# STABILITY OF COMPLEMENT VALUE PROBLEMS FOR $p$-LÉVY OPERATORS 

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#### Abstract

We set-up a general framework tailor made to solve complement value problems governed by symmetric nonlinear integrodifferential $p$-Lévy operators. A prototypical example of integrodifferential $p$-Lévy operators is the well-known fractional $p$-Laplace operator. Our main focus is on nonlinear IDEs in presence of Dirichlet, Neumann and Robin conditions and we show well posedness results. Several results are new even for the fractional $p$-Laplace operator but we develop the approach for general translation-invariant nonlocal operators. We also bridge a gap from nonlocal to local, by showing that solutions to the local Dirichlet and Neumann boundary value problems associated with $p$-Laplacian are strong limits of the nonlocal ones.


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## 1. Introduction

In this article we study certain class of nonlinear IntegroDifferential Equations (IDEs) associated with symmetric nonlinear nonlocal operators of $p$-Lévy operators, $1<p<\infty$ which are integrodifferential operators of the form

$$
\begin{align*}
L u(x) & =2 \text { p.v. } \int_{\mathbb{R}^{d}}|u(x)-u(y)|^{p-2}(u(x)-u(y)) \nu(x-y) \mathrm{d} y, \quad\left(x \in \mathbb{R}^{d}\right) \\
& =2 \text { p.v. } \int_{\mathbb{R}^{d}} \psi(u(x)-u(y)) \nu(x-y) \mathrm{d} y \tag{1.1}
\end{align*}
$$

Throughout, we assume that $d \geq 1, \psi(t)=|t|^{p-2} t, t \in \mathbb{R}$ and the function $\nu: \mathbb{R}^{d} \backslash\{0\} \rightarrow[0, \infty)$ is the symmetric density of a symmetric $p$-Lévy measure, i.e., $\nu$ satisfies

$$
\begin{equation*}
\nu(h)=\nu(-h) \quad h \neq 0 \text { and } \quad \int_{\mathbb{R}^{d}}\left(1 \wedge|h|^{p}\right) \nu(h) \mathrm{d} h<\infty . \tag{L}
\end{equation*}
$$

Moreover, if $\nu$ is radial we use same notation for its radial profile, i.e., $\nu(h)=\nu(|h|)$. The notation p.v. stands for the principal value, whereas $a \wedge b$ denotes $\min (a, b), a, b \in \mathbb{R}$. More succinctly, given an open set $\Omega \subset \mathbb{R}^{d}$ we are interested in the problems of the forms

$$
L u=f \quad \text { in } \Omega \quad \text { and } \quad \tau \mathcal{N} u+(1-\tau) \beta|u|^{p-2} u=g \text { on } \Omega^{c}
$$

[^0]where $\tau \in\left\{0, \frac{1}{2}, 1\right\}$ and the data $f: \Omega \rightarrow \mathbb{R}, g, \beta: \Omega^{c} \rightarrow \mathbb{R}$ are given. The operator $u \mapsto \mathcal{N} u$ is the nonlocal $p$-normal derivative across the boundary of $\Omega$,
\[

$$
\begin{equation*}
\mathcal{N} u(y)=2 \int_{\Omega} \psi(u(y)-u(x)) \nu(x-y) \mathrm{d} x \quad\left(y \in \Omega^{c}\right) \tag{1.2}
\end{equation*}
$$

\]

The nonlinear operator $u \mapsto \mathcal{N} u$ appears naturally while deriving the nonlocal Gauss-Green formula (see Appendix B.2) and was introduce in the linear setting in [DROV17]. We point out that another type of such an operator appeared earlier in literature see for instance [DGLZ12]. In view of problem $\left(P_{\nu, \tau}\right)$, it is necessary and sufficient to prescribe the complement condition on $\Omega_{e}:=\Omega_{\nu} \backslash \Omega$ with $\Omega_{\nu}=\Omega+\operatorname{supp} \nu$ which we name as the nonlocal boundary of the exterior boundary of $\Omega$ with respect to $\nu$. As well, we also call $\Omega_{\nu}=\Omega+\operatorname{supp} \nu$ to be the nonlocal hull of $\Omega$ with respect to $\nu$; see in Section 8 for more details. Genuinely speaking, PDEs oriented folks may view the problem $\left(P_{\nu, \tau}\right)$ as the nonlocal counterpart of the nonlinear local problem of the form

$$
-\Delta_{p} u=f \text { in } \Omega \quad \text { and } \quad \tau \partial_{n, p} u+(1-\tau) \beta|u|^{p-2} u=g \text { on } \partial \Omega
$$

where $\Delta_{p} u=\operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right)$ denotes the $p$-Laplace operator and when $\Omega$ is smooth, $\partial_{n, p} u=|\nabla u|^{p-2} \nabla u \cdot n$ is the $p$-normal derivative on the boundary $\partial \Omega$ of $\Omega$, viz., at a point $x \in \partial \Omega, n(x)$ is the outward normal derivative on the boundary $\partial \Omega$. Of course, one immediately recognizes that the complement (resp. boundary) condition the problem $\left(P_{\nu, \tau}\right)$ (resp. $\left(P_{\tau}\right)$ ) is the Neumann condition $\mathcal{N} u=g$ (resp. $\partial_{n, p} u=g$ ) for $\tau=1$, the Robin condition $\mathcal{N} u+\beta|u|^{p-2} u=2 g$ (resp. $\partial_{n, p} u+\beta|u|^{p-2} u=2 g$ ) for $\tau=\frac{1}{2}$ and the Dirichlet condition $|u|^{p-2} u=g$ or equivalently $u=|g|^{p^{\prime}-2} g$ for $\tau=0$ and $\beta=1$ where we have $p p^{\prime}=p+p^{\prime}$. Actually, our second goal is to show that weak solutions to the problem $\left(P_{\tau}\right) \tau=0,1$ are strong limit of those of problems of the forms $\left(P_{\nu, \tau}\right)$. Before we explain our strategy, let us discuss the assumption in (L) which is utterly decisive and give a prototypical example.
It remarkably appears that the assumption in (L) is unavoidable. Indeed, on the one hand, the symmetry condition is natural in the sense that one has

$$
\mathcal{E}_{\mathbb{R}^{d}}(u, u)=\iint_{\mathbb{R}^{d} \mathbb{R}^{d}}|u(x)-u(y)|^{p} \nu(x-y) \mathrm{d} y \mathrm{~d} x=\iint_{\mathbb{R}^{d} \mathbb{R}^{d}}|u(x)-u(y)|^{p} \nu^{\text {sym }}(x-y) \mathrm{d} y \mathrm{~d} x .
$$

where $\nu^{\operatorname{sym}}(h)=\frac{1}{2}(\nu(h)+\nu(-h))$ is the symmetric part of $\nu$. On the other hand, it appears that $\mathcal{E}_{\mathbb{R}^{d}}(u, u)<\infty$ for all $u \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$ if and only if $\nu \in L^{1}\left(\mathbb{R}^{d}, 1 \wedge|h|^{p}\right)$; cf. Section 4 for the details. Literally, it turns out that the symmetry and the $p$-Lévy condition in (L) can be self-generated through the energy form $\mathcal{E}_{\mathbb{R}^{d}}(u, u)$. Furthermore, a heuristic computation reveals that the first variation of the operator $u \mapsto Q(u)=\frac{1}{p} \mathcal{E}_{\mathbb{R}^{d}}(u, u)$ on the Banach space $W_{\nu}^{p}\left(\mathbb{R}^{d}\right)=\left\{u \in L^{p}\left(\mathbb{R}^{d}\right): \mathcal{E}_{\mathbb{R}^{d}}(u, u)<\infty\right\}$ with the norm $\|u\|_{W_{\nu}^{p}\left(\mathbb{R}^{d}\right)}=\left(\|u\|_{L^{p}\left(\mathbb{R}^{d}\right)}^{p}+\mathcal{E}_{\mathbb{R}^{d}}(u, u)\right)^{1 / p}$ gives rise to the $p$-Lévy operator $L$. Moreover, we have $\mathcal{E}_{\mathbb{R}^{d}}(u, u)=\langle L u, u\rangle=Q^{\prime}(u)(u), u \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$. The density $\nu$ is somewhat the "order" of the operator $L$, which becomes apparent in the case of fractional kernel $\nu(h)=|h|^{-d-s p}, s \in(0,1)$ is fixed. For case $p=2$, one immediately recognizes the so call Lévy condition in (1.7) while the operator $L$ is translation invariant and generates a symmetric Lévy process. In general, the pointwise evaluation of $L u(x)$ and $-\Delta_{p} u(x)$ for $u \in C_{b}^{2}\left(\mathbb{R}^{d}\right)$ is warranted in the degenerate case (also often called the superquadratic case), i.e. $p \geq 2$. However, in the situation singular case (also often called the subquadratic case), i.e., $1<p<2$, the pointwise definition of $L u(x)$ and $-\Delta_{p} u(x)$ might not exist even for a bona fide function $u \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$. In many situations, the operators $L$ and $-\Delta_{p}$ do not always act on functions in a reasonable pointwise sense. We refer interested reader in Appendix B. 1 for some sketchy examples. In both cases, i.e., $1<p<\infty$, a reasonable alternative is rather to evaluate $L u$ and $\Delta_{p} u$ in the generalized sense, i.e., in the weak sense or via their respective associated energies forms. For instance by duality (see Section 3.1), one finds that the nonlinear operators $L: W_{\nu}^{p}\left(\mathbb{R}^{d}\right) \rightarrow\left(W_{\nu}^{p}\left(\mathbb{R}^{d}\right)\right)^{\prime}$ and $\Delta_{p}: W^{1, p}\left(\mathbb{R}^{d}\right) \rightarrow\left(W^{1, p}\left(\mathbb{R}^{d}\right)\right)^{\prime}$ are well-defined. In particular, $L u$ and $\Delta_{p} u$ are distributions. Morally, the nonlocal operator $L$ may be as good or as bad the local operator $-\Delta_{p}$. In fact, the operator $L$ can be seen as a prototype of a nonlinear nonlocal operator of divergence just as the $-\Delta_{p}$ is a prototype of a nonlinear local operator of divergence.
For prototypical example, consider $s \in \mathbb{R}$ then an effortless computation reveals that $\nu(h)=\frac{C_{d, p, s}}{2}|h|^{-d-s p}$ belongs to $L^{1}\left(\mathbb{R}^{d}, 1 \wedge|h|^{p}\right)$, i.e., $\nu$ satisfies $(\mathrm{L})$ if and only if $s \in(0,1)$. In this case, the resulting Banach space $W_{\nu}^{p}\left(\mathbb{R}^{d}\right)=$ $W^{s, p}\left(\mathbb{R}^{d}\right)$ is the usual fractional Sobolev-Slobodeckij space, whereas the associated integrodifferential operator $L$ is the well-known fractional $p$-Laplace operator $(-\Delta)_{p}^{s}$,

$$
(-\Delta)_{p}^{s} u(x)=C_{d, p, s} \text { p.v. } \int_{\mathbb{R}^{d}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{d+s p}} \mathrm{~d} y
$$

where $C_{d, p, s}$ is a normalizing constant of the fractional $p$-Laplacian which we define by

$$
C_{d, p, s}=\frac{s(1-2 s) \Gamma\left(\frac{d+s p}{2}\right)}{\pi^{\frac{d-1}{2}} \Gamma\left(\frac{s p+1}{2}\right) \Gamma(p(1-s)) \cos (s \pi)}
$$

Our choice of the constant $C_{d, p, s}$, see Section 9.4, guaranties the following properties:

- For $p=2$, by [Fog20, Proposition 2.21], $C_{d, 2, s}$ is the unique normalizing constant of the fractional Laplacian such that $\widehat{(-\Delta)^{s}} u(\xi)=|\xi|^{2 s} \widehat{u}(\xi), \xi \in \mathbb{R}^{d}$ for all $u \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$, where $\widehat{u}(\xi)=\int_{\mathbb{R}^{d}} e^{-i x \cdot \xi} u(x) \mathrm{d} x$ denotes the in Fourier transform of $u$. Namely, we have

$$
C_{d, 2, s}=\frac{s(1-2 s) \Gamma\left(\frac{d+2 s}{2}\right)}{\pi^{\frac{d-1}{2}} \Gamma\left(\frac{2 s+1}{2}\right) \Gamma(2(1-s)) \cos (s \pi)}=\frac{s 2^{2 s} \Gamma\left(\frac{d+2 s}{2}\right)}{\pi^{d / 2} \Gamma(1-s)}
$$

- For any $u \in L^{\infty}\left(\mathbb{R}^{d}\right) \cap C^{2}\left(B_{1}(x)\right), \nabla u(x) \neq 0$ we have $(-\Delta)_{p}^{s} u(x) \xrightarrow{s \rightarrow 1}-\Delta_{p} u(x)$.
- For all $u \in W^{1, p}\left(\mathbb{R}^{d}\right)$ we have $\|u\|_{W^{s, p}\left(\mathbb{R}^{d}\right)} \xrightarrow{s \rightarrow 1}\|u\|_{W^{1, p}\left(\mathbb{R}^{d}\right)}$; see Section 9.5.
- Moreover we have the following asymptotic behaviors

$$
\begin{equation*}
\lim _{s \rightarrow 0} \frac{C_{d, p, s}}{s(1-s)}=\frac{2}{\left|\mathbb{S}^{d-1}\right| \Gamma(p)} \quad \text { and } \quad \lim _{s \rightarrow 1} \frac{C_{d, p, s}}{s(1-s)}=\frac{2 p}{\left|\mathbb{S}^{d-1}\right| K_{d, p}} \tag{1.3}
\end{equation*}
$$

Here, we emphasize that the constant $K_{d, p}$, see [Fog23, Fog20] for the computation, plays a crucial role in our asymptotic analysis and is given by

$$
\begin{equation*}
K_{d, p}=f_{\mathbb{S}^{d-1}}\left|w_{d}\right|^{p} \mathrm{~d} \sigma_{d-1}(w)=\frac{\Gamma\left(\frac{d}{2}\right) \Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{d+p}{2}\right) \Gamma\left(\frac{1}{2}\right)} \tag{1.4}
\end{equation*}
$$

The asymptotic $s \rightarrow 1$, highlighting the factor $K_{d, p}$ is already anticipated in [Fog20, Eq: 2.38] in the case $p=2$ since $K_{d, 2}=\frac{1}{d}$. The above asymptotic in (1.3) perfectly lines up with the case $p=2$ as obtained in [DNPV12]. Despite the amusing fact of this asymptotic, it is important for the reader to keep in mind that $C_{d, p, s}$ is purely artificial and that only the case $p=2$ naturally appears as the unique constant for which $\widehat{(-\Delta)^{s}} u(\xi)=|\xi|^{2 s} \widehat{u}(\xi)$. Another different normalizing constant for the fractional $p$-Laplacian is proposed in [DGV21] wherein, one also finds other representations of the fractional $p$-Laplace operator.
The main purpose of this article is twofold. The primary objective is to study the well-posedness of the nonlocal problem $\left(P_{\nu, \tau}\right)$ under additional mild assumptions. For instance, as an avant-goût, let us illuminate what we do with the particular case of the Neumann problem, i.e., $\tau=1$ and for the particular fractional kernel $\nu(h)=\frac{C_{d, p, s}}{2}|h|^{-d-s p}$. We refer the reader to Section 8 for the general setting and more details. We point that $u$ is a weak solution to Neumann problem $\left(P_{\nu, 1}\right)$ if $u \in W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ and satisfies

$$
\begin{equation*}
\mathcal{E}(u, v)=\int_{\Omega} f(x) v(x) \mathrm{d} x+\int_{\Omega^{c}} g(y) v(y) \mathrm{d} y \quad \text { for all } u \in W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right) \tag{1.5}
\end{equation*}
$$

Here we consider $W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)=\left\{u: \mathbb{R}^{d} \rightarrow \mathbb{R}\right.$ meas. : $\left.\left.u\right|_{\Omega} \in L^{p}(\Omega), \mathcal{E}(u, u)<\infty\right\}$ with energy form $\mathcal{E}(\cdot, \cdot)$,

$$
\mathcal{E}(u, v)=\frac{C_{d, p, s}}{2} \iint_{\left(\Omega^{c} \times \Omega^{c}\right)^{c}} \frac{\psi(u(x)-u(y))(v(x)-v(y))}{|x-y|^{d+s p}} \mathrm{~d} y \mathrm{~d} x
$$

In Theorem 8.12, we establish the well-posedness up to additive constants of the variational problem (1.5) and hence of the Neumann problem $\left(P_{\nu, 1}\right)$ on the space $W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ whenever $f \in L^{p^{\prime}}(\Omega)$ and $g \in L^{p^{\prime}}\left(\Omega^{c}, \omega^{1-p^{\prime}}\right)$ with $\omega(h)=(1+|h|)^{-d-s p}$ are compatible, i.e.,

$$
\int_{\Omega} f(x) \mathrm{d} x+\int_{\Omega^{c}} g(y) \mathrm{d} y=0 .
$$

Let us highlight two important observations regarding the weak formulation (1.5) at this stage. First of all, one observes in this particular that the weight $\omega^{1-p^{\prime}}$ is rapidly increasing, hence the Neumann data $g \in L^{p^{\prime}}\left(\Omega^{c}, \omega^{1-p^{\prime}}\right)$ is required to decay rapidly at infinity. Although this may seems restrictive, it is however counterintuitive since the space $L^{p^{\prime}}\left(\Omega^{c}, \omega^{1-p^{\prime}}\right)$ is in fact the largest admissible function space for the Neumann data $g$. Indeed, we establish a non-existence result in Theorem 8.14 where we exhibit some examples of Neumann data $g$ not belonging to $L^{p^{\prime}}\left(\Omega^{c}, \omega^{1-p^{\prime}}\right)$ and compatible with $f=0$ the variational Neumann problem (1.5) has no solution in $W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$. As the second observation, one notices that [DROV17, Definition 3.6] and subsequent definitions like [MPL19, Definition 2.7] look very similar to (1.5) at first glance; for the general case, we refer the reader to Definition 8.6. However, the test space defined in [DROV17, Eq. (3.1)], [MPL19, Section 2] depends on the Neumann data $g$,
which is somewhat unaccustomed. We emphasize that our test space $W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ in the weak formulation (1.5) does not depend on the Neumann data $g$. Next, we would like to mention that our strategy of studying the problem $\left(P_{\nu, \tau}\right)$ in the general setting also includes a Dirichlet and Robin complement problems, requires to bring into play certain crucial tools. These include nonlocal functions spaces, nonlocal Poincaré types inequalities and nonlocal trace spaces. On the one hand, the on Poincaré inequalities we establish in Section 7 encoding the coercivity of the nonlocal energies, include the Poincaré type inequality when $\Omega$ is not necessarily connected but has finitely many connected components and the Poincaré-Friedrichs type inequality when $\Omega$ is bounded only in one direction. To the best of our knowledge, such generalization of the Poincaré inequalities do not exist in the literature so far. On the other hand, the nonlocal trace spaces truly embody the Dirichlet, Neumann and/or Robin complement data $g$ of the problem $\left(P_{\nu, \tau}\right)$. There are several new findings on nonlocal trace spaces, we refer interested reader for instance to [GH22, DTWY22, BGPR20, Rut20, DK19]. It is worth mentioning that the result [GH22] provides a robust nonlocal fractional trace space, viz., one is able to recover the classical local trace from the nonlocal one.
The secondary objective is to prove the $L^{p}$-convergence of nonlocal to local weak solutions; cf Section 11. Let us explain for which family of problems we are able to prove the convergence. Given a family $\left(\nu_{\varepsilon}\right)_{\varepsilon}, \nu_{\varepsilon} \geq 0$, of radial $p$-Lévy kernel, we prove in Theorem 9.6 that, there holds the formula

$$
\begin{equation*}
\lim _{\varepsilon \rightarrow 0} \iint_{\mathbb{R}^{d} \mathbb{R}^{d}}|u(x)-u(y)|^{p} \nu_{\varepsilon}(x-y) \mathrm{d} y \mathrm{~d} x=K_{d, p} \int_{\mathbb{R}^{d}}|\nabla u(x)|^{p} \mathrm{~d} x \tag{1.6}
\end{equation*}
$$

for all $u \in W^{1, p}\left(\mathbb{R}^{d}\right)$ if and only if the family $\left(\nu_{\varepsilon}\right)_{\varepsilon}$ satisfies

$$
\begin{equation*}
\lim _{\varepsilon \rightarrow 0} \int_{\mathbb{R}^{d}}\left(1 \wedge|h|^{p}\right) \nu_{\varepsilon}(h) \mathrm{d} h=1 \quad \text { and for all } \delta>0, \quad \lim _{\varepsilon \rightarrow 0} \int_{|h|>\delta}\left(1 \wedge|h|^{p}\right) \nu_{\varepsilon}(h) \mathrm{d} h=0 . \tag{1.7}
\end{equation*}
$$

In other words, condition (1.7) is the sharpest for which in the formula (1.6) (a.k.a in literature the BBM-formula) holds true. Note in passing that a typical example of family $\left(\nu_{\varepsilon}\right)_{\varepsilon}$ satisfying (1.7), is to consider the normalized fractional kernels $\nu_{\varepsilon}(h)=a_{\varepsilon, d, p}|h|^{-d-(1-\varepsilon) p}$ with $\varepsilon=1-s$ and $a_{\varepsilon, d, p}=\frac{p \varepsilon(1-\varepsilon)}{\left|S^{d-1}\right|}$; see Example 9.4. Another fascinating example of $\left(\nu_{\varepsilon}\right)_{\varepsilon}$ satisfying (1.7) is obtained from any radial function $\nu: \mathbb{R}^{d} \backslash\{0\} \rightarrow[0, \infty)$ that is $p$-Lévy normalized, i.e., $\int_{\mathbb{R}^{d}}\left(1 \wedge|h|^{p}\right) \nu(h) \mathrm{d} h=1$, by considering $\nu_{\varepsilon}$ defined as the rescaled version of $\nu$ with

$$
\nu_{\varepsilon}(h)= \begin{cases}\varepsilon^{-d-p} \nu(h / \varepsilon) & \text { if }|h| \leq \varepsilon, \\ \varepsilon^{-d}|h|^{-p} \nu(h / \varepsilon) & \text { if } \varepsilon<|h| \leq 1, \\ \varepsilon^{-d} \nu(h / \varepsilon) & \text { if }|h|>1 .\end{cases}
$$

Let $L_{\varepsilon}$ and $\mathcal{N}_{\varepsilon}$ be the nonlocal operators associated with $\nu_{\varepsilon}$ and put $\mu=K_{d, p}^{-1}$. Consider the problem

$$
\mu L_{\varepsilon} u=f_{\varepsilon} \text { in } \Omega \quad \text { and } \quad \tau \mu \mathcal{N}_{\varepsilon} u+(1-\tau)|u|^{p-2} u=g_{\varepsilon} \text { on } \Omega^{c}, \quad\left(P_{\nu_{\varepsilon}, \tau}\right)
$$

Surprisingly, under the condition in(1.7) and some mild conditions on $\Omega, f_{\varepsilon}$ and $g_{\varepsilon}$ we prove, cf. Section 11, that weak solutions to the problem ( $P_{\nu_{\varepsilon}, \tau}$ ) with $\tau=\{0,1\}$ strongly converge in $L^{p}(\Omega)$ as $\varepsilon \rightarrow 0$, to the weak solutions of the corresponding local problem $\left(P_{\tau}\right)$. We also prove the convergence of weak associated with regional type operators. Another appealing effect of the approximation family $\left(\nu_{\varepsilon}\right)_{\varepsilon}$ is that for $u \in L^{\infty}\left(\mathbb{R}^{d}\right) \cap C^{2}\left(B_{1}(x)\right)$ with $\nabla u(x) \neq 0$ there holds,

$$
\lim _{\varepsilon \rightarrow 0} L_{\varepsilon} u(x)=\lim _{\varepsilon \rightarrow 0} 2 \text { p.v. } \int_{\mathbb{R}^{d}}|u(x)-u(y)|^{p-2}(u(x)-u(y)) \nu(x-y) \mathrm{d} y=-K_{d, p} \Delta_{p} u(x) .
$$

In particular, we find that $(-\Delta)_{p}^{s} u(x) \rightarrow-\Delta_{p} u(x)$ as $s \rightarrow 1$. A similar pointwise asymptotic from the fractional $p$-Laplacian to the $p$-Laplacian $-\Delta_{p}$ can be found in [BS22,DL21, IN10]. We refer to Section 9 for more details. In our strategy of proving the convergence of solutions we need, cf. Section 10, to establish the robust Poincaré type inequalities à la A. Ponce [Pon04a] including in the situation where $\Omega$ is only bounded in one direction.
Let us comment on related works in the literature. For an outstanding reference on basics related to the $p$ Laplace operator we refer the reader to the classical lecture notes [Lin19]. The study of nonlocal operators driven by Lévy kernel are becoming popular. For recent studies of IntegroDifferential Equations(IDEs) involving the Lé vy operator of type $L$ (for the case $\mathrm{p}=2$ ) see [FK22, DFK22]. See also [Rut18, Rut20] and [ROV16] for the studies of Dirichlet problems associated Lé vy operator driven by singular Lévy measure. For problems related to general nonlocal elliptic type operators of Lévy see [Fog20, Voi17]. We also point out [BGPR23,Rut20] where the nonlinear Douglas identity for $p$-Lévy type operators are investigated. The fractional $p$-Laplacian is one of the most studied $p$-Lévy operator with the framework of IDEs. For instance, the study of Dirichlet problems can be found in [FP14, DKP16, ILPS16, Pal18, QX16, BP16, FI22] and for the study of Neumann problems see [MPL19, MPL21]. Note that [MPL19] is somewhat the extension of the idea and the set up from [DROV17] to the nonlinear setting.

We also refer to [MRT16, AMRT10] for the studies of problems related to the regional type operators. For regularity of solutions associated to the fractional $p$-Laplacian and its alike see [IMS16a,IMS16b,BLS18, CK22, CKW22]. It is noteworthy to emphasize that our setting is sufficiently general and includes kernel with bounded support. Nonlocal problems driven by integrodifferential operators associated with kernels of bounded support appear as core models to several problems in peridynamic. In Section 8, we explain how to deal with such type of kernels. For recent studies of nonlocal problem aiming at the application to peridynamics see [FK22,FVV22,DTZ22], several additional references can be found therein.
Let us comment about convergence from nonlocal to local. In fact under the assumption in (1.7), the convergence in (1.6) remains true with $\mathbb{R}^{d}$ replaced by any extension domain $\Omega \subset \mathbb{R}^{d}$; see for instance [Fog23]. Interestingly, the latter convergence also holds in the sense of the gamma convergence or Mosco convergence; see for instance [Fog20, FKV20, Pon04b]. This type of limit was originally studied for Lipschitz bounded domain in [BBM01] and several generalizations have recently emerged, see for instance [BMR20, DB22, DD22, Fog23, PS17] along with the references contained there. To the best of our knowledge, rigorous proofs of convergence of weak solutions to nonlocal problems with complement Neumann condition to the local ones appear in [Fog20,FK22,GH22]. A heuristic proof of the convergence for fractional Laplacian $(-\Delta)^{s}$ was investigated in [DROV17]. Note however that the convergence of nonlocal Neumann problems associated with regional type operators can be found in [AMRT08,AMRT10,DPS15]. In contrast to the Neumann problems, there is a substantial amount of works treating the convergence from nonlocal to local of weak solutions of complement boundary Dirichlet problems associated with the fractional p-Laplacian $(-\Delta)_{p}^{s}$; we refer interested reader for instance to [BPS16, FBS20, BO21a, BO21b, SV22]. The convergence of weak solutions of Dirichlet problems for fractional $g$-Laplacian where $g$ is an Orlicz function see [BS19]. For convergence of solutions to elliptic problems, see [Fog20, Voi17]. Last, a uniform convergence of viscosity solutions to the constrained fractional $p$-Laplace operator is established in [IN10]. For more on viscosity solutions associated with the $p$-Laplacian and the fractional $p$-Laplacian see for instance the recent results [Lin16, Lin19, DL21] as well as the references therein.
The rest of this article is organized as follows. We introduce some basics concepts regarding the support of $\nu$ in Section 2. In Section 3 we study nonlocal Sobolev spaces, whereas Section 5 is dedicated to the study of the corresponding nonlocal trace spaces. Meanwhile we give an analytic characterization of $p$-Lévy integrability in Section 4. In Section 6 we give some compact embeddings. Nonlocal Poincaré and nonlocal Poincaré-Friedrichs inequalities are established in Section 7, while their robust versions are provided in Section 10. Section 9 and Section 11 focuses on convergences of from nonlocal to local objects; this includes convergence of forms, weak solutions, nonlinear operators. We also characterize the family $\left(\nu_{\varepsilon}\right)_{\varepsilon}$ satisfying (1.7). In Appendix A and Appendix B we provide some basics results such as nonlocal Gauss-Green formula, pointwise evaluations, elementary inequalities that are useful for the study and the understanding of nonlinear operators under considerations namely, $L$ and $\mathcal{N}$.

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## 2. BASIC CONCEPTS AND NOTATIONS

2.1. Notations. Throughout this article unless otherwise stated, we assume that $\Omega \subset \mathbb{R}^{d}$ is an open set, $1<$ $p^{\prime}, p<\infty$ with $p+p^{\prime}=p p^{\prime}$ and $\nu$ is a symmetric kernel satisfying ( L ). We frequently use the convex inequality $(a+b)^{p} \leq 2^{p-1}\left(a^{p}+b^{p}\right)$ for $a>0, b>0$. If there is no specific mention, all functions and sets are assumed to be at least Borel measurable and are understood in the almost everywhere sense. Given two quantities $F$ and $G$ the relation $F \asymp G$ means that there are positive constants $C_{1}$ and $C_{2}$ such that $C_{1} \leq F / G \leq C_{2}$. In general, $C>0$ denotes a generic constant depending on the local and $\varepsilon>0$ is a small quantity tending to 0 . The Euclidean scalar product of $x=\left(x_{1}, x_{2}, \cdots, x_{d}\right) \in \mathbb{R}^{d}$ and $y=\left(y_{1}, y_{2}, \cdots, y_{d}\right) \in \mathbb{R}^{d}$ is $x \cdot y=x_{1} x_{1}+x_{2} y_{2}+\cdots+x_{d} y_{d}$ and denote the norm of $x$ by $|x|=\sqrt{x \cdot x}$.
2.2. Support and nonlocal boundary. We introduce some basics definitions in connection with the measure theoretic support of a measurable function.

Definition 2.1. The support of a measurable function $\omega: \mathbb{R}^{d} \rightarrow \mathbb{R}$ is defined as

$$
\begin{aligned}
\operatorname{supp} \omega & =\mathbb{R}^{d} \backslash \mathcal{O} \text { with } \mathcal{O}=\bigcup\{O: O \text { is open and } \omega=0 \text { a.e. on } O\} \\
& =\left\{x \in \mathbb{R}^{d}: \omega>0 \text { a.e. on any open set } O \text { s.t. } x \in O\right\} .
\end{aligned}
$$

Note that $\operatorname{supp} \omega$ is a closed set and $\omega=0$ a.e. $\mathbb{R}^{d} \backslash \operatorname{supp} \omega$. In particular, $\omega=0$ a.e. if and only if $\operatorname{supp} \omega=\emptyset$. If $\omega$ is a continuous function then $\operatorname{supp} \omega=\overline{\left\{h \in \mathbb{R}^{d}: \omega(h) \neq 0\right\}}$. The latter is not true in general. For instance, on the real line we put $\omega=\mathbb{1}_{\mathbb{Q}}$, where $\mathbb{Q}$ is the set of rational numbers, then $\overline{\{h \in \mathbb{R}: \omega(h) \neq 0\}}=\mathbb{R}$ but $\operatorname{supp} \omega=\emptyset$.

Definition 2.2. We say that $\nu: \mathbb{R}^{d} \backslash\{0\} \rightarrow[0, \infty)$ has full support if $\operatorname{supp} \nu=\mathbb{R}^{d}$, equivalently $\nu(h)>0$ for almost every $h \in \mathbb{R}^{d}$ that is $|\{\nu=0\}|=0$.

Remark 2.3. It is very tempting to think that the set of zeros of $\nu$ in $\operatorname{supp} \nu$ is a null set, i.e., $|\operatorname{supp} \nu \cap\{\nu=0\}|=0$. In other words is true that $\nu>0$ a.e. on supp $\nu$ ? This is not always true, indeed consider $O \subset \mathbb{R}^{d}$ a non-empty open set whose boundary has positive measure, i.e., $|\partial O|>0$. The function $g(x)=\operatorname{dist}\left(x, \mathbb{R}^{d} \backslash O\right)$ is continuous. Note that $\{g \neq 0\}=O$ and thus $\operatorname{supp} g=\bar{O}$. However, we have $|\operatorname{supp} g \cap\{g=0\}|=|\partial O|>0$.
Definition 2.4 (Nonlocal hull and nonlocal boundary). Given a measurable set $S \subset \mathbb{R}^{d}$ :
(i) We define the nonlocal hull with respect to $\nu$ (or simply the $\nu$-nonlocal hull) of $S$ to be the set $S_{\nu}=S+\operatorname{supp} \nu$. Note that if $0 \in \operatorname{supp} \nu$ then $S \subset S_{\nu}$.
(ii) We define the exterior boundary (or the nonlocal boundary) of $S$ with respect to $\nu$ denoted $S_{e}$ or $\partial_{\nu} S$, to be the set $S_{e}=S_{\nu} \backslash S=(S+\operatorname{supp} \nu) \backslash S\left(\right.$ or $\left.\partial_{\nu} S=S_{\nu} \backslash S\right)$.

The terminologies are justified by the following facts. From an analysis point of view, the set $S_{\nu}=S+\operatorname{supp} \nu$ is smallest set needed to evaluate $L u$ on $S$ that is we have

$$
\int_{\mathbb{R}^{d}} \psi(u(x)-u(y)) \nu(x-y) \mathrm{d} y=\int_{S_{\nu}} \psi(u(x)-u(y)) \nu(x-y) \mathrm{d} y \quad \text { for all } x \in S
$$

In other words, the values $L u(x)$ for $x \in S$ solely depend of the values of $u$ in $S_{\nu}$. Therefore, to solve a nonlocal equation of the form $L u=f$ in $S$, it is sufficient to prescribe exterior boundary condition on $S_{\nu} \backslash S$. For instance if we consider $S=\Omega$, then the Neumann condition $\mathcal{N} u=g$, only makes if $g=0$ on $\mathbb{R}^{d} \backslash \Omega_{\nu}$; see Section 8 for additional details. Indeed, for $x \in \Omega$ and $y \in \mathbb{R}^{d} \backslash \Omega_{\nu}$ we have $x-y \notin \operatorname{supp} \nu$ so that $\nu(x-y)=0$, and hence

$$
\mathcal{N} u(y):=\int_{\Omega} \psi(u(y)-u(x)) \nu(x-y) \mathrm{d} x=0 \quad \text { for all } y \in \mathbb{R}^{d} \backslash \Omega_{\nu}
$$

From a probabilistic point of the view, nonlocal hull $S_{\nu}$ can be seen as the reachable area of a jump process starting from a point in $S$. In some sense, any jump starting in $S$ cannot reach farther beyond the set $S_{\nu}$. Therefore, if a jump process is censored (restricted) at the $S$ then the whole universe the process is $S_{\nu}$. In other words any censored process on $\Omega$ is never aware of anything happening on $\mathbb{R}^{d} \backslash \Omega_{\nu}$.

Example 2.5. Let us mention two classes of examples that are well-known in the literature.
(i) If $\nu$ has full support, i.e. $\operatorname{supp} \nu=\mathbb{R}^{d}$ then $S_{\nu}=S+\mathbb{R}^{d}=\mathbb{R}^{d}$ and $S_{e}=\mathbb{R}^{d} \backslash S$. That is the exterior boundary of $S$ with respect to $\nu$ coincides with its whole complement. A typical example include $\nu(h)=|h|^{-d-s p}$. IntegroDifferential Equation (IDEs) associated with nonlocal operators driven kernels $\nu$ with $\operatorname{supp} \nu=\mathbb{R}^{d}$, are often called complement values problems in the modern literature.
(ii) If $\operatorname{supp} \nu=\overline{B_{\delta}(0)}, \delta>0$ then $S_{\nu}=S+\overline{B_{\delta}(0)}=\left\{x \in \mathbb{R}^{d}: \operatorname{dist}(x, S) \leq \delta\right\}$, i.e., $S_{\nu}$ is the $\delta$-tubular thickening neighborhood of $S$. On other hand we have $S_{e}=S_{\nu} \backslash S=\left\{x \in \mathbb{R}^{d} \backslash S: \operatorname{dist}(x, S) \leq \delta\right\}$.
For instance, if $S=B_{r}(a), r>0$ and $a \in \mathbb{R}^{d}$ then $S_{\nu}=B_{r+\delta}(a)$ and $S_{e}=B_{r+\delta}(a) \backslash \overline{B_{r}(a)}$. For a concrete examples we have $\nu(h)=\mathbb{1}_{B_{\delta}(0)}(h)$ or $\nu(h)=|h|^{-d-s p} \mathbb{1}_{B_{\delta}(0)}(h)$. These types of kernels often appear in the area of peridynamics, wherein the exterior boundary $S_{e}=S_{\nu} \backslash S$ is also known as the volume constraint. Thus, IntegroDifferential Equation (IDEs) associated with nonlocal operators driven kernels $\nu$ with $\operatorname{supp} \nu=\overline{B_{\delta}(0)}$, are called in the area of peridynamic as volume constrained problems. See for instance the recent works [FK22,FVV22, DTZ22] and several additional references therein.

Remark 2.6. Let us highlight some remarks concerning $\operatorname{supp} \nu$ and $S_{\nu}$.
(i) If $0 \in \operatorname{supp} \nu$ then $S \subset S_{\nu}$. This property is sometimes important to makes of many IDEs.
(ii) If the set $S$ is open then $S_{\nu}$ is also open. This stems from the fact that each $S+h, h \in \operatorname{supp} \nu$ is open and $S_{\nu}=S+\operatorname{supp} \nu=\bigcup_{h \in \operatorname{supp} \nu} S+h$.
(iii) If the set $S$ is compact or $\operatorname{supp} \nu$ is compact then $S_{\nu}$ is a closed set.

## 3. Nonlocal function spaces

In this section we introduce generalized Sobolev-Slobodeckij-like function spaces with respect to a Lévy measure $\nu$ and an open subset $\Omega \subset \mathbb{R}^{d}$, in particular the space $W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ and its nonlocal trace space $T_{\nu}^{p}\left(\Omega^{c}\right)$. The function spaces are tailor-made for IntegroDifferential Equations (IDEs) with Neumann and Dirichlet complement conditions. Recall that $\nu: \mathbb{R}^{d} \backslash\{0\} \rightarrow[0, \infty)$ is the density of a symmetric $p$-Lévy measure, that is,

$$
\begin{equation*}
\nu(h)=\nu(-h) \quad h \neq 0 \quad \text { and } \quad \int_{\mathbb{R}^{d}}\left(1 \wedge|h|^{p}\right) \nu(h) \mathrm{d} h<\infty . \tag{L}
\end{equation*}
$$

In fact $\nu$ is said to be $p$-Lévy integrable, i.e. $\nu \in L^{1}\left(\mathbb{R}^{d}, 1 \wedge|h|^{p}\right)$.
3.1. Nonlocal energy forms. Recall $\psi(t)=|t|^{p-2} t$. Given a symmetric kernel $k:\left(\mathbb{R}^{d} \times \mathbb{R}^{d}\right) \backslash$ diag $\rightarrow[0, \infty)$ we consider the energy form $\mathcal{E}^{k}(\cdot, \cdot)$ by

$$
\mathcal{E}^{k}(u, v)=\iint_{\mathbb{R}^{d} \mathbb{R}^{d}} \psi(u(x)-u(y))(v(x)-v(y)) k(x, y) \mathrm{d} y \mathrm{~d} x
$$

Proposition 3.1.
(i) If $\mathcal{E}^{k}(u, u)$ and $\mathcal{E}^{k}(v, v)$ are finite so is $\mathcal{E}^{k}(u, v)$ and we have

$$
\left|\mathcal{E}^{k}(u, v)\right| \leq \mathcal{E}^{k}(u, u)^{1 / p^{\prime}} \mathcal{E}^{k}(v, v)^{1 / p}
$$

(ii) Consider the Banach space $W_{k}^{p}\left(\mathbb{R}^{d}\right)=\left\{u \in L^{p}\left(\mathbb{R}^{d}\right): \mathcal{E}^{k}(u, u)<\infty\right\}$ with $\|u\|_{W_{k}^{p}\left(\mathbb{R}^{d}\right)}^{p}=\|u\|_{L^{p}\left(\mathbb{R}^{d}\right)}^{p}+\mathcal{E}^{k}(u, u)$ . Then the operator $L: W_{k}^{p}\left(\mathbb{R}^{d}\right) \rightarrow\left(W_{k}^{p}\left(\mathbb{R}^{d}\right)\right)^{\prime}$ with $L u:=\mathcal{E}^{k}(u, \cdot)$ is nonlinear and bounded with

$$
\begin{aligned}
& |\langle L u, v\rangle|=\left|\mathcal{E}^{k}(u, v)\right| \leq \mathcal{E}^{k}(u, u)^{1 / p^{\prime}} \mathcal{E}^{k}(v, v)^{1 / p} \\
& \|L u\|_{\left(W_{k}^{p}\left(\mathbb{R}^{d}\right)\right)^{\prime}} \leq \mathcal{E}^{k}(u, u)^{1 / p^{\prime}} \leq\|u\|_{W_{k}^{p}\left(\mathbb{R}^{d}\right)}^{p-1}
\end{aligned}
$$

Proof. (ii) follows from $(i)$ whereas, $(i)$ is an immediate consequence of the Hölder inequality.
Remark 3.2. Formally, the operators $L$ is denoted as

$$
L u(x):=2 \text { p.v. } \int_{\mathbb{R}^{d}} \psi(u(x)-u(y)) k(x, y) \mathrm{d} y
$$

whenever this expression makes sense. For instance if $k(x, y)=\nu(x-y)$ with $\nu \in L^{1}\left(\mathbb{R}^{d}, 1 \wedge|h|^{p}\right)$ then under additional mild conditions $L u(x)$ can be defined for smooth functions (see Appendix B.1). Beside this, a curious reader easily verifies that, a similar analysis can be achieved with the spaces $W_{k}^{p}\left(\mathbb{R}^{d}\right)$ and the nonlocal operator $L$ replaced with the space $W^{1, p}\left(\mathbb{R}^{d}\right)$ and the local $p$-Laplace operator $-\Delta_{p}$.

From now we assume that $\mathcal{E}^{k}(u, u)<\infty$ and $\mathcal{E}^{k}(v, v)<\infty$. We will consider special cases of the $k(x, y)$ that are of paramount interest. First we consider the form $\mathcal{E}_{\Omega}(\cdot, \cdot)$ given by

$$
\begin{aligned}
\mathcal{E}_{\Omega}(u, v) & =\iint_{\Omega \Omega} \psi(u(x)-u(y))(v(x)-v(y)) \nu(x-y) \mathrm{d} y \mathrm{~d} x \\
& =\iint_{\mathbb{R}^{d} \mathbb{R}^{d}} \psi(u(x)-u(y))(v(x)-v(y)) \min \left(\mathbb{1}_{\Omega}(x), \mathbb{1}_{\Omega}(y)\right) \nu(x-y) \mathrm{d} y \mathrm{~d} x .
\end{aligned}
$$

That is, $k(x, y)=\min \left(\mathbb{1}_{\Omega}(x), \mathbb{1}_{\Omega}(y)\right) \nu(x-y)$. Next, the Gauss-Green formula (see Appendix B.2) motivates us to consider the special energy form $\mathcal{E}(\cdot, \cdot)$, define by

$$
\begin{aligned}
\mathcal{E}(u, v) & =\iint_{\left(\Omega^{c} \times \Omega^{c}\right)^{c}} \psi(u(x)-u(y))(v(x)-v(y)) \nu(x-y) \mathrm{d} y \mathrm{~d} x \\
& =\iint_{\mathbb{R}^{d} \mathbb{R}^{d}} \psi(u(x)-u(y))(v(x)-v(y)) \max \left(\mathbb{1}_{\Omega}(x), \mathbb{1}_{\Omega}(y)\right) \nu(x-y) \mathrm{d} y \mathrm{~d} x .
\end{aligned}
$$

That is, in this case we have $k(x, y)=\max \left(\mathbb{1}_{\Omega}(x), \mathbb{1}_{\Omega}(y)\right) \nu(x-y)$. Taking $k(x, y)=\frac{1}{2}\left(\mathbb{1}_{\Omega}(x), \mathbb{1}_{\Omega}(y)\right) \nu(x-y)$ we obtain another related form $\mathcal{E}_{+}(\cdot, \cdot)$, that is

$$
\begin{aligned}
\mathcal{E}_{+}(u, v) & =\iint_{\Omega \mathbb{R}^{d}} \psi(u(x)-u(y))(v(x)-v(y)) \nu(x-y) \mathrm{d} y \mathrm{~d} x \\
& =\frac{1}{2} \iint_{\mathbb{R}^{d} \mathbb{R}^{d}} \psi(u(x)-u(y))(v(x)-v(y))\left(\mathbb{1}_{\Omega}(x)+\mathbb{1}_{\Omega}(y)\right) \nu(x-y) \mathrm{d} y \mathrm{~d} x .
\end{aligned}
$$

As shown in the result, the forms $\mathcal{E}(\cdot, \cdot)$ and $\mathcal{E}_{+}(\cdot, \cdot)$ are equivalents.
Proposition 3.3. We have $\mathcal{E}(u, u) \leq 2 \mathcal{E}_{+}(u, u) \leq 2 \mathcal{E}(u, u)$ and $\mathcal{E}_{\Omega}(u, u) \leq \mathcal{E}(u, u)$.
Proof. The proof is immediate since $\min (a, b) \leq \max (a, b) \leq a+b \leq 2 \max (a, b)$.

Remark 3.4. For brevity let us denote $\chi(x, y)=|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))$. Recall that $\Omega_{e}=\Omega_{\nu} \backslash \Omega$ and $\Omega_{\nu}=\Omega+\operatorname{supp} \nu$. Assume $0 \in \operatorname{supp} \nu$ so that $\Omega \subset \Omega_{\nu}$, then

$$
\begin{aligned}
\mathcal{E}(u, v) & =\iint_{\left(\Omega^{c} \times \Omega^{c}\right)^{c}} \chi(x, y) \nu(x-y) \mathrm{d} y \mathrm{~d} x=\left(\iint_{\Omega \Omega}+2 \iint_{\Omega^{c} \Omega}\right) \\
& \chi(x, y) \nu(x-y) \mathrm{d} y \mathrm{~d} x \\
& =\left(\iint_{\Omega \Omega}+2 \iint_{\Omega_{e} \Omega}\right) \chi(x, y) \nu(x-y) \mathrm{d} y \mathrm{~d} x=\underset{\left(\Omega_{\nu} \times \Omega_{\nu}\right) \backslash\left(\Omega_{e} \times \Omega_{e}\right)}{ } \chi \int_{\substack{ \\
\Omega_{e}}} \chi(x, y) \nu(x-y) \mathrm{d} y \mathrm{~d} x .
\end{aligned}
$$

Analogously we have

$$
\begin{aligned}
\mathcal{E}_{+}(u, v) & =\iint_{\Omega \mathbb{R}^{d}} \chi(x, y) \nu(x-y) \mathrm{d} y \mathrm{~d} x=\iint_{\Omega \Omega_{\nu}} \chi(x, y) \nu(x-y) \mathrm{d} y \mathrm{~d} x \\
& =\iint_{\Omega \mathbb{R}^{d}} \chi(x, x+h) \nu(h) \mathrm{d} h \mathrm{~d} x=\iint_{\Omega \operatorname{supp} \nu} \chi(x, x+h) \nu(h) \mathrm{d} h \mathrm{~d} x .
\end{aligned}
$$

3.2. Nonlocal Sobolev-Slobodeckij-like spaces. We define the space $W_{\nu}^{p}(\Omega)$ by

$$
\begin{equation*}
W_{\nu}^{p}(\Omega)=\left\{u \in L^{p}(\Omega):|u|_{W_{\nu}^{p}(\Omega)}^{p}:=\mathcal{E}_{\Omega}(u, u)<\infty\right\} \tag{3.1}
\end{equation*}
$$

equipped with the norm

$$
\|u\|_{W_{\nu}^{p}(\Omega)}=\left(\|u\|_{L^{p}(\Omega)}^{p}+|u|_{W_{\nu}^{p}(\Omega)}^{p}\right)^{1 / p}=\left(\|u\|_{L^{p}(\Omega)}^{p}+\mathcal{E}_{\Omega}(u, u)\right)^{1 / p} .
$$

For $\nu(h)=|h|^{-d-s p}, s \in(0,1)$, then $W_{\nu}^{p}(\Omega)=W^{s, p}(\Omega)$ is the classical fractional Sobolev-Slobodeckij space. However if $\nu \in L^{1}\left(\mathbb{R}^{d}\right)$ then $W_{\nu}^{p}(\Omega)=L^{p}(\Omega)$. Indeed,

$$
\iint_{\Omega \Omega}|u(x)-u(y)|^{p} \nu(x-y) \mathrm{d} x \mathrm{~d} y \leq 2^{p} \iint_{\Omega \mathbb{R}^{d}}|u(x)|^{p} \nu(x-y) \mathrm{d} x \mathrm{~d} y=2^{p}\|\nu\|_{L^{1}\left(\mathbb{R}^{d}\right)}\|u\|_{L^{p}(\Omega)}^{p}
$$

For various choices of $\nu$ it appears (see Section 7) that, $\mathcal{E}(u, u)<\infty$ implies $u \in L^{p}(\Omega)$. Motivated by this remark and following [SV13, FKV15, Fog20] we introduce the space $W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$, which is crucial,

$$
\begin{align*}
W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right) & =\left\{u: \mathbb{R}^{d} \rightarrow \mathbb{R} \text { meas. : }\left.u\right|_{\Omega} \in L^{p}(\Omega) \text { and }|u|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}^{p}:=\mathcal{E}_{+}(u, u)<\infty\right\}  \tag{3.2}\\
& =\left\{u: \mathbb{R}^{d} \rightarrow \mathbb{R} \text { meas. : }\left.u\right|_{\Omega} \in L^{p}(\Omega) \text { and } \mathcal{E}(u, u)<\infty\right\}
\end{align*}
$$

The equality in (3.2) follows from Proposition 3.3. The space $W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ is a seminormed spaces equipped with the seminorm defined by

$$
\|u\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}=\left(\|u\|_{L^{p}(\Omega)}^{p}+|u|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}^{p}\right)^{1 / p} \asymp\left(\|u\|_{L^{p}(\Omega)}^{p}+\mathcal{E}(u, u)\right)^{1 / p} .
$$

Last, we define $W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ as the space of functions that vanish on the complement of $\Omega$ :

$$
\begin{align*}
W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right) & =\left\{u \in W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right): u=0 \text { a.e. on } \mathbb{R}^{d} \backslash \Omega\right\} \\
& =\left\{u \in W_{\nu}^{p}\left(\mathbb{R}^{d}\right): u=0 \text { a.e. on } \mathbb{R}^{d} \backslash \Omega\right\} \tag{3.3}
\end{align*}
$$

Naturally, $W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ is endowed with the $\|\cdot\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}$ which is equivalent therein with the norm $\|\cdot\|_{W_{\nu}^{p}\left(\mathbb{R}^{d}\right)}$, viz. for $u \in W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ we have

$$
\|u\|_{W_{\nu}^{p}\left(\mathbb{R}^{d}\right)}=\left(\|u\|_{L^{p}(\Omega)}^{p}+\mathcal{E}(u, u)\right)^{1 / p} \asymp\|u\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)} .
$$

It is not difficult to check that $W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ is a closed subspace of $W_{\nu}^{p}\left(\mathbb{R}^{d}\right)$ and $W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$. Another closely related closed subspace of $W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ is defined as

$$
W_{\nu, 0}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)={\overline{C_{c}^{\infty}(\Omega)}}^{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}=\left\{\text { the closure of } C_{c}^{\infty}(\Omega) \text { in } W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)\right\}
$$

It is worth emphasizing that if $\Omega$ has a continuous boundary then $W_{\nu, 0}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)=W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ (see [Fog20, Theorem 3.76] and also Theorem 3.11). However this equality does not hold true in general. For $\nu(h)=|h|^{-d-s p}, s \in(0,1)$ we denote $W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)=W^{s, p}\left(\Omega \mid \mathbb{R}^{d}\right)$, $W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)=W_{\Omega}^{s, p}\left(\Omega \mid \mathbb{R}^{d}\right)$ and $W_{\nu, 0}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)=W_{0}^{s, p}\left(\Omega \mid \mathbb{R}^{d}\right)$.
Remark 3.5. As a direct consequence of the Proposition 3.1 we have the following.
(i) If $u, v \in W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ then $|\mathcal{E}(u, v)| \leq \mathcal{E}(u, u)^{1 / p^{\prime}} \mathcal{E}(v, v)^{1 / p}$ and $\mathcal{E}(u, \cdot) \in\left(W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)\right)^{\prime}$.
(ii) If $u, v \in W_{\nu}^{p}(\Omega)$ then $\left|\mathcal{E}_{\Omega}(u, v)\right| \leq \mathcal{E}_{\Omega}(u, u)^{1 / p^{\prime}} \mathcal{E}_{\Omega}(v, v)^{1 / p}$ and $\mathcal{E}_{\Omega}(u, \cdot) \in\left(W_{\nu}^{p}(\Omega)\right)^{\prime}$.

