_1},t^{p_2})$.
The next example exhibits a situation where the growth condition (C) fails but still the inequality (1.10) holds true with a possibly different constant.

Example 2.10. Fix $0<s_1<s_2<1$, following the notations of Example 2.5, we define the Young function $\phi(t)=\max(t^{p^*_{s_1}}, t^{p^*_{s_2}})$, with $1/p^*_{s_i}>0$, $i=1,2$ so that $L^{\phi}(\mathbb{R}^d) = L^{p^*_{s_1}}(\mathbb{R}^d)\cap L^{p^*_{s_2}}(\mathbb{R}^d)$. Clearly, $\phi_p(t)=\phi(t^{1/p})$ is convex since $p^*_{s_2}>p^*_{s_1}\geq p$. Moreover, the relationB gives

$$w^p(r) = \frac{1}{\phi_p^{-1}(1/r)} = \max(r^{p/p^*_{s_1}}, r^{p/p^*_{s_2}}).$$

Now, we differentiate the relation (1.5) and put $\rho=\eta(r)$, equally $r=c_d\rho^d$ to obtain that

$$\nu(\rho)=\nu(\eta(r)) = -\frac{d}{dr}\Big(\frac{w^p(r)}{r}\Big) = \begin{cases} \frac{s_2p}{d}r^{-1-\frac{s_2p}{d}} & \text{if } r<1,\\ \frac{s_1p}{d}r^{-1-\frac{s_1p}{d}} & \text{if } r\geq 1\end{cases} = \begin{cases}\frac{1}{\gamma_{s_2}}\rho^{-d-s_2p} & \text{if } \rho<\eta(1),\\ \frac{1}{\gamma_{s_1}}\rho^{-d-s_1p} & \text{if } \rho\geq\eta(1).\end{cases}$$

Whence, the kernel ν is given by

$$\nu(h) = \frac{1}{\gamma_{s_2}}\mathbb{1}_{B(0,\eta(1))}(h)\,|h|^{-d-s_2p} + \frac{1}{\gamma_{s_1}}\mathbb{1}_{B^c(0,\eta(1))}(h)\,|h|^{-d-s_1p}.$$

One easily finds that $c_1\nu(h)\leq \max(|h|^{-d-s_1p},|h|^{-d-s_2p})\leq |h|^{-d-s_1p}+|h|^{-d-s_2p}\leq c_2\nu(h)$ for some constants $c_1,c_2>0$ and hence that $W^p_\nu(\mathbb{R}^d) = W^{s_1,p}(\mathbb{R}^d)\cap W^{s_2,p}(\mathbb{R}^d) = W^{s_2,p}(\mathbb{R}^d)$. Note however, that the growth condition (C) cannot hold here, i.e., there is no constant $\theta>0$ such that

$$\phi(\theta\frac{s}{t})\leq\frac{\phi(s)}{\phi(t)} \quad \text{for all } s\leq t.$$

It suffices to take $s=1$ and tend $t\to\infty$ to observe a contradiction. Nevertheless, according to Example 2.5, Theorem 1.1 applies on the kernels $|h|^{-d-s_ip}\leq c_2\nu(h)$, $i=1,2$ and since by Theorem 2.9, $L^{\phi}(\mathbb{R}^d) = L^{p^*_{s_1}}(\mathbb{R}^d)\cap L^{p^*_{s_2}}(\mathbb{R}^d)$ and $\|u\|_{L^{\phi}(\mathbb{R}^d)}\leq 2\max\big(\|u\|_{L^{p^*_{s_1}}(\mathbb{R}^d)}, \|u\|_{L^{p^*_{s_2}}(\mathbb{R}^d)}\big)$, we get

$$\|u\|_{L^{\phi}(\mathbb{R}^d)}\leq C\Big(\iint\limits_{\mathbb{R}^d\,\mathbb{R}^d}|u(x)-u(y)|^p\nu(x-y)\,\mathrm{d}y\,\mathrm{d}x\Big)^{1/p} \quad \text{for all} \quad u\in L^{\phi}(\mathbb{R}^d).$$

In other words, inequality (1.10) still holds despite the failure of the growth condition (C).

The next example shows that the lack of convexity can sometime be rectified.

Example 2.11. Fix $0 < s_1 < s_2 < 1$, considering the notations of Example 2.5 define $\phi(t) = \min(t^{p^*_{s_1}}, t^{p^*_{s_2}})$, with $p^*_{s_i} > 0$, $i = 1, 2$. Note that $\phi_p(t) = \phi(t^{1/p})$ is not necessarily convex. However one rectifies this deficiency by defining

$$\phi_{\min}(t) = \int_0^t \frac{\min(s^{p^*_{s_1}}, s^{p^*_{s_2}})}{s}\,\mathrm{d}s \quad \text{so that} \quad \phi_{\min}(t^{1/p}) = \frac{1}{p}\int_0^t \frac{\min(s^{p^*_{s_1}/p}, s^{p^*_{s_2}/p})}{s}\,\mathrm{d}s.$$

Since, $p^*_{s_2} \geq p^*_{s_1} \geq p$, one readily obtains that $t \mapsto \phi_{\min}(t^{1/p})$ is convex. Furthermore,

$$\phi\big(\frac{s}{t}\big) \leq \frac{\phi(s)}{\phi(t)} \quad \text{for all } s \leq t \quad \text{and} \quad \phi_{\min}(t) \leq \phi(t) \leq \phi_{\min}(2t) \quad \text{for all } t \geq 0.$$

Combining altogether implies that

$$\phi_{\min}\big(\frac{s}{2t}\big) \leq \phi\big(\frac{s}{2t}\big) \leq \frac{\phi(s/2)}{\phi(t)} \leq \frac{\phi_{\min}(s)}{\phi_{\min}(t)} \quad \text{for } s \leq t.$$

It turns out that, $\phi_{\min}$ satisfies (C) with $\theta = \frac{1}{2}$ and thus the **Assumption** C. Therefore, it is fair to identify ϕ with $\phi_{\min}$ so that $L^{\phi}(\mathbb{R}^d) = L^{\phi_{\min}}(\mathbb{R}^d) = L^{p^*_{s_1}}(\mathbb{R}^d) + L^{p^*_{s_2}}(\mathbb{R}^d)$. Next, we find the kernel associated with $\phi(t) = \min(t^{p^*_{s_1}}, t^{p^*_{s_2}})$. The relation (B) yields

$$w^p(r) = \frac{1}{\phi_p^{-1}(1/r)} = \min(r^{p/p^*_{s_1}}, r^{p/p^*_{s_2}}).$$

Using the relation (2.5) and put $\rho = \eta(r)$, equally $r = c_d\rho^d$ to obtain

$$\nu(\rho) = \nu(\eta(r)) = -\frac{d}{dr}\Big(\frac{w^p(r)}{r}\Big) = \begin{cases} \frac{s_2 p}{d} r^{-1-\frac{s_2 p}{d}} & \text{if } r \geq 1, \\ \frac{s_1 p}{d} r^{-1-\frac{s_1 p}{d}} & \text{if } r < 1 \end{cases} = \begin{cases} \frac{1}{\gamma_{s_2}} \rho^{-d-s_2 p} & \text{if } \rho \geq \eta(1), \\ \frac{1}{\gamma_{s_1}} \rho^{-d-s_1 p} & \text{if } \rho < \eta(1). \end{cases}$$

Whence, the kernel ν is given by

$$\nu(h) = \frac{1}{\gamma_{s_1}} \mathbb{1}_{B(0,\eta(1))}(h)\,|h|^{-d-s_1 p} + \frac{1}{\gamma_{s_2}} \mathbb{1}_{B^c(0,\eta(1))}(h)\,|h|^{-d-s_2 p}.$$

One easily finds that $c_1\nu(h) \leq \min(|h|^{-d-s_1p}, |h|^{-d-s_2p}) \leq c_2\nu(h)$ for some constants $c_1, c_2 > 0$ and hence that $W^{s_1,p}(\mathbb{R}^d) + W^{s_2,p}(\mathbb{R}^d) = W^p_\nu(\mathbb{R}^d) = W^{s_1,p}(\mathbb{R}^d)$. Thus, identifying ϕ and $\phi_{\min}$ **Assumption** A , **Assumption** B and **Assumption** C are satisfied.

