Onset of fracture in random heterogeneous particle chains

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Abstract

In mechanical systems it is of interest to know the onset of fracture in dependence of the boundary conditions. Here we study a one-dimensional model which allows for an underlying heterogeneous structure in the discrete setting. Such models have recently been studied in the passage to the continuum by means of variational convergence (Γ-convergence). The Γ-limit results determine thresholds of the boundary condition, which mark a transition from purely elastic behaviour to the occurrence of a crack. In this article we provide a notion of fracture in the discrete setting and show that its continuum limit yields the same threshold as that obtained from the Γ-limit. Since the calculation of the fracture threshold is much easier with the new method, we see a good chance that this new approach will turn out useful in applications.

Key Words: Fracture, discrete system, stochastic homogenization, Γ-convergence, Lennard-Jones potentials. fracture.

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1 Introduction

The mechanical behaviour of one-dimensional systems has been of interest for decades. Such systems serve as toy models for higher-dimensional theoretical investigations and are of interest with respect to one-dimensional structures, see, e.g., [7, 8, 10, 11, 19]. In order to understand the effective behaviour of materials, the systems are studied as the number of particles tends to infinity.

In this article we focus on the occurrence of cracks and continue a mathematical analysis of the effective behaviour of one-dimensional discrete systems in the passage to the continuum. In particular we strive for insight into the threshold for the overall prescribed length ℓ of a chain. If ℓ is smaller than the threshold, the system will show elastic behaviour, whereas cracks are energetically favoured if ℓ is larger than the threshold. The interaction potentials between the particles or atoms of the discrete chain are allowed to be in a large class of convex-concave potentials, which include for instance the classical Lennard-Jones potentials. The system is then modeled with the help of an energy functional that is the sum of all the interaction potentials, see (1). Here we restrict to the

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interactions of nearest neighbours; for related studies with interactions beyond nearest neighbours we refer to [4, 5, 18].

In view of misplaced atoms or of chains consisting of several different kind of particles, we allow for a random distribution of the interaction potentials, see Assumption 2.1 and (2) for details. The limit passage is then also referred to as stochastic homogenization, cf., e.g., [1, 6, 9, 16]. As a special case, also materials with a periodic heterostructure are included, cf. also [15].

An appropriate mathematical technique for the passage of energy functionals from discrete to continuous systems is based on the notion of Γ-convergence, which is a notion of a variational convergence and (under coercivity assumptions) ensures that minimizers of the discrete system converge to minimizers of the system in the continuum limit, see, e.g., [2, 3, 17] and references cited therein. As the number of particles tends to infinity, the energy functional converges to a functional that shows the occurrence of cracks in dependence of the boundary conditions that determine the length of the one-dimensional structure. We stress that, on the discrete level, no crack has been identified so far. Instead, the mathematical analysis works with a piecewise affine and thus continuous deformation of the discrete system. Still, a deeper understanding of the crack formation is also of interest on the discrete level.

In this article, which is partially based on the PhD thesis [12, Chapter 7] of L. Lauerbach, we focus on the emergence of cracks in atomistic chains. On the level of the continuum limiting model of the chain, "crack" has a clear meaning – it is the point where the continuum deformation features a jump and there is no interaction between the different segments separated by the jump. In contrary, on the level of a discrete chain with $n+1$ particles, the notion of "crack" cannot be unambigously defined, since always neighbouring particles interact. In the present paper we introduce a notion of "onset of a crack" at the discrete level for a chain with $n+1$ particles. For simplicity we discuss the key idea in the case of a chain with $n+1$ particles that is composed of (random) potentials that are convex around its ground state and otherwise concave, i. e. for deformations larger than an inflection point z_{frac}. We call a deformation u *elastic*, if the individual interaction potentials along the chain are only evaluated in their convex region. In contrary, a deformation that is *not* elastic invokes at least one bond that "lives" in the concave region of the corresponding potential. Next, we consider the energy minimizers u_n of the chain with $n+1$ particles and prescribed total length $\ell > 0$. If the minimizers u_n are elastic for all $n \in \mathbb{N}$, then we do not expect the occurance of crack in the continuum limit; while in the other case, we expect that minimizing sequeces show a concentration of strain on a finite number of weak bonds and thus a "crack" emerges in the continuum limit. Based on this heuristics we intrduce a "critical stretch" ℓ_n^* for random chains with $n+1$ particles. Firstly, we prove that it converges, for $n \to \infty$, to the jump-threshold predicted by the zeroth-order Γ-limit of the discrete energy, which has been obtained earlier in [13]. Secondly, we establish a first order expansion of the critical stretch and show that the coefficients of the expansion term agree with the values predicted by the first-order Γ-limit of the discrete energy derived in [13]. Since the proofs in [13] are technically quite involved, it is interesting to learn that there is a much simpler method for the derivation of the jump threshold in the continuum limit. We expect that the new notions of a fracture point and of a jump threshold in the discrete setting turn out to be useful also in a wider class of applications.

The outline of this article is as follows: In the second section we introduce the model in the discrete setting, including the assumptions on the large class of interaction potentials in the random setting. Further we provide the definition of a critical stretch (Definition 2.1), which corresponds to the
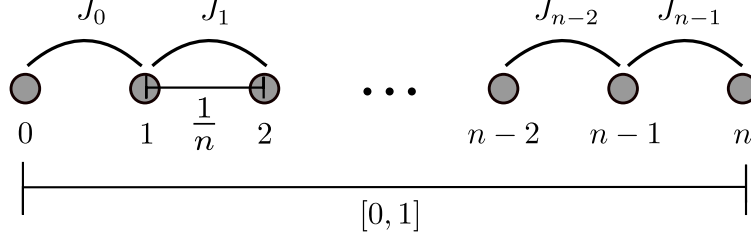
Figure 1: Chain of $n+1$ atoms with reference position $\frac{i}{n}$. The potential J_i describes the nearest neighbour interaction of atom i and $i+1$, $i=0,...,n-1$. The characteristic length scale is $\frac{1}{n}$ and the interval is $[0,1]$.

jump threshold. We assert the asymptotic behaviour of the critical stretch as the number of particles tends to infinity (Theorem 2.1) and compare the limit to the corresponding Γ-convergence results. Moreover, we consider a rescaled setting, define the rescaled jump threshold and assert its asymptotic behaviour as n tends to infinity (Theorem 2.2). Finally we compare also this result with the corresponding Γ-convergences result. All proofs are provided in Section 3.