A simple proof of the following Theorem can be found in [Fog20, Theorem 3.46].
Theorem 3.6. The spaces $W_{\nu}^{p}(\Omega)$ and $W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ are separable reflexive Banach spaces.
Remark 3.7. It is worthwhile noticing that $\|\cdot\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}$ is always a norm on $W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$, but not in general a norm on $W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ if $\nu$ is not fully supported. A simple counterexample is given by $\nu(h)=\mathbb{1}_{B_{1}(0)}(h)$ and $\Omega=B_{1}(0)$. For the function $u(x)=\mathbb{1}_{B_{2}^{c}(0)}(x)$ we have $\|u\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}=0$ whereas $u \neq 0$. More generally, assume that $\Omega$ is bounded and $\nu$ has a compact support. Let $S=\mathbb{R}^{d} \backslash(\Omega \cup \operatorname{supp} \nu+\Omega)$ and consider the function $u(x)=\mathbb{1}_{S}(x)$. A routine verification shows that $\|u\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}=0$ but $u \neq 0$. This means that $\left(W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right),\|\cdot\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}\right)$ cannot be a normed space.
However $\left(W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right),\|\cdot\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}\right)$ is a Banach space when if $\nu$ is of full support; cf. [Fog20, FK22] for the proof. Let us say some few words for the general case where $\nu$ is not necessarily fully supported. In view of the Remark 3.4 we introduce another version of the space $W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$. Namely, recall $\Omega_{\nu}=\operatorname{supp} \nu+\Omega$, we consider the space

$$
W_{\nu}^{p}\left(\Omega \mid \Omega_{\nu}\right)==\left\{u: \Omega \cup \Omega_{\nu} \rightarrow \mathbb{R} \text { meas. }:\left.u\right|_{\Omega} \in L^{p}(\Omega), \iint_{\Omega \Omega_{\nu}}|u(x)-u(y)|^{p} \nu(x-y) \mathrm{d} y \mathrm{~d} x<\infty\right\}
$$

equipped with the seminorm

$$
\|u\|_{W_{\nu}^{p}\left(\Omega \mid \Omega_{\nu}\right)}=\left(\|u\|_{L^{p}(\Omega)}^{p}+\iint_{\Omega \Omega_{\nu}}|u(x)-u(y)|^{p} \nu(x-y) \mathrm{d} y \mathrm{~d} x\right)^{1 / p}
$$

It worth noticing that if $0 \in \operatorname{supp} \nu$ then $\Omega \subset \Omega_{\nu}$ so that $\Omega \cup \Omega_{\nu}=\Omega_{\nu}$.
Remark 3.8. We have $W_{\nu}^{p}\left(\Omega \mid \Omega_{\nu}\right) \equiv W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ in the sense that both spaces are isometric isomorphic. Clearly if $u \in W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ then $u_{\nu}=\left.u\right|_{\Omega_{\nu}} \in W_{\nu}^{p}\left(\Omega \mid \Omega_{\nu}\right)$ and we have $\left\|u_{\nu}\right\|_{W_{\nu}^{p}\left(\Omega \mid \Omega_{\nu}\right)}=\|u\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}$. Conversely, if $u \in$ $W_{\nu}^{p}\left(\Omega \mid \Omega_{\nu}\right)$ then $\widetilde{u} \in W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ where $\widetilde{u}$ is the zero extension of $u$ off $\Omega_{\nu}$ and $\|u\|_{W_{\nu}^{p}\left(\Omega \mid \Omega_{\nu}\right)}=\|\widetilde{u}\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}$. Note that $u \in W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ is the function null function if and only if $u=0$ a.e. on $\Omega_{\nu}$. The notation $W^{p}\left(\Omega \mid \Omega_{\nu}\right)$ instead of $W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ should be also appropriate. However we deliberately keep the latter notation for simplicity.
In view of the Remark 2.3, it is legitimate to assume that $|\operatorname{supp} \nu \cap\{\nu=0\}|=0$. The latter seems not enough to show that $\|\cdot\|_{W_{\nu}^{p}\left(\Omega \mid \Omega_{\nu}\right)}$ is norm. More precisely:
Open Question: Assume that $\nu(h)>0$ for all $h \in \operatorname{supp} \nu$. Prove or disprove that is $\|\cdot\|_{W_{\nu}^{p}\left(\Omega \mid \Omega_{\nu}\right)}$ defines a norm on $W_{\nu}^{p}\left(\Omega \mid \Omega_{\nu}\right)$.
Next we provide a condition under which we are able to show that $W_{\nu}^{p}\left(\Omega \mid \Omega_{\nu}\right)$ is a Banach space.
Theorem 3.9. The space $W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right) \equiv W_{\nu}^{p}\left(\Omega \mid \Omega_{\nu}\right)$ is a separable and reflexive Banach space, whenever $\Omega$ and $\nu$ satisfy the condition

$$
\begin{equation*}
\omega(x):=\int_{\Omega} 1 \wedge \nu(x-y) \mathrm{d} y>0 \quad \text { for almost all } x \in \Omega_{\nu} \backslash \Omega \tag{1}
\end{equation*}
$$

In particular, if $\nu$ is of full support.

Proof. Observe that if $x \notin \Omega_{\nu}$, for all $y \in \Omega$ we have $x-y \notin \operatorname{supp} \nu$. It follows that $\omega(x)=0$ on $\mathbb{R}^{d} \backslash \Omega_{\nu}$. By Theorem 5.8 we have $W_{\nu}^{p}\left(\Omega \mid \Omega_{\nu}\right) \equiv W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right) \hookrightarrow L^{p}\left(\mathbb{R}^{d}, \omega\right) \equiv L^{p}\left(\Omega_{\nu}, \omega\right)$. Whence, there is $C>0$ such that

$$
\begin{equation*}
\|u\|_{L^{p}\left(\Omega_{\nu}, \omega\right)} \leq C\|u\|_{W_{\nu}^{p}\left(\Omega \mid \Omega_{\nu}\right)} \quad \text { for all } u \in W_{\nu}^{p}\left(\Omega \mid \Omega_{\nu}\right) \tag{3.4}
\end{equation*}
$$

Assume that $\left(S_{1}\right)$ holds, if $\|u\|_{W_{\nu}^{p}\left(\Omega \mid \Omega_{\nu}\right)}=0$ then $u=0$ a.e. on $\Omega$ and by (3.4), $u \omega=0$ a.e. on $\Omega_{\nu} \backslash \Omega$. There holds that $u=0$ a.e. $\Omega_{\nu}$, since $\omega>0$ a.e. on $\Omega_{\nu} \backslash \Omega$. Whence the seminorm $\|\cdot\|_{W_{\nu}^{p}\left(\Omega \mid \Omega_{\nu}\right)}$ is a norm on $W_{\nu}^{p}\left(\Omega \mid \Omega_{\nu}\right)$. Now, consider $\left(u_{n}\right)_{n}$ be a Cauchy sequence in $W_{\nu}^{p}\left(\Omega \mid \Omega_{\nu}\right)$, then taking into account the estimate (3.4), the sequences $\left(u_{n}\right)_{n}$ and $\left(w_{n}\right)_{n}$ with $w_{n}=u_{n} \omega$ are Cauchy sequence in $L^{p}(\Omega)$ and $L^{p}\left(\Omega_{\nu}\right)$ respectively. Thus, passing to subsequence if necessary, we can assume there exist $u \in L^{p}(\Omega)$ and $w \in L^{p}\left(\Omega_{\nu}\right)$ such that

- $\left(u_{n}\right)$ converges in $L^{p}(\Omega)$ and pointwise almost everywhere on $\Omega$ to some $u \in L^{p}(\Omega)$,
- $\left(w_{n}\right)$ converges in $L^{p}\left(\Omega_{\nu}\right)$ and pointwise almost everywhere in $\Omega_{\nu}$ to some $w \in L^{p}\left(\Omega_{\nu}\right)$.

We extend $u$ for $x \in \Omega_{\nu} \backslash \Omega$ by $u(x)=w(x) / \omega(x)$ which is well defined since $\omega>0$ a.e. on $\Omega_{\nu} \backslash \Omega$. It follows that $u: \Omega \cup \Omega_{\nu} \rightarrow \mathbb{R}$ is measurable since $u_{n} \rightarrow u$ a.e. on $\Omega$ and for almost all $x \in \Omega_{\nu} \backslash \Omega$ we have $u_{n}(x)=$ $w_{n}(x) / \omega(x) \xrightarrow{n \rightarrow \infty} w(x) / \omega(x)=u(x)$. It turns out that $\left(u_{n}\right)_{n}$ is a Cauchy sequence in $W_{\nu}^{p}\left(\Omega \mid \Omega_{\nu}\right)$ converging to $u$ on $L^{p}(\Omega)$ and a.e. on $\Omega_{\nu}$. This together with Fatou's lemma implies that

$$
\left|u_{n}-u\right|_{W_{\nu}^{p}\left(\Omega \mid \Omega_{\nu}\right)}^{p} \leq \liminf _{m \rightarrow \infty} \iint_{\Omega \Omega_{\nu}}\left|\left[u_{n}-u_{m}\right](x)-\left[u_{n}-u_{m}\right](y)\right|^{p} \nu(x-y) \mathrm{d} y \mathrm{~d} x \xrightarrow{n \rightarrow \infty} 0
$$

We can therefore easily conclude that $\left\|u_{n}-u\right\|_{W_{\nu}^{p}\left(\Omega \mid \Omega_{\nu}\right)} \xrightarrow{n \rightarrow \infty} 0$ and $u \in W_{\nu}^{p}\left(\Omega \mid \Omega_{\nu}\right)$ which show that completeness of $W_{\nu}^{p}\left(\Omega \mid \Omega_{\nu}\right)$. The completeness is proved. Now, consider the isometry $\mathcal{I}: W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right) \rightarrow L^{p}(\Omega) \times L^{p}\left(\Omega \times \mathbb{R}^{d}\right)$ with $\mathcal{I}(u)=\left(u(x),(u(x)-u(y)) \nu^{1 / p}(x-y)\right)$. From its Banach structure, the space $\left(W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right),\|\cdot\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}\right)$, identified with $\mathcal{I}\left(W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)\right)$, is separable and reflexive as a closed subspace of the separable and reflexive space $L^{p}(\Omega) \times L^{p}\left(\Omega \times \mathbb{R}^{d}\right)$.

Example 3.10. For a typical situation where $\nu>0$ a.e. on $\operatorname{supp} \nu$ with $\operatorname{supp} \nu=\overline{B_{1}(0)}$ and $\Omega=B_{1}(0)$, for instance $\nu(h)=\mathbb{1}_{B_{1}(0)}$ or $\nu(h)=\mathbb{1}_{B_{1}(0)}|h|^{-d-s p}, s \in(0,1)$ we have $\Omega_{\nu}=B_{2}(0)$ and $W_{\nu}^{p}\left(\Omega \mid \Omega_{\nu}\right)=W_{\nu}^{p}\left(B_{1}(0) \mid B_{2}(0)\right)$ is a Banach space.
3.3. Density of smooth functions. For many results it is crucial that smooth functions with compact support are dense in the function space under consideration. Let us summarize some important results in this direction.
Theorem 3.11. Let $\nu$ satisfies $(\mathrm{L})$ with full support and let $\Omega \subset \mathbb{R}^{d}$ be open.
(i) $C^{\infty}(\Omega) \cap W_{\nu}^{p}(\Omega)$ is dense in $W_{\nu}^{p}(\Omega)$.
(ii) If $\Omega$ has a continuous boundary $\partial \Omega$, then $C_{c}^{\infty}(\bar{\Omega})$ is dense in $W_{\nu}^{p}(\Omega)$.
(iii) If $\Omega$ has a continuous boundary $\partial \Omega$, then $C_{c}^{\infty}(\Omega)$ is dense in $W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$.
(iv) If $\Omega$ has a Lipschitz boundary $\partial \Omega$, then $C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$ is dense in $W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$.
(v) $C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$ is dense in $W_{\nu}^{p}\left(\mathbb{R}^{d}\right)$.

The proofs of the first and second statement can be found in [Fog20] and [DK21]. The first statement reminisces a sort of Meyer-Serrin density type result (see [Fog20, Theorem 3.67]). Note that $C_{c}^{\infty}(\bar{\Omega})$ is defined as $\left\{\left.v\right|_{\bar{\Omega}}: v \in\right.$ $\left.C_{c}^{\infty}\left(\mathbb{R}^{d}\right)\right\}$. The proof of the third statement is given in [FSV15] for a special choice of $\nu$ and in [Fog20], [BGPR20] for the general case. Note however that the third may fail if $\partial \Omega$ is not continuous see for instance [FSV15, Remark 7]. The proof of the fourth and the fifth assertions are given in [FKV20, Fog20].
3.4. Connection with classical Sobolev spaces. There are connections between the nonlocal and the local Sobolev spaces. Recall that $W^{1, p}(\Omega)$ denotes the classical Sobolev space endowed with the norm

$$
\|u\|_{W^{1, p}(\Omega)}^{p}=\|u\|_{L^{p}(\Omega)}^{p}+\|\nabla u\|_{L^{p}(\Omega)}^{p}
$$

While, $W_{0}^{1, p}(\Omega)$ is the closure of $C_{c}^{\infty}(\Omega)$ with respect to the $W^{1, p}(\Omega)$. We emphasize that $W_{0}^{1, p}(\Omega)$ also coincides with the closure of $C_{c}^{\infty}(\Omega)$ in $W^{1, p}\left(\mathbb{R}^{d}\right)$. Let $W_{\nu, 0}^{p}(\Omega)$ be the closure of $C_{c}^{\infty}(\Omega)$ in $W_{\nu}^{p}(\Omega)$. Note that, the zero extension to $\mathbb{R}^{d}$ of any function in $W_{0}^{1, p}(\Omega)$ belongs to $W^{1, p}\left(\mathbb{R}^{d}\right)$. Recall that $\Omega$ is an $W^{1, p}$-extension domain if there is an operator $E: W^{1, p}(\Omega) \rightarrow W^{1, p}\left(\mathbb{R}^{d}\right)$ and $C>0$ such that $\left.E u\right|_{\Omega}=u$ and $\|E u\|_{W^{1, p}\left(\mathbb{R}^{d}\right)} \leq C\|u\|_{W^{1, p}(\Omega)}$ for all $u \in W^{1, p}(\Omega)$.
Proposition 3.12. Let $\Omega \subset \mathbb{R}^{d}$ be open. The following embeddings hold true.
(i) $W^{1, p}\left(\mathbb{R}^{d}\right) \hookrightarrow W_{\nu}^{p}\left(\mathbb{R}^{d}\right)$.
(ii) If $\Omega \subset \mathbb{R}^{d}$ is an $W^{1, p}$-extension domain then $W^{1, p}(\Omega) \hookrightarrow W_{\nu}^{p}(\Omega)$.
(iii) $W_{\nu}^{p}\left(\mathbb{R}^{d}\right) \hookrightarrow W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right) \hookrightarrow W_{\nu}^{p}(\Omega) \hookrightarrow L^{p}(\Omega)$.
(iv) $W_{0}^{1, p}(\Omega) \hookrightarrow W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right) \hookrightarrow W_{\nu}^{p}(\Omega)$.
(v) If $\partial \Omega$ continuous then $W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right) \hookrightarrow W_{\nu, 0}^{p}(\Omega) \hookrightarrow L^{p}(\Omega)$.

Proof. Note that (ii) is implied by (i). Whereas (v) follows from the fact that $C_{c}^{\infty}(\Omega)$ is dense in $W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$; see Theorem 3.11. We only prove (i), i.e., $W^{1, p}\left(\mathbb{R}^{d}\right) \hookrightarrow W_{\nu}^{p}\left(\mathbb{R}^{d}\right)$ since the remaining ones are trivial. A routine argument yields that, for $u \in W^{1, p}\left(\mathbb{R}^{d}\right)$ and $h \in \mathbb{R}^{d}$

$$
\int_{\mathbb{R}^{d}}|u(x+h)-u(x)|^{p} \mathrm{~d} x \leq|h|^{p}\|\nabla u\|_{L^{p}\left(\mathbb{R}^{d}\right)}^{p},
$$

whereas $\|u(\cdot+h)-u\|_{L^{p}\left(\mathbb{R}^{d}\right)}^{p} \leq 2^{p}\|u\|_{L^{p}\left(\mathbb{R}^{d}\right)}^{p}$. Therefore we get

$$
\int_{\mathbb{R}^{d}}|u(x+h)-u(x)|^{p} \mathrm{~d} x \leq 2^{p}\left(1 \wedge|h|^{p}\right)\|u\|_{W^{1, p}\left(\mathbb{R}^{d}\right)}^{p} .
$$

Integrating both side over $\mathbb{R}^{d}$ with respect to the measure $\nu(h) \mathrm{d} h$ yields

$$
\begin{equation*}
\iint_{\mathbb{R}^{d} \mathbb{R}^{d}}|u(x)-u(y)|^{p} \nu(x-y) \mathrm{d} y \mathrm{~d} x \leq 2^{p}\|\nu\|_{L^{1}\left(\mathbb{R}^{d}, 1 \wedge|h| p \mathrm{~d} h\right)}\|u\|_{W^{1, p}\left(\mathbb{R}^{d}\right)}^{p} . \tag{3.5}
\end{equation*}
$$

The desired embedding readily follows.
Remark 3.13. The embedding $W^{1, p}(\Omega) \hookrightarrow W_{\nu}^{p}(\Omega)$ may fail if $\Omega$ is not an extension domain (see [Fog23, Counterexample 3.10] or [DNPV12, Example 9.1]). More importantly, [Fog23,Fog20] $W^{1, p}(\Omega)$ can be viewed as limiting space of the nonlocal space of type $W_{\nu}^{p}(\Omega)$ and $W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$; see for instance [Fog23].

## 4. Characterization of the $p$-Lévy integrability

We will now see that the $p$-Lévy integrability condition (L) is optimal and can be self-generated from the associated energy form. In fact the $p$-Lévy integrability condition (L) renders the space $W_{\nu}^{p}\left(\mathbb{R}^{d}\right)$ more consistent, in a sense that it warrants the space $W_{\nu}^{p}\left(\mathbb{R}^{d}\right)$ to contain smooth functions.

Theorem 4.1. Assume $\nu \geq 0$ is symmetric. The following assertions are equivalent.
(i) The p-Lévy condition ( L ) holds, i.e. $\nu \in L^{1}\left(\mathbb{R}^{d}, 1 \wedge|h|^{p}\right)$.
(ii) The embedding $W^{1, p}\left(\mathbb{R}^{d}\right) \hookrightarrow W_{\nu}^{p}\left(\mathbb{R}^{d}\right)$ is continuous.
(iii) $\mathcal{E}_{\mathbb{R}^{d}}(u, u)<\infty$ for all $u \in W^{1, p}\left(\mathbb{R}^{d}\right)$.
(iv) $\mathcal{E}_{\mathbb{R}^{d}}(u, u)<\infty$ for all $u \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$.

Moreover, this also remains true when $p=1$ with $B V\left(\mathbb{R}^{d}\right)$ in place of $W^{1,1}\left(\mathbb{R}^{d}\right)$.
Proof. (i) $\Longrightarrow$ (ii) The estimate (3.5) readily yields the continuity of the embedding $W^{1, p}\left(\mathbb{R}^{d}\right) \hookrightarrow W_{\nu}^{p}\left(\mathbb{R}^{d}\right)$. The implications $(i i) \Longrightarrow(i i i)$ and $(i i i) \Longrightarrow(i v)$ are straightforward. It remains to prove that $(i v) \Longrightarrow(i)$. To this end, assume that $\mathcal{E}_{\mathbb{R}^{d}}(u, u)<\infty$ for all $u \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$. Let $\zeta \in C_{c}^{\infty}(\mathbb{R})$ with $\zeta \not \equiv 0$ and $\operatorname{supp} \zeta \subset(0,1)$ and $\vartheta \in C_{c}^{\infty}\left(\mathbb{R}^{d-1}\right)$ such that $\vartheta=1$ on $Q^{\prime}$ with $Q^{\prime}=[0,1]^{d-1}$. Consider $u_{i}(x)=\zeta \otimes \vartheta\left(x_{i}, x_{i}^{\prime}\right)=\zeta\left(x_{i}\right) \vartheta\left(x_{i}^{\prime}\right)$ where $x_{i}^{\prime}=\left(x_{1}, \cdots x_{i-1}, x_{i+1} \cdots, x_{d}\right)$. Clearly we have $u_{i} \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$ and $\nabla u_{i}(x)=\zeta^{\prime}\left(x_{i}\right) e_{i}$ for $x \in Q=[0,1]^{d}$. By the continuity of the shift, for each $\varepsilon>0$, there is $\delta>0$ such that

$$
\|\nabla u(\cdot+h)-\nabla u\|_{L^{p}(Q)}^{p}<\varepsilon \quad \text { for all }|h| \leq \delta .
$$

Using this and the inequality $b^{p} \geq 2^{1-p}(a+b)^{p}-a^{p}, a, b \geq 0$ we find that

$$
\begin{aligned}
\mathcal{E}_{\mathbb{R}^{d}}\left(u_{i}, u_{i}\right) & \geq \int_{Q} \int_{B_{\delta}(0)}\left|\int_{0}^{1} \nabla u_{i}(x+t h) \cdot h \mathrm{~d} t\right|^{p} \nu(h) \mathrm{d} h \mathrm{~d} x \\
& \geq 2^{1-p} \int_{Q} \int_{B_{\delta}(0)}\left|\nabla u_{i}(x) \cdot h\right|^{p} \nu(h) \mathrm{d} h \mathrm{~d} x-\varepsilon \int_{B_{\delta}(0)}|h|^{p} \nu(h) \mathrm{d} h \\
& =2^{1-p} \int_{Q}\left|\zeta^{\prime}\left(x_{i}\right)\right|^{p} \mathrm{~d} x \int_{B_{\delta}(0)}\left|h_{i}\right|^{p} \nu(h) \mathrm{d} h-\varepsilon \int_{B_{\delta}(0)}|h|^{p} \nu(h) \mathrm{d} h .
\end{aligned}
$$

Note that $\int_{Q}\left|\zeta^{\prime}\left(x_{i}\right)\right|^{p} \mathrm{~d} x=\left\|\zeta^{\prime}\right\|_{L^{p}(0,1)}^{p}$. We have $\left|h_{1}\right|^{p}+\cdots+\left|h_{d}\right|^{p} \geq c_{d, p}|h|^{p}$ for some $c_{d, p}>0$, by equivalence of Euclidean norms. Taking $\varepsilon=\frac{C}{2 d}$ with $C=2^{1-p} c_{d, p}\left\|\zeta^{\prime}\right\|_{L^{p}(0,1)}^{p}$ we get

$$
\begin{equation*}
\sum_{i=1}^{d} \mathcal{E}_{\mathbb{R}^{d}}\left(u_{i}, u_{i}\right) \geq(C-d \varepsilon) \int_{B_{\delta}(0)}|h|^{p} \nu(h) \mathrm{d} h=\frac{C}{2} \int_{B_{\delta}(0)}|h|^{p} \nu(h) \mathrm{d} h \tag{4.1}
\end{equation*}
$$

Now, consider $u \in C_{c}^{\infty}\left(B_{\delta / 2}(0)\right), u \not \equiv 0$. Since $B_{\delta / 2}(0) \subset B_{\delta}(x)$ for all $x \in B_{\delta / 2}(0)$ we get

$$
\begin{equation*}
\mathcal{E}_{\mathbb{R}^{d}}(u, u) \geq 2 \int_{B_{\delta / 2}(0)}|u(x)|^{p} \int_{B_{\delta / 2}^{c}(0)} \nu(x-y) \mathrm{d} y \mathrm{~d} x \geq 2\|u\|_{L^{p}\left(B_{\delta / 2}(0)\right)}^{p} \int_{B_{\delta}^{c}(0)} \nu(h) \mathrm{d} h \tag{4.2}
\end{equation*}
$$

Combing (4.1) and (4.2) yields $\nu \in L^{1}\left(\mathbb{R}^{d}, 1 \wedge|h|^{p}\right)$ that is $\nu$ satisfies (L). The case $p=1$ follows analogously.
Lemma 4.2. Let $\phi \in L^{1}\left(\mathbb{R}^{d}\right)$ such that $\phi \geq 0$ and $\int_{\mathbb{R}^{d}} \phi(z) \mathrm{d} z=1$. For all $u \in L_{\mathrm{loc}}^{1}\left(\mathbb{R}^{d}\right)$,

$$
\mathcal{E}_{\mathbb{R}^{d}}(u * \phi, u * \phi) \leq \mathcal{E}_{\mathbb{R}^{d}}(u, u)
$$

Proof. From Jensen's inequality and simple change of variables we find that

$$
\begin{aligned}
\mathcal{E}_{\mathbb{R}^{d}}(u, u) & =\int_{\mathbb{R}^{d}} \phi(z) \iint_{\mathbb{R}^{d} \mathbb{R}^{d}}|u(x-z)-u(y-z)|^{p} \nu(x-y) \mathrm{d} y \mathrm{~d} x \mathrm{~d} z \\
& \geq \iint_{\mathbb{R}^{d} \mathbb{R}^{d}}\left|\int_{\mathbb{R}^{d}} \phi(z)(u(x-z)-u(y-z)) \mathrm{d} z\right|^{p} \nu(x-y) \mathrm{d} y \mathrm{~d} x \\
& =\iint_{\mathbb{R}^{d} \mathbb{R}^{d}}|u * \phi(x)-u * \phi(y)|^{p} \nu(x-y) \mathrm{d} y \mathrm{~d} x=\mathcal{E}_{\mathbb{R}^{d}}(u * \phi, u * \phi) .
\end{aligned}
$$

Theorem 4.3. Assume $\nu$ is radial, then $\nu$ satisfies $(\mathrm{L})$ if and only if $W_{\nu}^{p}\left(\mathbb{R}^{d}\right) \neq\{0\}$.
Proof. In view of Theorem 4.1 we know $C_{c}^{\infty}\left(\mathbb{R}^{d}\right) \subset W_{\nu}^{p}\left(\mathbb{R}^{d}\right)$ if $\nu$ satisfies (L). Conversely, assume there is $v \in$ $W_{\nu}^{p}\left(\mathbb{R}^{d}\right)$ such that $v \not \equiv 0$ then also $|v| \in W_{\nu}^{p}\left(\mathbb{R}^{d}\right)$ with $\mathcal{E}(|v|,|v|) \leq \mathcal{E}(v, v)$, since $\|v(x)|-|v(y) \| \leq|v(x)-v(y)|$. Let $\phi(x)=\kappa e^{-|x|^{2}}$ satisfies $\|\phi\|_{L^{1}\left(\mathbb{R}^{d}\right)}=1$ for some $\kappa>0$. By Young's inequality and Lemma 4.2 we get $u=|v| * \phi \in$ $W_{\nu}^{p}\left(\mathbb{R}^{d}\right)$. Moreover,

$$
\mathcal{E}(u, u) \leq \mathcal{E}(|v|,|v|) \leq \mathcal{E}(v, v)<\infty
$$

Clearly $u \in C^{\infty}\left(\mathbb{R}^{d}\right) \cap L^{\infty}\left(\mathbb{R}^{d}\right)$ and $u(x)>0$ for each $x \in \mathbb{R}^{d}$, i.e., $\operatorname{supp} u=\mathbb{R}^{d}$. On the other hand, Young's inequality implies $u \in W^{1, p}\left(\mathbb{R}^{d}\right)$. In particular we have $\|\nabla u\|_{L^{p}\left(\mathbb{R}^{d}\right)} \neq 0$.
For each $\varepsilon>0$, there is $\delta=\delta(\varepsilon)>0$ such that $\|\nabla u(\cdot+h)-\nabla u\|_{L^{p}\left(\mathbb{R}^{d}\right)}<\varepsilon$ for all $|h| \leq \delta$. Consider $C_{\delta}=$ $\int_{B_{\delta}(0)}|h|^{p} \nu(h) \mathrm{d} h$, then Minkowski's inequality implies

$$
\left(\iint_{\mathbb{R}^{d}}|\nabla u(x) \cdot h|^{p} \nu(h) \mathrm{d} h \mathrm{~d} x\right)^{1 / p} \leq\left(\iint_{\mathbb{R}^{d}}|(0)| \int_{B_{\delta}(0)}|\nabla u(x+t h) \cdot h \mathrm{~d} t|^{p} \nu(h) \mathrm{d} h \mathrm{~d} x\right)^{1 / p}+\varepsilon C_{\delta}^{1 / p}
$$

The fundamental theorem of calculus and a passage to polar coordinates yield

$$
\begin{aligned}
\mathcal{E}(u, u)^{1 / p} & \geq\left(\iint_{\mathbb{R}^{d}}\left|\int_{B_{\delta}(0)}^{1} \nabla u(x+t h) \cdot h \mathrm{~d} t\right|^{p} \nu(h) \mathrm{d} h \mathrm{~d} x\right)^{1 / p} \\
& \geq\left(\int_{\mathbb{R}^{d}} \int_{\mathbb{S}^{d}-1}|\nabla u(x) \cdot w|^{p} \mathrm{~d} \sigma_{d-1}(w) \int_{0}^{\delta} r^{p+d-1} \nu(r) \mathrm{d} r\right)^{1 / p}-\varepsilon\left(\int_{B_{\delta}(0)}|h|^{p} \nu(h) \mathrm{d} h\right)^{1 / p} \\
& =\left(K_{d, p}^{1 / p}\|\nabla u\|_{L^{p}\left(\mathbb{R}^{d}\right)}-\varepsilon\right)\left(\int_{B_{\delta}(0)}|h|^{p} \nu(h) \mathrm{d} h\right)^{1 / p}
\end{aligned}
$$

Necessarily, we get $\int_{B_{\delta}(0)}|h|^{p} \nu(h) \mathrm{d} h<\infty$ since, taking $\varepsilon=\frac{K_{d, p}^{1 / p}}{2}\|\nabla u\|_{L^{p}\left(\mathbb{R}^{d}\right)}$ we get

$$
\mathcal{E}(u, u) \geq \frac{K_{d, p}}{2}\|\nabla u\|_{L^{p}\left(\mathbb{R}^{d}\right)}^{p} \int_{B_{\delta}(0)}|h|^{p} \nu(h) \mathrm{d} h .
$$

Next, it is not difficult to show that $u(x) \rightarrow 0$ as $|x| \rightarrow \infty$. Thus for $r>0$ there is $R>\delta>0$ such that $0<u(x) \leq r$ whenever $|x|>\frac{R}{2}$. Now consider the 1 -Lipschitz function

$$
\zeta_{r}(s)=\max (-r, \min (r, s))= \begin{cases}s & |s| \leq r \\ r \operatorname{sgn}(s) & |s| \geq r\end{cases}
$$

It is readily seen that $\left|\zeta_{r}(u(x))-\zeta_{r}(u(y))\right| \leq|u(x)-u(y)|$ and $0 \leq \zeta_{r}(u(x)) \leq u(x)$. In particular, $\mathcal{E}(u, u) \geq$ $\mathcal{E}\left(\zeta_{r}(u), \zeta_{r}(u)\right)$. If we set $w_{r}(x)=u(x)-\zeta_{r}(u(x))$ then $w_{r}(x)=0$ whenever $|x|>R / 2$. Since $B_{R / 2}(x) \subset B_{R}(0)$ for all $x \in B_{R / 2}(0)$, we find that

$$
2^{p} \mathcal{E}(u, u) \geq \mathcal{E}\left(w_{r}, w_{r}\right) \geq \int_{B_{R / 2}(0)}\left|w_{r}(x)\right|^{p} \int_{\mathbb{R}^{d} \backslash B_{R / 2}(0)} \nu(x-y) \mathrm{d} y \mathrm{~d} x \geq\left\|w_{r}\right\|_{L^{p}\left(\mathbb{R}^{d}\right)}^{p} \int_{|h| \geq R} \nu(h) \mathrm{d} h
$$

Given that $u$ is not constant we get $\left\|w_{r}\right\|_{L^{p}\left(\mathbb{R}^{d}\right)}^{p} \neq 0$. Therefore we deduce that

$$
\int_{|h| \geq R} \nu(h) \mathrm{d} h<\infty .
$$

Last, note that for $\delta \leq|h| \leq R$ and $x \in B_{\frac{\delta}{4}}(0)$ we have $|x|<\frac{\delta}{4}<\frac{\delta}{2} \leq|x+h| \leq R+\frac{\delta}{2}$. Since $u$ is smooth with full support we put

$$
M=\min _{\frac{\delta}{2} \leq z \leq R+\frac{\delta}{2}} \int_{B_{\frac{\delta}{4}}(0)}|u(x)-u(z)|^{p} \mathrm{~d} x>0
$$

From this we find that

$$
\mathcal{E}(u, u) \geq \int_{B_{R}(0) \backslash B_{\delta}(0)} \int_{B_{\frac{\delta}{4}}(0)}|u(x)-u(x+h)|^{p} \mathrm{~d} x \nu(h) \mathrm{d} h \geq M \int_{B_{R}(0) \backslash B_{\delta}(0)} \nu(h) \mathrm{d} h .
$$

Hence, it follows that

$$
\int_{B_{R}(0) \backslash B_{\delta}(0)} \nu(h) \mathrm{d} h<\infty
$$

Finally, altogether we find that $\nu$ is $p$-Lévy integrable, i.e., $\nu \in L^{1}\left(\mathbb{R}^{d}, 1 \wedge|h|^{p}\right)$.
The following result is a straightforwards consequence of Theorem 4.1 and Theorem 4.3.
Theorem 4.4. Assume $\nu \geq 0$ is radial. The following assertions are equivalent.
(i) The p-Lévy condition $(\mathrm{L})$ holds, i.e. $\nu \in L^{1}\left(\mathbb{R}^{d}, 1 \wedge|h|^{p}\right)$.
(ii) The embedding $W^{1, p}\left(\mathbb{R}^{d}\right) \hookrightarrow W_{\nu}^{p}\left(\mathbb{R}^{d}\right)$ is continuous.
(iii) $\mathcal{E}_{\mathbb{R}^{d}}(u, u)<\infty$ for all $u \in W^{1, p}\left(\mathbb{R}^{d}\right)$.
(iv) $\mathcal{E}_{\mathbb{R}^{d}}(u, u)<\infty$ for all $u \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$.
(v) The space $W_{\nu}^{p}\left(\mathbb{R}^{d}\right)$ is nontrivial, i.e., $W_{\nu}^{p}\left(\mathbb{R}^{d}\right) \neq\{0\}$.

Moreover, this also remains true when $p=1$ with $B V\left(\mathbb{R}^{d}\right)$ in place of $W^{1,1}\left(\mathbb{R}^{d}\right)$.
As illustrated in the next result, the Lévy integrability (L) draws the borderline for which a space of type $W_{\nu}^{p}\left(\mathbb{R}^{d}\right)$ is trivial or not, see [Fog23, Proposition 2.15] or [Fog20, Proposition 3.46] for a general setting.

Proposition 4.5. Assume $\nu \geq 0$ is symmetric. The following assertions hold true.
(i) If $\nu \in L^{1}\left(\mathbb{R}^{d}\right)$, then $W_{\nu}^{p}\left(\mathbb{R}^{d}\right)=L^{p}\left(\mathbb{R}^{d}\right)$ with equivalence in norms.
(ii) If $\nu \in L^{1}\left(\mathbb{R}^{d}, 1 \wedge|h|^{p} \mathrm{~d} h\right)$ and $\Omega$ is bounded, then $W_{\nu}^{p}(\Omega)$ and $W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ contain all bounded Lipschitz functions.
(iii) Consider $C_{\delta}=\int_{B_{\delta}(0)}|h|^{p} \nu(h) \mathrm{d} h$. If $\nu$ is radial, $\Omega$ is connected and $C_{\delta}=\infty \forall \delta>0$, in particular $\nu \notin L^{1}\left(\mathbb{R}^{d}, 1 \wedge|h|^{p} \mathrm{~d} h\right)$, then $u \in W^{1, p}(\Omega) \cap W_{\nu}^{p}(\Omega)$ or $u \in C^{1}(\Omega) \cap W_{\nu}^{p}(\Omega)$ if and only if $u=c$ is a constant function.
(iv) If $\nu$ is radial, then for any $u \in W^{1, p}\left(\mathbb{R}^{d}\right)$ there is $\delta=\delta(u)>0$, such that

$$
\frac{K_{d, p}}{2} C_{\delta}\|\nabla u\|_{L^{p}\left(\mathbb{R}^{d}\right)}^{p} \leq|u|_{W_{\nu}^{p}\left(\mathbb{R}^{d}\right)}^{p} \leq 2^{p}\|\nu\|_{L^{1}\left(\mathbb{R}^{d}, 1 \wedge|h|^{p} \mathrm{~d} h\right)}\|u\|_{W^{1, p}\left(\mathbb{R}^{d}\right)}^{p} .
$$

## 5. Nonlocal Trace Theorem

5.1. Weighted $L^{p}$-spaces. In order to set up the Dirichlet and Neumann problem in $L^{p}$-spaces over $\mathbb{R}^{d}$, we introduce Borel measures on $\mathbb{R}^{d}$ that capture the behavior of $\nu$ at infinity. We will need the notion of unimodality.

Definition 5.1. $\nu$ is called unimodal if is radial with an almost decreasing profile, i.e., there is $c>0$ such that $\nu(|x|) \geq c \nu(|y|)$ for all whenever $|x| \leq|y|$.

Definition 5.2. Let $\nu$ satisfies the $p$-Lévy integrability condition (L) and $B \subset \mathbb{R}^{d}$ be open with positive measure, i.e., $|B|>0$. Define the Borel measures $\bar{\nu}_{B}, \widetilde{\nu}_{B}: \mathbb{R}^{d} \rightarrow[0, \infty)$ by

$$
\begin{aligned}
& \widetilde{\nu}_{B}(x)=\int_{B}(1 \wedge \nu(x-y)) \mathrm{d} y \\
& \bar{\nu}_{B}(x)=\operatorname{ess} \inf _{y \in B} \nu(x-y)
\end{aligned}
$$

If $\nu$ is a unimodal, then we define the Borel measure $\widehat{\nu}_{R}: \mathbb{R}^{d} \rightarrow[0, \infty)$ by

$$
\widehat{\nu}_{R}(x)=\nu(R(1+|x|)),
$$

where $R>1$ is an arbitrary fixed number.
It is worthwhile noticing that $\nu$ needs not be unimodal for the definition of $\widetilde{\nu}_{B}$ and $\bar{\nu}_{B}$. Let us discuss important properties of the three measures $\widetilde{\nu}_{B}, \bar{\nu}_{B}$, and $\widehat{\nu}_{R}$.

Proposition 5.3. The following assertion are true.
(i) $\widehat{\nu}_{R}, 1 \wedge \nu \in L^{1}\left(\mathbb{R}^{d}\right) \cap L^{\infty}\left(\mathbb{R}^{d}\right)$.
(ii) $\widetilde{\nu}_{B} \in L_{\mathrm{loc}}^{1}\left(\mathbb{R}^{d}\right) \cap L^{\infty}\left(\mathbb{R}^{d}\right)$. If $|B|<\infty$, then $\widetilde{\nu}_{B} \in L^{1}\left(\mathbb{R}^{d}\right) \cap L^{\infty}\left(\mathbb{R}^{d}\right)$.
(iii) $\bar{\nu}_{B} \in L^{1}\left(\mathbb{R}^{d}\right)$. If $\nu$ unimodal then $\bar{\nu}_{B} \in L^{1}\left(\mathbb{R}^{d}\right) \cap L^{\infty}\left(\mathbb{R}^{d}\right)$.
(iv) $\bar{\nu}_{B}(x)=\widetilde{\nu}_{B}(x)=0$ for all $x \notin B_{\nu}:=B+\operatorname{supp} \nu$.
(v) For $B_{1} \subset B_{2}$ we have $\bar{\nu}_{B_{2}} \leq \bar{\nu}_{B_{1}}$ and $\widetilde{\nu}_{B_{1}} \leq \widetilde{\nu}_{B_{2}}$.

Proof. The proof of $(i)-($ iii $)$ can be found in [Fog20, Chapter 3] or adapted from [FK22, Section 2.3]. To prove (iv), let $x \notin B+\operatorname{supp} \nu$ then $x-y \notin \operatorname{supp} \nu$ for all $y \in B$. In other words $\nu(x-y)=0$ for almost all $y \in B$. Whence, $\bar{\nu}_{B}(x)=\widetilde{\nu}_{B}(x)=0 .(v)$ is blatantly obvious.

The weights $\widehat{\nu}_{R}, \widetilde{\nu}_{B}$ and $\bar{\nu}_{B}$ turn out to be comparable when $\nu$ is unimodal and satisfies the doubling scaling condition at infinity (5.2).

Definition 5.4. We say that a radial kernel $\nu$ satisfies a doubling condition at infinity if:

$$
\begin{equation*}
\text { For every } \theta \geq 1 \text { there exist } c_{1}, c_{2}>0 \text { with } c_{1} \nu(r) \leq \nu(\theta r) \leq c_{2} \nu(r) \text { for all } r \geq 1 \text {. } \tag{5.1}
\end{equation*}
$$

Not that the property (5.1) is indeed equivalent to say that

$$
\begin{equation*}
\text { There exist } c_{1}, c_{2}>0 \text { with } c_{1} \nu(r) \leq \nu(2 r) \leq c_{2} \nu(r) \text { for all } r \geq 1 \tag{5.2}
\end{equation*}
$$

Obviously, the doubling condition is only relevant when the support of $\nu$ is large enough.
Theorem 5.5. Assume $\nu$ is unimodal and $B$ is bounded, e.g. say $B \subset B_{R}(0)$ then

$$
\begin{aligned}
& \sigma_{R}^{*} C^{-1}(1 \wedge \nu(2 R x)) \leq \widehat{\nu}_{R}(x) \leq C(1 \wedge \nu(x)) \\
& C^{-1} \widehat{\nu}_{R}(x) \leq \bar{\nu}_{B}(x) \leq C \widetilde{\nu}_{B}(x) \\
& \widetilde{\nu}_{B}(x) \leq \sigma_{R} C\left(1 \wedge \nu\left(\frac{x}{2}\right)\right)
\end{aligned}
$$

where $\sigma_{R}=\left(1+\nu^{-1}(2 R)\right)$ and $\sigma_{R}^{*}=1 \wedge \nu(2 R)$. In addition, if $\nu$ is of full support and satisfies the doubling condition (5.1) then

$$
\widetilde{\nu}_{B}(x) \asymp \bar{\nu}_{B}(x) \asymp \widehat{\nu}_{R}(x) \asymp 1 \wedge \nu(x) .
$$

Here, the constant $C>0$ and the constants behind the relation $\asymp$ are generic and depend on $B, R$ and $\nu$. Moreover, one notes that $\sigma_{R}^{*}=\sigma_{R}^{-1}=0$ if $\nu(2 R)=0$.

Proof. Let us fix $x \in \mathbb{R}^{d}$. Since $R \leq R(1+|x|)$ and $|x| \leq R(1+|x|)$, by unimodality,

$$
\widehat{\nu}_{R}(x) \leq C 1 \wedge \nu(x) \quad \text { with } \quad C=c(1+\nu(R)) .
$$

For $|x| \leq 1$ we have $R(1+|x|) \leq 2 R$, the unimodality implies

$$
c \widehat{\nu}_{R}(x) \geq \nu(2 R) \geq \nu(2 R)(1 \wedge \nu(2 R x)) \geq \sigma_{R}^{*}(1 \wedge \nu(2 R x))
$$

For $|x| \geq 1$ we have $R(1+|x|) \leq 2 R|x|$ the unimodality implies $c \widehat{\nu}_{R}(x) \geq \nu(2 R x) \geq \sigma_{R}^{*}(1 \wedge \nu(2 R x))$. In any case, taking $C=c(1+\nu(R))$, we find that

$$
\sigma_{R}^{*} C^{-1}(1 \wedge \nu(2 R x)) \leq \widehat{\nu}_{R}(x) \leq C(1 \wedge \nu(x))
$$

Next, according to Proposition $5.3 \bar{\nu}_{B}$ is bounded, thus by definition of $\bar{\nu}_{B}$

$$
\bar{\nu}_{B}(x) \leq\left(1+\left\|\bar{\nu}_{B}\right\|_{L^{\infty}\left(\mathbb{R}^{d}\right)}\right)(1 \wedge \nu(x-y)) \quad \text { for almost all } y \in B
$$

whence letting $C=|B|^{-1}\left(1+\left\|\bar{\nu}_{B}\right\|_{L^{\infty}\left(\mathbb{R}^{d}\right)}\right)$ we find that

$$
\bar{\nu}_{B}(x) \leq C \int_{B} 1 \wedge \nu(x-y) \mathrm{d} y=C \widetilde{\nu}_{B}(x)
$$

We find that $|x-y| \leq R(1+|x|)$ for $y \in B$, the unimodality implies $\widehat{\nu}_{R}(x) \leq c \nu(x-y)$. Hence we get

$$
c^{-1} \widehat{\nu}_{R}(x) \leq \underset{y \in B}{\operatorname{essinf}} \nu(x-y)=\bar{\nu}_{B}(x)
$$

Now, for $\frac{|x|}{2} \leq 2 R$ the unimodality implies $1 \leq c \nu^{-1}(2 R) \nu\left(\frac{x}{2}\right) \leq c \sigma_{R} \nu\left(\frac{x}{2}\right)$. Since by definition of $\widetilde{\nu}_{B}(x)$ we have $\widetilde{\nu}_{B}(x) \leq\|1 \wedge \nu\|_{L^{1}\left(\mathbb{R}^{d}\right)}$, we deduce that

$$
\widetilde{\nu}_{B}(x) \leq \sigma_{R}(c+1)\|1 \wedge \nu\|_{L^{1}\left(\mathbb{R}^{d}\right)}\left(1 \wedge \nu\left(\frac{x}{2}\right)\right)
$$

If $\frac{|x|}{2} \geq 2 R$ then $\frac{|x|}{2} \leq|x-y|$ for all $y \in B \subset B_{R}(0)$. By unimodality we get $\nu(x-y) \leq c \nu\left(\frac{x}{2}\right)$ which implies $1 \wedge \nu(x-y) \leq(c+1) 1 \wedge \nu\left(\frac{x}{2}\right)$ for all $y \in B$. whence we get

$$
\widetilde{\nu}_{B}(x)=\int_{B} 1 \wedge \nu(x-y) \mathrm{d} y \leq \sigma_{R}(c+1)|B|\left(1 \wedge \nu\left(\frac{x}{2}\right)\right)
$$

In any case, taking $C=(c+1)\left(|B|+\|1 \wedge \nu\|_{L^{1}\left(\mathbb{R}^{d}\right)}\right)$ we find that

$$
\widetilde{\nu}_{B}(x) \leq \sigma_{R} C\left(1 \wedge \nu\left(\frac{x}{2}\right)\right)
$$

Last, assume that $\nu$ has full support and satisfies the doubling condition (5.1). In view of the previous estimates, it sufficient to show that $1 \wedge \nu\left(\frac{x}{2}\right) \asymp 1 \wedge \nu(x) \asymp 1 \wedge \nu(2 R x)$. For $|x| \leq 2$ then unimodality implies $c \nu(2 R x) \geq \nu(2 R) \geq$ $\nu(2 R)(1 \wedge \nu(x))$. Analogously, $c \nu(x) \geq \nu(2)\left(1 \wedge \nu\left(\frac{x}{2}\right)\right)$. Thus, deduce that

$$
\left(1+c \nu^{-1}(2)\right)^{-1} 1 \wedge \nu\left(\frac{x}{2}\right) \leq 1 \wedge \nu(x) \leq\left(1+c \nu^{-1}(2 R)\right) 1 \wedge \nu(2 R x)
$$

For $|x| \geq 2$, the doubling condition (5.1) implies $c_{1} \nu\left(\frac{x}{2}\right) \leq \nu(x) \leq c_{2} \nu(2 R x)$, so that

$$
\left(c_{1}^{-1}+1\right)^{-1} 1 \wedge \nu\left(\frac{x}{2}\right) \leq 1 \wedge \nu(x) \leq\left(c_{2}+1\right) 1 \wedge \nu(2 R x)
$$

Finally, we put $C=\max \left(c_{1}^{-1}+1, c_{2}+1,1+c \nu^{-1}(2 R), 1+c \nu^{-1}(2)\right)$ so that $0<C<\infty$ since $\nu$ has full support and hence we get

$$
C^{-1}\left(1 \wedge \nu\left(\frac{x}{2}\right)\right) \leq 1 \wedge \nu(x) \leq C(1 \wedge \nu(2 R x)
$$

This achieves the proof.
Example 5.6. Consider $\nu(h)=|h|^{-d-s p}, s \in(0,1)$ then $\widetilde{\nu} \asymp 1 \wedge \nu$. In this case one can take $\widetilde{\nu}(h)=(1+|h|)^{-d-s p}$. In this case space $W_{\nu}^{p}(\Omega)$ equals the classical Sobolev-Slobodeckij space $W^{s, p}(\Omega)$. For the same choice of $\nu$ we define $W^{s, p}\left(\Omega \mid \mathbb{R}^{d}\right)$ as the space $W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$.

Lemma 5.7. Assume that $B \subset \Omega$ where $B$ is bounded, say $B \subset B_{R}(0)$. Let $\omega \in\left\{\widetilde{\nu}_{B}, \bar{\nu}_{B}, \widehat{\nu}_{R}\right\}$. If $\nu$ is unimodal with full support then on $W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$, the following norms $\|\cdot\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}^{\#}$ and $\|\cdot\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}^{*}$ are equivalent.

$$
\begin{aligned}
\|u\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}^{* p} & =\int_{\mathbb{R}^{d}}|u(x)|^{p} \omega(x) \mathrm{d} x+\iint_{\left(\Omega^{c} \times \Omega^{c}\right)^{c}}|u(x)-u(y)|^{p} \nu(x-y) \mathrm{d} y \mathrm{~d} x \\
\|u\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}^{\# p} & =\iint_{\Omega}|u(x)|^{p} \omega(x) \mathrm{d} x+\iint_{\left(\Omega^{c} \times \Omega^{c}\right)^{c}}|u(x)-u(y)|^{p} \nu(x-y) \mathrm{d} y \mathrm{~d} x
\end{aligned}
$$

Furthermore, if $\Omega$ is bounded then the norms $\|\cdot\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}$ and $\|\cdot\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}^{*}$ are also equivalent.
Proof. In virtue of Theorem 5.5 we have $C^{-1} \widehat{\nu}_{R}(x) \leq \omega(x) \leq C \widetilde{\nu}_{B}(x)$ for some $C>0$. Whence, since $B$ is bounded and $\nu$ is of full support, the unimodality implies that there is $c^{\prime}>0$ such that $\omega(x) \geq c^{\prime}$ for almost all $x \in B$. This together with the facts that $B \subset \Omega$ and $\int_{\Omega^{c}}(1 \wedge \nu(x-y)) \mathrm{d} y \leq\|1 \wedge \nu\|_{L^{1}\left(\mathbb{R}^{d}\right)}$ implies

$$
\begin{aligned}
& \int_{\Omega}|u(x)|^{p} \omega(x) \mathrm{d} x+\iint_{\Omega \Omega^{c}}|u(x)-u(y)|^{p} \nu(x-y) \mathrm{d} y \mathrm{~d} x \\
& \quad \geq c^{\prime} \int_{B}|u(x)|^{p} \mathrm{~d} x+\iint_{\Omega \Omega^{c}}|u(x)-u(y)|^{p} \nu(x-y) \mathrm{d} y \mathrm{~d} x \\
& \quad \geq c^{\prime}\|1 \wedge \nu\|_{L^{1}\left(\mathbb{R}^{d}\right)}^{-1} \iint_{B \Omega^{c}}|u(x)|^{p}(1 \wedge \nu(x-y)) \mathrm{d} y \mathrm{~d} x+\iint_{B \Omega^{c}}|u(x)-u(y)|^{p} \nu(x-y) \mathrm{d} y \mathrm{~d} x \\
& \quad \geq\left(1 \wedge c^{\prime}\|1 \wedge \nu\|_{L^{1}\left(\mathbb{R}^{d}\right)}^{-1}\right) \iint_{\Omega^{c} B}\left[|u(x)|^{p}+|u(x)-u(y)|^{p}\right] 1 \wedge \nu(x-y) \mathrm{d} x \mathrm{~d} y .
\end{aligned}
$$

Using $b^{p} \geq 2^{1-p}(a+b)^{p}-a^{p}, a, b \geq 0$ and $\omega(y) \leq C \widetilde{\nu}_{B}(y)$ we get

$$
\int_{\Omega}|u(x)|^{p} \omega(x) \mathrm{d} x+\iint_{\Omega \Omega^{c}}|u(x)-u(y)|^{p} \nu(x-y) \mathrm{d} y \mathrm{~d} x \geq C_{1} \int_{\Omega^{c}}|u(y)|^{p} \omega(y) \mathrm{d} y
$$

with $C_{1}=\frac{C^{-1}}{2^{p-1}}\left(1 \wedge c^{\prime}\|1 \wedge \nu\|_{L^{1}\left(\mathbb{R}^{d}\right)}^{-1}\right)$. This clearly implies that $\|u\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}^{*} \leq C\|u\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}^{\#}$ for some $C>0$. Thus the norms $\|\cdot\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}^{*}$ and $\|\cdot\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}^{\#}$ are equivalent.
Now if $\Omega$ is bounded, then by unimodality and boundedness of $\omega$ (see Proposition 5.3) we also have $\|\omega\|_{L^{\infty}\left(\mathbb{R}^{d}\right)} \geq$ $\omega(x) \geq c^{\prime}, x \in \Omega$ for some $c^{\prime}>0$. The equivalence of the norms $\|\cdot\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}$ and $\|\cdot\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}^{\#}$ readily follows.

The next result shows that functions in $W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ possess certain weighted integrability.
Theorem 5.8. Assume $B \subset \Omega$ and $R \geq 1$. Let $\omega \in\left\{\widetilde{\nu}_{B}, \bar{\nu}_{B}, \widehat{\nu}_{R}\right\}$.
(i) For $\omega \in\left\{\widetilde{\nu}_{B}, \bar{\nu}_{B}\right\}$ we have the continuous embedding

$$
W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right) \hookrightarrow L^{p}\left(\mathbb{R}^{d}, \omega\right)
$$

The same holds for $\omega=\widehat{\nu}_{R}$ if in addition $\left|B_{R}(0) \cap \Omega\right|>0$.
(ii) For $\omega \in\left\{\bar{\nu}_{B}, \widehat{\nu}_{R}\right\}$ we have the continuous embeddings

$$
\begin{equation*}
L^{p}\left(\mathbb{R}^{d}, \omega\right) \hookrightarrow L^{p-1}\left(\mathbb{R}^{d}, \omega\right) \cap L^{1}\left(\mathbb{R}^{d}, \omega\right) \tag{5.3}
\end{equation*}
$$

equivalently for any $q \in[\min (1, p-1), p]$ we have

$$
\begin{equation*}
L^{p}\left(\mathbb{R}^{d}, \omega\right) \hookrightarrow L^{q}\left(\mathbb{R}^{d}, \omega\right) \hookrightarrow L^{\min (1, p-1)}\left(\mathbb{R}^{d}, \omega\right) \tag{5.4}
\end{equation*}
$$

The same holds for $\omega=\widetilde{\nu}_{B}$ if in addition $|B|<\infty$.
Proof. (i) The proof of the embedding $W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right) \hookrightarrow L^{p}\left(\mathbb{R}^{d}, \omega\right)$ can be adapted from [FK22, Section 2.3] and [Fog20, Lemma 3.24].
(ii) For $\omega \in\left\{\widetilde{\nu}_{B}, \bar{\nu}_{B}, \widehat{\nu}_{R}\right\}$ by assumptions, Proposition 5.3 implies $\omega \in L^{1}\left(\mathbb{R}^{d}\right)$. Thus Hölder's inequality implies $\|u\|_{L^{r}\left(\mathbb{R}^{d}, \omega\right)} \leq\|\omega\|_{L^{1}\left(\mathbb{R}^{d}\right)}^{1 / r-1 / q}\|u\|_{L^{q}\left(\mathbb{R}^{d}, \omega\right)}$ whenever $q \geq r>0$. The desired embeddings immediately follow.

