# GRADIENT FLOW SOLUTIONS FOR POROUS MEDIUM EQUATIONS WITH NONLOCAL LÉVY-TYPE PRESSURE 

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#### Abstract

We study a porous medium-type equation whose pressure is given by a nonlocal Lévy operator associated to a symmetric jump Lévy kernel. The class of nonlocal operators under consideration appears as a generalization of the classical fractional Laplace operator. For the class of Lévy-operators, we construct weak solutions using a variational minimizing movement scheme. The lack of interpolation techniques is ensued by technical challenges that render our setting more challenging than the one known for fractional operators.


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## 1. Introduction

This paper is dedicated to studying the nonlocal continuity equation

$$
\begin{cases}\partial_{t} u+\operatorname{div}(u \nabla v)=0 & \text { in } \mathbb{R}^{d} \times(0, \infty),  \tag{1.1}\\ L v=u & \text { in } \mathbb{R}^{d} \times(0, \infty), \\ u(0)=u_{0} & \text { in } \mathbb{R}^{d} \times\{0\},\end{cases}
$$

for some initial data $u_{0} \in \mathcal{P}_{2}\left(\mathbb{R}^{d}\right)$, where $\mathcal{P}_{2}\left(\mathbb{R}^{d}\right)$ is space of probability measures with finite second moment where $u=u(x, t)$ denotes the density at a point $x \in \mathbb{R}^{d}$ at time $t>0$. Here, the pressure, $v=v(x, t)$, is coupled to the density via a linear, nonlocal, pseudo-differential operator

$$
\begin{equation*}
L v=u . \tag{1.2}
\end{equation*}
$$

This includes the fractional Laplace operator, $L=(-\Delta)^{s}$, which has been receiving a lot of attention in the last decade. Yet, in all generality, little attention has been paid to the study of more general pseudo-differential operators. In this paper, we want to address this dearth in the literature and assume the operator $L$ be a symmetric integrodifferential operator of Lévy type, i.e.,

$$
\begin{equation*}
L u(x):=2 \text { p.v. } \int_{\mathbb{R}^{d}}(u(x)-u(y)) \nu(x-y) \mathrm{d} y, \tag{1.3}
\end{equation*}
$$

[^0]where the kernel $\nu \geq 0$ is assumed to be the Lebesgue density of a symmetric Lévy measure, $\nu$, defined by
\[

$$
\begin{equation*}
\nu(h)=\nu(-h), \quad \text { and } \quad \int_{\mathbb{R}^{d}}\left(1 \wedge|h|^{2}\right) \nu(h) \mathrm{d} h<\infty . \tag{L}
\end{equation*}
$$

\]

Lévy operators as in Eq. (1.3) arise naturally in probability theory as the generator of Lévy processes with pure jumps, whose jumps interactions are regulated by the measure $\nu(h) \mathrm{d} h, c f$. [Sat13, App09, Ber96] for more details on Lévy processes. Recently, the study of nonlocal problems driven by Lévy operators has gained increasing popularity and we refer to [FK22, DFK22, BE23, Fog23b, Rut18, FG20], and references therein, for very recent results in the study of problems involving pseudodifferential operators of Lévy type.
Allowing for Lévy operators in Eq. (1.2) comes at the price of losing interpolation inequalities which used to play a crucial role in known results. In this work, we remedy the lack of interpolation techniques by obtaining surrogate estimates from the energy and its associated energy-dissipation functional as well as the study of fine properties of the symbol of the nonlocal operator $L$. Then, in order to construct solutions to Eq. (1.1), we employ a minimizing movement scheme à la De Giorgi [DG93].
In fact, in the last two decades, this topic has experienced a renaissance due to the intimate link between the continuity equation,

$$
\begin{equation*}
\partial_{t} u+\operatorname{div}(u \mathbf{v})=0, \tag{1.4a}
\end{equation*}
$$

governing the evolution of the density, $u$, and absolutely continuous curves in the set of probability measures, [AGS08].. At least formally, when

$$
\begin{equation*}
\mathbf{v}=-\nabla \frac{\delta \mathcal{E}}{\delta u} \tag{1.4b}
\end{equation*}
$$

solutions to the continuity equation can be understood as gradient flows in the set of probability measures, [AGS08, JKO98, Ott01]. In the subsequent discussion, three choices of functionals, $\mathcal{E} \in$ $\{\mathcal{F}, \mathcal{G}, \mathcal{H}\}$, will play a prominent role:

$$
\mathcal{G}_{m}(u):=\frac{1}{m-1} \int_{\mathbb{R}^{d}} u^{m} \mathrm{~d} x
$$

which gives rise to the porous medium equation [Váz07, Ott01], as well as the entropy

$$
\mathcal{H}(u):=\int_{\mathbb{R}^{d}} u \log u \mathrm{~d} x
$$

which gives rise to the linear diffusion equation [Ott01,JKO98], and the nonlocal interaction energy

$$
\begin{equation*}
\mathcal{F}(u):=\frac{1}{2} \int u K * u \mathrm{~d} x, \tag{1.5}
\end{equation*}
$$

for some kernel $K$. In our work, formally the kernel $K$ can be related to a symmetric Lévy measure, $\nu$, mentioned above, denoted by $K_{\nu}$. In the case of the inverse fractional Laplacian, $K_{\nu}$, coincides with the Riesz kernel (for instance, see [Ste70]) and the resulting equation reads

$$
\partial_{t} u=\nabla \cdot(u \nabla p)
$$

with $p=K_{\nu} * u=(-\Delta)^{-s} u$, and is referred to as porous medium equation with fractional pressure in the literature. It has been been proposed and studied in [CV11] as a generalization of the classical porous medium equation, where the pressure-density closure relation is local and given by $p=u^{\gamma}$, for some $\gamma \geq 1$. More precisely, the existence of weak solutions in the sense

$$
\int_{0}^{T} \int_{\Omega} u\left(\partial_{t} \varphi-\nabla \varphi \cdot \nabla K_{s} * u\right) \mathrm{d} x \mathrm{~d} t+\int_{\Omega} \varphi(x, 0) u_{0}(x) \mathrm{d} x=0
$$

where $K_{s}$ is the Riesz kernel. The argument hinges on a sophisticated approximation argument that consists of adding small viscosity, mollifying the Riesz kernel, and removing the degeneracy in the mobility. Subsequently, the authors show Hölder-regularity and boundedness of weak solutions in [CSV13] and extended the existence result to a wider class of initial data [SV14]. The limit $s \rightarrow 1$, in which case $K$ becomes the Newtonian potential, has been considered in [LZ00, SV14]. The other limit case, $s \rightarrow 0$, corresponding to the local porous medium equation as the limit of a porous medium equation with fractional pressure has been established in [LMS18]. We also refer to [LMG01, BE22, HDPP23] to nonlocal approximation of the porous medium equation with exponent two, and [CEW23] for arbitrary exponent. In these works, however, the nonlocal approximation is smooth and integrable unlike the Riesz kernel. Similarly, [DDMS23, ES23], consider a system of porous medium type with the pseudodifferential operator is $K_{s}=(\mathrm{id}-s \Delta)^{-1}$. In this vein, it worth mentioning that the works [Fog23b, FK22, FG20, FKV20, Voi17] discuss the nonlocal-to-local limit for 'elliptic' problems in the context of Lévy kernels.
Returning to Eq. (1.1), let us recall that the operator $L$ is a pseudo-differential operator, and we can associate the a symbol to it, which is given by

$$
\psi(\xi)=2 \int_{\mathbb{R}^{d}}(1-\cos (\xi \cdot h)) \nu(h) \mathrm{d} h,
$$

that is, in other words,

$$
\widehat{L u}(\xi)=\psi(\xi) \widehat{u}(\xi),
$$

for any $\xi \in \mathbb{R}^{d}$, and any Schwartz function, $u \in \mathcal{S}\left(\mathbb{R}^{d}\right)$. Here, we have used the common notation, $\widehat{u}$, to denote the Fourier transform of $u$, i.e.,

$$
\widehat{u}(\xi)=\frac{1}{(2 \pi)^{d / 2}} \int_{\mathbb{R}^{d}} e^{-i \xi \cdot x} u(x) \mathrm{d} x
$$

for any $\xi \in \mathbb{R}^{d}$. Therefore, formally, we may choose $v=L^{-1} u=K_{\nu} * u$, where $K_{\nu}$ is the potential (defining a tempered distribution) whose Fourier transform is given by $\widehat{K_{\nu}}=\psi^{-1}$, motivating the interest in the nonlocal energy functional, Eq. (1.5). In light of this formal link, we can cast Eq. (1.1) into the form of a formal 2-Wasserstein gradient flow, cf. Eq. (1.4), for the nonlocal energy given in Eq. (1.5), i.e.,

$$
\partial_{t} u=\operatorname{div}\left(u \nabla \frac{\delta \mathcal{F}}{\delta u}\right) .
$$

1.1. Relation to Fractional Laplacian. The epitome of a Lévy operator is obtained by setting

$$
\nu(h)=\frac{C_{d, s}}{2}|h|^{-d-2 s},
$$

for $s \in(0,1)$ and operator resulting from this choice is the well-known fractional Laplace operator,

$$
\begin{equation*}
(-\Delta)^{s} u(x)=C_{d, s} \text { p.v. } \int_{\mathbb{R}^{d}} \frac{(u(x)-u(y))}{|x-y|^{d+2 s}} \mathrm{~d} y, \tag{1.6}
\end{equation*}
$$

for any $x \in \mathbb{R}^{d}$, one of the most widely studied integrodifferential operators, see, for instance, [BV16, CS07, Kwa17, FG20, CLM20, Gar19]. Here, the constant $C_{d, s}$ guaranties the relation

$$
\widehat{(-\Delta)^{s}} u(\xi)=|\xi|^{2 s} \widehat{u}(\xi)
$$

for all $u$ in $C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$, and it can be shown that the it is given by

$$
C_{d, s}=\frac{2^{2 s} \Gamma\left(\frac{d+2 s}{2}\right)}{\pi^{d / 2}|\Gamma(-s)|}=\frac{2^{2 s} s \Gamma\left(\frac{d+2 s}{2}\right)}{\pi^{d / 2} \Gamma(1-s)},
$$

see [FG20, Buc16, ST10, AS61]. Interestingly, the constant $C_{d,-s}$ guarantees a similar relation for the inverse of the fractional Laplacian $(-\Delta)^{-s}$, also known for the general range $s \in(0, d / 2)$ as the Riesz potential. Namely, for $s \in(0, d / 2)$, we have

$$
\begin{equation*}
\left(\widehat{\Delta)^{-s} u}(\xi)=|\xi|^{-2 s} \widehat{u}(\xi)\right. \tag{1.7}
\end{equation*}
$$

in the distributional sense for all $u \in \mathcal{S}\left(\mathbb{R}^{d}\right)$ if and only if

$$
\begin{equation*}
(-\Delta)^{-s} u(x)=K_{s} * u(x)=C_{d,-s} \int_{\mathbb{R}^{d}} \frac{u(y)}{|x-y|^{d-2 s}} \mathrm{~d} y, \tag{1.8}
\end{equation*}
$$

for any $x \in \mathbb{R}^{d}$, see, for instance, [Rie38, Ste70, Put04, AS61]. Here $K_{s}(x)=C_{d,-s}|x|^{2 s-d}$ is the Riesz kernel. Let us foreshadow that, below, we shall provide further, non-trivial, examples of Lévy operators to which our existence results apply.
1.2. Our contribution. In this article, we focus on problems of the form (1.1), for radial Lévy kernels, $\nu$. Our interest in this equation was sparked by the recent work [LMS18], in which the special case of Eq. (1.1), with $L=(-\Delta)^{s}$, referred to as 'porous medium equation with fractional pressure' was investigated. Upon casting the problem into a gradient-flow setting, the authors prove an existence result of weak solutions and furnish decay estimates for $L^{p}$-norms of solutions to the evolution problem, Eq. (1.1), which was first studied in [SDTV19] for the general porous medium equation with fractional pressure. Following the strategy of [LMS18], we construct weak solutions to Eq. (1.1) employing the minimizing movement scheme

$$
u_{\tau}^{k}=\underset{u \in \mathcal{P}_{2}\left(\mathbb{R}^{d}\right)}{\operatorname{argmin}}\left\{\frac{1}{2 \tau} W^{2}\left(u, u_{\tau}^{k-1}\right)+\mathcal{F}(u)\right\},
$$

see Definition 3.3, below. In our work, we prove existence of solutions for a much wider class of Lévy kernels which comes at the price of losing the homogeneity of the Riesz kernel associated to the fractional Laplacian. Mainly, there are two main difficulties. The first one lies in the derivation of the Euler-Lagrange equations which is significantly more challenging for more general kernels. The second challenge is to obtain the appropriate compactness of the sequence obtained from the minimizing movement scheme which is needed to identify its limit as a weak solution. Specifically, interpolation inequalities break down in the nonlocal energy spaces that act as surrogates of the usual fractional Sobolev spaces. A drawback of our new methodology is that we are currently unable to establish decay estimates similar to those in [LMS18]. Indeed, [LMS18] makes heavy use of interpolation inequalities and Sobolev inequalities. We point out that a Sobolev-type inequality associated to Lévy kernels was established in [Fog21]. Therein, the Sobolev exponent is obtained in terms of a Young function, giving rise to embeddings into Orlicz-type spaces. However, the resulting Young function lacks sufficient regularity to use it as a generalized entropy. Such inequalities for Lévy kernels are interesting in their own right, would have several other applications, and they are the object of ongoing investigations. In particular, for special (not necessarily Riesz-type) kernels, the resulting Young function may have sufficient regularity already. This is kept as another future avenue to explore.
1.3. Main result and additional comment. Before stating the main result of this paper, let us first briefly introduce some necessary notation. Throughout, we consider spaces

$$
\dot{H}^{\phi}\left(\mathbb{R}^{d}\right)=\left\{u \in \mathcal{S}^{\prime}\left(\mathbb{R}^{d}\right): \widehat{u} \phi^{1 / 2} \in L^{2}\left(\mathbb{R}^{d}\right)\right\},
$$

and

$$
H^{\phi}\left(\mathbb{R}^{d}\right)=\dot{H}^{\phi}\left(\mathbb{R}^{d}\right) \cap L^{2}\left(\mathbb{R}^{d}\right),
$$

which act as generalizations of the usual (homogeneous) fractional Sobolev spaces. Here, $\phi$ : $\mathbb{R}^{d} \backslash\{0\} \rightarrow(0, \infty)$ is a symmetric and continuous function which we refer to as 'symbol', and we point the reader to Section 2, for further details on their construction. Specifically, the special
symbol (or regularizing symbol) $\widetilde{\psi}(\xi)=|\xi|^{2} \psi^{-1}(\xi)$ will play a prominent role as it determines the regularity of the weak solution to Eq. (1.1). We shall assume on the symbol that there exists a constant $c_{\nu}>0$ such that, for all $\xi \in \mathbb{R}^{d}$ there holds the lower-bound condition

$$
\psi(\xi) \geq c_{\nu}\left(1 \wedge|\xi|^{2}\right)
$$

which is required to establish the convergence of the velocity, point (iii), in the main theorem, Theorem 1.1. We show in Theorem 2.3 that Condition $\left(C_{\nu}\right)$ holds whenever $\nu$ is unimodal, radial, and nontrivial. Here, we call a Lévy kernel $\nu$ unimodal if there is constant $c>0$ such that

$$
\begin{equation*}
\nu(x) \leq c \nu(y), \quad \text { whenever }|x| \geq|y| \tag{1.9}
\end{equation*}
$$

Having introduced all necessary notations, we can now present the main result of this paper.
Theorem 1.1. Assume $u_{0} \in \dot{H}^{\psi^{-1}}\left(\mathbb{R}^{d}\right) \cap \mathcal{P}_{2}\left(\mathbb{R}^{d}\right)$. Consider the special symbol $\widetilde{\psi}(\xi)=|\xi|^{2} \psi^{-1}(\xi)$. The following hold.
(i) Existence of discrete solutions. There is a unique sequence $\left(u_{\tau}^{k}\right)_{k}$ with $u_{\tau}^{k} \in \dot{H}^{\tilde{\psi}}\left(\mathbb{R}^{d}\right) \cap$ $\dot{H}^{\psi^{-1}}\left(\mathbb{R}^{d}\right) \cap \mathcal{P}_{2}\left(\mathbb{R}^{d}\right)$ satisfying the minimization problem (3.1).
(ii) Convergence and regularity. Define the discrete curve $u_{\tau}(t)=u_{\tau}^{\lceil t / \tau\rceil}$ with $u_{\tau}(0)=u_{\tau}^{0}$. There exists a curve $u \in A C^{2}\left([0, \infty),\left(\mathcal{P}_{2}\left(\mathbb{R}^{d}\right), W\right)\right)$ and a subsequence $\left(\tau_{n}\right)_{n}$ tending to 0 , such that

$$
u_{\tau_{n}}(t) \rightarrow u(t) \text { narrowly as } n \rightarrow \infty \text { for each } t \in \mathbb{R}
$$

Let $\mathcal{H}$ be the standard Boltzmann entropy. If $u_{0} \in D(\mathcal{H})$ then

$$
u_{\tau_{n}}(t) \rightharpoonup u(t) \text { weakly in } L^{2}\left((0, T), H^{\tilde{\psi}}\left(\mathbb{R}^{d}\right)\right) \text { as } n \rightarrow \infty
$$

In addition, if we assume that

$$
\begin{equation*}
\sup _{\xi \in \mathbb{R}^{d}} \frac{1}{\widetilde{\psi}(\xi)}\left|e^{i \xi \cdot h}-1\right|^{2}=\sup _{\xi \in \mathbb{R}^{d}} \frac{\psi(\xi)}{|\xi|^{2}}\left|e^{i \xi \cdot h}-1\right|^{2} \xrightarrow{|h| \rightarrow 0} 0 \tag{1.10}
\end{equation*}
$$

then we have

$$
u_{\tau_{n}}(t) \rightarrow u(t) \text { strongly in } L^{2}\left((0, T), L_{\mathrm{loc}}^{2}\left(\mathbb{R}^{d}\right)\right) \text { as } n \rightarrow \infty
$$

(iii) Convergence of the velocity Define $v_{\tau}(t)=L^{-1} u_{\tau}(t)$ and $v(t)=L^{-1} u(t)$. Assume that condition $\left(C_{\nu}\right)$ holds and that $u_{0} \in D(\mathcal{H})$. Then we have $\nabla v \in L^{2}\left((0, T), L^{2}\left(\mathbb{R}^{d}\right)\right)$, and

$$
\nabla v_{\tau_{n}}(t) \rightharpoonup \nabla v(t) \text { weakly in } L^{2}\left((0, T), L^{2}\left(\mathbb{R}^{d}\right)\right) \text { as } n \rightarrow \infty
$$

(iv) Solution of the limiting equation. Assume that $\nu \notin L^{1}\left(\mathbb{R}^{d}\right)$ and satisfies the condition $\left(C_{\nu}\right)$. Assume also that $u_{0} \in D(\mathcal{H})$. Moreover, assume the symbol $\widetilde{\psi}$ is associated with a unimodal Lévy kernel $\widetilde{\nu}$ satisfying the following condition

For any $0<\lambda<1$ there is $c_{\lambda}>0$ such that $\widetilde{\nu}(\lambda h) \leq c_{\lambda} \widetilde{\nu}(h)$, whenever $|h| \leq 1$.
Then the limiting curve $u$ is a weak solution of the Eq. (1.1), viz., the functions $u$, $v$ defined by items (ii) and (iii) satisfy

$$
\begin{equation*}
\int_{0}^{\infty} \int_{\mathbb{R}^{d}}\left(\partial_{t} \varphi-\nabla \varphi \cdot \nabla v\right) u \mathrm{~d} x \mathrm{~d} t=0, \quad \forall \varphi \in C_{c}^{\infty}\left(\mathbb{R}^{d} \times(0, \infty)\right) \tag{1.12}
\end{equation*}
$$

(v) Energy dissipation inequality. Let $u$ satisfy the weak formulation from point (iv), there holds

$$
\mathcal{F}(u(T))+\int_{0}^{T} \int_{\mathbb{R}^{d}} u(t)|\nabla v(t)|^{2} \mathrm{~d} x \mathrm{~d} t \leq \mathcal{F}\left(u_{0}\right)
$$

where $\mathcal{F}(u)=\frac{1}{2}\|u\|_{\dot{H}^{\psi^{-1}}\left(\mathbb{R}^{d}\right)}^{2}$.
(vi) Entropy and $L^{p}$ Boundedness. If $u_{0} \in D(\mathcal{H})$ then

$$
\mathcal{H}(u(t)) \leq \mathcal{H}\left(u_{0}\right)
$$

If in addition, $\widetilde{\psi}$ is the symbol associated with a Lévy-integrable kernel $\widetilde{\nu}$ then for $1<p<\infty$ and $u_{0} \in L^{p}\left(\mathbb{R}^{d}\right.$ we have

$$
\|u(t)\|_{L^{p}\left(\mathbb{R}^{d}\right)} \leq\left\|u_{0}\right\|_{L^{p}\left(\mathbb{R}^{d}\right)}
$$

For the readers' convenience, we give a short overview of where to find the individual statements. Item (i) follows from Theorem 3.2, which establishes the lower semi-continuity of the Yosidapenalization. The narrow convergence in (ii) is a consequence of Theorem 3.6, and the weak and strong convergence is proven in Theorem 5.2. In Theorem 6.2, we identify the limit curve as a weak solution, thereby establishing point $(i v)$. The dissipation of the energy, $(v)$, is proven in Theorem 6.5 , and finally, the control of convex entropies, (vi), is established in Theorem 4.10 and Theorem 4.11.

The rest of this paper is organized as follows. In Section 2, we introduce fundamental results on Lévy operators, different formulations thereof, and their link to stochastic jump processes. Furthermore, we introduce energy spaces associated to this class of kernels and discuss compactness criteria in these spaces. We conclude Section 2, with a look at these kernels through the Fourier lens, which will play a crucial ingredient in the subsequent analysis. Section 3 is dedicated to introducing the variational framework in the set of probability measures seminally introduced by Jordan, Kinderlehrer \& Otto [JKO98]. Using this minimizing movement scheme, we will construct a sequence of probability measures and show its narrow compactness. The limit curve is a candidate of a weak solution to our main Eq. (1.1). In order to identify the limit curve as a weak solution, we employ the flow-interchange technique à la Matthes, McCann \& Savaré [MMS09] in Section 4. Using the Boltzmann-Shannon entropy and $L^{p}$-norms as auxiliary functionals, we obtain additional regularity, which we exploit in Section 5, to obtain convergence in better spaces. Indeed, this is sufficient to pass to the limit in the Euler-Lagrange equations derived in Section 6, which concludes the existence result.

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## 2. Symmetric Lévy operators and nonlocal function spaces

This section is dedicated to acquainting the reader with certain fundamental properties of symmetric Lévy operators, $L$, and the nonlocal Sobolev-type space associated to them. We refer the reader to [FG20,Fog23c, Fog23b], which contain a comprehensive summary of recent findings concerning these spaces. Moreover, Gagliardo-Nirenberg-Sobolev-type inequalities were recently obtained in [Fog21] and the notion of nonlocal trace spaces are discussed in [FK22, Fog23b].
2.1. Lévy operator. There are several possible ways to characterize a Lévy operator. Here we point out the most common ones and refer to [FG20] for further characterizations.

Second order difference. First of all, the change of variables, $y=x \pm h$, in (1.3), yields

$$
L u(x)=2 \text { p.v. } \int_{\mathbb{R}^{d}}(u(x)-u(x \pm h)) \nu(h) \mathrm{d} h,
$$

which, upon summing up the the two expressions, yields

$$
\begin{equation*}
L u(x)=\int_{\mathbb{R}^{d}}(2 u(x)-u(x+h)-u(x-h)) \nu(h) \mathrm{d} h . \tag{2.1}
\end{equation*}
$$

It is worth noting that the expression on the right-hand side of (2.1) is well-defined if $u \in L^{\infty}\left(\mathbb{R}^{d}\right) \cap$ $C^{2}\left(B_{\delta}(x)\right)$ for some $\delta>0$. In this case, the principal value may be dropped in the definition.
Pseudo-differential operator. Next, we show that the integrodifferential operator $L$ can be realized as a pseudo-differential operator, as foreshadowed in the introduction, indeed its definition using the Fourier symbol can be justified rigorously.
Theorem 2.1. For $u \in \mathcal{S}\left(\mathbb{R}^{d}\right)$ and $\xi \in \mathbb{R}^{d}$ the following relation holds:

$$
\widehat{L u}(\xi)=\psi(\xi) \widehat{u}(\xi) .
$$

Here $\psi$ is the Fourier symbol of $L$, which is given by

$$
\psi(\xi)=2 \int_{\mathbb{R}^{d}}(1-\cos (\xi \cdot h)) \nu(h) \mathrm{d} h .
$$

Proof. Observe that for each $h \in \mathbb{R}^{d}$, we have

$$
\begin{aligned}
\int_{\mathbb{R}^{d}}|u(x+h)+u(x-h)-2 u(x)| \mathrm{d} x & =\int_{\mathbb{R}^{d}}\left|\int_{0}^{1} \int_{0}^{1} 2 t\left[D^{2} u(x-t h+2 s t h) \cdot h\right] \cdot h \mathrm{~d} s \mathrm{~d} t\right| \mathrm{d} x \\
& \leq|h|^{2} \int_{\mathbb{R}^{d}}\left|D^{2} u(x)\right| \mathrm{d} x,
\end{aligned}
$$

proving that the integral is finite. On the other hand, we have

$$
\int_{\mathbb{R}^{d}}|u(x+h)+u(x-h)-2 u(x)| \mathrm{d} x \leq 4 \int_{\mathbb{R}^{d}}|u(x)| \mathrm{d} x<\infty .
$$

Combining the two preceding estimates, we readily find that

$$
\int_{\mathbb{R}^{d}}|u(x+h)+u(x-h)-2 u(x)| \mathrm{d} x \leq C\left(1 \wedge|h|^{2}\right),
$$

with $C=4\|u\|_{L^{1}\left(\mathbb{R}^{d}\right)}+\left\|\left|D^{2} u\right|\right\|_{L^{1}\left(\mathbb{R}^{d}\right)}$. Setting

$$
\Lambda(x, h)=|u(x+h)+u(x-h)-2 u(x)| \nu(h),
$$

we note that $\Lambda \in L^{1}\left(\mathbb{R}^{d} \times \mathbb{R}^{d}\right)$ since

$$
\iint_{\mathbb{R}^{d} \mathbb{R}^{d}} \Lambda(x, h) \mathrm{d} x \nu(h) \mathrm{d} h \leq C \int_{\mathbb{R}^{d}}\left(1 \wedge|h|^{2}\right) \nu(h) \mathrm{d} h .
$$

Therefore, using the identity $\widehat{u(\cdot+h})(\xi)=\widehat{u}(\xi) e^{i \xi \cdot h}$, along with Fubini's Theorem we get the desired result as follows

$$
\begin{aligned}
\widehat{L u}(\xi) & =-\int_{\mathbb{R}^{d}} e^{-i \xi \cdot x} \int_{\mathbb{R}^{d}}(u(x+h)+u(x-h)-2 u(x)) \nu(h) \mathrm{d} h \mathrm{~d} x \\
& =-\int_{\mathbb{R}^{d}} \nu(h) \int_{\mathbb{R}^{d}} e^{-i \xi \cdot x}(u(x+h)+u(x-h)-2 u(x)) \mathrm{d} x \mathrm{~d} h \\
& =-\widehat{u}(\xi) \int_{\mathbb{R}^{d}}\left(e^{i \xi \cdot h}+e^{-i \xi \cdot h}-2\right) \nu(h) \mathrm{d} h \\
& =2 \widehat{u}(\xi) \int_{\mathbb{R}^{d}}(1-\cos (\xi \cdot h)) \nu(h) \mathrm{d} h=\widehat{u}(\xi) \psi(\xi),
\end{aligned}
$$

which concludes the proof.

Throughout this work, we shall write, by an abuse of notation, $v=L^{-1} u$ to refer to $\hat{v}(\xi)=$ $\psi^{-1}(\xi) \hat{u}(\xi)$.