2   Setup and main results

We consider a chain of $n+1$ atoms that in a reference configuration are placed at the sites in $\frac{1}{n}\mathbb{Z} \cap [0,1]$, see Figure 1. The deformation of the atoms is referred to as $u_n : \frac{1}{n}\mathbb{Z} \cap [0,1] \to \mathbb{R}$. For the passage from discrete systems to their continuous counterparts it is useful to identify the discrete functions with their piecewise affine interpolations, i.e., with the functions in

$$\mathcal{A}_n := \left\{ u \in C([0,1]) : u \text{ is affine on } (i, i+1)\frac{1}{n}, \ i \in \{0, 1, ..., n-1\}, \text{monotonically increasing} \right\}.$$

We shall also consider clamped boundary conditions for the chain and thus introduce for $\ell > 0$ the set

$$\mathcal{A}_{n,\ell} := \{u \in \mathcal{A}_n : u(0) = 0, \ u(1) = \ell\}.$$

We consider a discrete energy functional of the form

$$\mathcal{A}_{n,\ell} \ni u \mapsto E_n(u) := \sum_{i=0}^{n-1} \frac{1}{n} J_i \left(\frac{u\left(\frac{i+1}{n}\right) - u\left(\frac{i}{n}\right)}{\frac{1}{n}} \right) = \sum_{i=0}^{n-1} \frac{1}{n} J_i \left(n \left(u\left(\frac{i+1}{n}\right) - u\left(\frac{i}{n}\right) \right) \right), \quad (1)$$

where $J_i : (0, \infty) \to \mathbb{R}$ is a potential describing the interaction between the ith atom and its neighbour to the right. We are interested in random heterogeneous chains of atoms, and thus assume that the potentials $\{J_i\}_{i\in\mathbb{Z}}$ are random with a distribution that is stationary and ergodic. We appeal to the following standard setup: Let $(\Omega, \mathcal{F}, \mathbb{P})$ denote a probability space and $(\tau_i)_{i\in\mathbb{Z}}$ a family of measurable maps $\tau_i : \Omega \to \Omega$ such that

- (Group property.) $\tau_0 \omega = \omega$ for all $\omega \in \Omega$ and $\tau_{i_1+i_2} = \tau_{i_1} \tau_{i_2}$ for all $i_1, i_2 \in \mathbb{Z}$,

- (Stationarity.) $\mathbb{P}(\tau_i B) = \mathbb{P}(B)$ for every $B \in \mathcal{F}$ $i \in \mathbb{Z}$,

- (Ergodicity.) For all $B \in \mathcal{F}$, it holds $(\tau_i(B) = B \ \forall i \in \mathbb{Z}) \Rightarrow \mathbb{P}(B) = 0$ or $\mathbb{P}(B) = 1$.

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We then consider the energy functional

$$E_n : \Omega \times \mathcal{A}_n \to \mathbb{R} \cup \{+\infty\}, \qquad E_n(\omega, u) := \sum_{i=0}^{n-1} \frac{1}{n} J\left(\tau_i \omega, n\left(u\left(\frac{i+1}{n}\right) - u\left(\frac{i}{n}\right)\right)\right), \qquad (2)$$

where the random potential satisfies the following assumptions:

Assumption 2.1. *Let $J : \Omega \times \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be jointly measurable with $J(z) = \infty$ if $z \le 0$. For $\mathbb{P}$-a.e. $\omega \in \Omega$ the following conditions hold true:*

(A1) (Regularity.) $J(\omega, \cdot) \in C^3(0, \infty)$.

(A2) (Behavior at 0 and ∞.) There exists functions $\psi^+, \psi^- \in C(0, \infty)$, independent of ω, such that

$$\lim_{z \to 0^+} \psi^-(z) = \infty \ \text{and} \ \lim_{z \to \infty} \psi^+(z) = 0,$$

and

$$J(\omega, z) \ge \psi^-(z) \text{ for all } 0 < z \le 1 \quad \text{and} \quad |J(\omega, z)| \le \psi^+(z) \text{ for all } z \ge 1.$$

(A3) (Convex-monotone-structure.) Suppose strict convexity close to 0 in form of

$$z_{\mathrm{frac}}(\omega) := \sup\left\{ z > 0 \ : \ J''(\omega, s) := \partial_s^2 J(\omega, s) > 0 \text{ for all } s \in (0, z) \right\} > 0,$$

and assume that $J(\omega, \cdot)$ is monotonically increasing on $[z_{\mathrm{frac}}(\omega), \infty)$.

By the previous assumptions $J(\omega, \cdot)$ is strictly convex in $(0, z_{\mathrm{frac}}]$ and admits a unique minimizer $\delta(\omega)$ that we call the ground state of $J(\omega, \cdot)$. We assume its non-degeneracy in the following sense:

(A4) (Non-degenerate ground state.) Suppose that there exists a constant $c > 0$, independent of ω, such that $\frac{1}{c} > \delta(\omega) > c$ and

$$\forall z \in \delta(\omega) + (-c, c) \ : \ c \le J''(\omega, z) \le \frac{1}{c} \text{ and } |J'''(\omega, z)| \le \frac{1}{c}.$$

Next, we introduce the following central quantities for a random heterogeneous chain with $n + 1$ particles:

Definition 2.1 (Critical stretch of a chain with $n + 1$ particles)**.** *Consider the situation of Assumption 2.1. Let $n \in \mathbb{N}$ and $\omega \in \Omega$. The critical stretch $\ell_n^*(\omega)$ is defined as the largest number such that*

$$\inf_{\mathcal{A}_n^{\mathrm{el}}(\omega) \cap \mathcal{A}_{n,\ell}} E_n(\omega, \cdot) = \inf_{\mathcal{A}_{n,\ell}} E_n(\omega, \cdot) \qquad \text{for all } 0 \le \ell < \ell_n^*(\omega),$$

where we denote by

$$\mathcal{A}_n^{\mathrm{el}}(\omega) := \left\{ u \in \mathcal{A}_n \ : \ \frac{u^{i+1} - u^i}{\frac{1}{n}} \le z_{\mathrm{frac}}(\tau_i \omega) \text{ for all } i = 0, \ldots, n \right\}$$

the set of purely elastic deformations.

The idea behind the above definition is the following: A deformation $u \in \mathcal{A}_n^{\text{el}}(\omega)$ only sees the strictly convex region of the interaction potentials. Thus, we could replace the potentials $J(\omega, i, z)$ in the definition of the energy function E_n by (globally) convex potentials with superlinear growth without changing the energy for deformations in $\mathcal{A}_n^{\text{el}}(\omega)$. As it is well-known, such energies do not allow for fracture in the continuum limit. The definition of the critical stretch implies that a prescribed macroscopic stretch (or compression) $\ell < \ell_n^*(\omega)$ can be realized by a deformation in $\mathcal{A}_n^{\text{el}}(\omega)$ and thus prohibits the formation of a jump, while, for $\ell > \ell_n^*(\omega)$, deformations with minimial energy are required to explore the non-convex region of at least one of the interaction potentials. We may refer to the bonds $[i, i+1]$ that are evaluated outside the convex region as "weak" bonds. If a jump occurs in the limit, then the minimizing sequence shows a concentration of strain in the weak bonds. We thus expect that $\ell_n^*(\omega)$ almost surely converges in the limit $n \to \infty$ to the continuum fracture threshold that can be defined on the level of the continuum Γ-limit, see below. In our first result we prove that ℓ_n^* indeed converges and we identify its limit, which is the statistical mean of the ground states:

Theorem 2.1. *Let Assumption 2.1 be fulfilled. Then,*

$$\lim_{n \to \infty} \ell_n^*(\omega) = \mathbb{E}[\delta] \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

(The proof of Theorem 2.1 can be found in Section 3.1.)