Corollary 5.9. Let $\omega \in\left\{\widetilde{\nu}_{B}, \bar{\nu}_{B}, \widehat{\nu}_{R}\right\}$ where $B \subset \Omega$. Assume $|B|<\infty$ and $\left|B_{R}(0) \cap \Omega\right|>0, R \geq 1$. The following embeddings are continuous

$$
W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right) \hookrightarrow L^{p}\left(\mathbb{R}^{d}, \omega\right) \hookrightarrow L^{p-1}\left(\mathbb{R}^{d}, \omega\right) \cap L^{1}\left(\mathbb{R}^{d}, \omega\right)
$$

Proof. This is a straightforward consequence of Theorem 5.8.
Corollary 5.10. Assume $\nu$ is unimodal with of full support and satisfies the doubling condition (5.1). The following embeddings are continuous

$$
W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right) \hookrightarrow L^{p}\left(\mathbb{R}^{d}, 1 \wedge \nu\right) \hookrightarrow L^{p-1}\left(\mathbb{R}^{d}, 1 \wedge \nu\right) \cap L^{1}\left(\mathbb{R}^{d}, 1 \wedge \nu\right)
$$

Proof. This is a direct consequence of Corollary 5.9 and Theorem 5.5.
In the fractional setting $L^{p-1}\left(\mathbb{R}^{d}, \omega\right)$ is also sometimes called the fraction tail space.
Corollary 5.11. Let $\nu(h)=|h|^{-d-s p}$ with $s \in(0,1)$, we put $W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)=W^{s, p}\left(\Omega \mid \mathbb{R}^{d}\right)$. The following embeddings are continuous

$$
\left.\left.\left.W^{s, p}\left(\Omega \mid \mathbb{R}^{d}\right) \hookrightarrow L^{p}\left(\mathbb{R}^{d},(1+|h|)^{-d-s p}\right)\right) \hookrightarrow L^{p-1}\left(\mathbb{R}^{d},(1+|h|)^{-d-s p}\right)\right) \cap L^{1}\left(\mathbb{R}^{d},(1+|h|)^{-d-s p}\right)\right) .
$$

Proof. This follows from Corollary 5.10 since $1 \wedge \nu(h) \asymp(1+|h|)^{-d-s p}$.
5.2. Trace space of $W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$. The main goal of this part is to design an abstract notion trace space of $W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ similarly as one does for the space $W^{1, p}(\Omega)$. The nonlocal trace space of $W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ assumes functions defined on $\mathbb{R}^{d} \backslash \Omega$, or strictly speaking on the nonlocal boundary $\Omega_{e}=\Omega_{\nu} \backslash \Omega$ of $\Omega$ with respect to $\nu$. The main reason is that elements of $W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ are essentially defined on $\mathbb{R}^{d}$, or strictly speaking on the nonlocal hull $\Omega_{\nu}=\Omega+\operatorname{supp} \nu$. This contrasts with the local situation, where the trace space of $W^{1, p}(\Omega)$ (with $\Omega$ smooth enough) are elements defined on the boundary $\partial \Omega$.
Definition 5.12 (Trace space of $W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ ). The trace space of $W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ denoted $T_{\nu}^{p}\left(\Omega^{c}\right)$ is the space of restrictions to $\mathbb{R}^{d} \backslash \Omega$ of functions of $W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$. More precisely,

$$
T_{\nu}^{p}\left(\Omega^{c}\right)=\left\{v: \Omega^{c} \rightarrow \mathbb{R} \text { meas. } \quad \text { such that } \quad v=\left.u\right|_{\Omega^{c}} \quad \text { with } \quad u \in W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)\right\}
$$

We endow $T_{\nu}^{p}\left(\Omega^{c}\right)$ with its natural norm,

$$
\|v\|_{T_{\nu}^{p}\left(\Omega^{c}\right)}=\inf \left\{\|u\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}: \quad u \in W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right) \quad \text { with } \quad v=\left.u\right|_{\Omega^{c}}\right\}
$$

If we identify $W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right) \equiv W_{\nu}^{p}\left(\Omega \mid \Omega_{\nu}\right)$ then we can also identify $T_{\nu}^{p}\left(\Omega^{c}\right) \equiv T_{\nu}^{p}\left(\Omega_{e}\right)$ where $\Omega_{e}=\Omega_{\nu} \backslash \Omega$ and

$$
T_{\nu}^{p}\left(\Omega_{e}\right)=\left\{v: \Omega_{e} \rightarrow \mathbb{R} \text { meas. } \quad \text { such that } \quad v=\left.u\right|_{\Omega_{e}} \quad \text { with } \quad u \in W_{\nu}^{p}\left(\Omega \mid \Omega_{\nu}\right)\right\}
$$

Theorem 5.13. If the space $W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ is a Banach space then so is the space $T_{\nu}^{p}\left(\Omega^{c}\right)$.
Proof. Noting that $T_{\nu}^{p}\left(\Omega^{c}\right)$ and the quotient space $W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right) / W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ are identical with equal norm in space and that $W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ is a closed subspace of $W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$, one concludes that $T_{\nu}^{p}\left(\Omega^{c}\right)$ is complete. For a more detailed proof see [Fog20, Chapter 3].

Theorem 5.14 (Nonlocal Trace Theorem). Let $\omega \in\left\{\widetilde{\nu}_{\Omega}, \bar{\nu}_{\Omega}, \widehat{\nu}_{R}\right\}$ where $\left|B_{R}(0) \cap \Omega\right|>0$ (see Definition 5.2). Define the trace operator $u \mapsto \operatorname{Tr}(u)=\left.u\right|_{\Omega^{c}}$,
(i) $\operatorname{ker}(\operatorname{Tr})=W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ and $\operatorname{Tr}\left(W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)\right)=T_{\nu}^{p}\left(\Omega^{c}\right)$.
(ii) The mappings $W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right) \xrightarrow{\operatorname{Tr}} T_{\nu}^{p}\left(\Omega^{c}\right) \xrightarrow{\mathrm{Id}} L^{p}\left(\Omega^{c}, \omega\right)$ are continuous

Proof. It is easy to check that $\operatorname{Tr}\left(W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)\right)=T_{\nu}^{p}\left(\Omega^{c}\right)$ and $\operatorname{ker}(\operatorname{Tr})=W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$. It immediately follows from the definition of $\|\cdot\|_{T_{\nu}^{p}\left(\Omega^{c}\right)}$ that $\|\operatorname{Tr}(u)\|_{T_{\nu}^{p}\left(\Omega^{c}\right)} \leq\|u\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}$ for all $u \in W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ whereas, by Theorem 5.8 there exists $C>0$ such that

$$
\|\operatorname{Tr}(u)\|_{L^{p}\left(\Omega^{c}, \omega\right)} \leq\|u\|_{L^{p}\left(\mathbb{R}^{d}, \omega\right)} \leq C\|u\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)} \quad \text { for all } \quad u \in W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)
$$

Inasmuch as the above estimate is true for $v \in W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ we deduce $\|u\|_{L^{p}\left(\Omega^{c}, \omega\right)} \leq C\|u\|_{T_{\nu}^{p}\left(\Omega^{c}\right)}$ for all $u \in$ $T_{\nu}^{p}\left(\Omega^{c}\right)$.

One may view the objects $L^{p}\left(\Omega^{c}, \omega\right), T_{\nu}^{p}\left(\Omega^{c}\right), W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ and $W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ respectively as the nonlocal counterpart of the local objects $L^{p}(\partial \Omega), W^{1-1 / p, p}(\partial \Omega), W^{1, p}(\Omega)$ and $W_{0}^{1, p}(\Omega)$.

Theorem 5.15 (Classical Trace Theorem, see [BF13, Chap III]). Assume $\Omega \subset \mathbb{R}^{d}$ is bounded Lipschitz. There exists a linear and continuous trace operator $\gamma_{0}: W^{1, p}(\Omega) \rightarrow L^{p}(\partial \Omega)$ such that $\gamma_{0} u=\left.u\right|_{\partial \Omega}$ for all $u \in C^{1}(\bar{\Omega}) \cap W^{1, p}(\Omega)$. Moreover, $\operatorname{ker}\left(\gamma_{0}\right)=W_{0}^{1, p}(\Omega)$ and $\gamma_{0}\left(W^{1, p}(\Omega)\right)=W^{1,1-1 / p}(\partial \Omega)$ (by definition).

Remark 5.16. Let us emphasize our nonlocal trace operator Tr does not need any special construction via functional analysis and density argument. Since $\Omega^{c}$ is still a d-dimensional manifold. Then it makes sense to consider hardcore restriction of functions on $\Omega^{c}$. Moreover no regularity on $\Omega$ is required nor on $u$. Whereas in the local situation (see Theorem 5.15), the trace of a Sobolev function $u$ on the boundary $\partial \Omega$ requires the smoothness of both $u$ and $\partial \Omega$.

It is natural to ask the following question: Can the space $T_{\nu}^{p}\left(\Omega^{c}\right)$ be self defined with an intrinsic norm preserving its initial Banach structure in a way that its trivial connection to $W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ is less seeable? In the local situation, it is possible to define a scalar product on the space $H^{1 / 2}(\partial \Omega)$ when $\Omega$ is a special Lipschitz domain (see [Din96]). This question is discussed in [FK22, GH22] when $p=2$, using the main result from [BGPR20]. Another treatment of nonlocal trace operator for the special fractional kernel $\nu(h)=|h|^{-d-s p}$ is encapsulated in [DK19, Theorem 3].

## 6. Compact embeddings

In this section we prove compact embeddings of the spaces $W_{\nu}^{p}(\Omega), W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ and $W_{\nu, \Omega}\left(\Omega \mid \mathbb{R}^{d}\right)$ into $L^{p}(\Omega)$. Our result on global compactness Theorem 6.6 requires some extra regularity assumptions on $\Omega$ compatible with $\nu$. We exploit some recent ideas from [JW20, DMT18]. However, extended details and discussions for the global compactness can be found in [Fog20, FK22].
6.1. Local and global compactness results. Given a Banach space $X$, we say that an operator $T: X \rightarrow L_{\mathrm{loc}}^{p}(\Omega)$ is compact if for any compact set $K \subset \Omega$, the operator $R_{K} T: X \rightarrow L_{\mathrm{loc}}^{p}(K), R_{K} T u=\left.T u\right|_{K}$ is compact.

Lemma 6.1 ( [Bre11, Corollary 4.28]). Let $w \in L^{1}\left(\mathbb{R}^{d}\right)$. The convolution operator $T_{w}: L^{p}\left(\mathbb{R}^{d}\right) \rightarrow L^{p}\left(\mathbb{R}^{d}\right)$, $T_{w} u=w * u$ is continuous with $\left\|T_{w}\right\|_{\mathcal{L}\left(L^{p}\left(\mathbb{R}^{d}\right), L^{p}\left(\mathbb{R}^{d}\right)\right)} \leq\|w\|_{L^{1}\left(\mathbb{R}^{d}\right)}$. Moreover, $R_{K} T_{w}: L^{p}\left(\mathbb{R}^{d}\right) \rightarrow L^{p}(K)$ is compact for any measurable set $K \subset \mathbb{R}^{d}$ with $|K|<\infty$.

Theorem 6.2. If $\nu$ satisfies (L). The following assertions are equivalent.
(i) $\nu$ is not integrable, i.e., $\nu \notin L^{1}\left(\mathbb{R}^{d}\right)$.
(ii) The embedding $W_{\nu}^{p}\left(\mathbb{R}^{d}\right) \hookrightarrow L^{p}(K)$ is compact, for any measurable set with $|K|<\infty$.
(iii) The embedding $W_{\nu}^{p}\left(\mathbb{R}^{d}\right) \hookrightarrow L_{\mathrm{loc}}^{p}\left(\mathbb{R}^{d}\right)$ is compact.
(iv) The embedding $W_{\nu}^{p}\left(\mathbb{R}^{d}\right) \hookrightarrow L^{p}(B(0,1))$ is compact,

Proof. We only prove $(i v) \Longrightarrow(i)$ and $(i) \Longrightarrow(i i)$ as the other implications are straightforward. Assume $\nu \in L^{1}\left(\mathbb{R}^{d}\right)$ and put $B=B(0,1)$ then $W_{\nu}^{p}\left(\mathbb{R}^{d}\right)=L^{2}\left(\mathbb{R}^{d}\right)$ and hence the mappings $L^{p}(B) \xrightarrow{E} W_{\nu}^{p}\left(\mathbb{R}^{d}\right) \xrightarrow{R_{B}} L^{p}(B)$, where $E u=\bar{u}$ is the zero extension of $u$ and $R_{B} u=\left.u\right|_{B}$, are continuous. Since is the identity map $I=R_{B} \circ E$ : $L^{p}(B) \rightarrow L^{p}(B)$ is not compact, necessarily, $R$ is not compact. Now assume $\nu \notin L^{1}\left(\mathbb{R}^{d}\right)$ and $\delta>0$ sufficiently small such that $0<\left\|\nu_{\delta}\right\|_{L^{1}\left(\mathbb{R}^{d}\right)}$. Consider $w_{\delta}(h)=\nu_{\delta}(h)\left\|\nu_{\delta}\right\|_{L^{1}\left(\mathbb{R}^{d}\right)}^{-1}$ so that $\left\|w_{\delta}\right\|_{L^{1}\left(\mathbb{R}^{d}\right)}=1$. Define $T_{w_{\delta}}=w_{\delta} * u$, $u \in L^{p}\left(\mathbb{R}^{d}\right)$. By the symmetry of $\nu$ we have, for all $x \in \mathbb{R}^{d}$

$$
T_{w_{\delta}} u(x)=\int_{\mathbb{R}^{d}} w_{\delta}(y) u(x-y) \mathrm{d} y=\int_{\mathbb{R}^{d}} w_{\delta}(y) u(x+y) \mathrm{d} y
$$

The Jensen's inequality implies

$$
\begin{aligned}
\left\|u-T_{w_{\delta}} u\right\|_{L^{p}\left(\mathbb{R}^{d}\right)}^{p} & =\int_{\mathbb{R}^{d}}\left|\int_{\mathbb{R}^{d}}[u(x)-u(x+h)] w_{\delta}(h) \mathrm{d} h\right|^{p} \mathrm{~d} x \\
& \leq\left\|\nu_{\delta}\right\|_{L^{1}\left(\mathbb{R}^{d}\right)}^{-1} \iint_{\mathbb{R}^{d} \mathbb{R}^{d}}|u(x)-u(x+h)|^{p} \nu(h) \mathrm{d} h \mathrm{~d} x \leq\left\|\nu_{\delta}\right\|_{L^{1}\left(\mathbb{R}^{d}\right)}^{-1}\|u\|_{W_{\nu}^{p}\left(\mathbb{R}^{d}\right)}^{p} .
\end{aligned}
$$

Accordingly, since $\nu \notin L^{1}\left(\mathbb{R}^{d}\right)$, for a subset $K \subset \mathbb{R}^{d}$ with $|K|<\infty$ we find that

$$
\left\|R_{K}-R_{K} T_{w_{\delta}}\right\|_{\mathcal{L}\left(W_{\nu}^{p}\left(\mathbb{R}^{d}\right), L^{p}(K)\right)} \leq\left\|\nu_{\delta}\right\|_{L^{1}\left(\mathbb{R}^{d}\right)}^{-1 / p} \xrightarrow{\delta \rightarrow 0} 0 .
$$

It follows that the operator $R_{K}: W_{\nu}^{p}\left(\mathbb{R}^{d}\right) \rightarrow L^{p}(K)$ with $R_{K} u=\left.u\right|_{K}$ is compact since each operator $R_{K} T_{w_{\delta}}$ is compact (by Lemma 6.1) and the set of compact operator is closed.

Another version of Theorem 6.2 is proved in [JW20, Theorem 1.1] for the case $p=2$. It is worth mentioning that earlier analogous results are provided in [PZ17, Proposition 6] and [BJ17, Proposition 1] for periodic functions on the torus. This technique of killing the singularity is also used in [BJ13, Lemma 3.1].

Corollary 6.3. Assume $\Omega \subset \mathbb{R}^{d}$ is open and $\nu \notin L^{1}\left(\mathbb{R}^{d}\right)$. The embedding $W_{\nu}^{p}(\Omega) \hookrightarrow L_{\mathrm{loc}}^{p}(\Omega)$ is compact. Moreover, if $|\Omega|<\infty$ then the embedding $W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right) \hookrightarrow L^{p}(\Omega)$, is compact.

Proof. For fixed $\varphi \in C_{c}^{\infty}(\Omega)$ the operator $J_{\varphi}: W_{\nu}^{p}(\Omega) \rightarrow W_{\nu}^{p}\left(\mathbb{R}^{d}\right), J_{\varphi} u=u \varphi$ is continuous. By Theorem 6.2 the embedding $W_{\nu}^{p}\left(\mathbb{R}^{d}\right) \hookrightarrow L^{p}(K), K=\operatorname{supp} \varphi \subset \Omega$, is compact. It follows that the embedding $W_{\nu}^{p}(\Omega) \hookrightarrow L^{p}(K)$ is compact. Hence the first claim is proved. The embeddings $W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right) \hookrightarrow W_{\nu}^{p}\left(\mathbb{R}^{d}\right) \hookrightarrow L^{p}(\Omega)$ are continuous and, by Theorem 6.2, the last one is compact when $|\Omega|<\infty$. Whence the second claim follows.

Another consequence of Theorem 6.2 is the local compactness of $W_{\nu}^{p}(\Omega)$ in $L^{p}(\Omega)$.
Corollary 6.4. Assume $\Omega \subset \mathbb{R}^{d}$ be open and $\nu \notin L^{1}\left(\mathbb{R}^{d}\right)$. The following assertions hold.
(i) For a bounded sequence $\left(u_{n}\right)_{n}$ in $W_{\nu}^{p}(\Omega)$ there exists $u \in W_{\nu}^{p}(\Omega)$ and subsequence $\left(u_{n_{j}}\right)_{j}$ converging to $u$ in $L_{\text {loc }}^{p}(\Omega)$. Moreover,

$$
|u|_{W_{\nu}^{p}(\Omega)} \leq \liminf _{n \rightarrow \infty}\left|u_{n}\right|_{W_{\nu}^{p}(\Omega)} .
$$

(ii) Assume $|\Omega|<\infty$ then for a bounded sequence $\left(u_{n}\right)_{n}$ in $W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ there exists $u \in W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ and subsequence $\left(u_{n_{j}}\right)_{j}$ converging to $u$ in $L^{p}(\Omega)$. Moreover,

$$
|u|_{W_{\nu}^{p}\left(\mathbb{R}^{d}\right)} \leq \liminf _{n \rightarrow \infty}\left|u_{n}\right|_{W_{\nu}^{p}\left(\mathbb{R}^{d}\right)}
$$

6.2. Global compactness. Let us introduce conditions on $\Omega$ and $\nu$ yielding global compactness results.

Definition 6.5. Assume $\Omega \subset \mathbb{R}^{d}$ is open and bounded, and $\nu \in L^{1}\left(\mathbb{R}^{d}, 1 \wedge|h|^{p}\right)$ and $\nu \notin L^{1}\left(\mathbb{R}^{d}\right)$. We say that $(\nu, \Omega)$ is in the class $\mathscr{A}_{i}, i \in\{1,2,3\}$, if in addition
$\left(\mathscr{A}_{1}\right) \ldots$ there exists an $W_{\nu}^{p}(\Omega)$-extension operator $E: W_{\nu}^{p}(\Omega) \rightarrow W_{\nu}^{p}\left(\mathbb{R}^{d}\right)$, i.e., there is $C(\nu, \Omega, d)>0$ such that for every $u \in W_{\nu}^{p}(\Omega),\|u\|_{W_{\nu}^{p}\left(\mathbb{R}^{d}\right)} \leq C\|u\|_{W_{\nu}^{p}(\Omega)}$ and $\left.E u\right|_{\Omega}=u$.
$\left(\mathscr{A}_{2}\right) \ldots \Omega$ has Lipschitz boundary, $\nu$ is radial and $q(\delta) \xrightarrow{\delta \rightarrow 0} \infty$ where

$$
\begin{equation*}
q(\delta):=\frac{1}{\delta^{p}} \int_{B_{\delta}(0)}|h|^{p} \nu(h) \mathrm{d} h . \tag{6.1}
\end{equation*}
$$

$\left(\mathscr{A}_{3}\right) \ldots$ the following condition holds true: $\widetilde{q}(\delta) \xrightarrow{\delta \rightarrow 0} \infty$ where

$$
\begin{equation*}
\widetilde{q}(\delta):=\inf _{a \in \partial \Omega} \int_{\Omega_{\delta}} \nu(h-a) \mathrm{d} h \tag{6.2}
\end{equation*}
$$

with $\Omega_{\delta}=\{x \in \Omega: \operatorname{dist}(x, \partial \Omega)>\delta\}$.
Here is our global compactness result; see [FK22] for the case $p=2$ or [Fog20, Theorem 3.89].
Theorem 6.6. If the couple $(\nu, \Omega)$ belongs to one of the class $\mathscr{A}_{i}, i=1,2,3$ then the embedding $W_{\nu}^{p}(\Omega) \hookrightarrow L^{p}(\Omega)$ is compact. In particular, the embedding $W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right) \hookrightarrow L^{p}(\Omega)$ is compact.

Remark 6.7. The well known Rellich-Kondrachov's compact embeddings $W_{0}^{1, p}(\Omega) \hookrightarrow L^{p}(\Omega)$ and $W^{1, p}(\Omega) \hookrightarrow$ $L^{p}(\Omega)$ when $\Omega$ is Lipschitz, respectively can be derived from Theorem 6.2 combined with the embedding $W_{0}^{1, p}(\Omega) \hookrightarrow$ $W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ and from Theorem 6.6 combined with the embedding $W^{1, p}(\Omega) \hookrightarrow W_{\nu}^{p}(\Omega)$ when $\Omega$ is Lipschitz.

## 7. Nonlocal Poincaré type inequalities

7.1. Nonlocal Poincaré-Friedrichs inequality. In this section $1 \leq p<\infty, \Omega \subset \mathbb{R}^{d}$ is open and that $\nu$ : $\mathbb{R}^{d} \backslash\{0\} \rightarrow[0, \infty)$ is symmetric satisfying $\nu \in L^{1}\left(\mathbb{R}^{d} \backslash B(0, r)\right)$ for every $r>0$. In particular, the latter holds true if $\nu \in L^{1}\left(\mathbb{R}^{d}, 1 \wedge|h|^{\gamma} \mathrm{d} h\right)$ for any $\gamma \geq 0$. Let us define the space $L_{\Omega}^{p}\left(\mathbb{R}^{d}\right)$ by

$$
L_{\Omega}^{p}\left(\mathbb{R}^{d}\right)=\left\{u \in L^{p}\left(\mathbb{R}^{d}\right): u=0 \text { a.e on } \Omega^{c}\right\} .
$$

Next we use a more refined argument to prove the above Poincaré-Friedrichs type inequalities with strictly positive constants, in a general setting only by assuming that $\nu$ is nontrivial that is $\nu \not \equiv 0$. We follow the strategy from [JW20] using the $2^{m}$-folded convolutions which is also used in [FKV15, proof of Lemma 2.7].

Lemma 7.1. Let $q \in L^{1}\left(\mathbb{R}^{d}\right) \cap L^{\infty}\left(\mathbb{R}^{d}\right)$ with $q \geq 0$ be symmetric, i.e., $q(h)=q(-h)$ and nontrivial, i.e., $\mid\{q>$ $0\} \mid>0$. Define $2^{m}$-fold convolution of $q$ as follows $q_{0}=q, q_{m}=q_{m-1} * q_{m-1}=\underbrace{q * \cdots * q}_{2^{m}-\text { times }}$. For every $m \geq 1$, the following assertions are true.
(i) Each $q_{m}$ belongs to $L^{1}\left(\mathbb{R}^{d}\right) \cap L^{\infty}\left(\mathbb{R}^{d}\right)$ and is uniformly continuous.
(ii) There exists $\delta>0$, such that $\inf _{B_{\delta_{m}}(0)} q_{m}>0$ with $\delta_{m}=\left(\frac{7}{4}\right)^{m} \delta$ so that $\delta_{m} \xrightarrow{m \rightarrow \infty} \infty$.
(iii) For $u: \mathbb{R}^{d} \rightarrow \mathbb{R}$ measurable, we have

$$
\begin{equation*}
\mathcal{Q}_{m}(u) \leq B_{m} \mathcal{Q}(u), \quad B_{m}=2^{p m}\|q\|_{L^{1}\left(\mathbb{R}^{d}\right)}^{2^{m}-1} \tag{7.1}
\end{equation*}
$$

where we set $\mathcal{Q}(u)=\mathcal{Q}_{0}(u)$ and

$$
\mathcal{Q}_{m}(u):=\iint_{\mathbb{R}^{d} \mathbb{R}^{d}}|u(x)-u(y)|^{p} q_{m}(x-y) \mathrm{d} y \mathrm{~d} x
$$

Proof. (i) Since $q_{m}=q_{m-1} * q_{m-1}$, by the Young's inequalities $q_{m} \in L^{1}\left(\mathbb{R}^{d}\right) \cap L^{\infty}\left(\mathbb{R}^{d}\right)$ and

$$
\begin{align*}
& \left\|q_{m}\right\|_{L^{1}\left(\mathbb{R}^{d}\right)} \leq\left\|q_{m-1}\right\|_{L^{1}\left(\mathbb{R}^{d}\right)}^{2} \leq\|q\|_{L^{1}\left(\mathbb{R}^{d}\right)}^{2^{m}}  \tag{7.2}\\
& \left\|q_{m}\right\|_{L^{\infty}\left(\mathbb{R}^{d}\right)} \leq\left\|q_{m-1}\right\|_{L^{\infty}\left(\mathbb{R}^{d}\right)}\left\|q_{m-1}\right\|_{L^{1}\left(\mathbb{R}^{d}\right)} \leq\|q\|_{L^{\infty}\left(\mathbb{R}^{d}\right)}\|q\|_{L^{1}\left(\mathbb{R}^{d}\right)}^{2^{m}-1} \tag{7.3}
\end{align*}
$$

The uniform continuity $q_{m}$ readily follows from the continuity of the shift in $L^{1}\left(\mathbb{R}^{d}\right)$.
(ii) Since $q$ is not identically vanishing, we have

$$
q_{1}(0)=q * q(0)=\int_{\mathbb{R}^{d}}|q(h)|^{2} \mathrm{~d} h>0
$$

wherefrom, the continuity of $q_{1}$ at 0 implies that $\theta=\inf _{B_{\delta}(0)} q_{1}>0$ for some $\delta>0$. We claim that $q_{2}(x)=$ $q_{1} * q_{1}(x)>0$ whenever $|x| \leq \frac{7 \delta}{4}$. Indeed, note that $B_{\frac{\delta}{16}}\left(\frac{x}{2}\right) \subset B_{\delta}(0) \cap B_{\delta}(x)$ since for $z \in B_{\frac{\delta}{16}}\left(\frac{x}{2}\right)$ we have

$$
\left|z-\frac{x}{2} \pm \frac{x}{2}\right| \leq\left|z-\frac{x}{2}\right|+\left|\frac{x}{2}\right| \leq \frac{15 \delta}{16}<\delta
$$

Given that $q_{1}(h) q_{1}(x-h)>\theta^{2}$ for all $h \in B_{\delta}(0) \cap B_{\delta}(x)$ we finally get

$$
q_{2}(x)=q_{1} * q_{1}(x) \geq \theta^{2}\left|B_{\frac{\delta}{16}}(0)\right|>0 \quad \text { for all }|x| \leq \frac{7 \delta}{4}
$$

Repeating this process, one easily retrieves that for $\theta_{m-1}=\inf _{B_{\delta_{m-1}}(0)} q_{m-1}(x)>0$

$$
q_{m}(x) \geq \theta_{m-1}^{2}\left|B_{\frac{\delta_{m-1}}{16}}(0)\right|>0 \quad \text { for all }|x| \leq \delta_{m}
$$

(iii) Using the inequality $|a+b|^{p} \leq 2^{p-1}\left(|a|^{p}+|b|^{p}\right)$ and Fubini's theorem yields

$$
\begin{aligned}
\mathcal{Q}_{m}(u) & =\iint_{\mathbb{R}^{d} \mathbb{R}^{d}}|u(x)-u(x+h)|^{p} \int_{\mathbb{R}^{d}} q_{m-1}(z) q_{m-1}(h-z) \mathrm{d} z \mathrm{~d} h \mathrm{~d} x \\
& \leq 2^{p-1} \iint_{\mathbb{R}^{d} \mathbb{R}^{d}}\left(|u(x)-u(x+z)|^{p}+|u(x+z)-u(x+h)|^{p}\right) \int_{\mathbb{R}^{d}} q_{m-1}(z) q_{m-1}(h-z) \mathrm{d} z \mathrm{~d} h \mathrm{~d} x \\
& =2^{p-1} \iint_{\mathbb{R}^{d}}|u(x)-u(x+z)|^{p}\left(\int_{\mathbb{R}^{d}} q_{m-1}(\xi) \mathrm{d} \xi\right) q_{m-1}(z) \mathrm{d} z \mathrm{~d} x \\
& +2^{p-1} \int_{\mathbb{R}^{d}} q_{m-1}(z) \mathrm{d} z \iint_{\mathbb{R}^{d} \mathbb{R}^{d}}|u(x+z)-u(x+z+\xi)|^{p} q_{m-1}(\xi) \mathrm{d} \xi \mathrm{~d} x \quad(\text { fix } z \text { and put } \xi=h-z) \\
& =2^{p}\left\|q_{m-1}\right\|_{L^{1}\left(\mathbb{R}^{d}\right)} \mathcal{Q}_{m-1}(u) .
\end{aligned}
$$

In short, this combined with inequality (7.2) implies

$$
\mathcal{Q}_{m}(u) \leq 2^{p}\left\|q_{m-1}\right\|_{L^{1}\left(\mathbb{R}^{d}\right)} \mathcal{Q}_{m-1}(u) \leq\left(\prod_{k=1}^{m} 2^{p}\|q\|_{L^{1}\left(\mathbb{R}^{d}\right)}^{2^{k-1}}\right) \mathcal{Q}(u)=2^{p m}\|q\|_{L^{1}\left(\mathbb{R}^{d}\right)}^{2^{m}-1} \mathcal{Q}(u)
$$

It is decisive to keep in mind that for $u \in L_{\Omega}^{p}\left(\mathbb{R}^{d}\right)$ we have

$$
\mathcal{E}(u, u)=\iint_{\left(\Omega^{c} \times \Omega^{c}\right)^{c}}|u(x)-u(y)|^{p} \nu(x-y) \mathrm{d} y \mathrm{~d} x=\iint_{\mathbb{R}^{d} \mathbb{R}^{d}}|u(x)-u(y)|^{p} \nu(x-y) \mathrm{d} y \mathrm{~d} x .
$$

Theorem 7.2 (Poincaré-Friedrichs inequality III). Assume $\nu \not \equiv 0$, i.e., $|\{\nu>0\}|>0, \nu(h)=\nu(-h)$ and $\nu \in$ $L^{1}\left(\mathbb{R}^{d} \backslash B(0, r)\right)$ for every $r>0$. Assume $\Omega$ is bounded one direction, i.e., there exist $R>0$ and $e \in \mathbb{R}^{d}$, $|e|=1$ such that $\Omega \subset H_{R}$ with $H_{R}=\left\{z \in \mathbb{R}^{d}:|z \cdot e| \leq R\right\}$. Then for $m \in \mathbb{N}$ large, there is $0<C_{R, m}:=C(d, p, R, m, \nu)<\infty$ such that

$$
\begin{equation*}
\|u\|_{L^{p}(\Omega)}^{p} \leq C_{R, m} \mathcal{E}(u, u) \quad \text { for all } u \in L_{\Omega}^{p}\left(\mathbb{R}^{d}\right) \tag{7.4}
\end{equation*}
$$

Moreover, with the notation $q_{m}=q_{m-1} * q_{m-1}, q_{0}=q, q:=1 \wedge \nu$ we can choose

$$
C_{R, m}=2^{p m}\|q\|_{L^{1}\left(\mathbb{R}^{d}\right)}^{2^{m}-1}\left(\int_{H_{2 R}^{c}} q_{m}(h) \mathrm{d} h\right)^{-1}
$$

Proof. We have $H_{R} \subset H_{2 R}(x)=\left\{z \in \mathbb{R}^{d}:|(z-x) \cdot e| \leq 2 R\right\}$ for $x \in H_{R}$. Accordingly, for $u \in L_{\Omega}^{p}\left(\mathbb{R}^{d}\right)$ there holds

$$
\begin{aligned}
\mathcal{Q}_{m}(u) & \geq 2 \int_{\Omega}|u(x)|^{p} \mathrm{~d} x \int_{H_{R}^{c}} q_{m}(x-y) \mathrm{d} y \\
& \geq \int_{\Omega}|u(x)|^{p} \mathrm{~d} x \int_{H_{2 R}^{c}(x)} q_{m}(x-y) \mathrm{d} y=\|u\|_{L^{p}(\Omega)}^{p} \int_{H_{2 R}^{c}} q_{m}(h) \mathrm{d} h
\end{aligned}
$$

The fact that $\nu \in L^{1}\left(\mathbb{R}^{d} \backslash B(0, r)\right)$ implies $q=1 \wedge \nu \in L^{1}\left(\mathbb{R}^{d}\right) \cap L^{\infty}\left(\mathbb{R}^{d}\right)$. In addition, $\nu \equiv 0$ if and only if $q \equiv 0$. Thus, $q=1 \wedge \nu \leq \nu$, the estimate above and Lemma 7.1 (iii) yield

$$
\mathcal{E}(u, u) \geq \mathcal{Q}(u) \geq B_{m}^{-1} \mathcal{Q}_{m}(u) \geq C_{R, m}^{-1}\|u\|_{L^{p}(\Omega)}^{p}
$$

According to Lemma 7.1 (i), $q_{m} \in L^{1}\left(\mathbb{R}^{d}\right) \cap L^{\infty}\left(\mathbb{R}^{d}\right)$ and by Lemma 7.1 (ii) for $m$ large we have $\inf _{B_{\delta_{m}}(0)} q_{m}>0$ and $\left|B_{\delta_{m}}(0) \backslash H_{2 R}^{c}\right|>0$ where we recall $\delta_{m} \xrightarrow{m \rightarrow \infty} \infty$. Hence

$$
0<\left|B_{\delta_{m}}(0) \backslash H_{2 R}^{c}\right|\left(\inf _{B_{\delta_{m}}(0)} q_{m}\right) \leq \int_{H_{2 R}^{c}} q_{m}(h) \mathrm{d} h \leq\left\|q_{m}\right\|_{L^{1}\left(\mathbb{R}^{d}\right)}<\infty
$$

wherefrom we deduce that $0<C_{R, m}<\infty$.
Next we want o consider the case where $|\Omega|<\infty$. We need the following Lemma.
Lemma 7.3. Let $E \subset \mathbb{R}^{d}$ be measurable with $|E|<\infty$. Define $r_{E}=\left(\frac{|E|}{\left|B_{1}\right|}\right)^{1 / d}$, then we have

$$
\begin{equation*}
\underset{x \in \mathbb{R}^{d}}{\operatorname{ess} \inf } \int_{E^{c}} \nu(x-y) \mathrm{d} y \geq \nu^{\#}(|E|), \quad \text { with } \quad \nu^{\#}(|E|)=\int_{\left\{\nu<\nu^{*}\left(r_{E}\right)\right\}} \nu(h) \mathrm{d} h, \tag{7.5}
\end{equation*}
$$

where $\nu^{*}$ is the symmetric rearrangement of $\nu$ defined by

$$
\begin{equation*}
\nu^{*}(x)=\nu^{*}(|x|)=\int_{0}^{\infty} \mathbb{1}_{\{|\nu|>s\}^{*}}(x) \mathrm{d} s=\inf \left\{s>0:|\{|\nu|>s\}| \leq c_{d}|x|^{d}\right\} \tag{7.6}
\end{equation*}
$$

It is worth noting that for the case $\nu(h)=|h|^{-d-s p}, h \neq 0$ we have $\nu^{*}=\nu$ and

$$
\nu^{\#}(|E|)=\int_{\left\{\nu<\nu^{*}\left(r_{E}\right)\right\}} \nu(h) \mathrm{d} h=\int_{|h|>r_{E}}|h|^{-d-s p} \mathrm{~d} h=\frac{1}{s p}\left(\frac{|E|}{\left|B_{1}(0)\right|}\right)^{-s p}\left|\mathbb{S}^{d-1}\right| .
$$

Accordingly, Lemma 7.3 implies the following; see also [DNPV12, Lemma 6.1]
Theorem 7.4 (Poincaré-Friedrichs inequality IV). Assume $\nu \not \equiv 0$, i.e., $|\{\nu>0\}|>0$ and $\nu \in L^{1}\left(\mathbb{R}^{d} \backslash B(0, r)\right)$ for every $r>0$. Assume $|\Omega|<\infty$. Then for every $m \in \mathbb{N}$

$$
\begin{equation*}
\|u\|_{L^{p}(\Omega)}^{p} \leq C_{m}^{\#} \mathcal{E}(u, u) \quad \text { for all } u \in L_{\Omega}^{p}\left(\mathbb{R}^{d}\right) \tag{7.7}
\end{equation*}
$$

where, for sufficiently large $m$, we have $0<C_{m}^{\#}=C(d, p,|\Omega|, m, \nu)<\infty$ with

$$
C_{m}^{\#}=2^{p m}\|q\|_{L^{1}\left(\mathbb{R}^{d}\right)}^{2^{m}}\left(q_{m}^{\#}(|\Omega|)\right)^{-1}
$$

Here, $q_{m}=q_{m-1} * q_{m-1}, q_{0}=q$ with $q:=1 \wedge \nu=\min (1, \nu)$ and $q_{m}^{\#}$ is defined as in (7.5).

Proof. Accordingly, for $u \in L_{\Omega}^{p}\left(\mathbb{R}^{d}\right)$ there holds that

$$
\mathcal{Q}_{m}(u) \geq 2 \int_{\Omega}|u(x)|^{p} \mathrm{~d} x \int_{\Omega^{c}} q_{m}(x-y) \mathrm{d} y \geq\|u\|_{L^{p}(\Omega)}^{p} q_{m}^{\#}(|\Omega|)
$$

The assumptions on $\nu$ imply that $q=1 \wedge \nu \in L^{1}\left(\mathbb{R}^{d}\right) \cap L^{\infty}\left(\mathbb{R}^{d}\right)$ and $\nu \equiv 0$ if and only if $q \equiv 0$. Moreover, $q=1 \wedge \nu \leq \nu$, the above estimate and Lemma 7.1 (iii) yield

$$
\mathcal{E}(u, u) \geq \mathcal{Q}(u) \geq B_{m}^{-1} \mathcal{Q}_{m}(u) \geq\left(C_{m}^{\#}\right)^{-1}\|u\|_{L^{p}(\Omega)}^{p}
$$

By Lemma 7.1 (i), $q_{m} \in L^{1}\left(\mathbb{R}^{d}\right) \cap L^{\infty}\left(\mathbb{R}^{d}\right)$ and hence $q_{m}^{\#}(|\Omega|)<\infty$. By Lemma 7.1 (ii) we have $\delta_{m}>|\Omega|$ for $m$ large so that $q_{m}^{\#}(|\Omega|) \geq \inf _{B_{\delta_{m}}(0)} q_{m}>0$. Therefore, $0<C_{m}^{\#}<\infty$.
7.2. Nonlocal Poincaré inequality. In this section $\Omega \subset \mathbb{R}^{d}$ is open and bounded, $1 \leq p<\infty$ and $\nu: \mathbb{R}^{d} \backslash\{0\} \rightarrow$ $[0, \infty)$ is symmetric. Our goal in this section is to find some conditions on $\Omega$ and $\nu$ under which the following Poincaré inequality holds true, i.e., we can find a constant $C=C(d, p, \Omega, \nu)>0$ such that

$$
\begin{equation*}
\left\|u-f_{\Omega} u\right\|_{L^{p}(\Omega)}^{p} \leq C \mathcal{E}_{\Omega}(u, u) \quad \text { for all } u \in L^{p}(\Omega) \tag{P}
\end{equation*}
$$

Here and in what follows the notation $f_{E} u=\frac{1}{|E|} \int_{E} u(x) \mathrm{d} x$. Recall that we define

$$
\mathcal{E}_{\Omega}(u, u)=\iint_{\Omega \Omega}|u(x)-u(y)|^{p} \nu(x-y) \mathrm{d} y \mathrm{~d} x .
$$

So that since $\mathcal{E}_{\Omega}(u, u) \leq \mathcal{E}(u, u),(P)$ would imply

$$
\begin{equation*}
\left\|u-f_{\Omega} u\right\|_{L^{p}(\Omega)}^{p} \leq C \mathcal{E}(u, u) \quad \text { for all } u \in L^{p}(\Omega) \tag{7.8}
\end{equation*}
$$

The following Lemma 7.5 is probably known in the literature. We however give a simple proof for the convenience of the reader.

Lemma 7.5 (Finite chain covering). Assume $\Omega \subset \mathbb{R}^{d}$ is connected. For every $r>0$ there is a finite family of balls $\left(B_{i}\right)_{1 \leq i \leq n}, B_{i}=B\left(x_{i}, r\right)$ covering $\Omega$ with $x_{i} \in \Omega$ such that $B_{i-1} \cap B_{i} \neq \emptyset, i=2,3, \cdots, n$.

Proof. It is readily seen that the balls $B(x, r), x \in \Omega$ covers the compact set $\bar{\Omega}$ (the closure of $\Omega$ ). Thus, there is a finite sub-cover $\mathcal{B}=\left\{B_{i}: i \in\{1,2, \cdots, n\}\right\}$ of $\Omega$ with $B_{i}=B\left(x_{i}, r\right), x_{i} \in \Omega$. Next, let us write $B_{i-1} \sim B_{i}$ to indicate that $B_{i-1} \cap B_{i} \neq \emptyset$. Since $\Omega$ is connected, there are $B, B^{\prime} \in \mathcal{B}$ such that $B \cap B^{\prime} \neq \emptyset$. Up to a relabeling, denote the chain $C_{2}=B_{1} \sim B_{2}$ with $B_{1}=B, B_{2}=B^{\prime}$. Assume, up to relabeling the indices, there is a chain

$$
C_{i}=B_{1} \sim B_{2} \sim \cdots \sim B_{i} .
$$

Given that $\Omega$ is connected, there is $B \in \mathcal{B} \backslash \bigcup_{j=1}^{i} B_{j}$ such that $B \cap \bigcup_{j=1}^{i} B_{j} \neq \emptyset$. Thus, we can consider $j_{0}=$ $\max \left\{j \in\{1,2, \cdots, i\}: B_{j} \sim B\right\}$ and define a chain $C_{i+1}$ as follows

$$
\begin{array}{ll}
C_{i+1}=B_{1} \sim \cdots \sim B_{j_{0}-1} \sim B \sim B_{j_{0}} \sim \cdots \sim B_{i} & \text { if } j_{0} \neq 1 \\
C_{i+1}=B \sim B_{1} \sim B_{2} \sim \cdots \sim B_{i} & \text { if } j_{0}=1
\end{array}
$$

In this manner, up to relabeling the indices, one gets a chain $C_{n}$ containing the whole $\mathcal{B}$.
We also need to consider the following generalization of the connected sets.
Definition 7.6. We say that $\Omega \subset \mathbb{R}^{d}$ is $\rho$-connected, $\rho \geq 0$, if $\Omega=\Omega_{1} \cup \Omega_{2} \cup \cdots \cup \Omega_{m}, m \geq 1$, where each $\Omega_{i}$ is open and connected such that $\operatorname{dist}\left(\Omega_{i}, \Omega_{i+1}\right)<\rho, i=1,2, \cdots, n-1$.

It is worthwhile to observe that every connected set is 0 -connected and the converse is not true. Consider $B_{ \pm}=$ $B_{1}(0) \cap\left\{x=\left(x^{\prime}, x_{d}\right): \pm x_{d}>0\right\}$ which are connected. Therefore, since $\operatorname{dist}\left(B_{-}, B_{+}\right)=0$ we see that $\Omega_{=} B_{-} \cup B_{+}$ is 0 -connected but not connected. Another observation is that if $\operatorname{dist}\left(\Omega_{i}, \Omega_{i+1}\right)<\rho$ one finds $a_{i} \in \Omega_{i}, b_{i} \in \Omega_{i+1}$ such that $\left|a_{i}-b_{i}\right|<\rho$.

Lemma 7.7. Assume $\nu>0$ a.e. on $B_{r}(0)$ for some $r>0$. Assume that one of the following conditions is true.
(i) $r \geq \operatorname{diam}(\Omega)$.
(ii) $\Omega$ is connected.
(iii) $\Omega$ is $\rho$-connected with $0 \leq \rho<r$.

There holds $\mathcal{E}_{\Omega}(u, u)=0$ if and only if $u=\lambda$ a.e. in $\Omega, \lambda \in \mathbb{R}$.

Proof. Clearly, if $u=\lambda$ a.e. in $\Omega$ then $\mathcal{E}_{\Omega}(u, u)=0$. Conversely, if $\mathcal{E}_{\Omega}(u, u)=0$, there is a null set $A \subset \Omega$ such that

$$
\int_{\Omega}|u(x)-u(y)|^{p} \nu(x-y) \mathrm{d} y=0 \quad \text { for all } x \in \Omega \backslash A
$$

For each $x \in \Omega \backslash A$ there is a null set $A_{x} \subset \Omega$ such that

$$
|u(x)-u(y)|^{p} \nu(x-y)=0 \quad \text { for all } y \in \Omega \backslash A_{x}
$$

Since $\nu>0$ a.e. in $B_{r}(0)$, we can assume up to renaming the null set $A_{x}$ that

$$
\begin{equation*}
u(y)=u(x) \quad \text { for } y \in B_{r}(x) \cap\left(\Omega \backslash A_{x}\right), x \in \Omega \backslash A \tag{7.9}
\end{equation*}
$$

(i) If $r \geq \operatorname{diam}(\Omega)$ then for fixed $x_{0} \in \Omega \backslash A$ we have $\Omega \subset B_{r}\left(x_{0}\right)$ so that $u(x)=u\left(x_{0}\right)$ for all $x \in \Omega \backslash A_{x_{0}}$. Hence, $u=u\left(x_{0}\right)$ almost everywhere in $\Omega$.
(ii) Assume $\Omega$ is connected. By Lemma 7.5, we can cover $\Omega$ with a family of balls $\left(B_{i}\right)_{1 \leq i \leq n}, B_{i}=B\left(x_{i}, r / 2\right)$ such that $B_{i-1} \cap B_{i} \neq \emptyset, i=2,3, \cdots, n$. Thus, there is $z_{i} \in\left(B_{i-1} \cap B_{i}\right) \backslash A$. Consider the null set $A^{\prime}=A_{z_{2}} \cup A_{z_{3}} \cup \cdots \cup A_{z_{n}}$. Given that $B_{i-1} \cup B_{i} \subset B_{r}\left(z_{i}\right)$, according to (7.9) we get $u(x)=u\left(z_{i}\right)$ for all $x \in\left(B_{r}\left(z_{i}\right) \cap \Omega\right) \backslash A_{z_{i}}$. In particular, we have $u(x)=u\left(z_{i}\right)$ for all $x \in\left(B_{i-1} \cup B_{i}\right) \cap\left(\Omega \backslash A^{\prime}\right)$. By the same token, $u(x)=u\left(z_{i+1}\right)$ for all $x \in\left(B_{i} \cup B_{i+1}\right) \cap\left(\Omega \backslash A^{\prime}\right)$. It turns out that $u(x)=u\left(z_{i}\right)=u\left(z_{i+1}\right)$ for all $x \in B_{i} \cap\left(\Omega \backslash A^{\prime}\right)$. Finally, we deduce that $u\left(z_{2}\right)=u\left(z_{3}\right)=\cdots u\left(z_{n}\right)=\lambda$. Since $\Omega=\bigcup_{i=1}^{n} B_{i} \cap \Omega$, we can conclude that $u(x)=\lambda$ for all $x \in \Omega \backslash A^{\prime}$ where we recall that $A^{\prime}$ is a null set.
(iii) Now assume $\Omega$ is $\rho$-connected with $0 \leq \rho<r$, i.e. $\Omega=\Omega_{1} \cup \Omega_{2} \cdots \Omega_{m}$ where each $\Omega_{i}$ is open bounded, connected and $\operatorname{dist}\left(\Omega_{i}, \Omega_{i+1}\right)<\rho$. The previous step implies that $u=\lambda_{i}$ a.e. on $\Omega_{i}, i=1,2, \cdots, m$. On the other hand, considering $0<\varepsilon<r-\rho$ such that $0 \leq \rho<\rho+2 \varepsilon<r$, we can find $a_{i} \in \Omega_{i}$ and $b_{i} \in \Omega_{i+1}$ such that

$$
\operatorname{dist}\left(\Omega_{i}, \Omega_{i+1}\right) \leq\left|a_{i}-b_{i}\right|<\rho+\varepsilon<r
$$

It follows that $B\left(a_{i}, \varepsilon\right) \cap \Omega_{i} \neq \emptyset$ and $B\left(a_{i}, \rho+\varepsilon\right) \cap \Omega_{i+1} \neq \emptyset$. Given that $A$ is a null set, we can find $\xi_{i} \in$ $\left(\Omega_{i+1} \cap B\left(a_{i}, \rho+\varepsilon\right)\right) \backslash A$. Since $u=\lambda_{i+1}$ a.e. in $\Omega_{i+1}$, by (7.9) we get

$$
u(x)=u\left(\xi_{i}\right)=\lambda_{i+1} \text { for all } x \in B_{r}\left(\xi_{i}\right) \cap\left(\Omega \backslash A_{\xi_{i}}\right)
$$

Note that, for $z \in B_{\varepsilon}\left(a_{i}\right)$ since $\xi_{i} \in B\left(a_{i}, \rho+\varepsilon\right)$ we have

$$
\left|z-\xi_{i}\right| \leq\left|z-a_{i}\right|+\left|a_{i}-\xi_{i}\right|<\rho+2 \varepsilon<r .
$$

This implies $B_{\varepsilon}\left(a_{i}\right) \subset B_{r}\left(\xi_{i}\right)$. Since $u=\lambda_{i}$ a.e. in $\Omega_{i}$, in particular we have

$$
\lambda_{i}=u(x)=u\left(\xi_{i}\right)=\lambda_{i+1} \quad \text { for all } \quad x \in B_{\varepsilon}\left(a_{i}\right) \cap\left(\Omega_{i} \backslash A_{\xi_{i}}\right) \subset B_{r}\left(\xi_{i}\right) \cap\left(\Omega \backslash A_{\xi_{i}}\right)
$$

This shows that $\lambda_{1}=\lambda_{2}=\cdots=\lambda_{n}$, and hence it follows that $u=\lambda_{1}$ a.e. in $\Omega$.
The following result is some general nonlocal Poincaré inequality.
Theorem 7.8. Assume that there is $r>0$ such that

$$
\begin{equation*}
\kappa:=\underset{h \in B_{r}(0)}{\operatorname{ess} \inf } \nu(h)>0 \tag{7.10}
\end{equation*}
$$

If $r \geq \operatorname{diam}(\Omega)$ or else $\Omega$ is $\rho$-connected with $r>\rho \geq 0$ then there is $C=C(d, p, \Omega, \nu)>0$ such that

$$
\begin{equation*}
\left\|u-f_{\Omega} u\right\|_{L^{p}(\Omega)}^{p} \leq C \mathcal{E}_{\Omega}(u, u) \quad \text { for all } u \in L^{p}(\Omega) \tag{7.11}
\end{equation*}
$$

Proof. If $r \geq \operatorname{diam}(\Omega)$ then $\nu(x-y) \geq \kappa$ for all $x, y \in \Omega$. Hence Jensen's inequality yields

$$
\mathcal{E}_{\Omega}(u, u) \geq \kappa \int_{\Omega} \int_{\Omega}|u(x)-u(y)|^{p} \mathrm{~d} y \mathrm{~d} x \geq \kappa|\Omega|\left\|u-f_{\Omega} u\right\|_{L^{p}(\Omega)}^{p}
$$

Now assume that $0<r<\operatorname{diam}(\Omega)$ and $\Omega$ is $\rho$-connected with $0<\rho \leq r$. Assume such $C$ does not exist. We can find a sequence $\left(u_{n}\right)_{n} \subset L^{p}(\Omega)$ such that $f_{\Omega} u_{n}=0,\left\|u_{n}\right\|_{L^{p}(\Omega)}=1$ and $\mathcal{E}_{\Omega}\left(u_{n}, u_{n}\right) \leq \frac{1}{2^{n}}$ and hence

$$
\kappa \iint_{\Omega \Omega}\left|u_{n}(x)-u_{n}(y)\right|^{p} \mathbb{1}_{B_{r}(0)}(x-y) \mathrm{d} y \mathrm{~d} x \leq \iint_{\Omega \Omega}\left|u_{n}(x)-u_{n}(y)\right|^{p} \nu(x-y) \mathrm{d} y \mathrm{~d} x \leq \frac{1}{2^{n}}
$$

Since $\left(u_{n}\right)_{n}$ is bounded in $L^{p}(\Omega)$, passing through a subsequence if necessary we can assume that $\left(u_{n}\right)_{n}$ weakly converges in $L^{p}(\Omega)$ to some $u$, where $u \in L^{p}(\Omega) p>1$ and $u$ is a signed Radon measure on $\Omega$ for $p=1$. In
particular, $f_{E} u_{n}(x) \mathrm{d} x \rightarrow f_{E} u(x) \mathrm{d} x$ for every Borel set $E \subset \Omega$. In addition, we can assume there is a null set $A \subset \Omega$ such that

$$
\begin{equation*}
\int_{B_{r}(x) \cap \Omega}\left|u_{n}(x)-u_{n}(y)\right|^{p} \mathrm{~d} y \xrightarrow{n \rightarrow \infty} 0 \quad \text { for all } x \in \Omega \backslash A . \tag{7.12}
\end{equation*}
$$

Moreover, for each $x \in \Omega \backslash A$ there is another null set $A_{x} \subset \Omega$ such that

$$
\begin{equation*}
u_{n}(x)-u_{n}(y) \xrightarrow{n \rightarrow \infty} 0 \quad \text { for all } y \in\left(B_{r}(x) \cap \Omega\right) \backslash A_{x} \tag{7.13}
\end{equation*}
$$

Consider $\left(B_{r / 2}\left(x_{i}\right)\right)_{1 \leq i \leq n}$ a finite cover of $\Omega$ with $x_{i} \in \Omega$. Choose $\xi_{i} \in\left(B_{r / 2}\left(x_{i}\right) \cap \Omega\right) \backslash A$. Hence $\left(E_{\xi_{i}}\right)_{1 \leq i \leq n}$ with $E_{\xi_{i}}=B_{r}\left(\xi_{i}\right) \cap \Omega$ is also a finite cover of $\Omega$. Note that

$$
\left|u_{n}\left(\xi_{i}\right)-f_{E_{\xi_{i}}} u(y) \mathrm{d} y\right|^{p} \leq 2^{p-1} f_{E_{\xi_{i}}}\left|u_{n}\left(\xi_{i}\right)-u_{n}(y)\right|^{p} \mathrm{~d} y+2^{p-1}\left|f_{E_{\xi_{i}}} u_{n}(y) \mathrm{d} y-f_{E_{\xi_{i}}} u(y) \mathrm{d} y\right|^{p}
$$

This together with the weak convergence and the convergence in (7.12) imply that

$$
u_{n}\left(\xi_{i}\right) \xrightarrow{n \rightarrow \infty} \lambda_{\xi_{i}}=f_{E_{\xi_{i}}} u(y) \mathrm{d} y
$$

Whence by (7.13) we get that

$$
u_{n}(y) \xrightarrow{n \rightarrow \infty} \lambda_{\xi_{i}} \quad \text { for all } y \in E_{\xi_{i}} \backslash A_{\xi_{i}}, \quad E_{\xi_{i}}=B_{r}\left(\xi_{i}\right) \cap \Omega
$$

In other words, $u_{n} \rightarrow v$ a.e. in $\Omega$ where $v(x)=\lambda_{\xi_{i}}$ for $x \in E_{\xi_{i}}$. The Fatou's Lemma yields

$$
\iint_{\Omega \Omega}|v(x)-v(y)|^{p} \nu(x-y) \mathrm{d} y \mathrm{~d} x \leq \liminf _{n \rightarrow \infty} \iint_{\Omega \Omega}\left|u_{n}(x)-u_{n}(y)\right|^{p} \nu(x-y) \mathrm{d} y \mathrm{~d} x=0 .
$$

It follows from Lemma 7.7 that $v=\lambda$ in $\Omega$, that is, $\lambda=\lambda_{\xi_{i}}=\cdots=\lambda_{\xi_{n}}$. On the other hand, since $\Omega=\bigcup_{i=1}^{n} E_{\xi_{i}}$, this combined with (7.12) gives

$$
\left\|u_{n}-v\right\|_{L^{p}(\Omega)} \leq \sum_{i=1}^{n}\left(\int_{E_{\xi_{i}}}\left|u_{n}\left(\xi_{i}\right)-u_{n}(y)\right|^{p} \mathrm{~d} y\right)^{1 / p}+\left|E_{\xi_{i}}\right|^{1 / p}\left|u_{n}\left(\xi_{i}\right)-\lambda_{\xi_{i}}\right| \xrightarrow{n \rightarrow \infty} 0
$$

That is, $\left(u_{n}\right)_{n}$ strongly converges to $v=\lambda$ in $L^{p}(\Omega)$. Taking into account the weak convergence we deduce that $u=v=\lambda$. Since $f_{\Omega} u_{n}=0$ and $\left\|u_{n}\right\|_{L^{p}(\Omega)}=1$, it follows that $u=f_{\Omega} u=0$ and $\|u\|_{L^{p}(\Omega)}=1$ which is a contradiction.