Proposition 2.2 (Upper bound on the symbol). There exists a constant, $C>0$ such that

$$
\begin{equation*}
\psi(\xi) \leq 2 \int_{\mathbb{R}^{d}}\left(1 \wedge|\xi|^{2}|h|^{2}\right) \nu(h) \mathrm{d} h \leq C\left(1+|\xi|^{2}\right), \tag{2.2}
\end{equation*}
$$

for any $\xi \in \mathbb{R}^{d}$. The constant can be chosen as $C=\kappa_{\nu} k$ where

$$
\kappa_{\nu}:=2\|\nu\|_{L^{1}\left(\mathbb{R}^{d}, 1 \wedge|h|^{2}\right)}
$$

Proof. From the elementary inequality $|\sin t| \leq 1 \wedge|t|$, for all $t \in \mathbb{R}$, we readily obtain

$$
|1-\cos (\xi \cdot h)|=\left|2 \sin ^{2} \frac{\xi \cdot h}{2}\right| \leq 2 \wedge \frac{1}{2}|\xi|^{2}|h|^{2} \leq 2\left(1 \wedge|\xi|^{2}|h|^{2}\right) .
$$

The desired estimates follow from the fact that $\left(1 \wedge|\xi|^{2}|h|^{2}\right) \leq\left(1+|\xi|^{2}\right)\left(1 \wedge|h|^{2}\right)$.
In the case of a radial Lévy kernel, a lower bound counterpart for the symbol can be obtained.
Theorem 2.3 (Lower bound on the symbol). Assume $\nu$ is radial. Then there exists a constant, $c>0$, such that

$$
c \int_{\mathbb{R}^{d}}\left(1 \wedge|h|^{2}|\xi|^{2}\right) \nu(h) \mathrm{d} h \leq \psi(\xi) \leq 2 \int_{\mathbb{R}^{d}}\left(1 \wedge|h|^{2}|\xi|^{2}\right) \nu(h) \mathrm{d} h,
$$

for all $\xi \in \mathbb{R}^{d}$. Moreover, using $\kappa_{\nu}=2\|\nu\|_{L^{1}\left(\mathbb{R}^{d}, 1 \wedge|h|^{2}\right)}$, we have

$$
\begin{equation*}
\frac{c \kappa_{\nu}}{2}\left(1 \wedge|\xi|^{2}\right) \leq \psi(\xi) \leq \kappa_{\nu}\left(1+|\xi|^{2}\right) \tag{2.3}
\end{equation*}
$$

for all $\xi \in \mathbb{R}^{d}$.
Proof. The upper bounds follow from Proposition 2.2. We only prove the lower bound. The Riemann-Lebesgue lemma implies

$$
\int_{\mathbb{S}^{d-1}}(1-\cos (\xi \cdot w)) \mathrm{d} \sigma_{d-1}(w)=\left|\mathbb{S}^{d-1}\right|-\int_{\mathbb{S}^{d-1}} \cos \left(|\xi| w_{1}\right) \mathrm{d} \sigma_{d-1}(w) \xrightarrow{|\xi| \rightarrow \infty}\left|\mathbb{S}^{d-1}\right|
$$

Using this fact in conjunction with the estimate

$$
1-\cos (t)=2 \sin ^{2}\left(\frac{t}{2}\right) \geq \frac{2 t^{2}}{\pi^{2}}
$$

for any $0 \leq t \leq \frac{\pi}{2}$, we can find a constant $c>0$ such that

$$
\int_{\mathbb{S}^{d-1}}(1-\cos (\xi \cdot w)) \mathrm{d} \sigma_{d-1}(w) \geq c\left|\mathbb{S}^{d-1}\right|\left(1 \wedge|\xi|^{2}\right)
$$

for all $\xi \in \mathbb{R}^{d}$. Switching to polar coordinates and using the above estimate we get

$$
\begin{aligned}
\psi(\xi) & =\int_{\mathbb{R}^{d}}(1-\cos (\xi \cdot h)) \nu(h) \mathrm{d} h \\
& =\int_{0}^{\infty} r^{d-1} \nu(r) \int_{\mathbb{S}^{d-1}}(1-\cos (\xi \cdot r w)) \mathrm{d} \sigma_{d-1}(w) \mathrm{d} r \\
& \geq c\left|\mathbb{S}^{d-1}\right| \int_{0}^{\infty}\left(1 \wedge|\xi|^{2} r^{2}\right) r^{d-1} \nu(r) \mathrm{d} r \\
& =c \int_{\mathbb{R}^{d}}\left(1 \wedge|\xi|^{2}|h|^{2}\right) \nu(h) \mathrm{d} h .
\end{aligned}
$$

Furthermore, since $(1 \wedge a)(1 \wedge b) \leq 1 \wedge a b$, for any $a, b>0$, we get

$$
\psi(\xi) \geq c \int_{\mathbb{R}^{d}}\left(1 \wedge|\xi|^{2}|h|^{2}\right) \nu(h) \mathrm{d} h \geq \frac{c \kappa_{\nu}}{2}\left(1 \wedge|\xi|^{2}\right)
$$

with $\kappa_{\nu}$ defined as above.
Remark 2.4 (Comparable growth). The symbols $\psi$ and $\widetilde{\psi}(\xi):=|\xi|^{2} \psi^{-1}(\xi)$ have a similar growth. Indeed it is not difficult to check that there exists $c_{1}, c_{2}>0$ such that

$$
c_{1}\left(1 \wedge|\xi|^{2}\right) \leq \psi(\xi) \leq c_{2}\left(1+|\xi|^{2}\right),
$$

for all $\xi \in \mathbb{R}^{d}$, if and only if there exists $c_{3}, c_{4}>0$ such that

$$
c_{3}\left(1 \wedge|\xi|^{2}\right) \leq \widetilde{\psi}(\xi) \leq c_{4}\left(1+|\xi|^{2}\right)
$$

for all $\xi \in \mathbb{R}^{d}$
Remark 2.5. Assume that $\nu$ is radial, in which case we write $\nu(h)=\nu(|h|)$, then $\psi$ is also a radial function. Indeed, by the rotation invariance of the Lebesgue measure we get that

$$
\begin{align*}
\psi(\xi) & =2 \int_{\mathbb{R}^{d}}(1-\cos (\xi \cdot h)) \nu(|h|) \mathrm{d} h \\
& =2 \int_{\mathbb{R}^{d}}\left(1-\cos \left(|\xi| e_{1} \cdot h^{\prime}\right)\right) \nu\left(\left|h^{\prime}\right|\right) \mathrm{d} h^{\prime} \\
& =2 \int_{\mathbb{R}^{d}}\left(1-\cos \left(h_{1}\right) \nu(h /|\xi|) \frac{\mathrm{d} h}{|\xi|^{d}} \quad(\xi \neq 0)\right.  \tag{2.4}\\
& =\psi\left(|\xi| e_{1}\right) .
\end{align*}
$$

In particular, if $\nu(h)=\frac{1}{2} C_{d, s}|h|^{-d-2 s}, s \in(0,1)$ then $\psi(\xi)=|\xi|^{2 s}$.
Generator of a symmetric Lévy process and of a semigroup. According to Bochner's Theorem for the Fourier transform (see [BF75]), for each $t \geq 0$, there exists a function $p_{t} \geq 0$ continuous on $\mathbb{R}^{d} \backslash\{0\}$ such that

$$
\widehat{p_{t}}(\xi)=\frac{1}{(2 \pi)^{d / 2}} \int_{\mathbb{R}^{d}} e^{-i \xi \cdot x} p_{t}(x) \mathrm{d} x=e^{-t \psi(\xi)},
$$

for any $\xi \in \mathbb{R}^{d}$. The convolution rule implies that $\widehat{p_{t+s}}=e^{-(t+s) \psi}=\widehat{p_{t}} \widehat{p_{s}}=\widehat{p_{t} * p_{s}}$, whence, we have $p_{t+s}=p_{t} * p_{s}=p_{s} * p_{t}$, for all $t, s \geq 0$. Therefore, the family of operators $\left(P_{t}\right)_{t}$ defined by

$$
P_{t} u(x)=u * p_{t}(x)=\int_{\mathbb{R}^{d}} u(y) p_{t}(x-y) \mathrm{d} y,
$$

with $x \in \mathbb{R}^{d}$, is a strongly continuous semigroup on $L^{2}\left(\mathbb{R}^{d}\right)$, i.e., $P_{t+s}=P_{t} \circ P_{s}=P_{s} \circ P_{t}$, for all $s, t \geq 0$, and $\left\|P_{t} u-u\right\|_{L^{2}\left(\mathbb{R}^{d}\right)} \xrightarrow{t \rightarrow 0^{+}} 0$. As we shall see next, it turns out that the generator of semigroup $\left(P_{t}\right)_{t}$ is the operator $-L$. To this end, let $u \in \mathcal{S}\left(\mathbb{R}^{d}\right)$, whence $\widehat{L u}(\xi)=\widehat{u}(\xi) \psi(\xi) \in L^{2}\left(\mathbb{R}^{d}\right)$, by (2.2). The Plancherel Theorem implies,

$$
\left\|\frac{P_{t} u-u}{t}-(-L u)\right\|_{L^{2}\left(\mathbb{R}^{d}\right)}=\left\|\frac{\widehat{p_{t}} \widehat{u}-\widehat{u}}{t}+\widehat{u} \psi\right\|_{L^{2}\left(\mathbb{R}^{d}\right)}=\left\|\widehat{u} \psi \frac{e^{-t \psi}-1+t \psi}{t \psi}\right\|_{L^{2}\left(\mathbb{R}^{d}\right)},
$$

having used the definition of $p_{t}$. Finally, we observe that the rightmost term goes to zero, as $t \rightarrow 0$, due to the fact that the function $\zeta: s \mapsto \frac{e^{-s}-1+s}{s}$ with $\zeta(0)=0$ is continuous and bounded on $[0, \infty)$. Indeed, an application of the Dominated Convergence Theorem suffices to establish

$$
\left\|\widehat{u} \psi \frac{e^{-t \psi}-1+t \psi}{t \psi}\right\|_{L^{2}\left(\mathbb{R}^{d}\right)} \xrightarrow{t \rightarrow 0} 0 .
$$

The Kolmogorov Extension Theorem (see [Sat13]) implies the existence of a stochastic process $\left(X_{t}\right)_{t}$ with the transition density is $p_{t}(x, y)=p_{t}(x-y)$, namely $\mathbb{P}^{x}\left(X_{t} \in A\right)=\mathbb{E}^{x}\left[\mathbb{1}_{A}\left(X_{t}\right)\right]$. More generally

$$
\mathbb{E}^{x}\left[u\left(X_{t}\right)\right]=\int_{\mathbb{R}^{d}} u(y) p_{t}(x, y) \mathrm{d} y
$$

Here $\mathbb{P}^{x}$ (resp. $\mathbb{E}^{x}$ ) is the probability (resp. the expectation) corresponding to a process $\left(X_{t}\right)_{t}$ starting from the position $x$, i.e. $\mathbb{P}^{x}\left(X_{0}=x\right)=1$. The generator of such a stochastic process turns out to be $-L$. Indeed for a smooth function $u$,

$$
\lim _{t \rightarrow 0} \frac{\mathbb{E}^{x}\left[u\left(X_{t}\right)\right]-u(x)}{t}=\lim _{t \rightarrow 0} \frac{P_{t} u(x)-u(x)}{t}=-L u(x) .
$$

In fact, $\left(X_{t}\right)_{t}$ is a pure-jump isotropic unimodal Lévy process in $\mathbb{R}^{d}$, i.e., a stochastic process with stationary and independent increments and càdlàg paths whose transition function $p_{t}(x)$ is isotropic and unimodal. We refer to [Sat13] for a more extensive study on Lévy processes.

Remark 2.6 (Particular cases). The above applies to the case where $\psi(\xi)=|\xi|^{2}$, in which case, $L=-\Delta$ and the process $\left(X_{t}\right)_{t}$ is the well-known Brownian motion. If $\psi(\xi)=|\xi|$, the corresponding process is a Cauchy process whose generator is $L=(-\Delta)^{1 / 2}$. More generally if $\psi(\xi)=|\xi|^{2 s}$, the corresponding process is a $2 s$-stable process whose generator is $L=(-\Delta)^{s}$.
Energy form. We now show that the integrodifferential operator $L$ is intimately related to a Hilbert space of great interest in its own right. Let $H_{\nu}\left(\mathbb{R}^{d}\right)$ be the space of functions $u \in L^{2}\left(\mathbb{R}^{d}\right)$ such that $\mathcal{E}_{\nu}(u, u)<\infty$ where the bilinear form $\mathcal{E}_{\nu}$ is defined as

$$
\begin{equation*}
\mathcal{E}_{\nu}(u, v)=\iint_{\mathbb{R}^{d} \mathbb{R}^{d}}(u(x)-u(y))(v(x)-v(y)) \nu(x-y) \mathrm{d} y \mathrm{~d} x . \tag{2.5}
\end{equation*}
$$

As we shall see below, $H_{\nu}\left(\mathbb{R}^{d}\right)$ is a Hilbert space.
Theorem 2.7. If $\mathcal{E}_{\nu}(u, u)<\infty$ then

$$
\mathcal{E}_{\nu}(u, u)=\int_{\mathbb{R}^{d}}|\widehat{u}(\xi)|^{2} \psi(\xi) \mathrm{d} \xi .
$$

Moreover, if $\mathcal{E}_{\nu}(u, u)<\infty$ and $\mathcal{E}_{\nu}(v, v)<\infty$ then $(u, v)_{\psi}=\mathcal{E}(u, v)$ which can be characterized as

$$
(u, v)_{\psi}:=\int_{\mathbb{R}^{d}} \widehat{u}(\xi) \overline{\widehat{v}}(\xi) \psi(\xi) \mathrm{d} \xi
$$

Proof. The second claim follows by applying the first one on $u+v$ and $u-v$. Therefore, we only prove the first claim. Note that $\left|1-e^{-i t}\right|^{2}=2(1-\cos t)$ for every $t \in \mathbb{R}$. Plancherel's Theorem yields,

$$
\begin{aligned}
\mathcal{E}_{\nu}(u, u) & =\iint_{\mathbb{R}^{d} \mathbb{R}^{d}}(u(x)-u(y))^{2} \nu(x-y) \mathrm{d} y \mathrm{~d} x=\int_{\mathbb{R}^{d}} \nu(h) \int_{\mathbb{R}^{d}}(u(x)-u(x+h))^{2} \mathrm{~d} x \mathrm{~d} h \\
& =\int_{\mathbb{R}^{d}} \nu(h) \int_{\mathbb{R}^{d}}|\widehat{u}(\xi)|^{2}\left|1-e^{-i \xi \cdot h}\right|^{2} \mathrm{~d} \xi \mathrm{~d} h=2 \int_{\mathbb{R}^{d}}|\widehat{u}(\xi)|^{2} \int_{\mathbb{R}^{d}}(1-\cos (\xi \cdot h)) \nu(h) \mathrm{d} h \mathrm{~d} \xi \\
& =\int_{\mathbb{R}^{d}}|\widehat{u}(\xi)|^{2} \psi(\xi) \mathrm{d} \xi,
\end{aligned}
$$

which concludes the proof.
Based on this, let us next argue that $L$ can be extended to a continuous linear operator, To this end, let $u, v \in \mathcal{S}\left(\mathbb{R}^{d}\right)$ and observe that

$$
\mathcal{E}_{\nu}(u, u)=\int_{\mathbb{R}^{d}}|\widehat{u}(\xi)|^{2} \psi(\xi) \mathrm{d} \xi=\int_{\mathbb{R}^{d}} \widehat{\operatorname{Lu}}(\xi) \overline{\widehat{u}(\xi)} \mathrm{d} \xi,
$$

since $\widehat{L u}(\xi)=\psi(\xi) \widehat{u}(\xi)$. Another application of Plancherel's Theorem to the last expression gives the relation

$$
\mathcal{E}_{\nu}(u, u)=\int_{\mathbb{R}^{d}} u(x) L u(x) \mathrm{d} x .
$$

Replacing $u$ by $u+v$ leads to the relation

$$
\mathcal{E}_{\nu}(u, v)=\int_{\mathbb{R}^{d}} v(x) L u(x) \mathrm{d} x=\int_{\mathbb{R}^{d}} u(x) L v(x) \mathrm{d} x .
$$

Therefore, due to the density of $\mathcal{S}\left(\mathbb{R}^{d}\right)$ in $H_{\nu}\left(\mathbb{R}^{d}\right)$, see [FG20, Chapter 3], $L u$ can extended to a continuous linear form on $H_{\nu}\left(\mathbb{R}^{d}\right)$. Moreover, through the dual pairing we have

$$
(L u, v)=\mathcal{E}_{\nu}(u, v),
$$

for all $v \in H_{\nu}\left(\mathbb{R}^{d}\right)$. The integrodifferential operator $L$ can be extended to functions $u$ in $H_{\nu}\left(\mathbb{R}^{d}\right)$. Thereupon, $L$ can legitimately be regarded as a linear bounded operator from $H_{\nu}\left(\mathbb{R}^{d}\right)$ into its dual, i.e. $L: H_{\nu}\left(\mathbb{R}^{d}\right) \rightarrow\left(H_{\nu}\left(\mathbb{R}^{d}\right)\right)^{\prime}$ where $\left(H_{\nu}\left(\mathbb{R}^{d}\right)\right)^{\prime}$ is the dual of $H_{\nu}\left(\mathbb{R}^{d}\right)$. Treating the operator $L$ this way, i.e., derived from an associated energy form, we observe that $H_{\nu}\left(\mathbb{R}^{d}\right)$ is a fairly large domain for $L$ compared to the definition second-order differences or as pseudo-differential operators. Of course, it is worthwhile stressing that $L$ may not always be evaluated in the classical sense if defined through the correspondence $L: H_{\nu}\left(\mathbb{R}^{d}\right) \rightarrow\left(H_{\nu}\left(\mathbb{R}^{d}\right)\right)^{\prime}$.
2.2. Sobolev-Slobodeckij-like spaces. In the last subsection, we have tied the operator $L$ to an associated nonlocal energy form. In doing so, we already got a glimpse at a bilinear form that can be derived from the quadratic from $\mathcal{E}_{\nu}$. Motivated by our treatise of the operator $L$ derived from the energy form, we shall now pursue a closer investigation of the associated spaces, $\dot{H}_{\nu}\left(\mathbb{R}^{d}\right)$ and $H_{\nu}\left(\mathbb{R}^{d}\right)$, given by

$$
\begin{equation*}
\dot{H}_{\nu}\left(\mathbb{R}^{d}\right)=\left\{u \in L_{\mathrm{loc}}^{1}\left(\mathbb{R}^{d}\right): \mathcal{E}_{\nu}(u, u)<\infty\right\}, \tag{2.6}
\end{equation*}
$$

and

$$
\begin{equation*}
H_{\nu}\left(\mathbb{R}^{d}\right)=\left\{u \in L^{2}\left(\mathbb{R}^{d}\right): \mathcal{E}_{\nu}(u, u)<\infty\right\} . \tag{2.7}
\end{equation*}
$$

We equip the space $\dot{H}_{\nu}\left(\mathbb{R}^{d}\right)$ with the seminorm

$$
|u|_{H_{\nu}\left(\mathbb{R}^{d}\right)}^{2}:=\mathcal{E}_{\nu}(u, u)=\left\|\widehat{u} \psi^{1 / 2}\right\|_{L^{2}\left(\mathbb{R}^{d}\right)}^{2},
$$

and, respectively, the space $H_{\nu}\left(\mathbb{R}^{d}\right)$ with the norm

$$
\|u\|_{H_{\nu}\left(\mathbb{R}^{d}\right)}^{2}=\|u\|_{L^{2}\left(\mathbb{R}^{d}\right)}^{2}+|u|_{H_{\nu}\left(\mathbb{R}^{d}\right)}^{2}=\|u\|_{L^{2}\left(\mathbb{R}^{d}\right)}^{2}+\mathcal{E}_{\nu}(u, u) .
$$

Remark 2.8 (Comparison with $\left.L^{2}\right)$. We claim that, if $\nu \in L^{1}\left(\mathbb{R}^{d}\right)$ then $L^{2}\left(\mathbb{R}^{d}\right) \subset \dot{H}_{\nu}\left(\mathbb{R}^{d}\right)$ and $H_{\nu}\left(\mathbb{R}^{d}\right)=L^{2}\left(\mathbb{R}^{d}\right)$. To prove this claim, we write

$$
\iint_{\mathbb{R}^{d} \mathbb{R}^{d}}(u(x)-u(y))^{2} \nu(x-y) \mathrm{d} y \mathrm{~d} x \leq 4 \iint_{\mathbb{R}^{d} \mathbb{R}^{d}}|u(x)|^{2} \nu(x-y) \mathrm{d} y \mathrm{~d} x=4\|\nu\|_{L^{1}\left(\mathbb{R}^{d}\right)}\|u\|_{L^{2}\left(\mathbb{R}^{d}\right)}^{2} .
$$

Now, we finally show that the space $H_{\nu}\left(\mathbb{R}^{d}\right)$ is a Hilbert space.
Theorem $2.9\left(H_{\nu}\right.$ is a Hilbert space $)$. The couple $\left(H_{\nu}\left(\mathbb{R}^{d}\right),\|\cdot\|_{H_{\nu}\left(\mathbb{R}^{d}\right)}\right)$ is a separable Hilbert space with the scalar product

$$
(u, v)_{H_{\nu}\left(\mathbb{R}^{d}\right)}=(u, v)_{L^{2}\left(\mathbb{R}^{d}\right)}+\mathcal{E}_{\nu}(u, v) .
$$

Proof. Clearly, $(\cdot, \cdot)_{H_{\nu}\left(\mathbb{R}^{d}\right)}$ is a scalar product on $H_{\nu}\left(\mathbb{R}^{d}\right)$ associated with the norm $\|\cdot\|_{H_{\nu}\left(\mathbb{R}^{d}\right)}$. Let $\left(u_{n}\right)_{n}$ be a Cauchy sequence in $H_{\nu}\left(\mathbb{R}^{d}\right)$, then a subsequence $\left(u_{n_{k}}\right)_{k}$ converges to some $u$ in $L^{2}\left(\mathbb{R}^{d}\right)$ and a.e. in $\mathbb{R}^{d}$. Fix $k$ large enough, the Fatou's lemma implies

$$
\left|u_{n_{k}}-u\right|_{H_{\nu}\left(\mathbb{R}^{d}\right)}^{2} \leq \liminf _{\ell \rightarrow \infty} \iint_{\mathbb{R}^{d} \mathbb{R}^{d}}\left|\left[u_{n_{k}}-u_{n_{\ell}}\right](x)-\left[u_{n_{k}}-u_{n_{\ell}}\right](y)\right|^{2} \nu(x-y) \mathrm{d} y \mathrm{~d} x .
$$

Since $\left(u_{n_{k}}\right)_{k}$ is a Cauchy sequence, the right-hand side is finite for any $k$ and tends to 0 as $k \rightarrow \infty$. This implies $u \in H_{\nu}\left(\mathbb{R}^{d}\right)$ and $\left|u_{n_{k}}-u\right|_{H_{\nu}\left(\mathbb{R}^{d}\right)}^{2} \xrightarrow{k \rightarrow \infty} 0$. Finally, $u_{n} \rightarrow u$ in $H_{\nu}\left(\mathbb{R}^{d}\right)$ and hence $H_{\nu}\left(\mathbb{R}^{d}\right)$ is a Hilbert space. The map $\mathcal{I}: H_{\nu}\left(\mathbb{R}^{d}\right) \rightarrow L^{2}\left(\mathbb{R}^{d}\right) \times L^{2}\left(\mathbb{R}^{d} \times \mathbb{R}^{d}\right)$ with

$$
\mathcal{I}(u)=\left(u(x),(u(x)-u(y)) \nu^{1 / 2}(x-y)\right)
$$

is an isometry. Hence, identifying $H_{\nu}\left(\mathbb{R}^{d}\right)$ with the closed subspace $\mathcal{I}\left(H_{\nu}\left(\mathbb{R}^{d}\right)\right)$ of $L^{2}\left(\mathbb{R}^{d}\right) \times$ $L^{2}\left(\mathbb{R}^{d} \times \mathbb{R}^{d}\right)$ implies that $H_{\nu}\left(\mathbb{R}^{d}\right)$ is separable.

Albeit not pertinent to the arguments in the proof of this paper's main result, the following properties highlight the importance of the Lévy condition $(L)$ by providing its analytic interpretation. This characterization is also true in the nonlinear setting; see [Fog23c, Fog23b].
Theorem 2.10. Let $\nu: \mathbb{R}^{d} \backslash\{0\} \rightarrow[0, \infty)$. The following assertions are equivalent.
(i) The Lévy condition ( $L$ ) holds.
(ii) The embedding $H^{1}\left(\mathbb{R}^{d}\right) \hookrightarrow H_{\nu}\left(\mathbb{R}^{d}\right)$ is continuous.
(iii) $\mathcal{E}_{\nu}(u, u)<\infty$ for all $u \in H^{1}\left(\mathbb{R}^{d}\right)$.
(iv) $\mathcal{E}_{\nu}(u, u)<\infty$ for all $u \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$.
(v) $H_{\nu}\left(\mathbb{R}^{d}\right) \neq\{0\}$ (if in addition $\nu$ is radial).

For a proof we refer to [Fog23b, Section 4] and [FK22, Section 2.1].
We will also need the following result inferring the stability of the space $H_{\nu}\left(\mathbb{R}^{d}\right)$ under a pushforward. This is a centrepiece in the establishment of the Euler-Lagrange equation which ultimately leads to the identification of the limit obtained from the minimising movement scheme as a weak solution of (1.1).

Theorem 2.11. Assume $\nu$ is a unimodal Lévy kernel, i.e., there exists $c \in(0,1)$ such that

$$
\begin{equation*}
\nu(x) \leq c \nu(y) \tag{2.8}
\end{equation*}
$$

whenever $|x| \geq|y|$, and such that the following scaling condition near the origin holds:
For every $\lambda>0$ there is $c_{\lambda}>0$ s.t. $\nu(\lambda h) \leq c_{\lambda} \nu(h)$, whenever $|h| \leq 1$.
Finally, let $\zeta: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$ be a bi-Lipschitz diffeomorphism on $\mathbb{R}^{d}$, i.e., there exists $\sigma>0$ such that

$$
\sigma|x-y| \leq|\zeta(x)-\zeta(y)| \leq \sigma^{-1}|x-y| .
$$

Then, for $u \in H_{\nu}\left(\mathbb{R}^{d}\right)$, we have $u \circ \zeta \in H_{\nu}\left(\mathbb{R}^{d}\right)$ and the following estimate holds

$$
\|u \circ \zeta\|_{H_{\nu}\left(\mathbb{R}^{d}\right)}^{2} \leq A_{\sigma, \nu}\left(1+\left\|\operatorname{det} D \zeta^{-1}\right\|_{L^{\infty}\left(\mathbb{R}^{d}\right)}\right)^{2}\|u\|_{H_{\nu}\left(\mathbb{R}^{d}\right)}^{2},
$$

for some constant $A_{\sigma, \nu}>0$ only depending on $\sigma$ and $\nu$.
Proof. Note that $\left|\zeta^{-1}(x)-\zeta^{-1}(y)\right| \geq \sigma|x-y|$ so that unimodality of $\nu$, see (2.8), implies

$$
\nu\left(\zeta^{-1}(x)-\zeta^{-1}(y)\right) \leq c \nu(\sigma(x-y))
$$

By a change of variables and the unimodality of $\nu$ we get

$$
\begin{aligned}
\mathcal{E}_{\nu}(u \circ \zeta, u \circ \zeta) & =\iint_{\mathbb{R}^{d} \mathbb{R}^{d}}|u \circ \zeta(x)-u \circ \zeta(y)|^{2} \nu(x-y) \mathrm{d} y \mathrm{~d} x \\
& =\iint_{\mathbb{R}^{d} \mathbb{R}^{d}}|u(x)-u(y)|^{2}\left|\operatorname{det} D \zeta^{-1}(x) \| \operatorname{det} D \zeta^{-1}(y)\right| \nu\left(\zeta^{-1}(x)-\zeta^{-1}(y)\right) \mathrm{d} y \mathrm{~d} x \\
& \leq c\left\|\left|\operatorname{det} D \zeta^{-1}\right|\right\|_{L^{\infty}\left(\mathbb{R}^{d}\right)}^{2} \iint_{\mathbb{R}^{d} \mathbb{R}^{d}}|u(x)-u(y)|^{2} \nu(\sigma(x-y)) \mathrm{d} y \mathrm{~d} x .
\end{aligned}
$$

Furthermore, using the scaling condition near the origin, (2.9), we find that

$$
\begin{aligned}
& \iint_{\mathbb{R}^{d} \mathbb{R}^{d}}|u(x)-u(y)|^{2} \nu(\sigma(x-y)) \mathrm{d} y \mathrm{~d} x \\
& \quad=\iint_{|x-y| \geq 1}|u(x)-u(y)|^{2} \nu(\sigma(x-y)) \mathrm{d} y \mathrm{~d} x+\iint_{|x-y| \leq 1}|u(x)-u(y)|^{2} \nu(\sigma(x-y)) \mathrm{d} y \mathrm{~d} x \\
& \quad \leq 4\|u\|_{L^{2}\left(\mathbb{R}^{d}\right)}^{2} \int_{|h|>1} \nu(\sigma h) \mathrm{d} h+c_{\sigma} \int_{\mathbb{R}^{d}} \int_{|h| \leq 1}|u(x)-u(x+h)|^{2} \nu(h) \mathrm{d} h \mathrm{~d} x \\
& \quad \leq 4 \sigma^{-d}\|u\|_{L^{2}\left(\mathbb{R}^{d}\right)}^{2} \int_{|h|>\sigma} \nu(h) \mathrm{d} h+c_{\sigma} \mathcal{E}_{\nu}(u, u) \\
& \quad \leq\left(4 \sigma^{-d} \int_{|h|>\sigma} \nu(h) \mathrm{d} h+c_{\sigma}\right)\|u\|_{H_{\nu}\left(\mathbb{R}^{d}\right)}^{2} .
\end{aligned}
$$

On the other hand, we have

$$
\int_{\mathbb{R}^{d}}|u \circ \zeta(x)|^{2} \mathrm{~d} x=\int_{\mathbb{R}^{d}}|u(x)|^{2}\left|\operatorname{det} D \zeta^{-1}(x)\right| \mathrm{d} x \leq\left\|\operatorname{det} D \zeta^{-1}\right\|_{L^{\infty}\left(\mathbb{R}^{d}\right)}\|u\|_{L^{2}\left(\mathbb{R}^{d}\right)}^{2} .
$$

Hence, combining the last two estimates, we obtain the desired estimate

$$
\|u \circ \zeta\|_{H_{\nu}\left(\mathbb{R}^{d}\right)}^{2} \leq A_{\sigma, \nu}\left(1+\left\|\operatorname{det} D \zeta^{-1}\right\|_{L^{\infty}\left(\mathbb{R}^{d}\right)}\right)^{2}\|u\|_{H_{\nu}\left(\mathbb{R}^{d}\right)}^{2}
$$

2.3. Compact embedding. In this section, we establish the compact embedding of $H_{\nu}\left(\mathbb{R}^{d}\right)$ into $L_{\text {loc }}^{2}\left(\mathbb{R}^{d}\right)$. We recall that $L_{\text {loc }}^{2}\left(\mathbb{R}^{d}\right)$ is equipped with the topology of $L^{2}$-convergence on compact sets. Namely, a sequence $\left(u_{n}\right)_{n}$ converges in $L_{\text {loc }}^{2}\left(\mathbb{R}^{d}\right)$ if for every compact set $K \subset \mathbb{R}^{d}$ there is $u_{K} \in L^{2}(K)$ such that $\left\|u_{n}-u_{K}\right\|_{L^{2}(K)} \rightarrow 0$ as $n \rightarrow \infty$. In this case, we say that $\left(u_{n}\right)_{n}$ converges in $L_{\mathrm{loc}}^{2}\left(\mathbb{R}^{d}\right)$ to the function $u$ defined by $\left.u\right|_{K}=u_{K}$ for every compact set.