Next we consider the special case when $\delta(\omega)$ is deterministic, say $\delta(\omega) = 1$ for $\mathbb{P}$-a.s. In that case we establish a first order expansion of $\ell_n^*(\omega)$ around its limit $\mathbb{E}[\delta] = 1$ of the form

$$\ell_n^*(\omega) \approx 1 + \sqrt{\frac{1}{n}} \sqrt{\frac{\beta}{\underline{\alpha}}},$$

where β is related to the maximal energy barrier among the random potentials J, and $1/\underline{\alpha}$ is the statistical mean of the curvatures of the random potentials at the ground state.

Theorem 2.2. *Let Assumption 2.1 be satisfied and assume $\delta(\omega) = 1$ for $\mathbb{P}$-a. e. $\omega \in \Omega$. Consider the rescaled jump threshold $\gamma_n^*(\omega) := \dfrac{\ell_n^*(\omega) - 1}{\sqrt{\frac{1}{n}}}$. Then*

$$\lim_{n \to \infty} \gamma_n^*(\omega) = \lim_{n \to \infty} \frac{\ell_n^*(\omega) - 1}{\sqrt{\frac{1}{n}}} = \sqrt{\frac{\beta}{\underline{\alpha}}} \qquad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega,$$

where

$$\underline{\alpha} := \left(\mathbb{E}\left[\left(\tfrac{1}{2} J''(\omega, 1) \right)^{-1} \right] \right)^{-1} \quad \text{and} \quad \beta := \operatorname*{ess\,inf}_{\omega \in \Omega} (-J(\omega, 1)). \tag{3}$$

(The proof of Theorem 2.2 can be found in Section 3.2.)

We finally relate the above results to the zeroth- and first-order Γ-limits of E_n subject to clamped boundary conditions, i.e.

$$E_n^\ell(\omega, \cdot) : L^1(0, 1) \to \mathbb{R} \cup \{+\infty\}, \quad E_n^\ell(\omega, u) := \begin{cases} E_n(\omega, u) & \text{if } u \in \mathcal{A}_{n, \ell}, \\ +\infty & \text{else.} \end{cases}$$

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The zeroth-order Γ-limit of the discrete energy yields a homogenized energy functional

$$E_{\text{hom}}^{\ell} = \int_0^1 J_{\text{hom}}(u'(x)),$$

recalled from [13], and adjusted to the stronger assumptions of this paper and $K = 1$, where the homogenized energy density map $z \mapsto J_{\text{hom}}(z)$ is convex, lower semicontinuous, monotonically decreasing and satisfies

$$\lim_{z \to 0^+} J_{\text{hom}}(z) = +\infty. \tag{4}$$

Moreover, the minimum values of $E_n^{\ell}(\omega, \cdot)$ and E_{hom}^{ℓ} satisfy

$$\lim_{n \to \infty} \inf_u E_n^{\ell}(\omega, u) = \min_u E_{\text{hom}}^{\ell}(u) = J_{\text{hom}}(\ell),$$

and therefore can be calculated as

$$\min_u E_{\text{hom}}^{\ell}(u) = J_{\text{hom}}(\ell) = \begin{cases} J_{\text{hom}}(\ell) & \text{for } \ell < \mathbb{E}[\delta], \\ J_{\text{hom}}(\mathbb{E}[\delta]) & \text{for } \ell \geq \mathbb{E}[\delta]. \end{cases}$$

Hence, the threshold between the elastic and the jump regimes is $\mathbb{E}[\delta]$, which is identical to the limit of $\ell_n^*(\omega)$, see Theorem 2.1. Secondly, we recall a Γ-limit result from [14] for the rescaled energy functional

$$H_n(\omega, v) := \sum_{i=0}^{n-1} \left(J\left(\tau_i \omega, \frac{v\left(\frac{i+1}{n}\right) - v\left(\frac{i}{n}\right)}{\sqrt{\frac{1}{n}}} + \delta(\tau_i \omega) \right) - J\left(\tau_i \omega, \delta(\tau_i \omega) \right) \right),$$

where the Γ-limit is given by

$$H^{\gamma}(v) = \underline{\alpha} \int_0^1 \left| v'(x) \right|^2 \, \mathrm{d}x + \beta \# S_v,$$

with homogenized elastic coefficient $\underline{\alpha}$, jump parameter β and $\# S_v$ being the number of jumps of v.

Moreover, for $\gamma > 0$ it holds true that

$$\lim_{n \to \infty} \inf_v H_n^{\gamma_n}(\omega, v) = \min_v H^{\gamma}(v) = \min\{\underline{\alpha}\gamma^2, \beta\},$$

which yields that the minima of the energy are given by

$$\min_v H^{\gamma}(v) = \min\{\underline{\alpha}\gamma^2, \beta\} = \begin{cases} \underline{\alpha}\gamma^2 & \text{if } \gamma < \sqrt{\frac{\beta}{\underline{\alpha}}}, \\ \beta & \text{if } \gamma \geq \sqrt{\frac{\beta}{\underline{\alpha}}}. \end{cases}$$

Hence the threshold between elasticity and fracture in the rescaled case is $\sqrt{\frac{\beta}{\underline{\alpha}}}$, which equals the limit of the jump threshold γ_n^* in Theorem 2.2.

In summary, although the techniques by which the results are calculated are completely different, they yield the same result regarding the jump threshold in the continuum setting. The derivation of the limiting jump threshold with help of the newly defined jump threshold in the discrete setting is, however, much easier and thus is of interest for applications. It remains an open problem to analyze corresponding questions in higher dimensional settings. In the following section we provide the proofs of the above theorems.

3 Proofs

For the upcoming analysis it is convenient to introduce the notation

$$M_n(\omega, \ell) := \min\left\{ \frac{1}{n}\sum_{i=1}^{n} J(\tau_i\omega, z^i) \;:\; \frac{1}{n}\sum_{i=1}^{n} z^i = \ell \right\}$$

to denote the minimum energy of a discrete chain of length ℓ. We begin with an elementary (yet, convenient) reformulation of the critical stretch ℓ_n^* (cf. Definition 2.1).

Lemma 3.1. *Consider the situation of Assumption 2.1. Let $n \in \mathbb{N}$ and $\omega \in \Omega$. Then, it holds*

$$M_n(\omega, \ell) = \min_{u \in \mathcal{A}_{n,\ell}} E_n(\omega, u). \tag{5}$$

Moreover, $\ell_n^*(\omega)$ is the largest number such that for all $0 < \ell < \ell_n^*(\omega)$ there exists $\bar{z} \in \mathbb{R}^n$ satisfying

$$M_n(\omega, \ell) = \frac{1}{n}\sum_{i=1}^{n} J(\tau_i\omega, \bar{z}^i), \quad \frac{1}{n}\sum_{i=1}^{n} \bar{z}^i = \ell, \quad \bar{z}^i \leq z_{\mathrm{frac}}(\tau_i\omega) \quad \forall i \in \{1, \dots, n\}. \tag{6}$$

Proof of Lemma 3.1. The identity (5) follows by a simple change of variables, that is by setting $z^i = n(u(\frac{i}{n}) - u(\frac{i-1}{n}))$, and the direct method of the calculus of variations.