Theorem 7.9. Assume $\Omega \subset \mathbb{R}^{d}$ is connected or 0 -connected. Assume $\nu$ is unimodal and $|\{\nu>0\}|>0$, i.e., $\nu \not \equiv 0$. Then there is $C=C(d, p, \Omega, \nu)>0$ such that

$$
\begin{equation*}
\left\|u-f_{\Omega} u\right\|_{L^{p}(\Omega)}^{p} \leq C \mathcal{E}_{\Omega}(u, u) \quad \text { for all } u \in L^{p}(\Omega) \tag{7.14}
\end{equation*}
$$

Proof. Since $\nu \not \equiv 0$ there is $\xi \in \mathbb{R}^{d}, \xi \neq 0$ such that $\nu(\xi)>0$. Let $r=|\xi|>0$, since $\nu$ is unimodal we have $\nu(h) \geq c \nu(\xi)$ for all $h \in B_{r}(0)$. Therefore, we have $\kappa=\operatorname{essinf}_{h \in B_{r}(0)} \nu(h)>0$ where $\kappa=c \nu(\xi)>0$. The desired result follows from Theorem 7.8.

Corollary 7.10 (Fractional Poincaré inequality). Assume $\Omega$ is connected or 0 -connected then for all $r>0$ there is $C=C(d, p, \Omega, s, r)>0$ such that

$$
\begin{equation*}
\left\|u-f_{\Omega} u\right\|_{L^{p}(\Omega)}^{p} \leq C \iint_{\Omega \Omega} \frac{|u(x)-u(y)|^{p}}{|x-y|^{d+s p}} \mathbb{1}_{B_{r}(0)}(x-y) \mathrm{d} y \mathrm{~d} x \quad \text { for all } u \in L^{p}(\Omega) \tag{7.15}
\end{equation*}
$$

Furthermore, the inequality is always true, whenever $r \geq \operatorname{diam}(\Omega)$.
Proof. This is a direct consequence of Theorem 7.9.
If the assumption (7.10) fails i.e., $\kappa=0$ for any $r$, we can balance this deficiency with the compactness.
Theorem 7.11. Let $r>\rho \geq 0$. Assume that

- $\nu>0$ a.e. on $B_{r}(0)$,
- $r \geq \operatorname{diam}(\Omega)$ or that $\Omega \subset \mathbb{R}^{d}$ is $\rho$-connected,
- the embedding $W_{\nu}^{p}(\Omega) \hookrightarrow L^{p}(\Omega)$ is compact.

Then there is $C=C(d, p, \Omega, \nu)>0$ such that

$$
\left\|u-f_{\Omega} u\right\|_{L^{p}(\Omega)}^{p} \leq C \iint_{\Omega \Omega}|u(x)-u(y)|^{p} \nu(x-y) \mathrm{d} y \mathrm{~d} x \quad \text { for all } u \in L^{p}(\Omega)
$$

Proof. Assume $C$ does not exist, then we can find a sequence $\left(u_{n}\right)_{n} \subset L^{p}(\Omega)$ such that $f_{\Omega} u_{n}=0,\left\|u_{n}\right\|_{L^{p}(\Omega)}=1$ and $\mathcal{E}_{\Omega}\left(u_{n}, u_{n}\right) \leq \frac{1}{2^{n}}$. The sequence $\left(u_{n}\right)_{n}$ is thus bounded in $W_{\nu}^{p}(\Omega)$. Since the embedding $W_{\nu}^{p}(\Omega) \hookrightarrow L^{p}(\Omega)$ is compact, passing through a subsequence if necessary we can assume that $\left(u_{n}\right)_{n}$ converges to some function $u$ in $L^{p}(\Omega)$. It clearly follows that $f_{\Omega} u=0$ and $\|u\|_{L^{p}(\Omega)}=1$. Moreover, by Fatou's lemma we have

$$
\iint_{\Omega \Omega}|u(x)-u(y)|^{p} \nu(x-y) \mathrm{d} y \mathrm{~d} x \leq \liminf _{n \rightarrow \infty} \iint_{\Omega \Omega}\left|u_{n}(x)-u_{n}(y)\right|^{p} \nu(x-y) \mathrm{d} y \mathrm{~d} x=0
$$

It follows from Lemma 7.7 that $u$ is constant and hence $u=f_{\Omega} u=0$ a.e. in $\Omega$. This goes against the fact that $\|u\|_{L^{p}(\Omega)}=1$ and hence our initial assumption is wrong.
The Poincaré inequality fails if $\Omega$ is $\rho$-connected with $\rho$ too large.
Counterexample 7.12. Assume $d \geq 2$ and consider $\Omega=\Omega_{1} \cup \Omega_{2}$ with $\Omega_{1}=B_{1}(0)$ and $\Omega_{2}=B_{4}(0) \backslash B_{3}(0)$ and $\nu(h)=\mathbb{1}_{B_{1}(0)}(h)$. The sets $\Omega_{1}$ and $\Omega_{2}$ are connected thus $\Omega$ is $\rho$-connected with $\rho>2$. However, the Poincaré inequality fails for the couple $(\Omega, \nu)$. Indeed, for $u(x)=\mathbb{1}_{\Omega_{1}}(x)-\mathbb{1}_{\Omega_{2}}(x)$. As $u$ is not constant on $\Omega$, it is readily seen that $\left\|u-f_{\Omega} u\right\|_{L^{p}(\Omega)}^{p} \neq 0$. On the other hand, $u=1$ on $\Omega_{1}$ and $u=-1$ on $\Omega_{2}$, moreover $\mathbb{1}_{B_{1}(0)}(x-y)=0$ if $x \in \Omega_{1}$ and $y \in \Omega_{2}$. Therefore, one easily verifies that $\mathcal{E}_{\Omega}(u, u)=0$.
In general, the Poincaré inequality fails if $\Omega=\Omega_{1} \cup \Omega_{2}$, $\operatorname{dist}\left(\Omega_{1}, \Omega_{2}\right)>r$ and $\operatorname{supp} \nu=\overline{B_{r}(0)}$. Indeed as above, for $u(x)=\mathbb{1}_{\Omega_{1}}(x)-\mathbb{1}_{\Omega_{2}}(x)$ we have $\left\|u-f_{\Omega} u\right\|_{L^{p}(\Omega)}^{p} \neq 0$ and $\mathcal{E}_{\Omega}(u, u)=0$.

## 8. Existence of weak solutions

In this section, we deal with the well-posedness of nonlocal problems $\left(P_{\nu, \tau}\right)$.
8.1. Basics on direct method. Our proofs of existence are based on the Direct Method of Calculus of Variations. Let us evoke some fundamental results of calculus of variation mainly collected from [BC17, ET76, Rin18].
Definition 8.1. Let $(X, \tau)$ be a topological space and $\mathcal{J}: X \rightarrow \mathbb{R} \cup\{\infty\}$ be a functional.

- $\mathcal{J}$ is called sequentially $\tau$-coercive (or simply coercive) if every lower level set of $\mathcal{J}$ is $\tau$-sequentially precompact, i.e., every sequence $\left(u_{n}\right)_{n} \subset\{u \in X: \mathcal{J}(u) \leq a\}, a \in \mathbb{R}$ has a $\tau$-converging subsequence in $X$.
- $\mathcal{J}$ is called sequentially $\tau$-lower semicontinuous (or simply lower semicontinuous) if for any sequence $\left(u_{n}\right)_{n}$ $\tau$-converging to $u$ in $X$, it holds that

$$
\mathcal{J}(u) \leq \liminf _{n \rightarrow \infty} \mathcal{J}\left(u_{n}\right)
$$

The direct method for the minimization problem is encapsulated in the following result.
Theorem 8.2. Let $(X, \tau)$ be a topological space. If the functional $\mathcal{J}: X \rightarrow \mathbb{R} \cup\{\infty\}$ is both coercive and lower semicontinuous, then there is $u_{*} \in X$ such that

$$
\mathcal{J}\left(u_{*}\right)=\min _{u \in X} \mathcal{J}(u)
$$

Next we see characterizations of the weak lower semicontinuity and the weak coercivity for the particular situation where $\left(X,\|\cdot\|_{X}\right)$ is a normed space. Recall on $X$ there is the strong topology induced by $\|\cdot\|_{X}$ and the weak topology induced by the weak convergence. Let us recall Mazur's Lemma beforehand.

Theorem 8.3 (Mazur's Lemma, [ET76, page 6]). If a sequence $\left(u_{n}\right)_{n}$ is weakly converge in $X$ to some $u \in X$, there is $\left(\theta_{k}^{n}\right)_{k=n, \cdots, N_{n}}$ such that, $\left\|v_{n}-u\right\|_{X} \xrightarrow{n \rightarrow \infty} 0$ where for each $n$,

$$
v_{n}=\sum_{k=n}^{N_{n}} \theta_{k}^{n} u_{k} \quad \text { with } \quad \sum_{k=n}^{N_{n}} \theta_{k}^{n}=1, \quad \theta_{k}^{n} \geq 0
$$

The Mazur's Lemma implies the following characterization of the weak lower semicontinuity.
Theorem 8.4. A weakly lower semicontinuous functional $\mathcal{J}: X \rightarrow \mathbb{R} \cup\{\infty\}$ is lower semicontinuous. The converse is true if in addition $\mathcal{J}$ is convex.
Theorem 8.5. If $\mathcal{J}: X \rightarrow \mathbb{R} \cup\{\infty\}$ is weakly coercive then $\mathcal{J}(u) \rightarrow \infty$ as $\|u\|_{X} \rightarrow \infty$. The converse holds if in addition $X$ is a reflexive Banach space.
8.2. Neumann boundary condition. The Neumann problem for the operator $L$ associated with the data $f$ : $\Omega \rightarrow \mathbb{R}$ and $g: \Omega^{c} \rightarrow \mathbb{R}$, is to find $u: \mathbb{R}^{d} \rightarrow \mathbb{R}$ such that

$$
\begin{equation*}
L u=f \quad \text { in } \quad \Omega \quad \text { and } \quad \mathcal{N} u=g \quad \text { on } \quad \mathbb{R}^{d} \backslash \Omega \tag{N}
\end{equation*}
$$

It is worth to emphasize that problem $(N)$ makes sense only if have $g=0$ on $\mathbb{R}^{d} \backslash \Omega_{\nu}=0$; where we recall that $\Omega_{\nu}=\Omega+\operatorname{supp} \nu$ is the nonlocal hull of $\Omega$. This is obviously due to the fact that, by nonlocality we have

$$
\mathcal{N} u(x)=\int_{\Omega} \psi(u(x)-u(y)) \nu(x-y) \mathrm{d} y=0 \quad \text { for all } x \in \mathbb{R}^{d} \backslash \Omega_{\nu}
$$

we recall $\psi(t)=|t|^{p-2} t$. By nonlocality, it turns out that $\Omega_{\nu}$ is the smallest set such that

$$
\begin{equation*}
L u(x)=\int_{\mathbb{R}^{d}} \psi(u(x)-u(y)) \nu(x-y) \mathrm{d} y=L_{\Omega_{\nu}} u(x):=\int_{\Omega_{\nu}} \psi(u(x)-u(y)) \nu(x-y) \mathrm{d} y \quad \text { for all } x \in \Omega \tag{8.1}
\end{equation*}
$$

Therefore, it is sufficient to prescribe the complement data on $\Omega_{e}$. It is rather, counter intuitive to see that prescribing the nonlocal boundary data on $\Omega_{e}$ is not all restrictive. Indeed, if we put $g_{e}=\left.g\right|_{\Omega_{e}}$, the restriction of $g$ on $\Omega_{e}$, the problem $(N)$ is the same as finding $u: \Omega_{\nu} \rightarrow \mathbb{R}$ such that

$$
L_{\Omega_{\nu}} u=f \quad \text { in } \quad \Omega \quad \text { and } \quad \mathcal{N} u=g_{e} \quad \text { on } \quad \Omega_{e}=\Omega_{\nu} \backslash \Omega
$$

where we recall $\Omega_{e}=\Omega_{\nu} \backslash \Omega$ is nonlocal boundary of $\Omega$ with respect to $\nu$. Actually, both the problems $(N)$ with $\left(N_{\nu}\right)$ are equivalent. Indeed, if $u$ solves $(N)$ then clearly $u_{\nu}=\left.u\right|_{\Omega_{\nu}}$ solves $\left(N_{\nu}\right)$. Conversely, if $u_{\nu}$ solves ( $N_{\nu}$ ), one verifies that $u=\widetilde{u}_{\nu}$ solves $(N)$ where $\widetilde{u}_{\nu}$ is the zero the extension of $u_{\nu}$ off $\Omega_{\nu}$, i.e., $\widetilde{u}_{\nu}=u_{\nu}$ on $\Omega_{\nu}$ and $\widetilde{u}_{\nu}=0$ on $\mathbb{R}^{d} \backslash \Omega_{\nu}$. From now on, the problem $(N)$ is understood in the sense of $\left(N_{\nu}\right)$. Motivated by the Gauss-Green formula (B.6) (see Appendix B.2) we define weak solutions of the Neumann problem as follows.

Definition 8.6. Let $f \in W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\prime}$ and $g \in T_{\nu}^{p}\left(\Omega^{c}\right)^{\prime}$. We say that $u \in W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ is a weak solution or the variational solution of Neumann problem $(N)$ if

$$
\mathcal{E}(u, v)=\langle f, v\rangle+\langle g, v\rangle \quad \text { for all } v \in W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)
$$

Note that the existence of a solution the compatibility condition $\langle f, 1\rangle+\langle g, 1\rangle=0$.
It is worth emphasizing that, the choice of $f$ and $g$ legitimately follows from the natural embeddings $W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right) \hookrightarrow$ $T_{\nu}^{p}\left(\Omega^{c}\right) \hookrightarrow L^{p}\left(\Omega^{c}, \omega\right)$. In particular, we have the following.
Definition 8.7. Let $f \in L^{p^{\prime}}(\Omega)$ and $g \in L^{p^{\prime}}\left(\Omega^{c}, \omega^{1-p^{\prime}}\right)$ where $\omega \in\left\{\widetilde{\nu}_{B}, \bar{\nu}_{B}, \widehat{\nu}_{R}\right\}, B \subset \Omega$ (see Definition 5.2). We say that $u \in W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ is a weak solution of $(N)$ if

$$
\begin{equation*}
\mathcal{E}(u, v)=\int_{\Omega} f(x) v(x) \mathrm{d} x+\int_{\Omega^{c}} g(y) v(y) \mathrm{d} y, \quad \text { for all } v \in W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right) \tag{V}
\end{equation*}
$$

In this case, taking $v=1$, the compatibility condition reads

$$
\begin{equation*}
\int_{\Omega} f(x) \mathrm{d} x+\int_{\Omega^{c}} g(y) \mathrm{d} y=0 \tag{C}
\end{equation*}
$$

Some comments and remarks about the Neumann problem may be helpful at this stage.
Remark 8.8. Note that [DROV17, Def. 3.6] and subsequent definitions like [MPL19, Definition 2.7] look very similar to $(V)$ at first glance. However, the test space defined in [DROV17, Eq. (3.1)], [MPL19, Section 2] depends on the Neumann data $g$, which is not natural. Our test space $W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ in the weak formulation $(V)$ does not depend on the Neumann data $g$.

Remark 8.9. The compatibility condition $\langle f, 1\rangle+\langle g, 1\rangle=0$ or $(C)$ is an implicit necessary requirement that the data $f$ and $g$ must fulfill before any attempt of solving the problems $(N),(V)$ or $\left(V^{\prime}\right)$. This is essentially due to fact that, the operators $L$ and $\mathcal{N}$ annihilate additive constants. An immediate effect, is that, if $f$ and $g$ are compatible, as long as $u$ is a solution to the problem $(N),(V)$ or $\left(V^{\prime}\right)$ so is any function $u+c$ with $c \in \mathbb{R}$. Accordingly, problems $(N),(V)$ or $\left(V^{\prime}\right)$ are ill-posed in the sense of Hadamard. One of the finest strategy to overcome this issue, is to introduce the reduced problem

$$
\mathcal{E}(u, v)=\langle f, v\rangle+\langle g, v\rangle, \quad \text { for all } v \in W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\perp}
$$

where the space $W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\perp}$ is given by

$$
W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\perp}:=\left\{u \in W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right): \int_{\Omega} u(x) \mathrm{d} x=0\right\}
$$

Note that $W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\perp}$ only discards non-zero constants from $W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$, and is a closed subspace of $W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$. It is worth to emphasize that, if $f$ and $g$ are compatible and $u$ is a solution to reduced problem $\left(V^{\prime \perp}\right)$ then each $u+c, c \in \mathbb{R}$ is also solution to $\left(V^{\prime}\right)$ and vice versa.

Remark 8.10. The nonlocal formulation the Neumann problem should be contrasted with the local one, where the operators $L$ and $\mathcal{N}$ respectively replaced by the operators $-\Delta_{p}$ and $\partial_{n, p} u(x)=|\nabla u(x)|^{p-2} \nabla u(x) \cdot n(x)$. Recall that in passing that $u \in W^{1, p}(\Omega)$ is a weak to the Neumann problem

$$
\begin{equation*}
-\Delta_{p} u=f \quad \text { in } \quad \Omega \quad \text { and } \quad \partial_{n, p} u=g \quad \text { on } \quad \partial \Omega \tag{8.2}
\end{equation*}
$$

if $u$ satisfies variational problem

$$
\begin{equation*}
\mathcal{E}^{0}(u, v)=\int_{\Omega} f(x) v(x) \mathrm{d} x+\int_{\partial \Omega} g(y) v(y) \mathrm{d} \sigma(y), \quad \text { for all } v \in W^{1, p}(\Omega) \tag{8.3}
\end{equation*}
$$

here we define

$$
\mathcal{E}^{0}(u, v)=\int_{\Omega}|\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla v(x) \mathrm{d} x .
$$

The local counterpart of the compatibility condition $(C)$ is given by

$$
\begin{equation*}
\int_{\Omega} f(x) \mathrm{d} x+\int_{\partial \Omega} g(y) \mathrm{d} \sigma(y)=0 . \tag{8.4}
\end{equation*}
$$

The local counterpart of the problem $\left(V^{\prime \perp}\right)$ reads as follows

$$
\begin{equation*}
\mathcal{E}^{0}(u, v)=\int_{\Omega} f(x) v(x) \mathrm{d} x+\int_{\partial \Omega} g(y) v(y) \mathrm{d} \sigma(y), \quad \text { for all } \quad v \in W^{1, p}(\Omega)^{\perp} \tag{8.5}
\end{equation*}
$$

where $W^{1, p}(\Omega)^{\perp}:=\left\{u \in W^{1, p}(\Omega): \int_{\Omega} u(x) \mathrm{d} x=0\right\}$.
For $\Omega$ and $u$ sufficiently regular, one can show that (see for instance [FK22,Fog20]), $u$ solves the problem ( $N$ ) if and only if it solves the problem $(V)$. This is in particular possible under the condition that and that the Gauss-Green formula (B.6) holds. To put it simply, under mild conditions on $\Omega$ and $\nu$ both problems ( $N$ ) and ( $V$ ) are equivalent when the solutions are regularity enough.
Actually, solutions to the variational problem $\left(V^{\prime}\right)$ are critical points of the functional

$$
\begin{equation*}
\mathcal{J}(v)=\frac{1}{p} \mathcal{E}(v, v)-\langle f, v\rangle-\langle g, v\rangle \tag{8.6}
\end{equation*}
$$

It is important to keep in mind that the Fréchet derivative of $\mathcal{J}$ on $W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ is given as

$$
\left\langle\mathcal{J}^{\prime}(u), v\right\rangle=\mathcal{E}(u, v)-\langle f, v\rangle-\langle g, v\rangle
$$

Proposition 8.11. The variational problem $\left(V^{\prime \perp}\right)$ is equivalent to the minimization problem

$$
\mathcal{J}(u)=\min _{v \in W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\perp}} \mathcal{J}(v) .
$$

Analogously, the variational problem $\left(V^{\prime}\right)$ is equivalent to the minimization problem

$$
\begin{equation*}
\mathcal{J}(u)=\min _{v \in W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)} \mathcal{J}(v) \tag{M}
\end{equation*}
$$

Proof. Let $u, v \in W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\perp}$. Assume $\left(V^{\prime \perp}\right)$ holds. Hölder's and Young's inequalities imply

$$
\mathcal{E}(u, v) \leq \mathcal{E}(u, u)^{1 / p^{\prime}} \mathcal{E}(v, v)^{1 / p} \leq \frac{1}{p^{\prime}} \mathcal{E}(u, u)+\frac{1}{p} \mathcal{E}(v, v)=\mathcal{E}(u, u)-\frac{1}{p} \mathcal{E}(u, u)+\frac{1}{p} \mathcal{E}(v, v)
$$

In virtue of $\left(V^{\prime \perp}\right)$ which holds for $u$ and $v$ we get $\mathcal{J}(u) \leq \mathcal{J}(v)$ and thus $u$ solves $\left(M^{\perp}\right)$.
Conversely let $u$ satisfies $\left(M^{\perp}\right)$, i.e., $\mathcal{J}(u) \leq \mathcal{J}(v)$ for $v \in W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\perp}$. In particular, $\mathcal{J}(u) \leq \mathcal{J}(u+t v)$ for all $t \in \mathbb{R}$. That is, the mapping $t \mapsto \mathcal{J}(u+t v)$ is differentiable and has a critical point at $t=0$. It follows that $u$ satisfies ( $V^{\prime \perp}$ ) since

$$
\mathcal{E}(u, v)-\langle f, v\rangle-\langle g, v\rangle=\left\langle\mathcal{J}^{\prime}(u), v\right\rangle=\lim _{t \rightarrow 0} \frac{\mathcal{J}(u+t v)-\mathcal{J}(u)}{t}=0
$$

The equivalence between $(V)$ and $(M)$ can be proven analogously.
We are now in position to state the well-posedness of the problems $\left(V^{\prime \perp}\right),\left(V^{\prime}\right)$ and $(V)$. Given a weight $\omega$, we opt for the convention that $\omega^{-1}(x)=0$ whenever $\omega(x)=0$.

Theorem 8.12. Let $\omega \in\left\{\widetilde{\nu}_{\Omega}, \bar{\nu}_{\Omega}, \widehat{\nu}_{R}\right\}$ where $\left|B_{R}(0) \cap \Omega\right|>0$ (see Definition 5.2). Assume that ( $S_{1}$ ) holds so that $W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right) \equiv W_{\nu}^{p}\left(\Omega \mid \Omega_{\nu}\right)$ is a reflexive Banach space and that Poincaré inequality $(P)$ holds (see page 22). Assume $f \in W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\prime}\left(\right.$ or $\left.f \in L^{p^{\prime}}(\Omega)\right)$ and $g \in T_{\nu}^{p}\left(\Omega^{c}\right)^{\prime}\left(\right.$ or $\left.g \in L^{p^{\prime}}\left(\Omega^{c}, \omega^{1-p^{\prime}}\right)\right)$. There is $C=C(d, p, \Omega, \nu)>0$ such that
( $i$ ) Existence. There exists a unique $u \in W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\perp}$ satisfying $\left(V^{\prime \perp}\right)$. The Neumann problem ( $V^{\prime}$ ) has a solution $w \in W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$, if and only if $f$ and $g$ are compatible, i.e., $\langle f, 1\rangle+\langle g, 1\rangle=0$ and $w$ is of the form $w=u+c, c \in \mathbb{R}$.
(ii) Boundedness. Any solution $w$ to $\left(V^{\prime}\right)$ satisfies

$$
\begin{equation*}
\left\|w-f_{\Omega} w\right\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)} \leq C\left(\|f\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\prime}}+\|g\|_{T_{\nu}^{p}\left(\Omega^{c}\right)^{\prime}}\right)^{1 /(p-1)} \tag{8.7}
\end{equation*}
$$

(iii) Continuity. If $u_{i}=w_{i}-f_{\Omega} w_{i}, i=1,2$ where $w_{i}$ satisfies $\left(V^{\prime}\right)$ with $f=f_{i}$ then

$$
\left\|u_{1}-u_{2}\right\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)} \leq \begin{cases}C\left(\left\|f_{1}-f_{2}\right\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\prime}}+\left\|g_{1}-g_{2}\right\|_{T_{\nu}^{p}\left(\Omega^{c}\right)^{\prime}}\right)^{1 /(p-1)} & p \geq 2  \tag{8.8}\\ C M\left(\left\|f_{1}-f_{2}\right\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\prime}}+\left\|g_{1}-g_{2}\right\|_{T_{\nu}^{p}\left(\Omega^{c}\right)^{\prime}}\right. & 1<p<2\end{cases}
$$

where we put $M=M\left(f_{1}, f_{2}, g_{1}, g_{2}\right)=\left(\sum_{i=1}^{2}\left\|f_{i}\right\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\prime}}+\left\|g_{i}\right\|_{T_{\nu}^{p}\left(\Omega^{c}\right)^{\prime}}\right)^{\frac{2-p}{p-1}}$.
(iv) Problem (V). In the case $f \in L^{p^{\prime}}(\Omega)$ and $g \in L^{p^{\prime}}\left(\Omega^{c}, \omega^{1-p^{\prime}}\right)$ identified with the forms $v \mapsto\langle f, v\rangle=$ $\int_{\Omega} f(x) v(x) \mathrm{d} x$ and $v \mapsto\langle g, v\rangle=\int_{\Omega^{c}} g(y) v(y) \mathrm{d} y$ respectively. Then $f \in W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\prime}, g \in T_{\nu}^{p}\left(\Omega^{c}\right)^{\prime}$ and there is $C_{1}=C_{1}(d, p, \Omega)$ such that

$$
\|f\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\prime}} \leq\|f\|_{L^{p^{\prime}}(\Omega)} \quad \text { and } \quad\|g\|_{T_{\nu}^{p}\left(\Omega^{c}\right)^{\prime}} \leq C_{1}\|g\|_{L^{p^{\prime}}\left(\Omega^{c}, \omega^{1-p^{\prime}}\right)}
$$

In other words, both problems $(V)$ and $\left(V^{\prime}\right)$ are identical.

Proof. We emphasize that throughout the proof, $C>0$ denotes a generic constant only depending on the constant from the Poincaré inequality and $p$.
(i) Since $g \in T_{\nu}^{p}\left(\Omega^{c}\right)^{\prime} \hookrightarrow W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\prime}$ we have $g \in W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\prime}$. Thus, the functional

$$
v \mapsto \mathcal{J}(v)=\frac{1}{p} \mathcal{E}(v, v)-\langle f, v\rangle-\langle g, v\rangle
$$

is clearly continuous (hence lower semicontinuous) and convex on $W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ a fortiori on $W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\perp}$. According to Theorem 8.4, $\mathcal{J}$ is weakly lower semicontinuous. On the other hand, in virtue of the Poincaré inequality $(P)$ one readily finds a constant $C>0$

$$
\begin{equation*}
C\left\|v-f_{\Omega} v\right\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}^{p} \leq \mathcal{E}(v, v) \quad \text { for all } v \in W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right) \tag{8.9}
\end{equation*}
$$

Therefore if $v \in W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\perp}$ then we have

$$
\mathcal{J}(v) \geq C\|v\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}^{p}-\|f\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\prime}}\|v\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}-\|g\|_{T_{\nu}^{p}\left(\Omega^{c}\right)^{\prime}}\|v\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}
$$

Since $p>1$ it follows that $\mathcal{J}(v) \rightarrow \infty$ as $\|v\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)} \rightarrow \infty$. In fact we have

$$
\frac{\mathcal{J}(v)}{\|v\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}} \rightarrow \infty, \quad \text { as }\|v\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)} \rightarrow \infty
$$

Since $W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ is Banach space, one deduces in view of Theorem 8.5 that $\mathcal{J}$ is weakly coercive on $W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\perp}$. Hence, by Theorem $8.2 \mathcal{J}$ possesses a minimizer $u \in W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\perp}$, i.e., $u$ solves $\left(M^{\perp}\right)$. By Proposition $8.11, u$ is also a solution to $\left(V^{\prime \perp}\right)$. The uniqueness follows from the strict convexity of $\mathcal{J}$ or merely from (iii); see the estimate (8.8).
Now if $f$ and $g$ are compatible then we easily observe that $\mathcal{J}(v+c)=\mathcal{J}(v)$ for every $v \in W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ and every $c \in \mathbb{R}$. For $v \in W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ we have $v-f_{\Omega} v \in W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\perp}$. Hence $\mathcal{J}(u+c)=\mathcal{J}(u) \leq \mathcal{J}\left(v-f_{\Omega} v\right)=\mathcal{J}(v)$. Thus, $u+c$ also solves $(M)$ which is equivalent to $\left(V^{\prime}\right)$. Conversely if $w$ solves $\left(V^{\prime}\right)$ then one verifies that $w-f_{\Omega} w \in W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\perp}$ solves $\left(V^{\prime \perp}\right)$. By uniqueness we get $u=w-f_{\Omega} w$ that is $w=u+c$ with $c=f_{\Omega} w$. Moreover taking $v=1$ in $\left(V^{\prime}\right)$ yields the compatibility condition.
(ii) If $w \in W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ is a solution to $\left(V^{\prime}\right)$ we have

$$
\begin{aligned}
\mathcal{E}(w, w) & =\mathcal{E}\left(w, w-f_{\Omega} w\right)=\left\langle f, w-f_{\Omega} w\right\rangle+\left\langle g, w-f_{\Omega} w\right\rangle \\
& \leq\left\|w-f_{\Omega} w\right\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}\left(\|f\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\prime}}+\|g\|_{T_{\nu}^{p}\left(\Omega^{c}\right)^{\prime}}\right) .
\end{aligned}
$$

Combining this together with the estimate (8.9) we find that

$$
\left\|w-f_{\Omega} w\right\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)} \leq C\left(\|f\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\prime}}+\|g\|_{\left.T_{\nu}^{p}\left(\Omega^{c}\right)^{\prime}\right)^{1 /(p-1)}}\right.
$$

(iii) Case $p \geq 2$. The estimate $\left(|b|^{p-2} b-|a|^{p-2} a \mid\right)(b-a) \geq A_{p}^{\prime}|b-a|^{p}$ (see(A.3)) implies

$$
\begin{equation*}
\left|\mathcal{E}\left(v, v-v^{\prime}\right)-\mathcal{E}\left(v^{\prime}, v-v^{\prime}\right)\right| \geq A_{p}^{\prime} \mathcal{E}\left(v-v^{\prime}, v-v^{\prime}\right) \tag{8.10}
\end{equation*}
$$

Put $w=w_{1}-w_{2}$. Since $u_{1}-u_{2}=w-f_{\Omega} w \in W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$, using (8.10) we have

$$
\begin{aligned}
A_{p}^{\prime} \mathcal{E}(w, w) & \leq\left|\mathcal{E}\left(w_{1}, w_{1}-w_{2}\right)-\mathcal{E}\left(w_{2}, w_{1}-w_{2}\right)\right| \\
& =\left|\mathcal{E}\left(w_{1}, w-f_{\Omega} w\right)-\mathcal{E}\left(w_{2}, w-f_{\Omega} w\right)\right| \\
& =\left|\left\langle f_{1}-f_{2}, w-f_{\Omega} w\right\rangle+\left\langle g_{1}-g_{2}, w-f_{\Omega} w\right\rangle\right| \\
& \leq\left\|w-f_{\Omega} w\right\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}\left(\left\|f_{1}-f_{2}\right\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\prime}}+\left\|g_{1}-g_{2}\right\|_{T_{\nu}^{p}\left(\Omega^{c}\right)^{\prime}}\right) .
\end{aligned}
$$

Inserting this in (8.9) implies

$$
\left\|w-f_{\Omega} w\right\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)} \leq C\left(\left\|f_{1}-f_{2}\right\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\prime}}+\left\|g_{1}-g_{2}\right\|_{T_{\nu}^{p}\left(\Omega^{c}\right)^{\prime}}\right)^{1 /(p-1)}
$$

Case $1<p<2$. The inequality $\left(|b|^{p-2} b-|a|^{p-2} a\right)(b-a) \geq A_{p}^{\prime}|b-a|^{2}\left(|a|^{p}+|b|^{p}\right)^{\frac{p-2}{p}}$ (see (A.4)) can be rewritten as

$$
c_{p}|b-a|^{p} \leq\left(\left(|b|^{p-2} b-|a|^{p-2} a\right)(b-a)\right)^{1 / q}\left(|a|^{p}+|b|^{p}\right)^{1 / q^{\prime}}
$$

where $q=\frac{2}{p}, q^{\prime}=\frac{2}{2-p}$ and $c_{p}=2^{\frac{2-p}{2}} A_{p}^{\prime \frac{p}{2}}$. This, together with Hölder inequality yields

$$
\begin{equation*}
\left(\mathcal{E}\left(v, v-v^{\prime}\right)-\mathcal{E}\left(v^{\prime}, v-v^{\prime}\right)\right)^{\frac{p}{2}}\left(\mathcal{E}(v, v)+\mathcal{E}\left(v^{\prime}, v^{\prime}\right)\right)^{\frac{2-p}{2}} \geq c_{p} \mathcal{E}\left(v-v^{\prime}, v-v^{\prime}\right) \tag{8.11}
\end{equation*}
$$

The same reasoning as above yields

$$
c_{p} \mathcal{E}(w, w) \leq\left\|w-f_{\Omega} w\right\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}^{\frac{p}{2}}\left(\left\|f_{1}-f_{2}\right\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\prime}}+\left\|g_{1}-g_{2}\right\|_{\left.T_{\nu}^{p}\left(\Omega^{c}\right)^{\prime}\right)^{\frac{p}{2}}}\left(\mathcal{E}\left(w_{1}, w_{1}\right)+\mathcal{E}\left(w_{2}, w_{2}\right)\right)^{\frac{2-p}{2}} .\right.
$$

On the other hand, by(8.7) we have

$$
\left(\mathcal{E}\left(w_{1}, w_{1}\right)+\mathcal{E}\left(w_{2}, w_{2}\right)\right)^{\frac{2-p}{2}} \leq C\left(\sum_{i=1}^{2}\left\|f_{i}\right\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\prime}}+\left\|g_{i}\right\|_{T_{\nu}^{p}\left(\Omega^{c}\right)^{\prime}}\right)^{\frac{2-p}{p-1}}=C M
$$

Altogether with the estimate (8.9) implies

$$
\left\|w-f_{\Omega} w\right\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)} \leq C M\left(\left\|f_{1}-f_{2}\right\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\prime}}+\left\|g_{1}-g_{2}\right\|_{T_{\nu}^{p}\left(\Omega^{c}\right)^{\prime}}\right)
$$

(iv) Clearly, Hölder inequality implies

$$
\left|\int_{\Omega} f(x) v(x) \mathrm{d} x\right| \leq\|f\|_{L^{p^{\prime}}(\Omega)}\|v\|_{L^{p}(\Omega)} \leq\|f\|_{L^{p^{\prime}}(\Omega)}\|v\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}
$$

By Hölder inequality and the continuity of $T_{\nu}^{p}\left(\Omega^{c}\right) \rightarrow L^{p}\left(\Omega^{c}, \omega\right)$ (see Theorem 5.14) we get

$$
\left|\int_{\Omega^{c}} g(y) v(y) \mathrm{d} y\right| \leq\|g\|_{L^{p^{\prime}\left(\Omega^{c}, \omega^{1-p^{\prime}}\right)}}\|v\|_{L^{p}\left(\Omega^{c}, \omega\right)} \leq C_{1}\|g\|_{L^{p^{\prime}}\left(\Omega^{c}, \omega^{1-p^{\prime}}\right)}\|v\|_{T_{\nu}^{p}\left(\Omega^{c}\right)}
$$

The remaining follows since in this case problems $(V)$ and $\left(V^{\prime}\right)$ are identical.
Remark 8.13. A modification of the Neumann data up a multiplicative weight; such as the substitution $g(y)=$ $g_{*}(y) \omega^{\beta}(y), \beta \in \mathbb{R}$ results in another variant of the Neumann problem. Naturally, one retrieves the following configuration:
(i) The Neumann problem $(N)$ becomes

$$
\begin{equation*}
L u=f \quad \text { in } \quad \Omega \quad \text { and } \quad \mathcal{N} u=g_{*} \omega^{\beta} \quad \text { on } \quad \mathbb{R}^{d} \backslash \Omega . \tag{*}
\end{equation*}
$$

(ii) The weak formulation $(V)$ becomes

$$
\begin{equation*}
\mathcal{E}(u, v)=\int_{\Omega} f(x) v(x) \mathrm{d} x+\int_{\Omega^{c}} g_{*}(y) v(y) \omega^{\beta}(y) \mathrm{d} y, \quad \text { for all } v \in W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right) \tag{*}
\end{equation*}
$$

whereas the compatibility condition $(C)$ becomes

$$
\begin{equation*}
\int_{\Omega} f(x) \mathrm{d} x+\int_{\Omega^{c}} g_{*}(y) \omega^{\beta}(y) \mathrm{d} y=0 \tag{*}
\end{equation*}
$$

(iii) Last, if $g_{*} \in L^{p^{\prime}}\left(\Omega^{c}, \omega^{\gamma}\right)$ with $\beta=\frac{1}{p}+\frac{\gamma}{p^{\prime}}$, the linear map $v \mapsto \int_{\Omega^{c}} g_{*}(y) \omega^{\beta}(y) \mathrm{d} y$ belongs to $T_{\nu}^{p}\left(\Omega^{c}\right)^{\prime}$. Some special couples are given by $(\gamma, \beta) \in\left\{\left(1-p^{\prime}, 0\right),(1,1),\left(0, \frac{1}{p}\right)\right\}$.
The next result concerns a non-existence of weak solution when the Neumann data $g$ is not in the weighted nonlocal trace space $L^{p^{\prime}}\left(\Omega^{c}, \omega^{1-p^{\prime}}\right)$. In other words, $L^{p^{\prime}}\left(\Omega^{c}, \omega^{1-p^{\prime}}\right)$ is a sufficiently large function space as the data space for the Neumann problem.
Theorem 8.14 (Non-existence of weak solution). Let $\Omega=B_{1}(0)$ and $\nu(h)=|h|^{-d-s p}$, $s \in(0,1)$ so that $\widetilde{\nu}(h) \asymp$ $(1+|h|)^{-d-s p}$. There exists $g \notin L^{p^{\prime}}\left(\Omega, \widetilde{\nu}^{1-p^{\prime}}\right)$ compatible, i.e. $\int_{\Omega^{c}} g(y) \mathrm{d} y=0$, for which the Neumann problem $L u=0$ on $\Omega$ and $\mathcal{N} u=g$ on $\mathbb{R}^{d} \backslash \Omega$ has no weak solution on $W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$.
Proof. Let us define $g_{\gamma}(x)=\frac{x_{1}}{|x|}(|x|-1)^{\gamma} \mathbb{1}_{B_{1}^{c}(0)}(x), \gamma \in \mathbb{R}$. Note that for $x \in B_{1}^{c}(0)$ we have $\operatorname{dist}(x, \partial \Omega)=$ $\operatorname{dist}\left(x, \mathbb{S}^{d-1}\right)=(|x|-1)$ and

$$
\int_{B_{1}(0)} \frac{\mathrm{d} y}{|x-y|^{d+s p}} \asymp(|x|-1)^{-s p} \wedge(|x|-1)^{-d-s p} .
$$

We have $g_{\gamma} \in W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ if and only if $\gamma \in\left(\frac{s p-1}{p}, s\right)$. Indeed, using to polar coordinates gives

$$
\begin{aligned}
\left\|g_{\gamma}\right\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}^{p} & =2 \int_{B_{1}^{c}(0)} \frac{\left|x_{1}\right|^{p}}{|x|^{p}}(|x|-1)^{p \gamma} \int_{B_{1}(0)}|x-y|^{-d-s p} \mathrm{~d} y \mathrm{~d} x \\
& \asymp 2 \int_{B_{1}^{c}(0)} \frac{\left|x_{1}\right|^{p}}{|x|^{p}}(|x|-1)^{p \gamma-s p}\left(1 \wedge(|x|-1)^{-d}\right) \mathrm{d} x \\
& =2\left|\mathbb{S}^{d-1}\right| K_{d, p}\left(\int_{0}^{1} r^{p \gamma-s p} \mathrm{~d} r+\int_{1}^{\infty} r^{p \gamma-s p-1} \mathrm{~d} r\right) .
\end{aligned}
$$

Analogously, $g_{\gamma+\beta} \in L^{1}\left(\Omega^{c}, \widetilde{\nu}\right)$ if and only if $\gamma+\beta \in(-1, s p)$. Since $g_{p^{\prime} \gamma+0}=g_{\gamma}^{p^{\prime}}$, it follows that $g_{\gamma} \in L^{p^{\prime}}\left(\Omega^{c}, \widetilde{\nu}\right)$ if and only if $\gamma \in\left(-\frac{1}{p^{\prime}}, \frac{s p}{p^{\prime}}\right)$. Indeed,

$$
\begin{aligned}
\left\|g_{\gamma+\beta}\right\|_{L^{1}\left(\Omega^{c}, \tilde{\nu}\right)} & =\int_{B_{1}^{c}(0)} \frac{\left|x_{1}\right|^{2}}{|x|^{2}}(|x|-1)^{\gamma+\beta}(1+|x|)^{-d-s p} \mathrm{~d} x \\
& \asymp\left(\int_{0}^{1} r^{\gamma+\beta} \mathrm{d} r+\int_{1}^{\infty} r^{\gamma+\beta-s p-1} \mathrm{~d} r\right) .
\end{aligned}
$$

Next we put $g=g_{\gamma} \widetilde{\nu}$. Clearly, $g \in L^{1}\left(\Omega^{c}\right) \backslash L^{p^{\prime}}\left(\Omega^{c}, \widetilde{\nu}^{1-p^{\prime}}\right)$ if and only if $g_{\gamma} \in L^{1}\left(\Omega^{c}, \widetilde{\nu}\right) \backslash L^{p^{\prime}}\left(\Omega^{c}, \widetilde{\nu}\right)$, i.e., if and only if $\gamma \in\left(-1,-\frac{1}{p^{\prime}}\right) \cup\left(\frac{s p}{p^{\prime}}, s p\right)$. Moreover, by symmetry of $g=g_{\gamma} \widetilde{\nu}$, satisfies the compatibility condition

$$
\int_{\Omega^{c}} g(y) \mathrm{d} y=\int_{\Omega^{c}} g_{\gamma}(y) \widetilde{\nu}(y) \mathrm{d} y=0 .
$$

Assume the Neumann problem has a weak solution $u$. That is

$$
\mathcal{E}(u, v)=\int_{\Omega^{c}} g_{\gamma}(y) v(y) \widetilde{\nu}(y) \mathrm{d} y \quad \text { for all } \quad v \in W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right) .
$$

It follows that, there is a constant $C>0$ such that

$$
\left|\int_{\Omega^{c}} g_{\gamma}(y) v(y) \widetilde{\nu}(y) \mathrm{d} y\right| \leq C\|v\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)} \quad \text { for all } \quad v \in W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right) .
$$

In particular taking $v=g_{\beta} \in W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ amounts the above estimate to

$$
\left\|g_{\gamma+\beta}\right\|_{L^{1}\left(\Omega^{c}, \widetilde{\nu}\right)}=\left|\int_{\Omega^{c}} g_{\gamma}(y) g_{\beta}(y) \widetilde{\nu}(y) \mathrm{d} y\right| \leq C\left\|g_{\beta}\right\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)} \quad \text { for all } \beta \in\left(\frac{s p-1}{p}, s\right) .
$$

Consider the particular choice $\beta=s p-\gamma$ with $\gamma \in\left(\frac{s p}{p^{\prime}}, \frac{1}{p}+\frac{s p}{p^{\prime}}\right)$ or $\beta=-\gamma-1$ with $\gamma \in\left(-(s+1),-\left(s+\frac{1}{p^{\prime}}\right)\right)$. In both case, $\gamma \in\left(-1,-\frac{1}{p^{\prime}}\right) \cup\left(\frac{s p}{p^{\prime}}, s p\right)$ and $\beta \in\left(\frac{s p-1}{p}, s\right)$ and $\gamma+\beta \in\{-1, s p\}$. In other words, $g_{\gamma} \in L^{1}\left(\Omega^{c}, \widetilde{\nu}\right) \backslash L^{p^{\prime}}\left(\Omega^{c}, \widetilde{\nu}\right)$ and $g_{\beta} \in W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$. Whence $\left\|g_{\gamma+\beta}\right\|_{L^{1}\left(\Omega^{c}, \tilde{\nu}\right)}=\infty$ and $\left\|g_{\beta}\right\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}<\infty$, which contradicts the above inequality.
The well posedness of the Neumann problem for the regional operator $L_{\Omega}$ can be derived analogously.

$$
L_{\Omega} u(x):=2 \text { p.v. } \int_{\Omega} \psi(u(x)-u(y)) \nu(x-y) \mathrm{d} y .
$$

Theorem 8.15. Assume that the Poincaré inequality $(P)$ holds (see page 22). Let $f \in W_{\nu}^{p}(\Omega)^{\prime}$. The following assertions are true.
(i) Existence. There is a unique $u \in W_{\nu}^{p}(\Omega)^{\perp}$ satisfying $\mathcal{E}_{\Omega}(u, v)=\langle f, v\rangle$ for all $v \in W_{\nu}^{p}(\Omega)^{\perp}$. A function $w \in W_{\nu}^{p}(\Omega)$, is weak solution to $L_{\Omega} u=f$ in $\Omega$, i.e., satisfies

$$
\begin{equation*}
\mathcal{E}_{\Omega}(u, v)=\langle f, v\rangle \quad \text { for all } v \in W_{\nu}^{p}(\Omega) \tag{8.12}
\end{equation*}
$$

if and only if $w$ is of the form $w=u+c, c \in \mathbb{R}$ and $\langle f, 1\rangle=0$.
(ii) Boundedness. Any solution $w$ to (8.12) satisfies

$$
\left\|w-f_{\Omega} w\right\|_{W_{\nu}^{p}(\Omega)} \leq C\|f\|_{W_{\nu}^{p}(\Omega)^{\prime}}^{1 /(p-1)}
$$

(iii) Continuity. If $u_{i}=w_{i}-f_{\Omega} w_{i}, i=1,2$ where $w_{i}$ satisfies (8.12) with $f=f_{i}$ then

$$
\left\|u_{1}-u_{2}\right\|_{W_{\nu}^{p}(\Omega)} \leq \begin{cases}C\left\|f_{1}-f_{2}\right\|_{W_{\nu}^{p}(\Omega)^{\prime}}^{1 /(p-1)} & p \geq 2 \\ C\left(\left\|f_{1}\right\|_{W_{\nu}^{p}(\Omega)^{\prime}}+\left\|f_{2}\right\|_{W_{\nu}^{p}(\Omega)^{\prime}}\right)^{\frac{2-p}{p-1}}\left\|f_{1}-f_{2}\right\|_{W_{\nu}^{p}(\Omega)^{\prime}} & 1<p<2\end{cases}
$$

Proof. The proof is analogous to that of Theorem 8.12.
8.3. Dirichlet problem. The Dirichlet problem associated with the data $f: \Omega \rightarrow \mathbb{R}$ and $g: \Omega^{c} \rightarrow \mathbb{R}$ is to find $u: \mathbb{R}^{d} \rightarrow \mathbb{R}$ such that

$$
\begin{equation*}
L u=f \quad \text { in } \quad \Omega \quad \text { and } \quad u=g \quad \text { on } \quad \mathbb{R}^{d} \backslash \Omega . \tag{D}
\end{equation*}
$$

In contrast to the Neumann condition $\left(\mathcal{N} u=g\right.$ on $\left.\mathbb{R}^{d} \backslash \Omega\right)$, the Dirichlet condition $\left(u=g\right.$ on $\left.\mathbb{R}^{d} \backslash \Omega\right)$ does not impose any constraint on $g$. Note however that the evaluation of $g$ on $\mathbb{R}^{d} \backslash \Omega_{\nu}$ with $\Omega_{\nu}=\Omega+\operatorname{supp} \nu$, does not influence the values of $u$ in $\Omega$. This is merely due to the fact that $L u(x)=L_{\Omega_{\nu}} u(x)$ for all $x \in \Omega$; see (8.1). It is therefore enough to prescribe the Dirichlet data only the exterior domain (the nonlocal boundary) $\Omega_{e}=\Omega_{\nu} \backslash \Omega$ where $\Omega_{\nu}=\Omega+\operatorname{supp} \nu$ is the nonlocal hull of $\Omega$. Accordingly, the problem $(D)$ is the same as finding $u: \Omega_{\nu} \rightarrow \mathbb{R}$ such that

$$
L u=f \quad \text { in } \quad \Omega \quad \text { and } \quad u=g_{e} \quad \text { on } \quad \Omega_{e} .
$$

where $g_{e}=\left.g\right|_{\Omega_{e}}$ is the restriction of $g$ on $\Omega_{e}$. Actually, both the problems $(D)$ with $\left(D_{\nu}\right)$ are equivalent. Indeed, if $u$ solves $(D)$ then clearly $u_{\nu}=\left.u\right|_{\Omega_{\nu}}$ solves $\left(D_{\nu}\right)$. Conversely, if $u_{\nu}$ solves $\left(D_{\nu}\right)$ then the function defined $u(x)=\widetilde{u}_{\nu}(x)$ for $x \in \Omega_{\nu}$ and $u(x)=g(x)$ for $x \in \mathbb{R}^{d} \backslash \Omega_{\nu}$ solves $(D)$. From now on, the problem $(D)$ is understood in the sense of $\left(D_{\nu}\right)$. Motivated by the Gauss-Green formula (B.6) we define weak solutions of the Dirichlet problem as follows.

Definition 8.16. Let $f \in W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\prime}$ and $g \in T_{\nu}^{p}\left(\Omega^{c}\right)$. We say that $u \in W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ is a weak solution or the variational solution of the Dirichlet problem ( $D$ ) if

$$
\begin{equation*}
u-g \in W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right) \quad \text { and } \quad \mathcal{E}(u, v)=\langle f, v\rangle \quad \text { for all } v \in W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right) \tag{0}
\end{equation*}
$$

Actually, for any extension $\bar{g} \in W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ of $g$, i.e., $g=\bar{g}$ a.e. on $\Omega^{c}$, solution to the variational problem $\left(V_{0}\right)$ are critical points of the functional

$$
\begin{equation*}
\mathcal{J}_{0}(v)=\frac{1}{p} \mathcal{E}(v, v)-\langle f, v-\bar{g}\rangle, \quad v \in g+W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right) \tag{8.13}
\end{equation*}
$$

It is decisive to keep in mind that the Fréchet derivative of $\mathcal{J}_{0}$ is given as

$$
\left\langle\mathcal{J}_{0}^{\prime}(u), v\right\rangle=\mathcal{E}(u, v)-\langle f, v\rangle \quad \text { for all } u, v \in W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)
$$

Proposition 8.17. Let $\bar{g} \in W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ and extension of $g$, i.e., $g=\bar{g}$ a.e. on $\Omega^{c}$. The variational problem $\left(V_{0}\right)$ is equivalent to the minimization problem

$$
\begin{equation*}
\mathcal{J}_{0}(u)=\min _{v-\bar{g} \in W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)} \mathcal{J}_{0}(v) . \tag{0}
\end{equation*}
$$

Moreover, if $\Omega \subset \mathbb{R}^{d}$ is bounded in one direction or $|\Omega|<\infty$ and $\nu \not \equiv 0$ then any solution to $\left(V_{0}\right)$ or $\left(M_{0}\right)$ is independent on the choice of $\bar{g}$.

