THE STATIONARY STOKES PROBLEM IN EXTERIOR DOMAINS: ESTIMATES OF THE DISTANCE TO SOLENOIDAL FIELDS AND FUNCTIONAL A POSTERIORI ERROR ESTIMATES

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ABSTRACT. This paper is concerned with the analysis of the inf-sup condition arising in the stationary Stokes problem in exterior domains. We deduce values of the constant in the stability lemma, which yields fully computable estimates of the distance to the set of divergence free fields defined in exterior domains. Using these estimates we obtain computable majorants of the difference between the exact solution of the Stokes problem in exterior domains and any approximation from the admissible (energy) class of functions satisfying the Dirichlet boundary condition exactly.

Dedicated to the 110th anniversary of Solomon Grigor'evich Mikhlin (April 23, 1908 – August 29, 1990)

Contents

1. Introduction	1
2. Preliminaries	5
3. The Stability Lemma for Exterior Domains	7
4. Applications for Exterior Domains	9
4.1. Inf-Sup Lemma and Estimates of the Distance to Solenoidal Fields	9
4.2. Solution Theory for the Stationary Stokes System	9
4.3. A Posteriori Error Estimates for Stationary Stokes Equations	10
4.3.1. A Posteriori Estimates for the Velocity Field: Solenoidal Approximations	11
4.3.2. A Posteriori Estimates for the Velocity Field: Non-Solenoidal Approximations	11
4.3.3. A Posteriori Estimates for the Pressure Function	12
4.3.4. A Posteriori Estimates for Non-Conforming Approximations	13
4.3.5. A Posteriori Estimates for the Stress Tensor Field	14
4.3.6. Lower Bounds for the Velocity Field	14
4.4. Applications for 2D Exterior Domains	15
4.4.1. A Posteriori Error Estimates for Stationary Stokes Equations in 2D	16
References	18

1. INTRODUCTION

Let $\omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded domain with Lipschitz boundary γ , which is composed of two open and disjoint parts $\gamma_D, \gamma_N \subset \gamma$ (Dirichlet and Neumann part) with $\overline{\gamma} = \overline{\gamma}_D \cup \overline{\gamma}_N$. Let the usual Lebesgue and Sobolev spaces (scalar, vector, or tensor valued) be introduced by $\mathsf{L}^2(\omega)$ and $\mathsf{H}^1(\omega)$, respectively. The standard inner product, norm, resp. orthogonality in $\mathsf{L}^2(\omega)$ will be denoted by $\langle \cdot, \cdot \rangle_{0,\omega}, \| \cdot \|_{0,\omega}$, resp. $\perp_{0,\omega}$. For $\gamma_D \neq \emptyset$ let $\mathsf{H}^1_{\gamma_D}(\omega)$ denote the subspace of $\mathsf{H}^1(\omega)$ with vanishing full traces on γ_D . Moreover, we define spaces with

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vanishing mean value byⁱ

$$\begin{split} \mathsf{L}^{2}_{\perp}(\omega) &:= \mathsf{L}^{2}(\omega) \cap \mathbb{R}^{\perp_{0,\omega}} = \big\{ \varphi \in \mathsf{L}^{2}(\omega) \, : \, \int_{\omega} \varphi = 0 \big\}, \\ \mathsf{H}^{1}_{\perp}(\omega) &:= \mathsf{H}^{1}(\omega) \cap \mathsf{L}^{2}_{\perp}(\omega) = \big\{ \varphi \in \mathsf{H}^{1}(\omega) \, : \, \int_{\omega} \varphi = 0 \big\} \end{split}$$

and introduce

$$\mathsf{L}^{2}_{\gamma_{D}}(\omega) := \begin{cases} \mathsf{L}^{2}(\omega) &, \text{ if } \gamma_{D} \neq \gamma, \\ \mathsf{L}^{2}_{\perp}(\omega) &, \text{ if } \gamma_{D} = \gamma, \end{cases} \qquad \qquad \mathsf{H}^{1}_{\gamma_{D}}(\omega) := \begin{cases} \mathsf{H}^{1}_{\gamma_{D}}(\omega) &, \text{ if } \gamma_{D} \neq \emptyset, \\ \mathsf{H}^{1}_{\perp}(\omega) &, \text{ if } \gamma_{D} = \emptyset. \end{cases}$$

Furthermore, let us define solenoidal (divergence free) subspaces of $H^1(\Omega)$ by

$$\mathsf{S}(\omega) := \big\{ \phi \in \mathsf{H}^1(\omega) \, : \, \operatorname{div} \phi = 0 \big\}, \qquad \mathsf{S}_{\gamma_D}(\omega) := \mathsf{H}^1_{\gamma_D}(\omega) \cap \mathsf{S}(\omega).$$