Remark 2.12 ( $L^{1}$-Lévy kernels). Let us observe that if $\nu \in L^{1}\left(\mathbb{R}^{d}\right)$, then $H_{\nu}\left(\mathbb{R}^{d}\right)=L^{2}\left(\mathbb{R}^{d}\right)$ which is not locally compactly embedded into $L_{\mathrm{loc}}^{2}\left(\mathbb{R}^{d}\right)$. Thus, imposing $\nu$ satisfy the Lévy condition, $(L)$, as well as the non-integrability condition, $\nu \notin L^{1}\left(\mathbb{R}^{d}\right)$, is paramount.

Let us start with the following lemma on the compactness of convolution operators.
Lemma 2.13. Let $w \in L^{1}\left(\mathbb{R}^{d}\right)$ and $K \subset \mathbb{R}^{d}$ be compact. The convolution operator $T_{w}: L^{2}\left(\mathbb{R}^{d}\right) \rightarrow$ $L^{2}(K)$, with $u \mapsto T_{w}=w * u$, is compact.

Proof. In virtue of Young's inequality, for very $u \in L^{2}\left(\mathbb{R}^{d}\right)$

$$
\begin{equation*}
\|w * u\|_{L^{2}\left(\mathbb{R}^{d}\right)} \leq\|w\|_{L^{1}\left(\mathbb{R}^{d}\right)}\|u\|_{L^{2}\left(\mathbb{R}^{d}\right)} . \tag{2.10}
\end{equation*}
$$

Let B be a bounded subset of $L^{2}\left(\mathbb{R}^{d}\right)$ and set $M=\sup _{u \in \mathrm{~B}}\|u\|_{L^{2}\left(\mathbb{R}^{d}\right)}$. Then, (2.10) implies that $T_{w}(\mathrm{~B})$ is a bounded subset of $L^{2}\left(\mathbb{R}^{d}\right)$, too, and we can control the shifts

$$
\sup _{u \in \mathrm{~B}} \int_{\mathbb{R}^{d}}\left|T_{w} u(x+h)-T_{w} u(x)\right|^{2} \mathrm{~d} x \leq M^{2}\|w(\cdot+h)-w(\cdot)\|_{L^{1}\left(\mathbb{R}^{d}\right)}^{2} \xrightarrow{|h| \rightarrow 0} 0 .
$$

The Riesz-Fréchet-Kolmogorov Theorem implies that $\left.T_{w}(\mathrm{~B})\right|_{K}$ is relatively compact in $L^{2}(K)$ and the desired result follows.

Theorem 2.14 (Characterization of compact embeddings). Any symmetric Lévy kernel $\nu$ satisfies $\nu \notin L^{1}\left(\mathbb{R}^{d}\right)$ if and only if the embedding $H_{\nu}\left(\mathbb{R}^{d}\right) \hookrightarrow L_{\mathrm{loc}}^{2}\left(\mathbb{R}^{d}\right)$ is compact.
Proof. Assume $\nu \in L^{1}\left(\mathbb{R}^{d}\right)$. Then, by Remarks 2.8 and 2.12, $H_{\nu}\left(\mathbb{R}^{d}\right)=L^{2}\left(\mathbb{R}^{d}\right)$. Of course, $L^{2}\left(\mathbb{R}^{d}\right) \hookrightarrow L_{\text {loc }}^{2}\left(\mathbb{R}^{d}\right)$ is not compact, which proves the first direction.
Next, we prove the converse, namely that any Lévy kernel lacking integrability at the origin gives rise to an energy space that compactly embeds into $L^{2}$ - which is perhaps more surprising. To prove the claim, we assume $\nu \notin L^{1}\left(\mathbb{R}^{d}\right)$ and show the embedding is compact, indeed. To this end, let $\delta>0$ be sufficiently small such that, upon removing the singularity close to the origin by introducing $\nu_{\delta}:=\nu \mathbb{1}_{B_{\delta}^{c}(0)}$, the resulting measure has finite mass, i.e., $0<\left\|\nu_{\delta}\right\|_{L^{1}\left(\mathbb{R}^{d}\right)}<\infty$. Rescaling its mass to unity, we introduce

$$
w_{\delta}(h):=\nu_{\delta}(h)\left\|\nu_{\delta}\right\|_{L^{1}\left(\mathbb{R}^{d}\right)}^{-1},
$$

which we shall use as a convolution kernel.
For fixed $u \in L^{2}\left(\mathbb{R}^{d}\right)$, by evenness of $\nu$ for all $x \in \mathbb{R}^{d}$ we have

$$
T_{w_{\delta}} u(x)=\int_{\mathbb{R}^{d}} w_{\delta}(y) u(x-y) \mathrm{d} y=\int_{\mathbb{R}^{d}} w_{\delta}(y) u(x+y) \mathrm{d} y
$$

Thus, by Jensen's inequality

$$
\begin{aligned}
\left\|u-T_{w_{\delta}} u\right\|_{L^{2}\left(\mathbb{R}^{d}\right)}^{2} & =\int_{\mathbb{R}^{d}}\left|\int_{\mathbb{R}^{d}}[u(x)-u(x+h)] w_{\delta}(h) \mathrm{d} h\right|^{2} \mathrm{~d} x \\
& \leq\left\|\nu_{\delta}\right\|_{L^{1}\left(\mathbb{R}^{d}\right)}^{-1} \iint_{\mathbb{R}^{d} \mathbb{R}^{d}}|u(x)-u(x+h)|^{2} \nu(h) \mathrm{d} h \mathrm{~d} x \\
& \leq\left\|\nu_{\delta}\right\|_{L^{1}\left(\mathbb{R}^{d}\right)}^{-1}\|u\|_{H_{\nu}\left(\mathbb{R}^{d}\right)}^{2} .
\end{aligned}
$$

For a compact set $K \subset \mathbb{R}^{d}$, we put $R_{K} u=\left.u\right|_{K}$. Since $\nu \notin L^{1}\left(\mathbb{R}^{d}\right)$ it follows that

$$
\left\|R_{K}-R_{K} T_{w_{\delta}}\right\|_{\mathcal{L}\left(H_{\nu}\left(\mathbb{R}^{d}\right), L^{2}(K)\right)} \leq\left\|\nu_{\delta}\right\|_{L^{1}\left(\mathbb{R}^{d}\right)}^{-1} \xrightarrow{\delta \rightarrow 0} 0 .
$$

Thus the operator $R_{K}: H_{\nu}\left(\mathbb{R}^{d}\right) \rightarrow L^{2}(K)$, is compact since by Lemma 2.13, the operator $R_{K} \circ T_{w_{\delta}}$ is also compact for every $\delta$.

As a straightforward consequence of Theorem 2.14 we have the following.
Corollary 2.15. Let $\nu \notin L^{1}\left(\mathbb{R}^{d}\right)$ be a symmetric, non-integrable Lévy kernel and $\left(u_{n}\right)_{n}$ be a bounded sequence in $H_{\nu}\left(\mathbb{R}^{d}\right)$. Then, there exist $u \in H_{\nu}\left(\mathbb{R}^{d}\right)$ and a subsequence $\left(u_{n_{j}}\right)_{j}$ converging to $u$ in $L_{\mathrm{loc}}^{2}\left(\mathbb{R}^{d}\right)$.

It is worth mentioning that Theorem 2.14 was first proved in [JW20, Theorem 1.1]. However, earlier results using similar techniques also appeared in [PZ17, Proposition 6] for periodic functions on the torus. The technique of removing the singularity is also used in [BJ13, Lemma 3.1] and [BJ17, Proposition 1].
2.4. Switching to Fourier notations. In the remainder of this section, we shall switch to the Fourier side setting up similar nonlocal spaces as above defined for a general Fourier symbol, $\phi$ : $\mathbb{R}^{d} \backslash\{0\} \rightarrow(0, \infty)$, which is assumed to be continuous and symmetric, i.e., $\phi(\xi)=\phi(-\xi)$.

Definition 2.16 (Nonlocal Spaces). We define the space

$$
\dot{H}^{\phi}\left(\mathbb{R}^{d}\right)=\left\{u \in \mathcal{S}^{\prime}\left(\mathbb{R}^{d}\right): \widehat{u} \phi^{1 / 2} \in L^{2}\left(\mathbb{R}^{d}\right)\right\},
$$

and $H^{\phi}\left(\mathbb{R}^{d}\right):=L^{2}\left(\mathbb{R}^{d}\right) \cap \dot{H}^{\phi}\left(\mathbb{R}^{d}\right)$ as

$$
H^{\phi}\left(\mathbb{R}^{d}\right)=\left\{u \in L^{2}\left(\mathbb{R}^{d}\right): \widehat{u} \phi^{1 / 2} \in L^{2}\left(\mathbb{R}^{d}\right)\right\}
$$

The two spaces $\dot{H}^{\phi}\left(\mathbb{R}^{d}\right)$ and $H^{\phi}\left(\mathbb{R}^{d}\right)$ are respectively equipped with the following (semi)norms

$$
\|u\|_{\dot{H}^{\phi}\left(\mathbb{R}^{d}\right)}^{2}=\int_{\mathbb{R}^{d}}|\widehat{u}(\xi)|^{2} \phi(\xi) \mathrm{d} \xi,
$$

as well as

$$
\|u\|_{H^{\phi}\left(\mathbb{R}^{d}\right)}^{2}=\int_{\mathbb{R}^{d}}|\widehat{u}(\xi)|^{2} \mathrm{~d} \xi+\int_{\mathbb{R}^{d}}|\widehat{u}(\xi)|^{2} \phi(\xi) \mathrm{d} \xi .
$$

It is an easy observation that, $\|\cdot\|_{\dot{H}^{\phi}\left(\mathbb{R}^{d}\right)}$ is the seminorm associated with the symmetric bilinear form $(\cdot, \cdot)_{\phi}$ given by

$$
(u, v)_{\phi}:=\int_{\mathbb{R}^{d}} \widehat{u}(\xi) \widehat{\hat{v}}(\xi) \phi(\xi) \mathrm{d} \xi
$$

In the same vein, we define the space $\dot{H}^{\phi^{-1}}\left(\mathbb{R}^{d}\right)$ as the dual space $\left(\dot{H}^{\phi}\left(\mathbb{R}^{d}\right)\right)^{\prime}$ endowed with the norm

$$
\|u\|_{\dot{H}^{\phi}}\left(\mathbb{R}^{d}\right)=\sup _{v \in \dot{H}^{\phi}\left(\mathbb{R}^{d}\right), v \neq 0} \frac{\langle u, v\rangle}{\|u\|_{\dot{H}^{\phi}\left(\mathbb{R}^{d}\right)}}=\left(\int_{\mathbb{R}^{d}}|\widehat{u}(\xi)|^{2} \phi^{-1}(\xi) \mathrm{d} \xi\right)^{1 / 2}
$$

where $\langle u, v\rangle:=\left(\widehat{u} \phi^{-1 / 2}, \widehat{v} \phi^{1 / 2}\right)_{L^{2}\left(\mathbb{R}^{d}\right)}$ for any $u \in \dot{H}^{\phi^{-1}}\left(\mathbb{R}^{d}\right)$ and $v \in \dot{H}^{\phi}\left(\mathbb{R}^{d}\right)$.
Let us recall that

$$
\psi(\xi)=2 \int_{\mathbb{R}^{d}}(1-\cos (\xi \cdot h)) \nu(h) \mathrm{d} h
$$

for a Lévy kernel $\nu \in L^{1}\left(\mathbb{R}^{d}, 1 \wedge|h|^{2}\right)$. It is apparent that the symbol $\psi$, respectively $\psi^{-1}$, will play a prominent role throughout the paper. Moreover, we stress the importance of the associated symbols $\widetilde{\psi}$, and $\psi^{*}$ given by

$$
\widetilde{\psi}(\xi)=|\xi|^{2} \psi^{-1}(\xi), \quad \text { and } \quad \psi^{*}(\xi)=|\xi|^{2} \psi^{-2}(\xi)=\psi^{-1}(\xi) \widetilde{\psi}(\xi)
$$

In particular, note that we have $H_{\nu}\left(\mathbb{R}^{d}\right)=H^{\psi}\left(\mathbb{R}^{d}\right)$.
Remark 2.17 (Relation to fractional Sobolev spaces). For the standard fractional case $\nu(h)=$ $\frac{C_{d, s}}{2}|h|^{-d-2 s}, s \in(0,1)$, the aforementioned symbols read

$$
\psi(\xi)=|\xi|^{2 s}, \quad \text { and } \quad \psi^{-1}(\xi)=|\xi|^{-2 s},
$$

i.e., the symbol of the fractional Laplacian and the associated Riesz kernel, respectively. Moreover,

$$
\widetilde{\psi}(\xi)=|\xi|^{2} \psi^{-1}(\xi)=|\xi|^{2(1-s)}, \quad \text { and } \quad \psi^{*}(\xi)=|\xi|^{2} \psi^{-2}(\xi)=|\xi|^{2(1-2 s)}
$$

The nonlocal spaces associated to these symbols simply become

$$
H_{\nu}\left(\mathbb{R}^{d}\right)=H^{\psi}\left(\mathbb{R}^{d}\right)=H^{s}\left(\mathbb{R}^{d}\right), \quad \text { and } \quad H^{\psi^{-1}}\left(\mathbb{R}^{d}\right)=\left(H^{\psi}\left(\mathbb{R}^{d}\right)\right)^{\prime}=H^{-s}\left(\mathbb{R}^{d}\right)
$$

as well as

$$
H^{\widetilde{\psi}}\left(\mathbb{R}^{d}\right)=H^{1-s}\left(\mathbb{R}^{d}\right), \quad \text { and } \quad H^{\psi^{*}}\left(\mathbb{R}^{d}\right)=H^{1-2 s}\left(\mathbb{R}^{d}\right)
$$

Let us now mention the compactness result with respect to the $H^{\phi}\left(\mathbb{R}^{d}\right)$.
Theorem 2.18. The embedding $H^{\phi}\left(\mathbb{R}^{d}\right) \hookrightarrow L_{\mathrm{loc}}^{2}\left(\mathbb{R}^{d}\right)$ is compact, provided that $\phi$ satisfies

$$
\begin{equation*}
\sup _{\xi \in \mathbb{R}^{d}} \frac{1}{\phi(\xi)}\left|e^{i \xi \cdot h}-1\right|^{2} \xrightarrow{|h| \rightarrow 0} 0 . \tag{2.11}
\end{equation*}
$$

Proof. Let $\mathbf{B} \subset H^{\phi}\left(\mathbb{R}^{d}\right)$ be a bounded subset, and let $M:=\sup _{u \in \mathbf{B}}\|u\|_{H^{\phi}\left(\mathbb{R}^{d}\right)}<\infty$. For $u \in \mathbf{B}$, using Plancherel we have

$$
\begin{aligned}
\|u(\cdot+h)-u\|_{L^{2}\left(\mathbb{R}^{d}\right)}^{2} & =\int_{\mathbb{R}^{d}}|\widehat{u}(\xi)|^{2}\left|e^{i \xi \cdot h}-1\right|^{2} \mathrm{~d} \xi \\
& \leq \sup _{\xi \in \mathbb{R}^{d}} \frac{1}{\phi(\xi)}\left|e^{i \xi \cdot h}-1\right|^{2} \int_{\mathbb{R}^{d}}|\widehat{u}(\xi)|^{2} \phi(\xi) \mathrm{d} \xi \\
& \leq M \sup _{\xi \in \mathbb{R}^{d}} \frac{1}{\phi(\xi)}\left|e^{i \xi \cdot h}-1\right|^{2} .
\end{aligned}
$$

Hence $\mathbf{B}$ is a bounded subset $L^{2}\left(\mathbb{R}^{d}\right)$ and we have

$$
\sup _{u \in \mathbf{B}}\|u(\cdot+h)-u\|_{L^{2}\left(\mathbb{R}^{d}\right)}^{2} \leq M \sup _{\xi \in \mathbb{R}^{d}} \frac{1}{\phi(\xi)}\left|e^{i \xi \cdot h}-1\right|^{2} \xrightarrow{|h| \rightarrow 0} 0 .
$$

The sought compactness thus follows from the Riesz-Fréchet-Kolmogorov.

## 3. Existence of solution to the minimizing movement scheme

As foreshadowed in the introduction, the construction of solutions to (1.1) is achieved by employing a minimizing movement scheme in the set of probability measures equipped with the 2 Wasserstein distance; we refer the reader to [Vil09, ABS21] for more details on the Wasserstein distance. Throughout, let us denote by $\mathcal{P}\left(\mathbb{R}^{d}\right)$ the set of Borel probability measures on $\mathbb{R}^{d}$. We say that a sequence $\left(u_{n}\right)_{n} \subset \mathcal{P}\left(\mathbb{R}^{d}\right)$ converges narrowly to $u \in \mathcal{P}\left(\mathbb{R}^{d}\right)$, if

$$
\lim _{n \rightarrow \infty} \int_{\mathbb{R}^{d}} \phi(x) \mathrm{d} u_{n}(x)=\int_{\mathbb{R}^{d}} \phi(x) \mathrm{d} u(x),
$$

for all $\phi \in C_{b}\left(\mathbb{R}^{d}\right)$, where $C_{b}\left(\mathbb{R}^{d}\right)$ is the space of continuous and bounded function on $\mathbb{R}^{d}$. Additionally, the space of Borel probability measures with finite second moment is defined as

$$
\mathcal{P}_{2}\left(\mathbb{R}^{d}\right):=\left\{u \in \mathcal{P}\left(\mathbb{R}^{d}\right): \int_{\mathbb{R}^{d}}|x|^{2} \mathrm{~d} u(x)<\infty\right\} .
$$

It is well-known that this set is a complete, separable metric space when equipped with the 2 Wasserstein distance, $W$, defined by

$$
W\left(u_{0}, u_{1}\right)=\min _{\gamma \in \Gamma\left(u_{0}, u_{1}\right)}\left\{\iint_{\mathbb{R}^{d} \times \mathbb{R}^{d}}|x-y|^{2} \mathrm{~d} \gamma(x, y)\right\}^{1 / 2}
$$

where $\Gamma\left(u_{0}, u_{1}\right)$ denotes the set of all transport plans between $u_{0}$ and $u_{1}$, that is, the set of probability measures on the product space $\mathbb{R}^{d} \times \mathbb{R}^{d}$ with marginals $u_{0}$ and $u_{1}$, i.e., $\left(\pi_{x}\right)_{\#} \gamma=u_{0}$ and $\left(\pi_{y}\right)_{\#} \gamma=$ $u_{1}$. Moreover [ABS21, Theorem 5.2], if $u_{0}$ is absolutely continuous with respect to the Lebesgue measure, then $\gamma=\left(\mathrm{I} \times T_{u_{0}}^{u_{1}}\right)_{\#} u$ is the minimizer for $W\left(u_{0}, u_{1}\right)$ where $T_{u_{0}}^{u_{1}}$ is the transport map such that $\left(T_{u_{0}}^{u_{1}}\right)_{\#} u_{0}=u_{1}$, i.e., $u_{1}$ is the push forward of $u_{0}$ through $T_{u_{0}}^{u_{1}}$. Moreover we have

$$
W\left(u_{0}, u_{1}\right)=\min _{S_{\#} u_{0}=u_{1}}\left\{\int_{\mathbb{R}^{d}}|S(x)-x|^{2} \mathrm{~d} u_{0}(x)\right\}^{1 / 2}:=\left\{\int_{\mathbb{R}^{d}}\left|T_{u_{1}}^{u_{0}}(x)-x\right|^{2} \mathrm{~d} u_{0}(x)\right\}^{1 / 2},
$$

Now, if both $u_{0}$ and $u_{1}$ have densities then

$$
T_{u_{1}}^{u_{0}} \circ T_{u_{0}}^{u_{1}}=\mathrm{I} \quad u_{0} \text {-a.e. } \quad \underset{16}{\text { and }} \quad T_{u_{0}}^{u_{1}} \circ T_{u_{1}}^{u_{0}}=\mathrm{I} \quad u_{1} \text {-a.e.. }
$$

Remark 3.1. Let us point out that the space $\dot{H}^{\psi^{-1}}\left(\mathbb{R}^{d}\right)$ is naturally associated to the operator $\mathcal{F}: \mathcal{P}_{2}\left(\mathbb{R}^{d}\right) \rightarrow \mathbb{R} \cup\{\infty\}$,

$$
u \mapsto \mathcal{F}(u)=\frac{1}{2}\left\|\psi^{-1 / 2} \widehat{u}\right\|_{L^{2}\left(\mathbb{R}^{d}\right)}^{2}=\frac{1}{2} \int_{\mathbb{R}^{d}}|\widehat{u}(\xi)|^{2} \psi^{-1}(\xi) \mathrm{d} \xi=\frac{1}{2}\|u\|_{\dot{H}^{\psi^{-1}}\left(\mathbb{R}^{d}\right)}^{2}
$$

Clearly, $\mathcal{F}(u)<\infty$ if and only if $u \in \dot{H}^{\psi^{-1}}\left(\mathbb{R}^{d}\right) \cap \mathcal{P}_{2}\left(\mathbb{R}^{d}\right)$, i.e., $D(\mathcal{F})=\dot{H}^{\psi^{-1}}\left(\mathbb{R}^{d}\right) \cap \mathcal{P}_{2}\left(\mathbb{R}^{d}\right)$.
Theorem 3.2. Let $u_{*} \in \mathcal{P}_{2}\left(\mathbb{R}^{d}\right)$ and $\tau>0$ be given. Then, the mapping

$$
u \mapsto \frac{1}{2 \tau} W^{2}\left(u, u_{*}\right)+\mathcal{F}(u),
$$

is lower semi-continuous with respect to the narrow convergence in $\mathcal{P}_{2}\left(\mathbb{R}^{d}\right)$. Moreover, there exists a unique minimizer, $u \in \mathcal{P}_{2}\left(\mathbb{R}^{d}\right)$.

Proof. We begin by establishing the lower semi-continuity and show the existence and uniqueness of a minimizer later. By the lower semi-continuity of the Wasserstein distance, cf. [AGS08, Proposition 7.1.3], it is sufficient to prove the lower semi-continuity of $u \mapsto \mathcal{F}(u)$, as the sum of two lower semicontinuous functions is lower semi-continuous. Now, let $\left(u_{n}\right)_{n} \subset \mathcal{P}_{2}\left(\mathbb{R}^{d}\right)$ be narrowly convergent to $u \in \mathcal{P}_{2}\left(\mathbb{R}^{d}\right)$. Without loss of generality, we assume that $\mathcal{F}\left(u_{n}\right)<\infty$, uniformly in $n \in \mathbb{N}$, as, otherwise, the liminf-inequality is trivially satisfied. Next, let us set $U_{n}:=\psi^{-1 / 2} \widehat{u}_{n}$ and observe, that the bound on $\left(\mathcal{F}\left(u_{n}\right)\right)_{n}$ implies the uniform $L^{2}$-bound $\sup _{n}\left\|U_{n}\right\|_{L^{2}\left(\mathbb{R}^{d}\right)}<\infty$. Consequently, up to a subsequence, $U_{n} \rightharpoonup U$, weakly converges in $L^{2}\left(\mathbb{R}^{d}\right)$. Next we show that $U(\xi)=\psi^{-1 / 2}(\xi) \widehat{u}(\xi)$. By the Banach-Saks Theorem, see [Fog23a] or [MR12, Appendix A], there exists a further subsequence still denoted $\left(U_{n}\right)_{n}$ whose Cesáro mean converges strongly in $L^{2}\left(\mathbb{R}^{d}\right)$, i.e.,

$$
V_{n}:=\frac{1}{n} \sum_{i=1}^{n} U_{i} \longrightarrow U .
$$

Passing to another subsequence, we have $V_{n} \rightarrow U$, almost everywhere in $\mathbb{R}^{d}$. Simultaneously, this narrow convergence implies the pointwise convergence of $\widehat{u}_{n}: \widehat{u}_{n}(\xi) \rightarrow \widehat{u}(\xi)$ for all $\xi \in \mathbb{R}^{d}$, as $n \rightarrow \infty$. Therefore, we get

$$
U(\xi)=\lim _{n \rightarrow \infty} V_{n}(\xi)=\frac{1}{n} \sum_{i=1}^{n} \psi^{-1 / 2} \hat{u}_{i}=\lim _{n \rightarrow \infty} \psi^{-1 / 2}(\xi) \widehat{u}_{n}(\xi)=\psi^{-1 / 2}(\xi) \widehat{u}(\xi),
$$

for almost every $\xi \in \mathbb{R}^{d}$. The weak lower semi-continuity of $\|\cdot\|_{L^{2}\left(\mathbb{R}^{d}\right)}$ with respect to pointwise a.e. convergence implies that

$$
\mathcal{F}(u)=\|U\|_{L^{2}\left(\mathbb{R}^{d}\right)}^{2} \leq \liminf _{n \rightarrow \infty}\left\|U_{n}\right\|_{L^{2}\left(\mathbb{R}^{d}\right)}^{2}=\liminf _{n \rightarrow \infty} \mathcal{F}\left(u_{n}\right),
$$

which proves the lower semi-continuity of the Moreau-Yosida penalization, as claimed. Now, the existence of a unique minimizer is a straightforward consequence of the direct method. Indeed, $u \mapsto \frac{1}{2 \tau} W^{2}\left(u, u_{*}\right)+\mathcal{F}(u)$ is lower semicontinuous. Moreover, it is coercive on $\mathcal{P}_{2}\left(\mathbb{R}^{d}\right)$ with respect to the narrow convergence - each sublevel set $\left\{u \in \mathcal{P}_{2}\left(\mathbb{R}^{d}\right): \frac{1}{2 \tau} W^{2}\left(u, u_{*}\right)+\mathcal{F}(u) \leq M\right\}, M \in \mathbb{R}$ is either empty or bounded in $\mathcal{P}_{2}\left(\mathbb{R}^{d}\right)$. It is well-known that any bounded set in $\mathcal{P}_{2}\left(\mathbb{R}^{d}\right)$ is relatively compact with respect to the narrow convergence. The uniqueness of the minimizer follows from the strict convexity of the functional $u \mapsto \frac{1}{2 \tau} W^{2}\left(u, u_{*}\right)+\mathcal{F}(u)$.
3.1. Minimizing movement scheme. The minimizing movements scheme in the set of probability measures, originally introduced in [JKO98] (see also [AGS08, Definition 2.0.2]), is defined as follows.