Next we give an argument regarding the characterization of ℓ_n^*. The definition of $\ell_n^*(\omega)$, see Definition 2.1, and (5) imply

$$\inf_{\mathcal{A}_n^{\mathrm{el}}(\omega) \cap \mathcal{A}_{n,\ell}} E_n(\omega, \cdot) = M_n(\omega, \ell) < \infty \qquad \forall \ell \in (0, \ell_n^*(\omega)).$$

Since $\mathcal{A}_n^{\mathrm{el}}(\omega) \cap \mathcal{A}_{n,\ell}$ is compact, there exists $\bar{u} \in \mathcal{A}_n^{\mathrm{el}}(\omega) \cap \mathcal{A}_{n,\ell}$ such that

$$E_n(\omega, \bar{u}) = \inf_{\mathcal{A}_n^{\mathrm{el}}(\omega) \cap \mathcal{A}_{n,\ell}} E_n(\omega, \cdot).$$

Clearly, $\bar{z} \in \mathbb{R}^n$ defined as $\bar{z}^i = n(\bar{u}(\frac{i}{n}) - \bar{u}(\frac{i-1}{i}))$ satisfies (6).

Now we suppose that for some $\ell \geq \ell_n^*$ there exists $\bar{z} \in \mathbb{R}^n$ satisfying (6). With help of the same change of variables as above, we find $\bar{u} \in \mathcal{A}_n^{\mathrm{el}}(\omega) \cap \mathcal{A}_{n,\ell}$ satisying $E_n(\omega, \bar{u}) = M_n(\omega, \ell)$ which contradicts the definition of ℓ_n^*. $\qquad\square$

Lemma 3.2. *Let Assumption 2.1 be satisfied. Then, $J(\omega, \cdot)$ is increasing on $[\delta(\omega), \infty)$ and it holds*

$$z_{\mathrm{frac}}^{\sup} := \sup\{z_{\mathrm{frac}}(\omega) \;:\; \omega \in \Omega\} < \infty. \tag{7}$$

Proof of Lemma 3.2. For convenience we drop the dependence on ω in our notation and simply write $J(z)$, δ and z_{frac} instead of $J(\omega, z)$, $\delta(\omega)$ and $z_{\mathrm{frac}}(\omega)$, respectively. We first prove that J is increasing on $[\delta, \infty)$. On $[z_{\mathrm{frac}}, \infty)$ this directly follows from (A3). On $[\delta, z_{\mathrm{frac}})$ this follows from the convexity of J on $(0, z_{\mathrm{frac}})$ and the fact that δ minimizes J. Next, we prove (7). We first note that (A2) and (A3) imply that

$$\forall z \in (\delta, \infty) \;:\; J(\delta) \leq J(z) \leq 0. \tag{8}$$

Moreover, (A4) implies that $z_{\text{frac}} \geq \delta + c$. Thus, for all $\eta \in (0, c)$ we obtain

$$0 \geq J(z_{\text{frac}}) = J(\delta + \eta) + \int_{\delta+\eta}^{z_{\text{frac}}} J'(t)\,dt \geq J(\delta + \eta) + J'(\delta + \eta)(z_{\text{frac}} - (\delta + \eta)), \tag{9}$$

where the second inequality holds, since J' is increasing on $(\delta + \eta, z_{\text{frac}})$ thanks to (A3). (A4) yields

$$J'(\delta + \eta) = J'(\delta + \eta) - J'(\delta) = \int_{\delta}^{\delta+\eta} J''(s)\,ds \geq c\eta.$$

Thus, by rearranging terms in (9) and appealing to (8) and the previous estimate we get

$$z_{\text{frac}} \leq \delta + \eta - \frac{J(\delta + \eta)}{J'(\delta + \eta)} \leq \delta + \eta - \frac{J(\delta)}{c\eta}. \tag{10}$$

It remains to bound $\delta = \delta(\omega)$ and $-J(\delta) = -J(\omega, \delta(\omega))$ by a constant that is independent of ω. From (A4) and (A2) we get

$$\delta \in (c, \frac{1}{c}) \quad \text{and} \quad -J(\delta) \leq \max_{z \in [c, \frac{1}{c}+\eta]} \max\{-\psi^-(z), |\psi^+(z)|\} =: d < \infty, \tag{11}$$

and thus, (10) yields $z_{\text{frac}} \leq \frac{1}{c} + \frac{d}{c\eta}$. $\qquad\square$

3.1 Proof of Theorem 2.1

Proof of Theorem 2.1. Note that $\omega \mapsto \delta(\omega)$ is (as a minimizer of a measurable function) measurable. Moreover, by (11) δ is a non-negative and bounded and thus an L^1-random variable. Thus the ergodic theorem yields

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \delta(\tau_i \omega) = \mathbb{E}[\delta], \quad \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} J(\tau_i \omega, \delta(\tau_i \omega)) = \mathbb{E}[J(\delta)] \tag{12}$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$. For the rest of the proof we consider $\omega \in \Omega$ such that (12) is valid and drop the dependence on ω. In particular, we set $\delta_i := \delta(\tau_i \omega)$, $z_{\text{frac}}^i := z_{\text{frac}}(\tau_i \omega)$ and $J_i(z) := J(\tau_i \omega, z)$.

Step 1. We show that $\overline{A} := \limsup_{n\to\infty} \ell_n^* \leq \mathbb{E}[\delta]$.
W.l.o.g. we suppose $\overline{A} = \lim_{n\to\infty} \ell_n^*$ and prove $\overline{A} \leq \mathbb{E}[\delta]$ by contradiction. Assume that there exists $\epsilon \in (0, c)$ such that $\overline{A} > \mathbb{E}[\delta] + 3\epsilon$. By (12), we find $\overline{N} \in \mathbb{N}$ such that

$$\ell_n^* > \frac{1}{n} \sum_{i=1}^{n} \delta_i + 2\epsilon =: k_n \quad \text{for } n > \overline{N}. \tag{13}$$

In view of Lemma 3.1 there exists a sequence $(\bar{z}_n)_n$ satisfying for $n \geq \overline{N}$

$$\frac{1}{n} \sum_{i=1}^{n} \bar{z}_n^i = k_n, \quad \frac{1}{n} \sum_{i=1}^{n} J_i(\bar{z}_n^i) = M_n(k_n), \quad \bar{z}_n^i \leq z_{\text{frac}}^i \quad \forall i \in \{1, \ldots, n\}. \tag{14}$$

We claim that

$$\limsup_{n\to\infty} M_n(k_n) \leq \mathbb{E}[J(\delta)], \tag{15}$$

$$\liminf_{n\to\infty} M_n(k_n) \geq \mathbb{E}[J(\delta)] + c_\epsilon, \tag{16}$$

for some $c_\epsilon > 0$. Clearly, (15) and (16) contradict (14)

Substep 1.1. Proof of (15). Let $z_n \in \mathbb{R}^n$ be given by $z_n^i := \delta_i$ for $i \geq 2$ and $z_n^1 := \delta_1 + 2n\epsilon$. Since $\frac{1}{n}\sum_{i=1}^n z_n^i = k_n$, we have

$$M_n(k_n) \leq \frac{1}{n}\sum_{i=2}^n J_i(\delta_i) + \frac{1}{n}J_1(\delta_1 + 2n\epsilon) = \frac{1}{n}\sum_{i=1}^n J_i(\delta_i) + \frac{1}{n}(J_1(\delta_1 + 2n\epsilon) - J_1(\delta_1)).$$

Hence, (15) follows by (A2) and (12).