For further notation we refer to Section 2. From results of Babuska and Aziz, Ladyzhenskaya and Solonnikov, Brezzi, Necas [1, 5, 13, 14, 18], for mixed boundary conditions see, e.g., the recent results in [2, 3], we have the following very important lemma in the theory of fluid dynamics and other fields of partial differential equations:

Lemma 1.1 (stability lemma). There exists c > 0 such that for any $g \in L^2_{\gamma_D}(\omega)$ there is a vector field $u \in H^1_{\gamma_D}(\omega)$ with div u = g and $\|\nabla u\|_{0,\omega} \leq c \|g\|_{0,\omega}$. The best constant c will be denoted by $\kappa(\omega, \gamma_D)$.

Remark 1.2. Let us note the following:

- (i) In the theory of electrodynamics u is called a regular potential as it admits for Maxwell's equations an unphysically high regularity and a very unphysical boundary condition, much stronger than the usual normal boundary condition related to the divergence operator.
- (ii) For $u \in \mathsf{H}^{1}_{\gamma_{D}}(\omega)$ we have the Friedrichs/Poincaré inequality $||u||_{0,\omega} \leq c||\nabla u||_{0,\omega}$. The best constant c is the Friedrichs/Poincaré constant and will be denoted by $c_{fp}(\omega, \gamma_{D})$. Hence we conclude for u from Lemma 1.1

$$\frac{1}{c_{fp}(\omega,\gamma_D)} \|u\|_{0,\omega} \le \|\nabla u\|_{0,\omega} \le \kappa(\omega,\gamma_D)\|\operatorname{div} u\|_{0,\omega}.$$

(iii) Note that $\kappa(\omega, \gamma_D)$ is the norm of the right inverse $g \mapsto u$.

Lemma 1.1 is a keystone fact in the theory of incompressible fluids. It generates several important corollaries. First of all, Lemma 1.1 guarantees the solvability of the stationary Stokes problem (in the velocity-pressure posing). Indeed by solving $g = \operatorname{div} u$ Lemma 1.1 yields immediately the following famous inf-sup of LBB condition:

Corollary 1.3 (inf-sup lemma). It holds

$$\inf_{g \in \mathsf{L}^2_{\gamma_D}(\omega)} \sup_{u \in \mathsf{H}^1_{\gamma_D}(\omega)} \frac{\langle g, \operatorname{div} u \rangle_{0,\omega}}{\|g\|_{0,\omega} \|\nabla u\|_{0,\omega}} \ge \frac{1}{\kappa(\omega, \gamma_D)}.$$

A solution theory for the Stokes problem follows. The stationary Stokes problem reads as follows: For given $\nu > 0$, $G \in L^2(\omega)$, $u_D \in S(\omega)$, σ_N find a velocity field u and a pressure function p solving the first order system

$$\int_{\omega} f = \int_{\omega} f d\lambda = \int_{\omega} f dx, \qquad \int_{\gamma} f = \int_{\gamma} f do = \int_{\gamma} f ds$$

 $^{^{\}rm i}{\rm Throughout}$ the paper we do not express the respective measure in the notation of integrals, so that, e.g., with often used notations

$-\operatorname{Div}\sigma = G$	in ω ,
$\sigma = \nu \nabla u - p \mathbb{I}$	in ω ,
$-\operatorname{div} u = 0$	in ω ,
$u = u_D$	on γ_D ,
$\sigma n = \sigma_N$	on γ_N .

Equivalently, by removing the additional stress tensor σ , we have the second order formulation

$$\begin{aligned} -\nu\Delta u + \nabla p &= G & \text{in } \omega, \\ -\operatorname{div} u &= 0 & \text{in } \omega, \\ u &= u_D & \text{on } \gamma_D, \\ (\nu \,\nabla \, u - p)n &= \sigma_N & \text{on } \gamma_N. \end{aligned}$$