Proof. Let $u, v \in \bar{g}+W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ then $v-u \in W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$. Assume $u$ solves $\left(V_{0}\right)$, then using Hölder's and Young's inequalities we get

$$
\langle f, v-u\rangle=\mathcal{E}(u, v-u) \leq \mathcal{E}(u, u)^{1 / p^{\prime}} \mathcal{E}(v, v)^{1 / p}-\mathcal{E}(u, u) \leq \frac{1}{p} \mathcal{E}(v, v)-\frac{1}{p} \mathcal{E}(u, u)
$$

Since $\langle f, v-u\rangle=\langle f, v-\bar{g}\rangle-\langle f, u-\bar{g}\rangle$, it follows that $\mathcal{J}_{0}(u) \leq \mathcal{J}_{0}(v)$ hence $u$ solves $\left(M_{0}\right)$.
Conversely let $u$ satisfies $\left(M_{0}\right)$, i.e., $u-g \in W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ and $\mathcal{J}_{0}(u) \leq \mathcal{J}_{0}(v)$ for $v \in g+W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$. In particular, since $u+t v \in g+W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ for all $v \in W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ and $t \in \mathbb{R}$ it follows that $\mathcal{J}_{0}(u) \leq \mathcal{J}_{0}(u+t v)$. Thus, the mapping $t \mapsto \mathcal{J}_{0}(u+t v)$ is differentiable and has a critical point at $t=0$. It follows that $u$ satisfies $\left(V_{0}\right)$ since

$$
\mathcal{E}(u, v)-\langle f, v\rangle=\left\langle\mathcal{J}_{0}^{\prime}(u), v\right\rangle=\lim _{t \rightarrow 0} \frac{\mathcal{J}_{0}(u+t v)-\mathcal{J}_{0}(u)}{t}=0
$$

Now, assume $g_{i} \in W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right), i=1,2$ are different extensions of $g$, i.e., $g_{1}=g_{2}=g$ a.e. on $\Omega^{c}$. Let $u_{i}$ be the associated solution to ( $V_{0}$ ) (or minimizer of $\mapsto \mathcal{J}_{0}\left(v+g_{i}\right)$ ). In particular $u_{1}=u_{2}=g$ a.e. on $\Omega^{c}$ and $u_{i}-g_{i} \in W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$. Hence testing with $u_{1}-u_{2} \in W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$, by definition of $u_{1}$ and $u_{2}$ we have $\mathcal{E}\left(u_{1}, u_{1}-u_{2}\right)=$ $\mathcal{E}\left(u_{1}, u_{1}-u_{2}\right)\left\langle f, u_{1}-u_{2}\right\rangle$. In virtue of the estimates (8.10) and (8.11) we deduce that $\mathcal{E}\left(u_{1}-u_{2}, u_{1}-u_{2}\right)=0$. According to Theorem 7.2 or Theorem 7.4 the Poincaré-Friedrichs inequality (7.4) or (7.7) holds, i.e.,

$$
\left\|u_{1}-u_{2}\right\|_{L^{p}(\Omega)}^{p} \leq C \mathcal{E}\left(u_{1}-u_{2}, u_{1}-u_{2}\right)=0
$$

Thus, $u_{1}=u_{2}$ a.e. on $\Omega$ and $u_{1}=u_{2}=g$ a.e. on $\Omega^{c}$, that is we get $u_{1}=u_{2}$ a.e. on $\mathbb{R}^{d}$.
We are now in position to state the well-posedness of the problem $\left(V_{0}\right)$.
Theorem 8.18. Assume $\nu \not \equiv 0$, i.e., $|\{\nu>0\}|>0$ and $\Omega \subset \mathbb{R}^{d}$ is bounded in one direction or $|\Omega|<\infty$. Let $f \in W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\prime}$ and $g \in T_{\nu}^{p}\left(\Omega^{c}\right)$. There is $C=C(d, p, \Omega, \nu)>0$ such that:
(i) Existence. The variational problem $\left(V_{0}\right)$ has a unique solution $u \in W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$.
(ii) Boundedness. Moreover, for any $\bar{g} \in W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ such that $\left.\bar{g}\right|_{\Omega^{c}}=g$, u satisfies

$$
\begin{align*}
& \mathcal{E}(u, u) \leq C\left(\|f\|_{W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\prime}}^{p^{\prime}}+\mathcal{E}(\bar{g}, \bar{g})\right)  \tag{8.14}\\
&\|u\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)} \leq C\left(\|f\|_{W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\prime}}^{p^{\prime}}+\|g\|_{T_{\nu}^{p}\left(\Omega^{c}\right)}^{p}\right)^{1 / p} . \tag{8.15}
\end{align*}
$$

(iii) Continuity. Let $u_{i}$ be the solution associated with $f=f_{i}$ and $g=g_{i}, i=1,2$. Let us put $R=$ $\left(D^{\frac{1}{p-1}}+D\right)^{1 / 2}$ with

$$
D=D\left(f_{1}, f_{2}, g_{1}, g_{2}\right)=\sum_{i=1}^{2}\left(\left\|f_{i}\right\|_{W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\prime}}^{p^{\prime}}+\left\|g_{i}\right\|_{T_{\nu}^{p}\left(\Omega^{c}\right)}^{p}\right)^{1 / p^{\prime}}
$$

The following estimates hold true. If $p \geq 2$ we have

$$
\left\|u_{1}-u_{2}\right\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)} \leq C\left(\left\|f_{1}-f_{2}\right\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}^{1 /(p-1)}+\left\|g_{1}-g_{2}\right\|_{T_{\nu}^{p}\left(\Omega^{c}\right)}+D^{1 / p}\left\|g_{1}-g_{2}\right\|_{T_{\nu}^{p}\left(\Omega^{c}\right)}^{1 / p}\right)
$$

If $1<p<2$ we have

$$
\left\|u_{1}-u_{2}\right\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)} \leq C\left(D^{\frac{2-p}{p-1}}\left\|f_{1}-f_{2}\right\|_{W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\prime}}+\left\|g_{1}-g_{2}\right\|_{T_{\nu}^{p}\left(\Omega^{c}\right)}+R\left\|g_{1}-g_{2}\right\|_{T_{\nu}^{p}\left(\Omega^{c}\right)}^{1 / 2}\right) .
$$

Remark 8.19. By definition of the trace space, for $g \in T_{\nu}^{p}\left(\Omega^{c}\right)$ there is $\bar{g} \in W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ such that $g=\bar{g}$ a.e on $\Omega^{c}$. Recall that by Proposition 8.17, any solution $u$ to the Dirichlet problem ( $V_{0}$ ) does not depend on the choice of $\bar{g}$. Furthermore we recall that

$$
\|g\|_{T_{\nu}^{p}\left(\Omega^{c}\right)}=\inf \left\{\|\bar{g}\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}: \bar{g} \in W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right), \bar{g}=g \text { a.e. on } \Omega^{c}\right\}
$$

It is sufficient to assume without lost of generality that all Dirichlet data $g \in W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$.

Proof. We emphasize that throughout the proof, $C>0$ denotes a generic constant only depending on the constant from the Poincaré inequality, $\Lambda$ and $p$.
(i) The functional $v \mapsto \mathcal{J}_{0}(v+g)=\frac{1}{p} \mathcal{E}(v+g, v+g)-\langle f, v\rangle$ is clearly convex and continuous (hence lower semicontinuous) on $W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$. According to Theorem $8.4 \mathcal{J}_{0}$ is weakly lower semicontinuous. On the other
hand, in virtue of Theorem 7.2 or Theorem 7.4 the Poincaré-Friedrichs inequality (7.4) or (7.7) holds. In any case one readily finds a constant $C>0$

$$
C\|v\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}^{p} \leq \mathcal{E}(v, v) \quad \text { for all } v \in W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)
$$

In particular we have

$$
\begin{equation*}
C\|v-g\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}^{p} \leq C \mathcal{E}(v-g, v-g) \quad \text { for all } v \in g+W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right) \tag{8.16}
\end{equation*}
$$

Therefore, since $\mathcal{E}(v, v) \leq 2^{p} \mathcal{E}(v+g, v+g)+2^{p} \mathcal{E}(g, g)$ we have

$$
\mathcal{J}_{0}(v+g) \geq 2^{-p} C\|v\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}^{p}-\mathcal{E}(g, g)-\|f\|_{W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\prime}}\|v\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}
$$

Since $p>1$ it follows that, $\mathcal{J}(v+g) \rightarrow \infty$ as $\|v\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)} \rightarrow \infty$. In fact we have

$$
\frac{\mathcal{J}_{0}(v+g)}{\|v\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}} \rightarrow \infty, \quad \text { as }\|v\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)} \rightarrow \infty, v \in W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)
$$

Since $W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ is always a reflexive Banach space as $p>1$, Theorem 8.5 implies that $v \mapsto \mathcal{J}_{0}(v+g)$ is weakly coercive on $W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$. Hence, by Theorem $8.2 \mathcal{J}_{0}(\cdot+g)$ possesses a minimizer $u_{0} \in W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$, that we have

$$
\mathcal{J}_{0}\left(u_{0}+g\right)=\min _{w \in W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)} \mathcal{J}_{0}(w+g)=\min _{v \in g+W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)} \mathcal{J}_{0}(v)
$$

In other words $u=u_{0}+g$ solves $\left(M_{0}\right)$ and by Proposition $8.17, u$ is also a solution to $\left(V_{0}\right)$. It is wroth emphasizing that $u$ is independent of the choice of the extension $g$; see Proposition 8.17. The uniqueness follows from the strict convexity of $\mathcal{J}_{0}(\cdot+g)$ or merely from the estimates in (iii).
(ii) Since $u$ is a solution to $\left(V_{0}\right)$ we have

$$
\begin{aligned}
\mathcal{E}(u, u) & =\mathcal{E}(u, u-g)+\mathcal{E}(u, g)=\langle f, u-g\rangle+\mathcal{E}(u, g) \\
& \leq\|u-g\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}\|f\|_{W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\prime}}+\mathcal{E}(u, u)^{1 / p^{\prime}} \mathcal{E}(g, g)^{1 / p}
\end{aligned}
$$

Since $u-g \in W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ the coercivity estimate (8.16) yields

$$
\|u-g\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)} \leq C \mathcal{E}(u-g, u-g)^{1 / p} \leq C \mathcal{E}(u, u)^{1 / p}+C \mathcal{E}(g, g)^{1 / p}
$$

Next, for $a, b \in \mathbb{R} \delta>0$, applying the Young's inequality on $a \delta$ and $\frac{b}{\delta}$ implies

$$
\begin{equation*}
|a b| \leq \frac{\delta^{p} a^{p}}{p}+\frac{b^{p^{\prime}}}{p^{\prime} \delta^{p^{\prime}}} \tag{8.17}
\end{equation*}
$$

Accordingly, by exploiting the Young inequality (8.17) we get

$$
\begin{gathered}
\mathcal{E}(u, u)^{1 / p^{\prime}} \mathcal{E}(g, g)^{1 / p} \leq \frac{\delta^{p} \mathcal{E}(u, u)}{p^{\prime}}+\frac{\mathcal{E}(g, g)}{p \delta^{\frac{p^{2}}{p^{\prime}}}} \\
C\|f\|_{W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\prime}} \mathcal{E}(u, u)^{1 / p} \leq \frac{\delta^{p} \mathcal{E}(u, u)}{p}+\frac{C^{p^{\prime}}\|f\|_{W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\prime}}^{p^{\prime}}}{p^{\prime} \delta^{p^{\prime}}} \\
C\|f\|_{W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\prime}} \mathcal{E}(g, g)^{1 / p} \leq \frac{\delta^{p} \mathcal{E}(g, g)}{p}+\frac{C^{p^{\prime}}\|f\|_{W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\prime}}^{p^{\prime}}}{p^{\prime} \delta^{p^{\prime}}}
\end{gathered}
$$

Inserting altogether in the previous estimate we obtain

$$
\begin{aligned}
\mathcal{E}(u, u) & \leq C\|f\|_{W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\prime}} \mathcal{E}(u, u)^{1 / p}+C\|f\|_{W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\prime}} \mathcal{E}(g, g)^{1 / p}+\mathcal{E}(u, u)^{1 / p^{\prime}} \mathcal{E}(g, g)^{1 / p} \\
& \leq \delta^{p} \mathcal{E}(u, u)+\left(\frac{1}{p \delta^{\frac{p^{2}}{p^{\prime}}}}+\frac{\delta^{p}}{p}\right) \mathcal{E}(g, g)+\frac{2 C^{p^{\prime}}}{p^{\prime} \delta^{p^{\prime}}}\|f\|_{W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\prime}}^{p^{\prime}}
\end{aligned}
$$

Taking in particular $\delta^{p}=\frac{1}{2}$ yields the desired estimate (8.14)

$$
\mathcal{E}(u, u) \leq C\left(\|f\|_{W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\prime}}^{p^{\prime}}+\mathcal{E}(g, g)\right)
$$

The coercivity estimate (8.16) implies

$$
\begin{equation*}
\|u\|_{L^{p}(\Omega)} \leq\|g\|_{L^{p}(\Omega)}+C \mathcal{E}(u-g, u-g)^{1 / p} \leq C\|g\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}+C \mathcal{E}(u, u)^{1 / p} \tag{8.18}
\end{equation*}
$$

This together with the penultimate estimate we deduce the desired inequality (8.15)

$$
\|u\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)} \leq C\left(\|f\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}^{p^{\prime}}+\|g\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}^{p}\right)^{1 / p} .
$$

(iii) Put $u=u_{1}-u_{2}, f=f_{1}-f_{2}$ and $g=g_{1}-g_{2}$. We clearly have $u-g \in W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$.

Case $p \geq 2$. By exploiting the estimate (8.16) and Young's inequality (8.17) we find that

$$
\begin{align*}
\|u-g\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}\|f\|_{W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\prime}} & \leq C \mathcal{E}(u-g, u-g)^{1 / p}\|f\|_{W_{, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\prime}} \\
& \leq C\|f\|_{W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\prime}}\left(\mathcal{E}(u, u)^{1 / p}+\mathcal{E}(g, g)^{1 / p}\right)  \tag{8.19}\\
& \leq \frac{\delta^{p} \mathcal{E}(u, u)}{p}+\frac{C}{p^{\prime} \delta^{p^{\prime}}}\|f\|_{W_{\nu, \Omega}}^{p^{\prime}}\left(\Omega \mid \mathbb{R}^{d}\right)^{\prime}
\end{align*}+C D \mathcal{E}(g, g)^{1 / p},
$$

where we also used $\|f\|_{W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\prime}} \leq D$. Analogously as for (8.14), we find that

$$
\begin{equation*}
\left|\mathcal{E}\left(u_{1}, g\right)-\mathcal{E}\left(u_{2}, g\right)\right| \leq\left(\mathcal{E}\left(u_{1}, u_{1}\right)^{1 / p^{\prime}}+\mathcal{E}\left(u_{2}, u_{2}\right)^{1 / p^{\prime}}\right) \mathcal{E}(g, g)^{1 / p} \leq C D \mathcal{E}(g, g)^{1 / p} . \tag{8.20}
\end{equation*}
$$

Using the definition of $u_{i}$ and the estimates (8.10), (8.19) and (8.20) we obtain

$$
\begin{aligned}
A_{p}^{\prime} \mathcal{E}(u, u) & \leq\left|\mathcal{E}\left(u_{1}, u-g\right)-\mathcal{E}\left(u_{2}, u-g\right)+\mathcal{E}\left(u_{1}, g\right)-\mathcal{E}\left(u_{2}, g\right)\right| \\
& =\left|\langle f, u-g\rangle+\mathcal{E}\left(u_{1}, g\right)-\mathcal{E}\left(u_{2}, g\right)\right| \\
& \leq\|u-g\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}\|f\|_{W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\prime}}+\left|\mathcal{E}\left(u_{1}, g\right)-\mathcal{E}\left(u_{2}, g\right)\right| \\
& \leq \frac{\delta^{p} \mathcal{E}(u, u)}{p}+\frac{C}{p^{\prime} \delta^{p^{\prime}}}\|f\|_{W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\prime}}^{p^{\prime}}+C D \mathcal{E}(g, g)^{1 / p} .
\end{aligned}
$$

Accordingly, taking $\delta^{p}=A_{p}^{\prime}$ yields

$$
\mathcal{E}(u, u) \leq C\|f\|_{W_{p, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\prime}}^{p^{\prime}}+C D \mathcal{E}(g, g)^{1 / p} .
$$

Since $u-g \in W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$, combining this with the estimate (8.18) it follows that

$$
\begin{aligned}
\|u\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)} & \leq C\left(\|f\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}^{p^{\prime}}+\|g\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}^{p}+D\|g\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}\right)^{1 / p} . \\
& \leq C\left(\|f\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}^{1 /(-1)}+\|g\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}+D^{1 / p}\|g\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}^{1 /}\right) .
\end{aligned}
$$

Case $1<p<2$. Using the definition of $u_{i}$ and the estimates (8.11) yields

$$
\begin{aligned}
c_{p} \mathcal{E}(u, u) & \leq\left(\mathcal{E}\left(u_{1}, u-g\right)-\mathcal{E}\left(u_{2}, u-g\right)+\mathcal{E}\left(u_{1}, g\right)-\mathcal{E}\left(u_{2}, g\right)\right)^{\frac{p}{2}}\left(\mathcal{E}\left(u_{1}, u_{1}\right)+\mathcal{E}\left(u_{2}, u_{2}\right)\right)^{\frac{2-p}{2}} \\
& =\left(\langle f, u-g\rangle+\mathcal{E}\left(u_{1}, g\right)-\mathcal{E}\left(u_{2}, g\right)\right)^{\frac{p}{2}}\left(\mathcal{E}\left(u_{1}, u_{1}\right)+\mathcal{E}\left(u_{2}, u_{2}\right)\right)^{\frac{2-p}{2}} \\
& \leq\left(\|u-g\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}\|f\|_{W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\prime}}+\left|\mathcal{E}\left(u_{1}, g\right)-\mathcal{E}\left(u_{2}, g\right)\right|\right)^{\frac{p}{2}}\left(\mathcal{E}\left(u_{1}, u_{1}\right)+\mathcal{E}\left(u_{2}, u_{2}\right)\right)^{\frac{2-p}{2}} .
\end{aligned}
$$

By exploiting once more, the estimates in (8.19) and (8.20) one readily arrives at the following

$$
\begin{aligned}
c_{p} \mathcal{E}(u, u) & \left.\leq C D^{\frac{p^{\prime}(2-p)}{2}}\left(\|f\|_{W_{p, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\prime}}\left(\mathcal{E}(u, u)^{1 / p}+\mathcal{E}(g, g)^{1 / p}\right)\right)^{\frac{p}{2}}+D^{p / 2} \mathcal{E}(g, g)^{1 / 2}\right) \\
& \leq C D^{\frac{p(2-p)}{2(p-1)}}\left(\|f\|_{W_{\nu, \Omega}^{p}}^{\frac{p}{2}\left(\Omega \mid \mathbb{R}^{d}\right)^{\prime}}\left(\mathcal{E}(u, u)^{1 / 2}+\mathcal{E}(g, g)^{1 / 2}\right)\right)+D^{p / 2} \mathcal{E}(g, g)^{1 / 2} \\
& \leq \frac{\delta^{2} \mathcal{E}(u, u)}{2}+\frac{C}{2 \delta^{2}}\|f\|_{W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\prime}}^{p} D^{\frac{p(2-p)}{p-1}}+C\left(D^{p^{\prime} / 2}+D^{p / 2}\right) \mathcal{E}(g, g)^{1 / 2},
\end{aligned}
$$

we used $\|f\|_{W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\prime}} \leq D$ and the Young inequality (8.17). Taking $\delta^{2}=c_{p}$ we get

$$
\mathcal{E}(u, u) \leq C D^{\frac{p(2-p)}{p-1}}\|f\|_{W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\prime}}^{p}+C\left(D^{p / 2(p-1)}+D^{p / 2}\right) \mathcal{E}(g, g)^{1 / 2} .
$$

Since $u-g \in W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$, combining this with the estimate (8.18) implies

$$
\|u\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)} \leq C\left(D^{\frac{2-p}{p-1}}\|f\|_{W_{\nu, \Omega}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\prime}}+\|g\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}+\left(D^{\frac{1}{p-1}}+D\right)^{1 / 2}\|g\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}^{1 / 2}\right) .
$$

Theorem 8.20 (Weak comparison principle). Assume that $\Omega \subset \mathbb{R}^{d}$ is bounded in one direction or that $|\Omega|<\infty$ and $\nu \not \equiv 0$. Let $u, v \in W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$. Assume that $u \leq v$ a.e. on $\Omega^{c}$ and $L u \leq L v$ in $\Omega$ in the weak sense, i.e.,

$$
\mathcal{E}(u, w) \leq \mathcal{E}(v, w) \quad \text { for all } w \in W_{\nu, 0}^{p}\left(\Omega \mid \mathbb{R}^{d}\right), w \geq 0
$$

Then we have $u \leq v$ a.e. in $\mathbb{R}^{d}$.
Proof. Recall that for $t \in \mathbb{R}$ we put $\psi(t)=|t|^{p-2} t$ and $t_{+}=\max (t, 0)$ and $t_{-}=\max (-t, 0)$ so that $t=t_{+}-t_{-}$. Note that by Corollary A. 7 we have

$$
(\psi(b)-\psi(a))\left(\left(b_{1}-a_{1}\right)_{+}-\left(b_{2}-a_{2}\right)_{+}\right) \geq \begin{cases}A_{p}^{\prime}\left|\left(b_{1}-a_{1}\right)_{+}-\left(b_{2}-a_{2}\right)_{+}\right|^{p} & p \geq 2 \\ A_{p}^{\prime}\left|\left(b_{1}-a_{1}\right)_{+}-\left(b_{2}-a_{2}\right)_{+}\right|^{2}(|b|+|a|)^{p-2} & 1<p<2\end{cases}
$$

Consider $w=(u-v)_{+}$so that $w=0$ on $\Omega^{c}$ since $u-v \leq 0$ on $\Omega^{c}$. Hence $w \in W_{\nu, 0}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ and $w \geq 0$. Thus taking $b_{1}=u(x), b_{2}=u(y), a_{1}=v(x), a_{2}=v(y)$ and proceeding as for the estimates (8.10) and (8.11) we get

$$
\begin{array}{rlrl}
0 \geq \mathcal{E}(u, w)-\mathcal{E}(v, w) & \geq A_{p}^{\prime} \mathcal{E}(w, w) & p \geq 2 \\
0 \geq(\mathcal{E}(u, w)-\mathcal{E}(v, w))(\mathcal{E}(v, v)+\mathcal{E}(u, u))^{\frac{2-p}{p}} \geq c_{p} \mathcal{E}(w, w)^{\frac{2}{p}} & 1<p<2
\end{array}
$$

In any case we deduce that $\mathcal{E}(w, w)=0$. In view of the Poincaré-Friedrichs inequality (see Theorem 7.2 and Theorem 7.4) we also have $\|w\|_{L^{p}(\Omega)}=0$ and hence $\|w\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}=0$. It follows that $w=(u-v)_{+}=0$ a.e on $\mathbb{R}^{d}$ equivalently $u \leq v$ a.e. on $\mathbb{R}^{d}$.
8.4. Robin boundary condition. In the classical setting for the $p$-Laplace operator, the Robin boundary problem - also known as Fourier boundary problem or third boundary problem -combines the Dirichlet and Neumann boundary problem as follows:

$$
-\Delta_{p} u=f \text { in } \Omega \quad \text { and } \quad \partial_{n, p} u+\beta|u|^{p-2} u=g \text { on } \partial \Omega .
$$

Here $f \in L^{p^{\prime}}(\Omega)$ and $\beta, g: \partial \Omega \rightarrow \mathbb{R}$ are given. Analogously, in the nonlocal set up, the Robin problem for $L$ with data $\beta, g: \Omega^{c} \rightarrow \mathbb{R}$ and $f \in L^{p^{\prime}}(\Omega)$ is to find $u: \mathbb{R}^{d} \rightarrow \mathbb{R}$ such that

$$
\begin{equation*}
L u=f \text { in } \Omega \quad \text { and } \quad \mathcal{N} u+\beta|u|^{p-2} u=g \text { on } \Omega^{c} . \tag{8.21}
\end{equation*}
$$

Note that, for $\beta=0$ one recovers the inhomogeneous Neumann problem. For $\beta \rightarrow \infty$ it leads to the homogeneous Dirichlet problem. Let

$$
Q_{\beta}(u, v)=\mathcal{E}(u, v)+\int_{\Omega^{c}}|u(y)|^{p-2} u(y) v(y) \beta(y) \mathrm{d} y
$$

As for the Neumann problem, we define a weak solution of (8.21) as follows.
Definition 8.21. We say that $u \in W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ is a weak solution (or variational solution) of the Robin problem (8.21) if

$$
Q_{\beta}(u, v)=\langle f, v\rangle+\langle g, v\rangle \text { for all } v \in W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)
$$

In fact, the problem $\left(V_{\beta}\right)$ is equivalent to the minimization problem

$$
\begin{align*}
\mathcal{J}_{\beta}(u) & =\min _{v \in W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)} \mathcal{J}_{\beta}(v) \\
\mathcal{J}_{\beta}(v) & =\frac{1}{p} Q_{\beta}(v, v)-\langle f, v\rangle-\langle g, v\rangle
\end{align*}
$$

Theorem 8.22. Let $\omega \in\left\{\widetilde{\nu}_{\Omega}, \bar{\nu}_{\Omega}, \widehat{\nu}_{R}\right\}$ where $\left|B_{R}(0) \cap \Omega\right|>0$ (see Definition 5.2). Assume that $\nu$ have full support, $\beta \omega^{-1} \in L^{\infty}\left(\Omega^{c}\right), \beta$ is nontrivial, i.e., $\left|\Omega^{c} \cap\{\beta>0\}\right|>0$ and the embedding $W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right) \hookrightarrow L^{p}(\Omega)$ is compact. Assume $f \in W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\prime}\left(\right.$ or $\left.f \in L^{p^{\prime}}(\Omega)\right)$ and $g \in T_{\nu}^{p}\left(\Omega^{c}\right)^{\prime}\left(\right.$ or $\left.g \in L^{p^{\prime}}\left(\Omega^{c}, \omega^{1-p^{\prime}}\right)\right)$. There exists $C=C(d, p, \Omega, \nu, \beta)>0$ such that
(i) Existence. There exists a unique $u \in W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ satisfying $\left(V_{\beta}\right)$.
(ii) Boundedness. Moreover u satisfies

$$
\begin{equation*}
\|u\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)} \leq C\left(\|f\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\prime}}+\|g\|_{T_{\nu}^{p}\left(\Omega^{c}\right)^{\prime}}\right)^{1 /(p-1)} \tag{8.22}
\end{equation*}
$$

(iii) Continuity. If $u_{i}, i=1,2$ satisfies $\left(V_{\beta}\right)$ with $f=f_{i}$ and $g=g_{i}$ then

$$
\left\|u_{1}-u_{2}\right\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)} \leq \begin{cases}C\left(\left\|f_{1}-f_{2}\right\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\prime}}+\left\|g_{1}-g_{2}\right\|_{T_{\nu}^{p}\left(\Omega^{c}\right)^{\prime}}\right)^{1 /(p-1)} & p \geq 2  \tag{8.23}\\ C M\left(\left\|f_{1}-f_{2}\right\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\prime}}+\left\|g_{1}-g_{2}\right\|_{T_{\nu}^{p}\left(\Omega^{c}\right)^{\prime}}\right. & 1<p<2\end{cases}
$$

where we put $M=M\left(f_{1}, f_{2}, g_{1}, g_{2}\right)=\left(\sum_{i=1}^{2}\left\|f_{i}\right\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\prime}}+\left\|g_{i}\right\|_{T_{\nu}^{p}\left(\Omega^{c}\right)^{\prime}}\right)^{\frac{2-p}{p-1}}$.

Proof. First of all we claim that the form $Q_{\beta}(\cdot, \cdot)$ is coercive on $W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$. To prove this, it is sufficient to prove that there exists a constant $C=C(d, p, \Omega, \nu, \beta)>0$ such that

$$
\begin{equation*}
Q_{\beta}(u, u) \geq C\|u\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}^{p} \quad \text { for all } u \in W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right) \tag{8.24}
\end{equation*}
$$

Assume not, then one finds $u_{n} \in W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ preferably $\left\|u_{n}\right\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}=1$ such that

$$
\mathcal{E}\left(u_{n}, u_{n}\right)+\int_{\Omega^{c}}\left|u_{n}(y)\right|^{p} \beta(y) \mathrm{d} y=Q_{\beta}\left(u_{n}, u_{n}\right)<\frac{1}{2^{n}} .
$$

In virtue of the compactness, $\left(u_{n}\right)_{n}$ converges to a subsequence in $L^{p}(\Omega)$ to some $u \in W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$. It turns out that $\|u\|_{L^{p}(\Omega)}=1$, since $\mathcal{E}\left(u_{n}, u_{n}\right) \xrightarrow{n \rightarrow \infty} 0$ and for all $n \geq 1,\left\|u_{n}\right\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}=1$. That $\mathcal{E}\left(u_{n}, u_{n}\right) \xrightarrow{n \rightarrow \infty} 0$ and $\left\|u_{n}-u\right\|_{L^{p}(\Omega)} \xrightarrow{n \rightarrow \infty} 0$ imply that $\left\|u_{n}-u\right\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)} \xrightarrow{n \rightarrow \infty} 0$ with $\mathcal{E}(u, u)=0$. Given that $\nu$ is of full support and $\mathcal{E}(u, u)=0$ it follows that $u$ is constant almost everywhere in $\mathbb{R}^{d}$. On the other hand, since $\beta \omega^{-1} \in L^{p^{\prime}}\left(\Omega^{c}\right)$ and the embedding $W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right) \hookrightarrow L^{p}\left(\Omega^{c}, \omega\right)$ is continuous (see Theorem 5.14), we have

$$
\begin{aligned}
\int_{\Omega^{c}}|u(y)|^{p} \beta(y) \mathrm{d} y & \leq 2^{p-1} \int_{\Omega^{c}}\left|u_{n}(y)\right|^{p} \beta(y) \mathrm{d} y+2^{p-1}\left\|\beta \omega^{-1}\right\|_{\infty} \int_{\Omega^{c}}\left|u_{n}(y)-u(y)\right|^{p} \omega(y) \mathrm{d} y \\
& \leq 2^{p-1} Q_{\beta}\left(u_{n}, u_{n}\right)+C\left\|u_{n}-u\right\|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}^{p} \xrightarrow{n \rightarrow \infty} 0
\end{aligned}
$$

It follows that $u=0$, since $u$ is constant a.e and $\left|\Omega^{c} \cap\{\beta>0\}\right|>0$. This negates the fact that $\|u\|_{L^{p}(\Omega)}=1$ and hence our initial assumption was wrong. The other details follow analogously as for the Neumann problem $\left(V^{\prime}\right)$, by replacing the form $\mathcal{E}(\cdot, \cdot)$ with $Q_{\beta}(\cdot, \cdot)$.

## 9. TRANSITION FROM NONLOCAL TO LOCAL

In this section we introduce and characterize what we name as $p$-Lévy approximation family; which serves as the main tool for the convergence of nonlocal objects to local ones. For instance, we show the convergence from nonlocal to local of energies forms, as well as the pointwise convergence of nonlocal $p$-Lévy operators to the $p$-Laplacian.
9.1. Basics on $p$-Lévy approximation family. We say that a family of radial p-Lévy integrable kernels $\left(\nu_{\varepsilon}\right)_{\varepsilon>0}$, $\nu_{\varepsilon}: \mathbb{R}^{d} \backslash\{0\} \rightarrow[0, \infty)$, is $p$-Lévy approximation family if it satisfies

$$
\begin{equation*}
\int_{\mathbb{R}^{d}}\left(1 \wedge|h|^{p}\right) \nu_{\varepsilon}(h) \mathrm{d} h=1 \quad \text { and for all } \delta>0, \quad \lim _{\varepsilon \rightarrow 0} \int_{|h|>\delta}\left(1 \wedge|h|^{p}\right) \nu_{\varepsilon}(h) \mathrm{d} h=0 . \tag{9.1}
\end{equation*}
$$

Proposition 9.1. Assume $\left(\nu_{\varepsilon}\right)_{\varepsilon}$ satisfies (9.1). For every $\beta \in \mathbb{R}$ and every $R>0$ we have

$$
\lim _{\varepsilon \rightarrow 0} \int_{|h| \leq R}\left(1 \wedge|h|^{p+\beta}\right) \nu_{\varepsilon}(h) \mathrm{d} h=\left\{\begin{array}{ll}
0 & \text { if } \beta>0 \\
1 & \text { if } \beta=0 \\
\infty & \text { if } \beta<0
\end{array} .\right.
$$

Proof. Fix $\delta \in(0, R)$ sufficiently small, by (9.1) we get

$$
\begin{aligned}
& \lim _{\varepsilon \rightarrow 0} \int_{\delta<|h| \leq R}\left(1 \wedge|h|^{p}\right) \nu_{\varepsilon}(h) \mathrm{d} h \leq \lim _{\varepsilon \rightarrow 0} \int_{|h|>\delta}\left(1 \wedge|h|^{p}\right) \nu_{\varepsilon}(h) \mathrm{d} h=0 \\
& \lim _{\varepsilon \rightarrow 0} \int_{|h|<\delta}\left(1 \wedge|h|^{p}\right) \nu_{\varepsilon}(h) \mathrm{d} h=1-\lim _{\varepsilon \rightarrow 0} \int_{|h| \geq \delta}\left(1 \wedge|h|^{p}\right) \nu_{\varepsilon}(h) \mathrm{d} h=1 .
\end{aligned}
$$

Thus if $\beta>0$ then we have

$$
\lim _{\varepsilon \rightarrow 0} \int_{|h| \leq R}\left(1 \wedge|h|^{p+\beta}\right) \nu_{\varepsilon}(h) \mathrm{d} h \leq \lim _{\varepsilon \rightarrow 0}\left(R^{\beta} \int_{\delta<|h| \leq R}\left(1 \wedge|h|^{p}\right) \nu_{\varepsilon}(h) \mathrm{d} h+\delta_{|h| \leq \delta}^{\beta} \int_{\mid \leq \delta}\left(1 \wedge|h|^{p}\right) \nu_{\varepsilon}(h) \mathrm{d} h\right)=\delta^{\beta} .
$$

Analogously, if $\beta<0$ then we have

$$
\lim _{\varepsilon \rightarrow 0} \int_{|h| \leq R}\left(1 \wedge|h|^{p+\beta}\right) \nu_{\varepsilon}(h) \mathrm{d} h \geq \lim _{\varepsilon \rightarrow 0}\left(R^{\beta} \int_{\delta<|h| \leq R}\left(1 \wedge|h|^{p}\right) \nu_{\varepsilon}(h) \mathrm{d} h+\delta_{|h| \leq \delta}^{\beta} \int_{\mid h}\left(1 \wedge|h|^{p}\right) \nu_{\varepsilon}(h) \mathrm{d} h\right)=\delta^{\beta} .
$$

In either case, letting $\delta \rightarrow 0$ provides the claim.
Remark 9.2. Assume the family $\left(\nu_{\varepsilon}\right)_{\varepsilon}$ satisfies (9.1). Note that the relation

$$
\begin{equation*}
\lim _{\varepsilon \rightarrow 0} \int_{|h|>\delta}\left(1 \wedge|h|^{p}\right) \nu_{\varepsilon}(h) \mathrm{d} h=0 \tag{9.2}
\end{equation*}
$$

is often known as the concentration property and is merely equivalent to

$$
\lim _{\varepsilon \rightarrow 0} \int_{|h|>\delta} \nu_{\varepsilon}(h) \mathrm{d} h=0 \quad \text { for all } \quad \delta>0
$$

Indeed, for all $\delta>0$ we have

$$
\int_{|h|>\delta}\left(1 \wedge|h|^{p}\right) \nu_{\varepsilon}(h) \mathrm{d} h \leq \int_{|h|>\delta} \nu_{\varepsilon}(h) \mathrm{d} h \leq\left(1 \wedge \delta^{p}\right)^{-1} \int_{|h|>\delta}\left(1 \wedge|h|^{p}\right) \nu_{\varepsilon}(h) \mathrm{d} h .
$$

Consequently, for all $\delta>0$ we also have

$$
\begin{equation*}
\lim _{\varepsilon \rightarrow 0} \int_{|h| \leq \delta}\left(1 \wedge|h|^{p}\right) \nu_{\varepsilon}(h) \mathrm{d} h=\lim _{\varepsilon \rightarrow 0} \int_{|h| \leq \delta}|h|^{p} \nu_{\varepsilon}(h) \mathrm{d} h=1 . \tag{9.3}
\end{equation*}
$$

Let us mention some prototypical examples of sequences $\left(\nu_{\varepsilon}\right)_{\varepsilon}$ satisfying (9.1) of interest here. For more examples we refer the reader to [Fog23, Fog20, FKV20].

Example 9.3. Assume $\nu: \mathbb{R}^{d} \backslash\{0\} \rightarrow[0, \infty)$ is radial and $p$-Lévy normalized, i.e.,

$$
\int_{\mathbb{R}^{d}}\left(1 \wedge|h|^{p}\right) \nu(h) \mathrm{d} h=1 .
$$

Consider the family $\left(\nu_{\varepsilon}\right)_{\varepsilon}$ defined as the rescaled version of $\nu$ with

$$
\nu_{\varepsilon}(h)= \begin{cases}\varepsilon^{-d-p} \nu(h / \varepsilon) & \text { if }|h| \leq \varepsilon \\ \varepsilon^{-d}|h|^{-p} \nu(h / \varepsilon) & \text { if } \varepsilon<|h| \leq 1 \\ \varepsilon^{-d} \nu(h / \varepsilon) & \text { if }|h|>1\end{cases}
$$

Example 9.4. Consider the sequence $\left(\nu_{\varepsilon}\right)_{\varepsilon}$ of fractional kernels defined by

$$
\nu_{\varepsilon}(h)=a_{\varepsilon, d, p}|h|^{-d-(1-\varepsilon) p} \quad \text { with } \quad a_{\varepsilon, d, p}=\frac{p \varepsilon(1-\varepsilon)}{\left|\mathbb{S}^{d-1}\right|} .
$$

Indeed, passing through polar coordinates yields

$$
\int_{\mathbb{R}^{d}}\left(1 \wedge|h|^{p}\right)|h|^{-d-(1-\varepsilon) p} \mathrm{~d} h=\left|\mathbb{S}^{d-1}\right|\left(\int_{0}^{1} r^{\varepsilon p-1} \mathrm{~d} r+\int_{1}^{\infty} r^{-1-(1-\varepsilon) p} \mathrm{~d} r\right)=a_{\varepsilon, d, p}^{-1}
$$

For $\delta>0$, a similar computation gives

$$
a_{\varepsilon, d, p} \int_{|h| \geq \delta}\left(1 \wedge|h|^{p}\right)|h|^{-d-(1-\varepsilon) p} \mathrm{~d} h \leq p \varepsilon(1-\varepsilon) \int_{\delta}^{\infty} r^{-1-(1-\varepsilon) p} \mathrm{~d} r=\varepsilon \delta^{-(1-\varepsilon) p} \xrightarrow{\varepsilon \rightarrow 0} 0 .
$$

The choice of $\nu_{\varepsilon}(h)=a_{\varepsilon, d, p}|h|^{-d-(1-\varepsilon) p}$ gives rise to a multiple of fractional $p$-Laplace operator, namely we have $L_{\varepsilon} u=\frac{2 a_{\varepsilon, d, p}}{C_{d, 1-\varepsilon, p}}(-\Delta)_{p}^{s} u, s=1-\varepsilon$. We emphasize that $C_{d, p, s}$ is the normalizing constant of $(-\Delta)_{p}^{s}$ and that $\frac{2 a_{\varepsilon, d, p}}{C_{d, 1-\varepsilon, p}} \rightarrow K_{d, p}$ as $\varepsilon \rightarrow 0$ (cf. Section 9.4).
Example 9.5. Let $0<\varepsilon<1$ and $\beta>-d$. Set

$$
\nu_{\varepsilon}(h)=\frac{d+\beta}{\left|\mathbb{S}^{d-1}\right| \varepsilon^{d+\beta}}|h|^{\beta-p} \mathbb{1}_{B_{\varepsilon}}(h)
$$

Some special cases are obtained with $\beta=0, \beta=p$ and $\beta=(1-s) p-d$ for $s \in(0,1)$. For the limiting case $\beta=-d$ consider $0<\varepsilon<\varepsilon_{0}<1$ and put

$$
\nu_{\varepsilon}(h)=\frac{1}{\left|\mathbb{S}^{d-1}\right| \log \left(\varepsilon_{0} / \varepsilon\right)}|h|^{-d-p} \mathbb{1}_{B_{\varepsilon_{0}} \backslash B_{\varepsilon}}(h)
$$

9.2. Characterization of $p$-Lévy approximation family. Now we characterize the class $\left(\nu_{\varepsilon}\right)_{\varepsilon}$ such that for all $u \in W^{1, p}\left(\mathbb{R}^{d}\right)$ we have

$$
\begin{equation*}
\lim _{\varepsilon \rightarrow 0} \iint_{\mathbb{R}^{d} \mathbb{R}^{d}}|u(x)-u(y)|^{p} \nu_{\varepsilon}(x-y) \mathrm{d} y \mathrm{~d} x=K_{d, p} \int_{\mathbb{R}^{d}}|\nabla u(x)|^{p} \mathrm{~d} x \tag{9.4}
\end{equation*}
$$

In fact this is equivalent to say that

$$
\begin{equation*}
\lim _{\varepsilon \rightarrow 0} \int_{\mathbb{R}^{d}}\left(1 \wedge|h|^{p}\right) \nu_{\varepsilon}(h) \mathrm{d} h=1 \quad \text { and for all } \delta>0, \quad \lim _{\varepsilon \rightarrow 0} \int_{|h|>\delta}\left(1 \wedge|h|^{p}\right) \nu_{\varepsilon}(h) \mathrm{d} h=0 \tag{9.5}
\end{equation*}
$$

To be more accurate, we have the following.
Theorem 9.6. Let $\left(\nu_{\varepsilon}\right)_{\varepsilon}$ be a family of radial functions. The following are equivalent.
(i) The family $\left(\nu_{\varepsilon}\right)_{\varepsilon}$ satisfies (9.5).
(ii) For all $u \in W^{1, p}\left(\mathbb{R}^{d}\right)$, the relation (9.4) holds.
(iii) For all $u \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$, the relation (9.4) holds.
(iv) There exists $u \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right) \backslash\{0\}$ such that the relation (9.4) holds for each $u_{\tau}(x)=\tau^{d} u(\tau x), \tau>0$.

Remark 9.7. It is worthwhile noticing that properties in (9.1) and (9.5) are equivalent in the sense that, using the normalizing factor $c_{\varepsilon}^{-1}=\int_{\mathbb{R}^{d}}\left(1 \wedge|h|^{p}\right) \nu_{\varepsilon}(h) \mathrm{d} h$ one readily recovers (9.1) from (9.5) vice-versa. Whence Theorem 9.6 also characterizes families satisfying (9.1).

Proof. Up to replacing $\nu_{\varepsilon}$ with $c_{\varepsilon} \nu_{\varepsilon}$ with $c_{\varepsilon}^{-1}=\int_{\mathbb{R}^{d}}\left(1 \wedge|h|^{p}\right) \nu_{\varepsilon}(h) \mathrm{d} h$, the implication $(i) \Longrightarrow$ (ii) follows from [Fog23] or [Fog20, Theorem 5.23]. We only prove that $(i v) \Longrightarrow(i)$, as the remaining implications are trivial. Note in passing that, since $u \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right) \backslash\{0\}$ we have $\left\|u_{\tau}\right\|_{L^{p}\left(\mathbb{R}^{d}\right)}=\|u\|_{L^{p}\left(\mathbb{R}^{d}\right)} \neq 0$ and $\left\|\nabla u_{\tau}\right\|_{L^{p}\left(\mathbb{R}^{d}\right)}=\tau^{1 / p}\|\nabla u\|_{L^{p}\left(\mathbb{R}^{d}\right)} \neq 0$. By continuity of the shift operator, for every $\eta \in(0,1)$ there is $0<\delta_{\eta}<\eta$ such that $\|\nabla u(\cdot+h)-\nabla u\|_{L^{p}\left(\mathbb{R}^{d}\right)}<\eta$ for all $|h| \leq \delta_{\eta}$. Thus for $\tau>0$ we find that

$$
\left\|\nabla u_{\tau}(\cdot+h)-\nabla u_{\tau}\right\|_{L^{p}\left(\mathbb{R}^{d}\right)}<\tau^{1 / p} \eta \quad \text { for all }|h| \leq \delta_{\eta} / \tau
$$

Minkowski's inequality implies

$$
\begin{aligned}
&\left(\int_{\mathbb{R}^{d}} \int_{B_{\delta_{\eta} / \tau}(0)}\left|\nabla u_{\tau}(x) \cdot h\right|^{p} \nu_{\varepsilon}(h) \mathrm{d} h \mathrm{~d} x\right)^{1 / p} \\
& \leq\left(\int_{\mathbb{R}^{d}} \int_{B_{\delta_{\eta} / \tau}(0)}\left|\int_{0}^{1} \nabla u_{\tau}(x+t h) \cdot h \mathrm{~d} t\right|^{p} \nu_{\varepsilon}(h) \mathrm{d} h \mathrm{~d} x\right)^{1 / p}+\tau^{1 / p} \eta\left(\int_{B_{\delta_{\eta} / \tau}(0)}|h|^{p} \nu_{\varepsilon}(h) \mathrm{d} h\right)^{1 / p}
\end{aligned}
$$

Observe that, since $\nu_{\varepsilon}$ is radial, using through polar coordinates yields,

$$
\begin{equation*}
\int_{B_{\delta}(0)}|\nabla u(x) \cdot h|^{p} \nu_{\varepsilon}(h) \mathrm{d} h=K_{d, p}|\nabla u(x)|^{p} \int_{B_{\delta}(0)}|h|^{p} \nu_{\varepsilon}(h) \mathrm{d} h . \tag{9.6}
\end{equation*}
$$

Accordingly, using the fundamental theorem of calculus and the foregoing yields

$$
\begin{gathered}
\iint_{\mathbb{R}^{d}}\left|u_{\tau}(x)-u_{\tau}(x+h)\right|^{p} \nu_{\varepsilon}(h) \mathrm{d} h \mathrm{~d} x=\int_{\mathbb{R}^{d}} \int_{B_{\delta_{\eta} / \tau}(0)}\left|\int_{0}^{1} \nabla u_{\tau}(x+t h) \cdot h \mathrm{~d} t\right|^{p} \nu_{\varepsilon}(h) \mathrm{d} h \mathrm{~d} x \\
\geq\left[\left(\int_{\mathbb{R}^{d}} \int_{B_{\delta_{\eta} / \tau}(0)}\left|\nabla u_{\tau}(x) \cdot h\right|^{p} \nu_{\varepsilon}(h) \mathrm{d} h \mathrm{~d} x\right)^{1 / p}-\eta\left(\int_{B_{\delta_{\eta} / \tau}(0)}|h|^{p} \nu_{\varepsilon}(h) \mathrm{d} h\right)^{1 / p}\right]^{p} \\
\quad=\tau\left(K_{d, p}^{1 / p}\|\nabla u\|_{L^{p}\left(\mathbb{R}^{d}\right)}-\eta\right)^{p} \int_{B_{\delta_{\eta} / \tau}(0)}|h|^{p} \nu_{\varepsilon}(h) \mathrm{d} h .
\end{gathered}
$$

For the sake of brevity, let us put

$$
\begin{aligned}
\mathcal{E}_{\mathbb{R}^{d}}^{\varepsilon}\left(u_{\tau}, u_{\tau}\right) & =\iint_{\mathbb{R}^{d}}\left|u_{\tau}(x)-u_{\tau}(x+h)\right|^{p} \nu_{\varepsilon}(h) \mathrm{d} h \mathrm{~d} x \\
R_{\tau}(\eta, \varepsilon) & =\iint_{\mathbb{R}^{d}} \int_{B_{\delta_{\eta} / \tau}^{c}(0)}\left|u_{\tau}(x)-u_{\tau}(x+h)\right|^{p} \nu_{\varepsilon}(h) \mathrm{d} h \mathrm{~d} x .
\end{aligned}
$$

Therefore, from the above we find that

$$
\begin{equation*}
\mathcal{E}_{\mathbb{R}^{d}}^{\varepsilon}\left(u_{\tau}, u_{\tau}\right) \geq \tau\left(K_{d, p}^{1 / p}\|\nabla u\|_{L^{p}\left(\mathbb{R}^{d}\right)}-\eta\right)^{p} \int_{B_{\delta_{\eta} / \tau}(0)}|h|^{p} \nu_{\varepsilon}(h) \mathrm{d} h+R_{\tau}(\eta, \varepsilon) \tag{9.7}
\end{equation*}
$$

Using once again the fundamental theorem of calculus and (9.6) we easily get

$$
\begin{equation*}
\mathcal{E}_{\mathbb{R}^{d}}^{\varepsilon}\left(u_{\tau}, u_{\tau}\right) \leq \tau K_{d, p}\|\nabla u\|_{L^{p}\left(\mathbb{R}^{d}\right)}^{p} \int_{B_{\delta_{\eta} / \tau}(0)}|h|^{p} \nu_{\varepsilon}(h) \mathrm{d} h+R_{\tau}(\eta, \varepsilon) . \tag{9.8}
\end{equation*}
$$

Now we consider the following quantities

$$
\begin{gathered}
\rho_{\tau}^{+}=\limsup \limsup _{\eta \rightarrow 0} \int_{\varepsilon \rightarrow 0}|h|^{p} \nu_{\varepsilon}(h) \mathrm{d} h, \quad \text { and } \quad \rho_{\tau}^{-}=\liminf _{\eta \rightarrow 0} \liminf _{\varepsilon \rightarrow 0} \int_{B_{\delta_{\eta} / \tau}(0)}|h|^{p} \nu_{\varepsilon}(h) \mathrm{d} h, \\
R_{\tau}^{+}=\limsup \\
\limsup _{\eta \rightarrow 0} \\
\limsup _{\tau \rightarrow 0}(\eta, \varepsilon) \quad \text { and } \quad R_{\tau}^{-}=\liminf _{\eta \rightarrow 0} \liminf _{\varepsilon \rightarrow 0} R_{\tau}(\eta, \varepsilon) .
\end{gathered}
$$

Passing to the limsup and liminf in (9.7) and (9.8) respectively, we get the following

$$
\begin{aligned}
\tau K_{d, p}\|\nabla u\|_{L^{p}\left(\mathbb{R}^{d}\right)}^{p} & \geq \tau \rho_{\tau}^{+} K_{d, p}\|\nabla u\|_{L^{p}\left(\mathbb{R}^{d}\right)}^{p}+R_{\tau}^{+} \\
\tau K_{d, p}\|\nabla u\|_{L^{p}\left(\mathbb{R}^{d}\right)}^{p} & \leq \tau \rho_{\tau}^{-} K_{d, p}\|\nabla u\|_{L^{p}\left(\mathbb{R}^{d}\right)}^{p}+R_{\tau}^{-} .
\end{aligned}
$$

It follows that $\rho_{\tau}^{+} \leq 1$. To show $\rho_{\tau}^{-} \geq 1$, we observe that analogously to (9.8), for all $\delta>0$ and $\theta>0$ we have

$$
\mathcal{E}_{\mathbb{R}^{d}}^{\varepsilon}\left(u_{\theta}, u_{\theta}\right) \leq \theta K_{d, p}\|\nabla u\|_{L^{p}\left(\mathbb{R}^{d}\right)}^{p} \int_{B_{\delta}(0)}|h|^{p} \nu_{\varepsilon}(h) \mathrm{d} h+2^{p}\left\|u_{\theta}\right\|_{L^{p}\left(\mathbb{R}^{d}\right)}^{p} \int_{|h| \geq \delta} \nu_{\varepsilon}(h) \mathrm{d} h .
$$

Then passing to the liminf like previously also yields that

$$
1 \leq \liminf _{\varepsilon \rightarrow 0} \int_{B_{\delta}(0)}|h|^{p} \nu_{\varepsilon}(h) \mathrm{d} h+\frac{1}{\theta} \frac{2^{p}\|u\|_{L^{p}\left(\mathbb{R}^{d}\right)}^{p}}{K_{d, p}\|\nabla u\|_{L^{p}\left(\mathbb{R}^{d}\right)}^{p}} \liminf _{\varepsilon \rightarrow 0} \int_{|h| \geq \delta} \nu_{\varepsilon}(h) \mathrm{d} h .
$$

Letting $\theta \rightarrow \infty$ implies that

$$
1 \leq \liminf _{\varepsilon \rightarrow 0} \int_{B_{\delta}(0)}|h|^{p} \nu_{\varepsilon}(h) \mathrm{d} h \quad \text { for all } \delta>0
$$

In particular, taking $\delta=\delta_{\eta} / \tau$ we obtain

$$
\rho_{\tau}^{-}=\liminf _{\eta \rightarrow 0} \liminf _{\varepsilon \rightarrow 0} \int_{B_{\delta_{\eta} / \tau}(0)}|h|^{p} \nu_{\varepsilon}(h) \mathrm{d} h \geq 1 .
$$

From the foregoing we get $\rho_{\tau}^{+} \leq 1 \leq \rho_{\tau}^{-}$, that is, we have

$$
\begin{equation*}
\rho_{\tau}^{+}=\rho_{\tau}^{-}=\lim _{\eta \rightarrow 0} \lim _{\varepsilon \rightarrow 0} \int_{B_{\delta_{\eta} / \tau}(0)}|h|^{p} \nu_{\varepsilon}(h) \mathrm{d} h=1 . \tag{9.9}
\end{equation*}
$$

Therefore, we also deduce that

$$
R_{\tau}^{+}=\limsup _{\eta \rightarrow 0} \limsup _{\varepsilon \rightarrow 0} \iint_{\mathbb{R}^{d}} \iint_{B_{\delta_{\eta} / \tau}^{c}(0)}\left|u_{\tau}(x)-u_{\tau}(x+h)\right|^{p} \mathrm{~d} h \mathrm{~d} x=0
$$

For fixed $\delta>0$ and $\eta<\delta \tau$ we have

$$
\iint_{\mathbb{R}^{d}}\left|u_{\tau}(x)-u_{\tau}^{c}(x+h)\right|^{p} \nu_{\varepsilon}(h) \mathrm{d} h \mathrm{~d} x \leq \iint_{\mathbb{R}^{d}} \int_{B_{\delta_{\eta} / \tau}^{c}(0)}\left|u_{\tau}(x)-u_{\tau}(x+h)\right|^{p} \nu_{\varepsilon}(h) \mathrm{d} h \mathrm{~d} x .
$$

This implies that, for all $\delta>0$ we have

$$
\limsup _{\varepsilon \rightarrow 0} \iint_{\mathbb{R}^{d}}\left|u_{B_{\delta}^{c}(0)}(x)-u_{\tau}(x+h)\right|^{p} \nu_{\varepsilon}(h) \mathrm{d} h \mathrm{~d} x=0 .
$$

Hence for fixed $\delta>0$ and $\tau<\frac{\delta}{2}$ so that $\operatorname{supp} u_{\tau} \subset B_{\delta / 2}(0)$. Note that $x \in B_{\delta / 2}(0)$ and $h \in B_{\delta}^{c}(0)$ we have $x+h \in B_{\delta / 2}^{c}(0)$ and hence $u_{\tau}(x+h)=0$. Therefore, it follows that

$$
0=\limsup _{\varepsilon \rightarrow 0} \iint_{\mathbb{R}^{d}}\left|u_{B_{\delta}^{c}(0)}(x)-u_{\tau}(x+h)\right|^{p} \nu_{\varepsilon}(h) \mathrm{d} h \mathrm{~d} x \geq\|u\|_{L^{p}\left(\mathbb{R}^{d}\right)}^{p} \limsup _{\varepsilon \rightarrow 0} \int_{|h| \geq \delta}\left(1 \wedge|h|^{p}\right) \nu_{\varepsilon}(h) \mathrm{d} h .
$$

Therefore we have,

$$
\lim _{\varepsilon \rightarrow 0} \int_{|h| \geq \delta}\left(1 \wedge|h|^{p}\right) \nu_{\varepsilon}(h) \mathrm{d} h=0 \quad \text { for all } \delta>0
$$

Finally, combining this and (9.9), with $\tau=1$, it follows that

$$
\lim _{\varepsilon \rightarrow 0} \int_{\mathbb{R}^{d}}\left(1 \wedge|h|^{p}\right) \nu_{\varepsilon}(h) \mathrm{d} h=\lim _{\eta \rightarrow 0} \lim _{\varepsilon \rightarrow 0} \int_{|h|<\delta_{\eta}}|h|^{p} \nu_{\varepsilon}(h) \mathrm{d} h=1
$$

The proof is now complete.
9.3. Pointwise asymptotic. In this section, we wish to establish the asymptotic of the nonlinear nonlocal operators $L_{\varepsilon} u$ and $\mathcal{N}_{\varepsilon} u$ of as $\varepsilon \rightarrow 0$, with

$$
\begin{aligned}
L_{\varepsilon} u(x) & =2 \text { p.v. } \int_{\mathbb{R}^{d}} \psi(u(x)-u(y)) \nu_{\varepsilon}(x-y) \mathrm{d} y, \quad\left(x \in \mathbb{R}^{d}\right) \\
\mathcal{N}_{\varepsilon} u(x) & =2 \int_{\Omega} \psi(u(x)-u(y)) \nu_{\varepsilon}(x-y) \mathrm{d} y \quad\left(x \in \Omega^{c}\right)
\end{aligned}
$$

Typically we show that $L_{\varepsilon} u$ converges to $K_{d, p} \Delta_{p} u$ pointwise and weakly and $\mathcal{N}_{\varepsilon} u$ converges to $K_{d, p} \partial_{n, p} u=$ $K_{d, p}|\nabla u|^{p-2} \nabla u \cdot n$ in a (sort of) weak sense, where $n$ is normal vector on $\partial \Omega$. In fact, here we extend the linear results [Fog23, Proposition 2.5] and [FK22, Lemma 5.3] which only deal with the particular case $p=2$. See also [Fog20, Lemma 5.75 \& Proposition 2.38] wherein the asymptotic for general nonlocal elliptic operators is treated. To begin with, let us establish the following spherical mean representation for the $p$-Laplacian.