Substep 1.2. Proof of (16). Let $\bar{z}_n$ be as in (14) and set

$$I_n := \left\{ i \in \{1,...,n\} : \bar{z}_n^i > \delta_i + \epsilon \right\}.$$

Obviously, it holds $0 \leq |I_n|/n \leq 1$ and we claim

$$\frac{|I_n|}{n} \geq \frac{\epsilon}{z_{\text{frac}}^{\sup}} > 0 \qquad \text{for all } n \in \mathbb{N}, \tag{17}$$

where $z_{\text{frac}}^{\sup} \in (0, \infty)$ is as in Lemma 3.2. Indeed,

$$\frac{1}{n}\sum_{i=1}^n \delta_i + 2\epsilon = k_n = \frac{1}{n}\sum_{i=1}^n \bar{z}_n^i = \frac{1}{n}\sum_{i \in I_n} \bar{z}_n^i + \frac{1}{n}\sum_{i \notin I_n} \bar{z}_n^i \overset{(14)}{\leq} \frac{|I_n|}{n}z_{\text{frac}}^{\sup} + \frac{1}{n}\sum_{i=1}^n (\delta_i + \epsilon)$$

implies (17). Finally, using the monotonicity of J_i on (δ_i, ∞) (see Lemma 3.2) and (A4), we obtain

$$\frac{1}{n}\sum_{i=1}^n J_i(\bar{z}_n^i) = \frac{1}{n}\sum_{i \in I_n} J_i(\bar{z}_n^i) + \frac{1}{n}\sum_{i \notin I_n} J_i(\bar{z}_n^i) \geq \frac{1}{n}\sum_{i \in I_n} J_i(\delta_i + \epsilon) + \frac{1}{n}\sum_{i \notin I_n} J_i(\delta_i)$$

$$\geq \frac{1}{n}\sum_{i \in I_n} \left(J_i(\delta_i) + \tfrac{1}{2}c\epsilon^2 \right) + \frac{1}{n}\sum_{i \notin I_n} J_i(\delta_i) = \frac{1}{n}\sum_{i=1}^n J_i(\delta_i) + \frac{|I_n|}{n}\frac{1}{2}c\epsilon^2,$$

where $c > 0$ is as in (A4). Sending $n \to \infty$, we obtain with help of (12) and (17) the claim (16).

Step 2. We claim $\underline{A} := \liminf_{n \to \infty} \ell_n^* \geq \mathbb{E}[\delta]$.

For all $\epsilon > 0$, we show

$$\ell_n^* \geq \frac{1}{n}\sum_{i=1}^n \delta_i - \epsilon =: k_n \qquad \forall n \in \mathbb{N}, \tag{18}$$

which in combination with (12) implies $\underline{A} := \liminf_{n \to \infty} \ell_n^* \geq \mathbb{E}[\delta]$ by the arbitrariness of $\epsilon > 0$. Let $\bar{z}_n$ be such that

$$\frac{1}{n}\sum_{i=1}^n \bar{z}_n^i = k_n, \quad \frac{1}{n}\sum_{i=1}^n J_i(\bar{z}_n^i) = M_n(k_n).$$

We show $\bar{z}_n^i \leq \delta_i < z_{\text{frac}}^i \; \forall i \in \{1, \ldots, n\}$, which obviously implies (18). Indeed, the optimality condition for $\bar{z}_n$ implies that there exists a Lagrange-Multiplier $\Lambda \in \mathbb{R}$ such that $\Lambda = J_i'(\bar{z}_n^i)$ for all $i \in \{1, \ldots, n\}$. Since $\frac{1}{n}\sum_{i=1}^n (\bar{z}_n^i - \delta_i) \leq -\epsilon$ there exists $\hat{i} \in \{1, \ldots, n\}$ such that $\bar{z}_n^{\hat{i}} \in (0, \delta_i)$ and thus $J_{\hat{i}}'(\bar{z}_n^{\hat{i}}) < 0$. Hence $J_i'(\bar{z}_n^i) < 0$ for all $i \in \{1, \ldots, n\}$. Since $J_i' \geq 0$ on (δ_i, ∞) by Lemma 3.2, we conclude that $\bar{z}_n^i \leq \delta_i \leq z_{\text{frac}}^i$ and thus $\ell_n^* \geq k_n$ by Lemma 3.1. $\qquad\square$

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3.2 Proof of Theorem 2.2

We begin with a preliminary structure result for minimizers of the minimum problem in the definition of $M_n(\omega, 1 + n^{-\frac{1}{2}}D)$ for some $D > 0$ see (5).

Proposition 3.3. *Let Assumption 2.1 be satisfied and assume $\delta(\omega) = 1$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. Fix $D > 0$. There exist $\bar{N} \in \mathbb{N}$ and a sequence (N_n) satisfying $N_n \to \infty$ such that the following statements hold true for $\mathbb{P}$-a.e. $\omega \in \Omega$ and $n \geq \bar{N}$:*
Let $\bar{z}_n \in \mathbb{R}^n$ be such that

$$\frac{1}{n} \sum_{i=1}^{n} \bar{z}_n^i = 1 + n^{-\frac{1}{2}}D \quad and \quad \frac{1}{n} \sum_{i=1}^{n} J(\tau_i \omega, \bar{z}_n^i) = M_n(\omega, 1 + n^{-\frac{1}{2}}D). \tag{19}$$

Then, it holds

$$\bar{z}_n^i \in [1, 1 + c^{-2} n^{-\frac{1}{2}}D] \cup [N_n, \infty) \quad for\ all\ i \in \{1, ..., n\}, \tag{20}$$

where $c > 0$ is as in (A4).

Proof of Proposition 3.3. We consider $\omega \in \Omega$ such that $\delta(\tau_i \omega) = 1 \ \forall i \in \mathbb{N}$ and drop the dependence on ω. Moreover, we use the shorthand notation $z_{\text{frac}}^i := z_{\text{frac}}(\tau_i \omega)$ and $J_i(z) := J(\tau_i \omega, z)$.

Step 1. We show that

$$0 \leq J'(\bar{z}_n^i) \leq \frac{1}{c} D n^{-\frac{1}{2}} \quad \text{for all } i \in \{1, ..., n\} \tag{21}$$

where $c > 0$ is as in (A4).