It is worth noting that the Dirichlet boundary term u_D satisfies

(1)
$$\int_{\gamma} n \cdot u_D = \int_{\omega} \operatorname{div} u_D = 0.$$

Hence, if the boundary datum is given by some $\tilde{u}_D \in \mathsf{H}^{1/2}(\gamma)$ any solenoidal extension u_D to ω of \tilde{u}_D must satisfy the normal mean value property

$$\int_{\gamma} n \cdot \tilde{u}_D = 0.$$

On the other hand, one can always find a continuous and solenoidal lifting of a boundary term $\tilde{u}_D \in \mathsf{H}^{1/2}(\gamma)$ as long as (1) holds, see also our more general Corollary 1.6. In the smooth case we have for $\phi \in \mathsf{C}^{\infty}_{\gamma_D}(\omega)$

$$\begin{split} &-\nu\langle\Delta u,\phi\rangle_{0,\omega}=\nu\langle\nabla \,u,\nabla\,\phi\rangle_{0,\omega}-\nu\big\langle(\nabla\,u)n,\phi\big\rangle_{0,\gamma_N}\\ &=\langle G,\phi\rangle_{0,\omega}-\langle\nabla\,p,\phi\rangle_{0,\omega}=\langle G,\phi\rangle_{0,\omega}+\langle p,\operatorname{div}\phi\rangle_{0,\omega}-\big\langle pn,\phi\big\rangle_{0,\gamma_N}.\end{split}$$

i.e.,

$$\nu \langle \nabla u, \nabla \phi \rangle_{0,\omega} = \langle G, \phi \rangle_{0,\omega} + \langle p, \operatorname{div} \phi \rangle_{0,\omega} + \langle \sigma_N, \phi \rangle_{0,\gamma_N}$$

Let us for simplicity assume $\sigma_N = 0$. A possible variational formulation is given by the following: Find $u \in u_D + \mathsf{S}_{\gamma_D}(\omega)$ such that for all $\phi \in \mathsf{S}_{\gamma_D}(\omega)$

$$\nu \langle \nabla \, u, \nabla \, \phi \rangle_{0,\omega} = \langle G, \phi \rangle_{0,\omega}.$$

Using the ansatz $u = u_D + \hat{u}$ with $\hat{u} \in \mathsf{S}_{\gamma_D}(\omega)$ we reduce this formulation to find $\hat{u} \in \mathsf{S}_{\gamma_D}(\omega)$ such that for all $\phi \in \mathsf{S}_{\gamma_D}(\omega)$

$$\nu \langle \nabla \hat{u}, \nabla \phi \rangle_{0,\omega} = \langle G, \phi \rangle_{0,\omega} - \nu \langle \nabla u_D, \nabla \phi \rangle_{0,\omega}.$$

Note that the pressure p is not involved in this formulation. Another formulation taking the pressure into account and removing the unpleasant solenoidal condition from the Hilbert space is the following saddle point formulation: Find $(\hat{u}, p) \in H^1_{\gamma_D}(\omega) \times L^2_{\gamma_D}(\omega)$ such that for all $(\phi, \varphi) \in H^1_{\gamma_D}(\omega) \times L^2_{\gamma_D}(\omega)$

$$\nu \langle \nabla \hat{u}, \nabla \phi \rangle_{0,\omega} - \langle p, \operatorname{div} \phi \rangle_{0,\omega} = \langle G, \phi \rangle_{0,\omega} - \nu \langle \nabla u_D, \nabla \phi \rangle_{0,\omega}, - \langle \operatorname{div} \hat{u}, \varphi \rangle_{0,\omega} = 0,$$

which reads in formal matrix notation (boundary conditions are indicated as subscripts) as

$$\begin{bmatrix} -\nu \operatorname{div}_{\gamma_N} \nabla_{\gamma_D} & \nabla_{\gamma_N} \\ -\operatorname{div}_{\gamma_D} & 0 \end{bmatrix} \begin{bmatrix} \hat{u} \\ p \end{bmatrix} = \begin{bmatrix} G + \nu \operatorname{div}_{\gamma_N} \nabla_{\gamma_D} u_D \\ 0 \end{bmatrix}$$

.

Corollary 1.4 (Stokes lemma). For $\nu > 0$, $G \in L^2_{\gamma_N}(\omega)$, $u_D \in S(\omega)$ the Stokes system is uniquely solvable with $u = u_D + \hat{u} \in u_D + S_{\gamma_D}(\omega) \subset S(\omega)$ and $p \in L^2_{\gamma_D}(\omega)$. Moreover,

$$\nu \| \nabla \hat{u} \|_{0,\omega} \le c_{fp}(\omega, \gamma_D) \| G \|_{0,\omega} + \nu \| \nabla u_D \|_{0,\omega},$$

$$\nu \| \nabla u \|_{0,\omega} \le c_{fp}(\omega, \gamma_D) \| G \|_{0,\omega} + 2\nu \| \nabla u_D \|_{0,\omega},$$

 $\|p\|_{0,\omega} \le 2\kappa(\omega,\gamma_D) \big(c_{fp}(\omega,\gamma_D) \|G\|_{0,\omega} + \nu \|\nabla u_D\|_{0,\omega} \big).$