Definition 3.3. Given $\tau>0$ and $u_{0} \in D(\mathcal{F})$, we consider the sequence of discrete approximations $\left\{u_{\tau}^{k}\right\}_{k \in \mathbb{N}}$ uniquely defined through the recursive scheme: $u_{\tau}^{0}:=G_{\omega(\tau)} * u_{0}$, where $G_{t}(x)=$ $\frac{1}{(4 t \pi)^{d / 2}} e^{-\frac{|x|^{2}}{4 t}}$ is the standard Gaussian heat kernel and $\omega(\tau)=-1 / \log (\tau)$ if $\tau \in(0,1 / 2)$ and $\omega(\tau)=1 / \log (2)$ if $\tau \in[1 / 2, \infty)$ and for $k \geq 1$,

$$
\begin{equation*}
u_{\tau}^{k} \in \underset{u \in \mathcal{P}_{2}\left(\mathbb{R}^{d}\right)}{\operatorname{argmin}}\left\{\frac{1}{2 \tau} W^{2}\left(u, u_{\tau}^{k-1}\right)+\mathcal{F}(u)\right\} \tag{3.1}
\end{equation*}
$$

In other words $u_{\tau}^{k}$ is the unique element such that

$$
\begin{equation*}
\frac{1}{2 \tau} W^{2}\left(u_{\tau}^{k}, u_{\tau}^{k-1}\right)+\mathcal{F}\left(u_{\tau}^{k}\right)=\min _{u \in \mathcal{P}_{2}\left(\mathbb{R}^{d}\right)}\left\{\frac{1}{2 \tau} W^{2}\left(u, u_{\tau}^{k-1}\right)+\mathcal{F}(u)\right\} \tag{3.2}
\end{equation*}
$$

We introduce the piecewise constant interpolation associated with the sequence of minimizers defined as follows $u_{\tau}:[0, \infty) \rightarrow \mathcal{P}_{2}\left(\mathbb{R}^{d}\right)$,

$$
u_{\tau}(0):=u_{\tau}^{0} \quad \text { and } \quad u_{\tau}(t):=u_{\tau}^{\lceil t / \tau\rceil} \quad \text { for } t>0
$$

Recall that the existence and uniqueness of each $u_{\tau}^{k}$ are guaranteed by Theorem 3.2. Keep in mind that given $a \in \mathbb{R}$ we denote $\lfloor a\rfloor=\max \{m \in \mathbb{Z}: m \leq a\}$ and $\lceil a\rceil=\min \{m \in \mathbb{Z}: a \leq m\}$ so that $\lceil a\rceil=\lfloor a\rfloor+1,\lfloor a\rfloor \leq a<\lfloor a\rfloor+1$ and $\lceil a\rceil-1<a \leq\lceil a\rceil$.

The next result follows the idea from [LMS18, Theorem 3.3].
Proposition 3.4. Let $u_{0} \in \mathcal{P}_{2}\left(\mathbb{R}^{d}\right), \tau>0$ and $\left(u_{\tau}^{k}\right)_{k}$ be the sequence defined as in Eq. (3.1).
(i) For all $k \geq 1$ we have

$$
\frac{1}{2 \tau} W^{2}\left(u_{\tau}^{k-1}, u_{\tau}^{k}\right) \leq \mathcal{F}\left(u_{\tau}^{k-1}\right)-\mathcal{F}\left(u_{\tau}^{k}\right)
$$

and therefore $\mathcal{F}\left(u_{\tau}^{k}\right) \leq \mathcal{F}\left(u_{\tau}^{k-1}\right) \leq \cdots \leq \mathcal{F}\left(u_{0}\right)$.
(ii) For all $N \geq 1$ we have

$$
\begin{equation*}
\int_{\mathbb{R}^{d}}|x|^{2} u_{\tau}^{N}(x) \mathrm{d} x \leq \frac{8 d}{\log (2)}\left(1+\tau N \mathcal{F}\left(u_{0}\right)+\int_{\mathbb{R}^{d}}|x|^{2} \mathrm{~d} u_{0}(x)\right) \tag{3.3}
\end{equation*}
$$

Proof. (i) Testing Eq. (3.2) with $u=u_{\tau}^{k-1}$ implies $W^{2}\left(u_{\tau}^{k-1}, u_{\tau}^{k}\right) \leq 2 \tau\left(\mathcal{F}\left(u_{\tau}^{k-1}\right)\right)-\mathcal{F}\left(u_{\tau}^{k}\right)$, whereas the estimate $\mathcal{F}\left(u_{\tau}^{0}\right) \leq \mathcal{F}\left(u_{0}\right)$ is an obvious consequence of the fact that $0<\widehat{G}_{\omega(\tau)} \leq 1$.
(ii) The triangle inequality and the estimates $\frac{1}{2 \tau} W^{2}\left(u_{\tau}^{k-1}, u_{\tau}^{k}\right) \leq \mathcal{F}\left(u_{\tau}^{k-1}\right)-\mathcal{F}\left(u_{\tau}^{k}\right)$ imply

$$
\begin{aligned}
\int_{\mathbb{R}^{d}}|x|^{2} u_{\tau}^{N}(x) \mathrm{d} x=W^{2}\left(u_{\tau}^{N}, \delta_{0}\right) & \leq 2 W^{2}\left(u_{\tau}^{0}, \delta_{0}\right)+2\left(\sum_{k=1}^{N} W\left(u_{\tau}^{k-1}, u_{\tau}^{k}\right)\right)^{2} \\
& \leq 2 W^{2}\left(u_{\tau}^{0}, \delta_{0}\right)+4 \tau N \sum_{k=1}^{N} \frac{1}{2 \tau} W^{2}\left(u_{\tau}^{k-1}, u_{\tau}^{k}\right) \\
& \leq 2 W^{2}\left(u_{\tau}^{0}, \delta_{0}\right)+4 \tau N \sum_{k=1}^{N} \mathcal{F}\left(u_{\tau}^{k-1}\right)-\mathcal{F}\left(u_{\tau}^{k}\right) \\
& =2 W^{2}\left(u_{\tau}^{0}, \delta_{0}\right)+4 \tau N\left(\mathcal{F}\left(u_{\tau}^{0}\right)-\mathcal{F}\left(u_{\tau}^{N}\right)\right) \\
& \leq 2 W^{2}\left(G_{\omega(\tau)} * u_{0}, \delta_{0}\right)+4 \tau N \mathcal{F}\left(u_{\tau}^{0}\right)
\end{aligned}
$$

A standard computation reveals that

$$
\begin{equation*}
W^{2}\left(G_{\tau} * u, \delta_{0}\right) \leq 2 W^{2}\left(u, \delta_{0}\right)+2 W^{2}\left(G_{\tau}, \delta_{0}\right)=2 \int_{\mathbb{R}^{d}}|x|^{2} \mathrm{~d} u(x)+4 \tau d \tag{3.4}
\end{equation*}
$$

We deduce from this, and the fact that $\omega(\tau) \leq \frac{1}{\log (2)}$, that

$$
\begin{aligned}
\int_{\mathbb{R}^{d}}|x|^{2} u_{\tau}^{N}(x) \mathrm{d} x=W^{2}\left(u_{\tau}^{N}, \delta_{0}\right) & \leq 4 \tau N \mathcal{F}\left(u_{0}\right)+4 \int_{\mathbb{R}^{d}}|x|^{2} \mathrm{~d} u_{0}(x)+8 d \omega(\tau) \\
& \leq \frac{8 d}{\log (2)}\left(1+\tau N \mathcal{F}\left(u_{0}\right)+\int_{\mathbb{R}^{d}}|x|^{2} \mathrm{~d} u_{0}(x)\right) .
\end{aligned}
$$

Next, we give the definition of an absolutely continuous curve.
Definition 3.5 ([AGS08, Definition 1.1.1], [ABS21, Definition 9.1]). We say that a curve $u$ : $[0, \infty) \rightarrow \mathcal{P}_{2}\left(\mathbb{R}^{d}\right)$ belongs to $A C^{2}\left([0, \infty),\left(\mathcal{P}_{2}\left(\mathbb{R}^{d}\right), W\right)\right)$ if there exist $m \in L^{2}((0, \infty))$ such that

$$
\begin{equation*}
W\left(u\left(t_{1}\right), u\left(t_{2}\right)\right) \leq \int_{t_{1}}^{t_{2}} m(t) \mathrm{d} t, \quad \text { for every } 0 \leq t_{1}<t_{2}<\infty . \tag{3.5}
\end{equation*}
$$

It is worth mentioning that, if $u \in A C^{2}\left([0, \infty),\left(\mathcal{P}_{2}\left(\mathbb{R}^{d}\right), W\right)\right)$ then (see [AGS08, Theorem 1.1.2] or [ABS21, Theorem 9.2]) $u$ has a metric derivative $\left|u^{\prime}\right|$ almost everywhere, that is, the following limit exists

$$
\left|u^{\prime}\right|(t):=\lim _{s \rightarrow 0} \frac{W(u(t+s), u(t))}{|s|}
$$

for almost all $t>0$. Accordingly, the Lebesgue Differentiation Theorem implies $\left|u^{\prime}\right|(t) \leq m(t)$ for almost all $t>0$. Furthermore, $t \mapsto\left|u^{\prime}\right|(t)$ turns out to be the minimal (smallest) among functions $m$ in $L^{2}((0, \infty))$ satisfying Eq. (3.5).

The next result establishes the existence of an absolute continuous curve $u:[0, \infty) \rightarrow\left(\mathcal{P}_{2}\left(\mathbb{R}^{d}\right), W\right)$, limit of the piecewise constant curves $\left(u_{\tau}\right)_{\tau}$ as $\tau \rightarrow 0$ and representing the minimizing movement (see [AGS08, Definition 2.0.6]) for the main Eq. (1.1).
Theorem 3.6 (Convergence of the Minimizing Movement Scheme). Let $u_{0} \in D(\mathcal{F})=\dot{H}^{\psi^{-1}}\left(\mathbb{R}^{d}\right) \cap$ $\mathcal{P}_{2}\left(\mathbb{R}^{d}\right)$. Let $u_{\tau}(t)=u_{\tau}^{[t / \tau\rceil}$ defined as in Definition 3.3. Then, there holds

$$
\lim _{\tau \rightarrow 0^{+}} \mathcal{F}\left(u_{\tau}^{0}\right)=\mathcal{F}\left(u_{0}\right), \quad \text { and } \quad \limsup _{\tau \rightarrow 0^{+}} W\left(u_{\tau}^{0}, u_{0}\right)<\infty
$$

as well as

$$
u_{\tau}^{0} \rightarrow u_{0}, \quad \text { narrowly as } \tau \rightarrow 0^{+}
$$

Moreover, there is a curve $u \in A C^{2}\left([0, \infty),\left(\mathcal{P}_{2}\left(\mathbb{R}^{d}\right), W\right)\right)$ and a subsequence $\tau_{n} \rightarrow 0^{+}$, as $n \rightarrow \infty$, such that $u\left(0^{+}\right)=u_{0}$ and

$$
u_{\tau_{n}}(t) \rightarrow u(t) \quad \text { narrowly as } n \rightarrow \infty
$$

for all $t>0$.
Proof. First of all, since $u_{\tau}^{0}=G_{\omega(\tau)} * u_{0}$ (recall that we denote by $G_{t}(x)$ the heat kernel at time $\left.t, G_{t}(x)=\frac{1}{(4 t \pi)^{d / 2}} e^{-\frac{|x|^{2}}{4 t}}\right)$ by definition, we have that $W^{2}\left(u_{\tau}^{0}, u_{0}\right) \leq 2 d \omega(\tau) \rightarrow 0$ as $\tau \rightarrow 0^{+}$. This implies on the one hand that $\lim \sup _{\tau \rightarrow 0^{+}} W\left(u_{\tau}^{0}, u_{0}\right)=0$, and a standard computation shows that $u_{\tau}^{0} \rightarrow u_{0}$ narrowly as $\tau \rightarrow 0^{+}$. Secondly, given that $\widehat{G}_{\omega(\tau)} \rightarrow 1$ and $0 \leq \widehat{G}_{\omega(\tau)} \leq 1$ the Dominated Convergence Theorem implies

$$
\int_{\mathbb{R}^{d}}\left|\widehat{u}_{0}(\xi)\right|^{2}\left|\widehat{G}_{\omega(\tau)}(\xi)\right|^{2} \psi^{-1}(\xi) \mathrm{d} \xi \rightarrow \int_{\mathbb{R}^{d}}\left|\widehat{u}_{0}(\xi)\right|^{2} \psi^{-1}(\xi) \mathrm{d} \xi, \quad \text { as } \tau \rightarrow 0^{+}
$$

Equivalently we get that $\mathcal{F}\left(u_{\tau}^{0}\right) \rightarrow \mathcal{F}\left(u_{0}\right)$ as $\tau \rightarrow 0^{+}$. Lastly, we want to show the existence of $u \in A C^{2}\left([0, \infty),\left(\mathcal{P}_{2}\left(\mathbb{R}^{d}\right), W\right)\right)$ which is a narrow limit of a subsequence $\left(u_{\tau_{n}}\right)_{n}$. Accordingly, for $t>0$ we define $m_{\tau}(t)$ by

$$
m_{\tau}(t):= \begin{cases}\frac{W\left(u_{\tau}(t), u_{\tau}(t-\tau)\right)}{\tau}, & t-\tau>0 \\ \frac{W\left(u_{\tau}(t), u_{\tau}(0)\right)}{\tau}, & t-\tau \leq 0\end{cases}
$$

The estimate $W^{2}\left(u_{\tau}^{k-1}, u_{\tau}^{k}\right) \leq 2 \tau\left(\mathcal{F}\left(u_{\tau}^{k-1}\right)-\mathcal{F}\left(u_{\tau}^{k}\right)\right)$ (see Proposition 3.4) implies

$$
\begin{aligned}
\int_{0}^{\infty} m_{\tau}^{2}(t) \mathrm{d} t & =\sum_{k=0}^{\infty} \int_{\tau k}^{\tau(k+1)} \frac{W^{2}\left(u_{\tau}(t), u_{\tau}(t-\tau)\right)}{\tau^{2}} \mathrm{~d} t=2 \sum_{k=0}^{\infty} \frac{W^{2}\left(u_{\tau}^{k+1}, u_{\tau}^{k}\right)}{2 \tau} \\
& \leq 2 \sum_{k=0}^{\infty} \mathcal{F}\left(u_{\tau}^{k}\right)-\mathcal{F}\left(u_{\tau}^{k+1}\right)=2 \lim _{N \rightarrow \infty} \mathcal{F}\left(u_{\tau}^{0}\right)-\mathcal{F}\left(u_{\tau}^{N+1}\right) \\
& \leq 2 \mathcal{F}\left(u_{\tau}^{0}\right) .
\end{aligned}
$$

Due to the uniform boundedness of the family $\left(m_{\tau}\right)_{\tau}$ in $L^{2}((0, \infty))$, it converges weakly, up to a subsequence, to some $m \in L^{2}([0, \infty))$, as $\tau \rightarrow 0^{+}$. Moreover, the weak lower semicontinuity of the $L^{2}$-norm implies

$$
\int_{0}^{\infty} m^{2}(t) \mathrm{d} t \leq \liminf _{\tau \rightarrow 0^{+}} \int_{0}^{\infty} m_{\tau}^{2}(t) \mathrm{d} t \leq 2 \mathcal{F}\left(u_{0}\right)
$$

Now, for $0 \leq t_{1}<t_{2}$ we set $N_{i}(\tau):=\left\lfloor t_{i} / \tau\right\rfloor$ so that $\tau N_{i}(\tau) \leq t_{i}<\tau\left(N_{i}(\tau)+1\right)$. Taking into account the fact that, $u_{\tau}\left(t_{i}\right)=u_{\tau}^{N_{i}}$ or $u_{\tau}\left(t_{i}\right)=u_{\tau}^{N_{i}+1}$ depending whether $t_{i} / \tau$ is an integer or not, the triangle inequality yields

$$
\begin{aligned}
W\left(u_{\tau}\left(t_{1}\right), u_{\tau}\left(t_{2}\right)\right) & \leq W\left(u_{\tau}\left(t_{1}\right), u_{\tau}^{N_{1}(\tau)+1}\right)+\sum_{k=N_{1}(\tau)+1}^{N_{2}(\tau)} W\left(u_{\tau}^{k-1}, u_{\tau}^{k}\right)+W\left(u_{\tau}^{N_{2}(\tau)}, u_{\tau}\left(t_{2}\right)\right) \\
& \leq \sum_{k=N_{1}(\tau)+1}^{N_{2}(\tau)+1} W\left(u_{\tau}^{k-1}, u_{\tau}^{k}\right)=\sum_{k=N_{1}(\tau)+1}^{N_{2}(\tau)+1} \int_{\tau k}^{\tau(k+1)} \frac{W\left(u_{\tau}(t), u_{\tau}(t-\tau)\right)}{\tau} \mathrm{d} t \\
& =\int_{\tau N_{1}(\tau)+\tau}^{\tau N_{2}(\tau)+\tau} m_{\tau}(t) \mathrm{d} t .
\end{aligned}
$$

Since $\tau N_{i}(\tau) \rightarrow t_{i}$, the weak convergence of $\left(m_{\tau}\right)_{\tau}$ entails

$$
\begin{equation*}
\limsup _{\tau \rightarrow 0^{+}} W\left(u_{\tau}\left(t_{1}\right), u_{\tau}\left(t_{2}\right)\right) \leq \lim _{\tau \rightarrow 0^{+}} \int_{\tau N_{1}(\tau)}^{\tau N_{2}(\tau)} m_{\tau}(t) \mathrm{d} t=\int_{t_{1}}^{t_{2}} m(t) \mathrm{d} t \tag{3.6}
\end{equation*}
$$

Combining this with the Cauchy-Schwartz inequality yields

$$
\begin{equation*}
\limsup _{\tau \rightarrow 0^{+}} W\left(u_{\tau}\left(t_{1}\right), u_{\tau}\left(t_{2}\right)\right) \leq\|m\|_{L^{2}((0, \infty))}\left|t_{2}-t_{1}\right|^{1 / 2} \tag{3.7}
\end{equation*}
$$

Next, let us fix $T>0$. Estimate (3.3) (see Proposition 3.4) and the fact that $\tau\lfloor t / \tau\rfloor \leq t$ yield

$$
W^{2}\left(u_{\tau}^{\lfloor t / \tau\rfloor}, \delta_{0}\right)=\int_{\mathbb{R}^{d}}|x|^{2} u_{\tau}^{\lfloor t / \tau\rfloor}(x) \mathrm{d} x \leq R_{T},
$$

for all $0 \leq t \leq T$, where

$$
R_{T}=\frac{8 d}{\log (2)}\left(1+T \mathcal{F}\left(u_{0}\right)+\int_{\mathbb{R}^{d}}|x|^{2} \mathrm{~d} u_{0}(x)\right)
$$

Consider $\mathcal{K}_{T}=\left\{u \in \mathcal{P}_{2}\left(\mathbb{R}^{d}\right): W^{2}\left(u, \delta_{0}\right) \leq R_{T}\right\}$ which is a bounded closed subset of $\mathcal{P}_{2}\left(\mathbb{R}^{d}\right)$, and recall that $u_{\tau}(t)=u_{\tau}^{\lfloor t / \tau\rfloor}$. Note that

$$
\begin{equation*}
\mathcal{K}_{T} \text { is narrowly compact in } \mathcal{P}_{2}\left(\mathbb{R}^{d}\right), \quad \text { and } \quad u_{\tau}(t) \in \mathcal{K}_{T}, \text { for all } 0 \leq t<T . \tag{3.8}
\end{equation*}
$$

In light of Eq. (3.7) and Eq. (3.8) and the refined version of the Arzelá-Ascoli Theorem (cf. [AGS08, Proposition 3.3.1]) there is subsequence $\tau_{n} \rightarrow 0^{+}$and a curve $u:[0, \infty) \rightarrow\left(\mathcal{P}_{2}\left(\mathbb{R}^{d}\right), W\right)$ such that $u_{\tau_{n}}(t) \rightarrow u(t)$ for almost all $t \geq 0$. Estimate (3.6) in conjunction with the narrow convergence of $\left(u_{\tau}(t)\right)$ and the lower semicontinuity of $W$, see [AGS08, Proposition 7.1.3], yield

$$
W\left(u\left(t_{1}\right), u\left(t_{2}\right)\right) \leq \liminf _{n \rightarrow \infty} W\left(u_{\tau_{n}}\left(t_{1}\right), u_{\tau_{n}}\left(t_{2}\right)\right) \leq \int_{t_{1}}^{t_{2}} m(t) \mathrm{d} t .
$$

It follows that $u \in A C^{2}\left([0, \infty),\left(\mathcal{P}_{2}\left(\mathbb{R}^{d}\right), W\right)\right)$ since $m \in L^{2}((0, \infty))$. In particular, since $u_{\tau}^{0}=$ $u_{\tau}(0) \rightarrow u_{0}$ narrowly as $\tau \rightarrow 0^{+}$up to a modification of $u$ at $t=0$ we get $u_{0}=u\left(0^{+}\right)$.

## 4. Improved regularity of discrete solutions

In this section, we aim to establish better regularity properties of the solutions to minimization problem, Eq. (3.1), using the flow interchange technique introduced by Matthes, McCann, and Savaré [MMS09]. Let us first recall some basics of this technique.

Definition 4.1 (Displacement convex entropy). Let $V \in C^{1}((0, \infty)) \cap C([0, \infty))$ with $V(0)=0$ be a convex function such that

$$
\lim _{x \rightarrow 0^{+}} \frac{V(x)}{x^{\alpha}}>-\infty, \quad \text { for some } \alpha>\frac{d}{d+2}
$$

and the McCann displacement convexity condition holds [McC97], i.e.,

$$
r \mapsto r^{d} V\left(r^{-d}\right) \quad \text { is convex and decreasing on }(0, \infty)
$$

In this case, the displacement convex entropy associated to $V$ is the functional $\mathcal{V}: \mathcal{P}_{2}\left(\mathbb{R}^{d}\right) \rightarrow$ $\mathbb{R} \cup\{\infty\}$ defined as follows

$$
\mathcal{V}(u)= \begin{cases}\int_{\mathbb{R}^{d}} V(u(x)) \mathrm{d} x, & \text { if } \mathrm{d} u(x)=u(x) \mathrm{d} x \\ \infty, & \text { otherwise }\end{cases}
$$

The effective domain of $\mathcal{V}$ is denoted by $D(\mathcal{V})=\left\{u \in \mathcal{P}_{2}\left(\mathbb{R}^{d}\right): \mathcal{V}(u)<\infty\right\}$. It is worth mentioning that $\mathcal{V}$ is lower semicontinuous with respect to the narrow convergence.

According to [AGS08, Theorem 11.1.4] see also [MS20], the displacement convex entropy $\mathcal{V}$ generates a continuous semigroup $S_{t}: D(\mathcal{V}) \rightarrow D(\mathcal{V})$ satisfying the following evolution variational inequality (EVI):

$$
\begin{equation*}
\frac{1}{2} W^{2}\left(S_{t} u, v\right)-\frac{1}{2} W^{2}(u, v) \leq t\left(\mathcal{V}(v)-\mathcal{V}\left(S_{t} u\right)\right), \quad \text { for all } u, v \in D(\mathcal{V}), t>0 \tag{4.1}
\end{equation*}
$$

where, by definition, $S_{t} u$ is the unique distributional (with respect to the narrow topology) solution of the Cauchy problem

$$
\begin{equation*}
\partial_{t} w=\Delta L_{V}(w), \quad w(0)=u \tag{4.2}
\end{equation*}
$$

with $L_{V}(u):=u V^{\prime}(u)-V(u)$. The flow associated to $\mathcal{V},\left(S_{t}\right)_{t}$, is a semigroup of contractions with respect to the Wasserstein distance $W$ and extends to $\overline{D(\mathcal{V})}=\mathcal{P}_{2}\left(\mathbb{R}^{d}\right)$.

Remark 4.2. It is worth emphasizing that the regularizing effect of the semigroup implies that $S_{t} u \in D(\mathcal{V})$ for all $u \in \mathcal{P}_{2}\left(\mathbb{R}^{d}\right)$. Analogously, if $u_{0} \in \mathcal{P}_{2}\left(\mathbb{R}^{d}\right)$ then the boundedness of $u_{\tau}^{0}=G_{\omega(\tau)} * u_{0}$ implies that we also have $u_{\tau}^{0} \in D(\mathcal{V})$ for any displacement convex entropy.

Example 4.3. The standard example of a displacement convex entropy is obtained by considering the convex function $H(x)=x \log x-x, x>0$ and $H(0)=0$, giving rise to the usual BoltzmannShannon entropy

$$
\mathcal{H}(u)= \begin{cases}\int_{\mathbb{R}^{d}} u(x) \log u(x)-u(x) \mathrm{d} x, & \text { if } \mathrm{d} u(x)=u(x) \mathrm{d} x \\ \infty, & \text { otherwise }\end{cases}
$$

Indeed it is clear that $H \in C^{1}((0, \infty)) \cap C([0, \infty))$, and it can be shown that $\lim _{x \rightarrow 0^{+}} \frac{H(x)}{x^{\alpha}}=0$, for all $\alpha \in\left(\frac{d}{d+2}, 1\right)$. We also find that, $r^{d} H\left(r^{-d}\right)=-d \log (r)-1$ is convex and decreasing on $(0, \infty)$. For completeness, let us point out the well-known fact that $L_{H}(u)=u H^{\prime}(u)-u H(u)=u$, which, according to Eq. (4.2), gives rise to heat semigroup since, here, $S_{t} u$ is the distributional solution to the heat equation $\partial_{t} w=\Delta w$ and $w(0)=u$.