Remark 2.12. There are two keys geometric observations emanating from Example 2.5, Example 2.10 and Example 2.11. Firstly, modifying the p-Lévy integrable kernel ν at the origin or at the infinity may not change the topology of the nonlocal Sobolev space $W^p_\nu(\mathbb{R}^d)$.
Secondly, the geometric behavior of the kernel ν at the origin or at the infinity truly governs that of the associated critical function ϕ and hence influences the topology of the Orlicz space $L^\phi(\mathbb{R}^d)$. In other words a perturbation of the kernel ν at the origin or at the infinity can drastically change the resulting associated Orlicz space (or associated critical function). This geometric behavior also reads the through the relation (2.3) which implies that

$$\lim_{t\to\infty} \frac{\phi(t)}{t^p} = \int_{|h| \geq \eta(\frac{1}{\phi(\infty)})} \nu(h)\,\mathrm{d}h \quad \text{and} \quad \lim_{t\to 0^+} \frac{\phi(t)}{t^p} = \int_{|h| \geq \eta(\frac{1}{\phi(0)})} \nu(h)\,\mathrm{d}h.$$

3. Main results

With a view to establish our main result, we need auxiliary results that are the milestones to prove Theorem 1.1. We begin with the following important lemma.

Lemma 3.1. *Assume ν is almost decreasing, i.e., satisfies* (A). *For a measurable set $E \subset \mathbb{R}^d$ such that $|E| < \infty$ the following estimate is true*

$$\int_{E^c} \nu(x-y)\,\mathrm{d}y \geq \kappa^2 \frac{w^p(|E|)}{|E|} \quad \textit{for all } x \in \mathbb{R}^d.$$

Proof. Let the ball $B(0, r_E)$ centered at the origin with radius r_E, be the symmetric rearrangement of E that is $|E| = |B(0, r_E)|$, equally we have the radius $r_E = (\frac{|E|}{c_d})^{1/d} = \eta(|E|)$ where $c_d = |B(0,1)|$. Noticing that $A \setminus B = A \setminus (A \cap B)$, one gets

$$\big|B(x, r_E) \setminus E\big| = \big|B(x, r_E)\big| - \big|E \cap B(x, r_E)\big| = \big|E\big| - \big|E \cap B(x, r_E)\big| = \big|E \setminus B(x, r_E)\big|.$$

Accordingly, using the fact that ν is almost decreasing, we get the sought estimate as follow

$$\begin{aligned}
\int_{E^c} \nu(x-y)\,\mathrm{d}y &= \int_{E^c \cap B(x, r_E)} \nu(x-y)\,\mathrm{d}y + \int_{E^c \cap B^c(x, r_E)} \nu(x-y)\,\mathrm{d}y \\
&\geq \kappa \nu(r_E) |E^c \cap B(0, r_E)| + \int_{E^c \cap B^c(x, r_E)} \nu(x-y)\,\mathrm{d}y \\
&= \kappa \nu(r_E) |E \cap B^c(0, r_E)| + \int_{E^c \cap B^c(x, r_E)} \nu(x-y)\,\mathrm{d}y \\
&\geq \kappa^2 \int_{E \cap B^c(x, r_E)} \nu(x-y)\,\mathrm{d}y + \int_{E^c \cap B^c(x, r_E)} \nu(x-y)\,\mathrm{d}y \\
&\geq \kappa^2 \int_{B^c(x, r_E)} \nu(x-y)\,\mathrm{d}y = \kappa^2 \frac{w^p(|E|)}{|E|}.
\end{aligned}$$

□

We also need the following lemma dealing with convexity of the critical function ϕ.

Lemma 3.2. *Assume that* ***Assumption*** *C is satisfied. Let $(a_k)_{k \in \mathbb{Z}}$ be a nonincreasing nonnegative sequence, i.e., $0 \leq a_{k+1} \leq a_k$, and $T > 0$. Then the following estimate holds true*

$$\phi_p(\frac{\theta^p}{T}) \sum_{k \in \mathbb{Z}} \Big(\frac{1}{\phi^{-1}(1/a_k)}\Big)^p T^k \leq \sum_{k \in \mathbb{Z}} \frac{a_{k+1}}{a_k} \Big(\frac{1}{\phi^{-1}(1/a_k)}\Big)^p T^k. \tag{3.1}$$

Proof. First, taking $s = \phi^{-1}(1/t')$ and $t = \phi^{-1}(1/s')$ the growth condition in (C) becomes

$$\phi_p^{-1}(\frac{s'}{t'}) w^p(t') \geq \theta^p w^p(s') \qquad \text{for all } s' \leq t'. \tag{3.2}$$

There is no loss of generality if we assume that the right hand side of (3.1) is finite and that,for $n \geq 1$ sufficiently large, $\lambda_n > 0$ with $\lambda_n = \sum_{|k| \leq n} w^p(a_k) T^k = \sum_{k \in \mathbb{Z}} w^p(a'_k) T^k$ where here, $a'_k = a_k$ if $|k| \leq n$ and 0 if $|k| > n$. This makes sense as $w(0) = \phi_p(0) = 0$. In virtue of the Jensen inequality and the estimate (3.2) we obtain the following estimates,

$$\begin{aligned}
\sum_{k \in \mathbb{Z}} \frac{a_{k+1}}{a_k} w^p(a_k) T^k &\geq \lambda_n \sum_{k \in \mathbb{Z}} \phi_p\big(\phi_p^{-1}(\frac{a'_{k+1}}{a_k})\big) \frac{1}{\lambda_n} w^p(a_k) T^k \\
&\geq \lambda_n \phi_p\Big(\frac{1}{\lambda_n} \sum_{k \in \mathbb{Z}} \phi_p^{-1}(\frac{a'_{k+1}}{a_k}) w^p(a_k) T^k\Big) \\
&\geq \lambda_n \phi_p\Big(\frac{\theta^p}{\lambda_n T} \sum_{k \in \mathbb{Z}} w^p(a'_k) T^k\Big) = \phi_p(\frac{\theta^p}{T}) \sum_{|k| \leq n} w^p(a_k) T^k.
\end{aligned}$$

Letting $n \to \infty$ gives the sought inequality since $w(t) = \frac{1}{\phi^{-1}(1/t)}$ as in (B). □

In connection with Lemma 3.2, we take $\phi(t) = t^q$, $q \geq 1$ to obtain the following particular result.

Lemma 3.3. *For a nonnegative sequence* $(a_k)_{k\in\mathbb{Z}}$, $T>0$ *and* $q\geq 1$ *we have*

$$\sum_{k\in\mathbb{Z}} a_k^{1/q}T^k \leq T^q \sum_{k\in\mathbb{Z}} \frac{a_{k+1}}{a_k} a_k^{1/q}T^k. \tag{3.3}$$

Proof. It suffices to assume that $0<\sum_{k\in\mathbb{Z}} a_k^{1/q}T^k<\infty$. Let the counting measure $\mathrm{d}\mu(k)=a_k^{1/q}T^k$ so that $\mathrm{d}\mu(k+1)=T\Big(\frac{a_{k+1}}{a_k}\Big)^{1/q}\mathrm{d}\mu(k)$. Jensen's inequality yields the sought inequality since

$$1=\Big(\fint_{\mathbb{Z}}\mathrm{d}\mu(k)\Big)^q=\Big(\fint_{\mathbb{Z}} T\Big(\frac{a_{k+1}}{a_k}\Big)^{1/q}\mathrm{d}\mu(k)\Big)^q\leq \fint_{\mathbb{Z}} T^q\frac{a_{k+1}}{a_k}\,\mathrm{d}\mu(k).$$

□

Consequently, taking $q=p_s^*/p\geq 1$ in Lemma 3.3 results with the following inequality; compare with [NPV12, Lemma 6.2] or [SV11, Lemma 5].