Note that here in the vector valued case

$$\mathsf{L}^{2}_{\gamma_{N}}(\omega) = \begin{cases} \mathsf{L}^{2}(\omega) &, \text{ if } \gamma_{D} \neq \emptyset, \\ \mathsf{L}^{2}_{\perp}(\omega) &, \text{ if } \gamma_{D} = \emptyset, \end{cases}$$
$$\mathsf{L}^{2}_{\perp}(\omega) = \mathsf{L}^{2}(\omega) \cap (\mathbb{R}^{d})^{\perp_{0,\omega}} = \big\{ \phi \in \mathsf{L}^{2}(\omega) \, : \, \int_{\omega} \phi_{i} = 0 \big\}.$$

Proof. Standard saddle point theory and the inf-sup lemma, Corollary 1.3, shows existence and the estimates follow by standard arguments, which provide also uniqueness. Note that we solve $p = \operatorname{div} \phi$ by Lemma 1.1 to get the estimates for the pressure p.

Another direct consequence of Lemma 1.1 is an estimate for the distance of vector fields to solenoidal fields, more precisely:

Corollary 1.5 (distance lemma). For any $u \in H^1_{\gamma_D}(\omega)$ there exists a solenoidal $u_0 \in S_{\gamma_D}(\omega)$ such that

$$\operatorname{dist}\left(u,\mathsf{S}_{\gamma_{D}}(\omega)\right) = \inf_{\phi\in\mathsf{S}_{\gamma_{D}}(\omega)} \left\|\nabla(u-\phi)\right\|_{0,\omega} \le \left\|\nabla(u-u_{0})\right\|_{0,\omega} \le \kappa(\omega,\gamma_{D})\|\operatorname{div} u\|_{0,\omega}.$$

Proof. For $u \in \mathsf{H}^1_{\gamma_D}(\omega)$ solve div $\tilde{u} = \operatorname{div} u \in \mathsf{L}^2_{\gamma_D}(\omega)$ with $\tilde{u} \in \mathsf{H}^1_{\gamma_D}(\omega)$ and the stability estimate $\|\nabla \tilde{u}\|_{0,\omega} \leq \kappa(\omega, \gamma_D) \|\operatorname{div} u\|_{0,\omega}$ by Lemma 1.1. Note that for $\gamma_D = \gamma$ it holds

$$\int_{\omega} \operatorname{div} u = \int_{\gamma} n \cdot u = 0.$$

Then $u_0 := u - \tilde{u} \in \mathsf{S}_{\gamma_D}(\omega)$ and $\|\nabla(u - u_0)\|_{0,\omega} = \|\nabla \tilde{u}\|_{0,\omega} \le \kappa(\omega, \gamma_D) \|\operatorname{div} u\|_{0,\omega}.$ \Box

This result can be extended to vector fields satisfying non-homogeneous Dirichlet boundary conditions provided that such a vector field u satisfies div $u \in L^2_{\gamma_D}(\Omega)$, the mean value condition, i.e.,

(2)
$$\int_{\omega} \operatorname{div} u = 0 \quad , \text{ if } \gamma_D = \gamma.$$

Corollary 1.6 (inhomogeneous distance lemma). For any $u \in H^1(\omega)$ with div $u \in L^2_{\gamma_D}(\omega)$ there exists a solenoidal $u_0 \in S(\omega)$ such that $u_0 - u \in H^1_{\gamma_D}(\omega)$, i.e., $u_0|_{\gamma_D} = u|_{\gamma_D}$, and

$$\left\| \nabla (u_0 - u) \right\|_{0,\omega} \le \kappa(\omega, \gamma_D) \|\operatorname{div} u\|_{0,\omega}.$$

Similar estimates for vector fields defined in $W^{1,q}(\Omega)$ spaces for $q \in (1,\infty)$ have been obtained in [36, 37]. In the literature, results like Corollary 1.6 are often called lifting lemmas, since a boundary datum $u|_{\gamma_D}$ is lifted to the domain ω , in this case with a solenoidal representative. Note that

$$\int_{\gamma} n \cdot u = \int_{\omega} \operatorname{div} u = 0 \quad \text{, if } \gamma_D = \gamma.$$