By the optimality condition for $\bar{z}_n$ there exists a Lagrange-Multiplier $\Lambda \in \mathbb{R}$ such that $\Lambda = J_i'(\bar{z}_n^i)$ for all $i \in \{1, \ldots, n\}$. Since $\frac{1}{n} \sum_{i=1}^{n} \bar{z}_n^i = 1 + n^{-\frac{1}{2}}D$, there exists $i_1 \in \{1, \ldots, n\}$ such that $\bar{z}_n^{i_1} \geq 1 + n^{-\frac{1}{2}}D > 1$. Lemma 3.2 and the assumption $\delta(\tau_i \omega) = 1$ imply that J_i is increasing on $(1, \infty)$ and thus we have $\Lambda \geq 0$. Moreover, there exists $i_2 \in \{1, \ldots, n\}$ such that $\bar{z}_n^{i_2} \leq 1 + n^{-\frac{1}{2}}D$. For n sufficiently large such that $n^{-\frac{1}{2}}D < c$, where $c > 0$ as in (A4), we have (using that $J_i'(1) = 0$)

$$0 \leq \Lambda = J_{i_2}'(\bar{z}_n^{i_2}) = \int_1^{\bar{z}_n^{i_2}} J_{i_2}''(t)\, dt \overset{(A4)}{\leq} \frac{1}{c} n^{-\frac{1}{2}}D.$$

Since $\Lambda = J_i'(\bar{z}_n^i)$ for all $i \in \{1, \ldots, n\}$ the claim (21) follows.

Step 2. Argument for (20).
We firstly observe that (21) implies $1 \leq \bar{z}_n^i$ for all $i \in \{1, \ldots, n\}$ (recall $J_i'(z) < 0$ on $(0,1)$). The remaining estimates of (20) are proven in three steps.

Substep 2.1. We claim that for n sufficiently large, $\bar{z}_n^i \leq z_{\text{frac}}^i$ implies $\bar{z}_n^i \leq 1 + c^{-2} n^{-\frac{1}{2}}D$, where $c > 0$ is as in (A4). Indeed, using $J_i''(s) > 0$ on $(0, z_{\text{frac}}^i)$ and (A4), we deduce from $\bar{z}_n^i \leq z_{\text{frac}}^i$ and n sufficiently large that

$$c^{-1} D n^{-\frac{1}{2}} \overset{(21)}{\geq} J_i'\left(\bar{z}_n^i\right) = \int_1^{\bar{z}_n^i} J_i''(t)\, dt \overset{(A4)}{\geq} c \min\{\bar{z}_n^i - 1, c\}.$$

From the above inequality we deduce that $\bar{z}_n^i - 1 \geq c$ implies $n \leq D^2/c^6$. Hence, $\bar{z}_n^i - 1 < c$ and thus $1 \leq \bar{z}_n^i \leq 1 + c^{-2} D n^{-\frac{1}{2}}$ for $n > D^2/c^6$.

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Substep 2.2. There exists $M < \infty$, depending only on $\psi^-(1)$ from (A2) and $c > 0$ from (A4), such that

$$\sup_{n \in \mathbb{N}} |I_n^w| \le M \qquad \text{where} \qquad I_n^w := \left\{ i \in \{1, ..., n\} : \bar{z}_n^i \ge z_{\text{frac}}^i \right\}. \tag{22}$$

Suppose $|I_n^\omega| \ge 2$ and consider some $i_n \in I_n^\omega$. Define

$$\hat{z}_n^i := \begin{cases} \bar{z}_n^i & \text{if } i \notin I_n^\omega, \\ 1 & \text{if } i \in I_n^\omega \setminus \{i_n\}, \\ 1 + \sum_{i \in I_n^\omega} (\bar{z}_n^i - 1) & \text{if } i = i_n. \end{cases} \tag{23}$$

By construction, we have $\sum_{i=1}^n \bar{z}_n^i = \sum_{i=1}^n \hat{z}_n^i$ and thus by (19)

$$0 \ge \sum_{i=1}^n \left(J_i(\bar{z}_n^i) - J_i(\hat{z}_n^i) \right) = \sum_{i \in I_n^\omega \setminus \{i_n\}} (J_i(\bar{z}_n^i) - J_i(1)) + J_{i_n}(\bar{z}_n^{i_n}) - J_{i_n}(\hat{z}_n^{i_n}). \tag{24}$$

By the monotonicity of J_i on $(1, \infty)$, (A3) and (A4), we find $\eta = \eta(c) > 0$, where $c > 0$ is as in (A4), such that

$$J_i(\bar{z}_n^i) - J_i(1) \ge J_i(z_{\text{frac}}^i) - J_i(1) \ge \eta \qquad \forall i \in I_n^w. \tag{25}$$

Moreover, using $\hat{z}_n^{i_n} \ge 1$ and thus $J_{i_n}(\hat{z}_n^{i_n}) \le 0$ (wich follows from the monotonicity of J_i on $(1, \infty)$ and (A2)) we obtain

$$J_{i_n}(\bar{z}_n^i) - J_{i_n}(\hat{z}_n^{i_n}) \ge J_{i_n}(1) \overset{(A2)}{\ge} \psi^-(1). \tag{26}$$

Combining (24)–(26), we deduce the uniform bound $|I_n^\omega| \le 1 - \eta^{-1} \psi^-(1)$.

Substep 2.3. We show that there exists (N_n) satisfying $N_n \to \infty$ as $n \to \infty$ such that $\bar{z}_n^i \ge N_n$ for all $i \in I_n^w$, where I_n^w is defined in (22).
We argue by contradiction and assume that there exists $A \in [1, \infty)$ and an index $\hat{i} \in I_n^w$ such that $\bar{z}_n^{\hat{i}} \le A$. For n sufficiently large, we show that this contradicts (19). Define

$$\tilde{z}_n^i := \begin{cases} 1 & \text{if } i = \hat{i}, \\ \bar{z}_n^i + (n - |I_n^w|)^{-1} (\bar{z}_n^{\hat{i}} - 1) & \text{if } i \notin I_n^\omega, \\ \bar{z}_n^i & \text{if } i \in I_n^w \setminus \{\hat{i}\}. \end{cases} \tag{27}$$

By construction, we have $\sum_{i=1}^n \tilde{z}_n^i = \sum_{i=1}^n \bar{z}_n^i$. Since $\bar{z}_n$ is a minimizer (see(19)),

$$0 \ge \sum_{i=1}^n (J_i(\bar{z}_n^i) - J_i(\tilde{z}_n^i)) = J_{\hat{i}}(\bar{z}_n^{\hat{i}}) - J_{\hat{i}}(1) + \sum_{i \notin I_n^\omega} (J_i(\bar{z}_n^i) - J_i(\tilde{z}_n^i)).$$