Corollary 3.4. *Let* $s\in(0,1)$ *and* $p\geq 1$ *be such that* $p_s^*>0$, *and* $T>0$. *For every nonnegative sequence* $(a_k)_{k\in\mathbb{Z}}$ *the following estimate holds*

$$\sum_{k\in\mathbb{Z}} a_k^{(d-sp)/d}T^k\leq T^{d/(d-sp)}\sum_{k\in\mathbb{Z}} a_{k+1}a_k^{-sp/d}T^k.$$

The next lemma is an immediate consequence of the relation in (1.7) and provides an interplay between a Luxemburg norm associated with a function ψ and that of the mapping $t\mapsto\psi(t^q)$.

Lemma 3.5. *Let* $\psi:[0,\infty]\to[0,\infty]$ *be a Young function and* $q>0$. *Assume* $\overline{\psi}_q(t)=\psi(t^q)$ *is also a Young function then* $u\in L^{\overline{\psi}_q}(\mathbb{R}^d)$ *if and only if* $u^q\in L^{\psi}(\mathbb{R}^d)$. *Moreover, we have*

$$\|u\|_{L^{\overline{\psi}_q}(\mathbb{R}^d)}=\|u^q\|^{1/q}_{L^{\psi}(\mathbb{R}^d)}.$$

We are now in position to prove Theorem 1.1. Our approach uses the measure theoretic decomposition of functions by level sets.

Proof of Theorem 1.1. Without loss of the generality assume that $u\geq 0$ and $|u|_{W^p_\nu(\mathbb{R}^d)}<\infty$. For each $k\in\mathbb{Z}$ define

$$\begin{aligned} A_k&=\big\{u>2^k\big\}\quad\text{and}\quad D_k=A_k\setminus A_{k+1}=\big\{2^k<u\leq 2^{k+1}\big\},\\ a_k&=\big|\{u>2^k\}\big|\quad\text{and}\quad d_k=\big|D_k\big|=a_k-a_{k+1}.\end{aligned}$$

Note that $A_{k+1}\subset A_k$ and hence $a_{k+1}\leq a_k$. Moreover, $D_k's$ are disjoints, cover $\mathbb{R}^d$ and we get

$$A^c_{k+1}=\bigcup_{\ell\leq k}D_\ell\quad\text{and}\quad A_k=\bigcup_{\ell\geq k}D_\ell. \tag{3.4}$$

Accordingly we find that

$$a_k=\sum_{\ell\geq k}d_\ell\quad\text{and}\quad d_k=a_k-\sum_{\ell\geq k+1}d_\ell. \tag{3.5}$$

Given $x \in D_i$ and $y \in D_j$ with $j \leq i-2$, we have $|u(x)-u(y)| \geq 2^i - 2^{j+1} \geq 2^{i-1}$. Therefore, according to (3.4) and Lemma 3.1, one deduces the following

$$\begin{aligned}\iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} |u(x)-u(y)|^p\nu(x-y)\,\mathrm{d}y\,\mathrm{d}x &= \sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} \iint\limits_{D_iD_j} |u(x)-u(y)|^p\nu(x-y)\,\mathrm{d}y\,\mathrm{d}x \\ &\geq 2\sum_{i\in\mathbb{Z}}\sum_{\substack{j\in\mathbb{Z}\\ i\leq j-2}} \iint\limits_{D_iD_j} 2^{p(i-1)}\nu(x-y)\,\mathrm{d}y\,\mathrm{d}x \\ &= 2\sum_{i\in\mathbb{Z}} 2^{p(i-1)}\int_{D_i}\int_{A^c_{i-1}} \nu(x-y)\,\mathrm{d}y\,\mathrm{d}x \\ &\geq 2\kappa^2\sum_{i\in\mathbb{Z}} 2^{p(i-1)} d_i \frac{w^p(a_{i-1})}{a_{i-1}}.\end{aligned}$$

In short, using the relation (B) we have

$$|u|^p_{W^p_\nu(\mathbb{R}^d)} \geq 2\kappa^2\sum_{k\in\mathbb{Z}} 2^{pk}\frac{d_{k+1}}{a_k}\Big(\frac{1}{\phi^{-1}(1/a_k)}\Big)^p. \tag{3.6}$$

Recalling (3.5) and that $d_i = a_i - a_{i+1}$, we get

$$|u|^p_{W^p_\nu(\mathbb{R}^d)} \geq 2\kappa^2\sum_{\substack{i\in\mathbb{Z}\\ a_{i-1}\neq 0}} 2^{p(i-1)} a_i\frac{w^p(a_{i-1})}{a_{i-1}} - 2\kappa^2\sum_{\substack{i\in\mathbb{Z}\\ a_{i-1}\neq 0}}\sum_{\substack{i\in\mathbb{Z}\\ j\geq i+1}} 2^{p(i-1)} d_j\frac{w^p(a_{i-1})}{a_{i-1}}. \tag{3.7}$$

Given that $d_j \leq a_j$, using Fubini's theorem and the formula $\sum_{\substack{i\in\mathbb{Z}\\ i\leq j-1}} c^{i-1} = \frac{c^{j-1}}{c-1}$ for $c>1$ implies

$$\begin{aligned}\sum_{\substack{i\in\mathbb{Z}\\ a_{i-1}\neq 0}}\sum_{\substack{j\in\mathbb{Z}\\ i+1\leq j}} 2^{p(i-1)} d_j\frac{w^p(a_{i-1})}{a_{i-1}} &= \sum_{\substack{j\in\mathbb{Z}\\ a_{j-1}\neq 0}}\sum_{\substack{i\in\mathbb{Z}\\ i\leq j-1}} 2^{p(i-1)} d_j\frac{w^p(a_{i-1})}{a_{i-1}} \\ &\leq \frac{1}{2^p-1}\sum_{\substack{j\in\mathbb{Z}\\ a_{j-1}\neq 0}} 2^{p(j-1)} a_j\frac{w^p(a_{j-1})}{a_{j-1}}.\end{aligned}$$

Combining this together with (3.6) and (3.7), recalling $C_p(t) = 2\kappa^2\frac{t^p-2}{t^p-1}$, $t>1$ yields

$$|u|^p_{W^p_\nu(\mathbb{R}^d)} \geq C_p(2)\sum_{k\in\mathbb{Z}} 2^{pk}\frac{a_{k+1}}{a_k}w^p(a_k) = C_p(2)\sum_{k\in\mathbb{Z}} 2^{pk}\frac{a_{k+1}}{a_k}\Big(\frac{1}{\phi^{-1}(1/a_k)}\Big)^p \tag{3.8}$$

In order to employ **Assumption** C, we emphasize that $d_{k+1} \leq a_{k+1} \leq a_k$. So that in virtue of Lemma 3.2 with $T = 2^p$ and the fact that ϕ^{-1} is nondecreasing, the following estimates hold true

$$\sum_{k\in\mathbb{Z}} 2^{pk}\frac{a_{k+1}}{a_k}\Big(\frac{1}{\phi^{-1}(1/a_k)}\Big)^p \geq \phi(\frac{\theta}{2})\sum_{k\in\mathbb{Z}} 2^{pk}\Big(\frac{1}{\phi^{-1}(1/a_k)}\Big)^p \geq \phi(\frac{\theta}{2})\sum_{k\in\mathbb{Z}} 2^{pk}\Big(\frac{1}{\phi^{-1}(1/d_k)}\Big)^p. \tag{3.9}$$

Finally, since $u^p = \sum_{k\in\mathbb{Z}} u^p\mathbb{1}_{D_k}$, in view of Lemma 3.5 and the relation (1.9) we find that

$$\|u\|^p_{L^\phi(\mathbb{R}^d)} = \|u^p\|_{L^{\phi_p}(\mathbb{R}^d)} \leq \sum_{k\in\mathbb{Z}} \|u^p\mathbb{1}_{D_k}\|_{L^{\phi_p}(\mathbb{R}^d)} \leq 2^p\sum_{k\in\mathbb{Z}} 2^{pk}\Big(\frac{1}{\phi^{-1}(1/d_k)}\Big)^p. \tag{3.10}$$

Merging together (3.8), (3.9) and (3.10) gives the desired inequality for $t=2$

$$|u|^p_{W^p_\nu(\mathbb{R}^d)} \geq \frac{1}{\Theta_2^p}\|u\|^p_{L^\phi(\mathbb{R}^d)}, \quad \Theta_2^p = 2^{p-1}[\kappa^2 C_p(2)\phi(\frac{\theta}{2})]^{-1}.$$

More generally, using the level sets decomposition $A_k(t) = \{u > t^k\}$ and $D_k(t) = \{t^k < u \leq t^{k+1}\}$, for $t \geq 2$, in place of A_k and D_k and repeating with a close look at the proof of the previous case, t=2, reveals the desired estimate (1.10). It immediately follows that the embedding $W^p_\nu(\mathbb{R}^d) \hookrightarrow L^\phi(\mathbb{R}^d)$ is continuous while the continuity of the embedding $W^{1,p}(\mathbb{R}^d) \hookrightarrow L^\phi(\mathbb{R}^d)$ stems from the continuous embedding $W^{1,p}(\mathbb{R}^d) \hookrightarrow W^p_\nu(\mathbb{R}^d)$; see the relation (2.1). This ends the proof of Theorem 1.1. □

In view of generalizing Theorem 1.1 we need to bring into play the symmetric rearrangement.