Lemma 9.8 (Spherical mean representation of $\left.\Delta_{p}\right)$. Assume $u \in C^{2}\left(B_{1}(x)\right)$ and $\nabla u(x) \neq 0$ when $1<p<2$ then for $d \geq 2$ we have

$$
f_{\mathbb{S}^{d-1}}|\nabla u(x) \cdot w|^{p-2} D^{2} u(x) w \cdot w \mathrm{~d} \sigma_{d-1}(w)=\frac{K_{d, p}}{p-1} \Delta_{p} u(x)
$$

Proof. Let us put $e(x)=\frac{\nabla u(x)}{|\nabla u(x)|}$ and consider $O: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$ be a rotation, i.e., $O^{t} O=I$ such that $e(x)=O e_{d}$ with $e_{d}=(0,0, \cdots, 1)$. Note that $O z \cdot O y=z \cdot y$. By rotationally invariance of the Lebesgue measure, the change variables $w=O \xi$ yields $\mathrm{d} \sigma_{d-1}(w)=\mathrm{d} \sigma_{d-1}(\xi)$ and

$$
\begin{aligned}
I u(x) & :=f_{\mathbb{S}^{d-1}}|\nabla u(x) \cdot w|^{p-2} D^{2} u(x) w \cdot w \mathrm{~d} \sigma_{d-1}(w) \\
& =|\nabla u(x)|^{p-2} f_{\mathbb{S}^{d-1}}|e(x) \cdot w|^{p-2} D^{2} u(x) w \cdot w \mathrm{~d} \sigma_{d-1}(w) \\
& =|\nabla u(x)|^{p-2} f_{\mathbb{S}^{d-1}}\left|\xi_{d}\right|^{p-2}\left[O^{t} D^{2} u(x) O\right] \xi \cdot \xi \mathrm{d} \sigma_{d-1}(\xi) \\
& =|\nabla u(x)|^{p-2} \sum_{i, j=1}^{d} q_{i j}(x) f_{\mathbb{S}^{d}-1}\left|w_{d}\right|^{p-2} w_{i} w_{j} \mathrm{~d} \sigma_{d-1}(w)
\end{aligned}
$$

Here $Q(x)=\left(q_{i j}(x)\right)_{1 \leq i, j \leq d}$ is the $d \times d$-matrix $Q(x)=O^{t} D^{2} u(x) O$. On the one hand by symmetry, we get

$$
\begin{equation*}
f_{\mathbb{S}^{d}-1}\left|w_{d}\right|^{p-2} w_{i} w_{j} \mathrm{~d} \sigma_{d-1}(w)=0 \quad \text { if } i \neq j \tag{9.10}
\end{equation*}
$$

Recall that (see [Fog23]) for $j=d$ we have the following formula,

$$
K_{d, p}=f_{\mathbb{S}^{d-1}}\left|w_{d}\right|^{p} \mathrm{~d} \sigma_{d-1}(w)=\frac{\Gamma\left(\frac{d}{2}\right) \Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{d+p}{2}\right) \Gamma\left(\frac{1}{2}\right)}
$$

Further, the quantity $f_{\mathbb{S}^{d-1}}\left|w_{d}\right|^{p-2} w_{j}^{2} \mathrm{~d} \sigma_{d-1}(w)$ is oblivious to the choice of $j=1,2, \cdots, d-1$. Indeed, let $O^{\prime}$ : $\mathbb{R}^{d-1} \rightarrow \mathbb{R}^{d-1}$ be a rotation such that $O^{\prime} e_{j}^{\prime}=e_{1}^{\prime}$ where we put $e_{i}=\left(e_{i}^{\prime}, 0\right)$ and $x=\left(x^{\prime}, x_{d}\right)$. Then $x \mapsto O x=$ $\left(O^{\prime} x^{\prime}, x_{d}\right)$ is a $x_{d}$-invariant rotation such that $O e_{j}=e_{1}$. Enforcing the change of variables $\xi=O w, \mathrm{~d} \sigma_{d-1}(w)=$ $\mathrm{d} \sigma_{d-1}(\xi)$ we obtain

$$
f_{\mathbb{S}^{d-1}}\left|w_{d}\right|^{p-2} w_{j}^{2} \mathrm{~d} \sigma_{d-1}(w)=f_{\mathbb{S}^{d-1}}\left|\xi_{d}\right|^{p-2} \xi_{1}^{2} \mathrm{~d} \sigma_{d-1}(\xi)
$$

Since $\left|w^{\prime}\right|^{2}=w_{1}^{2}+w_{2}^{2}+\cdots+w_{d-1}^{2}=1-\left|w_{d}\right|^{2}$, for $w \in \mathbb{S}^{d-1}$, this implies that

$$
f_{\mathbb{S}^{d}-1}\left|w_{d}\right|^{p-2} w_{j}^{2} \mathrm{~d} \sigma_{d-1}(w)=\frac{1}{d-1} f_{\mathbb{S}^{d-1}}\left|w_{d}\right|^{p-2}\left(1-\left|w_{d}\right|^{2}\right) \mathrm{d} \sigma_{d-1}(w)=\frac{1}{d-1}\left(K_{d, p-2}-K_{d, p}\right)
$$

Applying the formula $\Gamma(x+1)=x \Gamma(x)$ one readily obtains

$$
K_{d, p-2}=\frac{\Gamma\left(\frac{d}{2}\right) \Gamma\left(\frac{p-1}{2}\right)}{\Gamma\left(\frac{d+p-2}{2}\right) \Gamma\left(\frac{1}{2}\right)}=\frac{d+p-2}{p-1} \frac{\Gamma\left(\frac{d}{2}\right) \Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{d+p}{2}\right) \Gamma\left(\frac{1}{2}\right)}=\frac{d+p-2}{p-1} K_{d, p}
$$

Inserting this in the previous expression yields that

$$
f_{\mathbb{S}^{d-1}}\left|w_{d}\right|^{p-2} w_{j}^{2} \mathrm{~d} \sigma_{d-1}(w)= \begin{cases}\frac{K_{d, p}}{p-1} & j \neq d  \tag{9.11}\\ K_{d, p} & j=d\end{cases}
$$

Beside this, we observe that

$$
\begin{equation*}
\sum_{j=1}^{d} q_{j j}(x)=\operatorname{Tr}(Q(x))=\operatorname{Tr}\left(O^{t} D^{2} u(x) O\right)=\operatorname{Tr}\left(D^{2} u(x)\right)=\Delta u(x) \tag{9.12}
\end{equation*}
$$

In addition, since $Q(x) e_{d} \cdot e_{d}=D^{2} u(x) O e_{d} \cdot O e_{d}$ and $\frac{\nabla u(x)}{|\nabla u(x)|}=e(x)=O e_{d}$ we obtain

$$
\begin{equation*}
q_{d d}(x)=Q(x) e_{d} \cdot e_{d}=|\nabla u(x)|^{-2} D^{2} u(x) \nabla u(x) \cdot \nabla u(x)=|\nabla u(x)|^{-2} \Delta_{\infty} u(x) \tag{9.13}
\end{equation*}
$$

Combing (9.10), (9.11), (9.12) and (9.13) we find that

$$
\begin{array}{ll}
\sum_{i, j=1}^{d} q_{i j}(x) f_{\mathbb{S}^{d-1}}\left|w_{d}\right|^{p-2} w_{i} w_{j} \mathrm{~d} \sigma_{d-1}(w) & =\sum_{j=1}^{d} q_{j j}(x) f_{\mathbb{S}^{d-1}}\left|w_{d}\right|^{p-2} w_{j}^{2} \mathrm{~d} \sigma_{d-1}(w) \\
=\frac{K_{d, p}}{p-1} \sum_{j=1}^{d-1} q_{j j}(x)+q_{d d}(x) K_{d, p} & =\frac{K_{d, p}}{p-1} \sum_{j=1}^{d} q_{j j}(x)+\frac{p-2}{p-1} q_{d d}(x) K_{d, p} \\
=\frac{K_{d, p}}{p-1}\left(\operatorname{Tr}(Q(x))+(p-2) q_{d d}(x)\right) & \\
=\frac{K_{d, p}}{p-1}\left(\Delta u(x)+(p-2)|\nabla u(x)|^{-2} \Delta_{\infty} u(x)\right)
\end{array}
$$

Finally, inserting this in the foregoing expression yields

$$
\begin{aligned}
I u(x) & =|\nabla u(x)|^{p-2} \sum_{i, j=1}^{d} q_{i j}(x) f_{\mathbb{S}^{d-1}}\left|w_{d}\right|^{p-2} w_{i} w_{j} \mathrm{~d} \sigma_{d-1}(w) \\
& =\frac{K_{d, p}}{p-1}|\nabla u(x)|^{p-2}\left(\Delta u(x)+(p-2)|\nabla u(x)|^{-2} \Delta_{\infty} u(x)\right) \\
& =\frac{K_{d, p}}{p-1} \operatorname{div}\left(|\nabla u(x)|^{p-2} \nabla u(x)\right)=\frac{K_{d, p}}{p-1} \Delta_{p} u(x)
\end{aligned}
$$

It is worth mentioning that the computations yielding the identities (9.12) and (9.13) is essentially adapted from the computations in [IN10, Section 7].

Theorem 9.9. Assume that $u \in L^{\infty}\left(\mathbb{R}^{d}\right) \cap C^{2}\left(B_{\tau}(x)\right)$ for some $\tau>0$ and that $\nabla u(x) \neq 0$ when $1<p<2$. Then we have

$$
\lim _{\varepsilon \rightarrow 0} L_{\varepsilon} u(x)=-K_{d, p} \Delta_{p} u(x)
$$

Proof. First of all, for every $\delta>0$ by boundedness of $u$ we have

$$
\begin{equation*}
2 \int_{|h| \geq \delta}|\psi(u(x+h)-u(x))| \nu_{\varepsilon}(h) \mathrm{d} h \leq 2^{p}\|u\|_{L^{\infty}\left(\mathbb{R}^{d}\right)}^{p-1} \int_{|h| \geq \delta} \nu_{\varepsilon}(h) \mathrm{d} h \xrightarrow{\varepsilon \rightarrow 0} 0 \tag{9.14}
\end{equation*}
$$

Since p.v. $\left.\int_{B_{1}(0)} \psi(\nabla u(x) \cdot h)\right] \nu_{\varepsilon}(h) \mathrm{d} h=0$, the claim reduces to the following

$$
\begin{aligned}
-\lim _{\varepsilon \rightarrow 0} L_{\varepsilon} u(x) & =\lim _{\varepsilon \rightarrow 0} 2 \int_{|h|<\delta} \psi(u(x+h)-u(x)) \nu_{\varepsilon}(h) \mathrm{d} h \\
& =\lim _{\varepsilon \rightarrow 0} 2 \int_{|h|<\delta}\left[\psi\left(\int_{0}^{1} \nabla u(x+t h) \cdot h\right)-\psi(\nabla u(x) \cdot h)\right] \nu_{\varepsilon}(h) \mathrm{d} h \\
& =\lim _{\varepsilon \rightarrow 0} 2 \int_{|h|<\delta} \psi^{\prime}(a)(b-a)+R(a, b) \nu_{\varepsilon}(h) \mathrm{d} h
\end{aligned}
$$

where using the fundamental theorem $u(x+h)-u(x)=\int_{0}^{1} \nabla u(x+t h) \cdot h \mathrm{~d} t$ we put

$$
a=\nabla u(x) \cdot h \quad \text { and } \quad b=\int_{0}^{1} \nabla u(x+t h) \cdot h \mathrm{~d} t .
$$

Furthermore, $\psi^{\prime}(t)=(p-1)|t|^{p-2}$ and the remainder $R(a, b)$ is given by

$$
R(a, b)=\psi(b)-\psi(a)-\psi^{\prime}(a)(b-a)=(b-a) \int_{0}^{1} \psi^{\prime}(a+t(b-a))-\psi^{\prime}(a) \mathrm{d} t
$$

Now, we assume without lost of generality that $\tau=1$, i.e., $u \in C^{2}\left(B_{1}(x)\right)$. Fixing $0<\eta<1$ there is $0<\delta<1$ such that

$$
\begin{equation*}
\left|D^{2} u(x+h)-D^{2} u(x)\right|<\eta, \quad \text { whenever }|h|<\delta \tag{9.15}
\end{equation*}
$$

The fundamental theorem of calculus yields

$$
\begin{align*}
b-a & =\int_{0}^{1} \nabla u(x+t h) \cdot h-\nabla u(x) \cdot h \mathrm{~d} t  \tag{9.16}\\
& =\frac{1}{2}\left[D^{2} u(x) \cdot h\right] \cdot h+\int_{0}^{1} t \int_{0}^{1}\left[D^{2} u(x+s t h) \cdot h-D^{2} u(x) \cdot h\right] \cdot h \mathrm{~d} s \mathrm{~d} t .
\end{align*}
$$

In particular the above implies

$$
|b-a| \leq C_{0}|h|^{2}, \quad C_{0}=\sup _{z \in B_{1}(x)}\left|D^{2} u(z)\right| .
$$

Next we estimate the remainder $R(a, b)$, by distinguishing 3 cases; $1<p<2,2<p<3$ and $p \geq 3$. Observe in passing that $R(a, b)=0$ when $p=2$. To this end, let $h \in B_{\delta}(0)$.
Case: $1<p<2$. In this case, $\nabla u(x) \neq 0$ and $\psi(t)=|t|^{p-2} t$ is $C^{1}$ on $\mathbb{R} \backslash\{0\}$ so that

$$
\lim _{b \rightarrow a} \frac{\psi(b)-\psi(a)-\psi^{\prime}(a)(b-a)}{b-a}=0 .
$$

Set $\sigma=2-p>0$. Since $b \rightarrow a$ as $h \rightarrow 0$, for $\nabla u(x) \cdot h \neq 0$ and $0<\delta<1$ small, we get

$$
|R(a, b)| \leq|b-a| \leq C_{0}|h|^{2}=C_{0}|h|^{p+\sigma} .
$$

Case: $2<p<3$. Since $1<p-1<2$, we deduce from (A.2) that

$$
\left||b|^{p-2}-|a|^{p-2}\right| \leq 2^{4-p}|b-a|^{p-2} .
$$

Set $\sigma=p-2>0$. Recall that $\psi^{\prime}(t)=(p-1)|t|^{p-2}$, for all $|h|<\delta$, we find

$$
\begin{aligned}
|R(a, b)| & \leq|b-a| \int_{0}^{1}\left|\psi^{\prime}(a+t(b-a))-\psi^{\prime}(a)\right| \mathrm{d} t \leq(p-1) 2^{4-p}|b-a|^{p-1} \\
& \leq C_{1}|h|^{2(p-1)}=C_{1}|h|^{p+\sigma}, \quad C_{1}=(p-1) 2^{4-p} C_{0}^{p-1}
\end{aligned}
$$

Case: $p \geq 3$. Since $p-1 \geq 2$, the estimate (A.1) implies

$$
\left||b|^{p-2}-|a|^{p-2}\right| \leq(p-2)|b-a|\left(|b|^{p-3}+|a|^{p-3}\right) \leq C_{3}|b-a||h|^{p-3}
$$

with $C_{3}=2(p-2) \sup _{z \in B_{1}(x)}|\nabla u(z)|^{p-3}$. It follows that

$$
\begin{aligned}
|R(a, b)| & \leq|b-a| \int_{0}^{1}\left|\psi^{\prime}(a+t(b-a))-\psi^{\prime}(a)\right| \mathrm{d} t \\
& \leq C_{3}(p-1)|b-a|^{2}|h|^{p-3} \leq C_{4}|h|^{p+1}, \quad C_{4}=C_{0}^{2} C_{3}(p-1)
\end{aligned}
$$

Altogether, for $\sigma=|p-2|$ and $0<\delta<1$ sufficiently small, we have shown that

$$
|R(a, b)| \leq \begin{cases}C|h|^{p+\sigma} & 0<|p-2|<1, \\ 0 & p=2, \\ C|h|^{p+1} & p \geq 3 .\end{cases}
$$

Note that the case $1<p<2$ is understood in almost everywhere sense, since $\nabla u(x) \neq 0$ so that $\mid B_{\delta}(0) \cap\{\nabla u(x) \cdot z=$ $0\} \mid=0$. Applying Proposition 9.1, in any case we get

$$
\begin{equation*}
\lim _{\varepsilon \rightarrow 0} \int_{|h|<\delta} R(a, b) \nu_{\varepsilon}(h) \mathrm{d} h=0 . \tag{9.17}
\end{equation*}
$$

From the foregoing, combining (9.16) and (9.17) we find that

$$
\begin{aligned}
-\lim _{\varepsilon \rightarrow 0} L_{\varepsilon} u(x) & =\lim _{\varepsilon \rightarrow 0} 2 \int_{|h|<\delta} \psi^{\prime}(a)(b-a) \nu_{\varepsilon}(h) \mathrm{d} h \\
& =\lim _{\varepsilon \rightarrow 0} 2 \int_{|h|<\delta}\left(\psi^{\prime}(\nabla u(x) \cdot h) \int_{0}^{1} \nabla u(x+t h) \cdot h-\nabla u(x) \cdot h \mathrm{~d} t\right) \nu_{\varepsilon}(h) \mathrm{d} h \\
& =\lim _{\varepsilon \rightarrow 0} \int_{|h|<\delta}\left(\psi^{\prime}(\nabla u(x) \cdot h) D^{2} u(x) h \cdot h+\psi^{\prime}(\nabla u(x) \cdot h) R_{1}(x, h) h \cdot h\right) \nu_{\varepsilon}(h) \mathrm{d} h .
\end{aligned}
$$

Here the remainder $R_{1}(x, h)$ is the matrix defined by

$$
R_{1}(x, h) h \cdot h=2 \int_{0}^{1} t \int_{0}^{1}\left[D^{2} u(x+s t h) \cdot h-D^{2} u(x) \cdot h\right] \cdot h \mathrm{~d} s \mathrm{~d} t .
$$

It clearly occurs that $\left.|\nabla u(x) \cdot h|^{p-2} h\left|\leq|\nabla u(x)|^{p-2}\right| h\right|^{p-1}$ with $\nabla u(x) \neq 0$ for $1<p<2$. Hence, for $|h|<\delta$, (9.15) yields

$$
\begin{aligned}
\lim _{\varepsilon \rightarrow 0} 2 \int_{|h|<\delta} \psi^{\prime}(\nabla u(x) \cdot h) R_{1}(x, h) h \cdot h \nu_{\varepsilon}(h) \mathrm{d} h & \leq \eta(p-1)|\nabla u(x)|^{p-2} \lim _{\varepsilon \rightarrow 0} \int_{|h|<\delta}|h|^{p} \nu_{\varepsilon}(h) \mathrm{d} h \\
& \leq \eta(p-1)|\nabla u(x)|^{p-2} \xrightarrow{\eta \rightarrow 0} 0 .
\end{aligned}
$$

Since $\eta>0$ is arbitrarily chosen we deduce that

$$
\begin{equation*}
-\lim _{\varepsilon \rightarrow 0} L_{\varepsilon} u(x)=\lim _{\varepsilon \rightarrow 0} \int_{|h|<\delta} \psi^{\prime}(\nabla u(x) \cdot h) D^{2} u(x) h \cdot h \nu_{\varepsilon}(h) \mathrm{d} h \tag{9.18}
\end{equation*}
$$

The case $d=1$ follows immediately by exploiting (9.18). Now assume $d \geq 2$, since $\nu_{\varepsilon}$ is radial, using the polar coordinates in (9.18) and Proposition 9.1 gives

$$
\begin{aligned}
-\lim _{\varepsilon \rightarrow 0} L_{\varepsilon} u(x) & =\lim _{\varepsilon \rightarrow 0}(p-1) \int_{|h|<\delta}|\nabla u(x) \cdot h|^{p-2} D^{2} u(x) h \cdot h \nu_{\varepsilon}(h) \mathrm{d} h \\
& =\lim _{\varepsilon \rightarrow 0}(p-1) \int_{0}^{\delta} \int_{\mathbb{S}^{d-1}}|\nabla u(x) \cdot w|^{p-2} D^{2} u(x) w \cdot w \mathrm{~d} \sigma_{d-1}(w) r^{d+p-1} \nu_{\varepsilon}(r) \mathrm{d} r \\
& =\frac{(p-1)}{\left|\mathbb{S}^{d-1}\right|} \int_{\mathbb{S}^{d-1}}|\nabla u(x) \cdot w|^{p-2} D^{2} u(x) w \cdot w \mathrm{~d} \sigma_{d-1}(w) \lim _{\varepsilon \rightarrow 0} \int_{|h|<\delta}|h|^{p} \nu_{\varepsilon}(h) \mathrm{d} h \\
& =\frac{(p-1)}{\left|\mathbb{S}^{d-1}\right|} \int_{\mathbb{S}^{d-1}}|\nabla u(x) \cdot w|^{p-2} D^{2} u(x) w \cdot w \mathrm{~d} \sigma_{d-1}(w) .
\end{aligned}
$$

The desired result follows from the spherical representation of $\Delta_{p}$ (see Lemma 9.8), viz.,

$$
\lim _{\varepsilon \rightarrow 0} L_{\varepsilon} u(x)=-\frac{(p-1)}{\left|\mathbb{S}^{d-1}\right|} \int_{\mathbb{S}^{d-1}}|\nabla u(x) \cdot w|^{p-2} D^{2} u(x) w \cdot w \mathrm{~d} \sigma_{d-1}(w)=-K_{d, p} \Delta_{p} u(x) .
$$

Using (9.14), one obtains the following straightforward variants of Theorem 9.9.
Theorem 9.10. Let $\Omega \subset \mathbb{R}^{d}$ be open and let $\delta>0$. Assume that $u \in L^{\infty}(\Omega) \cap C^{2}\left(B_{\tau}(x)\right), x \in \Omega, 0<\tau<$ $\operatorname{dist}(x, \partial \Omega)$ and that $\nabla u(x) \neq 0$ when $1<p<2$. We also have

$$
\lim _{\varepsilon \rightarrow 0} L_{\Omega, \varepsilon} u(x)=\lim _{\varepsilon \rightarrow 0} L_{\Omega, \delta, \varepsilon} u(x)=-K_{d, p} \Delta_{p} u(x)
$$

where $L_{\Omega, \varepsilon}$ and $L_{\Omega, \delta, \varepsilon}$ are respectively the regional and the constrained operator

$$
\begin{aligned}
L_{\Omega, \varepsilon} u(x) & =2 \text { p.v. } \int_{\Omega} \psi(u(x)-u(y)) \nu_{\varepsilon}(x-y) \mathrm{d} y \\
L_{\Omega, \delta, \varepsilon} u(x) & =2 \text { p.v. } \int_{\Omega \cap B(x, \delta)} \psi(u(x)-u(y)) \nu_{\varepsilon}(x-y) \mathrm{d} y .
\end{aligned}
$$

By definition, $\mathcal{N}_{\varepsilon} u(x)=L_{\Omega, \varepsilon} u(x)$ for all $x \in \mathbb{R}^{d} \backslash \bar{\Omega}$ and (9.14) also implies the following.
Theorem 9.11. Let $\Omega \subset \mathbb{R}^{d}$ be open. If $u: \mathbb{R}^{d} \rightarrow \mathbb{R}$ is measurable and $\left.u\right|_{\Omega} \in L^{\infty}(\Omega)$, then

$$
\lim _{\varepsilon \rightarrow 0} \mathcal{N}_{\varepsilon} u(x)=0 \quad \text { for } x \in \mathbb{R}^{d} \backslash \bar{\Omega}
$$

Let us point out some particular cases of Theorem 9.9 already appeared in the literature, viz., [BS22, Theorem 2.8], [DL21, Corollary 6.2] and the variant in [IN10, Section 7].

Corollary 9.12. Under the assumptions of Theorem 9.9 we have

$$
\lim _{s \rightarrow 1} 2 s(1-s) \text { p.v. } \int_{\mathbb{R}^{d}} \frac{\psi(u(x)-u(y))}{|x-y|^{d+s p}} \mathrm{~d} y=-\frac{\left|\mathbb{S}^{d-1}\right|}{p} K_{d, p} \Delta_{p} u(x)
$$

Proof. Apply Theorem 9.9 with $\varepsilon=1-s$ and $\nu_{\varepsilon}(h)=\frac{p \varepsilon(1-\varepsilon)}{\left|\mathbb{S}^{d-1}\right|}|h|^{-d-(1-\varepsilon) p}$.
Corollary 9.13. If $u \in C^{2}\left(B_{\tau}(x)\right), \tau>0$ then we have

$$
\lim _{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^{d+p}} \int_{B_{\varepsilon}(x)}|u(x)-u(y)|^{p-2}(u(x)-u(y)) \mathrm{d} y=-\frac{\left|\mathbb{S}^{d-1}\right|}{d+p} K_{d, p} \Delta_{p} u(x)
$$

Proof. Apply Theorem 9.9 with $\nu_{\varepsilon}(h)=\frac{(d+p) \varepsilon^{-d-p}}{\left|\mathbb{S}^{d-1}\right|} \mathbb{1}_{B_{\varepsilon}(0)}(h)$.
9.4. A normalization constant for the fractional $p$-Laplacian. In light of Corollary 9.12 we define a suitable normalizing constant $C_{d, p, s}$ for the fractional $p$-Laplacian $(-\Delta)_{P}^{s}$ such that $C_{d, 2, s}$ is the normalizing constant of the fractional Laplacian $(-\Delta)^{s}$ and that for appropriate $u \in L^{\infty}\left(\mathbb{R}^{d}\right) \cap C^{2}\left(B_{1}(x)\right)$ we have $(-\Delta)_{p}^{s} u(x) \xrightarrow{s \rightarrow 1}-\Delta_{p} u(x)$, i.e., we have

$$
\begin{equation*}
(-\Delta)_{p}^{s} u(x):=C_{d, p, s} \text { p.v. } \int_{\mathbb{R}^{d}} \frac{\psi(u(x)-u(y))}{|x-y|^{d+s p}} \mathrm{~d} y \xrightarrow{s \rightarrow 1}-\Delta_{p} u(x) \tag{9.19}
\end{equation*}
$$

In view of Corollary 9.12 we find that

$$
\lim _{s \rightarrow 1} 2 s(1-s) B_{p} \text { p.v. } \int_{\mathbb{R}^{d}} \frac{\psi(u(x)-u(y))}{|x-y|^{d+s p}} \mathrm{~d} y=-\Delta_{p} u(x)
$$

where, since $\left|\mathbb{S}^{d-1}\right|=\frac{2 \pi^{d / 2}}{\Gamma(d / 2)}$ and $\Gamma(1 / 2)=\pi^{1 / 2}$, the constant $B_{p}$ is given by

$$
\frac{1}{B_{p}}:=\frac{\left|\mathbb{S}^{d-1}\right|}{p} K_{d, p}=\frac{2 \pi^{d / 2}}{p \Gamma\left(\frac{d}{2}\right)} \frac{\Gamma\left(\frac{d}{2}\right) \Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{d+p}{2}\right) \Gamma\left(\frac{1}{2}\right)}=\frac{2 \pi^{\frac{d-1}{2}} \Gamma\left(\frac{p+1}{2}\right)}{p \Gamma\left(\frac{d+p}{2}\right)}
$$

On the other hand we know from [Fog20, Proposition 2.21] that the normalizing constant of the fractional Laplacian $(-\Delta)^{s}$ is given by

$$
\frac{1}{C_{d, 2, s}}=\frac{\pi^{\frac{d-1}{2}} \Gamma\left(\frac{2 s+1}{2}\right) \Gamma(2(1-s))}{s(1-2 s) \Gamma\left(\frac{d+2 s}{2}\right)} \cos (s \pi)
$$

Alternatively, be aware that a common representation formula of $C_{d, 2, s}$ is as follows

$$
\frac{1}{C_{d, 2, s}}=\frac{\pi^{d / 2} \Gamma(1-s)}{s 2^{2 s} \Gamma\left(\frac{d+2 s}{2}\right)}
$$

Our normalizing constant, mimics the first expression of $\frac{1}{C_{d, 2, s}}$ and $\frac{1}{B_{p}}$. To wit, we define

$$
\frac{1}{C_{d, p, s}}=\frac{\pi^{\frac{d-1}{2}} \Gamma\left(\frac{s p+1}{2}\right) \Gamma(p(1-s))}{s(1-2 s) \Gamma\left(\frac{d+s p}{2}\right)} \cos (s \pi)
$$

Namely, we have

$$
C_{d, p, s}=\frac{s p(1-s)(1-2 s) \Gamma\left(\frac{d+s p}{2}\right)}{\pi^{\frac{d-1}{2}} \Gamma\left(\frac{s p+1}{2}\right) \Gamma(p(1-s)+1) \cos (s \pi)} .
$$

Clearly (9.19), holds true since the asymptotic $s \rightarrow 1$ can be rewritten with help of

$$
\lim _{s \rightarrow 1} \frac{C_{d, p, s}}{2 s(1-s) B_{p}}=1
$$

It is straightforward to verify that our choice of the constant $C_{d, p s}$ guaranties the properties:

- For $p=2, C_{d, 2, s}$ is the unique constant such that, in Fourier variables we have $\widehat{(-\Delta)^{s}} u(\xi)=|\xi|^{2 s} \widehat{u}(\xi)$, $\xi \in \mathbb{R}^{d}$ for all $u \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$.
- For any $u \in L^{\infty}\left(\mathbb{R}^{d}\right) \cap C^{2}\left(B_{1}(x)\right)$ we have $(-\Delta)_{p}^{s} u(x) \xrightarrow{s \rightarrow 1}-\Delta_{p} u(x)$.
- Moreover we have the following asymptotic behaviors

$$
\begin{equation*}
\lim _{s \rightarrow 0} \frac{C_{d, p, s}}{s(1-s)}=\frac{2}{\left|\mathbb{S}^{d-1}\right| \Gamma(p)} \quad \text { and } \quad \lim _{s \rightarrow 1} \frac{C_{d, p, s}}{s(1-s)}=\frac{2 p}{\left|\mathbb{S}^{d-1}\right| K_{d, p}} \tag{9.20}
\end{equation*}
$$

The asymptotic $s \rightarrow 1$, highlighting the factor $K_{d, p}$ is already anticipated in [Fog20, Eq: 2.38] in the case $p=2$. Despite the amusing fact of this asymptotic, it is important for the reader to remember $C_{d, p, s}$ is purely artificial and that only the case $p=2$ naturally appears as the unique normalizing constant for which $\left(\widehat{-\Delta)^{s}} u(\xi)=|\xi|^{2 s} \widehat{u}(\xi)\right.$ for all $u \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$.
9.5. Convergence of forms. We are interested in the asymptotic of the energy forms

$$
\begin{aligned}
& \mathcal{E}_{\Omega}^{\varepsilon}(u, v)=\iint_{\Omega \Omega}|u(y)-u(x)|^{p-2}(u(y)-u(x))(v(y)-v(x)) \nu_{\varepsilon}(x-y) \mathrm{d} y \mathrm{~d} x, \\
& \mathcal{E}^{\varepsilon}(u, v)=\iint_{\left(\Omega^{c} \times \Omega^{c}\right)^{c}}|u(y)-u(x)|^{p-2}(u(y)-u(x))(v(y)-v(x)) \nu_{\varepsilon}(x-y) \mathrm{d} y \mathrm{~d} x, \\
& \mathcal{E}_{+}^{\varepsilon}(u, v)=\iint_{\Omega \mathbb{R}^{d}}|u(y)-u(x)|^{p-2}(u(y)-u(x))(v(x)-v(y)) \nu_{\varepsilon}(x-y) \mathrm{d} y \mathrm{~d} x, \\
& \mathcal{E}_{c r}^{\varepsilon}(u, v)=\iint_{\Omega \Omega^{c}}|u(y)-u(x)|^{p-2}(u(y)-u(x))(v(x)-v(y)) \nu_{\varepsilon}(x-y) \mathrm{d} y \mathrm{~d} x, \\
& \mathcal{E}^{0}(u, v)=\int_{\Omega}|\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla v(x) \mathrm{d} x .
\end{aligned}
$$

Note in passing that $\mathcal{E}^{\varepsilon}(u, v)=\mathcal{E}_{+}^{\varepsilon}(u, v)=\mathcal{E}_{\Omega}^{\varepsilon}(u, v)$ when $\Omega=\mathbb{R}^{d}$. Moreover, by [Fog20, Theorem 5.23], [Fog23, Theorem 1.3] see also the variant in [Bre02, BBM01] we have

$$
\begin{equation*}
\lim _{\varepsilon \rightarrow 0} \mathcal{E}_{\mathbb{R}^{d}}^{\varepsilon}(u, u)=K_{d, p} \int_{\mathbb{R}^{d}}|\nabla u(x)|^{p} \mathrm{~d} x \tag{9.21}
\end{equation*}
$$

Note that since $\frac{C_{d, p, s}}{2 s(1-s)} \rightarrow \frac{p}{\left|\mathbb{S}^{d-1}\right|}$, see the the asymptotic in (9.20), for the particular fractional case $\nu_{\varepsilon}(h)=$ $\frac{C_{d, p, s}}{2}|h|^{-d-s p}, s=1-\varepsilon$ we have

$$
\lim _{s \rightarrow 1} \frac{C_{d, p, s}}{2} \iint_{\mathbb{R}^{d} \mathbb{R}^{d}} \frac{|u(x)-u(y)|^{p}}{|x-y|^{d+s p}} \mathrm{~d} y \mathrm{~d} x=\int_{\mathbb{R}^{d}}|\nabla u(x)|^{p} \mathrm{~d} x
$$

In short, up to a multiplicative factor, we have $|u|_{W^{s, p}\left(\mathbb{R}^{d}\right)} \xrightarrow{s \rightarrow 1}\|\nabla u\|_{L^{p}\left(\mathbb{R}^{d}\right)}$. Next, we need the following result involving the collapse across the boundary.
Lemma 9.14. Assume $\Omega \subset \mathbb{R}^{d}$ satisfies $|\partial \Omega|=0$. For any $u, v \in W^{1, p}\left(\mathbb{R}^{d}\right)$ we have

$$
\lim _{\varepsilon \rightarrow 0} \iint_{\Omega \Omega^{c}} \psi(u(x)-u(y))(v(x)-v(y)) \nu_{\varepsilon}(x-y) \mathrm{d} y \mathrm{~d} x=0
$$

Proof. Consider, $U_{\delta}=\left\{x \in \mathbb{R}^{d}: \operatorname{dist}(x, \Omega)>\delta\right\}, \delta>0$, so that $U_{\delta} \subset \Omega^{c}$. Since $\Omega$ and $U_{\delta}$ are open, by [Fog23, Theorem 3.3] or [Fog20, Theorem 5.16] we find that

$$
\begin{aligned}
& K_{d, p} \int_{\Omega}|\nabla u(x)|^{p} \mathrm{~d} x \leq \liminf _{\varepsilon \rightarrow 0} \mathcal{E}_{\Omega}(u, u) \\
& K_{d, p} \int_{U_{\delta}}|\nabla u(x)|^{p} \mathrm{~d} x \leq \liminf _{\varepsilon \rightarrow 0} \mathcal{E}_{U_{\delta}}(u, u) \leq \liminf _{\varepsilon \rightarrow 0} \mathcal{E}_{\Omega^{c}}(u, u)
\end{aligned}
$$

By convergence dominated theorem we obtain

$$
K_{d, p} \int_{\Omega^{c}}|\nabla u(x)|^{p} \mathrm{~d} x \leq \liminf _{\varepsilon \rightarrow 0} \mathcal{E}_{\Omega^{c}}(u, u)
$$

Accordingly, together with (9.21) since $|\partial \Omega|=0$, we get the desired result as follows

$$
\begin{aligned}
\limsup _{\varepsilon \rightarrow 0} \iint_{\Omega \Omega^{c}}|u(x)-u(y)|^{p} \nu_{\varepsilon}(x-y) \mathrm{d} y \mathrm{~d} x & =\frac{1}{2} \limsup _{\varepsilon \rightarrow 0}\left(\mathcal{E}_{\mathbb{R}^{d}}^{\varepsilon}(u, u)-\mathcal{E}_{\Omega}^{\varepsilon}(u, u)-\mathcal{E}_{\Omega^{c}}^{\varepsilon}(u, u)\right) \\
& \leq \limsup _{\varepsilon \rightarrow 0} \mathcal{E}_{\mathbb{R}^{d}}^{\varepsilon}(u, u)-\liminf _{\varepsilon \rightarrow 0} \mathcal{E}_{\Omega}^{\varepsilon}(u, u)-\liminf _{\varepsilon \rightarrow 0} \mathcal{E}_{\Omega^{c}}^{\varepsilon}(u, u) \\
& \leq K_{d, p}\left(\|\nabla u\|_{L^{p}\left(\mathbb{R}^{d}\right)}^{p}-\|\nabla u\|_{L^{p}(\Omega)}^{p}-\|\nabla u\|_{L^{p}\left(\Omega^{c}\right)}^{p}\right)=0
\end{aligned}
$$

The case $u \neq v$ follows this by applying the Hölder inequality.
The next result combines the ideas of [Fog23, Theorem 1.5] and [FBS20, Lemma 2.8].
Theorem 9.15. Assume that $\Omega \subset \mathbb{R}^{d}$ is open satisfying (i) or (ii),
(i) $\Omega$ is an $W^{1, p}$-extension domain,
(ii) $\partial \Omega=\partial \bar{\Omega}$ and $\mathbb{R}^{d} \backslash \bar{\Omega}$ is an $W^{1, p}$-extension domain.

Then for $u, v \in W^{1, p}\left(\mathbb{R}^{d}\right)$, we have

$$
\begin{align*}
& \lim _{\varepsilon \rightarrow 0} \mathcal{E}_{\Omega}^{\varepsilon}(u, v)=K_{d, p} \mathcal{E}^{0}(u, v)  \tag{9.22}\\
& \lim _{\varepsilon \rightarrow 0} \mathcal{E}^{\varepsilon}(u, v)=K_{d, p} \mathcal{E}^{0}(u, v)  \tag{9.23}\\
& \lim _{\varepsilon \rightarrow 0} \mathcal{E}_{+}^{\varepsilon}(u, v)=K_{d, p} \mathcal{E}^{0}(u, v) \tag{9.24}
\end{align*}
$$

Proof. The elementary inequality $|b|^{p}-|a|^{p}-p|a|^{p-2} a(b-a) \geq 0$ (see Corollary A.5) yields, for $t>0$ and $\sigma \in \mathbb{R}$,

$$
\mathcal{E}_{\Omega}^{\varepsilon}(u+t \sigma v, u+t \sigma v)-\mathcal{E}_{\Omega}^{\varepsilon}(u, u)-p t \mathcal{E}_{\Omega}^{\varepsilon}(u, \sigma v) \geq 0
$$

If $\Omega$ is a $W^{1, p}$-extension domain then $\mathcal{E}_{\Omega}^{\varepsilon}(w, w) \xrightarrow{\varepsilon \rightarrow 0} K_{d, p} \mathcal{E}^{0}(w, w)$ for $w \in W^{1, p}(\Omega)$; see [Fog20, Theorem 5.23], [Fog23, Theorem 1.3] or the variant in [BBM01]. Accordingly, passing to the limsup yields

$$
K_{d, p} \frac{\|\nabla(u+t \sigma v)\|_{L^{p}(\Omega)}^{p}-\|\nabla u\|_{L^{p}(\Omega)}^{p}}{t} \geq p \limsup _{\varepsilon \rightarrow 0} \mathcal{E}_{\Omega}^{\varepsilon}(u, \sigma v)
$$

Letting $t \rightarrow 0$ and taking $\sigma= \pm 1$ yields $\sigma p K_{d, p} \mathcal{E}^{0}(u, v) \geq p \lim \sup _{\varepsilon \rightarrow 0} \sigma \mathcal{E}_{\Omega}^{\varepsilon}(u, v)$ and hence

$$
\lim _{\varepsilon \rightarrow 0} \mathcal{E}_{\Omega}^{\varepsilon}(u, v)=K_{d, p} \mathcal{E}^{0}(u, v)
$$

This remains true with $\Omega$ replaced by $\mathbb{R}^{d}$ or by $\mathbb{R}^{d} \backslash \bar{\Omega}$ when $\mathbb{R}^{d} \backslash \bar{\Omega}$ is a $W^{1, p}$-extension domain. Note hat in both cases we have $|\partial \Omega|=|\partial \bar{\Omega}|=0$, since the boundary of an extension domains is a null set. Thus, all claims follow combing Lemma 9.14 and the situation where $\mathbb{R}^{d}, \Omega$ and/or $\mathbb{R}^{d} \backslash \bar{\Omega}$ is a $W^{1, p}$-extension domain. Indeed, ( $i$ ) follows since $\left(\Omega^{c} \times \Omega^{c}\right)^{c}=\Omega \times \Omega \cup \Omega \times \Omega^{c} \cup \Omega^{c} \times \Omega$ and $\Omega \times \mathbb{R}^{d}=\Omega \times \Omega \cup \Omega \times \Omega^{c}$. The case (ii) follows analogously since $\Omega \times \Omega=\left(\mathbb{R}^{d} \times \mathbb{R}^{d}\right) \backslash\left[\Omega^{c} \times \Omega \cup \Omega \times \Omega^{c} \cup \Omega^{c} \times \Omega^{c}\right]$ and $\left(\Omega^{c} \times \Omega^{c}\right)^{c}=\left(\mathbb{R}^{d} \times \mathbb{R}^{d}\right) \backslash\left(\Omega^{c} \times \Omega^{c}\right)$.

As illustrated in the next result, $\partial \Omega$ need no be regular when $u$ or $v$ vanishes on $\partial \Omega$.
Theorem 9.16. Assume $\Omega \subset \mathbb{R}^{d}$ is any open set. Let $u, v \in W^{1, p}\left(\mathbb{R}^{d}\right)$. If $u \in W_{0}^{1, p}(\Omega)$ or $v \in W_{0}^{1, p}(\Omega)$ then the convergence (9.22), (9.23) and (9.24) hold.

Proof. By density it is sufficient to assume $u \in C_{c}^{\infty}(\Omega)$ or $v \in C_{c}^{\infty}(\Omega)$. In any case $\nabla u(x) \cdot \nabla v(x)=0$ a.e. on $\Omega^{c}$ and $(u(x)-u(y))(v(x)-v(y))=0$, a.e. on $\Omega^{c} \times \Omega^{c}$. The result follows by combining Lemma 9.14 and Theorem 9.15 for $\Omega=\mathbb{R}^{d}$, i.e., $\mathcal{E}_{\mathbb{R}^{d}}^{\varepsilon}(u, v) \xrightarrow{\varepsilon \rightarrow 0} K_{d, p} \mathcal{E}^{0}(u, v)$.

Theorem 9.17 ([Fog20, Theorem 3.37]). Let $\Omega \subset \mathbb{R}^{d}$ is open and $u,\left(u_{\varepsilon}\right)_{\varepsilon} \subset L^{p}(\Omega)$ such that

$$
\sup _{\varepsilon>0}\left(\left\|u_{\varepsilon}\right\|_{L^{p}(\Omega)}^{p}+\iint_{\Omega \Omega}\left|u_{\varepsilon}(x)-u_{\varepsilon}(y)\right|^{p} \nu_{\varepsilon}(x-y) \mathrm{d} y \mathrm{~d} x\right)<\infty
$$

If $\left\|u_{\varepsilon}-u\right\|_{L^{p}(\Omega)} \xrightarrow{\varepsilon \rightarrow 0} 0$ then $u \in W^{1, p}(\Omega)$ and we have

$$
K_{d, p}\|\nabla u\|_{L^{p}(\Omega)}^{p} \leq \liminf _{\varepsilon \rightarrow 0} \iint_{\Omega \Omega}\left|u_{\varepsilon}(x)-u_{\varepsilon}(y)\right|^{p} \nu_{\varepsilon}(x-y) \mathrm{d} y \mathrm{~d} x
$$

A refinement of [Fog20, Theorem $5.35 \& 5.40]$ and also [Pon04a] yields the following result.
Theorem 9.18. Assume $\Omega \subset \mathbb{R}^{d}$ is open. Let $\left(u_{\varepsilon}\right)_{\varepsilon} \subset L^{p}(\Omega)$ such that

$$
\sup _{\varepsilon>0}\left(\left\|u_{\varepsilon}\right\|_{L^{p}(\Omega)}^{p}+\iint_{\Omega \Omega}\left|u_{\varepsilon}(x)-u_{\varepsilon}(y)\right|^{p} \nu_{\varepsilon}(x-y) \mathrm{d} y \mathrm{~d} x\right)<\infty
$$

There exist $u \in W^{1, p}(\Omega)$ and a subsequence $\left(\varepsilon_{n}\right)_{n}$ with $\varepsilon_{n} \rightarrow 0^{+}$as $n \rightarrow \infty$ such that $\left(u_{\varepsilon_{n}}\right)_{n}$ converges to $u$ in $L_{\mathrm{loc}}^{p}(\Omega)$. Moreover, we have

$$
K_{d, p}\|\nabla u\|_{L^{p}(\Omega)}^{p} \leq \liminf _{\varepsilon \rightarrow 0} \iint_{\Omega \Omega}\left|u_{\varepsilon}(x)-u_{\varepsilon}(y)\right|^{p} \nu_{\varepsilon}(x-y) \mathrm{d} y \mathrm{~d} x
$$

In addition, we have following strong convergences.
(i) If $\Omega$ bounded and Lipschitz then we have $\left\|u_{\varepsilon_{n}}-u\right\|_{L^{p}(\Omega)} \xrightarrow{n \rightarrow \infty} 0$,
(ii) If $\Omega=\mathbb{R}^{d}$ then we have $\left\|u_{\varepsilon_{n}}-u\right\|_{L^{p}\left(\Omega^{\prime}\right)} \xrightarrow{n \rightarrow \infty} 0$ whenever $\left|\Omega^{\prime}\right|<\infty$.

## 10. Robust Poincaré inequality

In this section, we establish robust Poincaré type inequalities. The robustness should be understood in the sense that within such inequalities, one is able to recover the corresponding classical Poincaré inequalities.
Theorem 10.1 (Robust Poincaré inequality). Assume $\Omega \subset \mathbb{R}^{d}$ is bounded Lipschitz and connected. Let $G \subset L^{p}(\Omega)$ be a nonempty subset satisfying:

$$
\begin{equation*}
G \text { is closed in } L^{p}(\Omega), 1 \notin G \text { and } G \text { is homogeneous, i.e., } \lambda u \in G \text { if } \lambda \in \mathbb{R}, u \in G \text {. } \tag{10.1}
\end{equation*}
$$

There exist $\varepsilon_{0}=\varepsilon_{0}(d, p, \Omega, G)>0$ and $B=B(d, p, \Omega, G)>0$ such that

$$
\begin{equation*}
\|u\|_{L^{p}(\Omega)}^{p} \leq B \mathcal{E}_{\Omega}^{\varepsilon}(u, u) \quad \text { for all } \varepsilon \in\left(0, \varepsilon_{0}\right) \text { and } u \in G \tag{10.2}
\end{equation*}
$$

Obvious examples of sets $G$ satisfying the condition (10.1) include the sets $G_{1}, G_{2}, G_{3}$ introduced in Section 7, i.e., for $0<\gamma \leq|\Omega|$ and $E \subset \Omega$ measurable such that $|E|>0$,

- $G_{1}=\left\{u \in L^{p}(\Omega):|\{u=0\}| \geq \gamma\right\}$,
- $G_{2}=\left\{u \in L^{p}(\Omega): f_{E} u=0\right\}$,
- $G_{3}=\left\{u \in L^{p}(\Omega): u=0\right.$ a.e on $\left.E\right\}$.

Proof. Assume $\varepsilon_{0}$ and $B$ do not exist. For each $n \geq 1$ taking $\varepsilon_{0}=\frac{1}{2^{n}}$ and $B=2^{n}$ there exist $\varepsilon_{n} \in\left(0, \frac{1}{2^{n}}\right)$ and $u_{n} \in G$ for which (10.2) fails, i.e., $\left\|u_{n}\right\|_{L^{p}(\Omega)}^{p}>2^{n} \mathcal{E}_{\Omega}^{\varepsilon_{n}}\left(u_{n}, u_{n}\right)$. By the homogeneity condition (10.1), we can assume without lost of generality that $u_{n} \in G,\left\|u_{n}\right\|_{L^{p}(\Omega)}=1$ so that $\mathcal{E}_{\Omega}^{\varepsilon_{n}}\left(u_{n}, u_{n}\right) \leq \frac{1}{2^{n}}$. According to Theorem 9.18 there is $u \in W^{1, p}(\Omega)$ and a subsequence still denoted $\left(u_{n}\right)_{n}$ converging to $u$ in $L^{p}(\Omega)$. Moreover, we have

$$
K_{d, p}\|\nabla u\|_{L^{p}(\Omega)}^{p} \leq \liminf _{n \rightarrow \infty} \mathcal{E}_{\Omega}^{\varepsilon_{n}}\left(u_{n}, u_{n}\right)=0
$$

This implies that $\nabla u=0$ almost everywhere on $\Omega$, which is a connected set. Necessarily, $u=c$ is a constant function. We find that $\|u\|_{L^{p}(\Omega)}=1$, hence $u=c \neq 0$ and $u \in G$ since $G$ is closed in $L^{p}(\Omega)$ and $\left\|u_{n}-u\right\|_{L^{p}(\Omega)} \xrightarrow{n \rightarrow \infty} 0$. By homogeneity we have $c^{-1} u=1 \in G$, but by assumption we know that $1 \notin G$. We have reached a contradiction.

Here is a direct consequence of Theorem 10.1; see also [Pon04a, Theorem 1.1].
Corollary 10.2. Under the conditions and notations of Theorem 10.1 we have

$$
\left\|u-f_{\Omega}\right\|_{L^{p}(\Omega)}^{p} \leq B \iint_{\Omega \Omega}|u(x)-u(y)|^{p} \nu_{\varepsilon}(x-y) \mathrm{d} y \mathrm{~d} x \quad \text { for all } \varepsilon \in\left(0, \varepsilon_{0}\right) \text { and } u \in L^{p}(\Omega)
$$

Noting that the constant $B=B(d, p, \Omega, G)$ is independent of $\varepsilon>0$, a noteworthy consequence of Theorem 10.1 is obtained letting $\varepsilon \rightarrow 0$; using Theorem 9.15, [Fog23, Theorem 1.3] or [Fog20, Theorem 5.23], we recover the classical Poincaré type inequality.

Corollary 10.3. Under the conditions and notations of Theorem 10.1 we have

$$
\begin{equation*}
\|u\|_{L^{p}(\Omega)}^{p} \leq B K_{d, p} \int_{\Omega}|\nabla u(x)|^{p} \mathrm{~d} x \quad \text { for every } u \in G . \tag{10.3}
\end{equation*}
$$

Corollary 10.4. Assume $\Omega \subset \mathbb{R}^{d}$ is open, bounded, Lipschitz and connected. Then there exists $C=C(p, d, \Omega)>0$ such that

$$
\begin{equation*}
\left\|u-f_{\Omega} u\right\|_{L^{p}(\Omega)}^{p} \leq C(1-s) \iint_{\Omega \Omega} \frac{|u(x)-u(y)|^{p}}{|x-y|^{d+s p}} d y d x \quad \text { for all } s \in(0,1), u \in L^{p}(\Omega) \tag{10.4}
\end{equation*}
$$

When $\Omega=Q$ is a cube, the robust inequality (10.4) is also proved in [BBM02] and improved in [MS02, HMPV22]. The approaches therein use techniques from harmonic analysis.

Proof. Take, $\nu_{\varepsilon}(h)=\frac{p \varepsilon(1-\varepsilon)}{\left|S^{d-1}\right|}|h|^{-d-(1-\varepsilon) p}$ where we put $\varepsilon=1-s$. By Theorem 10.1 , there exist $s_{0} \in(0,1)$ and $C=B>0$ such that the inequality (10.4) holds for all $s \in\left(s_{0}, 1\right)$ and $u \in L^{p}(\Omega)$. Now, let $R-1=\operatorname{diam}(\Omega)$ be the diameter of $\Omega$ so that $R \geq 1$. For $s \in\left(0, s_{0}\right)$ we have $1-s_{0}<1-s$ and $|x-y|^{-d-s p} \geq R^{-d-s_{0} p}$ for all $x, y \in \Omega$. This, together with Jensen's inequality yield

$$
\begin{aligned}
(1-s) \iint_{\Omega \Omega} \frac{|u(x)-u(y)|^{p}}{|x-y|^{d+s p}} \mathrm{~d} y \mathrm{~d} x & \geq\left(1-s_{0}\right) R^{-d-s_{0} p} \iint_{\Omega \Omega}|u(x)-u(y)|^{p} \mathrm{~d} y d x \\
& \geq\left(1-s_{0}\right) R^{-d-s_{0} p}|\Omega| \int_{\Omega}\left|u(x)-f_{\Omega} u\right|^{p} d x
\end{aligned}
$$

Up to a relabeling, it suffices to take $C=\max \left(B, \frac{R^{d+s_{0} p}}{|\Omega|\left(1-s_{0}\right)}\right)$.
The following variant of Theorem 10.1, encapsulates a sort of double robustness (bi-robustness) in parameters for the fractional type Poincaré inequality

Theorem 10.5 (Double robustness for fractional Poincaré inequality). Under the conditions of Theorem 10.1, there exist $B=B(d, p, \Omega)>0, r_{0}=r_{0}(d, p, \Omega)>0$ and $s_{0}=s_{0}(d, p, \Omega)>0$ such that, for every $r \in\left(0, r_{0}\right)$, $s \in\left(s_{0}, 1\right)$ and $u \in G$ we have

$$
\begin{equation*}
\|u\|_{L^{p}(\Omega)}^{p} \leq B \frac{(1-s)}{r^{p(1-s)}} \iint_{\Omega \Omega} \frac{|u(x)-u(y)|^{p}}{|x-y|^{d+s p}} \mathbb{1}_{B_{r}(0)}(x-y) \mathrm{d} y \mathrm{~d} x . \tag{10.5}
\end{equation*}
$$

Remark 10.6. The double robustness (bi-robustness) in (10.5) is well understood, since letting $s \rightarrow 1$ and/or $r \rightarrow 0$ in (10.5) one recovers the local Poincaré inequality (10.3). Indeed, consider

$$
\begin{equation*}
\rho_{r, s}(h)=p(1-s) r^{-p(1-s)}\left|\mathbb{S}^{d-1}\right|^{-1}|h|^{-d-s p} \mathbb{1}_{B_{r}(0)}(h) . \tag{10.6}
\end{equation*}
$$

Then $\nu_{\varepsilon}(h)=\rho_{\varepsilon, s}(h)$ for fixed $s \in(0,1)$ or $\nu_{\varepsilon}(h)=\rho_{r, 1-\varepsilon}(h)$ for fixed $r>0$ satisfies the condition (9.1). More importantly, $\nu_{\varepsilon}(h)=\rho_{\varepsilon, 1-\varepsilon}(h)$ also satisfies the condition (9.1).