Definition 4.4 (Dissipation along the flow). The dissipation of $\mathcal{F}$ along the flow $S_{t}$ associated to $\mathcal{V}$ at the point $u \in D(\mathcal{F})$ is defined as

$$
\mathscr{D} \mathcal{V}[\mathcal{F}](u):=\limsup _{t \rightarrow 0^{+}} \frac{\mathcal{F}(u)-\mathcal{F}\left(S_{t} u\right)}{t} .
$$

Theorem 4.5 (Flow interchange). Let $\left(u_{\tau}^{k}\right)_{k}$ be a sequence solving Eq. (3.1) and $\mathcal{V}$ be a displacement convex entropy. If, for all $k \in \mathbb{N}, \mathscr{D} \mathcal{V}[\mathcal{F}]\left(u_{\tau}^{k}\right)>-\infty$, then $u_{\tau}^{k} \in D(\mathcal{V})$ and

$$
\mathscr{D} \mathcal{V}[\mathcal{F}]\left(u_{\tau}^{k}\right) \leq \frac{\mathcal{V}\left(u_{\tau}^{k-1}\right)-\mathcal{V}\left(u_{\tau}^{k}\right)}{\tau} \quad \text { for all } k \geq 1
$$

The statement and proof can be found, for example, in [MMS09]. The next result infers that $\mathscr{D}_{\mathcal{H}}\left(u_{\tau}^{k}\right)>-\infty$, for any $k \in \mathbb{N}$, that is, we consider the flow interchange for the particular case $\mathcal{V}=\mathcal{H}$.
Lemma 4.6. Let $u_{0} \in D(\mathcal{F})$ and $\left(u_{\tau}^{k}\right)$ be defined as in Eq. (3.1). For any $k \geq 0$ we have $u_{\tau}^{k} \in H^{\widetilde{\psi}}\left(\mathbb{R}^{d}\right) \cap D(\mathcal{H})$, (recalling that $H^{\widetilde{\psi}}\left(\mathbb{R}^{d}\right)=\dot{H}^{\tilde{\psi}}\left(\mathbb{R}^{d}\right) \cap L^{2}\left(\mathbb{R}^{d}\right)$ ). Moreover we have

$$
\left\|u_{\tau}^{k}\right\|_{\dot{H}_{\tilde{\psi}}\left(\mathbb{R}^{d}\right)}^{2} \leq \mathscr{D}_{\mathcal{H}}[\mathcal{F}]\left(u_{\tau}^{k}\right), \quad \text { for any } k \geq 0
$$

Proof. First, we need to show that $t \mapsto \mathcal{F}\left(w_{t}\right)$ belongs to $C^{1}((0, \infty)) \cap C([0, \infty))$. By definition $u_{\tau}^{0}=G_{\omega(\tau)} * u_{0}$, and therefore $u_{\tau}^{0} \in D(\mathcal{F})$. Recall that the flow associated to $\mathcal{H}$ is the heat semigroup, viz., $w_{t}:=S_{t} u_{\tau}^{k}=G_{t} * u_{\tau}^{k}$. In the Fourier variables, we have $\widehat{w}_{t}(\xi)=\widehat{G}_{t}(\xi) \widehat{u}_{\tau}^{k}(\xi)=\widehat{u}_{\tau}^{k}(\xi) e^{-t|\xi|^{2}}$. In particular, we have $\widehat{w}_{t} \in C^{1}((0, \infty)) \cap C([0, \infty))$ and $\partial_{t} \widehat{w}_{t}(\xi)=-|\xi|^{2} \widehat{w}_{t}(\xi)$. It is not difficult to see that

$$
\left.\left.\left|\partial_{t}\right| \widehat{w}_{t}(\xi)\right|^{2}\left|\psi^{-1}(\xi)=2\right| \widehat{w}_{t}(\xi)\right|^{2}|\xi|^{2} \psi^{-1}(\xi)=2\left|\widehat{w}_{t}(\xi)\right|^{2} \widetilde{\psi}(\xi) .
$$

For fixed $\varepsilon>0$ and all $t>\varepsilon$, we can estimate

$$
\begin{aligned}
\left|\widehat{w}_{t}(\xi)\right|^{2} \widetilde{\psi}(\xi) & \leq \max _{r \geq 0} r e^{-2 r t}\left|\widehat{u}_{\tau}^{k}(\xi)\right|^{2} \psi^{-1}(\xi) \\
& =\frac{1}{2 t e}\left|\widehat{u}_{\tau}^{k}(\xi)\right|^{2} \psi^{-1}(\xi) \\
& \leq \frac{1}{2 \varepsilon e}\left|\widehat{u}_{\tau}^{k}(\xi)\right|^{2} \psi^{-1}(\xi) .
\end{aligned}
$$

We know that $\mathcal{F}\left(u_{\tau}^{k}\right) \leq \mathcal{F}\left(u_{\tau}^{k-1}\right) \leq \cdots \leq \mathcal{F}\left(u_{\tau}^{0}\right)<\infty$, i.e., $\left\|\left|\widehat{u}_{\tau}^{k}\right|^{2} \psi^{-1}\right\|_{L^{1}\left(\mathbb{R}^{d}\right)}=2 \mathcal{F}\left(u_{\tau}^{k}\right)<\infty$. In particular, we get $w_{t}=S_{t} u_{\tau}^{k} \in \dot{H}^{\widetilde{\psi}}\left(\mathbb{R}^{d}\right)$, since the previous estimate implies

$$
\left\|w_{t}\right\|_{\dot{H}^{\tilde{\psi}}\left(\mathbb{R}^{d}\right)}^{2} \leq \frac{1}{2 t e}\left\|u_{\tau}^{k}\right\|_{\dot{H}^{\psi^{-1}}\left(\mathbb{R}^{d}\right)}^{2}=\frac{1}{t e} \mathcal{F}\left(u_{\tau}^{k}\right)<\infty .
$$

Next, again for $\varepsilon>0$ fixed, we find that $\left|\widehat{u}_{\tau}^{k}\right|^{2} \psi^{-1} \in L^{1}\left(\mathbb{R}^{d}\right)$ and, by combining the two preceding estimates, we have

$$
\begin{equation*}
\left.\left.\left|\partial_{t}\right| \widehat{w}_{t}(\xi)\right|^{2}\left|\psi^{-1}(\xi)=\left|\widehat{w}_{t}(\xi)\right|^{2} \widetilde{\psi}(\xi) \leq \frac{1}{2 \varepsilon e}\right| \widehat{u}_{\tau}^{k}(\xi)\right|^{2} \psi^{-1}(\xi), \tag{4.3}
\end{equation*}
$$

for all $t>\varepsilon, \xi \in \mathbb{R}^{d}$ On the one hand, Leibniz rule together with the estimate in Eq. (4.3) implies that $t \mapsto \mathcal{F}\left(w_{t}\right)$ is differentiable on $(\varepsilon, \infty)$ and we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} t} \mathcal{F}\left(w_{t}\right) & =\frac{1}{2} \frac{\mathrm{~d}}{\mathrm{~d} t} \int_{\mathbb{R}^{d}}\left|\widehat{w}_{t}(\xi)\right|^{2} \psi^{-1}(\xi) \mathrm{d} \xi=\int_{\mathbb{R}^{d}} \mathcal{R e}\left(\widehat{\widehat{w}}_{t}(\xi) \partial_{t} \widehat{w}_{t}(\xi)\right) \psi^{-1}(\xi) \mathrm{d} \xi \\
& =-\int_{\mathbb{R}^{d}}\left|\widehat{w}_{t}(\xi)\right|^{2}|\xi|^{2} \psi^{-1}(\xi) \mathrm{d} \xi=-\left\|w_{t}\right\|_{\dot{H}^{\tilde{\psi}}\left(\mathbb{R}^{d}\right)^{2}}^{2} .
\end{aligned}
$$

On the other hand, by the dominated convergence theorem, Eq. (4.3) also implies that $t \mapsto$ $-\left\|w_{t}\right\|_{\dot{H}^{\tilde{\psi}}\left(\mathbb{R}^{d}\right)}^{2}=\frac{\mathrm{d}}{\mathrm{d} t} \mathcal{F}\left(w_{t}\right)$ is continuous on $(\varepsilon, \infty)$. Since $\varepsilon>0$ is arbitrarily chosen we deduce that $t \mapsto \mathcal{F}\left(w_{t}\right)$ belongs to $C^{1}((0, \infty))$ with

$$
\frac{\mathrm{d}}{\mathrm{~d} t} \mathcal{F}\left(w_{t}\right)=-\left\|w_{t}\right\|_{\dot{H} \tilde{\psi}\left(\mathbb{R}^{d}\right)}^{2}=-\left\|S_{t} u_{\tau}^{k}\right\|_{\dot{H} \tilde{\psi}\left(\mathbb{R}^{d}\right)}^{2}
$$

On the other side, $\left|\widehat{w}_{t}(\xi)\right|^{2}=e^{-2 t|\xi|^{2}}\left|\widehat{u}_{\tau}^{k}(\xi)\right|^{2} \rightarrow\left|\widehat{u}_{\tau}^{k}(\xi)\right|^{2}$ as $t \rightarrow 0^{+}$and $\left|\widehat{w}_{t}(\xi)\right|^{2} \leq\left|\widehat{u}_{\tau}^{k}(\xi)\right|^{2}$. The dominated convergence theorem implies $\mathcal{F}\left(w_{t}\right) \rightarrow \mathcal{F}\left(u_{\tau}^{k}\right)$ as $t \rightarrow 0^{+}$; which proves the continuity of $t \mapsto \mathcal{F}\left(w_{t}\right)$ at $t=0$. Therefore, $t \mapsto \mathcal{F}\left(w_{t}\right)$ belongs to $C^{1}((0, \infty)) \cap C([0, \infty))$.
Next, the fundamental theorem of calculus implies

$$
\frac{\mathcal{F}\left(u_{\tau}^{k}\right)-\mathcal{F}\left(S_{t} u_{\tau}^{k}\right)}{t}=\frac{\mathcal{F}\left(w_{0}\right)-\mathcal{F}\left(w_{t}\right)}{t}=\frac{1}{t} \int_{0}^{t}\left\|w_{r}\right\|_{\dot{H}^{\tilde{\psi}}\left(\mathbb{R}^{d}\right)}^{2} \mathrm{~d} r .
$$

Additionally, by Fatou's Lemma there holds

$$
\left\|u_{\tau}^{k}\right\|_{\dot{H}^{\tilde{\psi}}\left(\mathbb{R}^{d}\right)}^{2} \leq \liminf _{t \rightarrow 0^{+}}\left\|w_{t}\right\|_{\dot{H}^{\tilde{\psi}}\left(\mathbb{R}^{d}\right)}^{2}
$$

Combining the two estimates, it follows that

$$
\mathscr{D}_{\mathcal{H}}[\mathcal{F}]\left(u_{\tau}^{k}\right)=\limsup _{t \rightarrow 0^{+}} \frac{\mathcal{F}\left(u_{\tau}^{k}\right)-\mathcal{F}\left(S_{t} u_{\tau}^{k}\right)}{t} \geq\left\|u_{\tau}^{k}\right\|_{\dot{H}^{\tilde{\psi}}\left(\mathbb{R}^{d}\right)}^{2} .
$$

We deduce from Theorem 4.5 that, for any $k \geq 1$, we have $u_{\tau}^{k} \in D(\mathcal{H})$ and

$$
\left\|u_{\tau}^{k}\right\|_{\dot{H}^{\tilde{\psi}}\left(\mathbb{R}^{d}\right)}^{2} \leq \mathscr{D}_{\mathcal{H}}[\mathcal{F}]\left(u_{\tau}^{k}\right) \leq \frac{\mathcal{H}\left(u_{\tau}^{k-1}\right)-\mathcal{H}\left(u_{\tau}^{k}\right)}{\tau} .
$$

Therefore, we may deduce $u_{\tau}^{k} \in \dot{H}^{\tilde{\psi}}\left(\mathbb{R}^{d}\right) \cap D(\mathcal{H})$. Finally, it remains to show that $u_{\tau}^{k} \in L^{2}\left(\mathbb{R}^{d}\right)$. For $|\xi| \geq 1$, it follows from Proposition 2.2 that $\psi(\xi) \leq \kappa_{\nu}\left(1+|\xi|^{2}\right) \leq 2 \kappa_{\nu}|\xi|^{2}$. Then, using Plancherel, we find

$$
\begin{aligned}
\int_{\mathbb{R}^{d}}\left|\widehat{u}_{\tau}^{k}(\xi)\right|^{2} \mathrm{~d} \xi & =\int_{|\xi| \leq 1}\left|\widehat{u}_{\tau}^{k}(\xi)\right|^{2} \mathrm{~d} \xi+\int_{|\xi|>1}\left|\widehat{u}_{\tau}^{k}(\xi)\right|^{2} \psi(\xi) \psi^{-1}(\xi) \mathrm{d} \xi \\
& \leq\left|B_{1}(0)\right|\left\|\widehat{u}_{\tau}^{k}\right\|_{L^{\infty}\left(\mathbb{R}^{d}\right)}^{2}+2 \kappa_{\nu} \int_{|\xi|>1}\left|\widehat{u}_{\tau}^{k}(\xi)\right|^{2}|\xi|^{2} \psi^{-1}(\xi) \mathrm{d} \xi \\
& \leq\left|B_{1}(0)\right|\left\|\widehat{u}_{\tau}^{k}\right\|_{L^{\infty}\left(\mathbb{R}^{d}\right)}^{2}+2 \kappa_{\nu}\left\|u_{\tau}^{k}\right\|_{\dot{H}^{\tilde{\psi}}\left(\mathbb{R}^{d}\right)}^{2}
\end{aligned}
$$

where we also used the fact that $u_{\tau}^{k}$ is a probability density. Hence, we deduce $u_{\tau}^{k} \in L^{2}\left(\mathbb{R}^{d}\right)$, and therefore $u_{\tau}^{k} \in H^{\tilde{\psi}}\left(\mathbb{R}^{d}\right)=\dot{H} \tilde{\psi}^{\tilde{\psi}}\left(\mathbb{R}^{d}\right) \cap L^{2}\left(\mathbb{R}^{d}\right)$.

The following result is an immediate consequence of Theorem 4.5 and Lemma 4.6.

Corollary 4.7. Let $u_{0} \in D(\mathcal{F})$ and $\left(u_{\tau}^{k}\right)_{k}$ be defined as in Eq. (3.1). Then $u_{\tau}^{k} \in H^{\tilde{\psi}}\left(\mathbb{R}^{d}\right) \cap D(\mathcal{H})$ for any $k \geq 0$. Moreover, we have

$$
\left\|u_{\tau}^{k}\right\|_{\dot{H}^{\tilde{\psi}}\left(\mathbb{R}^{d}\right)}^{2} \leq \frac{\mathcal{H}\left(u_{\tau}^{k-1}\right)-\mathcal{H}\left(u_{\tau}^{k}\right)}{\tau}, \quad \text { for any } k \geq 1
$$

In particular $\mathcal{H}\left(u_{\tau}^{k}\right) \leq \mathcal{H}\left(u_{\tau}^{k-1}\right)$.
Next, we use the lifting (perturbation) of the entropy technique introduced in [MMS09] to show that $u_{\tau}^{k}$ is in the domain of every displacement convex entropy.
Theorem 4.8. Assume that $\widetilde{\psi}(\xi)=|\xi|^{2} \psi^{-1}(\xi)$ is the symbol associated with a radial Lévy kernel $\widetilde{\nu}$. Let $\left(u_{\tau}^{k}\right)_{k}$ be the sequence of Definition 3.3. Let $\mathcal{G}$ be a displacement convex entropy with density function $G$. Then, for any $k \geq 0, u_{\tau}^{k} \in D(\mathcal{G})$ and we have

$$
0 \leq\left(u_{\tau}^{k}, L_{G}\left(u_{\tau}^{k}\right)\right)_{\tilde{\psi}} \leq \mathscr{D}_{\mathcal{G}}[\mathcal{F}]\left(u_{\tau}^{k}\right) \leq \frac{\mathcal{G}\left(u_{\tau}^{k-1}\right)-\mathcal{G}\left(u_{\tau}^{k}\right)}{\tau} .
$$

Proof. Note that $u_{\tau}^{0}=G_{\omega(\tau)} * u$ clearly belongs to $D(\mathcal{G})$. For fixed $\varepsilon>0$, consider the perturbed displacement convex entropy

$$
\mathcal{V}(u)=\mathcal{G}(u)+\varepsilon \mathcal{H}(u) .
$$

Let us denote by $S_{t}$ the flow associated to $\mathcal{V}$. For fixed $k \geq 1, \tau>0$, and $\varepsilon>0$ we set $w_{t}=S_{t} u_{\tau}^{k}$ which is the unique solution to the generalized porous medium equation

$$
\partial_{t} w_{t}=\Delta \Phi\left(w_{t}\right)=\operatorname{div}\left(\Phi^{\prime}\left(w_{t}\right) \nabla w_{t}\right), \quad \text { and } \quad w_{0}=u_{\tau}^{k},
$$

where $\Phi(v)=L_{G}(v)+\varepsilon v$, and $L_{G}(v)=v G^{\prime}(v)-G(v)$ as before. Since $G$ is convex we have $\Phi^{\prime}(r)=r G^{\prime \prime}(r)+\varepsilon \geq \varepsilon$, for $r>0$, that is $\Phi$ is monotone increasing, and the above equation nondegenerate. Note that since $u_{\tau}^{k} \in L^{1}\left(\mathbb{R}^{d}\right)$ and $u_{\tau}^{k} \geq 0$, each $w_{t}$ is strictly positive, see [Váz07, Chapter 3] and bounded since $\Phi^{\prime}(r) \geq \varepsilon>0$, see for instance [Bén78, V7́9, BB85, Váz06, Váz05]. The facts that $w_{t} \geq 0, \Phi^{\prime}\left(w_{t}\right) \geq 0$, and $\partial_{t} w_{t}=\Delta \Phi\left(w_{t}\right)$ imply that

$$
\frac{\mathrm{d}}{\mathrm{~d} t} \int_{\mathbb{R}^{d}}\left|w_{t}(x)\right|^{2} \mathrm{~d} x=2 \int_{\mathbb{R}^{d}} w_{t}(x) \Delta \Phi\left(w_{t}(x)\right) \mathrm{d} x=-2 \int_{\mathbb{R}^{d}}\left|\nabla w_{t}(x)\right|^{2} \Phi^{\prime}\left(w_{t}(x)\right) \mathrm{d} x \leq 0 .
$$

Since $w_{0}=u_{\tau}^{k} \in L^{2}\left(\mathbb{R}^{d}\right) \cap \dot{H} \tilde{\psi}^{( }\left(\mathbb{R}^{d}\right)$, by Lemma 4.6 , we find that $w_{t} \in L^{2}\left(\mathbb{R}^{d}\right)$ and

$$
\left\|w_{t}\right\|_{L^{2}\left(\mathbb{R}^{d}\right)} \leq\left\|u_{\tau}^{k}\right\|_{L^{2}\left(\mathbb{R}^{d}\right)}
$$

Note that the narrow convergence of $w_{t}$ to $u_{\tau}^{k}$ implies that $\widehat{w}_{t}(\xi) \rightarrow \widehat{u}_{\tau}^{k}(\xi)$ as $t \rightarrow 0$ for all $\xi$. Furthermore, Fatou's Lemma yields $\left\|w_{t}\right\|_{L^{2}\left(\mathbb{R}^{d}\right)} \rightarrow\left\|u_{\tau}^{k}\right\|_{L^{2}\left(\mathbb{R}^{d}\right)}$ as $t \rightarrow 0$. These two results together with the pointwise convergence $\widehat{w}_{t} \rightarrow \widehat{u}_{\tau}^{k}$, as $t \rightarrow 0$, yield

$$
\lim _{t \rightarrow 0}\left\|w_{t}-u_{\tau}^{k}\right\|_{L^{2}\left(\mathbb{R}^{d}\right)}=0
$$

Therefore, a subsequence (not relabeled) satisfies that $w_{t} \rightarrow u_{\tau}^{k}$, a.e. as $t \rightarrow 0$. According to Theorem 2.7, for $u, v \in H^{\tilde{\psi}}\left(\mathbb{R}^{d}\right)$ we have

$$
(u, v)_{\tilde{\psi}}:=\int_{\mathbb{R}^{d}} \widehat{u}(\xi) \overline{\hat{v}}(\xi) \widetilde{\psi}(\xi) \mathrm{d} \xi=\iint_{\mathbb{R}^{d} \mathbb{R}^{d}}(u(x)-u(y))(v(x)-v(y)) \widetilde{\nu}(x-y) \mathrm{d} y \mathrm{~d} x .
$$

Since $L_{G}$ increases, $(u(x)-u(y))\left(L_{G} \circ u(x)-L_{G} \circ u(y)\right) \geq 0$, so that Fatou's Lemma yields

$$
\liminf _{t \rightarrow 0}\left(L_{G} \circ w_{t}, w_{t}\right)_{\tilde{\psi}}+\varepsilon\left(w_{t}, w_{t}\right)_{\tilde{\psi}} \geq\left(L_{G} \circ u_{\tau}^{k}, u_{\tau}^{k}\right)_{\tilde{\psi}}+\varepsilon\left(u_{\tau}^{k}, u_{\tau}^{k}\right)_{\tilde{\psi}} .
$$

Recall that $L_{G}$ is locally Lipschitz and increasing. For every function $u \in L^{\infty}\left(\mathbb{R}^{d}\right)$,

$$
0 \leq\left(L_{G} \circ u(x)-L_{G} \circ u(y)\right)^{2} \leq C_{L}(u(x)-u(y))\left(L_{G} \circ u(x)-L_{G} \circ u(y)\right) \leq C_{L}^{2}(u(x)-u(y))^{2},
$$

where $C_{L}>0$ is a Lipschitz constant of $L_{G}$ depending on $u$. This immediately implies

$$
\begin{equation*}
0 \leq\left(L_{G} \circ u, L_{G} \circ u\right)_{\tilde{\psi}} \leq C_{L}\left(L_{G} \circ u, u\right)_{\tilde{\psi}} \leq C_{L}^{2}(u, u)_{\tilde{\psi}} . \tag{4.4}
\end{equation*}
$$

Next, we exploit these estimates with $u=w_{t} \in L^{\infty}\left(\mathbb{R}^{d}\right)$. Differentiating $t \mapsto \mathcal{F}\left(w_{t}\right)=\frac{1}{2}\left\|w_{t}\right\|_{\dot{H}^{\psi^{-1}}\left(\mathbb{R}^{d}\right)}^{2}$ gives

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} t} \mathcal{F}\left(w_{t}\right) & =\int_{\mathbb{R}^{d}} \mathcal{R e}\left(\overline{\widehat{w}_{t}}(\xi) \partial_{t} \widehat{w}_{t}(\xi)\right) \psi^{-1}(\xi) \mathrm{d} \xi \\
& =-\int_{\mathbb{R}^{d}}\left(\widehat{L_{G} \circ w_{t}}(\xi) \widehat{\widehat{w}}_{t}(\xi)+\varepsilon\left|\widehat{w}_{t}(\xi)\right|^{2}\right)|\xi|^{2} \psi^{-1}(\xi) \mathrm{d} \xi \\
& =-\left(L_{G} \circ w_{t}, w_{t}\right)_{\tilde{\psi}}-\varepsilon\left(w_{t}, w_{t}\right)_{\tilde{\psi}} \leq 0 .
\end{aligned}
$$

Accordingly, we have $\mathcal{F}\left(w_{t}\right) \leq \mathcal{F}\left(u_{\tau}^{k}\right)$ and hence $w_{t} \in \dot{H}^{\psi^{-1}}\left(\mathbb{R}^{d}\right)$. Let us recall that the narrow convergence implies $\widehat{w}_{t}(\xi) \rightarrow \widehat{u}_{\tau}^{k}(\xi)$, as $t \rightarrow 0$ for all $\xi$, and that by Fatou's Lemma we deduce $\left\|w_{t}\right\|_{\dot{H}^{\psi^{-1}}\left(\mathbb{R}^{d}\right)} \rightarrow\left\|u_{\tau}^{k}\right\|_{\dot{H}^{\psi^{-1}}\left(\mathbb{R}^{d}\right)}$ as $t \rightarrow 0$, that is $t \mapsto \mathcal{F}\left(w_{t}\right)$ is continuous at $t=0$. Then, the fundamental theorem of calculus yields

$$
\frac{\mathcal{F}\left(u_{\tau}^{k}\right)-\mathcal{F}\left(w_{t}\right)}{t}=\frac{1}{t} \int_{0}^{t}\left(L_{G} \circ w_{r}, w_{r}\right)_{\tilde{\psi}}+\varepsilon\left(w_{r}, w_{r}\right)_{\tilde{\psi}} \mathrm{d} r \geq \frac{\varepsilon}{t} \int_{0}^{t}\left(w_{r}, w_{r}\right)_{\tilde{\psi}} \mathrm{d} r .
$$

Without loss of generality, we can assume that $w_{t} \in \dot{H} \widetilde{\psi}\left(\mathbb{R}^{d}\right)$ and hence by the estimates given in Eq. (4.4) that $L_{G} \circ w_{t} \in \dot{H}^{\widetilde{\psi}}\left(\mathbb{R}^{d}\right)$. By definition we have

$$
\mathscr{D}[\mathcal{F}]\left(u_{\tau}^{k}\right)=\limsup _{t \rightarrow 0} \frac{\mathcal{F}\left(u_{\tau}^{k}\right)-\mathcal{F}\left(S_{t} u_{\tau}^{k}\right)}{t} \geq\left(L_{G} \circ u_{\tau}^{k}, u_{\tau}^{k}\right)_{\tilde{\psi}}+\varepsilon\left(u_{\tau}^{k}, u_{\tau}^{k}\right)_{\tilde{\psi}}
$$

In particular, $\mathscr{D}_{\mathcal{G}}[\mathcal{F}]\left(u_{\tau}^{k}\right) \geq 0$. Furthermore, if $S_{t}^{\prime}$ is the flow associated to $\mathcal{G}$ then

$$
\frac{\mathcal{F}\left(u_{\tau}^{k}\right)-\mathcal{F}\left(S_{t}^{\prime} u_{\tau}^{k}\right)}{t}=\frac{1}{t} \int_{0}^{t}\left(L_{G} \circ S_{r}^{\prime} u_{\tau}^{k}, S_{r}^{\prime} u_{\tau}^{k}\right)_{\tilde{\psi}} \mathrm{d} r \geq 0
$$

By Theorem 4.5, we have that $u_{\tau}^{k} \in D(\mathcal{V})=D(\mathcal{H}) \cap D(\mathcal{G})$ for any $k \geq 1$ and

$$
\begin{aligned}
0 \leq\left(L_{G} \circ u_{\tau}^{k}, u_{\tau}^{k}\right)_{\tilde{\psi}} & \leq \mathscr{X}[\mathcal{F}]\left(u_{\tau}^{k}\right) \leq \frac{\mathcal{V}\left(u_{\tau}^{k-1}\right)-\mathcal{V}\left(u_{\tau}^{k}\right)}{\tau} \\
& =\frac{\mathcal{G}\left(u_{\tau}^{k-1}\right)-\mathcal{G}\left(u_{\tau}^{k}\right)}{\tau}+\varepsilon \frac{\mathcal{H}\left(u_{\tau}^{k-1}\right)-\mathcal{H}\left(u_{\tau}^{k}\right)}{\tau}
\end{aligned}
$$

and also

$$
0 \leq \mathscr{D}_{\mathcal{G}}[\mathcal{F}]\left(u_{\tau}^{k}\right) \leq \frac{\mathcal{G}\left(u_{\tau}^{k-1}\right)-\mathcal{G}\left(u_{\tau}^{k}\right)}{\tau}
$$

Thus, we get $0 \leq\left(L_{G} \circ u_{\tau}^{k}, u_{\tau}^{k}\right)_{\tilde{\psi}}<\infty$ and, upon letting $\varepsilon \rightarrow 0$, we obtain

$$
0 \leq\left(L_{G} \circ u_{\tau}^{k}, u_{\tau}^{k}\right)_{\tilde{\psi}} \leq \mathscr{D} \mathcal{V}[\mathcal{F}]\left(u_{\tau}^{k}\right) \leq \frac{\mathcal{G}\left(u_{\tau}^{k-1}\right)-\mathcal{G}\left(u_{\tau}^{k}\right)}{\tau}
$$

In particular, for the specific class of functionals $G(u)=\frac{u^{p}}{p-1}$ we have $L_{G}(u)=u^{p}$, which we will use to get $L^{p}$-control.
Corollary 4.9. Assume $u_{0} \in L^{p}\left(\mathbb{R}^{d}\right)$ and let $\left(u_{\tau}^{k}\right)_{k}$ be the sequence of Definition 3.3. Assume that $\widetilde{\psi}(\xi)=|\xi|^{2} \psi^{-1}(\xi)$ is the symbol associated with a Lévy kernel $\widetilde{\nu}$. Then $u_{\tau}^{k} \in L^{p}\left(\mathbb{R}^{d}\right)$ for any $k \geq 0$ and for all $k \geq 1$ we have

$$
\left\|u_{\tau}^{k}\right\|_{L^{p}\left(\mathbb{R}^{d}\right)} \leq\left\|u_{\tau}^{k-1}\right\|_{L^{p}\left(\mathbb{R}^{d}\right)}
$$

As a consequence, we are able to prove Theorem 1.1, item (vi).
Theorem 4.10. Assume $u_{0} \in D(\mathcal{H})$ and let $u \in A C^{2}\left([0, \infty),\left(\mathcal{P}_{2}, W\right)\right)$ be the limit curve obtained in Theorem 3.6. Then, for all $t \geq 0$, we have

$$
\mathcal{H}(u(t)) \leq \mathcal{H}\left(u_{0}\right)
$$

Proof. Recalling that $u_{\tau}^{0}=G_{\omega(\tau)} * u_{0}$ and $u_{\tau}(t)=u_{\tau}^{[t / \tau]}$, by Corollary 4.7 we have

$$
\mathcal{H}\left(u_{\tau}(t)\right) \leq \mathcal{H}\left(u_{\tau}^{0}\right) \leq \mathcal{H}\left(u_{0}\right) .
$$

The lower semicontinuity of $\mathcal{H}$ with respect to the narrow convergence implies

$$
\mathcal{H}(u(t)) \leq \liminf _{\tau \rightarrow 0} \mathcal{H}\left(u_{\tau}^{0}\right) \leq \mathcal{H}\left(u_{0}\right)
$$

Theorem 4.11. Assume that $\widetilde{\psi}(\xi)=|\xi|^{2} \psi^{-1}(\xi)$ is the symbol associated with a Lévy kernel $\widetilde{\nu}$. Assume $u_{0} \in L^{p}\left(\mathbb{R}^{d}\right)$ and let $u \in A C^{2}\left([0, \infty),\left(\mathcal{P}_{2}, W\right)\right)$ be the limit curve obtained in Theorem 3.6. Then, for all $t \geq 0$, we have

$$
\|u(t)\|_{L^{p}\left(\mathbb{R}^{d}\right)} \leq\left\|u_{0}\right\|_{L^{p}\left(\mathbb{R}^{d}\right)}
$$

Proof. Recalling that $u_{\tau}^{0}=G_{\omega(\tau)} * u_{0}$ and $u_{\tau}(t)=u_{\tau}^{[t / \tau]}$, by Corollary 4.9 we have

$$
\left\|u_{\tau}(t)\right\|_{L^{p}\left(\mathbb{R}^{d}\right)} \leq\left\|u_{\tau}^{0}\right\|_{L^{p}\left(\mathbb{R}^{d}\right)} \leq\left\|u_{0}\right\|_{L^{p}\left(\mathbb{R}^{d}\right)}
$$

The lower semicontinuity of $\|\cdot\|_{L^{p}\left(\mathbb{R}^{d}\right)}$ with respect to the narrow convergence implies

$$
\|u(t)\|_{L^{p}\left(\mathbb{R}^{d}\right)} \leq \liminf _{\tau \rightarrow 0}\left\|u_{\tau}(t)\right\|_{L^{p}\left(\mathbb{R}^{d}\right)} \leq\left\|u_{0}\right\|_{L^{p}\left(\mathbb{R}^{d}\right)}
$$