Symmetric rearrangement: Next, we want to weaken the assumptions on ν, by possibly enlarging them to the class of non-radial kernels. Ultimately, we recall some essential notions of symmetric decreasing rearrangement; see for instance [Bae19, Gra08, LL01] for more details. Let $E \subset \mathbb{R}^d$ be a measurable set with $|E| < \infty$. The symmetric rearrangement of E denoted $E^* = B(0, r_E)$ is the open ball having the same volume with E, i.e., $r_E = \eta(|E|) = \big(\frac{|E|}{c_d}\big)^{1/d}$. The symmetric rearrangement of a measurable function $u : \mathbb{R}^d \to \mathbb{R}$, is the measurable function denoted $u^* : \mathbb{R}^d \to [0, \infty]$ and defined by

$$u^*(x) = u^*(|x|) = \int_0^\infty \mathbb{1}_{\{|u|>s\}^*}(x)\,\mathrm{d}s = \inf\{s > 0 :\ |\{|u| > s\}| \le c_d|x|^d\}. \tag{3.11}$$

It is a routine to check that the identity in (3.11) holds true. Obviously the function u^* is radial and radially nonincreasing, i.e., $u^*(x) \le u^*(y)$ whenever $|x| \ge |y|$. Furthermore, $\{|u| > s\}^* = \{u^* > s\}$, for all $s > 0$. This implies that u^* and u are equimeasurables, i.e., $|\{|u| > s\}| = |\{u^* > s\}|$ for all $s > 0$ and that u^* is lower semi-continuous. Next, assume $u^*(r) < \infty$, $r > 0$ and let $s_n = u^*(r) - \frac{1}{n}$, for $n \ge 1$ large. The inf characterization in (3.11) implies $s_n \notin \{s > 0 : |\{|u| > s\}| \le c_d r^d\}$. Thus, $|\{|u| > u^*(r) - \frac{1}{n}\}| > c_d r^d$ and hence letting $n \to \infty$ we get

$$|\{|u| \ge u^*(r)\}| \ge c_d r^d = |B(0, r)|.$$

Since u^* is radially nonincreasing, $\{u^* > u^*(r)\} \subset B(0, r) \subset \{u^* \ge u^*(r)\} \subset \overline{B}(0, r)$. Hence we get

$$|\{|u| > u^*(r)\}| = |\{u^* > u^*(r)\}| \le |B(0, r)| = |\{u^* \ge u^*(r)\}|.$$

This combined with the previous inequality yields

$$|\{|u| > u^*(r)\}| \le |B(0, r)| \le |\{|u| \ge u^*(r)\}|. \tag{3.12}$$

The equalities hold if u^* is decreasing. The next result, compare with [JW19, Lemma 3.1], is a fair generalization of Lemma 3.1.

Theorem 3.6. *Let $\nu : \mathbb{R}^d \setminus \{0\} \to [0, \infty]$ be measurable and note ν^* be its symmetric rearrangement. For $x \in \mathbb{R}^d$ and a measurable $E \subset \mathbb{R}^d$ such that $|E| < \infty$, the following inequality holds*

$$\int_{E^c} \nu(x - y)\,\mathrm{d}y \ge \nu^{\#}(|E|), \quad \textit{with} \quad \nu^{\#}(|E|) = \int_{\{\nu<\nu^*(r_E)\}} \nu(h)\,\mathrm{d}h. \tag{3.13}$$

Recall that $r_E = \eta(|E|) = \big(\frac{|E|}{c_d}\big)^{1/d}$. Moreover, $\nu^{\#}(|E|) \to \int_{\mathbb{R}^d} \nu(h)\,\mathrm{d}h$ as $|E| \to 0$ and if $\nu \in L^1(\mathbb{R}^d \setminus B(0, \delta))$ for some $\delta > 0$ then $\nu^{\#}(|E|) \to 0$ as $|E| \to \infty$.

Proof. It is sufficient to assume $\int_{E^c} \nu(x - y)\,\mathrm{d}y < \infty$. Let $E_x = x + E$ so that $|E_x| = |E|$. We get

$$\begin{aligned}
\int_{E^c} \nu(x - y)\,\mathrm{d}y &= \int_{E_x^c} \nu(h)\,\mathrm{d}h = \Big[\int_{\{\nu<\nu^*(r_E)\}} - \int_{E_x\cap\{\nu<\nu^*(r_E)\}} + \int_{E_x^c\cap\{\nu\ge\nu^*(r_E)\}}\Big]\nu(h)\,\mathrm{d}h \\
&\ge \int_{\{\nu<\nu^*(r_E)\}} \nu(h)\,\mathrm{d}h - \nu^*(r_E)|E_x \cap \{\nu < \nu^*(r_E)\}| + \nu^*(r_E)|E_x^c \cap \{\nu \ge \nu^*(r_E)\}| \\
&= \int_{\{\nu<\nu^*(r_E)\}} \nu(h)\,\mathrm{d}h + \nu^*(r_E)\big(|E_x^c \cap \{\nu \ge \nu^*(r_E)\}| - |E_x \cap \{\nu < \nu^*(r_E)\}|\big) \\
&\ge \int_{\{\nu<\nu^*(r_E)\}} \nu(h)\,\mathrm{d}h.
\end{aligned}$$

The last inequality follows since, as inequality (3.12) implies $|\{\nu \ge \nu^*(r_E)\}| \ge |E|$, we have

$$\begin{aligned}
&|E_x^c \cap \{\nu \ge \nu^*(r_E)\}| - |E_x \cap \{\nu < \nu^*(r_E)\}| \\
&\qquad = (|\{\nu \ge \nu^*(r_E)\}| - |E_x \cap \{\nu \ge \nu^*(r_E)\}|) - (|E| - |E_x \cap \{\nu \ge \nu^*(r_E)\}|) \\
&\qquad = |\{\nu \ge \nu^*(r_E)\}| - |E| \ge |B(0, r_E)| - |E| = 0.
\end{aligned}$$

Meanwhile, $\nu^*(0) = \inf\{s > 0 : \ |\{\nu > s\}| = 0\} = \operatorname{ess\,sup}\nu = \|\nu\|_{L^\infty(\mathbb{R}^d)}$, hence we find that $\nu^{\#}(|E|) \to \int_{\mathbb{R}^d} \nu(h)\,\mathrm{d}h$ as $|E| \to 0$. If $\in L^1(\mathbb{R}^d \setminus B(0,\delta))$, then a convergence argument implies $\nu^{\#}(|E|) \to 0$ as $|E| \to \infty$. □

We mention in passing that Lemma 3.1 or Theorem 3.6 generalizes [NPV12, Lemma 6.1] focusing on the particular kernel, $\nu(h) = |h|^{-d-sp}$, $h \neq 0$ and $s \in (0,1)$. Indeed, in this case $\nu = \nu^*$ and hence for all $x \in \mathbb{R}^d$

$$\int_{E^c} \frac{\mathrm{d}y}{|x-y|^{d+sp}} \geq \nu^{\#}(|E|) = \gamma_s |E|^{-sp/d}, \quad \text{with } \gamma_s = \frac{dc_d^{1+\frac{sp}{d}}}{sp}. \tag{3.14}$$

We are now ready to state a refined version of Theorem 1.1 under weaker assumptions.