Proof. The proof is analogous to that of Theorem 10.1 with the slight difference that $\nu_{\varepsilon_{n}}(h)$ is replaced by $\rho_{r_{n}, 1-\varepsilon_{n}}$, $r_{n}, \varepsilon_{n} \in\left(0, \frac{1}{2^{n}}\right)$, with $\rho_{r, s}$ given in (10.6). Indeed, just for $\nu_{\varepsilon_{n}}$, it is easy to check that, $\rho_{r_{n}, 1-\varepsilon_{n}}$ is Dirac p-Lévy approximation sequence in the sense of (9.1).
10.1. Robust Poincaré-Friedrichs inequality. The analog robust Poincaré-Friedrichs inequality, is delicate and deserves a different slightly formulation. Here we identify $C_{c}^{\infty}(\Omega)$ as a subspace of $C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$ whose element vanish on $\mathbb{R}^{d} \backslash \Omega$. We also need the following notation of the $\delta$-tubular thickening neighborhood of $\Omega$, with For $\delta \in(0, \infty]$,

$$
\Omega(\delta)=\Omega+B_{\delta}(0)=\left\{x \in \mathbb{R}^{d}: \operatorname{dist}(x, \Omega)<\delta\right\} \text { and by convention } \Omega(\infty)=\mathbb{R}^{d} .
$$

Theorem 10.7 (Robust Poincaré-Friedrichs inequality I). Assume $\Omega \subset \mathbb{R}^{d}$ is bounded. For each $\delta \in(0, \infty]$, there exist $\varepsilon_{0}=\varepsilon_{0}(d, p, \Omega, \delta)>0$ and $B=B(d, p, \Omega, \delta)>0$ such that

$$
\begin{equation*}
\|u\|_{L^{p}(\Omega)}^{p} \leq B \iint_{\Omega(\delta) \Omega(\delta)}|u(x)-u(y)|^{p} \nu_{\varepsilon}(x-y) \mathrm{d} y \mathrm{~d} x \quad \text { for all } \varepsilon \in\left(0, \varepsilon_{0}\right), u \in C_{c}^{\infty}(\Omega) \tag{10.7}
\end{equation*}
$$

Proof. Let $\Omega \subset B_{R^{\prime}}\left(x_{0}\right) \subset B_{R}\left(x_{0}\right) \subset \Omega(\delta)$ for some $x_{0} \in \Omega, 0<R^{\prime}<R<\operatorname{diam}(\Omega)+\delta$. The set $G=\{u \in$ $L^{p}\left(B_{R}\left(x_{0}\right)\right): u=0$ a.e. on $\left.B_{R}\left(x_{0}\right) \backslash B_{R^{\prime}}\left(x_{0}\right)\right\}$ satisfies the condition (10.1). Thus, the relation (10.7) follows from Theorem 10.1 since for $u \in C_{c}^{\infty}(\Omega) \subset G,\|u\|_{L^{p}\left(B_{R}\left(x_{0}\right)\right)}^{p}=\|u\|_{L^{p}(\Omega)}^{p}$ and $\mathcal{E}_{B_{R}\left(x_{0}\right)}^{\varepsilon}(u, u) \leq \mathcal{E}_{\Omega(\delta)}^{\varepsilon}(u, u)$.

Remark 10.8. When $\Omega$ is open bounded, it is highly tempting to think that the analog robust inequality (10.7) holds true for $\delta=0$, namely that

$$
\begin{equation*}
\|u\|_{L^{p}(\Omega)}^{p} \leq B \iint_{\Omega \Omega}|u(x)-u(y)|^{p} \nu_{\varepsilon}(x-y) \mathrm{d} y \mathrm{~d} x \quad \text { for all } u \in C_{c}^{\infty}(\Omega), \quad \varepsilon \in\left(0, \varepsilon_{0}\right) \tag{10.8}
\end{equation*}
$$

This however, not fully satisfactory because for fixed $0<\varepsilon<\varepsilon_{0}$, the kernel $\nu_{\varepsilon}$ is also allowed to be integrable.
Next, we deal with the situation where $\Omega$ is bounded in only one direction.
Theorem 10.9 (Robust Poincaré-Friedrichs inequality II). Assume $\Omega \subset \mathbb{R}^{d}$ is bounded in one direction, say $\Omega \subset H_{R}$ with $H_{R}=\left\{x \in \mathbb{R}^{d}:|x \cdot e| \leq R\right\}$ with $R>0$ and $|e|=1$. There exist $\varepsilon_{0}=\varepsilon_{0}(d, p, \Omega)>0$ and $B=B(d, p, \Omega)$ such that

$$
\begin{equation*}
\|u\|_{L^{p}(\Omega)}^{p} \leq B \iint_{\mathbb{R}^{d} \mathbb{R}^{d}}|u(x)-u(y)|^{p} \nu_{\varepsilon}(x-y) \mathrm{d} y \mathrm{~d} x \quad \text { for all } \varepsilon \in\left(0, \varepsilon_{0}\right) \text { and } u \in C_{c}^{\infty}(\Omega) \tag{10.9}
\end{equation*}
$$

Proof. First proof. Assume $\varepsilon_{0}$ and $B$ do not exist. For $\varepsilon_{0}=\frac{1}{2^{n}}$ and $B=2^{n}, n \geq 1$, there exist $\varepsilon_{n} \in\left(0, \frac{1}{2^{n}}\right)$ and $u_{n} \in C_{c}^{\infty}(\Omega)$ for which (10.2) fails, i.e., $\left\|u_{n}\right\|_{L^{p}(\Omega)}^{p}>2^{n} \mathcal{E}_{\mathbb{R}^{d}}^{\varepsilon_{n}}\left(u_{n}, u_{n}\right)$. We can assume without lost of generality that $\left\|u_{n}\right\|_{L^{p}(\Omega)}=1$ so that $\mathcal{E}_{\mathbb{R}^{d}}^{\varepsilon_{n}}\left(u_{n}, u_{n}\right) \leq \frac{1}{2^{n}}$.
Since $u_{n} \in C_{c}^{\infty}(\Omega)$, we find $x_{n} \in \Omega$ and $r_{n}>0$ such that $\operatorname{supp} u_{n} \subset \overline{B_{2 r_{n}}\left(x_{n}\right)} \subset \Omega$. Next, we have $z_{n}=x_{n}+2 r_{n} e \in$ $\overline{B_{2 r_{n}}\left(x_{n}\right)} \subset \Omega \subset H_{R}$. We find that

$$
R \geq z_{n} \cdot e=2 r_{n}+x_{n} \cdot e \geq 2 r_{n}-R \quad \text { hence }, \quad r_{n} \leq R
$$

That is, we have supp $u_{n} \subset \overline{B_{R}\left(x_{n}\right)}$. Set $\tau_{x_{n}} u_{n}(x):=u_{n}\left(x+x_{n}\right)$ then $\tau_{x_{n}} u_{n} \in C_{c}^{\infty}\left(B_{2 R}(0)\right)$ and $\tau_{x_{n}} u_{n}=0$ on $B_{2 R}(0) \backslash B_{R}(0)$. Now, since $u_{n} \in C_{c}^{\infty}(\Omega)$ we have

$$
\begin{gathered}
\left\|\tau_{x_{n}} u_{n}\right\|_{L^{p}\left(B_{2 R}(0)\right)}=\left\|u_{n}\right\|_{L^{p}\left(B_{2 r_{n}}\left(x_{n}\right)\right)}=\left\|u_{n}\right\|_{L^{p}(\Omega)}=1 \\
\mathcal{E}_{B_{2 R}(0)}^{\varepsilon_{n}}\left(\tau_{x_{n}} u_{n}, \tau_{x_{n}} u_{n}\right) \leq \mathcal{E}_{\mathbb{R}^{d}}^{\varepsilon_{n}}\left(\tau_{x_{n}} u_{n}, \tau_{x_{n}} u_{n}\right)=\mathcal{E}_{\mathbb{R}^{d}}^{\varepsilon_{n}}\left(u_{n}, u_{n}\right) \leq \frac{1}{2^{n}}
\end{gathered}
$$

According to Theorem 9.18 there is $u \in W^{1, p}\left(B_{2 R}(0)\right)$ and a subsequence still denoted $\left(\tau_{x_{n}} u_{n}\right)_{n}$ converging to $u$ in $L^{p}\left(B_{2 R}(0)\right)$. Moreover, we have

$$
K_{d, p}\|\nabla u\|_{L^{p}\left(B_{2 R}(0)\right)}^{p} \leq \liminf _{n \rightarrow \infty} \mathcal{E}_{B_{2 R}(0)}^{\varepsilon_{n}}\left(u_{n}, u_{n}\right)=0
$$

This implies that $\nabla u=0$ almost everywhere on $B_{2 R}(0)$, which is a connected set. Necessarily, $u=c$ is a constant function on $B_{R}(0)$. However, since $\tau_{x_{n}} u_{n}=0$ on $B_{2 R}(0) \backslash B_{R}(0)$ we deduce, via the convergence $\| \tau_{x_{n}} u_{n}-$ $u \|_{L^{p}\left(B_{2 R}\right)} \xrightarrow{n \rightarrow \infty} 0$, that $u=c=0$ and $\|u\|_{L^{p}\left(B_{2 R}(0)\right)}=1$ since $\left\|\tau_{x_{n}} u_{n}\right\|_{L^{p}\left(B_{2 R}(0)\right)}=1$. We have reached a contradiction.
Second proof. Claim. First we prove the following claim.

$$
\begin{equation*}
\text { For each } u \in C_{c}^{\infty}(\Omega) \text { there exists } x_{0} \in \Omega \text { such that supp } u \subset B_{R}\left(x_{0}\right) \tag{10.10}
\end{equation*}
$$

Indeed, if $u \in C_{c}^{\infty}(\Omega)$, we can find $x_{0}=x_{0}(u) \in \Omega$ and $r=r(u)>0$ so that $\operatorname{supp} u \subset B_{R}\left(x_{0}\right)$ and $\overline{B_{2 r}\left(x_{0}\right)} \subset \Omega$. Clearly, we have $z:=x_{0}+2 r e \in \overline{B_{2 r}\left(x_{0}\right)} \subset \Omega \subset H_{R}$. We find that

$$
R \geq z \cdot e=2 r+x_{0} \cdot e \geq 2 r-R \quad \text { hence }, \quad r \leq R
$$

It follows that $\operatorname{supp} u \subset B_{R}\left(x_{0}\right)$. Hence $\operatorname{supp} \tau_{x_{0}} u \in C_{c}^{\infty}\left(B_{R}(0)\right)$ with $\tau_{x_{0}} u(x):=u\left(x+x_{0}\right)$.

Now, the set $G=\left\{u \in L^{p}\left(B_{2 R}(0)\right): u=0\right.$ a.e. on $\left.B_{2 R}(0) \backslash B_{R}(0)\right\}$ is a closed subset of $L^{p}\left(B_{2 R}(0)\right)$ such that $1 \notin G$ and $G$ is homogeneous, i.e., $\lambda u \in G$ whenever $u \in G$ and $\lambda \in \mathbb{R}$. Accordingly, by Theorem 10.1 we find $\varepsilon_{0}=\varepsilon_{0}(d, p, R)$ and $B=B(d, p, R)$ so that

$$
\|v\|_{L^{p}\left(B_{2 R}(0)\right)}^{p} \leq B \mathcal{E}_{B_{2 R}(0)}^{\varepsilon_{n}}(v, v) \quad \text { for all } v \in G, \varepsilon \in\left(0, \varepsilon_{0}\right)
$$

In particular, taking $v=\tau_{x_{0}} u \in C_{c}^{\infty}\left(B_{R}(0)\right) \subset G$, the result follows since

$$
\begin{aligned}
\left\|\tau_{x_{0}} u\right\|_{L^{p}\left(B_{2 R}(0)\right)} & =\|u\|_{L^{p}\left(\mathbb{R}^{d}\right)}=\|u\|_{L^{p}(\Omega)} \\
\mathcal{E}_{B_{2 R}(0)}^{\varepsilon_{n}}\left(\tau_{x_{0}} u, \tau_{x_{0}} u\right) & \leq \mathcal{E}_{\mathbb{R}^{d}}^{\varepsilon_{n}}\left(\tau_{x_{0}} u, \tau_{x_{0}} u\right)=\mathcal{E}_{\mathbb{R}^{d}}^{\varepsilon_{n}}(u, u)
\end{aligned}
$$

Theorem 10.10 (Robust Poincaré-Friedrichs inequality III). Assume $\Omega \subset \mathbb{R}^{d}$ has finite measure, i.e., $|\Omega|<\infty$. There exist $\varepsilon_{0}=\varepsilon_{0}(d, p, \Omega)>0$ and $B=B(d, p, \Omega)$ such that

$$
\begin{equation*}
\|u\|_{L^{p}(\Omega)}^{p} \leq B \iint_{\mathbb{R}^{d} \mathbb{R}^{d}}|u(x)-u(y)|^{p} \nu_{\varepsilon}(x-y) \mathrm{d} y \mathrm{~d} x \quad \text { for all } \varepsilon \in\left(0, \varepsilon_{0}\right) \text { and } u \in C_{c}^{\infty}(\Omega) \tag{10.11}
\end{equation*}
$$

Proof. Assume $\varepsilon_{0}$ and $B$ do not exist. For $\varepsilon_{0}=\frac{1}{2^{n}}$ and $B=2^{n}, n \geq 1$, there exist $\varepsilon_{n} \in\left(0, \frac{1}{2^{n}}\right)$ and $u_{n} \in C_{c}^{\infty}(\Omega)$ for which (10.2) fails, i.e., $\left\|u_{n}\right\|_{L^{p}(\Omega)}^{p}>2^{n} \mathcal{E}_{\mathbb{R}^{d}}^{\varepsilon_{n}}\left(u_{n}, u_{n}\right)$. We can assume without lost of generality that $\left\|u_{n}\right\|_{L^{p}\left(\mathbb{R}^{d}\right)}=$ $\left\|u_{n}\right\|_{L^{p}(\Omega)}=1$ so that $\mathcal{E}_{\mathbb{R}^{d}}^{\varepsilon_{n}}\left(u_{n}, u_{n}\right) \leq \frac{1}{2^{n}}$.
Consider $\Omega^{\prime}=\Omega \cup B$ where $B$ is, for instance, an arbitrary nonempty ball such that $B \subset \mathbb{R}^{d} \backslash \Omega$. Observe that $\left|\Omega^{\prime}\right|<\infty$, and thus by Theorem 9.18 we find $u \in W^{1, p}\left(\mathbb{R}^{d}\right)$ and a subsequence still denoted $\left(u_{n}\right)_{n}$ strongly converging to $u$ in $L^{p}\left(\Omega^{\prime}\right)$ and hence $\|u\|_{L^{p}\left(\Omega^{\prime}\right)}=1$ since $\left\|u_{n}\right\|_{L^{p}\left(\Omega^{\prime}\right)}=\left\|u_{n}\right\|_{L^{p}(\Omega)}=1$. Moreover, we have

$$
K_{d, p}\|\nabla u\|_{L^{p}\left(\mathbb{R}^{d}\right)}^{p} \leq \liminf _{n \rightarrow \infty} \mathcal{E}_{\mathbb{R}^{d}}^{\varepsilon_{n}}\left(u_{n}, u_{n}\right)=0
$$

This implies that $u=c$ is constant a.e. on $\mathbb{R}^{d}$. Since $u_{n}=0$ on $B$ as $u_{n} \in C_{c}^{\infty}(\Omega)$, we find $\|u\|_{L^{p}(B)} \leq$ $\left\|u_{n}-u\right\|_{L^{p}\left(\Omega^{\prime}\right)} \xrightarrow{n \rightarrow \infty} 0$. Wherefrom, we deduce that $u=c=0$ a.e. on $\mathbb{R}^{d}$. This contradict $\|u\|_{L^{p}(\Omega)}=1$.

By letting $\varepsilon \rightarrow 0$ in Theorem 10.7, Theorem 10.10 and/or Theorem 10.9 one recovers the following classical Poincaré-Friedrichs inequality.
Corollary 10.11 (Classical Poincaré-Friedrichs inequality). If $\Omega \subset \mathbb{R}^{d}$ is bounded in one direction or has finite measure, i.e., $|\Omega|<\infty$, then

$$
\begin{equation*}
\|u\|_{L^{p}(\Omega)}^{p} \leq B K_{d, p} \int_{\Omega}|\nabla u(x)|^{p} \mathrm{~d} x \quad \text { for every } u \in W_{0}^{1, p}(\Omega) \tag{10.12}
\end{equation*}
$$

Next results is combined consequence of Theorem 10.9 and Theorem 10.10.
Corollary 10.12. Assume that $\Omega \subset \mathbb{R}^{d}$ is bounded in one direction or that $|\Omega|<\infty$. There are $C=C(d, p, \Omega)$ and $s_{0}=s_{0}(d, p, \Omega)$ such that for every $s \in\left(s_{0}, 1\right)$ and every $u \in C_{c}^{\infty}(\Omega)$

$$
\begin{equation*}
\|u\|_{L^{p}(\Omega)}^{p} \leq C s(1-s) \iint_{\mathbb{R}^{d} \mathbb{R}^{d}} \frac{|u(x)-u(y)|^{p}}{|x-y|^{d+s p}} d y d x \tag{10.13}
\end{equation*}
$$

Proof. It suffices to put $\nu_{\varepsilon}(h)=\frac{p \varepsilon(1-\varepsilon)}{\left|\mathbb{S}^{d-1}\right|}|h|^{-d-(1-\varepsilon) p}$ with $\varepsilon=1-s$ and apply Theorem 10.9 and Theorem 10.10.

## 11. Convergence of weak solutions

In this section we establish the convergence in $L^{p}(\Omega)$ of weak solutions of nonlocal Dirichlet and Neumann problems to the local ones. We will need the following Lemma.
Lemma 11.1. Let $\Omega \subset \mathbb{R}^{d}$ is bounded Lipschitz. Let $v \in W_{\nu_{\varepsilon}}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ and $\varphi \in C_{b}^{2}\left(\mathbb{R}^{d}\right)$. Assume that

$$
\kappa_{\varphi}:=\sup _{\varepsilon>0}\left\|L_{\varepsilon} \varphi\right\|_{L^{\infty}\left(\mathbb{R}^{d}\right)}<\infty \quad \text { for } 1<p<2
$$

The following assertions hold true.
(i) There is a constant $C_{\varphi}>0$ independent of $\varepsilon$ such that

$$
\left|\int_{\Omega^{c}} \mathcal{N}_{\varepsilon} \varphi(y) v(y) \mathrm{d} y\right| \leq C_{\varphi}\|v\|_{W_{\nu_{\varepsilon}}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}
$$

(ii) Assume $v \in W^{1, p}\left(\mathbb{R}^{d}\right)$, recall $\partial_{n, p} \varphi(x)=|\nabla \varphi(x)|^{p-2} \nabla \varphi(x) \cdot n(x)$, then

$$
\lim _{\varepsilon \rightarrow 0} \int_{\Omega^{c}} \mathcal{N}_{\varepsilon} \varphi(y) v(y) \mathrm{d} y=K_{d, p} \int_{\partial \Omega} \partial_{n, p} \varphi(x) v(x) \mathrm{d} \sigma(x)
$$

Proof. (i) In view of the estimate (B.2) for $p \geq 2$ we have

$$
|\psi(u(x+h)-u(x))+\psi(u(x-h)-u(x))| \leq C\|u\|_{C_{b}^{2}\left(\mathbb{R}^{d}\right)}^{p-1}\left(1 \wedge|h|^{p}\right)
$$

with $\psi(t)=|t|^{p-2} t$, which implies

$$
\left|L_{\varepsilon} \varphi\right| \leq C\|\varphi\|_{C_{b}^{2}\left(\mathbb{R}^{d}\right)}^{p-1} \quad \text { for } p \geq 2
$$

Therefore taking into account the assumption, in case we have

$$
\begin{equation*}
\kappa_{\varphi}:=\sup _{\varepsilon>0}\left\|L_{\varepsilon} \varphi\right\|_{L^{\infty}\left(\mathbb{R}^{d}\right)}<\infty \quad \text { for } 1<p<\infty \tag{11.1}
\end{equation*}
$$

Since $|\varphi(x+h)-\varphi(x)| \leq 2\|u\|_{C_{b}^{1}\left(\mathbb{R}^{d}\right)}(1 \wedge|h|)$ we find that

$$
\begin{equation*}
\mathcal{E}^{\varepsilon}(\varphi, \varphi) \leq 2^{p+1}|\Omega|\|\varphi\|_{C_{b}^{1}\left(\mathbb{R}^{d}\right)}^{p} \quad \text { for all } \quad \varepsilon>0 \tag{11.2}
\end{equation*}
$$

Next, assume that $v \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$. The nonlocal Gauss-Green formula (B.6) yields

$$
\begin{aligned}
\left|\int_{\Omega^{c}} \mathcal{N}_{\varepsilon} \varphi(y) v(y) \mathrm{d} y\right| & =\left|\mathcal{E}^{\varepsilon}(\varphi, v)-\int_{\Omega} L_{\varepsilon} \varphi(x) v(x) \mathrm{d} x\right| \\
& \leq \mathcal{E}^{\varepsilon}(\varphi, \varphi)^{1 / p^{\prime}} \mathcal{E}^{\varepsilon}(v, v)^{1 / p}+\left\|L_{\varepsilon} \varphi\right\|_{L^{\infty}\left(\mathbb{R}^{d}\right)}|\Omega|^{1 / p^{\prime}}\|v\|_{L^{p}(\Omega)} \\
& \leq C_{\varphi}\|v\|_{W_{\nu_{\varepsilon}}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}, \quad C_{\varphi}=|\Omega|^{1 / p^{\prime}}\left(2^{(p+1) / p^{\prime}}\|\varphi\|_{C_{b}^{1}\left(\mathbb{R}^{d}\right)}^{p-1}+\kappa_{\varphi}\right)
\end{aligned}
$$

Note that $C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$ is dense in $W_{\nu_{\varepsilon}}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ (see [Fog20, Theorem 3.70]). By the continuity of the linear mapping $v \mapsto \mathcal{E}^{\varepsilon}(\varphi, v)-\int_{\Omega} L_{\varepsilon} \varphi(x) v(x) \mathrm{d} x$, the Gauss-Green formula (B.6) is applicable for $\varphi \in C_{b}^{2}\left(\mathbb{R}^{d}\right)$ and $v \in W_{\nu_{\varepsilon}}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$. Therefore, the above estimate yields (i).
(ii) By Theorem 9.9, $L_{\varepsilon} \varphi(x) \xrightarrow{\varepsilon \rightarrow 0}-K_{d, p} \Delta_{p} \varphi(x)$ (a.e for $1<p<2$ ). Together with (11.1) and the fact that $v \in L^{p}(\Omega) \subset L^{1}(\Omega)$, the dominated convergence theorem yields

$$
\int_{\Omega} L_{\varepsilon} \varphi(x) v(x) \mathrm{d} x \xrightarrow{\varepsilon \rightarrow 0}-K_{d, p} \int_{\Omega} \Delta_{p} \varphi(x) v(x) \mathrm{d} x .
$$

If $v \in W^{1, p}\left(\mathbb{R}^{d}\right)$, then $\left.v\right|_{\Omega},\left.\varphi\right|_{\Omega} \in W^{1, p}(\Omega)$. Since $\partial \Omega$ is Lipschitz, combing Theorem 9.15, Lemma 11.1 and the fact that $\mathcal{E}^{\varepsilon}(\varphi, \varphi)^{1 / p^{\prime}} \leq C_{\varphi}$ (by estimate (11.2)), we get

$$
\mathcal{E}^{\varepsilon}(\varphi, v) \xrightarrow{\varepsilon \rightarrow 0} K_{d, p} \mathcal{E}^{0}(\varphi, v)
$$

Finally from the foregoing and the (non)local Gauss-Green formula we obtain (ii) as follows

$$
\begin{aligned}
\lim _{\varepsilon \rightarrow 0} \int_{\Omega^{c}} \mathcal{N}_{\varepsilon} \varphi(y) v(y) \mathrm{d} y & =\lim _{\varepsilon \rightarrow 0} \mathcal{E}^{\varepsilon}(\varphi, v)-\lim _{\varepsilon \rightarrow 0} \int_{\Omega} L_{\varepsilon} \varphi(x) v(x) \mathrm{d} x \\
& =K_{d, p} \int_{\Omega}|\nabla \varphi(x)|^{p-2} \nabla \varphi(x) \cdot \nabla v(x) \mathrm{d} x-K_{d, p} \int_{\Omega} \Delta_{p} \varphi(x) v(x) \mathrm{d} x \\
& =K_{d, p} \int_{\partial \Omega}|\nabla \varphi(x)|^{p-2} \nabla \varphi(x) \cdot n(x) v(x) \mathrm{d} \sigma(x)
\end{aligned}
$$

Theorem 11.2 (Convergence of Neumann problem I). Assume $\Omega \subset \mathbb{R}^{d}$ is open bounded and connected with Lipschitz boundary. Let $f, f_{\varepsilon} \in L^{p^{\prime}}(\Omega)$ be such that $\left(f_{\varepsilon}\right)_{\varepsilon}$ converges weakly sense to $f$. Let $g_{\varepsilon}=\mathcal{N}_{\varepsilon} \varphi$ and $g=\partial_{n, p} \varphi$ where $\varphi \in C_{b}^{2}\left(\mathbb{R}^{d}\right)$. In addition we assume

$$
\kappa_{\varphi}:=\sup _{\varepsilon>0}\left\|L_{\varepsilon} \varphi\right\|_{L^{\infty}\left(\mathbb{R}^{d}\right)}<\infty \quad \text { for } 1<p<2
$$

Assume $u_{\varepsilon} \in W_{\nu_{\varepsilon}}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\perp}$ is a weak solution of Neumann problem $L_{\varepsilon} u=f_{\varepsilon}$ in $\Omega$ and $\mathcal{N}_{\varepsilon} u=g_{\varepsilon}$ on $\Omega^{c}$ that is,

$$
\mathcal{E}^{\varepsilon}\left(u_{\varepsilon}, v\right)=\int_{\Omega} f_{\varepsilon}(x) v(x) \mathrm{d} x+\int_{\Omega^{c}} g_{\varepsilon}(y) v(y) \mathrm{d} y \quad \text { for all } \quad v \in W_{\nu_{\varepsilon}}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\perp}
$$

Let $u \in W^{1, p}(\Omega)^{\perp}$ be the weak solution of $-K_{d, p} \Delta_{p} u=f$ in $\Omega$ and $K_{d, p} \partial_{n, p} u=g$ on $\partial \Omega$ i.e.

$$
K_{d, p} \mathcal{E}^{0}(u, v)=\int_{\Omega} f(x) v(x) \mathrm{d} x+K_{d, p} \int_{\partial \Omega} g(x) v(x) \mathrm{d} \sigma(x) \quad \text { for all } u \in W^{1, p}(\Omega)^{\perp}
$$

Then $\left(u_{\varepsilon}\right)_{\varepsilon}$ strongly converges to $u$ in $L^{p}(\Omega)$, i.e., $\left\|u_{\varepsilon}-u\right\|_{L^{p}(\Omega)} \xrightarrow{\varepsilon \rightarrow 0} 0$. Moreover, the following weak convergence of the energies forms holds true

$$
\mathcal{E}^{\varepsilon}\left(u_{\varepsilon}, v\right) \xrightarrow{\varepsilon \rightarrow 0} K_{d, p} \mathcal{E}^{0}(u, v) \quad \text { for all } v \in W^{1, p}\left(\mathbb{R}^{d}\right)
$$

Proof. The robust Poincaré inequality (see Corollary 10.2) implies the existence of $\varepsilon_{0} \in(0,1)$ and $C=C(d, p, \Omega)>0$ such that for all $\varepsilon \in\left(0, \varepsilon_{0}\right)$ and all $v \in W_{\nu_{\varepsilon}}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ we have

$$
\begin{equation*}
\left\|v-f_{\Omega} v\right\|_{W_{\nu_{\varepsilon}}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}^{p} \leq C \mathcal{E}^{\varepsilon}(v, v) \tag{11.3}
\end{equation*}
$$

In virtue of the weak convergence of $\left(f_{\varepsilon}\right)_{\varepsilon}$, up to relabeling $\varepsilon_{0}>0$, we can assume that $M:=\sup _{\varepsilon \in\left(0, \varepsilon_{0}\right)}\left\|f_{\varepsilon}\right\|_{L^{p^{\prime}}(\Omega)}<$ $\infty$, so that

$$
\left|\int_{\Omega^{c}} f_{\varepsilon}(x) u_{\varepsilon}(x) \mathrm{d} x\right| \leq M\left\|u_{\varepsilon}\right\|_{W_{\nu_{\varepsilon}}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}
$$

whereas, Lemma 11.1 ( $i$ ) yields

$$
\left|\int_{\Omega^{c}} g_{\varepsilon}(y) u_{\varepsilon}(y) \mathrm{d} y\right|=\left|\int_{\Omega^{c}} \mathcal{N}_{\varepsilon} \varphi(y) u_{\varepsilon}(y) \mathrm{d} y\right| \leq C_{\varphi}\left\|u_{\varepsilon}\right\|_{W_{\nu_{\varepsilon}}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}
$$

Since $u_{\varepsilon} \in W_{\nu_{\varepsilon}}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\perp}$, by definition of $u_{\varepsilon}$ we have

$$
\begin{aligned}
\mathcal{E}^{\varepsilon}\left(u_{\varepsilon}, u_{\varepsilon}\right) & =\int_{\Omega} f_{\varepsilon}(x) u_{\varepsilon}(x) \mathrm{d} x+\int_{\Omega^{c}} g_{\varepsilon}(y) u_{\varepsilon}(y) \mathrm{d} y \\
& \leq\left\|u_{\varepsilon}\right\|_{W_{\nu_{\varepsilon}}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}\left(\left\|f_{\varepsilon}\right\|_{L^{p^{\prime}}(\Omega)}+C_{\varphi}\right) \leq C\left\|u_{\varepsilon}\right\|_{W_{\nu_{\varepsilon}}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}
\end{aligned}
$$

Combining this with (11.3) yields the following uniform boundedness for a generic constant $C>0$ independent of $\varepsilon$

$$
\begin{equation*}
\left\|u_{\varepsilon}\right\|_{W_{\nu_{\varepsilon}}^{p}(\Omega)}^{p-1} \leq\left\|u_{\varepsilon}\right\|_{W_{\nu_{\varepsilon}}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}^{p-1} \leq C \quad \text { for all } \varepsilon \in\left(0, \varepsilon_{0}\right) \tag{11.4}
\end{equation*}
$$

Accordingly, by the asymptotic compactness Theorem 9.18 , there is $u \in W^{1, p}(\Omega)$ and subsequence $\varepsilon_{n} \rightarrow 0$ such that $\left(u_{\varepsilon_{n}}\right)_{n}$ converges to $u$ in $L^{p}(\Omega)$ and we have

$$
K_{d, p} \mathcal{E}^{0}(u, u) \leq \liminf _{\varepsilon \rightarrow 0} \mathcal{E}^{\varepsilon}\left(u_{\varepsilon}, u_{\varepsilon}\right)
$$

In particular we get $u \in W^{1, p}(\Omega)^{\perp}$ since each $u_{\varepsilon} \in L^{p}(\Omega)^{\perp}$. Next, we show that $u$ is in fact the unique weak solution in to the Neumann problem $-K_{d, p} \Delta_{p} u=f$ in $\Omega$ and $K_{d, p} \partial_{n, p} u=g$ on $\partial \Omega$. To this end, it is sufficient to show that $\mathcal{J}(u)=\min \left\{\mathcal{J}(v): v \in W^{1, p}(\Omega)^{\perp}\right\}$,

$$
\mathcal{J}(v)=\frac{1}{p} K_{d, p} \mathcal{E}^{0}(v, v)-\int_{\Omega} f(x) v(x) \mathrm{d} x-K_{d, p} \int_{\partial \Omega} g(x) v(x) \mathrm{d} \sigma(x)
$$

Recall that, by Proposition 8.11, each $u_{\varepsilon}$ satisfies $\mathcal{J}^{\varepsilon}(u)=\min \left\{\mathcal{J}^{\varepsilon}(v): v \in W_{\nu_{\varepsilon}}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\perp}\right\}$,

$$
\mathcal{J}^{\varepsilon}(v)=\frac{1}{p} \mathcal{E}^{\varepsilon}(v, v)-\int_{\Omega} f_{\varepsilon}(x) v(x) \mathrm{d} x-\int_{\Omega^{c}} g_{\varepsilon}(x) v(x) \mathrm{d} x .
$$

Now we consider $v \in W^{1, p}(\Omega)^{\perp}$. Given that $\partial \Omega$ is Lipschitz, i.e., $\Omega$ is an extension domain, we let $\bar{v} \in W^{1, p}\left(\mathbb{R}^{d}\right)$ be an extension of $v$. In view of Lemma 11.1, Theorem 9.15 and the weak convergence we have

$$
\begin{aligned}
\lim _{\varepsilon \rightarrow 0} \mathcal{J}^{\varepsilon}(\bar{v}) & =\lim _{\varepsilon \rightarrow 0}\left(\frac{1}{p} \mathcal{E}^{\varepsilon}(\bar{v}, \bar{v})-\int_{\Omega} f_{\varepsilon}(x) v(x) \mathrm{d} x-\int_{\Omega^{c}} g_{\varepsilon}(y) v(y) \mathrm{d} y\right) \\
& =\frac{1}{p} K_{d, p} \mathcal{E}^{0}(v, v)-\int_{\Omega} f(x) v(x) \mathrm{d} x-K_{d, p} \int_{\partial \Omega} g(x) v(x) \mathrm{d} \sigma(x)=\mathcal{J}(v)
\end{aligned}
$$

The strong convergence of $\left(u_{\varepsilon_{n}}\right)_{n}$ and the weak convergence of $\left(f_{\varepsilon_{n}}\right)_{n}$ in $L^{p}(\Omega)$ yield

$$
\lim _{n \rightarrow \infty} \int_{\Omega} f_{\varepsilon_{n}}(x) u_{\varepsilon_{n}}(x) \mathrm{d} x=\int_{\Omega} f(x) u(x) \mathrm{d} x
$$

Analogously, by further taking into account the uniform boundedness of $\left(L_{\varepsilon_{n}} \varphi\right)_{n}$ (see (11.1)), the pointwise convergence $L_{\varepsilon_{n}} \varphi(x) \rightarrow-K_{d, p} \Delta_{p} \varphi(x)$ (see Theorem 9.9) and the Gauss-Green formula (see Appendix B.2) we get

$$
\begin{aligned}
& \lim _{n \rightarrow \infty} \int_{\Omega^{c}} g_{\varepsilon_{n}}(y) u_{\varepsilon_{n}}(y) \mathrm{d} y=\lim _{n \rightarrow \infty} \int_{\Omega^{c}} \mathcal{N}_{\varepsilon_{n}} \varphi(y) u_{\varepsilon_{n}}(y) \mathrm{d} y \\
& =\lim _{n \rightarrow \infty} \mathcal{E}^{\varepsilon_{n}}\left(\varphi, u_{\varepsilon_{n}}\right)-\lim _{n \rightarrow \infty} \int_{\Omega} L_{\varepsilon_{n}} \varphi(y) u_{\varepsilon_{n}}(y) \mathrm{d} y \\
& =K_{d, p} \mathcal{E}^{0}(\varphi, u)-K_{d, p} \int_{\Omega} \Delta_{p} \varphi(x) u(x) \mathrm{d} x \\
& =K_{d, p} \int_{\partial \Omega} \partial_{n, p} \varphi(x) u(x) \mathrm{d} \sigma(x)=K_{d, p} \int_{\partial \Omega} g(x) u(x) \mathrm{d} \sigma(x) .
\end{aligned}
$$

Together with the lower estimate $K_{d, p} \mathcal{E}^{0}(u, u) \leq \liminf _{n \rightarrow \infty} \mathcal{E}^{\varepsilon_{n}}\left(u_{\varepsilon_{n}}, u_{\varepsilon_{n}}\right)$, we obtain

$$
\mathcal{J}(u) \leq \liminf _{n \rightarrow \infty} \mathcal{J}^{\varepsilon_{n}}\left(u_{\varepsilon_{n}}\right)
$$

Since, each $u_{\varepsilon_{n}}$ minimizes $\mathcal{J}^{\varepsilon_{n}}, \int_{\Omega} \bar{v}(x) \mathrm{d} x=0$ and $\bar{v} \in W^{1, p}\left(\mathbb{R}^{d}\right) \subset W_{\nu_{\varepsilon}}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$, we get

$$
\mathcal{J}(u) \leq \liminf _{n \rightarrow \infty} \mathcal{J}^{\varepsilon_{n}}\left(u_{\varepsilon_{n}}\right) \leq \liminf _{n \rightarrow \infty} \mathcal{J}^{\varepsilon_{n}}(\bar{v})=\mathcal{J}(v)
$$

It turns out that $\left\|u_{\varepsilon_{n}}-u\right\|_{L^{p}(\Omega)} \rightarrow 0$ as $n \rightarrow \infty$ and $u$ minimizes $\mathcal{J}$ that is,

$$
\mathcal{J}(u)=\min _{v \in W^{1, p}(\Omega)^{\perp}} \mathcal{J}(v)
$$

Whence $u$ is the unique weak solution to the Neumann on $W^{1, p}(\Omega)^{\perp}$ that is

$$
\mathcal{E}^{0}(u, v)=\int_{\Omega} f(x) v(x) \mathrm{d} x+\int_{\partial \Omega} g(x) v(x) \mathrm{d} \sigma(x) \quad \text { for all } v \in W^{1, p}(\Omega)^{\perp}
$$

The uniqueness of $u$ implies that $\left\|u_{\varepsilon}-u\right\|_{L^{p}(\Omega)} \rightarrow 0$ as $\varepsilon \rightarrow 0$. Moreover, for $v \in W^{1, p}\left(\mathbb{R}^{d}\right)$ we have $v-c \in$ $W^{1, p}(\Omega)^{\perp} \cap W_{\nu_{\varepsilon}}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\perp}$ with $c=f_{\Omega} v$ and hence

$$
\begin{aligned}
\lim _{\varepsilon \rightarrow 0} \mathcal{E}^{\varepsilon}\left(u_{\varepsilon}, v\right)=\lim _{\varepsilon \rightarrow 0} \mathcal{E}^{\varepsilon}\left(u_{\varepsilon}, v-c\right) & =\lim _{\varepsilon \rightarrow 0} \int_{\Omega} f_{\varepsilon}(x)(v(x)-c) \mathrm{d} x+\lim _{\varepsilon \rightarrow 0} \int_{\Omega^{c}} g_{\varepsilon}(y)(v(y)-c) \mathrm{d} y \\
& =\int_{\Omega} f(x)(v(x)-c) \mathrm{d} x+K_{d, p} \int_{\partial \Omega} g(x)(v(x)-c) \mathrm{d} x \\
& =K_{d, p} \mathcal{E}^{0}(u, v-c)=K_{d, p} \mathcal{E}^{0}(u, v)
\end{aligned}
$$

The above convergence remains for weak solutions associated with the regional operators;

$$
L_{\Omega, \varepsilon} u(x)=2 \text { p.v. } \int_{\Omega} \psi(u(x)-u(y)) \nu_{\varepsilon}(x-y) \mathrm{d} y
$$

Theorem 11.3 (Convergence of Neumann problem II). Let the assumptions of Theorem 11.2 be in force. Assume $u_{\varepsilon} \in W_{\nu_{\varepsilon}}^{p}(\Omega)^{\perp}$ is a weak solution to the regional Neumann problem $L_{\Omega, \varepsilon} u=f_{\varepsilon}$ on $\Omega$ and $\int_{\Omega} u=0$ that is $u_{\varepsilon}$ satisfies

$$
\mathcal{E}_{\Omega}^{\varepsilon}\left(u_{\varepsilon}, v\right)=\int_{\Omega} f_{\varepsilon}(x) v(x) \mathrm{d} x \quad \text { for all } \quad v \in W_{\nu_{\varepsilon}}^{p}(\Omega)^{\perp}
$$

Let $u \in W^{1, p}(\Omega)^{\perp}$ be the weak solution of $-K_{d, p} \Delta_{p} u=f$ in $\Omega$ and $\partial_{n, p} u=0$ on $\partial \Omega$ i.e.

$$
K_{d, p} \mathcal{E}^{0}(u, v)=\int_{\Omega} f(x) v(x) \mathrm{d} x \quad \text { for all } u \in W^{1, p}(\Omega)^{\perp}
$$

Then $\left(u_{\varepsilon}\right)_{\varepsilon}$ strongly converges to $u$ in $L^{p}(\Omega)$, i.e., $\left\|u_{\varepsilon}-u\right\|_{L^{p}(\Omega)} \xrightarrow{\varepsilon \rightarrow 0} 0$. Moreover, the following weak convergence of the energies forms holds true

$$
\mathcal{E}^{\varepsilon}\left(u_{\varepsilon}, v\right) \xrightarrow{\varepsilon \rightarrow 0} K_{d, p} \mathcal{E}^{0}(u, v) \quad \text { for all } v \in W^{1, p}(\Omega) .
$$

Proof. The proof is analogous to that of Theorem 11.2.

Recall (see Section 9.4) the fractional $p$-Laplacian and the corresponding normal derivative

$$
\begin{aligned}
(-\Delta)_{p}^{s} u(x) & :=C_{d, p, s} \text { p.v. } \int_{\mathbb{R}^{d}} \frac{\psi(u(x)-u(y))}{|x-y|^{d+s p}} \mathrm{~d} y \\
\mathcal{N}_{s} u(y) & :=C_{d, p, s} \int_{\Omega} \frac{\psi(u(y)-u(x))}{|x-y|^{d+s p}} \mathrm{~d} x
\end{aligned}
$$

Theorem 11.4. Assume the assumptions of Theorem 11.2 hold. Let $f, f_{s} \in L^{p^{\prime}}(\Omega)$ be such that $\left(f_{s}\right)_{s}$ converges weakly sense to $f$ as $s \rightarrow 1$ and we put $g_{s}=\mathcal{N}_{s} \varphi$ and $\partial_{n, p} \varphi$. Let $u_{s} \in W^{s, p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\perp}, s \in(0,1)$ be the weak solution to the Neumann problem

$$
(-\Delta)_{p}^{s} u=f_{s} \text { on } \Omega \text { and } \mathcal{N}_{s} u=g_{s} \text { on } \Omega^{c}
$$

Let $u \in W^{1, p}(\Omega)^{\perp}$ be the weak solution of the Neumann problem

$$
-\Delta_{p} u=f \text { in } \Omega \text { and } \partial_{n, p} u=g \text { on } \partial \Omega
$$

Then $\left(u_{s}\right)_{s}$ strongly converges to $u$ in $L^{p}(\Omega)$, i.e., $\left\|u_{s}-u\right\|_{L^{p}(\Omega)} \xrightarrow{s \rightarrow 1} 0$.
Proof. It is sufficient to consider $\nu_{\varepsilon}(h)=a_{d, p, \varepsilon}|h|^{-d-(1-\varepsilon) p}, a_{d, p, \varepsilon}=\frac{p \varepsilon(1-\varepsilon)}{\left|\mathbb{S}^{d-1}\right|}$ in Theorem 11.2, accounting the fact that the asymptotic of the normalizing constant $C_{d, p, s}$ yields

$$
\lim _{\varepsilon \rightarrow 0} \frac{C_{d, p, 1-\varepsilon}}{a_{d, p, \varepsilon}}=\lim _{s \rightarrow 1} \frac{C_{d, p, s}\left|\mathbb{S}^{d-1}\right|}{p s(1-s)}=\frac{2}{K_{d, p}}
$$

Theorem 11.5 (Convergence of Dirichlet problem). Assume $\Omega \subset \mathbb{R}^{d}$ is open with a continuous boundary and in addition that $|\Omega|<\infty$ or that $\Omega$ is bounded in one direction. Let $g \in W^{1, p}\left(\mathbb{R}^{d}\right)$ and $f, f_{\varepsilon} \in L^{p^{\prime}}(\Omega)$ for which $\left(f_{\varepsilon}\right)_{\varepsilon}$ converges weakly to $f$. Let $u_{\varepsilon} \in W_{\nu_{\varepsilon}, 0}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ the weak solution of Dirichlet problem $L_{\varepsilon} u=f_{\varepsilon}$ on $\Omega$ and $u=g$ on $\Omega^{c}$ that is,

$$
u-g \in W_{\nu_{\varepsilon}, 0}^{p}\left(\Omega \mid \mathbb{R}^{d}\right) \quad \text { and } \quad \mathcal{E}^{\varepsilon}\left(u_{\varepsilon}, v\right)=\int_{\Omega} f_{\varepsilon}(x) v(x) \mathrm{d} x \quad \text { for all } \quad v \in W_{\nu_{\varepsilon}, 0}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)
$$

Let $u \in W^{1, p}(\Omega)$ be the weak solution of $-K_{d, p} \Delta_{p} u=f$ in $\Omega$ and $u=g$ on $\partial \Omega$ i.e.

$$
u-g \in W_{0}^{1, p}(\Omega) \quad \text { and } \quad K_{d, p} \mathcal{E}^{0}(u, v)=\int_{\Omega} f(x) v(x) \mathrm{d} x \quad \text { for all } u \in W_{0}^{1, p}(\Omega)
$$

Then $\left(u_{\varepsilon}\right)_{\varepsilon}$ strongly converges to $u$ in $L_{\mathrm{loc}}^{p}\left(\mathbb{R}^{d}\right)$, where we put $u=g$ on $\Omega^{c}$, i.e., we have $\left\|u_{\varepsilon}-u\right\|_{L^{p}(B)} \xrightarrow{\varepsilon \rightarrow 0} 0$ for any bounded set $B \subset \mathbb{R}^{d}$. If in addition $|\Omega|<\infty$ then we have $\left\|u_{\varepsilon}-u\right\|_{L^{p}\left(\mathbb{R}^{d}\right)} \xrightarrow{\varepsilon \rightarrow 0} 0$. Moreover, the weak convergence of the energies forms holds, i.e.,

$$
\mathcal{E}^{\varepsilon}\left(u_{\varepsilon}, v\right) \xrightarrow{\varepsilon \rightarrow 0} K_{d, p} \mathcal{E}^{0}(u, v) \quad \text { for all } v \in W_{0}^{1, p}(\Omega)
$$

Proof. By the robust Poincaré-Friedrichs inequality (see Theorem 10.9 and Theorem 10.10) there exist $\varepsilon_{0} \in(0,1)$ and $C=C(d, p, \Omega)>0$ such that for all $\varepsilon \in\left(0, \varepsilon_{0}\right)$ and all $v \in W_{\nu_{\varepsilon}, 0}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ we have

$$
\begin{equation*}
\|v\|_{W_{\nu_{\varepsilon}}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}^{p} \leq C \mathcal{E}^{\varepsilon}(v, v) \tag{11.5}
\end{equation*}
$$

In virtue of the weak convergence of $\left(f_{\varepsilon}\right)_{\varepsilon}$, up to relabeling $\varepsilon_{0}>0$, we can assume that $M:=\sup _{\varepsilon \in\left(0, \varepsilon_{0}\right)}\left\|f_{\varepsilon}\right\|_{L^{p^{\prime}}(\Omega)}<$ $\infty$. Beside this since $g \in W^{1, p}\left(\mathbb{R}^{d}\right)$ and $\int_{\mathbb{R}^{d}}\left(1 \wedge|h|^{p}\right) \nu_{\varepsilon}(h) \mathrm{d} h=1$, by the estimate (3.5) we have

$$
\int_{\mathbb{R}^{d}}|g(x)|^{p} \mathrm{~d} x+\iint_{\mathbb{R}^{d} \mathbb{R}^{d}}|g(x)-g(y)|^{p} \nu_{\varepsilon}(x-y) \mathrm{d} y \mathrm{~d} x \leq 2^{p+1}\|g\|_{W^{1, p}\left(\mathbb{R}^{d}\right)}^{p}
$$

In virtue of the coercivity estimate (11.5), by proceeding as for the proof of (8.15), one finds a constant $C>0$ independent of $\varepsilon$ such that, for all $\varepsilon \in\left(0, \varepsilon_{0}\right)$ we have

$$
\begin{equation*}
\left\|u_{\varepsilon}\right\|_{W_{\nu_{\varepsilon}}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)} \leq C\left(\left\|f_{\varepsilon}\right\|_{L^{p^{\prime}}(\Omega)}+\left\|g_{\varepsilon}\right\|_{W_{\nu_{\varepsilon}}^{p}\left(\mathbb{R}^{d}\right)}\right) \leq C\left(M+2^{p+1}\|g\|_{W^{1, p}\left(\mathbb{R}^{d}\right)}^{p}\right) \tag{11.6}
\end{equation*}
$$

Therefore taking into account that $u_{\varepsilon}=g$ on $\Omega^{c}$ and $\|g\|_{W_{\nu_{\varepsilon}\left(\Omega^{c}\right)}} \leq 2^{p+1}\|g\|_{W^{1, p}\left(\mathbb{R}^{d}\right)}$, we deduce from the estimate (11.6) that for some $C=C(d, p, \Omega, M, g)>0$ we have

$$
\left\|u_{\varepsilon}\right\|_{W_{\nu_{\varepsilon}}^{p}\left(\mathbb{R}^{d}\right)}=\left(\|g\|_{W_{\nu_{\varepsilon}}^{p}\left(\Omega^{c}\right)}^{p}+\left\|u_{\varepsilon}\right\|_{W_{\nu_{\varepsilon}}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}^{p}\right)^{1 / p} \leq C \quad \text { for all } \varepsilon \in\left(0, \varepsilon_{0}\right)
$$

Accordingly, by the asymptotic compactness Theorem 9.18 , there is $u \in W^{1, p}\left(\mathbb{R}^{d}\right)$ and subsequence $\varepsilon_{n} \rightarrow 0$ such that $\left(u_{\varepsilon_{n}}\right)_{n}$ converges to $u$ in $L_{\text {loc }}^{p}\left(\mathbb{R}^{d}\right)$. Moreover, there holds

$$
\begin{aligned}
K_{d, p} \int_{\mathbb{R}^{d}}|\nabla u(x)|^{p} \mathrm{~d} x & \leq \liminf _{\varepsilon \rightarrow 0} \mathcal{E}_{\mathbb{R}^{d}}^{\varepsilon}\left(u_{\varepsilon}, u_{\varepsilon}\right) \\
K_{d, p} \mathcal{E}^{0}(u, u) & \leq \liminf _{\varepsilon \rightarrow 0} \mathcal{E}^{\varepsilon}\left(u_{\varepsilon}, u_{\varepsilon}\right)
\end{aligned}
$$

In particular we have $u=g$ on $\Omega^{c}$ since $u_{\varepsilon}=g$ on $\Omega^{c}$. Therefore, since $\partial \Omega$ is continuous, we get $u-g \in W_{0}^{1, p}(\Omega)$. Next, we show that $u$ is in fact the unique weak solution in to the Dirichlet problem $-K_{d, p} \Delta_{p} u=f$ in $\Omega$ and $u=g$ on $\partial \Omega$. To this end, it is sufficient to show that $\mathcal{J}(u)=\min \left\{\mathcal{J}(v): v-g \in W_{0}^{1, p}(\Omega)\right\}$,

$$
\mathcal{J}(v)=\frac{1}{p} K_{d, p} \mathcal{E}^{0}(v, v)-\int_{\Omega} f(x) v(x) \mathrm{d} x .
$$

Recall that, by Proposition 8.17, each $u_{\varepsilon}$ satisfies $\mathcal{J}_{0}^{\varepsilon}(u)=\min \left\{\mathcal{J}_{0}^{\varepsilon}(v): v-g \in W_{\nu_{\varepsilon}, 0}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)\right\}$,

$$
\mathcal{J}_{0}^{\varepsilon}(v)=\frac{1}{p} \mathcal{E}^{\varepsilon}(v, v)-\int_{\Omega} f_{\varepsilon}(x) v(x) \mathrm{d} x .
$$

Now we consider $v \in W^{1, p}(\Omega)$ such that $v-g \in W_{0}^{1, p}(\Omega)$. Thus, we can consider $\bar{v}$ be the extension of $v$ by $\bar{v}=g$ on $\Omega^{c}$ so that $\bar{v} \in W^{1, p}\left(\mathbb{R}^{d}\right.$ (since $v$ and $g$ have the same trace on $\left.\partial \Omega\right)$ and $v-g \in W_{\nu_{\varepsilon}, 0}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$. Since, each $u_{\varepsilon_{n}}$ minimizes $\mathcal{J}_{0}^{\varepsilon_{n}}$ and $\bar{v}-g \in W_{\nu_{\varepsilon}, 0}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$, we have $\mathcal{J}_{0}^{\varepsilon_{n}}\left(u_{\varepsilon_{n}}\right) \leq \mathcal{J}_{0}^{\varepsilon_{n}}(\bar{v})$. In view of Theorem 9.15 and the weak convergence we have

$$
\begin{aligned}
\lim _{\varepsilon \rightarrow 0} \mathcal{J}_{0}^{\varepsilon}(\bar{v}) & =\lim _{\varepsilon \rightarrow 0}\left(\frac{1}{p} \mathcal{E}^{\varepsilon}(\bar{v}, \bar{v})-\int_{\Omega} f_{\varepsilon}(x) v(x) \mathrm{d} x\right) \\
& =\frac{1}{p} K_{d, p} \mathcal{E}^{0}(v, v)-\int_{\Omega} f(x) v(x) \mathrm{d} x=\mathcal{J}_{0}(v)
\end{aligned}
$$

The strong convergence of $\left(u_{\varepsilon_{n}}\right)_{n}$ and the weak convergence of $\left(f_{\varepsilon_{n}}\right)_{n}$ in $L^{p}(\Omega)$ yield

$$
\lim _{n \rightarrow \infty} \int_{\Omega} f_{\varepsilon_{n}}(x) u_{\varepsilon_{n}}(x) \mathrm{d} x=\int_{\Omega} f(x) u(x) \mathrm{d} x
$$

Whence, we deduce that

$$
\mathcal{J}_{0}(u) \leq \liminf _{n \rightarrow \infty} \mathcal{J}_{0}^{\varepsilon_{n}}\left(u_{\varepsilon_{n}}\right) \leq \liminf _{n \rightarrow \infty} \mathcal{J}_{0}^{\varepsilon_{n}}(\bar{v})=\mathcal{J}_{0}(v)
$$

It turns out that $\left\|u_{\varepsilon_{n}}-u\right\|_{L^{p}(\Omega)} \rightarrow 0$ as $n \rightarrow \infty, u-g \in W_{0}^{1, p}(\Omega)$ and

$$
\mathcal{J}_{0}(u)=\min _{v-g \in W_{0}^{1, p}(\Omega)} \mathcal{J}_{0}(v)
$$

In other words, $u$ is the unique weak solution to the Dirichlet on $W^{1, p}(\Omega)$ that is

$$
u-g \in W_{0}^{1, p}(\Omega) \quad \text { and } \quad \mathcal{E}^{0}(u, v)=\int_{\Omega} f(x) v(x) \mathrm{d} x \quad \text { for all } v \in W_{0}^{1, p}(\Omega)
$$

The uniqueness of $u$ implies that $\left\|u_{\varepsilon}-u\right\|_{L^{p}(\Omega)} \rightarrow 0$ as $\varepsilon \rightarrow 0$. Moreover, for $v \in W_{0}^{1, p}(\Omega)$, if assume $v=0$ on $\Omega^{c}$ then we have $v \in W_{\nu_{\varepsilon}, 0}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ and hence

$$
\lim _{\varepsilon \rightarrow 0} \mathcal{E}^{\varepsilon}\left(u_{\varepsilon}, v\right)=\lim _{\varepsilon \rightarrow 0} \int_{\Omega} f_{\varepsilon}(x) v(x) \mathrm{d} x=\int_{\Omega} f(x) v(x) \mathrm{d} x=K_{d, p} \mathcal{E}^{0}(u, v)
$$

Theorem 11.6. Assume the assumptions of Theorem 11.5 hold. Let $f, f_{s} \in L^{p^{\prime}}(\Omega)$ be such that $\left(f_{s}\right)_{s}$ converges weakly sense to $f$ as $s \rightarrow 1$. Let $u_{s} \in W_{0}^{s, p}\left(\Omega \mid \mathbb{R}^{d}\right)^{\perp}, s \in(0,1)$ be the weak solution to the Dirichlet problem

$$
(-\Delta)_{p}^{s} u=f_{s} \text { on } \Omega \text { and } u=g_{s} \text { on } \Omega^{c} .
$$

Let $u \in W^{1, p}(\Omega)$ be the weak solution of the Dirichlet problem

$$
-\Delta_{p} u=f \text { in } \Omega \text { and } u=g \text { on } \partial \Omega
$$

Then $\left(u_{s}\right)_{s}$ strongly converges to $u$ in $L_{\mathrm{loc}}^{p}\left(\mathbb{R}^{d}\right)$, where we put $u=g$ on $\Omega^{c}$. If in addition $|\Omega|<\infty$ then we have $\left\|u_{s}-u\right\|_{L^{p}\left(\mathbb{R}^{d}\right)} \xrightarrow{s \rightarrow 1} 0$.