Corollary 4.12. Let $u_{0} \in D(\mathcal{F})$ and $\left(u_{\tau}^{k}\right)$ defined as in Eq. (3.1) and consider the piecewise constant approximation $u_{\tau}(t)=u_{\tau}^{[t / \tau\rceil}$. For every $t>0, u_{\tau}(t) \in \dot{H}^{\tilde{\psi}}\left(\mathbb{R}^{d}\right)$. Moreover, for $T>T_{0} \geq \tau>0$, we have

$$
\int_{T_{0}}^{T}\left\|u_{\tau}(t)\right\|_{\dot{H}^{\tilde{\psi}}\left(\mathbb{R}^{d}\right)}^{2} \mathrm{~d} t \leq \mathcal{H}\left(u_{\tau}^{N_{0}(\tau)}\right)+\frac{8 d \kappa}{\log (2)}\left(1+2 T \mathcal{F}\left(u_{0}\right)+\int_{\mathbb{R}^{d}}|x|^{2} \mathrm{~d} u_{0}(x)\right)
$$

where $N_{0}(\tau)=\left\lfloor T_{0} / \tau\right\rfloor$ and $\kappa$ is a constant only depending on $d$.
Proof. Set $N=\lfloor T / \tau\rfloor$ and $N_{0}=N_{0}(\tau)=\left\lfloor T_{0} / \tau\right\rfloor$ so that $\left(T_{0}, T\right) \subset\left(\tau N_{0}, \tau(N+1)\right)$. Thus,

$$
\begin{aligned}
\int_{T_{0}}^{T}\left\|u_{\tau}(t)\right\|_{\dot{H} \tilde{\psi}\left(\mathbb{R}^{d}\right)}^{2} \mathrm{~d} t & \leq \int_{\tau N_{0}}^{\tau(N+1)}\left\|u_{\tau}(t)\right\|_{\dot{H}^{\tilde{\psi}}\left(\mathbb{R}^{d}\right)}^{2} \mathrm{~d} t \\
& =\sum_{k=N_{0}}^{N} \int_{\tau k}^{\tau(k+1)}\left|u_{\tau}(t)\right|_{H^{\tilde{\psi}}\left(\mathbb{R}^{d}\right)}^{2} \mathrm{~d} t=\sum_{k=N_{0}}^{N} \tau\left\|u_{\tau}^{k+1}\right\|_{\dot{H}^{\tilde{\psi}}\left(\mathbb{R}^{d}\right)}^{2} \\
& \leq \sum_{k=N_{0}}^{N}\left(\mathcal{H}\left(u_{\tau}^{k}\right)-\mathcal{H}\left(u_{\tau}^{k+1}\right)\right)=\mathcal{H}\left(u_{\tau}^{N_{0}}\right)-\mathcal{H}\left(u_{\tau}^{N+1}\right)
\end{aligned}
$$

The Carleman's type inequality implies that

$$
\begin{equation*}
-\mathcal{H}\left(u_{\tau}^{N+1}\right) \leq \kappa\left(1+\int_{\mathbb{R}^{d}}|x|^{2} u_{\tau}^{N+1}(x) \mathrm{d} x\right) . \tag{4.5}
\end{equation*}
$$

This, together with the estimate in Eq. (3.3) gives

$$
\begin{aligned}
-\mathcal{H}\left(u_{\tau}^{N+1}\right) & \leq \frac{8 d \kappa}{\log (2)}\left(1+\tau(N+1) \mathcal{F}\left(u_{0}\right)+\int_{\mathbb{R}^{d}}|x|^{2} \mathrm{~d} u_{0}(x)\right) \\
& \leq \frac{8 d \kappa}{\log (2)}\left(1+2 T \mathcal{F}\left(u_{0}\right)+\int_{\mathbb{R}^{d}}|x|^{2} \mathrm{~d} u_{0}(x)\right) .
\end{aligned}
$$

We need to prove Inequality (4.5) to complete the proof. To this end, let us proceed as follows

$$
\begin{aligned}
-\int_{\mathbb{R}^{d}} u_{\tau}^{N+1} \log u_{\tau}^{N+1} \mathrm{~d} x= & -\int_{\mathbb{R}^{d}} u_{\tau}^{N+1} \log u_{\tau}^{N+1} \mathbb{1}_{\left\{u_{\tau}^{N+1} \geq \exp \left(-|x|^{2}\right)\right\}} \mathrm{d} x \\
& -\int_{|x| \leq 1} u_{\tau}^{N+1} \log u_{\tau}^{N+1} \mathbb{1}_{\left\{u_{\tau}^{N+1} \leq \exp \left(-|x|^{2}\right)\right\}} \mathrm{d} x \\
& -\int_{|x| \geq 1} u_{\tau}^{N+1} \log u_{\tau}^{N+1} \mathbb{1}_{\left\{u_{\tau}^{N+1} \leq \exp \left(-|x|^{2}\right)\right\}} \mathrm{d} x .
\end{aligned}
$$

First of all, we note that if $u_{\tau}^{N+1} \geq \exp \left(-|x|^{2}\right)$ then $-\log u_{\tau}^{N+1} \leq|x|^{2}$, so that

$$
-\int_{\mathbb{R}^{d}} u_{\tau}^{N+1} \log u_{\tau}^{N+1} \mathbb{1}_{\left\{u_{\tau}^{N+1} \geq \exp \left(-|x|^{2}\right)\right\}} \mathrm{d} x \leq \int_{\mathbb{R}^{d}} u_{\tau}^{N+1}|x|^{2} \mathrm{~d} x
$$

For the second term we have

$$
-\int_{|x| \leq 1} u_{\tau}^{N+1} \log u_{\tau}^{N+1} \mathbb{1}_{\left\{u_{\tau}^{N+1} \leq \exp \left(-|x|^{2}\right)\right\}} \mathrm{d} x \leq\left|B_{1}(0)\right| \max _{t \in[0,1]} t|\log t| \leq C .
$$

Last, the monotonicity of $t \mapsto t \log t$ implies

$$
\begin{aligned}
-\int_{|x| \geq 1} u_{\tau}^{N+1} \log u_{\tau}^{N+1} \mathbb{1}_{\left\{u_{\tau}^{N+1} \leq \exp \left(-|x|^{2}\right)\right\}} \mathrm{d} x & \leq \int_{|x| \geq 1} u_{\tau}^{N+1}\left|\log u_{\tau}^{N+1}\right| \mathbb{1}_{\left\{u_{\tau}^{N+1} \leq \exp \left(-|x|^{2}\right)\right\}} \mathrm{d} x \\
& \leq \int_{\mathbb{R}^{d}}|x|^{2} \exp \left(-|x|^{2}\right) \mathrm{d} x \leq C .
\end{aligned}
$$

Hence, we obtain the desired bound for $u_{\tau}^{N+1}$.

## 5. Convergence of discrete approximations

In this section, we establish the convergence of the piecewise constant interpolations of the approximate solutions $\left(u_{\tau}\right)_{\tau}$ and the associated discrete pressures $\left(v_{\tau}\right)_{\tau}$ (recalling $v_{\tau}=L^{-1} u_{\tau}$, i.e., $v_{\tau}$ is defined so that $\left.\widehat{v}_{\tau}(\xi)=\psi^{-1}(\xi) \widehat{u}_{\tau}(\xi)\right)$ in appropriate spaces. In Section 6, we prove that the limit curve obtained in Theorem 3.6 satisfies Eq. (1.1) in the sense of Eq. (1.12). Let us start with the following observation.

Theorem 5.1. Let $u \in \dot{H}^{\psi^{-1}}\left(\mathbb{R}^{d}\right) \cap \dot{H}^{\tilde{\psi}}\left(\mathbb{R}^{d}\right)$ then we have

$$
\|u\|_{L^{2}\left(\mathbb{R}^{d}\right)}^{2} \leq 2 \kappa_{\nu}\left(\|u\|_{\dot{H}^{\tilde{\psi}}\left(\mathbb{R}^{d}\right)}^{2}+\|u\|_{\dot{H}^{\psi^{-1}}\left(\mathbb{R}^{d}\right)}^{2}\right) .
$$

If, in addition, Condition $\left(C_{\nu}\right)$ is satisfied, then we have

$$
\|u\|_{\dot{H}^{\psi^{*}}\left(\mathbb{R}^{d}\right)}^{2} \leq c_{\nu}^{-1}\left(\|u\|_{\dot{H}^{\tilde{\psi}}\left(\mathbb{R}^{d}\right)}^{2}+\|u\|_{\dot{H}^{\psi^{-1}}\left(\mathbb{R}^{d}\right)}^{2}\right)
$$

In particular, the embedding $H^{\psi^{-1}}\left(\mathbb{R}^{d}\right) \cap H^{\tilde{\psi}}\left(\mathbb{R}^{d}\right) \hookrightarrow H^{\psi^{*}}\left(\mathbb{R}^{d}\right)$ is continuous.

Proof. Note that $\psi(\xi) \leq 2 \kappa_{\nu}|\xi|^{2}$ for $|\xi|>1$, and $\psi(\xi) \leq 2 \kappa_{\nu}$ for $|\xi| \leq 1$. Using this observation, we find

$$
\begin{aligned}
\|u\|_{L^{2}\left(\mathbb{R}^{d}\right)}^{2} & =\int_{|\xi|>1}|\widehat{u}(\xi)|^{2} \psi(\xi) \psi^{-1}(\xi) \mathrm{d} \xi+\int_{|\xi| \leq 1}|\widehat{u}(\xi)|^{2} \psi(\xi) \psi^{-1}(\xi) \mathrm{d} \xi \\
& \leq 2 \kappa_{\nu} \int_{|\xi|>1}|\widehat{u}(\xi)|^{2} \widetilde{\psi}(\xi) \mathrm{d} \xi+2 \kappa_{\nu} \int_{|\xi| \leq 1}|\widehat{u}(\xi)|^{2} \psi^{-1}(\xi) \mathrm{d} \xi \\
& \leq 2 \kappa_{\nu}\left(\|u\|_{\dot{H^{\tilde{\psi}}\left(\mathbb{R}^{d}\right)}}^{2}+\|u\|_{\dot{H}^{\psi^{-1}\left(\mathbb{R}^{d}\right)}}^{2}\right)
\end{aligned}
$$

which proves the first statement. Concerning the second statement, let us use Condition $\left(C_{\nu}\right)$, i.e., $\psi^{-1}(\xi) \leq \frac{c_{\nu}^{-1}}{1 \Lambda|\xi|^{2}}$. Again, decomposing the domain of integration into high and low frequencies, we may estimate both regimes separately and obtain

$$
\begin{aligned}
\|u\|_{\dot{H}^{\psi^{*}\left(\mathbb{R}^{d}\right)}}^{2} & =\int_{|\xi|>1}|\widehat{u}(\xi)|^{2} \psi^{*}(\xi) \mathrm{d} \xi+\int_{|\xi| \leq 1}|\widehat{u}(\xi)|^{2} \psi^{*}(\xi) \mathrm{d} \xi \\
& =\int_{|\xi|>1}|\widehat{u}(\xi)|^{2} \psi^{-1}(\xi) \widetilde{\psi}(\xi) \mathrm{d} \xi+\int_{|\xi| \leq 1}|\widehat{u}(\xi)|^{2}|\xi|^{2} \psi^{-1}(\xi) \psi^{-1}(\xi) \mathrm{d} \xi \\
& \leq c_{\nu}^{-1} \int_{|\xi|>1}|\widehat{u}(\xi)|^{2} \frac{1}{1 \wedge|\xi|^{2}} \widetilde{\psi}(\xi) \mathrm{d} \xi+c_{\nu}^{-1} \int_{|\xi| \leq 1}|\widehat{u}(\xi)|^{2} \frac{|\xi|^{2}}{1 \wedge|\xi|^{2}} \psi^{-1}(\xi) \mathrm{d} \xi \\
& =c^{-1} \int_{|\xi|>1}|\widehat{u}(\xi)|^{2} \widetilde{\psi}(\xi) \mathrm{d} \xi+c_{\nu}^{-1} \int_{|\xi| \leq 1}|\widehat{u}(\xi)|^{2} \psi^{-1}(\xi) \mathrm{d} \xi \\
& \leq c_{\nu}^{-1}\left(\|u\|_{\dot{H}^{\tilde{\psi}}\left(\mathbb{R}^{d}\right)}^{2}+\|u\|_{\dot{H}^{\psi-1}\left(\mathbb{R}^{d}\right)}^{2}\right) .
\end{aligned}
$$

From the two estimates the continuity of the embedding $H^{\psi^{-1}}\left(\mathbb{R}^{d}\right) \cap H^{\tilde{\psi}}\left(\mathbb{R}^{d}\right) \hookrightarrow H^{\psi^{*}}\left(\mathbb{R}^{d}\right)$ follows.
Theorem 5.2. Let $u_{0} \in \dot{H}^{\psi^{-1}}\left(\mathbb{R}^{d}\right) \cap \mathcal{P}_{2}\left(\mathbb{R}^{d}\right)$, $\left(u_{\tau}\right)_{\tau}$ be the piecewise constant approximations in Definition 3.3 and $u$ its limit curve obtained in Theorem 3.6. Define $v_{\tau}=L^{-1} u_{\tau}$ and $v=L^{-1} u$. Then, there is a subsequence (not relabeled) $\left(\tau_{n}\right)_{n}$, such that following hold:
(i) We have $u \in L^{2}\left(0, T ; H^{\tilde{\psi}}\left(\mathbb{R}^{d}\right)\right)$ and

$$
u_{\tau_{n}} \rightharpoonup u, \quad \text { weakly in } L^{2}\left(0, T ; H^{\tilde{\psi}}\left(\mathbb{R}^{d}\right)\right) .
$$

If, in addition, Condition (1.10) is met, i.e., there holds

$$
\begin{equation*}
\sup _{\xi \in \mathbb{R}^{d}} \frac{1}{\widetilde{\psi}(\xi)}\left|e^{i \xi \cdot h}-1\right|^{2}=\sup _{\xi \in \mathbb{R}^{d}} \frac{\psi(\xi)}{|\xi|^{2}}\left|e^{i \xi \cdot h}-1\right|^{2} \xrightarrow{|h| \rightarrow 0} 0, \tag{1.10}
\end{equation*}
$$

then, for any $0<T_{0}<T$, we have

$$
u_{\tau_{n}} \rightarrow u, \quad \text { strongly in } L^{2}\left(T_{0}, T ; L_{\mathrm{loc}}^{2}\left(\mathbb{R}^{d}\right)\right)
$$

(ii) If Condition $\left(C_{\nu}\right)$ is met, we have $\nabla v \in L^{2}\left(0, T ; L^{2}\left(\mathbb{R}^{d}\right)\right)$, and

$$
\nabla v_{\tau_{n}} \rightharpoonup \nabla v, \quad \text { weakly in } L^{2}\left(0, T ; L^{2}\left(\mathbb{R}^{d}\right)\right) .
$$

Proof. By Plancherel's Theorem, we deduce that

$$
\left\|\nabla v_{\tau}\right\|_{L^{2}\left(\mathbb{R}^{d}\right)}^{2}=\int_{\mathbb{R}^{d}}\left|\widehat{v}_{\tau}(\xi)\right|^{2}|\xi|^{2} \mathrm{~d} \xi=\int_{\mathbb{R}^{d}}\left|\widehat{u}_{\tau}(\xi)\right|^{2}|\xi|^{2} \psi^{-2}(\xi) \mathrm{d} \xi=\left\|u_{\tau}\right\|_{\dot{H} \psi^{*}\left(\mathbb{R}^{d}\right)}^{2}
$$

Moreover, by Theorem 5.1, we know that

$$
\left\|u_{\tau}\right\|_{L^{2}\left(\mathbb{R}^{d}\right)}^{2} \leq C\left(\left\|u_{\tau}\right\|_{\dot{H} \tilde{\psi}\left(\mathbb{R}^{d}\right)}^{2}+\left\|u_{\tau}\right\|_{\dot{H}^{\psi}\left(\mathbb{R}^{d}\right)}^{2}\right),
$$

and, using Condition $\left(C_{\nu}\right)$, we also have

$$
\left\|u_{\tau}\right\|_{\dot{H} \psi^{*}\left(\mathbb{R}^{d}\right)}^{2} \leq C\left(\left\|u_{\tau}\right\|_{\dot{H}^{\tilde{\psi}}\left(\mathbb{R}^{d}\right)}^{2}+\left\|u_{\tau}\right\|_{\dot{H}^{\psi^{-1}}\left(\mathbb{R}^{d}\right)}^{2}\right) .
$$

Note that the narrow convergence $u_{\tau_{n}} \rightarrow u$ implies the pointwise convergence of the Fourier transforms, i.e., $\widehat{u}_{\tau_{n}} \rightarrow \widehat{u}$. Similarly, $\widehat{v}_{\tau_{n}}(\xi) \rightarrow \widehat{v}(\xi)$, for all $\xi_{\sim} \in \mathbb{R}^{d}$. Therefore, all claimed weak convergences are true once the boundedness of $\left(u_{\tau}\right)_{\tau}$ in $\dot{H}^{\widetilde{\psi}}\left(\mathbb{R}^{d}\right) \cap \dot{H}^{\psi^{-1}}\left(\mathbb{R}^{d}\right)$ is established. By Proposition 3.4, it follows that $\mathcal{F}\left(u_{\tau}\right) \leq \mathcal{F}\left(u_{0}\right)$. That is, for any $T_{0} \in(0, T)$

$$
\int_{T_{0}}^{T}\left\|u_{\tau}(t)\right\|_{\dot{H}^{\psi^{-1}}\left(\mathbb{R}^{d}\right)}^{2} \mathrm{~d} t \leq\left(T-T_{0}\right)\left\|u_{0}\right\|_{\dot{H}^{\psi^{-1}}\left(\mathbb{R}^{d}\right)^{2}}^{2}
$$

By Corollary 4.12, we know that for $T>T_{0} \geq \tau>0$,

$$
\int_{T_{0}}^{T}\left\|u_{\tau}(t)\right\|_{\dot{H}^{\tilde{\psi}}\left(\mathbb{R}^{d}\right)}^{2} \mathrm{~d} t \leq \mathcal{H}\left(u_{\tau}^{N_{0}(\tau)}\right)+\frac{8 d \kappa}{\log (2)}\left(1+2 T \mathcal{F}\left(u_{0}\right)+\int_{\mathbb{R}^{d}}|x|^{2} \mathrm{~d} u_{0}(x)\right) .
$$

where $N_{0}(\tau)=\left\lfloor T_{0} / \tau\right\rfloor$ and $\kappa$ is a constant only depending on $d$.
Since $u_{0} \in D(\mathcal{H})$ we have $\mathcal{H}\left(u_{\tau}^{N_{0}(\tau)}\right) \leq \mathcal{H}\left(u_{0}\right)<\infty$. Moreover, under Condition (1.10), the embed$\operatorname{ding} H^{\widetilde{\psi}}\left(\mathbb{R}^{d}\right) \hookrightarrow L_{\mathrm{loc}}^{2}\left(\mathbb{R}^{d}\right)$ is compact in virtue of Theorem 2.18. Thence, the strong convergence of $\left(u_{\tau_{n}}\right)_{n}$ in $L^{2}\left(T_{0}, T ; L_{\mathrm{loc}}^{2}\left(\mathbb{R}^{d}\right)\right)$ follows.

## 6. Weak solution of the equation and energy dissipation inequality

This section is dedicated to establishing the energy dissipation inequality, item $(v)$ of Theorem 1.1. Moreover, we identify the limit curve as a weak solution of Eq. (1.1), Theorem 1.1 (iv), in the sense of Eq. (1.12). To this end, let us begin by deriving the associated Euler-Lagrange equations.
Theorem 6.1. Assume that $\nu \notin L^{1}\left(\mathbb{R}^{d}\right)$ and satisfies Condition $\left(C_{\nu}\right)$. Moreover, assume the symbol $\widetilde{\psi}$ is associated with a unimodal Lévy kernel $\widetilde{\nu}$ satisfying the following condition:

$$
\begin{equation*}
\text { For any } 0<\lambda<1 \text { there is } c_{\lambda}>0 \text { s.t. } \widetilde{\nu}(\lambda h) \leq c_{\lambda} \widetilde{\nu}(h) \text { whenever }|h| \leq 1 \text {. } \tag{6.1}
\end{equation*}
$$

Let $u_{0} \in \dot{H}^{\psi^{-1}}\left(\mathbb{R}^{d}\right) \cap \mathcal{P}_{2}\left(\mathbb{R}^{d}\right)$ and let $\left(u_{\tau}^{k}\right)_{k}$ be the associated solution to the minimizing movement scheme, Eq. (3.1). Define $\left(v_{\tau}^{k}\right)_{k}$ by $v_{\tau}^{k}:=L^{-1} u_{\tau}^{k}$. Then, for any $k \geq 0$, we have

$$
\begin{equation*}
\int_{\mathbb{R}^{d}} \nabla v_{\tau}^{k} \cdot \eta u_{\tau}^{k} \mathrm{~d} x=\frac{1}{\tau} \int_{\mathbb{R}^{d}}\left(T_{u_{\tau}^{k}}^{u_{\tau}^{k-1}}-\mathrm{I}\right) \cdot \eta u_{\tau}^{k} \mathrm{~d} x, \quad \text { for all } \quad \eta \in C_{c}^{\infty}\left(\mathbb{R}^{d}, \mathbb{R}^{d}\right) \tag{6.2}
\end{equation*}
$$

Moreover, we have

$$
\begin{equation*}
\int_{\mathbb{R}^{d}}\left|\nabla v_{\tau}^{k}\right|^{2} u_{\tau}^{k} \mathrm{~d} x=\frac{1}{\tau^{2}} W^{2}\left(u_{\tau}^{k}, u_{\tau}^{k-1}\right) \tag{6.3}
\end{equation*}
$$

## Proof. Step 1. - Perturbation of minimizers.

For ease of notation, throughout the proof we shall simply write $u:=u_{\tau}^{k}$ and $v:=v_{\tau}^{k}$. Given $\delta>0$, we define $\phi_{\delta}: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$ as $\phi_{\delta}(x):=x+\delta \eta(x)$. Clearly, for $\delta_{0}>0$ small enough, we have

$$
\begin{equation*}
\frac{1}{2} \leq \operatorname{det}\left(D \phi_{\delta}(x)\right) \leq \frac{3}{2} \tag{6.4}
\end{equation*}
$$

for all $x \in \mathbb{R}^{d}, \delta \in\left[0, \delta_{0}\right]$. Define $u_{\delta}:=\phi_{\delta \#} u_{\tau}^{k}=\operatorname{det}\left(D \phi_{\delta}\right)^{-1} u_{\tau}^{k} \circ \phi_{\delta}^{-1}$ and $\widehat{v}_{\delta}:=\psi^{-1} \widehat{u_{\delta}}$. Since $u$ is optimal in Eq. (3.1), there holds

$$
0 \leq \frac{1}{\delta}\left[\mathcal{F}\left(u_{\delta}\right)-\mathcal{F}(u)+\frac{1}{2 \tau}\left(W^{2}\left(u_{\delta}, u_{\tau}^{k-1}\right)-W^{2}\left(u, u_{\tau}^{k-1}\right)\right)\right] .
$$

The second term is classical which is known to satisfy

$$
\lim _{\delta \rightarrow 0} \frac{1}{\delta}\left[\frac{1}{2 \tau}\left(W^{2}\left(u_{\delta}, u_{\tau}^{k-1}\right)-W^{2}\left(u, u_{\tau}^{k-1}\right)\right)\right]=\frac{1}{\tau} \int_{\mathbb{R}^{d}}\left(T_{u_{\tau}^{k}}^{u_{\tau}^{k-1}}-\mathrm{I}\right) \cdot \eta u_{\tau}^{k} \mathrm{~d} x
$$

which can be adapted from [Vil03, Theorem 8.13], see also [San15, Section 7.2.2]. The rest of the proof focuses on the treating the limit

$$
\lim _{\delta \rightarrow 0} \frac{1}{\delta}\left[\mathcal{F}\left(u_{\delta}\right)-\mathcal{F}(u)\right]
$$

Switching to Fourier, let us rewrite the energy

$$
\begin{equation*}
\frac{1}{\delta}\left[\mathcal{F}\left(u_{\delta}\right)-\mathcal{F}(u)\right]=\frac{1}{2 \delta} \int_{\mathbb{R}^{d}}\left(\left|\widehat{u_{\delta}}(\xi)\right|^{2}-|\widehat{u}(\xi)|^{2}\right) \psi^{-1}(\xi) \mathrm{d} \xi . \tag{6.5}
\end{equation*}
$$

Using the identity

$$
\left|\widehat{u}_{\delta}(\xi)\right|^{2}-|\widehat{u}(\xi)|^{2}=\left(\overline{\widehat{u}_{\delta}}(\xi)-\overline{\widehat{u}}(\xi)\right)\left(\widehat{u}_{\delta}(\xi)+\widehat{u}(\xi)\right)+\overline{\widehat{u}_{\delta}}(\xi) \widehat{u}(\xi)-\widehat{u}_{\delta}(\xi) \overline{\widehat{u}}(\xi),
$$

in conjunction with $\widehat{v}(\xi)=\psi^{-1}(\xi) \widehat{u}(\xi)$ and $\widehat{v}_{\delta}(\xi)=\psi^{-1}(\xi) \widehat{u}_{\delta}(\xi)$, the variation of the energy can be simplified such that

$$
\begin{equation*}
\frac{1}{\delta}\left[\mathcal{F}\left(u_{\delta}\right)-\mathcal{F}(u)\right]=\frac{1}{2} \int_{\mathbb{R}^{d}}|\xi|\left(\widehat{v}_{\delta}(-\xi)+\widehat{v}(-\xi)\right) \cdot \frac{1}{\delta}|\xi|^{-1}\left(\widehat{u}_{\delta}(\xi)-\widehat{u}(\xi)\right) \mathrm{d} \xi . \tag{6.6}
\end{equation*}
$$

Next, let $R>1$ be sufficiently large such that $\operatorname{supp} \eta \subset B(0, R)$. By definition of $u_{\delta}$, we have

$$
\begin{aligned}
\widehat{u}_{\delta}(\xi)-\widehat{u}(\xi) & =\int_{\mathbb{R}^{d}} \exp (-i \xi \cdot(x+\delta \eta(x))) u(x) \mathrm{d} x-\int_{\mathbb{R}^{d}} \exp (-i \xi \cdot x) u(x) \mathrm{d} x \\
& =\int_{B(0, R)} \exp (-i \xi \cdot x)(\exp (-i \xi \cdot \delta \eta(x))-1) u(x) \mathrm{d} x
\end{aligned}
$$

Then, the dominated convergence theorem implies the pointwise convergence $\widehat{u}_{\delta}(\xi) \rightarrow \widehat{u}(\xi)$, as $\delta \rightarrow 0$. Therefore, we also have $|\xi| \widehat{v}_{\delta}(-\xi) \rightarrow|\xi| \widehat{v}(-\xi)$ as $\delta \rightarrow 0$, pointwise.
Step 2. - Boundedness of $\left(u_{\delta}\right)_{\delta}$ in $H^{\widetilde{\psi}}\left(\mathbb{R}^{d}\right)$.
Note that by Corollary 4.7 we have $u \in H^{\widetilde{\psi}}\left(\mathbb{R}^{d}\right)$. According to Theorem 2.7, the existence of $\widetilde{\nu}$ implies that $H_{\widetilde{\nu}}\left(\mathbb{R}^{d}\right)=H^{\widetilde{\psi}}\left(\mathbb{R}^{d}\right)$, and we have to estimate

$$
\begin{aligned}
\left\|u_{\delta}\right\|_{H^{\tilde{\psi}}\left(\mathbb{R}^{d}\right)}^{2} & =\iint_{\mathbb{R}^{d} \mathbb{R}^{d}}\left|u_{\delta}(x)-u_{\delta}(y)\right|^{2} \widetilde{\nu}(x-y) \mathrm{d} y \mathrm{~d} x \\
& \leq \iint_{\mathbb{R}^{d} \mathbb{R}^{d}}\left[2\left|u \circ \phi_{\delta}^{-1}(x)-u \circ \phi_{\delta}^{-1}(y)\right|^{2}\left|h_{\delta}(x)\right|^{2}+2\left|h_{\delta}(x)-h_{\delta}(y)\right|^{2}\left|u_{\delta}(y)\right|^{2}\right] \widetilde{\nu}(x-y) \mathrm{d} y \mathrm{~d} x,
\end{aligned}
$$

having used $u_{\delta}=h_{\delta} u \circ \phi_{\delta}^{-1}$, where $h_{\delta}:=\operatorname{det}\left(D \phi_{\delta}\right)^{-1}$. Since $h_{\delta}(x)=\operatorname{det}\left((I+\delta D \eta)^{-1}(x)\right)$ is at least $W^{1, \infty}\left(\mathbb{R}^{d}\right)$, there exists $A>0$ independent of $\delta>0$ (once $\delta$ is sufficiently small) such that $\left\|h_{\delta}\right\|_{W^{1, \infty}\left(\mathbb{R}^{d}\right)} \leq A$. Hence, for all $x, y \in \mathbb{R}^{d}$ we have

$$
\left|h_{\delta}(x)\right| \leq A, \quad \text { as well as } \quad\left|h_{\delta}(x)-h_{\delta}(y)\right| \leq 2 A(1 \wedge|x-y|) .
$$