Proof. For $u \in \mathsf{H}^{1}(\omega)$ solve by Lemma 1.1 div $\tilde{u} = \operatorname{div} u \in \mathsf{L}^{2}_{\gamma_{D}}(\omega)$ with $\tilde{u} \in \mathsf{H}^{1}_{\gamma_{D}}(\omega)$ and $\|\nabla \tilde{u}\|_{0,\omega} \leq \kappa(\omega,\gamma_{D})\|\operatorname{div} u\|_{0,\omega}$. Note that (2) holds for $\gamma_{D} = \gamma$. Then $u_{0} := u - \tilde{u} \in \mathsf{S}(\omega)$ with $u - u_{0} = \tilde{u} \in \mathsf{H}^{1}_{\gamma_{D}}(\omega)$ and $\|\nabla(u_{0} - u)\|_{0,\omega} = \|\nabla \tilde{u}\|_{0,\omega} \leq \kappa(\omega,\gamma_{D})\|\operatorname{div} u\|_{0,\omega}$. \Box

Estimates of the distance to $S_{\gamma_D}(\omega)$ have not only theoretical meaning. They are also important for the a posteriori analysis of numerical solutions which usually satisfy the divergence free condition only approximately. If the constant $\kappa(\omega, \gamma_D)$ is known, then by using Corollary 1.5 we can deduce guaranteed and fully computable error bounds for approximations of problems arising in the theory of viscous incompressible fluids. For problems in bounded Lipschitz domains the respective results are presented in [33, 34].

In this contribution we extend Lemma 1.1 and its corollaries to the case of exterior domains $\Omega \subset \mathbb{R}^d$ and investigate applications to estimate the distance of vector fields to solenoidal fields. These estimates allows us to deduce new functional a posteriori error

estimates valid for a wide class of approximate solutions to the stationary Stokes problem in exterior domains.

2. Preliminaries

Let $\mathcal{D} \subset \mathbb{R}^d$, $d \geq 2$, be a domain (an open and connected set) with Lipschitz boundary \mathcal{B} , which is composed of two open and disjoint parts $\mathcal{B}_D, \mathcal{B}_N \subset \mathcal{B}$ with $\overline{\mathcal{B}} = \overline{\mathcal{B}}_D \cup \overline{\mathcal{B}}_N$ (Dirichlet and Neumann part). We note that \mathcal{D} can be bounded or unbounded, especially an exterior domain (a domain with compact complement). We introduce the usual Lebesgue and Sobolev spaces of square integrable functions or vector/tensor fields by $\mathsf{L}^2(\mathcal{D})$ and $\mathsf{H}^1(\mathcal{D})$, respectively. The standard inner product, norm, resp. orthogonality in $\mathsf{L}^2(\mathcal{D})$ are denoted by $\langle \cdot, \cdot \rangle_{0,\mathcal{D}}$, $\| \cdot \|_{0,\mathcal{D}}$, resp. $\bot_{0,\mathcal{D}}$. Moreover, let

$$\mathsf{L}^{2}_{\mathcal{B}_{D}}(\mathcal{D}) := \begin{cases} \mathsf{L}^{2}(\mathcal{D}) \cap \mathbb{R}^{\perp_{0}, \mathcal{D}} &, \text{ if } \mathcal{B}_{D} = \mathcal{B}, \\ \mathsf{L}^{2}(\mathcal{D}) &, \text{ else,} \end{cases}$$

provided that \mathcal{D} is bounded. If $\mathcal{B}_D \neq \emptyset$, homogeneous Dirichlet boundary conditions are encoded in $H^1_{\mathcal{B}_D}(\mathcal{D})$, defined as closure of

$$\mathsf{C}^{\infty}_{\mathcal{B}_D}(\mathcal{D}) := \left\{ \phi|_{\mathcal{D}} : u \in \mathsf{C}^{\infty}(\mathbb{R}^d), \operatorname{supp} \phi \operatorname{ compact}, \operatorname{ dist}(\operatorname{supp} \phi, \mathcal{B}_D) > 0 \right\}$$

in $H^1(\mathcal{D})$. Moreover, we introduce the polynomially weighted spaces

$$\begin{split} \mathsf{L}^{2}_{\pm 1}(\mathcal{D}) &:= \big\{ \phi \in \mathsf{L}^{2}_{\mathsf{loc}}(\mathcal{D}) \,:\, \rho^{\pm 1} \phi \in \mathsf{L}^{2}(\mathcal{D}) \big\},\\ \mathsf{H}^{1}_{-1}(\mathcal{D}) &:= \big\{ \phi \in \mathsf{L}^{2}_{-1}(\mathcal{D}) \,:\, \nabla \phi \in \mathsf{L}^{2}(\mathcal{D}) \big\}, \end{split}$$