Theorem 3.7. *Let $\nu : \mathbb{R}^d \setminus \{0\} \to [0,\infty]$ be measurable and note ν^* its symmetric rearrangement. Assume that $w : [0,\infty] \to [0,\infty]$ with $w(r) = (r\nu^{\#}(r))^{1/p}$ is invertible. We emphasize that*

$$w(r) = \Big(|B(0,\eta(r))| \int_{\{\nu<\nu^*(\eta(r))\}} \nu(h)\,\mathrm{d}h\Big)^{1/p}, \qquad \eta(r) = \big(\frac{r}{c_d}\big)^{1/d}.$$

Moreover, denote the critical function by $\phi(t) = 1/w^{-1}(1/t)$ and assume that $t \mapsto \phi_p(t) = \phi(t^{1/p})$ is a Young function and there is $\theta > 0$ such that

$$\phi(\theta \frac{s}{t}) \leq \frac{\phi(s)}{\phi(t)} \qquad \text{for all } \ 0 \leq s \leq t.$$

For $t \geq 2$, define $\Theta_t = t[2C_p(t)\phi(\frac{\theta}{t})]^{-1/p}$ with $C_p(t) = \frac{t^p-2}{t^p-1}$. Then the following inequality holds

$$\|u\|_{L^\phi(\mathbb{R}^d)} \leq \Theta_t \Big(\iint_{\mathbb{R}^d\,\mathbb{R}^d} |u(x)-u(y)|^p \nu(x-y)\,\mathrm{d}y\,\mathrm{d}x \Big)^{1/p} \quad \text{for all} \quad u \in L^\phi(\mathbb{R}^d).$$

Proof. The proof follows exactly the lines of the proof of Theorem 1.1, as the only major change is the analog of the estimate (3.6) which easily derives from Theorem 3.6. □

Remark 3.8. The assertions $(i) - (vii)$ of Proposition 2.2 remain true for a general kernel ν such that $w(r) = (r\nu^{\#}(r))^{1/p}$ and $\phi(t) = 1/w^{-1}(1/t)$ exist.

We now present two contexts in which, it is possible to eschew the lack of certain assumptions of Theorem 3.7 with a similar conclusion. The first context implies that lack of growth condition (C) may sometime not be a direct obstacle in order to the Sobolev inequality and the second context implies that one can correct the lack of convexity. This is goal of the next result.

Theorem 3.9. *Let $\nu : \mathbb{R}^d \setminus \{0\} \to [0,\infty]$ be measurable and ϕ be the corresponding critical function. Then the following inequality remains valid*

$$\|u\|_{L^\phi(\mathbb{R}^d)} \leq C \Big(\iint_{\mathbb{R}^d\,\mathbb{R}^d} |u(x)-u(y)|^p \nu(x-y)\,\mathrm{d}y\,\mathrm{d}x \Big)^{1/p} \quad \text{for all} \quad u \in L^\phi(\mathbb{R}^d),$$

if there exist $\nu_i : \mathbb{R}^d \setminus \{0\} \to [0,\infty]$ associated with ϕ_i, $i = 1,2$, each $t \mapsto \phi_i(t^{1/p})$ is convex and satisfies the growth condition: $\phi_i(t)\phi_i\big(\theta_i \frac{s}{t}\big) \leq \phi_i(s)$ for all $s \leq t$; such that (i) or (ii) below holds.

- (i) *The function $\phi(t) = \max(\phi_1(t), \phi_2(t))$, i.e. does not necessarily satisfy the growth condition* (C)*, and there are constants $c_2 \geq c_1 > 0$ such that $c_1\nu(h) \leq \max(\nu_1(h), \nu_2(h)) \leq c_2\nu(h)$.*
- (ii) *The function $\phi(t) = \min(\phi_1(t), \phi_2(t))$, i.e. ϕ might not be convex, is identified $\phi_{\min}$ with*

$$\phi_{\min}(t) = \int_0^t \frac{\min(\phi_1(s), \phi_2(s))}{s}\,\mathrm{d}s.$$

Proof. To prove (i) assume $\phi(t) = \max(\phi_1(t), \phi_2(t))$, which, for instance as in Example 2.10 $t \mapsto \max(t^{p^*_{s_1}}, t^{p^*_{s_2}})$ with $p^*_{s_2} > p^*_{s_1} \geq p$ does not necessarily satisfy the growth condition (C). As Theorem 3.7 is true for each kernel ν_i, the desired inequality follows since by Theorem 2.9 $\|u\|_{L^\phi(\mathbb{R}^d)} \leq 2\max(\|u\|_{L^{\phi_1}(\mathbb{R}^d)}, \|u\|_{L^{\phi_2}(\mathbb{R}^d)})$ and then for $u \in L^\phi(\mathbb{R}^d) = L^{\phi_1}(\mathbb{R}^d) \cap L^{\phi_2}(\mathbb{R}^d)$ we get

$$\begin{aligned}\|u\|_{L^{\phi_i}(\mathbb{R}^d)} &\leq C_i\Big(\iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} |u(x)-u(y)|^p\nu_i(x-y)\,\mathrm{d}y\,\mathrm{d}x\Big)^{1/p}\\ &\leq C\Big(\iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} |u(x)-u(y)|^p\nu(x-y)\,\mathrm{d}y\,\mathrm{d}x\Big)^{1/p},\ C = \max(C_1, C_2)c_2^{1/p}.\end{aligned}$$

Next, to prove (ii) assume $\phi(t) = \min(\phi_1(t), \phi_2(t))$, which, for instance as in Example 2.11 $t \mapsto \min(t^{p^*_{s_1}}, t^{p^*_{s_2}})$ with $p^*_{s_2} > p^*_{s_1} \geq p$, is not necessarily convex. In virtue of Proposition 2.2 each mapping $t \mapsto \frac{\phi_i(t^{1/p})}{t}$, $i = 1, 2$ is increasing and hence, $t \mapsto \phi_{\min}(t^{1/p})$ is convex since

$$\phi_{\min}(t^{1/p}) = \frac{1}{p}\int_0^t \frac{\min(\phi_1(s^{1/p}), \phi_2(s^{1/p}))}{s}\,\mathrm{d}s.$$

Moreover, putting $\theta' = \min(\theta_1, \theta_2)$ one easily checks that

$$\phi\big(\theta'\frac{s}{t}\big) \leq \frac{\phi(s)}{\phi(t)} \quad \text{for all } s \leq t \quad \text{ and } \quad \phi_{\min}(t) \leq \phi(t) \leq \phi_{\min}(2t) \quad \text{for all } t > 0.$$

Altogether, this implies that $\phi_{\min}$ satisfies the growth condition (C) with $\theta = \frac{1}{2}\min(\theta_1, \theta_2)$. Indeed,

$$\phi_{\min}\big(\frac{\theta' s}{2t}\big) \leq \phi\big(\frac{\theta' s}{2t}\big) \leq \frac{\phi(s/2)}{\phi(t)} \leq \frac{\phi_{\min}(s)}{\phi_{\min}(t)} \quad \text{for } s \leq t.$$

It turns out that Lemma 3.2 applies to $\phi_{\min}$ and hence, since $\phi^{-1}(1/r) \leq \phi^{-1}_{\min}(1/r) \leq 2\phi^{-1}(1/r)$, as a substitute for the inequality(3.1) one readily obtains that

$$\phi_{\min}\big(\frac{\theta}{T^{1/p}}\big)\sum_{k\in\mathbb{Z}}\Big(\frac{1}{\phi^{-1}(1/d_k)}\Big)^p T^k \leq 2^p\sum_{k\in\mathbb{Z}}\frac{d_{k+1}}{a_k}\Big(\frac{1}{\phi^{-1}(1/a_k)}\Big)^p T^k. \tag{3.1'}$$

Therefore, a mere adaptation of the proof Theorem 1.1 provides the desired inequality. □

It is worth recalling that by Example 2.5, the fractional Gagliardo-Nirenberg-Sobolev inequality (1.4) is a direct consequence of Theorem 1.1, where, $\nu(h) = |h|^{-d-sp}$ is associated with the critical Young function $\phi(t) = \gamma_s^{p^*_s/p} t^{p^*_s}$. For the reader convenience we however offer an alternative proof incorporated in [Pon16, Proposition 15.5].