Therefore,

$$
\begin{align*}
\left\|u_{\delta}\right\|_{H^{\tilde{\psi}}\left(\mathbb{R}^{d}\right)}^{2} \leq & 2 A^{2} \iint_{\mathbb{R}^{d} \mathbb{R}^{d}}\left|u \circ \phi_{\delta}^{-1}(x)-u \circ \phi_{\delta}^{-1}(y)\right|^{2} \widetilde{\nu}(x-y) \mathrm{d} y \mathrm{~d} x \\
& +4 A^{4} \iint_{\mathbb{R}^{d} \mathbb{R}^{d}}\left(1 \wedge|x-y|^{2}\right)\left|u \circ \phi_{\delta}^{-1}(y)\right|^{2} \widetilde{\nu}(x-y) \mathrm{d} y \mathrm{~d} x, \tag{6.7}
\end{align*}
$$

Now, by Theorem 2.11, we find

$$
\begin{equation*}
\left\|u \circ \phi_{\delta}^{-1}\right\|_{H^{\tilde{\psi}}\left(\mathbb{R}^{d}\right)}^{2} \leq C\left(1+\left\|\operatorname{det} D \phi_{\delta}\right\|_{L^{\infty}\left(\mathbb{R}^{d}\right)}\right)^{2}\|u\|_{H^{\tilde{\psi}}\left(\mathbb{R}^{d}\right)}^{2} \leq C\|u\|_{H^{\tilde{\psi}}\left(\mathbb{R}^{d}\right)}^{2}, \tag{6.8}
\end{equation*}
$$

where $C>0$ is independent of $\delta$ by Eq. (6.4). Using Eq. (6.8) in Eq. (6.7), we finally have

$$
\begin{align*}
\left\|u_{\delta}\right\|_{H^{\tilde{\psi}}\left(\mathbb{R}^{d}\right)}^{2} & \leq 4 A^{2}\left(1+A^{2} \int_{\mathbb{R}^{d}} 1 \wedge|h|^{2} \widetilde{\nu}(h) \mathrm{d} h\right)\left\|u \circ \phi_{\delta}^{-1}\right\|_{H^{\tilde{\psi}}\left(\mathbb{R}^{d}\right)}^{2}  \tag{6.9}\\
& \leq C\|u\|_{H^{\tilde{\psi}}\left(\mathbb{R}^{d}\right)}^{2},
\end{align*}
$$

where $C>0$ is independent of $\delta$.
Step 3. - Strong convergence $\nabla v_{\delta} \rightarrow \nabla v$ in $L_{\text {loc }}^{2}\left(\mathbb{R}^{d}\right)$.
The goal is to apply the compactness Theorem 2.14. Therefore, it is sufficient to establish the boundedness of $\nabla\left(v_{\delta}-v\right)$ in $H_{\nu}\left(\mathbb{R}^{d}\right)=H^{\psi}\left(\mathbb{R}^{d}\right)$, see Remark 2.17. Under Condition $\left(C_{\nu}\right)$ and by proceeding as in the proof of Theorem 5.1, we obtain

$$
\begin{align*}
\left\|\nabla v_{\delta}-\nabla v\right\|_{L^{2}\left(\mathbb{R}^{d}\right)}^{2} & =\left\|u_{\delta}-u\right\|_{\dot{H}^{\psi^{*}}\left(\mathbb{R}^{d}\right)}^{2} \\
& \leq c_{\nu}^{-1} \int_{|\xi|>1}\left|\widehat{u_{\delta}}(\xi)-\widehat{u}(\xi)\right|^{2} \widetilde{\psi}(\xi) \mathrm{d} \xi+c_{\nu}^{-1} \int_{|\xi| \leq 1}\left|\widehat{u_{\delta}}(\xi)-\widehat{u}(\xi)\right|^{2} \psi^{-1}(\xi) \mathrm{d} \xi \\
& \leq c_{\nu}^{-1} \int_{|\xi|>1}\left|\widehat{u_{\delta}}(\xi)-\widehat{u}(\xi)\right|^{2} \widetilde{\psi}(\xi) \mathrm{d} \xi+c_{\nu}^{-1} \int_{|\xi| \leq 1}\left|\widehat{u_{\delta}}(\xi)-\widehat{u}(\xi)\right|^{2}|\xi|^{-2} \mathrm{~d} \xi  \tag{6.10}\\
& \leq c_{\nu}^{-1}\left(\left\|u_{\delta}-u\right\|_{\dot{H}^{\tilde{\psi}}\left(\mathbb{R}^{d}\right)}^{2}+\left\|u_{\delta}-u\right\|_{\dot{H}^{-1}\left(\mathbb{R}^{d}\right)}^{2}\right) .
\end{align*}
$$

Next, let us show that $\left\|u_{\delta}-u\right\|_{\dot{H}^{-1}}^{2} \lesssim\left\|u_{\delta}-u\right\|_{L^{2}\left(\mathbb{R}^{d}\right)}^{2}$, which then implies

$$
\left\|\nabla v_{\delta}-\nabla v\right\|_{L^{2}\left(\mathbb{R}^{d}\right)}^{2} \lesssim\left\|u_{\delta}-u\right\|_{H}^{2}{ }_{H}\left(\mathbb{R}^{d}\right) .
$$

Let us recall that the Riesz kernel

$$
K_{1}(x):= \begin{cases}\frac{1}{2 \pi} \log |x|, & d=2 \\ C_{d,-1}|x|^{2-d}, & d \geq 3\end{cases}
$$

satisfies $\widehat{K_{1}}(\xi)=|\xi|^{-2}$, see for instance Eq. (1.7)-(1.8). As the one-dimensional case is particular, we will treat it separately below and focus on $d \geq 2$. Now, we may estimate

$$
\left\|u_{\delta}-u\right\|_{\dot{H}^{-1}\left(\mathbb{R}^{d}\right)}^{2}=\int_{\mathbb{R}^{d}}\left|\widehat{u}_{\delta}(\xi)-\widehat{u}(\xi)\right|^{2} \widehat{K_{1}}(\xi) \mathrm{d} \xi=\int_{B(0, R)}\left(u_{\delta}-u\right)(x) K_{1} *\left(u_{\delta}-u\right)(x) \mathrm{d} x .
$$

In the last equality, we exploited the fact that $\operatorname{supp}\left(u_{\delta}-u\right) \subset B(0, R)$ and $R>1$ independent of $\delta>0$. Moreover, we have

$$
K_{1} *\left(u_{\delta}-u\right)(x)=\int_{\mathbb{R}^{d}}\left(u_{\delta}(y)-u(y)\right) \mathbb{1}_{B(0, R)}(y) K_{1}(x-y) \mathrm{d} y
$$

whence, for $x \in B(0, R)$, we obtain

$$
\left|K_{1} *\left(u_{\delta}-u\right)(x)\right| \leq \int_{\mathbb{R}^{d}}\left|u_{\delta}(y)-u(y)\right| \mathbb{1}_{B(0,2 R)}(x-y) K_{1}(x-y) \mathrm{d} y
$$

This combined with Young's convolution inequality gives

$$
\left\|K_{1} *\left(u_{\delta}-u\right)\right\|_{L^{2}(B(0, R))} \leq\left\|K_{1}\right\|_{L^{1}(B(0,2 R))}\left\|u_{\delta}-u\right\|_{L^{2}\left(\mathbb{R}^{d}\right)} .
$$

By Cauchy-Schwartz inequality,

$$
\begin{align*}
\left\|u_{\delta}-u\right\|_{\dot{H}^{-1}\left(\mathbb{R}^{d}\right)}^{2} & \leq\left\|u_{\delta}-u\right\|_{L^{2}\left(\mathbb{R}^{d}\right)}\left\|\left(u_{\delta}-u\right) * K_{1}\right\|_{L^{2}(B(0, R))} \\
& \leq\left\|K_{1}\right\|_{L^{1}(B(0,2 R))}\left\|u_{\delta}-u\right\|_{L^{2}\left(\mathbb{R}^{d}\right)}^{2} . \tag{6.11}
\end{align*}
$$

Now, let us address the one-dimensional case and begin by introducing the anti-derivative of $u-u_{\delta}$, which is given by

$$
\begin{equation*}
U(x)=\frac{1}{2} \int_{\mathbb{R}}\left(u-u_{\delta}\right)(y) \operatorname{sgn}(x-y) \mathrm{d} y=\frac{1}{2} \int_{B(0, R)}\left(u-u_{\delta}\right)(y) \operatorname{sgn}(x-y) \mathrm{d} y \tag{6.12}
\end{equation*}
$$

Here, the notation $\operatorname{sgn}(a)=a /|a|$ denotes the sign of $a \neq 0$. Since $\operatorname{supp}\left(u-u_{\delta}\right) \subset B(0, R)$ and $\int_{\mathbb{R}} u-u_{\delta} \mathrm{d} x=0$, it follows that $\operatorname{supp} U \subset B(0, R)$ and $\widehat{U}(\xi)=-i \xi^{-1}\left(\widehat{u}-\widehat{u}_{\delta}\right)(\xi), \xi \in \mathbb{R}$. Moreover, we have

$$
|U(x)|^{2} \leq R\left\|u-u_{\delta}\right\|_{L^{2}(\mathbb{R})}^{2}
$$

Hence we get

$$
\begin{equation*}
\left\|u-u_{\delta}\right\|_{\dot{H}^{-1}(\mathbb{R})}^{2}=\int_{\mathbb{R}}|\widehat{U}(\xi)|^{2} \mathrm{~d} \xi=\int_{B(0, R)}|U(x)|^{2} \mathrm{~d} x \leq 2 R^{2}\left\|u-u_{\delta}\right\|_{L^{2}(\mathbb{R})}^{2} \tag{6.13}
\end{equation*}
$$

which established the control of $\left\|u-u_{\delta}\right\|_{\dot{H}^{-1}(\mathbb{R})}^{2}$ in terms of $\left\|u-u_{\delta}\right\|_{L^{2}}^{2}$.
Substituting Eq. (6.11) (resp. Eq. (6.13) for $d=1$ ) into Eq. (6.10), we obtain

$$
\left\|\nabla v_{\delta}-\nabla v\right\|_{L^{2}\left(\mathbb{R}^{d}\right)}^{2} \leq c_{\nu}^{-1}\left(\left\|u_{\delta}-u\right\|_{\dot{H}^{\tilde{\psi}}\left(\mathbb{R}^{d}\right)}^{2}+\left\|u_{\delta}-u\right\|_{\dot{H}^{-1}\left(\mathbb{R}^{d}\right)}^{2}\right) \leq C\left\|u_{\delta}-u\right\|_{H^{\tilde{\psi}}\left(\mathbb{R}^{d}\right)}^{2} .
$$

In particular, Estimate (6.9) implies

$$
\begin{equation*}
\left\|\nabla v_{\delta}-\nabla v\right\|_{L^{2}\left(\mathbb{R}^{d}\right)} \leq C\|u\|_{H^{\tilde{\psi}}\left(\mathbb{R}^{d}\right)} \tag{6.14}
\end{equation*}
$$

Since $\left|\widehat{\nabla v}_{\delta}(\xi)\right|=|\xi| \psi^{-1}(\xi)\left|\widehat{u}_{\delta}(\xi)\right|$, it is readily seen that

$$
\left\|\nabla v_{\delta}-\nabla v\right\|_{\dot{H}^{\psi}\left(\mathbb{R}^{d}\right)}^{2}=\left\|u_{\delta}-u\right\|_{\dot{H}^{\tilde{\psi}}\left(\mathbb{R}^{d}\right)}^{2} .
$$

By the boundedness of $\left(u_{\delta}\right)_{\delta}$ in $H^{\tilde{\psi}}\left(\mathbb{R}^{d}\right)$ from Step 2., we find that

$$
\left\|\nabla v_{\delta}-\nabla v\right\|_{H_{\nu}\left(\mathbb{R}^{d}\right)}=\left\|\nabla v_{\delta}-\nabla v\right\|_{H^{\psi}\left(\mathbb{R}^{d}\right)} \leq C\left\|u_{\delta}-u\right\|_{H^{\tilde{\psi}}\left(\mathbb{R}^{d}\right)} \leq C\|u\|_{H^{\tilde{\psi}}\left(\mathbb{R}^{d}\right)}
$$

Taking into account the pointwise convergence, $|\xi| \widehat{v}_{\delta}(-\xi) \rightarrow|\xi| \widehat{v}(-\xi)$ as $\delta \rightarrow 0$, the compactness Theorem 2.14 implies that $\nabla v_{\delta} \rightarrow \nabla v$ in $L_{\mathrm{loc}}^{2}\left(\mathbb{R}^{d}\right)$.
Step 4. - Weak convergence of $\frac{1}{\delta}|\xi|^{-1}\left(\widehat{u}_{\delta}(\xi)-\widehat{u}(\xi)\right)$.
This step is dedicated to showing

$$
\frac{1}{\delta}|\xi|^{-1}\left(\widehat{u}_{\delta}(\xi)-\widehat{u}(\xi)\right) \rightharpoonup-i|\xi|^{-1} \xi \cdot \widehat{\eta u}(\xi)
$$

weakly in $L^{2}\left(\mathbb{R}^{d}\right)$, as $\delta \rightarrow 0$. To prove this claim, note that, by the dominated convergence theorem, we get

$$
\begin{align*}
\lim _{\delta \rightarrow 0} \frac{1}{\delta}(\widehat{u} \delta(\xi)-\widehat{u}(\xi)) & =\lim _{\delta \rightarrow 0} \int_{B(0, R)} \frac{1}{\delta}(\exp (-i \xi \cdot \delta \eta(x))-1) \exp (-i \xi \cdot x) u(x) \mathrm{d} x \\
& =-i \xi \cdot \int_{\mathbb{R}^{d}} \exp (-i \xi \cdot x) \eta(x) u(x) \mathrm{d} x  \tag{6.15}\\
& =-i \xi \cdot \widehat{(\eta u)}(\xi)
\end{align*}
$$

In other words, for all $\xi \in \mathbb{R}^{d}$, letting $\delta \rightarrow 0$ gives

$$
\frac{1}{\delta}|\xi|^{-1}\left(\widehat{u}_{\delta}(\xi)-\widehat{u}(\xi)\right) \rightarrow-i|\xi|^{-1} \xi \cdot \widehat{\eta u}(\xi)
$$

Now, for fixed $\xi \in \mathbb{R}^{d}$, let us define the function $g_{\xi}:[0, \infty) \rightarrow \mathbb{R}, g_{\xi}: \delta \mapsto g_{\xi}(\delta):=\widehat{u}_{\delta}(\xi)$ and $(\eta u)_{\delta}:=\phi_{\delta \#}(\eta u)$. Once again, by the dominated convergence theorem, we obtain

$$
\begin{aligned}
g_{\xi}^{\prime}(\delta) & =\lim _{h \rightarrow 0} \frac{1}{h}\left(\widehat{u}_{\delta+h}(\xi)-\widehat{u}_{\delta}(\xi)\right) \\
& =\lim _{h \rightarrow 0} \int_{B(0, R)} \frac{1}{h}(\exp (-i \xi \cdot h \eta(x))-1) \exp (-i \xi \cdot(x+\delta \eta(x))) u(x) \mathrm{d} x \\
& =-i \xi \cdot \int_{\mathbb{R}^{d}} \exp (-i \xi \cdot(x+\delta \eta(x))) \eta(x) u(x) \mathrm{d} x \\
& =-i \xi \cdot \widehat{(\eta u)}_{\delta}(\xi) .
\end{aligned}
$$

Moreover, note that also by dominated convergence theorem, $g_{\xi}^{\prime}$ is continuous, and therefore $g_{\xi}$ is of class $C^{1}$. Therefore by the intermediate value theorem, there is some $\delta_{\xi} \in(0, \delta)$ such that

$$
\frac{1}{\delta}\left(\widehat{u}_{\delta}(\xi)-\widehat{u}(\xi)\right)=g_{\xi}^{\prime}\left(\delta_{\xi}\right) .
$$

Furthermore, for $\delta \in\left(0, \delta_{0}\right)$,

$$
\left\|\widehat{(\eta u)}_{\delta}\right\|_{L^{2}\left(\mathbb{R}^{d}\right)}=\left\|(\eta u)_{\delta}\right\|_{L^{2}\left(\mathbb{R}^{d}\right)} \leq 2\|\eta u\|_{L^{2}\left(\mathbb{R}^{d}\right)} .
$$

Therefore $\frac{1}{\delta}|\xi|^{-1}\left(\widehat{u}_{\delta}-\widehat{u}\right)$ is bounded in $L^{2}\left(\mathbb{R}^{d}\right)$ uniformly in $\delta$, i.e.,

$$
\begin{equation*}
\frac{1}{\delta}\left\|\left\|\left.\xi\right|^{-1}\left(\hat{u}_{\delta}-\hat{u}\right)\right\|_{L^{2}\left(\mathbb{R}^{d}\right)}=\frac{1}{\delta}\right\| u_{\delta}-u \|_{\dot{H}^{-1}\left(\mathbb{R}^{d}\right)} \leq C, \tag{6.16}
\end{equation*}
$$

for some constant $C>0$ independent of $\delta>0$. In combination with the pointwise convergence, we have that, as $\delta \rightarrow 0, \frac{1}{\delta}|\xi|^{-1}(\widehat{u} \delta(\xi)-\widehat{u}(\xi))$ converges to $-i|\xi|^{-1} \xi \cdot \widehat{\eta u}(\xi)$ weakly in $L^{2}\left(\mathbb{R}^{d}\right)$.

## Step 5. - Variations of $\mathcal{F}$.

We claim that, for any probability density $u \in H^{\tilde{\psi}}\left(\mathbb{R}^{d}\right) \cap \dot{H}^{\psi^{-1}}\left(\mathbb{R}^{d}\right)$,

$$
\begin{equation*}
\lim _{\delta \rightarrow 0} \frac{1}{\delta}\left[\mathcal{F}\left(u_{\delta}\right)-\mathcal{F}(u)\right]=\int_{\mathbb{R}^{d}} \nabla v(x) \cdot \eta u(x) \mathrm{d} x . \tag{6.17}
\end{equation*}
$$

This step is divided into 2 parts, where the first part is dedicated to smooth densities and the second part less regular densities, respectively.

Substep 5.1 - Smooth densities. First, we prove Eq. (6.17) for a probability density $u \in C^{1}\left(\mathbb{R}^{d}\right) \cap$ $L^{\infty}\left(\mathbb{R}^{d}\right)$. By Plancherel's Theorem and Eq. (6.6), we have, for any $\rho>0$

$$
\begin{aligned}
\frac{1}{\delta}\left(\mathcal{F}\left(u_{\delta}\right)-\mathcal{F}(u)\right)= & \frac{1}{2} \int_{\mathbb{R}^{d}}-i \xi\left(\widehat{v}_{\delta}(-\xi)+\widehat{v}(-\xi)\right) \cdot \frac{1}{\delta} i \xi|\xi|^{-2}\left(\widehat{u}_{\delta}(\xi)-\widehat{u}(\xi)\right) \mathrm{d} \xi \\
= & \frac{1}{2} \int_{\mathbb{R}^{d}} \nabla\left(v+v_{\delta}\right)(x) \cdot \frac{1}{\delta} \nabla K_{1} *\left(u-u_{\delta}\right)(x) \mathrm{d} x \\
= & \frac{1}{2} \int_{\mathbb{R}^{d} \backslash B(0, \rho)} \nabla\left(v+v_{\delta}\right)(x) \cdot \frac{1}{\delta} \nabla K_{1} *\left(u-u_{\delta}\right)(x) \mathrm{d} x \\
& +\frac{1}{2} \int_{B(0, \rho)} \nabla\left(v+v_{\delta}\right)(x) \cdot \frac{1}{\delta} \nabla K_{1} *\left(u-u_{\delta}\right)(x) \mathrm{d} x .
\end{aligned}
$$

Since supp $\eta \subset B(0, R)$, by weak-strong lemma and the previous steps, for any $\rho>R$,

$$
\begin{aligned}
\lim _{\delta \rightarrow 0} \frac{1}{2} \int_{B(0, \rho)} \nabla\left(v+v_{\delta}\right)(x) \cdot \frac{1}{\delta} \nabla K_{1} *\left(u-u_{\delta}\right)(x) \mathrm{d} x & =\int_{B(0, \rho)} \nabla v(x) \cdot \eta u(x) \mathrm{d} x \\
& =\int_{\mathbb{R}^{d}} \nabla v(x) \cdot \eta u(x) \mathrm{d} x .
\end{aligned}
$$

Therefore, it is sufficient to prove that the term over $\mathbb{R}^{d} \backslash B(0, \rho)$ goes to 0 uniformly in $\delta$ as $\rho \rightarrow \infty$. Since $u \in C^{1}\left(\mathbb{R}^{d}\right) \cap L^{\infty}\left(\mathbb{R}^{d}\right)$, for some constant $C$ depending on $\|u\|_{C^{1}\left(\mathbb{R}^{d}\right)}$ and $\|\eta\|_{C^{1}\left(\mathbb{R}^{d}\right)}$, we get

$$
\left|u(x)-u_{\delta}(x)\right| \leq\left|u(x)-u\left(\phi_{\delta}(x)\right)\right|+\left|u\left(\phi_{\delta}(x)\right)\left(\operatorname{det} \nabla \phi_{\delta}-\mathrm{I}\right)\right| \leq C \delta,
$$

Since $\operatorname{supp}\left(u-u_{\delta}\right) \subset B(0, R)$, for $|x|>\rho$ with $\rho>2 R$ we get

$$
\nabla K_{1} *\left(u-u_{\delta}\right)(x)=c_{d} \int_{B(0, R)} \frac{\left(u-u_{\delta}\right)(y)(x-y)}{|x-y|^{d}} \mathrm{~d} y .
$$

Thus, since $|x| \leq 2|x-y|$ for $y \in B(0, R)$ we have

$$
\left|\nabla K_{1} *\left(u-u_{\delta}\right)(x)\right| \leq C \delta \int_{B(0, R)} \frac{\mathrm{d} y}{|x|^{d-1}}=C \delta|x|^{1-d}
$$

and therefore, for $d \geq 2$, we obtain

$$
\begin{aligned}
\sup _{\delta \in\left(0, \delta_{0}\right)} \mid \int_{\mathbb{R}^{d} \backslash B(0, \rho)} & \left.\nabla\left(v+v_{\delta}\right)(x) \cdot \frac{1}{\delta} \nabla K_{1} *\left(u-u_{\delta}\right)(x) \mathrm{d} x \right\rvert\, \\
& \leq \sup _{\delta \in\left(0, \delta_{0}\right)}\left\{\left\|\nabla\left(v+v_{\delta}\right)\right\|_{L^{2}\left(\mathbb{R}^{d}\right)}\left(\int_{\mathbb{R}^{d} \backslash B(0, \rho)}\left|\frac{1}{\delta} \nabla K_{1} *\left(u-u_{\delta}\right)(x)\right|^{2} \mathrm{~d} x\right)^{1 / 2}\right\} \\
& \leq C \int_{\mathbb{R}^{d} \backslash B(0, \rho)} \frac{\mathrm{d} x}{|x|^{2(d-1)}} \xrightarrow{\rho \rightarrow \infty} 0
\end{aligned}
$$

In the one-dimensional case $d=1$, observing that $i \xi|\xi|^{-2}=i \xi^{-1}, \xi \in \mathbb{R}$, we can legitimately identify $\nabla K_{1} *\left(u-u_{\delta}\right)(x)=U(x)$ where $U(x)$ is given in Eq. (6.12), that is,

$$
\nabla K_{1} *\left(u-u_{\delta}\right)(x)=\frac{1}{2} \int_{\mathbb{R}}\left(u-u_{\delta}\right)(y) \operatorname{sgn}(x-y) \mathrm{d} y=\frac{1}{2} \int_{B(0, R)} \frac{\left(u-u_{\delta}\right)(y)(x-y)}{|x-y|} \mathrm{d} y .
$$

Note that $|x|>R$ and $y \in B(0, R)$ we have $\operatorname{sgn}(x-y)=\operatorname{sgn}(x)$. So that

$$
\nabla K_{1} *\left(u-u_{\delta}\right)(x)=\operatorname{sgn}(x) \int_{B(0, R)}\left(u-u_{\delta}\right)(y) \mathrm{d} y=0
$$

Substep 5.2 - General case. Let probability density $u \in H^{\widetilde{\psi}}\left(\mathbb{R}^{d}\right) \cap \dot{H}^{\psi^{-1}}\left(\mathbb{R}^{d}\right)$ be given and let $u^{\varepsilon} \in$
 Chapter 3]. We define the push forward $u_{\delta}^{\varepsilon}:=\phi_{\delta \#} u^{\varepsilon}$, and the associated pressure $v_{\delta}^{\varepsilon}$ by $\widehat{v}_{\delta}^{\varepsilon}:=L^{-1} \widehat{u}_{\delta}^{\varepsilon}$. Using Cauchy-Schwartz and Eq. (6.6), we may write

$$
\begin{aligned}
& \left.\frac{1}{\delta} \right\rvert\,\left(\mathcal{F}\left(u_{\delta}\right)-\right.\mathcal{F}(u))-\left(\mathcal{F}\left(u_{\delta}^{\varepsilon}\right)-\mathcal{F}\left(u^{\varepsilon}\right)\right) \mid \\
& \left.=\frac{1}{2 \delta} \right\rvert\, \int_{\mathbb{R}^{d}}\left(\left(\widehat{v}_{\delta}-\widehat{v}_{\delta}^{\varepsilon}\right)(-\xi)+\left(\widehat{v}-\widehat{v}^{\varepsilon}\right)(-\xi)\right)\left(\widehat{u}_{\delta}(\xi)-\widehat{u}(\xi)\right) \mathrm{d} \xi \\
& \left.\quad-\int_{\mathbb{R}^{d}}\left(\widehat{v}_{\delta}^{\varepsilon}(-\xi)+\widehat{v}^{\varepsilon}(-\xi)\right) \frac{1}{\delta}\left(\left(\widehat{u}_{\delta}^{\varepsilon}-\widehat{u}_{\delta}\right)(\xi)-\left(\widehat{u}^{\varepsilon}-\widehat{u}\right)(\xi)\right) \mathrm{d} \xi \right\rvert\, \\
& \leq C\left(\left\|u_{\delta}-u_{\delta}^{\varepsilon}\right\|_{\dot{H}^{\psi^{*}}\left(\mathbb{R}^{d}\right)}+\left\|u-u^{\varepsilon}\right\|_{\dot{H}^{\psi^{*}}\left(\mathbb{R}^{d}\right)}\right) \frac{1}{\delta}\left\|u_{\delta}-u\right\|_{\dot{H}^{-1}\left(\mathbb{R}^{d}\right)} \\
& \quad+C\left\|u^{\epsilon}+u_{\delta}^{\varepsilon}\right\|_{\dot{H}^{\psi^{*}}\left(\mathbb{R}^{d}\right)} \frac{1}{\delta}\left\|\left(u_{\delta}^{\varepsilon}-u_{\delta}\right)-\left(u^{\varepsilon}-u\right)\right\|_{\dot{H}^{-1}\left(\mathbb{R}^{d}\right)} .
\end{aligned}
$$

Proceeding as in Step 3, Eq. (6.10), we have that

$$
\begin{aligned}
\left\|u_{\delta}-u_{\delta}^{\varepsilon}\right\|_{\dot{H} \psi^{*}\left(\mathbb{R}^{d}\right)} & \leq C\left(\left\|u_{\delta}-u_{\delta}^{\varepsilon}\right\|_{H^{\tilde{\psi}}\left(\mathbb{R}^{d}\right)}+\left\|u_{\delta}-u_{\delta}^{\varepsilon}\right\|_{\dot{H}^{-1}\left(\mathbb{R}^{d}\right)}\right) \\
& \leq C\left(\left\|u-u^{\varepsilon}\right\|_{H^{\tilde{\psi}}\left(\mathbb{R}^{d}\right)}+\left\|u-u^{\varepsilon}\right\|_{\dot{H}^{-1}\left(\mathbb{R}^{d}\right)}\right),
\end{aligned}
$$

where $C>0$ is independent of $\delta>0$. Let us stress that the first term vanishes, i.e., $\left\|u-u^{\varepsilon}\right\|_{H^{\tilde{\psi}}\left(\mathbb{R}^{d}\right)} \rightarrow$ 0 , as $\varepsilon \rightarrow 0$, by definition. Next, by the continuous embedding of Theorem 2.10 (ii), we have $\|u\|_{\dot{H}^{-1}} \leq C\|u\|_{\dot{H}^{\psi^{-1}}}$, which is bounded by assumption, whence we conclude that $u \in \dot{H}^{-1}\left(\mathbb{R}^{d}\right)$. Moreover, we have that $\left\|u-u^{\varepsilon}\right\|_{H^{-1}\left(\mathbb{R}^{d}\right)} \rightarrow 0$, as $\varepsilon \rightarrow 0$. By Eq. (6.16) in Step 3, we have that $\frac{1}{\delta}\left\|u_{\delta}-u\right\|_{H^{-1}\left(\mathbb{R}^{d}\right)}$ is bounded uniformly in $\delta$. Furthermore, proceeding as in Step 3, we obtain a bound on $\left\|u_{\delta}+u_{\delta}^{\varepsilon}\right\|_{\dot{H}^{\psi^{*}}\left(\mathbb{R}^{d}\right)}$, uniform in $\delta$ and $\varepsilon$, see Eq. (6.14) and, additionally, we obtain that $\frac{1}{\delta}\left\|\left(u-u^{\varepsilon}\right)-\left(u_{\delta}-u_{\delta}^{\varepsilon}\right)\right\|_{\dot{H}^{-1}\left(\mathbb{R}^{d}\right)} \leq C\left\|u-u^{\varepsilon}\right\|_{L^{2}\left(\mathbb{R}^{d}\right)}$, see Eq. (6.9).
Finally, since $\left\|u-u^{\varepsilon}\right\|_{L^{2}\left(\mathbb{R}^{d}\right)}+\left\|\nabla v-\nabla v^{\varepsilon}\right\|_{L^{2}\left(\mathbb{R}^{d}\right)} \rightarrow 0$, as $\varepsilon \rightarrow 0$, we have that

$$
\int_{\mathbb{R}^{d}} \nabla v^{\varepsilon}(x) \cdot \eta u^{\varepsilon}(x) \mathrm{d} x \rightarrow \int_{\mathbb{R}^{d}} \nabla v(x) \cdot \eta u(x) \mathrm{d} x .
$$

Putting everything together, we have that, for every $\varepsilon>0$,

$$
\begin{align*}
\lim _{\delta \rightarrow 0} \left\lvert\, \frac{1}{\delta}\left(\mathcal{F}\left(u_{\delta}\right)\right.\right. & -\mathcal{F}(u))-\int_{\mathbb{R}^{d}} \nabla v(x) \cdot \eta u(x) \mathrm{d} x \mid \\
\leq & \lim _{\delta \rightarrow 0}\left|\frac{1}{\delta}\left(\mathcal{F}\left(u_{\delta}\right)-\mathcal{F}(u)\right)-\frac{1}{\delta}\left(\mathcal{F}\left(u_{\delta}^{\varepsilon}\right)-\mathcal{F}\left(u^{\varepsilon}\right)\right)\right| \\
& +\lim _{\delta \rightarrow 0}\left|-\frac{1}{\delta}\left(\mathcal{F}\left(u_{\delta}^{\varepsilon}\right)-\mathcal{F}\left(u^{\varepsilon}\right)\right)-\int_{\mathbb{R}^{d}} \nabla v^{\varepsilon}(x) \cdot \eta u^{\varepsilon}(x) \mathrm{d} x\right|  \tag{6.18}\\
& +\left|\int_{\mathbb{R}^{d}} \nabla v^{\varepsilon}(x) \cdot \eta u^{\varepsilon}(x) \mathrm{d} x-\int_{\mathbb{R}^{d}} \nabla v(x) \cdot \eta u(x) \mathrm{d} x\right| \\
\leq & C\left(\left\|u-u^{\varepsilon}\right\|_{H^{\tilde{\psi}}\left(\mathbb{R}^{d}\right)}+\left\|u-u^{\varepsilon}\right\|_{H^{-1}\left(\mathbb{R}^{d}\right)}+\left\|u-u^{\varepsilon}\right\|_{\left.L^{2}\left(\mathbb{R}^{d}\right)\right)}\right) \\
& +\left|\int_{\mathbb{R}^{d}} \nabla v^{\varepsilon}(x) \cdot \eta u^{\varepsilon}(x) \mathrm{d} x-\int_{\mathbb{R}^{d}} \nabla v(x) \cdot \eta u(x) \mathrm{d} x\right| .
\end{align*}
$$

Since Eq. (6.18) is valid for any $\varepsilon>0$, we have shown claim of the second part.