where the weight function ρ is defined by $\rho(r) := (1 + r^2)^{1/2}$, r(x) := |x|. Inner product, norm, resp. orthogonality in $\mathsf{L}^2_{\pm 1}(\mathcal{D})$ is defined and denoted by $\langle \cdot, \cdot \rangle_{\pm 1,\mathcal{D}} := \langle \rho^{\pm 2} \cdot, \cdot \rangle_{0,\mathcal{D}}$, $\| \cdot \|_{\pm 1,\mathcal{D}}$, resp. $\perp_{\pm 1,\mathcal{D}}$. As before, if $\mathcal{B}_D \neq \emptyset$, homogeneous (full, tangential, resp. normal) Dirichlet boundary conditions are introduced in $\mathsf{H}^1_{-1,\mathcal{B}_D}(\mathcal{D})$, the closure of $\mathsf{C}^\infty_{\mathcal{B}_D}(\mathcal{D})$ in $\mathsf{H}^1_{-1}(\mathcal{D})$. Finally, in particular for the Stokes equations, we introduce spaces of solenoidal fields

$$\begin{split} \mathsf{S}(\mathcal{D}) &:= \big\{ \phi \in \mathsf{H}^1(\mathcal{D}) \, : \, \operatorname{div} \phi = 0 \big\}, \qquad \qquad \mathsf{S}_{\mathcal{B}_D}(\mathcal{D}) &:= \mathsf{H}^1_{\mathcal{B}_D}(\mathcal{D}) \cap \mathsf{S}(\mathcal{D}), \\ \mathsf{S}_{-1}(\mathcal{D}) &:= \big\{ \phi \in \mathsf{H}^1_{-1}(\mathcal{D}) \, : \, \operatorname{div} \phi = 0 \big\}, \qquad \qquad \mathsf{S}_{-1,\mathcal{B}_D}(\mathcal{D}) &:= \mathsf{H}^1_{-1,\mathcal{B}_D}(\mathcal{D}) \cap \mathsf{S}_{-1}(\mathcal{D}). \end{split}$$

Note that in the case of a bounded domain, there is no difference between the unweighted and weighted spaces, meaning that the spaces coincide as sets and possess different inner products.

Throughout the paper we assume that $\Omega \subset \mathbb{R}^d$, where $d \geq 3$ (the special case d = 2 is considered in Section 4.4 and in Appendix II), is an exterior domain with a Lipschitz boundary Γ , which is composed of two open and disjoint parts $\Gamma_D, \Gamma_N \subset \Gamma$ (Dirichlet and Neumann part) with $\overline{\Gamma} = \overline{\Gamma}_D \cup \overline{\Gamma}_N$. Moreover, let $\mathbb{R}^d \setminus \Omega \subset B_{r_1}$ for some $r_2 > r_1 > 0$ and

$$\omega := \Omega_{r_2} := \Omega \cap B_{r_2}, \quad \gamma = \Gamma \cup S_{r_2}, \quad \gamma_D := \Gamma_D \cup S_{r_2},$$

where B_r and S_r denote the open ball and the sphere of radius r centered at the origin in \mathbb{R}^d , respectively. We also pick some cut-off Lipschitz continuous function $\xi \in W^{1,\infty}(\mathbb{R}; [0, 1])$ satisfying $\xi|_{(-\infty,0]} = 0$ and $\xi|_{[1,\infty)} = 1$ and set

$$\xi'_{\infty} := \operatorname{ess\,sup}_{[0,1]} |\xi'|.$$

Then the function $\tilde{\xi}$ defined by $\tilde{\xi}(z) := \xi((z-r_1)/(r_2-r_1))$ belongs to $\mathsf{W}^{1,\infty}(\mathbb{R};[0,1])$ as well and satisfies $\tilde{\xi}|_{(-\infty,r_1]} = 0$ and $\tilde{\xi}|_{[r_2,\infty)} = 1$. Thus

(3)
$$\eta := \tilde{\xi} \circ r \in \mathsf{W}^{1,\infty}(\mathbb{R}^d)$$

with $\eta|_{\overline{B}_{r_1}} = 0$ and $\eta|_{\mathbb{R}^d \setminus B_{r_2}} = 1$. Finally, we define the constant

$$c_d := \frac{2}{d-2}.$$



FIGURE 1. $\mathbb{R}^d \setminus \overline{\Omega}$ (gray) surrounded by the boundary Γ (thin black lines), the boundary part Γ_D (thick black lines), and the artificial boundary spheres (dashed lines)

The two main ingredients for our proofs are Lemma 1.1 and a few results from the theory of rot-div-systems in exterior domains, which can be summarised in the two subsequent lemmas as follows:

Lemma 2.1 (Friedrichs/Poincaré lemma for exterior domains). The following weighted Friedrichs/Poincaré estimates hold:

(i) There exists c > 0 such that for all $v \in H^1_{-1,\Gamma_D}(\Omega)$ it holds

$$||v||_{-1,\Omega} \le c ||\nabla v||_{0,\Omega}.$$

The best constant c is the Friedrichs/Poincaré constant and is denoted by $c_{fp}(\Omega, \Gamma_D)$.