Theorem 3.10 (Gagliardo-Nirenberg-Sobolev)**.** *Let $s \in (0,1)$ such that $\frac{1}{p^*_s} = \frac{1}{p} - \frac{s}{d} > 0$ then*

$$\|u\|_{L^{p^*_s}(\mathbb{R}^d)} \leq 2^{p^*_s/p}|B(0,1)|^{-1/p-s/d}\Big(\iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} \frac{|u(x)-u(y)|^p}{|x-y|^{d+sp}}\,\mathrm{d}x\,\mathrm{d}y\Big)^{1/p} \quad \textit{for all} \quad u \in L^{p^*_s}(\mathbb{R}^d).$$

Proof. Fix $x \in \mathbb{R}^d$ and $r > 0$. Integrating the inequality $|u(x)| \leq |u(y)| + |u(x) - u(y)|$ over $y \in B(x,r)$ and using Jensen's inequality implies

$$\begin{aligned}|u(x)| &\leq \fint_{B(x,r)} |u(y)|\,\mathrm{d}y + \fint_{B(x,r)} |u(x)-u(y)|\,\mathrm{d}y\\ &\leq \Big(\fint_{B(x,r)} |u(y)|^{p^*_s}\,\mathrm{d}y\Big)^{1/p^*_s} + \Big(\fint_{B(x,r)} |u(x)-u(y)|^p\,\mathrm{d}y\Big)^{1/p}\\ &\leq \Big(\fint_{B(x,r)} |u(y)|^{p^*_s}\,\mathrm{d}y\Big)^{1/p^*_s} + \Big(r^{d+sp}\fint_{B(x,r)} \frac{|u(x)-u(y)|^p}{|x-y|^{d+sp}}\,\mathrm{d}y\Big)^{1/p}\\ &\leq r^{-d/p^*_s}|B(0,1)|^{-1/p^*_s}\Big(\int_{\mathbb{R}^d} |u(y)|^{p^*_s}\,\mathrm{d}y\Big)^{1/p^*_s} + r^s|B(0,1)|^{-1/p}\Big(\int_{\mathbb{R}^d} \frac{|u(x)-u(y)|^p}{|x-y|^{d+sp}}\,\mathrm{d}y\Big)^{1/p}.\end{aligned}$$

Now we choose r such that both summands of the last inequality are equals. To more be precised,

$$r(x) = r = |B(0,1)|^{1/d-p/dp_s^*}\Big(\int_{\mathbb{R}^d}|u(y)|^{p_s^*}\,\mathrm{d}y\Big)^{p/dp_s^*}\Big(\int_{\mathbb{R}^d}\frac{|u(x)-u(y)|^p}{|x-y|^{d+sp}}\,\mathrm{d}y\Big)^{-1/d}.$$

Substituting this specific $r(x)$ in the preceding estimate leads to

$$\begin{aligned}|u(x)|^{p_s^*} &\le 2^{p_s^*}r^{-d}(x)|B(0,1)|^{-1}\Big(\int_{\mathbb{R}^d}|u(y)|^{p_s^*}\,\mathrm{d}y\Big)\\ &= 2^{p_s^*}|B(0,1)|^{-2+p/p_s^*}\Big(\int_{\mathbb{R}^d}|u(y)|^{p_s^*}\,\mathrm{d}y\Big)^{1-p/p_s^*}\Big(\int_{\mathbb{R}^d}\frac{|u(x)-u(y)|^p}{|x-y|^{d+sp}}\,\mathrm{d}y\Big).\end{aligned}$$

This implies an equivalent of the sought inequality, as integrating with respect to x yields,

$$\int_{\mathbb{R}^d}|u(x)|^{p_s^*}\,\mathrm{d}x \le 2^{p_s^*}|B(0,1)|^{-1-sp/d}\Big(\int_{\mathbb{R}^d}|u(y)|^{p_s^*}\,\mathrm{d}y\Big)^{1-p/p_s^*}\Big(\iint_{\mathbb{R}^d\,\mathbb{R}^d}\frac{|u(x)-u(y)|^p}{|x-y|^{d+sp}}\,\mathrm{d}y\,\mathrm{d}x\Big).$$

□

An immediate consequence of Theorem 3.7 is given by the following Sobolev type embeddings.

Corollary 3.11. *Assume the assumptions of Theorem 3.7 are in force, with $\nu \in L^1(\mathbb{R}^d, 1\wedge|h|^p\,\mathrm{d}h)$. Let ψ be a Young function such that $\psi(ct) \le \max(t^p, \phi(t))$ for all $t > 0$ and for some constant $c > 0$. The embeddings $W^p_\nu(\mathbb{R}^d) \hookrightarrow L^\psi(\mathbb{R}^d)$ and $W^{1,p}(\mathbb{R}^d) \hookrightarrow L^\psi(\mathbb{R}^d)$ are continuous.*

In particular, we obtain the classical fractional Sobolev embedding, that is for $s \in (0,1)$, if $p_s^* > 0$ then for every $q \in [p, p_s^*]$ the embedding $W^{s,p}(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d)$ is continuous.

More generally, in order to capture the above embeddings in Corollary 3.11 on arbitrarily open sets, we need to introduce extension domain with respect to the kernel ν.

Definition 3.12. An open set $\Omega \subset \mathbb{R}^d$ will be called an W^p_ν-extension domain if there exists a linear operator $E : W^p_\nu(\Omega) \to W^p_\nu(\mathbb{R}^d)$ and a constant $C := C(\Omega, \nu, d, p)$ such that

$$Eu\mid_\Omega = u \qquad \text{and} \qquad \|Eu\|_{W^p_\nu(\mathbb{R}^d)} \le C\|u\|_{W^p_\nu(\Omega)}, \quad \text{for all} \quad u \in W^p_\nu(\Omega).$$

According to [Zho15], an open set $\Omega \subset \mathbb{R}^d$ is an $W^{s,p}$-extension domain if and only if Ω satisfies the measure density condition, i.e., there is $c > 0$ such that for every $x \in \Omega$ and $r > 0$ we have $|\Omega \cap B(x,r)| > cr^d$. For some authors this condition also means that Ω is a d-set. If ν is radially decreasing then bounded bi-Lipschitz domain $\Omega \subset \mathbb{R}^d$ is an W^p_ν-extension domain; see [FG20, Theorem 3.78].

Corollary 3.13. *Let the assumptions of Theorem 3.7 be in force. Assume $\Omega \subset \mathbb{R}^d$ is an W^p_ν-extension domain. Let ψ be a Young function such that $\psi(ct) \le \max(t^p, \phi(t))$ for all $t > 0$ and for some constant $c > 0$. The embeddings $W^p_\nu(\Omega) \hookrightarrow L^\psi(\Omega)$ is continuous. Moreover, if $|\Omega| < \infty$, then $W^p_\nu(\Omega) \hookrightarrow L^\psi(\Omega)$ is continuous when $\psi(ct) \le \max(t, \phi(t))$ for all $t \ge 1$. In particular, if $\nu(h) = |h|^{-d-sp}$, $s \in (0,1)$ with $p_s^* > 0$ then $W^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$ is continuous for every $q \in [p, p_s^*]$ and for every $q \in [1, p_s^*]$ if $|\Omega| < \infty$.*