## Step 6. - Conclusion.

Combining all pieces of the jigsaw, we obtain

$$
\begin{equation*}
0 \leq \int_{\mathbb{R}^{d}} \nabla v(x) \cdot \eta u(x) \mathrm{d} x-\frac{1}{\tau} \int_{\mathbb{R}^{d}}\left(T_{u_{\tau}^{k}}^{u_{\tau}^{k-1}}-\mathrm{I}\right) \cdot \eta u_{\tau}^{k} \mathrm{~d} x . \tag{6.19}
\end{equation*}
$$

Replacing $\eta$ by $-\eta$ in Eq. (6.19), we have that

$$
\begin{equation*}
0=\int_{\mathbb{R}^{d}} \nabla v(x) \cdot \eta u(x) \mathrm{d} x-\frac{1}{\tau} \int_{\mathbb{R}^{d}}\left(T_{u_{\tau}^{k}}^{u_{\tau}^{k-1}}-\mathrm{I}\right) \cdot \eta u_{\tau}^{k} \mathrm{~d} x . \tag{6.20}
\end{equation*}
$$

We have thus proven Eq. (6.2). Taking a sequence $\eta$ converging to $\nabla v$, and using the fact that

$$
\int_{\mathbb{R}^{d}}\left|T_{u_{\tau}^{k}}^{u_{\tau}^{k-1}}-\mathrm{I}\right|^{2} u_{\tau}^{k} \mathrm{~d} x=W^{2}\left(u, u_{\tau}^{k-1}\right),
$$

we obtain Eq. (6.3).
Finally, we show that the limiting curve $u$ is a weak solution, i.e., $u$ satisfies Eq. (1.1), hence giving the proof of Theorem 1.1 (iv).

Theorem 6.2 (Weak solution). Assume that $\nu \notin L^{1}\left(\mathbb{R}^{d}\right)$ and satisfies Condition $\left(C_{\nu}\right)$. Moreover, assume the symbol $\widetilde{\psi}$ is associated with a unimodal Lévy kernel $\widetilde{\nu}$ satisfying the following condition

$$
\begin{equation*}
\text { For any } 0<\lambda<1 \text { there is } c_{\lambda}>0 \text { s.t. } \widetilde{\nu}(\lambda h) \leq c_{\lambda} \widetilde{\nu}(h) \text { whenever }|h| \leq 1 \text {. } \tag{6.21}
\end{equation*}
$$

Let $u_{0} \in \dot{H}^{\psi^{-1}}\left(\mathbb{R}^{d}\right) \cap \mathcal{P}_{2}\left(\mathbb{R}^{d}\right)$. Let $v=L^{-1} u$ where $u$ is the limiting curve defined by Theorem 3.6 and $v$ its associated pressure. Then, $u$ is a weak solution to Eq. (1.1) in the following sense

$$
\begin{equation*}
\int_{0}^{\infty} \int_{\mathbb{R}^{d}}\left(\partial_{t} \varphi-\nabla \varphi \cdot \nabla v\right) u \mathrm{~d} x \mathrm{~d} t=0, \quad \forall \varphi \in C_{c}^{\infty}\left(\mathbb{R}^{d} \times(0, \infty)\right) \tag{6.22}
\end{equation*}
$$

Proof. Given $\varphi \in C_{c}^{\infty}\left(\mathbb{R}^{d} \times(0, \infty)\right)$, we use $\eta=\nabla_{x} \varphi$ in the Euler-Lagrange equation, Eq. (6.2), in Theorem 6.1, and integrate in time to get

$$
\begin{equation*}
\sum_{k \in \mathbb{N}} \int_{(k-1) \tau}^{k \tau}\left\{\int_{\mathbb{R}^{d}} \nabla v_{\tau}^{k} \cdot \eta u_{\tau}^{k}-\frac{1}{\tau}\left(T_{u_{\tau}^{k}}^{u_{\tau}^{k-1}}-\mathrm{I}\right) \cdot \eta u_{\tau}^{k} \mathrm{~d} x\right\} \mathrm{d} t=0 . \tag{6.23}
\end{equation*}
$$

Using the definition of the piecewise constant interpolation, Theorem 5.2 allows us to pass of the limit in the first term of Eq. (6.23), which converges to

$$
\int_{0}^{+\infty} \int_{\mathbb{R}^{d}} \nabla v \cdot \nabla_{x} \varphi u \mathrm{~d} x \mathrm{~d} t
$$

The convergence of the second term associated to the time derivative is classical, and its limit is

$$
\int_{0}^{+\infty} \int_{\mathbb{R}^{d}} \partial_{t} \varphi u \mathrm{~d} x \mathrm{~d} t
$$

see, for example, [AGS05, Theorem 11.1.6], which concludes the proof.
Finally, let us address the remaining outstanding item, Theorem 1.1, (iv), that is, the energy dissipation inequality. To this end, we follow the usual argument relying on the De Giorgi interpolation.

Definition 6.3 (De Giorgi variational interpolation). We define the De Giorgi interpolant $\widetilde{u}_{\tau} \in$ $A C^{2}\left([0, T),\left(\mathcal{P}_{2}\left(\mathbb{R}^{d}\right), W\right)\right)$ as $\widetilde{u}_{\tau}(k \tau):=u_{\tau}(k \tau)$ for $k=1,2, \ldots$, and

$$
\begin{equation*}
\widetilde{u}_{\tau}:=\underset{u}{\operatorname{argmin}}\left\{\frac{1}{2(t-(k-1) \tau)} W^{2}\left(u, u_{\tau}^{k-1}\right)+\mathcal{F}(u)\right\}, \tag{6.24}
\end{equation*}
$$

for $t \in((k-1) \tau, k \tau)$ and $k \in \mathbb{N}$. We observe that Eq. (6.24) has a unique solution $\widetilde{u} \in H^{\tilde{\psi}}\left(\mathbb{R}^{d}\right)$ which can be established as in the proof of Theorem 1.1 (i) and (ii). Moreover, the De Giorgi interpolation coincides with the discrete minimizers at multiples of $\tau$.

We will now prove a proposition, analog to Proposition 6.3 in [LMS18], regarding De Giorgi variational interpolation, which we will use to prove the energy dissipation inequality, Theorem 1.1, (iv).

Proposition 6.4. Let $v_{\tau}$ be given by $\widehat{\tilde{v}_{\tau}}=L^{-1} \widehat{\widetilde{u}_{\tau}}$. Then, for all $N \in \mathbb{N}$, and $\tau>0$,

$$
\begin{equation*}
\frac{1}{2} \int_{0}^{N \tau} \int_{\mathbb{R}^{d}}\left|\nabla v_{\tau}\right|^{2} u_{\tau} \mathrm{d} x \mathrm{~d} t+\frac{1}{2} \int_{0}^{N \tau} \int_{\mathbb{R}^{d}}\left|\nabla \widetilde{v}_{\tau}\right|^{2} \widetilde{u}_{\tau} \mathrm{d} x \mathrm{~d} t+\mathcal{F}\left(u_{\tau}(N \tau)\right) \leq \mathcal{F}\left(u_{0}\right) \tag{6.25}
\end{equation*}
$$

Moreover, for any $t \in[0, T]$, there holds

$$
\begin{equation*}
W^{2}\left(\widetilde{u}_{\tau}(t), u_{\tau}(t)\right) \leq 8 \tau \mathcal{F}\left(u_{0}\right) . \tag{6.26}
\end{equation*}
$$

Proof. The proof of this energy dissipation inequality is classical and we give it here for completeness. First, proceeding as in the proof of Theorem 6.1, we show that

$$
\begin{equation*}
\int_{\mathbb{R}^{d}}\left|\nabla \widetilde{v}_{\tau}^{k}\right|^{2} \widetilde{u}_{\tau} \mathrm{d} x=\frac{1}{(t-(k-1) \tau)^{2}} W^{2}\left(\widetilde{u}_{\tau}, u_{\tau}^{k-1}\right) \tag{6.27}
\end{equation*}
$$

From [AGS05, Lemma 3.2.2], we infer the energy identity

$$
\begin{equation*}
\frac{1}{2 \tau} W^{2}\left(u_{\tau}^{k}, u_{\tau}^{k-1}\right)+\frac{1}{2} \int_{(k-1) \tau}^{k \tau} \frac{1}{(t-(k-1) \tau)^{2}} W^{2}\left(\widetilde{u}_{\tau}, u_{\tau}^{k}\right) \mathrm{d} t+\mathcal{F}\left(u_{\tau}^{k}\right) \leq \mathcal{F}\left(u_{\tau}^{k-1}\right) \tag{6.28}
\end{equation*}
$$

Summing over $k \in 1, \ldots, N$, we obtain

$$
\begin{equation*}
\frac{1}{2 \tau} \sum_{k=1}^{N} W^{2}\left(u_{\tau}^{k}, u_{\tau}^{k-1}\right)+\frac{1}{2} \sum_{k=1}^{N} \int_{(k-1) \tau}^{k \tau} \frac{1}{(t-(k-1) \tau)^{2}} W^{2}\left(\widetilde{u}_{\tau}, u_{\tau}^{k}\right) \mathrm{d} t+\mathcal{F}\left(u_{\tau}^{N}\right) \leq \mathcal{F}\left(u_{\tau}^{0}\right) \tag{6.29}
\end{equation*}
$$

Substituting Eqs. (6.3, 6.27) into Eq. (6.29), we obtain

$$
\frac{\tau}{2} \sum_{k=1}^{N} \int_{\mathbb{R}^{d}}\left|\nabla v_{\tau}^{k}\right|^{2} u_{\tau}^{k} \mathrm{~d} x+\frac{1}{2} \sum_{k=1}^{N} \int_{(k-1) \tau}^{k \tau} \int_{\mathbb{R}^{d}}\left|\nabla \widetilde{v}_{\tau}^{k}\right|^{2} \widetilde{u}_{\tau} \mathrm{d} x \mathrm{~d} t+\mathcal{F}\left(u_{\tau}^{N}\right) \leq \mathcal{F}\left(u_{\tau}^{0}\right)
$$

which yields the first statement, Eq. (6.25), after replacing the sums over $k$ by integrals. The second statement, Eq. (6.26), follows from the triangle inequality and Proposition 3.4 (i).

We now give the brief proof of the energy dissipation inequality, Theorem $1.1(v)$.
Theorem 6.5. Let $u$ be a weak solution of Problem (1.1), obtained as the limit of the minimizing movement scheme. Then, there holds the following energy estimate

$$
\begin{equation*}
\mathcal{F}(u(T))+\int_{0}^{T} \int_{\mathbb{R}^{d}} u(t)|\nabla v(t)|^{2} \mathrm{~d} x \mathrm{~d} t \leq \mathcal{F}\left(u_{0}\right) \tag{6.30}
\end{equation*}
$$

Proof. With the regularity of $\left(u_{\tau}\right)_{\tau}$ and $\left(\nabla v_{\tau}\right)_{\tau}$ established above, we may use the lower semicontinuity result [AGS05, Theorem 5.4.4] to the limit in Eq. (6.25), as $\tau$ tends to 0. This way we obtain the energy dissipation inequality, Eq. (6.30).

## 7. EXAMPLES OF KERNELS

Here, we provide various examples of kernels that satisfy the conditions of our main result. Let us recall the notations

$$
\psi^{-1}(\xi)=\frac{1}{\psi(\xi)}, \quad \widetilde{\psi}(\xi)=|\xi|^{2} \psi^{-1}(\xi), \quad \text { and } \quad \psi^{*}(\xi)=|\xi|^{2} \psi^{-2}(\xi)=\widetilde{\psi}(\xi) \psi^{-1}(\xi)
$$

where $\psi(\xi)$ is the symbol of a Lévy operator $L$, see Theorem 2.1. We will also denote the Lévy kernel corresponding to the symbol $\widetilde{\psi}$ by $\widetilde{\nu}$, whenever the latter exists.
Example 7.1 (Standard example). For $s \in[0,1]$ consider $\psi(\xi)=|\xi|^{2 s}$. As highlighted in the introduction the corresponding Lévy kernel $\nu(h)=\frac{C_{d, s}}{2}|h|^{-d-2 s}$ is the standard interaction kernel of the fraction Laplacian $L=(-\Delta)^{s}$ when $s \in(0,1)$. In the two extreme cases, $s=1, \psi(\xi)=|\xi|^{2}$ corresponds to the symbol of usual Laplacian, and $s=0, \psi(\xi)=1$ corresponds to the symbol of the identity operator $L=I$. In either case, we have the following.

$$
\psi(\xi)=|\xi|^{2 s}, \quad \psi^{-1}(\xi)=|\xi|^{-2 s}, \quad \widetilde{\psi}(\xi)=|\xi|^{2(1-s)}, \quad \text { and } \quad \psi^{*}(\xi)=|\xi|^{2(1-2 s)}
$$

The nonlocal spaces associated to $\psi$ are $H^{\psi}\left(\mathbb{R}^{d}\right)=H^{s}\left(\mathbb{R}^{d}\right), \quad \dot{H}^{\psi}\left(\mathbb{R}^{d}\right)=\dot{H}^{s}\left(\mathbb{R}^{d}\right)$, and

$$
\dot{H}^{\psi^{-1}}\left(\mathbb{R}^{d}\right)=\dot{H}^{-s}\left(\mathbb{R}^{d}\right), \quad H^{\tilde{\psi}}\left(\mathbb{R}^{d}\right)=H^{1-s}\left(\mathbb{R}^{d}\right), \quad \text { and } \quad H^{\psi^{*}}\left(\mathbb{R}^{d}\right)=H^{1-2 s}\left(\mathbb{R}^{d}\right)
$$

Moreover, we readily see that $\widetilde{\psi}$ is the symbol associated with the operator $(-\Delta)^{1-s}$ whose kernel is given by $\widetilde{\nu}(h)=\frac{C_{d, 1-s}}{2}|h|^{-d-2(1-s)}$.
Example 7.2. Consider the Lévy operator associated with the kernel

$$
\nu(h)=\int_{0}^{\infty} e^{-t} G_{t}(h) \mathrm{d} t=\int_{0}^{\infty} \frac{e^{-t}}{(4 \pi t)^{d / 2}} e^{\frac{|h|^{2}}{4 t}} \mathrm{~d} t .
$$

The corresponding Lévy symbol is given by

$$
\psi(\xi)=\int_{\mathbb{R}^{d}}(1-\cos (\xi \cdot h)) \nu(h) \mathrm{d} h=1-\int_{0}^{\infty} \widehat{G}_{t}(\xi) \mathrm{d} t=\frac{|\xi|^{2}}{1+|\xi|^{2}} .
$$

The symbol $\psi$ clearly satisfies

$$
\frac{1}{2}\left(1 \wedge|\xi|^{2}\right) \leq \psi(\xi) \leq\left(1 \wedge|\xi|^{2}\right)
$$

and we also have

$$
\widetilde{\psi}(\xi)=1+|\xi|^{2}, \quad \psi^{-1}(\xi)=1+|\xi|^{-2}, \quad \text { and } \quad \psi^{*}(\xi)=2+|\xi|^{-2}+|\xi|^{2} .
$$

Therefore, we get the following functions spaces

$$
\dot{H}^{1}\left(\mathbb{R}^{d}\right)+L^{2}\left(\mathbb{R}^{d}\right) \subset \dot{H}^{\psi}\left(\mathbb{R}^{d}\right), \quad H^{\psi}\left(\mathbb{R}^{d}\right)=L^{2}\left(\mathbb{R}^{d}\right), \quad \text { and } \quad \dot{H}^{\psi^{-1}}\left(\mathbb{R}^{d}\right)=\dot{H}^{-1}\left(\mathbb{R}^{d}\right) \cap L^{2}\left(\mathbb{R}^{d}\right)
$$

as well as

$$
H^{\tilde{\psi}}\left(\mathbb{R}^{d}\right)=\dot{H}^{\tilde{\psi}}\left(\mathbb{R}^{d}\right)=H^{1}\left(\mathbb{R}^{d}\right), \quad \text { and } \quad H^{\psi^{*}}\left(\mathbb{R}^{d}\right)=\dot{H}^{\psi^{*}}\left(\mathbb{R}^{d}\right)=\dot{H}^{-1}\left(\mathbb{R}^{d}\right) \cap H^{1}\left(\mathbb{R}^{d}\right)
$$

Example 7.3. Consider $s \in(0,1]$ and $\psi(\xi)=\left(|\xi|^{2}+1\right)^{s}-1$. This is the symbol of the Lévy type denoted $L=(-\Delta+I)^{s}-I$, usually referred to as fractional Schrödinger operator or the fractional relativistic operator. According to [FF14], see also [JKS23], it can be shown that

$$
L u(x)=(-\Delta+I)^{s} u(x)-u(x)=S_{d, s} \int_{\mathbb{R}^{d}} \frac{(u(x)-u(y))}{|x-y|^{\frac{d+2 s}{2}}} K_{\frac{d+2 s}{2}}(|x-y|) \mathrm{d} y,
$$

where $K_{\beta}$ is the modified Bessel function of the second kind $^{1}$ of order $\beta \in \mathbb{R}$ [AS61,EMOT81, Wat95] given as

$$
K_{\beta}(r)=2^{-\beta-1} r^{\beta} \int_{0}^{\infty} e^{-t} e^{-\frac{r^{2}}{4 t}} t^{-\beta-1} \mathrm{~d} t .
$$

In the definition of $L$, the constant

$$
S_{d, s}=\frac{2^{-\frac{d-2 s}{2}+1}}{\pi^{d / 2}|\Gamma(-s)|}=\frac{2^{-\frac{d+2 s}{2}+1} C_{d, s}}{\Gamma\left(\frac{d+2 s}{2}\right)}
$$

is a normalizing constant such that $\widehat{L u}(\xi)=\psi(\xi) \widehat{u}(\xi)$, for all $u \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$, see, for instance, [FF14, JKS23, AS61]. One readily checks that

$$
\lim _{|\xi| \rightarrow \infty} \frac{\psi(\xi)}{|\xi|^{2 s}}=\lim _{|\xi| \rightarrow \infty} \frac{\left(|\xi|^{2}+1\right)^{s}-1}{|\xi|^{2 s}}=1, \quad \text { and } \quad \lim _{|\xi| \rightarrow 0} \frac{\psi(\xi)}{|\xi|^{2}}=\lim _{|\xi| \rightarrow 0} \frac{\left(|\xi|^{2}+1\right)^{s}-1}{|\xi|^{2}}=s
$$

That is $\psi(\xi) \sim|\xi|^{2 s}$, as $|\xi| \rightarrow \infty$ and $\psi(\xi) \sim|\xi|^{2}$, as $|\xi| \rightarrow 0$. Therefore, there exists some $c>0$ such that

$$
c^{-1} \min \left(|\xi|^{2},|\xi|^{2 s}\right) \leq \psi(\xi) \leq c \min \left(|\xi|^{2},|\xi|^{2 s}\right) .
$$

It turns out that,

$$
\psi(\xi) \asymp \min \left(|\xi|^{2},|\xi|^{2 s}\right), \quad \text { and } \quad \psi^{-1}(\xi) \asymp \max \left(|\xi|^{-2},|\xi|^{-2 s}\right),
$$

[^1]as well as
$$
\widetilde{\psi}(\xi) \asymp \max \left(1,|\xi|^{2(1-s)}\right), \quad \text { and } \quad \psi^{*}(\xi) \asymp \max \left(|\xi|^{-2},|\xi|^{2-4 s}\right) .
$$

Moreover, we have

$$
\psi(\xi) \geq c_{s}\left(1 \wedge|\xi|^{2}\right), \quad \text { with } \quad c_{s}=\min \left(2^{s}-1, s 2^{s-1}\right)
$$

Indeed, for $|\xi|^{2} \geq 1$ we have $\psi(\xi) \geq\left(2^{s}-1\right) \geq\left(2^{s}-1\right)\left(1 \wedge|\xi|^{2}\right)$, whereas for $|\xi|^{2} \leq 1$ we get

$$
\psi(\xi)=s \int_{0}^{|\xi|^{2}}(1+t)^{s-1} \mathrm{~d} t \geq s 2^{s-1}|\xi|^{2} \geq s 2^{s-1}\left(1 \wedge|\xi|^{2}\right)
$$

The corresponding functions spaces are given by

$$
\dot{H}^{\psi}\left(\mathbb{R}^{d}\right)=\dot{H}^{1}\left(\mathbb{R}^{d}\right)+\dot{H}^{s}\left(\mathbb{R}^{d}\right), \quad H^{\psi}\left(\mathbb{R}^{d}\right)=H^{s}\left(\mathbb{R}^{d}\right), \quad \text { and } \quad \dot{H}^{\psi^{-1}}\left(\mathbb{R}^{d}\right)=\dot{H}^{-1}\left(\mathbb{R}^{d}\right) \cap \dot{H}^{-s}\left(\mathbb{R}^{d}\right)
$$

as well as

$$
H^{\tilde{\psi}}\left(\mathbb{R}^{d}\right)=\dot{H}^{\tilde{\psi}}\left(\mathbb{R}^{d}\right)=H^{1-s}\left(\mathbb{R}^{d}\right), \quad \text { and } \quad \dot{H}^{\psi^{*}}\left(\mathbb{R}^{d}\right)=\dot{H}^{-1}\left(\mathbb{R}^{d}\right) \cap \dot{H}^{1-2 s}\left(\mathbb{R}^{d}\right)
$$

Example 7.4. Consider $\psi(\xi)=\log \left(|\xi|^{2}+1\right)$. This symbol of Lévy type is associated to $L=$ $\log (-\Delta+I)$, usually referred to as the logarithmic Schröndiger operator. More precisely, it can be shown that

$$
L u(x)=\log (-\Delta+I) u(x)=S_{d, 0} \int_{\mathbb{R}^{d}} \frac{(u(x)-u(y))}{|x-y|^{\frac{d}{2}}} K_{\frac{d}{2}}(|x-y|) \mathrm{d} y,
$$

where the constant $S_{d, 0}=2^{-\frac{d}{2}+1} \pi^{-d / 2}$ is a normalizing constant such that $\widehat{L u}(\xi)=\psi(\xi) \widehat{u}(\xi)$, for all $u \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$. Observe that

$$
\lim _{|\xi| \rightarrow \infty} \frac{\psi(\xi)}{\log |\xi|^{2}}=\lim _{|\xi| \rightarrow \infty} \frac{\log \left(|\xi|^{2}+1\right)}{\log |\xi|^{2}}=1, \quad \text { and } \quad \lim _{|\xi| \rightarrow 0} \frac{\psi(\xi)}{|\xi|^{2}}=\lim _{|\xi| \rightarrow 0} \frac{\log \left(|\xi|^{2}+1\right)}{|\xi|^{2}}=1 .
$$

That is $\psi(\xi) \sim \log |\xi|^{2}$, as $|\xi| \rightarrow \infty$ and $\psi(\xi) \sim|\xi|^{2}$, as $|\xi| \rightarrow 0$. Therefore, there is $c>0$ such that

$$
c^{-1} \min \left(\left.|\log | \xi\right|^{2}\left|,|\xi|^{2}\right) \leq \psi(\xi) \leq c \min \left(\left.|\log | \xi\right|^{2}\left|,|\xi|^{2}\right)\right.\right.
$$

As in the previous example, we have

$$
\psi(\xi) \geq c_{0}\left(1 \wedge|\xi|^{2}\right) \quad \text { with } \quad c_{0}=\min \left(\log 2,2^{-1}\right)
$$

Moreover, for any $0 \leq \eta<1$, there exists $c=c(\eta)>0$, such that

$$
\psi(\xi) \asymp \min \left(\left.|\log | \xi\right|^{2}\left|,|\xi|^{2}\right) \leq c \min \left(|\xi|^{2},|\xi|^{2(1-\eta)}\right), \quad \text { and } \quad \psi^{-1}(\xi) \asymp \max \left(\log ^{-1}|\xi|^{2},|\xi|^{-2}\right),\right.
$$

as well as

$$
\widetilde{\psi}(\xi) \asymp \max \left(1,|\xi|^{2} \log ^{-1}|\xi|^{2}\right) \geq c^{-1}\left(|\xi|^{2 \eta}+1\right), \quad \text { and } \quad \psi^{*}(\xi) \asymp \max \left(|\xi|^{-2},|\xi|^{2} \log ^{-2}|\xi|^{2}\right) .
$$

Thus, for any $0 \leq \eta<1$, we get

$$
H^{1-\eta} \subset H^{\psi}\left(\mathbb{R}^{d}\right) \quad \text { and } \quad H_{39}^{\tilde{\psi}}\left(\mathbb{R}^{d}\right)=\dot{H}^{\tilde{\psi}}\left(\mathbb{R}^{d}\right) \subset H^{\eta}\left(\mathbb{R}^{d}\right)
$$

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[^1]:    ${ }^{1}$ Modified Bessel function of the second kind are also called modified Bessel function of the third kind as for instance in [AS61].