(ii) If $\Gamma_D = \Gamma$, then $c_{fp}(\Omega, \Gamma)$ is the Friedrichs constant $c_f(\Omega)$ and can be estimated by

$$c_{fp}(\Omega,\Gamma) = c_f(\Omega) \le c_d.$$

Especially, for all $v \in \mathsf{H}^{1}_{-1,\Gamma}(\Omega)$ it holds $\|v\|_{-1,\Omega} \leq c_{d} \|\nabla v\|_{0,\Omega}$.

- (ii') If $\Gamma_D = \emptyset$, then $c_{fp}(\Omega, \emptyset)$ is the Poincaré constant $c_p(\Omega)$. Particularly, for all $v \in \mathsf{H}^1_{-1}(\Omega)$ it holds $\|v\|_{-1,\Omega} \leq c_p(\Omega) \|\nabla v\|_{0,\Omega}$.
- (iii) If $\Omega = \mathbb{R}^d$, then the Friedrichs and Poincaré constants coincide and, moreover,

$$c_{fp}(\mathbb{R}^d) = c_f(\mathbb{R}^d) = c_p(\mathbb{R}^d) \le c_d.$$

Especially, for all $v \in H^1_{-1}(\mathbb{R}^d)$ it holds

$$||v||_{-1,\mathbb{R}^d} \le c_d ||\nabla v||_{0,\mathbb{R}^d}.$$

Note that no boundary or mean value conditions are needed in Lemma 2.1, since the constants are not integrableⁱⁱ in $L^2_{-1}(\Omega)$ for $d \ge 3$. For d = 3, Lemma 2.1 follows immediately from [15, Poincaré's estimate IV, p. 62] by approximation. Nevertheless, we present a simple and self-contained proof.

Proof. From [24, Appendix 4.2, Lemma 4.1, Corollary 4.2, Remark 4.3], see also [15, Poincaré's estimate III, p. 57] and [38, Lemma 4.1], we have for allⁱⁱⁱ $u \in C_{\Gamma}^{\infty}(\Omega)$

$$||u||_{-1,\Omega} \le ||r^{-1}u||_{0,\Omega} \le c_d ||\nabla u||_{0,\Omega}$$

ⁱⁱMore precisely, it holds $(1 + r^2)^{t/2} \in L^2_{-1}(\Omega)$, if and only if t - 1 < -d/2. Putting t = 0 shows the assertion.

ⁱⁱⁱNote that $r^{-1} \in L^2(B_1)$ if and only if $d \ge 3$.

and hence by density and continuity for all $u \in \mathsf{H}^1_{-1,\Gamma}(\Omega)$

(4)
$$||u||_{-1,\Omega} \le c_d ||\nabla u||_{0,\Omega}.$$

For all $u \in \mathsf{H}^{1}_{-1}(\Omega)$ we see $\eta u \in \mathsf{H}^{1}_{-1,\Gamma}(\Omega)$ and $\|\eta u\|_{-1,\Omega} \leq c_{d} \|\nabla(\eta u)\|_{0,\Omega}$ by (4). Hence

(5)
$$||u||_{-1,\Omega} \le c_d ||\nabla u||_{0,\Omega} + c_d ||u\nabla \eta||_{0,\Omega} + ||(1-\eta)u||_{-1,\Omega} \le c_d ||\nabla u||_{0,\Omega} + \tilde{c}_d ||u||_{0,\omega},$$

where^{iv} $\tilde{c}_d := c_d \xi'_{\infty}/(r_2 - r_1) + 1$. Now we can prove (i), even the stronger result (ii'). If the estimate in (ii') is false, there is a sequence $(u_n) \subset H^1_{-1}(\Omega)$ with $||u_n||_{-1,\Omega} = 1$ and $||\nabla u_n||_{0,\Omega} < 1/n$. Hence, (u_n) is bounded in $H^1(\omega)$ as well. By Rellich's selection theorem we can assume w.l.o.g. that (u_n) already converges in $L^2(\omega)$. Thus, by (5) (u_n) is a Cauchy sequence in $L^2_{-1}(\Omega)$ and hence also in $H^1_{-1}(\Omega)$. Therefore, (u_n) converges in $H^1_{-1}(\Omega)$ to some $u \in H^1_{-1}(\Omega)$ with $\nabla u = 0$. We conclude that u is constant. But then $u \in L^2_{-1}(\Omega)$ must vanish, which implies a contradiction by $1 = ||u_n||_{-1,\Omega} \to 0$.