Proof. It is sufficient to consider $\nu_{\varepsilon}(h)=a_{d, p, \varepsilon}|h|^{-d-(1-\varepsilon) p}, a_{d, p, \varepsilon}=\frac{p \varepsilon(1-\varepsilon)}{\left|\mathbb{S}^{d-1}\right|}$ in Theorem 11.5, accounting the fact that the asymptotic of the normalizing constant $C_{d, p, s}$ yields

$$
\lim _{\varepsilon \rightarrow 0} \frac{C_{d, p, 1-\varepsilon}}{a_{d, p, \varepsilon}}=\lim _{s \rightarrow 1} \frac{C_{d, p, s}\left|\mathbb{S}^{d-1}\right|}{p s(1-s)}=\frac{2}{K_{d, p}}
$$

## Appendix A.

A.1. Elementary estimates for the $p$-Laplacian and $p$-Lévy operators. We establish elementary estimates involving the mapping $x \mapsto|x|^{p-2} x, x \in \mathbb{R}^{d}, p \geq 1$, useful in the study of the $p$-Laplacian and $p$-Lévy operators. We adopt the convention $|x|^{p-2} x=0$ if $x=0$.

Lemma A.1. For $x, y \in \mathbb{R}^{d}$, there hold the following inequalities

$$
\begin{aligned}
& \left||x|^{p-2} x-|y|^{p-2} y\right| \leq A_{p}|x-y|(|x|+|y|)^{p-2} \\
& \left(|x|^{p-2} x-|y|^{p-2} y\right) \cdot(x-y) \geq A_{p}^{\prime}|x-y|^{2}(|x|+|y|)^{p-2} .
\end{aligned}
$$

Here, we are able to get $A_{p}=p-1, A_{p}^{\prime}=\min \left(2^{-1}, 2^{2-p}\right)$ if $p \geq 2$ and $A_{p}=2^{2-p}(3-p) \leq 2^{3-p}, A_{p}^{\prime}=p-1$ if $1 \leq p<2$. One easily verifies that $A_{p} \leq 2^{1+|p-2|}$ and $A_{p}^{\prime} \geq \min \left(p-1,2^{1-p}\right)$.

Proof. If $p \geq 2$ then by monotonicity, $\left(|y|^{p-2}-|x|^{p-2}\right)\left(|y|^{2}-|x|^{2}\right) \geq 0$ and hence using $(a+b)^{q} \leq \max \left(1,2^{q-1}\right)\left(a^{q}+\right.$ $b^{q}$ ) for all $q, a, b \in(0, \infty)$ we get

$$
\begin{aligned}
\left(|x|^{p-2} x-|y|^{p-2} y\right) \cdot(x-y) & =\frac{1}{2}\left(|x|^{p-2}+|y|^{p-2}\right)|x-y|^{2}+\frac{1}{2}\left(|x|^{p-2}-|y|^{p-2}\right)\left(|x|^{2}-|y|^{2}\right) \\
& \geq \min \left(2^{-1}, 2^{2-p}\right)(|x|+|y|)^{p-2}|x-y|^{2}
\end{aligned}
$$

On the other hand the fundamental theorem of calculus implies

$$
\begin{aligned}
|x|^{p-2} x-|y|^{p-2} y & =(x-y) \int_{0}^{1}|t x+(1-t) y|^{p-2} \mathrm{~d} t \\
& +(p-2) \int_{0}^{1}|t x+(1-t) y|^{p-4}[(t x+(1-t) y) \cdot(x-y)](t x+(1-t) y) \mathrm{d} t
\end{aligned}
$$

Thus, if $-1<p-2<0$, by Cauchy-Schwartz inequality and $|t x+(1-t) y|^{2-p} \leq(|x|+|y|)^{2-p}$,

$$
\begin{aligned}
\left(\left||x|^{p-2} x-|y|^{p-2} y\right|\right) \cdot(x-y) & \geq(p-1)|x-y|^{2} \int_{0}^{1}|t x+(1-t) y|^{p-2} \mathrm{~d} t \\
& \geq(p-1)|x-y|^{2}(|x|+|y|)^{p-2}
\end{aligned}
$$

This completes the proof of the second inequality. Analogously, if $p \geq 2$ then

$$
\begin{aligned}
\|\left. x\right|^{p-2} x-|y|^{p-2} y| | & \leq(p-1)|x-y| \int_{0}^{1}(t|x|+(1-t)|y|)^{p-2} \mathrm{~d} t \\
& \leq(p-1)|x-y|(|x|+|y|)^{p-2}
\end{aligned}
$$

It remains to prove the first inequality when $1<p<2$. Without lost of generality, assume $x \neq y,|y| \leq|x|$ and $y=|y| e$ with $|e|=1$. Put $z=x|y|^{-1}$, it suffices to bound $F(z)$ where

$$
F(z)=\frac{\left||z|^{p-2} z-e\right|}{|z-e|(1+|z|)^{p-2}}, \quad \text { for } z \in \mathbb{R}^{d} \backslash\{e\},|z| \geq 1
$$

Put $r=|z-e|$ so that $|z| \leq 1+r$. Since $|z|^{p-2} z-e=|z|^{p-2}\left((z-e)+\left(1-|z|^{2-p}\right) e\right)$ we get

$$
\begin{aligned}
\frac{\|\left. z\right|^{p-2} z-e \mid}{|z-e|(1+|z|)^{p-2}} & \leq\left(1+|z|^{-1}\right)^{2-p}\left(1+\frac{\left(|z|^{2-p}-1\right)}{|z-e|}\right) \\
& \leq 2^{2-p}\left(1+\frac{1}{r}\left((1+r)^{2-p}-1\right)\right) \\
& =2^{2-p}\left(1+\frac{(2-p)}{r} \int_{0}^{r} \frac{\mathrm{~d} \tau}{(1+\tau)^{p-1}}\right) \leq 2^{2-p}(3-p) .
\end{aligned}
$$

A direct consequence of Lemma A. 1 is the following; see also [BL94, Lemma 2.2].
Corollary A.2. Let $x, y \in \mathbb{R}^{d}$ and $\beta \geq 0$. There hold the following inequalities

$$
\begin{aligned}
& \left||x|^{p-2} x-|y|^{p-2} y\right| \leq A_{p}|x-y|^{1-\beta}(|x|+|y|)^{p-2+\beta} \\
& \left(|x|^{p-2} x-|y|^{p-2} y\right) \cdot(x-y) \geq A_{p}^{\prime}|x-y|^{2+\beta}(|x|+|y|)^{p-2-\beta}
\end{aligned}
$$

Proof. It suffices to observe that $|x-y|(|x|+|y|)^{-1} \leq 1$ and use Lemma A.1.
Lemma A.3. The following inequalities hold true for all $x, y \in \mathbb{R}^{d}$.

$$
\begin{gather*}
\left||x|^{p-2} x-|y|^{p-2} y\right| \leq \begin{cases}A_{p}|x-y|(|x|+|y|)^{p-2} & p \geq 2 \\
A_{p}|x-y|^{p-1} & 1 \leq p<2\end{cases}  \tag{A.1}\\
\left(|x|^{p-2} x-|y|^{p-2} y\right) \cdot(x-y) \geq \begin{cases}A_{p}^{\prime}|x-y|^{p} & p \geq 2 \\
A_{p}^{\prime}|x-y|^{2}(|x|+|y|)^{p-2} & 1 \leq p<2 .\end{cases} \tag{A.2}
\end{gather*}
$$

Proof. The claims follow from Corollary A. 2 by taking $\beta=0$ for (A.1) and (A.4) and $\beta=|p-2|$ for (A.2) and (A.3). Note however that, (A.2) also follows from (A.3) by duality. Indeed, if $1<p<2$ then $p^{\prime}>2$ where $p+p^{\prime}=p p^{\prime}$. Keeping in mind that by duality $z^{\prime}=|z|^{p-2} z$ if and only if $z=\left|z^{\prime}\right|^{p^{\prime}-2} z^{\prime}$ (see also [Lin19]), the estimate (A.3) implies

$$
\begin{aligned}
\left|x^{\prime}-y^{\prime}\right|^{p^{\prime}} & \leq \max \left(2,2^{p^{\prime}-2}\right)\left(\left|x^{\prime}\right|^{p^{\prime}-2} x^{\prime}-\left|y^{\prime}\right|^{p^{\prime}-2} y^{\prime}\right) \cdot\left(x^{\prime}-y^{\prime}\right) \\
& \leq \max \left(2,2^{p^{\prime}-2}\right)\left|x-y \| x^{\prime}-y^{\prime}\right|
\end{aligned}
$$

Since $(p-1)\left(p^{\prime}-1\right)=1$ and $\max \left(2^{p-1}, 2^{2-p}\right)<2<2^{3-p}$ it follows that

$$
\begin{equation*}
\left||x|^{p-2} x-|y|^{p-2} y\right| \leq \max \left(2^{p-1}, 2^{2-p}\right)|x-y|^{p-1} \leq 2^{3-p}|x-y|^{p-1} \tag{A.5}
\end{equation*}
$$

The constant $2^{2-p}$ is expected in (A.2) as this is the case in one dimension.
Lemma A.4. If $1 \leq p<2$, there hold following inequality

$$
\begin{equation*}
\left||b|^{p-2} b-|a|^{p-2} a\right| \leq 2^{2-p}|b-a|^{p-1} \quad \text { for all } a, b \in \mathbb{R} \tag{A.6}
\end{equation*}
$$

Proof. We only prove for $1<p<2$, i.e., $\sigma=p-1 \in(0,1)$. For $t \in \mathbb{R}$, we have

$$
\left||t|^{\sigma-1} t-1\right|= \begin{cases}t^{\sigma}-1 & \text { if } t \geq 1 \\ 1-t^{\sigma} & \text { if } 0 \leq t<1 \\ (-t)^{\sigma}+1 & \text { if } t<0\end{cases}
$$

Since $t \mapsto t^{\sigma-1}$ is decreasing on $(0, \infty)$ it follows that $t^{\sigma}=(t-1) t^{\sigma-1}+t^{\sigma-1} \leq(t-1)^{\sigma}+1$ for $t \geq 1$, that is $t^{\sigma}-1 \leq(t-1)^{\sigma}$ and $1=(1-t) \times 1^{\sigma-1}+t \times 1^{\sigma-1} \leq(1-t)^{\sigma}+t^{\sigma}$ for $0<t<1$, that is $1-t^{\sigma} \leq(1-t)^{\sigma}$. The concavity of $t \mapsto t^{\sigma}$ implies $(-t)^{\sigma}+1 \leq 2^{1-\sigma}(-t+1)^{\sigma}$ for $t<0$. Altogether, we obtain the following inequality

$$
\left||t|^{\sigma-1} t-1\right| \leq 2^{1-\sigma}|t-1|^{\sigma} \quad \text { for all } t \in \mathbb{R} \text { and } \sigma \in[0,1]
$$

The desired inequality (A.6) is inherited from the one above by taking $t=\frac{b}{a}, a \neq 0$.
Let us see some useful consequences of Lemma A.3.
Corollary A.5. The following inequalities hold true for all $x, y \in \mathbb{R}^{d}$.

$$
\begin{aligned}
& |x|^{p}-|y|^{p}-p|y|^{p-2} \cdot y(x-y) \leq \begin{cases}A_{p}|x-y|^{p} & 1 \leq p<2 \\
\frac{p}{2} A_{p}|x-y|^{2}(|x-y|+2|y|)^{p-2} & p \geq 2\end{cases} \\
& |x|^{p}-|y|^{p}-p|y|^{p-2} y \cdot(x-y) \geq \begin{cases}A_{p}^{\prime}|x-y|^{p} & p \geq 2 \\
\frac{p}{2} A_{p}^{\prime}|x-y|^{2}(|x-y|+2|y|)^{p-2} & 1 \leq p<2 . \\
57\end{cases}
\end{aligned}
$$

Proof. Applying the fundamental theorem of calculus on $t \mapsto\left|z_{t}\right|, z_{t}=y+t(x-y)$ implies

$$
|x|^{p}-|y|^{p}-p|y|^{p-2} y \cdot(x-y)=p \int_{0}^{1}\left(\left|z_{t}\right|^{p-2} z_{t}-|y|^{p-2} y\right) \cdot\left(z_{t}-y\right) \frac{\mathrm{d} t}{t}
$$

The result is inherited from Lemma A. 3 and the estimate $|y|+\left|z_{t}\right| \leq(2|y|+|x-y|)$.

Corollary A.6. For $R>0, q>0$ there is $c_{q, R}>0$ such that

$$
|a|^{q-1} a-|a-b|^{q-1}(a-b) \leq c_{q, R} \max \left(b, b^{q}\right) \quad \text { for all }|a| \leq R, b \geq 0
$$

Proof. If $b>R$ then since $t \mapsto|t|^{q-1} t$ is increasing, and $|a| \leq R$ we have

$$
-|a-b|^{q-1}(a-b) \leq-|a-R|^{q-1}(a-R) \leq|a-R|^{q} \leq 2^{q} R^{q} .
$$

Therefore, since $R^{q} \leq \max \left(b, b^{q}\right)$, we get

$$
a^{q-1} a-|a-b|^{q-1}(a-b) \leq|a|^{q}+2^{q} R^{q} \leq 2^{q+1} R^{q} \max \left(b, b^{q}\right)
$$

Now if $q \in(0,1]$ then since $\left||x|^{q-1} x-|y|^{q-1} y\right| \leq 2^{2-q}|x-y|^{q}$ (see (A.2)) it follows that

$$
|a|^{q-1} a-|a-b|^{q-1}(a-b) \leq 2^{2-q} \max \left(b, b^{q}\right)
$$

Last, if $q>1$ and $b \leq R$ then $|a-b|^{q-1} \leq 2^{q-1} R^{q-1}$. The estimate (A.1) for $p=q+1$ yields

$$
|a|^{q-1} a-|a-b|^{q-1}(a-b) \mid \leq q b\left(|a|^{q-1}+(|a|+b)^{q-1}\right) \leq q 2^{q} R^{q-1} \max \left(b, b^{q}\right)
$$

Corollary A.7. Let $b=b_{1}-b_{2}$, and $a=a_{1}-a_{2}$ where $a_{1}, a_{2}, b_{1}, b_{2} \in \mathbb{R}$ then we have

$$
(\psi(b)-\psi(a))\left(\left(b_{1}-a_{1}\right)_{+}-\left(b_{2}-a_{2}\right)_{+}\right) \geq \begin{cases}A_{p}^{\prime}\left|\left(b_{1}-a_{1}\right)_{+}-\left(b_{2}-a_{2}\right)_{+}\right|^{p} & p \geq 2 \\ A_{p}^{\prime}\left|\left(b_{1}-a_{1}\right)_{+}-\left(b_{2}-a_{2}\right)_{+}\right|^{2}(|b|+|a|)^{p-2} & 1 \leq p<2\end{cases}
$$

where we put $\psi(t)=|t|^{p-2} t$ and $t_{+}=\max (t, 0)$ and $t_{-}=\max (-t, 0)$ so that $t=t_{+}-t_{-}$.
Proof. First of all, as $t \mapsto \psi(t)$ is increasing $\left(\psi^{\prime}(t)=(p-1)|t|^{p-2}\right)$ we easily get that

$$
\begin{equation*}
\left.(\psi(b)-\psi(a))\left(\left(b_{1}-a_{1}\right)_{+}-\left(b_{2}-a_{2}\right)_{+}\right)\right)=|\psi(b)-\psi(a)|\left|\left(b_{1}-a_{1}\right)_{+}-\left(b_{2}-a_{2}\right)_{+}\right| . \tag{A.7}
\end{equation*}
$$

For instance, if $\left(b_{2}-a_{2}\right) \leq 0$ and $\left(b_{1}-a_{1}\right) \geq 0$ then we have

$$
b-a=\left(b_{1}-a_{1}\right)-\left(b_{2}-a_{2}\right) \geq\left(b_{1}-a_{1}\right)_{+}=\left(b_{1}-a_{1}\right)_{+}-\left(b_{2}-a_{2}\right)_{+} \geq 0
$$

In particular, $\psi(a) \leq \psi(b)$ and hence the relation (A.7) follows. The cases can be derived analogously. On the other hand, from (A.3) and (A.4) we get

$$
|\psi(b)-\psi(a)| \geq \begin{cases}A_{p}^{\prime}|b-a|^{p-1} & p \geq 2 \\ A_{p}^{\prime}|b-a|(|b|+|a|)^{p-2} & 1 \leq p<2\end{cases}
$$

Using the fact that $\left|t_{+}-s_{+}\right| \leq|t-s|$, the desired estimates follow from the relation (A.7).
Another important consequence of Lemma A. 1 is the following.
Theorem A.8. Let $p, q \in[1, \infty)$. Under the notations of Lemma A. 1 for $x, y \in \mathbb{R}^{d}$, we get

$$
\begin{aligned}
& \left||x|^{p-2} x-|y|^{p-2} y\right| \leq\left. A_{p}\left[A_{\frac{p-2}{q}+2}^{\prime}\right]^{-q}|x-y|^{1-q}| | x\right|^{\frac{p-2}{q}} x-\left.|y|^{\frac{p-2}{q}} y\right|^{q}, \\
& \left(|x|^{p-2} x-|y|^{p-2} y\right) \cdot(x-y) \geq\left. A_{p}^{\prime}\left[A_{\frac{p-2}{q}+2}\right]^{-q}|x-y|^{2-q}| | x\right|^{\frac{p-2}{q}} x-\left.|y|^{\frac{p-2}{q}} y\right|^{q} .
\end{aligned}
$$

Proof. Nothing that $\frac{p-2}{q}+2 \geq 1$, it suffices to apply Lemma A. 1 for $p$ and $\frac{p-2}{q}+2$.

$$
\begin{aligned}
& (|x|+|y|)^{p-2}=\left[(|x|+|y|)^{\frac{p-2}{q}}\right]^{q} \leq\left.\left[A_{\frac{p-2}{q}+2}^{\prime}\right]^{-q}|x-y|^{-q}| | x\right|^{\frac{p-2}{q}} x-\left.|y|^{\frac{p-2}{q}} y\right|^{q}, \\
& (|x|+|y|)^{p-2}=\left[(|x|+|y|)^{\frac{p-2}{q}}\right]^{q} \geq\left.\left[A_{\frac{p-2}{q}+2}\right]^{-q}|x-y|^{-q}| | x\right|^{\frac{p-2}{q}} x-\left.|y|^{\frac{p-2}{q}} y\right|^{q} .
\end{aligned}
$$

## Appendix B.

B.1. Pointwise evaluation of the operator $L$ and $\mathcal{N}$. We aim in this appendix to provide ancillary results about the of a translation-invariant p-Lévy operator that enter the scope at hand. As usual we assume $\nu$ is symmetric and $p$-Lévy integrable, i.e., $\nu(h)=\nu(-h)$ and $\nu \in L^{1}\left(1 \wedge|h|^{p} \mathrm{~d} h\right)$.

$$
\begin{aligned}
L u(x) & =2 \text { p.v. } \int_{\mathbb{R}^{d}} \psi(u(x)-u(y)) \nu(x-y) \mathrm{d} y=\lim _{\varepsilon \rightarrow 0} L_{\varepsilon} u(x), \\
L_{\varepsilon} u(x) & =2 \int_{\mathbb{R}^{d} \backslash B_{\varepsilon}(x)} \psi(u(x)-u(y)) \nu(x-y) \mathrm{d} y \quad\left(x \in \mathbb{R}^{d} ; \varepsilon>0\right) .
\end{aligned}
$$

The pointwise definition of $L u(x)$ and $-\Delta_{p} u(x)$ for $u \in C_{b}^{2}\left(\mathbb{R}^{d}\right)$ is warranted in the degenerate case (also often called the superquadratic case), i.e. $p \geq 2$. However, in the situation singular case (also often called the subquadratic case), i.e., $1<p<2$, the pointwise definition of $L u(x)$ and $-\Delta_{p} u(x)$ might not exist even for a bona fide function $u \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$. Actually, in general it is difficult to characterize a set of all functions on which the operators $L$ and $-\Delta_{p}$ act in a reasonable pointwise sense. As a reasonable alternative in both cases, i.e., $1<p<\infty$, is rather to evaluate $L u$ and $\Delta_{p} u$ in the generalized sense, i.e., in the weak sense or via their respective associated energies forms for instance by duality the following identifications are well-defined;

$$
\begin{aligned}
\langle L u, \varphi\rangle & =\mathcal{E}_{\mathbb{R}^{d}}(u, \varphi) \quad u, \varphi \in W_{\nu}^{p}\left(\mathbb{R}^{d}\right) \\
\left\langle\Delta_{p} u, \varphi\right\rangle & =\int_{\mathbb{R}^{d}}|\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla \varphi(x) \mathrm{d} x \quad u, \varphi \in W^{1, p}\left(\mathbb{R}^{d}\right)
\end{aligned}
$$

From the duality point of view, one obtains that $L u \in\left(W_{\nu}^{p}\left(\mathbb{R}^{d}\right)\right)^{\prime}$ and $\Delta_{p} u \in\left(W^{1, p}\left(\mathbb{R}^{d}\right)\right)^{\prime}$. It turns out that, the operators $L: W_{\nu}^{p}\left(\mathbb{R}^{d}\right) \rightarrow\left(W_{\nu}^{p}\left(\mathbb{R}^{d}\right)\right)^{\prime}$ and $\Delta_{p}: W^{1, p}\left(\mathbb{R}^{d}\right) \rightarrow\left(W^{1, p}\left(\mathbb{R}^{d}\right)\right)^{\prime}$ are well-defined. We emphasize that $\langle L u, \varphi\rangle$ and $\left\langle\Delta_{p} u, \varphi\right\rangle$ are given as above. Morally, the nonlocal operator $L$ may be as good or as bad the local operator $-\Delta_{p}$. In fact, the operator $L$ can be seen as a prototype of a nonlinear nonlocal operator of divergence just as the $-\Delta_{p}$ is a prototype of a nonlinear local operator of divergence.
Next, in order to investigate the pointwise evaluation of $L u$, we need to introduce consider Hölder spaces. Denote the space $C_{b}^{k+\gamma}\left(\mathbb{R}^{d}\right)=\left\{u \in C^{k}\left(\mathbb{R}^{d}\right):\|u\|_{C_{b}^{k, \gamma}\left(\mathbb{R}^{d}\right)}<\infty\right\}, k \in \mathbb{N}$ and $0 \leq \gamma \leq 1$ with the norm

$$
\|u\|_{C_{b}^{k, \gamma}\left(\mathbb{R}^{d}\right)}=\sum_{|\beta| \leq k} \sup _{x \in \mathbb{R}^{d}}\left|\partial^{\beta} u(x)\right|+\sum_{|\beta|=k} \sup _{x \neq y \in \mathbb{R}^{d}} \frac{\left|\partial^{\beta} u(x)-\partial^{\beta} u(y)\right|}{|x-y|^{\gamma}} .
$$

For $0 \leq \gamma \leq 1$, we define

$$
\widehat{p}_{\gamma}= \begin{cases}p+\gamma-1 & \text { if } \quad p \geq 2  \tag{B.1}\\ (\gamma+1)(p-1) & \text { if } \quad 1<p<2\end{cases}
$$

Remark B.1. Note that $\widehat{p}_{\gamma} \leq p$ and hence we find that $L^{1}\left(\mathbb{R}^{d}, 1 \wedge|h|^{\hat{p}_{\gamma}}\right) \subset L^{1}\left(\mathbb{R}^{d}, 1 \wedge|h|^{p}\right)$. For the particular case $\gamma=1$ we have

$$
\widehat{p}_{1}= \begin{cases}p & \text { if } p \geq 2 \\ 2(p-1) & \text { if } 1<p<2\end{cases}
$$

For the subclass of kernel $\nu \in L^{1}\left(\mathbb{R}^{d}, 1 \wedge|h|^{\widehat{p}_{\gamma}}\right)$, it is possible to evaluate $L u$ pointwise when $u$ is sufficiently smooth. Here are some basic properties of the operator $L$.

Proposition B.2. Assume $u \in C_{b}^{1+\gamma}\left(\mathbb{R}^{d}\right), 0<\gamma \leq 1$, and $\nu \in L^{1}\left(\mathbb{R}^{d}, 1 \wedge|h|^{\widehat{p}_{\gamma}}\right)$.
(i) The map $x \mapsto L u(x)$ is bounded and uniformly continuous. Moreover,

$$
\begin{aligned}
L u(x) & =-\int_{\mathbb{R}^{d}}[\psi(u(x+h)-u(x))+\psi(u(x-h)-u(x))] \nu(h) \mathrm{d} h \\
& =-2 \int_{\mathbb{R}^{d}}\left[\psi(u(x+h)-u(x))-\mathbb{1}_{B_{1}}(h) \psi(\nabla u(x) \cdot h)\right] \nu(h) \mathrm{d} h .
\end{aligned}
$$

(ii) The map $x \mapsto L_{\varepsilon} u(x), 0<\varepsilon<1$, is uniformly continuous.
(iii) The family $\left(L_{\varepsilon} u(x)\right)_{\varepsilon}$ is uniformly bounded and uniformly converges to Lu, i.e.

$$
\left\|L_{\varepsilon} u-L u\right\|_{L^{\infty}\left(\mathbb{R}^{d}\right)} \xrightarrow{\varepsilon \rightarrow 0} 0 .
$$

Proof. Fix $x \in \mathbb{R}^{d}$, since $h \mapsto \psi(\nabla u(x) \cdot h) \nu(h)$ has vanishing integral over $B_{1} \backslash B_{\varepsilon}$, i.e.,

$$
\int_{B_{1}(0) \backslash B_{\varepsilon}(0)}[\nabla u(x) \cdot h] \nu(h) \mathrm{d} h=0 \quad \text { for all } 0<\varepsilon<1 .
$$

Thus, we get

$$
L_{\varepsilon} u(x)=-2 \int_{\mathbb{R}^{d} \backslash B_{\varepsilon}(0)}\left[\psi(u(x+h)-u(x))-\mathbb{1}_{B_{1}}(h) \psi(\nabla u(x) \cdot h)\right] \nu(h) \mathrm{d} h
$$

whereas, the simple $y=x \pm h$ hange of variables gives

$$
L_{\varepsilon} u(x)=-\int_{\mathbb{R}^{d} \backslash B_{\varepsilon}(0)}[\psi(u(x+h)-u(x))+\psi(u(x-h)-u(x))] \nu(h) \mathrm{d} h .
$$

We emphasize that $L_{\varepsilon} u(x)$ exists since $\nu \in L^{1}\left(\mathbb{R}^{d} \backslash B_{\varepsilon}(0)\right)$. For $h \in \mathbb{R}^{d}$, by the mean value Theorem, $u(x+h)-u(x)=$ $\nabla u(x+\tau h) \cdot h$ for some $\tau \in(0,1)$. Thus the estimates (A.1) and (A.2) with $a=\nabla u\left(x-\tau_{1} h\right) \cdot h$ and $b=\nabla u\left(x+\tau_{2} h\right) \cdot h$ for suitable $\tau_{1}, \tau_{2} \in[0,1]$ yield

$$
\begin{align*}
& |\psi(u(x+h)-u(x))+\psi(u(x-h)-u(x))| \leq C\|u\|_{C_{b}^{1+\gamma}\left(\mathbb{R}^{d}\right)}^{p-1}\left(1 \wedge|h|^{\widehat{p}_{\gamma}}\right)  \tag{B.2}\\
& |\psi(u(x+h)-u(x))-\psi(\nabla u(x) \cdot h)| \leq C\|u\|_{C_{b}^{1+\gamma}\left(\mathbb{R}^{d}\right)}^{p-1}\left(1 \wedge|h|^{\widehat{p}_{\gamma}}\right) \tag{B.3}
\end{align*}
$$

Here, $C>0$ is a generic constant only depending on $p$ and $d$. In view of the estimates (B.2) and (B.3), since $h \mapsto\left(1 \wedge|h|^{\widehat{\gamma}_{\gamma}}\right) \nu(h)$ is integrable, the boundedness of $x \mapsto L u(x)$ and the uniform boundedness of $x \mapsto L_{\varepsilon} u(x)$ follow and one also gets rid of the principal value. In addition, the uniform convergence of $\left(L_{\varepsilon} u\right)_{\varepsilon}$ to $L u$ follows since

$$
\left\|L_{\varepsilon} u-L u\right\|_{L^{\infty}\left(\mathbb{R}^{d}\right)} \leq C\|u\|_{C_{b}^{1+\gamma}\left(\mathbb{R}^{d}\right)}^{p-1} \int_{B_{\varepsilon}(0)}\left(1 \wedge|h|^{\widehat{p}_{\gamma}}\right) \nu(h) \mathrm{d} h \xrightarrow{\varepsilon \rightarrow 0} 0
$$

Turning to the uniform continuity, we fix $x, z \in \mathbb{R}^{d}$ such that $|x-z| \leq \delta$ with $0<\delta<1$. For every $\mathrm{h} \in \mathbb{R}^{d}, h \neq 0$, the estimates (A.1) and (A.2) imply

$$
\begin{equation*}
|\psi(u(x)-u(x \pm h))-\psi(u(z)-u(z \pm h))| \leq C \delta^{\gamma \kappa_{p}}\|u\|_{C_{b}^{1+\gamma}\left(\mathbb{R}^{d}\right)}^{p-1} \tag{B.4}
\end{equation*}
$$

where $\kappa_{p}=1$ if $p \geq 2$ and $\kappa_{p}=p-1$ if $1<p<2$. This combined with (B.2) yields

$$
|\psi(u(x)-u(x \pm h))-\psi(u(z)-u(z \pm h))| \leq C\|u\|_{C_{b}^{1+\gamma}\left(\mathbb{R}^{d}\right)}^{p-1}\left(\delta^{\gamma \kappa_{p}} \wedge|h|^{\widehat{p}_{\gamma}}\right)
$$

Therefore, the integrability of $h \mapsto\left(1 \wedge|h|^{\widehat{p}_{\gamma}}\right) \nu(h)$ implies the uniform continuity as follows

$$
\|L u(x)-L u(z)\|_{L^{\infty}\left(\mathbb{R}^{d}\right)} \leq C\|u\|_{C_{b}^{1+\gamma}\left(\mathbb{R}^{d}\right)}^{p-1} \int_{\mathbb{R}^{d}}\left(\delta^{\gamma \kappa_{p}} \wedge|h|^{\widehat{p}_{\gamma}}\right) \nu(h) \mathrm{d} h \xrightarrow{\delta \rightarrow 0} 0 .
$$

The uniform continuity of $x \mapsto L_{\varepsilon} u(x)$ follows analogously.
It is natural to seek for a larger functional space on which $L u$ is defined. In an attempt to answer this question, assume in addition that $\nu: \mathbb{R}^{d} \backslash\{0\} \rightarrow[0, \infty)$ is unimodal, i.e. $\nu$ is radial and almost decreasing, i.e., there is a constant $c$ such that $\nu(y) \leq c \nu(x)$ whenever $|y| \geq|x|$. Let us define the function

$$
\widehat{\nu}(x)=\nu\left(\frac{1}{2}(1+|x|)\right)
$$

Remark B.3. Assume $\nu \in L^{1}\left(\mathbb{R}^{d} \backslash B_{\varepsilon}(0)\right), \varepsilon>0$ and $\nu$ is unimodal.
(i) It is not difficult to show that $\widehat{\nu} \in L^{1}\left(\mathbb{R}^{d}\right) \cap L^{\infty}\left(\mathbb{R}^{d}\right)$.
(ii) The space $L^{p-1}\left(\mathbb{R}^{d}, \widehat{\nu}\right)$ contains $C_{b}^{1+\gamma}\left(\mathbb{R}^{d}\right), L^{\infty}\left(\mathbb{R}^{d}\right)$ and $C_{\mathrm{loc}}^{1+\gamma}\left(\mathbb{R}^{d}\right) \cap L^{\infty}\left(\mathbb{R}^{d}\right)$.
(iii) If $\nu(h)=|h|^{-d-s p}, s \in(0,1)$, then $\widehat{\nu}(h) \asymp(1+|h|)^{-d-s p}$.

Proposition B.4. Assume that $\nu \in L^{1}\left(\mathbb{R}^{d}, 1 \wedge|h|^{\widehat{p}_{\gamma}}\right)$ and that $\nu$ is almost decreasing, i.e., there is $c>0$ such that $\nu(x) \geq c \nu(y)$ whenever $|x| \leq|y|$.
(i) If $u \in C_{\mathrm{loc}}^{1+\gamma}\left(\mathbb{R}^{d}\right) \cap L^{p-1}\left(\mathbb{R}^{d}, \widehat{\nu}\right)$ then $L u(x), x \in \mathbb{R}^{d}$, is well defined.
(ii) If $u \in C_{b}^{1+\gamma}\left(\mathbb{R}^{d}\right)$ and $\operatorname{supp} u \subset B_{R}(0), R \geq 1$ then there is $C=C(R, d, \nu)>0$

$$
\begin{equation*}
|L u(x)| \leq C\|u\|_{C_{b}^{2}\left(\mathbb{R}^{d}\right)}^{p} \widehat{\nu}(x) \quad \text { for all } x \in \mathbb{R}^{d} \tag{B.5}
\end{equation*}
$$

In particular, $L u \in L^{1}\left(\mathbb{R}^{d}\right) \cap L^{\infty}\left(\mathbb{R}^{d}\right)$.

Proof. (i) Set $R=2|x|+1, x \in \mathbb{R}^{d}$, then for $y \in B_{R}^{c}(0)$ we have $|x-y| \geq \frac{1}{2}(1+|y|)$ so that $\nu(x-y) \leq c \widehat{\nu}(y)$. Consequently, for $u \in L^{1}\left(\mathbb{R}^{d}, \widehat{\nu}\right)$ we have

$$
\left|\int_{B_{R}^{c}(0)} \psi(u(x)-u(y)) \nu(x-y) \mathrm{d} y\right| \leq C|u(x)|^{p-1}\|\widehat{\nu}\|_{L^{1}\left(\mathbb{R}^{d}\right)}+C \int_{B_{R}^{c}(0)}|u(y)|^{p-1} \widehat{\nu}(y) \mathrm{d} y<\infty
$$

We conclude that $L u(x)$ exists since by exploiting (B.2) we get

$$
\begin{aligned}
\left|\int_{B_{R}(0)} \psi(u(x)-u(y)) \nu(x-y) \mathrm{d} y\right| & =\frac{1}{2}\left|\int_{B_{R}(0)} \psi(u(x)-u(x-h))+\psi(u(x)-u(x+h)) \nu(h) \mathrm{d} h\right| \\
& \leq C\|u\|_{C^{1+\gamma}\left(B_{R}(0)\right)}^{p-1} \int_{\mathbb{R}^{d}}\left(1 \wedge|h|^{\widehat{p}_{\gamma}}\right) \nu(h) \mathrm{d} h<\infty
\end{aligned}
$$

(ii) Assume $\operatorname{supp} u \subset B_{R}(0)$ for some $R \geq 1$. If $|x| \geq 4 R$ then $u(x)=0$. For $y \in B_{R}(0)$, we get $\nu(x-y) \leq c \widehat{\nu}(x)$ since $|x-y| \geq \frac{|x|}{2}+R \geq \frac{1}{2}(1+|x|)$. Accordingly,

$$
|L u(x)| \leq \int_{B_{R}(0)}|u(y)|^{p-1} \nu(x-y) \mathrm{d} y \leq c\left|B_{R}(0)\right|\|u\|_{C_{b}^{1+\gamma}\left(\mathbb{R}^{d}\right)}^{p-1} \widehat{\nu}(x)
$$

Whereas, if $|x| \leq 4 R$ then $\frac{1}{2}(1+|x|) \leq 4 R$ and hence $\widehat{\nu}(x) \geq c_{1}$ with $c_{1}=c \nu(4 R)>0$. The proof of (B.5) is complete as follows using (B.2),

$$
|L u(x)| \leq c_{1}^{-1} C\|\nu\|_{L^{1}\left(\mathbb{R}^{d}, 1 \wedge|h|^{\hat{p}_{\gamma} \gamma}\right)}\|u\|_{C_{b}^{1+\gamma}\left(\mathbb{R}^{d}\right)}^{p-1} \widehat{\nu}(x) .
$$

Next we show that $L u$ can still be continuous under less regularity on $u$.
Theorem B.5. Let $u \in C_{b}^{\gamma}\left(\mathbb{R}^{d}\right), 0<\gamma \leq 1$. Assume that $\nu \in L^{1}\left(\mathbb{R}^{d}, 1 \wedge|h|^{\gamma(p-1)}\right)$ then Lu $\in C_{b}^{0}\left(\mathbb{R}^{d}\right)$ is uniform continuous. Moreover, for all $x, z \in \mathbb{R}^{d}$,

$$
|L(x)-L(z)| \leq C\|u\|_{C^{\gamma}\left(\mathbb{R}^{d}\right)}^{p-1} \omega(|x-z|)
$$

where $\omega:(0, \infty) \rightarrow(0, \infty)$ is an increasing modulus of continuity such that $\omega(r) \xrightarrow{r \rightarrow 0} 0$.
Proof. Recall that $\kappa_{p}=1$ if $p \geq 2$ and $\kappa_{p}=p-1$ if $1<p<2$. For $r>0$, the estimate (B.4) gives

$$
\left|\int_{B_{r}^{c}(0)} \psi(u(x)-u(x+h))-\psi(u(z)-u(z+h)) \nu(h) \mathrm{d} h\right| \leq C\|u\|_{C^{\gamma}\left(\mathbb{R}^{d}\right)}^{p-1} \int_{B_{r}^{c}(0)}|x-z|^{\gamma_{p}} \nu(h) \mathrm{d} h
$$

Using $|\psi(u(\cdot)-u(\cdot+h))| \leq|h|^{\gamma(p-1)}$ we get

$$
\left|\int_{B_{r}(0)} \psi(u(x)-u(x+h))-\psi(u(z)-u(z+h)) \nu(h) \mathrm{d} h\right| \leq C\|u\|_{C^{\gamma}\left(\mathbb{R}^{d}\right)}^{p-1} \int_{B_{r}(0)}|h|^{\gamma(p-1)} \nu(h) \mathrm{d} h
$$

Summing the previous inequalities yields

$$
|L(x)-L(z)| \leq C\|u\|_{C^{\gamma}\left(\mathbb{R}^{d}\right)}^{p-1}\left[g(r)|x-z|^{\gamma \kappa_{p}}+h(r)\right] .
$$

where $g$ and $h$ are the monotone functions,

$$
g(r)=\int_{B_{r}^{c}(0)} \nu(h) \mathrm{d} h, \quad h(r)=\int_{B_{r}(0)}|h|^{\gamma(p-1)} \nu(h) \mathrm{d} h .
$$

Note that $\rho(r)=\left(\frac{h(r)}{g(r)}\right)^{1 / \gamma \kappa_{p}}$ is increasing and $\rho(r) \xrightarrow{r \rightarrow 0} 0$. Taking $r=\rho^{-1}(r)$ we obtain

$$
|L(x)-L(z)| \leq C\|u\|_{C^{\gamma}\left(\mathbb{R}^{d}\right)}^{p-1} \omega(|x-z|), \quad \omega(|x-z|)=h \circ \rho^{-1}(|x-z|)
$$

The nonlocal normal derivative $\mathcal{N} u$ of a function measurable $u: \mathbb{R}^{d} \rightarrow \mathbb{R}$ can be thought of as the restriction on $\mathbb{R}^{d} \backslash \Omega$ of the regional operator on $\Omega$, namely,

$$
L_{\Omega} u(x)=-\mathcal{N} u(x)=\int_{\Omega} \psi(u(x)-u(y)) \nu(x-y) \mathrm{d} y \quad x \in \mathbb{R}^{d} \backslash \Omega
$$

It might be interesting to know some situations where $\mathcal{N} u(x)$ makes sense at least almost everywhere.
Proposition B.6. Assume $\Omega \subset \mathbb{R}^{d}$ is open. The following assertions are true.
(i) If $\left.u\right|_{\Omega} \in L^{\infty}(\Omega)$ then $\mathcal{N} u(x)$ exists for almost all $x \in \mathbb{R}^{d} \backslash \bar{\Omega}$.
(ii) If $u \in W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ then $\mathcal{N} u \in L_{\mathrm{loc}}^{q}\left(\mathbb{R}^{d} \backslash \bar{\Omega}\right)$ for any $1 \leq q \leq p^{\prime}$.
(iii) If $u \in W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)$ then $\mathcal{N} u \in L^{p^{\prime}}\left(\mathbb{R}^{d}, w^{-1}(x)\right), w(x)=\int_{\Omega} \nu(x-y) \mathrm{d} y$.

Proof. (i) Observing that $\delta_{x}=\operatorname{dist}(x, \partial \Omega)>0$ we get

$$
|\mathcal{N} u(x)| \leq\left(\|u\|_{L^{\infty}(\Omega)}+|u(x)|\right)^{p-1} \int_{|h|>\delta_{x}} \nu(h) \mathrm{d} h<\infty .
$$

(ii) Let $K \subset \mathbb{R}^{d} \backslash \bar{\Omega}$ be compact and put $\delta=\operatorname{dist}(K, \partial \Omega)>0$ so that $|x-y|>\delta$ whenever $x \in K$ and $y \in \Omega$. The Hölder inequality implies

$$
\begin{aligned}
\int_{K}|\mathcal{N} u(x)|^{q} d x & \leq \int_{K}\left(\int_{\Omega}|u(x)-u(y)|^{p-1} \nu(x-y) \mathrm{d} y\right)^{q} \mathrm{~d} x \\
& \leq \int_{K}\left(\int_{\Omega}|u(x)-u(y)|^{p} \nu(x-y) \mathrm{d} y\right)^{\frac{q}{p^{\prime}}}\left(\int_{\Omega} \nu(x-y) \mathrm{d} y\right)^{\frac{q}{p}} d x \\
& \leq|K|^{1-\frac{q}{p^{\prime}}}\left(\int_{|h|>\delta} \nu(h) \mathrm{d} h\right)^{\frac{q}{p}}\left(\int_{K} \int_{\Omega}|u(x)-u(y)|^{p} \nu(x-y) \mathrm{d} y \mathrm{~d} x\right)^{\frac{q}{p^{\prime}}} \\
& \leq|K|^{1-\frac{q}{p^{\prime}}}\left(\int_{|h|>\delta} \nu(h) \mathrm{d} h\right)^{\frac{q}{p}}|u|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)^{q}}^{q(p-1)}
\end{aligned}
$$

(iii) The Hölder inequality implies

$$
\begin{aligned}
\int_{\mathbb{R}^{d}}|\mathcal{N} u(x)|^{p^{\prime}} w(x)^{-1} d x & \leq \int_{\mathbb{R}^{d}}\left(\int_{\Omega}|u(x)-u(y)|^{p-1} \nu(x-y) \mathrm{d} y\right)^{p^{\prime}} \mathrm{d} x \\
& \leq \int_{\mathbb{R}^{d}} \int_{\Omega}|u(x)-u(y)|^{p} \nu(x-y) \mathrm{d} y d x \leq|u|_{W_{\nu}^{p}\left(\Omega \mid \mathbb{R}^{d}\right)}^{p}
\end{aligned}
$$

B.2. Gauss-Green type formula. In this section we establish the nonlocal Gauss-Green type formula associated with the operator $L$. We start with the following general formula.
Theorem B.7. Assume $\Omega \subset \mathbb{R}^{d}$ is open bounded. Let $k: \mathbb{R}^{d} \times \mathbb{R}^{d} \backslash \operatorname{diag} \rightarrow[0, \infty)$ be measurable, anti-symmetric, i.e., $k(x, y)=-k(y, x)$, satisfying

$$
\sup _{x \in \mathbb{R}^{d}} \int_{\mathbb{R}^{d}}|k(x, y)| \mathrm{d} y<\infty
$$

For every $v \in L^{\infty}\left(\mathbb{R}^{d}\right)$ the following identity holds

$$
\frac{1}{2} \iint_{\left(\Omega^{c} \times \Omega^{c}\right)^{c}}(v(x)-v(y)) k(x, y) \mathrm{d} y \mathrm{~d} x=\iint_{\Omega \mathbb{R}^{d}} v(x) k(x, y) \mathrm{d} y \mathrm{~d} x-\iint_{\Omega^{c} \Omega} v(y) k(x, y) \mathrm{d} x \mathrm{~d} y
$$

Proof. First of all, observe that $(x, y) \mapsto v(x) k(x, y)$ belongs to $L^{1}\left(\Omega \times \mathbb{R}^{d}\right)$ since

$$
\iint_{\Omega \mathbb{R}^{d}}\left|v ( x ) \left\|k(x, y)\left|\mathrm{d} y \mathrm{~d} x \leq\|v\|_{L^{1}(\Omega)} \sup _{x \in \mathbb{R}^{d}} \int_{\mathbb{R}^{d}}\right| k(x, y) \mid \mathrm{d} y<\infty .\right.\right.
$$

Thus Fubini's theorem and the anti-symmetry $k(x, y)=-k(y, x)$ imply

$$
2 \iint_{\Omega \Omega} v(x) k(x, y) \mathrm{d} y \mathrm{~d} x=\iint_{\Omega \Omega}(v(x)-v(y)) k(x, y) \mathrm{d} y \mathrm{~d} x
$$

Likewise, it follows that

$$
\begin{aligned}
& 2 \iint_{\Omega \Omega^{c}} v(x) k(x, y) \mathrm{d} y \mathrm{~d} x=2 \iint_{\Omega \Omega^{c}}(v(x)-v(y)) k(x, y) \mathrm{d} y \mathrm{~d} x+2 \iint_{\Omega \Omega^{c}} v(y) k(x, y) \mathrm{d} y \mathrm{~d} x \\
& =\iint_{\Omega_{\Omega^{c}}}(v(x)-v(y)) k(x, y) \mathrm{d} y \mathrm{~d} x+\iint_{\Omega^{c} \Omega}(v(x)-v(y)) k(x, y) \mathrm{d} y \mathrm{~d} x+2 \iint_{\Omega^{c} \Omega} v(y) k(x, y) \mathrm{d} x \mathrm{~d} y
\end{aligned}
$$

Summing up altogether, yields the sought result since $\Omega \times \mathbb{R}^{d}=\Omega \times \Omega \cup \Omega \times \Omega^{c}$.
Next, recall that if $p \geq 2$ then $\widehat{p}_{1}=p$ and hence $L^{1}\left(\mathbb{R}^{d}, 1 \wedge|h|^{\widehat{p}_{1}}\right)=L^{1}\left(\mathbb{R}^{d}, 1 \wedge|h|^{p}\right)$ and if $1<p<2$ then $\widehat{p}_{1}=2(p-1)$ and $L^{1}\left(\mathbb{R}^{d}, 1 \wedge|h|^{\widehat{p}_{1}}\right) \subset L^{1}\left(\mathbb{R}^{d}, 1 \wedge|h|^{p}\right)$.

Theorem B. 8 (Gauss-Green formula). Assume $\Omega \subset \mathbb{R}^{d}$ open bounded. Let $u \in C_{b}^{2}\left(\mathbb{R}^{d}\right)$ and $v \in C_{b}^{1}\left(\mathbb{R}^{d}\right)$. Assume either $(i) p \geq 2$, (ii) $1<p<2$ and $\left(L_{\varepsilon} u\right)_{\varepsilon}$ is uniformly bounded or (iii) $\nu \in L^{1}\left(\mathbb{R}^{d}, 1 \wedge|h|^{\widehat{p}_{1}}\right)$. There holds that

$$
\begin{equation*}
\int_{\Omega} L u(x) v(x) \mathrm{d} x=\mathcal{E}(u, v)-\int_{\Omega^{c}} \mathcal{N} u(y) v(y) \mathrm{d} y \tag{B.6}
\end{equation*}
$$

In particular, for $v=1$, one gets the integration by part formula

$$
\begin{equation*}
\int_{\Omega} L u(x) \mathrm{d} x=-\int_{\Omega^{c}} \mathcal{N} u(y) \mathrm{d} y \tag{B.7}
\end{equation*}
$$

Proof. Since $\nu \in L^{1}\left(\mathbb{R}^{d} \backslash B_{\varepsilon}(0)\right), \varepsilon>0$ and $u, v$ are bounded, Theorem B. 7 implies that

$$
\begin{equation*}
\frac{1}{2} \iint_{\left(\Omega^{c} \times \Omega^{c}\right)^{c}}(v(x)-v(y)) k_{\varepsilon}(x, y) \mathrm{d} y \mathrm{~d} x=\iint_{\Omega \mathbb{R}^{d}} v(x) k_{\varepsilon}(x, y) \mathrm{d} y \mathrm{~d} x-\iint_{\Omega^{c} \Omega} v(y) k_{\varepsilon}(x, y) \mathrm{d} x \mathrm{~d} y \tag{B.8}
\end{equation*}
$$

where $k_{\varepsilon}(x, y)$ is the anti-symmetric kernel defined by

$$
k_{\varepsilon}(x, y)=2|u(x)-u(y)|^{p-2}(u(x)-u(y)) \nu(x-y) \mathbb{1}_{B_{\varepsilon}^{c}}(x-y)
$$

Note that $\widehat{p}_{1}=p, p \geq 2$ and $\widehat{p}_{1}=2(p-1), 1<p<2$ so that $\nu \in L^{1}\left(\mathbb{R}^{d}, 1 \wedge|h|^{\widehat{p}_{1}}\right) \subset L^{1}\left(\mathbb{R}^{d}, 1 \wedge|h|^{p}\right)$. In any case, by Proposition B.2, $\left(L_{\varepsilon} u\right)_{\varepsilon}$ is uniformly bounded. Whence,

$$
\int_{\Omega} L u(x) v(x) \mathrm{d} x=\lim _{\varepsilon \rightarrow 0} \int_{\Omega} L_{\varepsilon} u(x) v(x) \mathrm{d} x=\lim _{\varepsilon \rightarrow 0} \iint_{\Omega \mathbb{R}^{d}} v(x) k_{\varepsilon}(x, y) \mathrm{d} y \mathrm{~d} x
$$

On the other hand, since $\Omega$ is bounded and $u, v \in C_{b}^{1}\left(\mathbb{R}^{d}\right)$ we have

$$
\left|(v(x)-v(y)) k_{\varepsilon}(x, y)\right| \leq C\left(1 \wedge|x-y|^{p}\right) \nu(x-y) \in L^{1}\left(\Omega \times \mathbb{R}^{d}\right)
$$

By the convergence dominated theorem we have

$$
\mathcal{E}(u, v)=\lim _{\varepsilon \rightarrow 0} \frac{1}{2} \iint_{\left(\Omega^{c} \times \Omega^{c}\right)^{c}}(v(x)-v(y)) k_{\varepsilon}(x, y) \mathrm{d} y \mathrm{~d} x .
$$

Necessarily, the desired formula is obtained by letting $\varepsilon \rightarrow 0$ in (B.8) as follows

$$
\begin{aligned}
\int_{\Omega} L u(x) v(x) \mathrm{d} x-\mathcal{E}(u, v) & =\lim _{\varepsilon \rightarrow 0} \iint_{\Omega \mathbb{R}^{d}} v(x) k_{\varepsilon}(x, y) \mathrm{d} y \mathrm{~d} x-\lim _{\varepsilon \rightarrow 0} \frac{1}{2} \iint_{\left(\Omega^{c} \times \Omega^{c}\right)^{c}}(v(x)-v(y)) k_{\varepsilon}(x, y) \mathrm{d} y \mathrm{~d} x \\
& =\lim _{\varepsilon \rightarrow 0} \iint_{\Omega^{c} \Omega} v(y) k_{\varepsilon}(x, y) \mathrm{d} x \mathrm{~d} y=-\int_{\Omega^{c}} \mathcal{N} u(y) v(y) \mathrm{d} y
\end{aligned}
$$

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