By (25) we have $J_{\hat{i}}(\bar{z}_n^{\hat{i}}) - J_{\hat{i}}(1) \ge \eta(c) > 0$. To obtain a contradiction, it suffices to show that the second term on the right-hand side vanishes as n tends to infinity. This can be seen as follows: On the one hand, we have $\bar{z}_n^i \in [1, 1 + c^{-2} n^{-\frac{1}{2}} D]$ for all $i \notin I_n^w$ by Substep 2.1, and on the other hand, we have $(n - |I_n^w|)^{-1} (\bar{z}_n^{\hat{i}} - 1) \le (n - M)^{-1}(A - 1)$ thanks to $|I_n^w| \le M$. Hence, $\bar{z}_n^i, \tilde{z}_n^i \in [1, 1 + \frac{c}{2}]$

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for n sufficiently large (depending only on c, D, M and A). Now, a quadratic Taylor expansion of J_i at $\bar{z}_n^i$ yields (using $|J''(z)| \le c^{-1}$ for $z \in [1, 1+c)$, see (A4))

$$\sum_{\substack{i=1 \\ i \notin I_n^\omega}}^n |J_i(\bar{z}_n^i) - J_i(\tilde{z}_n^i)| \le \sum_{i=1}^n \left(|J_i'(\bar{z}_n^i)|(n-M)^{-1}(A-1) + c^{-1}(n-M)^{-2}(A-1)^2 \right)$$

$$\overset{(21)}{\le} n(n-M)^{-1}c^{-1}(A-1)\left(n^{-\frac{1}{2}}D + (A-1)(n-M)^{-1} \right) \le Cn^{-\frac{1}{2}},$$

where $C < \infty$ depends only on A, c, D and M. $\qquad\square$

Proof of Theorem 2.2. By the ergodic theorem, it holds

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n J''(\tau_i\omega, 1)^{-1} = \mathbb{E}[J''(1)^{-1}], \qquad \lim_{n\to\infty} \beta_n(\omega) = \beta \qquad (28)$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$, where β is defined in (3) and

$$\beta_n(\omega) := \min\{-J(\tau_i\omega, 1) : i \in \{1, \dots, n\}\}. \qquad (29)$$

In Step 3 below we provide an argument for the limit $\beta_n \to \beta$.

In Step 1 and Step 2, we consider $\omega \in \Omega$ such that (28) and the conclusion of Proposition 3.3 are valid. Moreover, we drop the dependence on ω and use the shorthand notation $z_{\text{frac}}^i := z_{\text{frac}}(\tau_i\omega)$ and $J_i(z) := J(\tau_i\omega, z)$.

Step 1. We prove $\overline{A} := \limsup_{n\to\infty} \gamma_n^* \le \sqrt{\dfrac{\beta}{\underline{\alpha}}}$ by contradiction: Assume that there exists $\epsilon > 0$ and $\overline{N} \in \mathbb{N}$ such that

$$\ell_n^* > 1 + n^{-\frac{1}{2}}\sqrt{\frac{\beta}{\underline{\alpha}}}(1+\epsilon) =: k_n \quad \text{for } n > \overline{N}. \qquad (30)$$

In view of Lemma 3.1 there exists $(\bar{z}_n)_n$ satisfying

$$\frac{1}{n}\sum_{i=1}^n \bar{z}_n^i = k_n, \quad \frac{1}{n}\sum_{i=1}^n J_i(\bar{z}_n^i) = M_n(k_n) \quad \bar{z}_n^i \le z_{\text{frac}}^i \quad \forall i \in \{1, \dots, n\}. \qquad (31)$$

We show

$$\limsup_{n\to\infty} n\left(M_n(k_n) - \frac{1}{n}\sum_{i=1}^n J_i(1) \right) \le \beta, \qquad (32)$$

$$\liminf_{n\to\infty} n\left(\frac{1}{n}\sum_{i=1}^n J_i(\bar{z}_n^i) - \frac{1}{n}\sum_{i=1}^n J_i(1) \right) \ge \beta(1+\epsilon)^2. \qquad (33)$$

Clearly, (32) and (33) contradict (31) for n sufficiently large.

Substep 1.1. Argument for (33).

We claim that there exists $K < \infty$ such that for all n sufficiently large

$$n\left(\frac{1}{n}\sum_{i=1}^{n} J_i(\bar{z}_n^i) - \frac{1}{n}\sum_{i=1}^{n} J_i(1)\right) \geq \left(\frac{1}{n}\sum_{i=1}^{n}\left(\tfrac{1}{2}J_i''(1)\right)^{-1}\right)^{-1}\frac{\beta}{\underline{\alpha}}(1+\epsilon)^2 - \frac{K}{\sqrt{n}}, \tag{34}$$

where $\bar{\alpha}$ and β are defined in (3). Note that (28) and (34) imply (33).

We prove (34). By (30), (31), and Proposition 3.3 (applied with $D = \sqrt{\frac{\beta}{\underline{\alpha}}(1+\epsilon)^2}$), we get

$$1 \leq z_n^i \leq 1 + n^{-\frac{1}{2}}C \tag{35}$$

for some $C < \infty$ independent of n. Hence, a Taylor expansion yields

$$\sum_{i=1}^{n} J_i(\bar{z}_n^i) = \sum_{i=1}^{n} J_i(1) + \frac{1}{2}\sum_{i=1}^{n} J_i''(1)\left(\bar{z}_n^i - 1\right)^2 + \frac{1}{6}\sum_{i=1}^{n} J_i'''(\xi_n^i)\left(\bar{z}_n^i - 1\right)^3, \tag{36}$$

where $\xi_n^i \in [1, \bar{z}_n^i]$. To estimate the second term on the right-hand side, note that Cauchy-Schwarz' inequality yields

$$\left(\sum_{i=1}^{n}(\bar{z}_n^i - 1)\right)^2 \leq \left(\frac{1}{2}\sum_{i=1}^{n} J_i''(1)(\bar{z}_n^i - 1)^2\right)\left(\sum_{i=1}^{n}(\tfrac{1}{2}J_i''(1))^{-1}\right).$$

Combined with the identity $\sum_{i=1}^{n}(\bar{z}_n^i - 1) = n(k_n - 1) = \sqrt{n}\sqrt{\frac{\beta}{\underline{\alpha}}}(1+\epsilon)$ we get

$$\left(\frac{1}{n}\sum_{i=1}^{n}\left(\tfrac{1}{2}J_i''(1)\right)^{-1}\right)^{-1}\frac{\beta}{\underline{\alpha}}(1+\epsilon)^2 \leq \frac{1}{2}\sum_{i=1}^{n} J_i''(1)\left(\bar{z}_n^i - 1\right)^2. \tag{37}$$

Moreover, (35) and (A4) imply for n sufficiently large

$$\frac{1}{6}\sum_{i=1}^{n} J_i'''(\xi_n^i)\left(\bar{z}_n^i - 1\right)^3 \geq -\frac{C^3}{6c\sqrt{n}}. \tag{38}$$

Clearly, (36)–(38) imply (34) (with $K = \frac{C^3}{6c}$).