Proof. Clearly, Theorem 3.7 and the extension property of Ω imply $W^p_\nu(\Omega) \hookrightarrow L^\phi(\Omega)$ and we naturally have $W^p_\nu(\Omega) \hookrightarrow L^p(\Omega)$. Hence $W^p_\nu(\Omega) \hookrightarrow L^p(\Omega) \cap L^\phi(\Omega) = L^{\max(t^p,\phi(t))}(\Omega)$, by Theorem 2.3. Since $\psi(ct) \le \max(t^p, \phi(t))$, Theorem 2.9 implies that $W^p_\nu(\Omega) \hookrightarrow L^\psi(\mathbb{R}^d)$. Analogously, if $|\Omega| < \infty$, then we have $L^p(\Omega) \hookrightarrow L^1(\Omega)$ so that we have $W^p_\nu(\Omega) \hookrightarrow L^p(\Omega) \cap L^\phi(\Omega) = L^{\max(t,\phi(t))}(\Omega)$ and if then by Theorem 2.3 we get $W^p_\nu(\Omega) \hookrightarrow L^p(\Omega) \cap L^\phi(\Omega) = L^{\psi(t)}(\Omega)$. Last, noting, $q \in [p, p_s^*]$ if and only if $t^q \le \max(t^p, t^{p_s^*})$, it follows that $W^{s,p}(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d)$ for every $q \in [p, p_s^*]$. The case $|\Omega| < \infty$ and $q \in [1, p_s^*]$ follows analogously. □

4. Poincaré-Sobolev inequality

In this section, we wish to establish Gagliardo-Nirenberg-Sobolev inequality for functions restricted on a domain $\Omega \subset \mathbb{R}^d$. First and foremost, observe that the inequality

$$\|u\|_{L^\phi(\Omega)} \leq C\Big(\iint\limits_{\Omega\Omega} |u(x)-u(y)|^p \nu(x-y)\,\mathrm{d}y\,\mathrm{d}x\Big)^{1/p} \quad \text{for all} \quad u \in L^\phi(\Omega),$$

cannot hold for an arbitrary bounded $\Omega \subset \mathbb{R}^d$. In fact, if u is a nonzero constant, then the right-hand side is zero but the left-hand side is not. Accordingly, we need to rule out the constant functions in this context. For instance, if we replace the integrand on the left-hand side by $u - \fint_\Omega u$ then it fully makes sense to think of an inequality of the form

$$\big\|u - \fint_\Omega u\big\|_{L^\phi(\Omega)} \leq C\Big(\iint\limits_{\Omega\Omega} |u(x)-u(y)|^p \nu(x-y)\,\mathrm{d}y\,\mathrm{d}x\Big)^{1/p} \quad \text{for all} \quad u \in L^\phi(\Omega),$$

where we write $\fint_\Omega u = \frac{1}{|\Omega|}\int_\Omega u(x)\,\mathrm{d}x$ to denote the mean value of u over Ω. This particular type of inequality is customarily well known as a Sobolev-Poincaré type inequality and turns out to have a strong reciprocity with the Poincare type inequalities. To be more precise, see Theorem 4.3, the validity of Sobolev-Poincaré type inequality implies that of Poincaré type inequality vice versa. Let us recall some Poincaré type inequalities of interest here. Another consequence of Theorem 3.6 is given by the Poincaré-Friedrichs type inequality.

Theorem 4.1 (Poincaré type inequality)**.** *Let $\Omega \subset \mathbb{R}^d$ be measurable with $|\Omega| < \infty$ and let $\nu : \mathbb{R}^d \setminus \{0\} \to [0,\infty]$ be measurable with full support.*

Poincaré-Friedrichs inequality: *Let $L^p_\Omega(\mathbb{R}^d) = \{u \in L^p(\mathbb{R}^d) :\ u = 0,\ a.e \text{ on } \Omega^c\}$ and let $\nu^\#$ be defined as in (3.13). Letting $C = [2\nu^\#(|\Omega|)]^{-1/p}$, the following inequality holds*

$$\|u\|_{L^p(\Omega)} \leq C\Big(\iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} |u(x)-u(y)|^p \nu(x-y)\,\mathrm{d}y\,\mathrm{d}x\Big)^{1/p} \quad \textit{for all} \quad u \in L^p_\Omega(\mathbb{R}^d).$$

Poincaré inequality: *Assume ν is radially almost decreasing, i.e. $\kappa\nu(|x|) \leq \nu(|y|)$ if $|x| \geq |y|$, then letting $C = [\kappa|\Omega|\nu(R)]^{-1/p}$ where $R = \operatorname{diam}(\Omega)$ is the diameter of Ω, we have*

$$\big\|u - \fint_\Omega u\big\|_{L^p(\Omega)} \leq C\Big(\iint\limits_{\Omega\Omega} |u(x)-u(y)|^p \nu(x-y)\,\mathrm{d}y\,\mathrm{d}x\Big)^{1/p} \quad \textit{for all } u \in L^p(\Omega).$$

Proof. If $u = 0$ a.e. on Ω^c, the Theorem 3.6 yields the Poincaré-Friedrichs inequality as follows

$$|u|^p_{W^p_\nu(\mathbb{R}^d)} \geq 2\int_\Omega |u(x)|^p\,\mathrm{d}x \int_{\Omega^c} \nu(x-y)\,\mathrm{d}y \geq 2\nu^\#(|\Omega|)\|u\|^p_{L^p(\Omega)}.$$

Next let $R = \operatorname{diam}(\Omega)$ and assume ν is almost decreasing, so that we get $\nu(x-y) \geq \kappa\nu(R)$ for all $x, y \in \Omega$. Set $C^p = \kappa|\Omega|\nu(R)$. For $u \in L^p(\Omega)$, Jensen's inequality yields,

$$\iint\limits_{\Omega\Omega} |u(x)-u(y)|^p \nu(x-y)\,\mathrm{d}y\,\mathrm{d}x \geq C^p \int_\Omega \fint_\Omega |u(x)-u(y)|^p\,\mathrm{d}y\,\mathrm{d}x \geq C^p \big\|u - \fint_\Omega u\big\|^p_{L^p(\Omega)}.$$

□

It is still possible to obtain the Poincaré inequality if the almost decreasing condition on ν is dropped, by requiring the embedding $W^p_\nu(\Omega) \hookrightarrow L^p(\Omega)$ to be compact. The latter holds when certain compatibility regularity conditions are imposed on both kernel ν and domain Ω. See for instance [FG20] for more on this topics. First, we immediately the following from Theorem 3.7

Corollary 4.2 (Poincaré-Friedrichs-Sobolev type inequality)**.** *Let* $\Omega \subset \mathbb{R}^d$ *be open and define* $L^{\phi}_{\Omega}(\mathbb{R}^d) = \{u \in L^{\phi}(\mathbb{R}^d) : \; u = 0, \; a.e \text{ on } \Omega^c\}$*. Under the assumptions of Theorem 3.7 we get that*

$$\|u\|_{L^{\phi}(\Omega)} \leq \Theta_t \Big(\iint\limits_{\mathbb{R}^d \mathbb{R}^d} |u(x) - u(y)|^p \nu(x-y) \, \mathrm{d}y \, \mathrm{d}x \Big)^{1/p} \quad \textit{for all } u \in L^{\phi}_{\Omega}(\mathbb{R}^d).$$

The next result, which is somewhat a side consequence of the definition of ϕ, shows the equivalence between the Sobolev inequality and the Poincaré-Sobolev inequality.

Theorem 4.3 (Poincaré-Sobolev inequality)**.** *Assume assumptions of Theorem 3.7 hold true. Let* $\Omega \subset \mathbb{R}^d$ *be measurable such that* $0 < |\Omega| < \infty$*. If the Poincaré-Sobolev inequality holds true* Ω*, i.e.,*

$$\big\|u - ⨍_{\Omega} u\big\|_{L^{\phi}(\Omega)} \leq C \Big(\iint\limits_{\Omega\Omega} |u(x) - u(y)|^p \nu(x-y) \, \mathrm{d}y \, \mathrm{d}x \Big)^{1/p} \quad \textit{for all } u \in L^{\phi}(\Omega), \tag{4.1}$$

then the Poincaré inequality is also holds true, i.e.,

$$\big\|u - ⨍_{\Omega} u\big\|_{L^{p}(\Omega)} \leq C \Big(\iint\limits_{\Omega\Omega} |u(x) - u(y)|^p \nu(x-y) \, \mathrm{d}y \, \mathrm{d}x \Big)^{1/p} \quad \textit{for all } u \in L^{p}(\Omega). \tag{4.2}$$