Substep 1.2. Argument for (32).
For every $n \in \mathbb{N}$, we choose $\hat{i}_n \in \{1,\ldots,n\}$ such that $-J_{\hat{i}_n}(1) = \beta_n$ (see (29)) and define $z_n \in \mathbb{R}^n$ as

$$z_n^i = \begin{cases} 1 & \text{if } i \in \{1,\ldots,n\} \setminus \{\hat{i}_n\} \\ 1 + n(k_n - 1) & \text{if } i = \hat{i}_n \end{cases}.$$

Since $\frac{1}{n}\sum_{i=1}^{n} z_n^i = k_n = 1 + n^{-\frac{1}{2}}\sqrt{\frac{\beta}{\underline{\alpha}}}(1+\epsilon)$, we have

$$n\left(M_n(k_n) - \frac{1}{n}\sum_{i=1}^{n} J_i(1)\right) \leq J_{\hat{i}_n}(1 + n(k_n - 1)) - J_{\hat{i}_n}(1)$$

$$\leq \psi^+\left(1 + \sqrt{n}\sqrt{\frac{\beta}{\underline{\alpha}}}(1+\epsilon)\right) + \beta_n,$$

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where the second inequality holds by (A2) and the choice of $\hat{i}_n$. Now, (32) follows from (28) and assumption (A2).

Step 2. Proof of $\underline{A} := \liminf_{n\to\infty} \gamma_n^* \geq \sqrt{\frac{\beta}{\alpha}}$.

We show that for every $\epsilon > 0$ there exists $\bar{N} \in \mathbb{N}$ such that

$$\ell_n^* \geq 1 + n^{-\frac{1}{2}}\sqrt{\frac{\beta}{\underline{\alpha}}}(1 - \epsilon) =: k_n \quad \text{for } n > \bar{N}. \tag{39}$$

Note that (39) implies $\liminf_{n\to\infty} \gamma_n^* \geq \sqrt{\frac{\beta}{\underline{\alpha}}}(1-\epsilon)$ for all $\epsilon > 0$, and thus the claim.

Let $(\bar{z}_n)_n$ be a sequence satisfying for all $n \in \mathbb{N}$,

$$\frac{1}{n}\sum_{i=1}^n \bar{z}_n^i = k_n, \qquad M_n(k_n) = \frac{1}{n}\sum_{i=1}^n J_i(\bar{z}_n^i). \tag{40}$$

To prove (39), we only need to show that

$$z_n^i \leq z_{\text{frac}}^i \text{ for all } i \in \{1, \ldots, n\} \text{ for } n \text{ sufficiently large}, \tag{41}$$

depending only on $\underline{\alpha}$ β, c, and $\epsilon > 0$.

Substep 2.1. We show that

$$\limsup_{n\to\infty} n\left(M_n(k_n) - \frac{1}{n}\sum_{i=1}^n J_i(1)\right) \leq \beta(1 - \epsilon). \tag{42}$$

Set

$$\hat{z}_n^i := 1 + n^{-\frac{1}{2}}\sqrt{\frac{\beta}{\underline{\alpha}}}(1 - \epsilon)\left(\frac{1}{n}\sum_{i=1}^n \frac{1}{\alpha_i}\right)^{-1}\frac{1}{\alpha_i},$$

where $\alpha_i := \frac{1}{2}J_i''(1)$. By construction, we have

$$\frac{1}{n}\sum_{i=1}^n \hat{z}_n^i = k_n, \qquad 0 \leq \hat{z}_n^i - 1 \leq n^{-\frac{1}{2}}C \tag{43}$$

where $C < \infty$ depends only on $\underline{\alpha}, \beta$ and $c > 0$ from (A4) (note that (A4) implies $\alpha_i \leq \frac{1}{2c}$ and $\frac{1}{\alpha_i} \leq \frac{2}{c}$). Hence, a Taylor expansion of J_i at 1 and (A4) yield for n sufficiently large

$$\sum_{i=1}^n \left(J_i(\hat{z}_n^i) - J_i(1)\right) \leq \sum_{i=1}^n \alpha_i\left(\hat{z}_n^i - 1\right)^2 + \frac{1}{6c}\sum_{i=1}^n \left(\hat{z}_n^i - 1\right)^3$$

$$\leq \frac{\beta}{\underline{\alpha}}(1 - \epsilon)^2\left(\frac{1}{n}\sum_{i=1}^n \frac{1}{\alpha_i}\right)^{-1} + \frac{C^3}{6c}n^{-\frac{1}{2}},$$

where $C < \infty$ is the same as in (43). Finally, (28) implies $(\frac{1}{n}\sum_{i=1}^n \frac{1}{\alpha_i})^{-1} \leq \underline{\alpha}(1+\epsilon)$ for n sufficiently large and thus (42) follows.

Substep 2.2. We now prove (41) by contraposition. Suppose $\bar{z}_n^{\hat{i}} > z_{\text{frac}}^{\hat{i}}$ for some $\hat{i} \in \{1, \ldots, n\}$. Then Proposition 3.3 yields $\bar{z}_n^{\hat{i}} \geq N_n$ for some (N_n) with $N_n \to \infty$, and thus $J_{\hat{i}}(\bar{z}_n^{\hat{i}}) \geq -\sup_{s \geq N_n} \psi^+(s)$ by (A2). Hence, with $J_i(\bar{z}_n^i) \geq J_i(1)$ and $-J_{\hat{i}}(1) \geq \beta$, we therefore get

$$\sum_{i=1}^n (J_i(\bar{z}_n^i) - J_i(1)) \geq J_{\hat{i}}(\bar{z}_n^{\hat{i}}) - J_{\hat{i}}(1) \geq \beta - \sup_{s \geq N_n} \psi^+(s).$$

Since $\sup_{s \geq N_n} \psi^+(s) \to 0$ for $n \to \infty$, the above lower bound combined with the upper bound (42) and (40) yields a contradiction for n sufficiently large, and thus (41) follows.

Step 3. Argument for $\beta_n \to \beta$ almost surely in (28).
The sequence $(\beta_n(\omega))_n \subset \mathbb{R}$ is decreasing and it holds $\beta_n(\omega) \geq \beta$ for all $n \in \mathbb{N}$. Hence, there exists $\hat{\beta}(\omega) \geq \beta$ such that
$$\lim_{n \to \infty} \beta_n(\omega) = \hat{\beta}(\omega) \geq \beta.$$

It remains to show that $\hat{\beta}(\omega) = \beta$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. We argue by contradiction and therefore suppose that there exist $\epsilon > 0$ and a set $\Omega' \subset \Omega$ with positive measure such that $\hat{\beta}(\omega) \geq \beta + \epsilon$ for all $\omega \in \Omega'$. Then we obtain for all $\omega \in \Omega'$ that

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{i=1}^n \chi_{\left\{-J(\tau_i\omega, 1) \leq \beta + \frac{1}{2}\epsilon\right\}}(\tau_i\omega) = 0,$$

where χ_A denotes the indicator function. Clearly this contradicts the ergodic theorem and the definition of β in the form

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^n \chi_{\left\{-J(\tau_i\omega, 1) \leq \beta + \frac{1}{2}\epsilon\right\}} = \mathbb{E}\left[\chi_{\left\{-J(1) \leq \beta + \frac{1}{2}\epsilon\right\}}\right] > 0 \qquad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

Hence the theorem is proven. $\qquad\square$

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