The converse holds true if in addition, Ω *is an* W^p_{ν}*-extension domain.*

Proof. Proposition 2.2 implies that $\phi(t) \geq \delta_0 t^p$ for all $t \geq t_0$ with fixed $t_0 > 0$. Thus, it follows from Theorem 2.3 that $L^{\phi}(\Omega) \hookrightarrow L^p(\Omega)$ is a continuous embedding by (2.6). Together with the Poincaré inequality we get

$$\big\|u - ⨍_{\Omega} u\big\|_{L^{p}(\Omega)} \leq C \big\|u - ⨍_{\Omega} u\big\|_{L^{\phi}(\Omega)} \leq C \Big(\iint\limits_{\Omega\Omega} |u(x) - u(y)|^p \nu(x-y) \, \mathrm{d}y \, \mathrm{d}x \Big)^{1/p}.$$

Conversely assume, the Poincaré inequality holds and Ω is an W^p_{ν}-extension domain. Let $\overline{u} \in W^p_{\nu}(\mathbb{R}^d)$ be an extension of $u_0 = u - ⨍_{\Omega} u$ with $u \in W^p_{\nu}(\Omega)$. Applying Theorem 3.7 reveals that $\overline{u} \in L^{\phi}(\mathbb{R}^d)$ and we deduce the Poincaré-Sobolev inequality as follows

$$\begin{aligned}
\big\|u - ⨍_{\Omega} u\big\|_{L^{\phi}(\Omega)} \leq \|\overline{u}\|_{L^{\phi}(\mathbb{R}^d)} &\leq C \Big(\iint\limits_{\mathbb{R}^d \mathbb{R}^d} |\overline{u}(x) - \overline{u}(y)|^p \nu(x-y) \, \mathrm{d}y \, \mathrm{d}x \Big)^{1/p} \\
&\leq C \Big(\int_{\Omega} |u(x) - ⨍_{\Omega} u|^p \, \mathrm{d}x + \iint\limits_{\Omega\Omega} |u_0(x) - u_0(y)|^p \nu(x-y) \, \mathrm{d}y \, \mathrm{d}x \Big)^{1/p} \\
&\leq C \Big(\iint\limits_{\Omega\Omega} |u(x) - u(y)|^p \nu(x-y) \, \mathrm{d}y \, \mathrm{d}x \Big)^{1/p}.
\end{aligned}$$

□

As a direct consequence of Theorem 3.7 and Theorem 4.3 combined with Theorem 4.1 we get.

Corollary 4.4. *Let the assumptions of Theorem 3.7 be in force. If* $\nu : \mathbb{R}^d \setminus \{0\} \to [0, \infty]$ *is almost decreasing and* $\Omega \subset \mathbb{R}^d$ *is an* W^p_{ν}*-extension domain then Poincaré-Sobolev inequality holds, that is, there is a the constant* $C = C(\Omega, \nu, p, d) > 0$ *only depends on* Ω, ν, d *and* p *such that*

$$\big\|u - ⨍_{\Omega} u\big\|_{L^{\phi}(\Omega)} \leq C \Big(\iint\limits_{\Omega\Omega} |u(x) - u(y)|^p \nu(x-y) \, \mathrm{d}y \, \mathrm{d}x \Big)^{1/p} \quad \textit{for all } \; u \in L^{\phi}(\Omega).$$

In particular, if $s \in (0,1)$ *and* $p^*_s > 0$ *then*

$$\big\|u - ⨍_{\Omega} u\big\|_{L^{p^*_s}(\Omega)} \leq C \Big(\iint\limits_{\Omega\Omega} \frac{|u(x) - u(y)|^p}{|x-y|^{d+sp}} \, \mathrm{d}y \, \mathrm{d}x \Big)^{1/p} \quad \textit{for all } \; u \in L^{p^*_s}(\Omega).$$

It turns out that, for the case of the fractional using a straightforwards scaling argument, there is constant $C = C(d,p,s)$ such that for all balls $B_r = B(0,r)$, $r>0$ we have

$$\|u - \fint_{B_r} u\|_{L^{p_s^*}(B_r)} \leq C\Big(\iint\limits_{B_r B_r} \frac{|u(x)-u(y)|^p}{|x-y|^{d+sp}} \,\mathrm{d}y\,\mathrm{d}x\Big)^{1/p} \quad \text{for all } u \in L^{p_s^*}(B_r). \tag{4.3}$$

In fact, letting $r \to \infty$, one can check that the uniform estimate in (4.3) implies the Gagliardo-Nirenberg-Sobolev inequality given in Theorem 3.10. More generally, we prove that the uniform Poincaré-Sobolev inequality on balls implies the nonlocal Gagliardo-Nirenberg-Sobolev inequality.

Theorem 4.5. *Let $\nu \in L^1(\mathbb{R}^d, 1\wedge |h|^p\,\mathrm{d}h)$ be nonnegative and let $\phi(t) = 1/w^{-1}(1/t)$ be defined as in Theorem 3.7, with $w(r) = (r\nu^{\#}(r))^{1/p}$. Assume there is a universal constant $\Theta > 0$ such that for all balls $B \subset \mathbb{R}^d$, the following Poincaré-Sobolev inequality holds true,*

$$\|u - \fint_B u\|_{L^\phi(B)} \leq \Theta\Big(\iint\limits_{BB} |u(x)-u(y)|^p \nu(x-y)\,\mathrm{d}y\,\mathrm{d}x\Big)^{1/p} \quad \textit{for all } u \in L^\phi(B). \tag{4.4}$$

Then following inequality holds true as well

$$\|u\|_{L^\phi(\mathbb{R}^d)} \leq \Theta\Big(\iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} |u(x)-u(y)|^p \nu(x-y)\,\mathrm{d}y\,\mathrm{d}x\Big)^{1/p} \quad \textit{for all } u \in L^\phi(\mathbb{R}^d). \tag{4.5}$$

Proof. Let $B_r = B(0,r)$, $r>0$ and recall that (1.9) implies $\|\mathbb{1}_{B_r}\|_{L^\phi(\mathbb{R}^d)} = 1/\phi^{-1}(1/|B_r|)$ then by assumption we have

$$\begin{aligned}
\|u\|_{L^\phi(B_r)} &\leq \|u - \fint_{B_r} u\|_{L^\phi(B_r)} + \|\mathbb{1}_{B_r}\|_{L^\phi(\mathbb{R}^d)}\Big|\fint_{B_r} u\Big| \\
&\leq \Theta\Big(\iint\limits_{B_r B_r} |u(x)-u(y)|^p \nu(x-y)\,\mathrm{d}y\,\mathrm{d}x\Big)^{1/p} + \|\mathbb{1}_{B_r}\|_{L^\phi(\mathbb{R}^d)} |B_r|^{-1/p}\Big(\int_{B_r} |u(x)|^p\,\mathrm{d}x\Big)^{1/p} \\
&\leq \Theta\Big(\iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} |u(x)-u(y)|^p \nu(x-y)\,\mathrm{d}y\,\mathrm{d}x\Big)^{1/p} + \frac{|B_r|^{-1/p}}{\phi^{-1}(1/|B_r|)}\Big(\int_{\mathbb{R}^d} |u(x)|^p\,\mathrm{d}x\Big)^{1/p}.
\end{aligned}$$

In virtue of Theorem 3.6, we know that $\nu^{\#}(r) \to 0$ as $r \to \infty$ so, by the definition of ϕ we obtain

$$\frac{|B_r|^{-1/p}}{\phi^{-1}(1/|B_r|)} = |B_r|^{-1/p} w(|B_r|) = (\nu^{\#}(r))^{1/p} \to 0 \quad \text{as} \quad r\to\infty.$$

Since $\Theta > 0$ is independent of r, tending $r \to \infty$ in the foregoing yields the desired inequality. ☐

Open Questions: (i) Regarding Theorem 3.9, can the growth condition (C) be improved? (ii) Is the critical function ϕ optimal? In a sense, if ψ is another Young function satisfying the inequality (1.10), then it also holds true for the Young function $t \mapsto \max(\phi(t), \psi(t))$. Are ϕ and ψ then comparable? Note that we call ϕ optimal if there is $c>0$ such that $\psi(ct) \leq \phi(t)$ for all $t>0$.

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