Lemma 2.2 (rot-div lemma for the whole space). For any $f \in L^2(\mathbb{R}^d)$ there exists a unique $v \in H^1_{-1}(\mathbb{R}^d)$ such that rot v = 0, div v = f, and

$$\frac{1}{c_d} \|v\|_{-1,\mathbb{R}^d} \le \|\nabla v\|_{0,\mathbb{R}^d} = \|\operatorname{div} v\|_{0,\mathbb{R}^d} = \|f\|_{0,\mathbb{R}^d}.$$

Note that the equation $-\Delta = \operatorname{rot}^* \operatorname{rot} - \nabla \operatorname{div}$ implies

(6)
$$\|\nabla\Phi\|_{0,\mathbb{R}^d}^2 = \|\operatorname{rot}\Phi\|_{0,\mathbb{R}^d}^2 + \|\operatorname{div}\Phi\|_{0,\mathbb{R}^d}^2$$

for all $\Phi \in C^{\infty}(\mathbb{R}^d)$ having compact support and extends to all $\Phi \in H^{1}_{-1}(\mathbb{R}^d)$ by density and continuity. Hence the equality $\|\nabla v\|_{0,\mathbb{R}^d} = \|\operatorname{div} v\|_{0,\mathbb{R}^d}$ in Lemma 2.2 follows immediately. The results of Lemma 2.2 are well known and can be found, e.g., in [26, 27, 28] or in [12, 21]. In particular, Lemma 2.2 follows from Lemma 2.1 (iii), (6), and [12, Theorem A.7, Theorem 3.2 (ii)], see also [21, Lemma 3.5, Lemma 3.6, Theorem 4.1].

3. The Stability Lemma for Exterior Domains

First we define our upper bound related to the geometry presented in Figure 1.

(7)
$$\hat{\kappa}(\Omega,\Gamma_D) := (1+\kappa) \left(1 + c_d \frac{\xi'_{\infty} \rho(r_2)}{r_2 - r_1} \right), \qquad \kappa := \min\left\{ \kappa(\omega,\gamma_D), \kappa(\omega,\gamma) \right\}.$$

Especially for $r_2 = r_1 + 1$ and $\xi'_{\infty} \leq 1$ we have the simple upper bound

$$\hat{\kappa}(\Omega, \Gamma_D) = (1+\kappa) \big(1 + c_d \rho(r_2) \big).$$

The above constants contain the stability constants $\kappa(\omega, \gamma_D)$, $\kappa(\omega, \gamma)$ associated with the bounded domain ω and respective parts of its boundary γ_D and γ .

Remark 3.1. $\kappa(\omega, \gamma_D)$ and $\kappa(\omega, \Gamma_D)$ depend on r_2 , so that the best value of r_2 (which minimises the constant) is not known a priori and has to be optimized by some algebraic procedure. We emphasize that

$$\kappa \leq \kappa(\omega, \gamma_D), \qquad \kappa \leq \kappa(\omega, \gamma).$$

and that a bound in the simple situation from above is given by

$$\hat{\kappa}(\Omega,\Gamma_D) = (1+\kappa) \left(1 + \frac{2\sqrt{2}}{d-2}r_2\right)$$

Now we can proceed to prove the stability lemma for exterior domains. First we observe a trivial case for compactly supported right hand sides:

^{iv}For $r_2 = r_1 + 1$ and $\xi'_{\infty} \leq 1$ we have $\tilde{c}_d \leq c_d + 1$.

Remark 3.2. There exists c > 0 such that for all $f \in L^2(\Omega)$ with $\operatorname{supp} f \subset \overline{\omega}$ and in the case $\Gamma_D = \Gamma$, additionally, $\int_{\Omega} f = 0$, there is a vector field $v \in H^1_{-1,\Gamma_D}(\Omega)$ with div v = f and $\|\nabla v\|_{0,\Omega} \leq c \|f\|_{0,\Omega}$. The best constant is denoted by $\kappa(\Omega, \Gamma_D)$. Moreover, v can be chosen with compact support in $\overline{\omega}$, in particular, $v \in H^1_{\gamma_D}(\omega) \subset H^1_{\Gamma_D}(\Omega)$. In this case, $\kappa(\Omega, \Gamma_D) = \kappa(\omega, \gamma_D)$. For a short proof, we set $g := f|_{\omega} \in L^2_{\gamma_D}(\omega)$ and by Lemma 1.1 there exist $\kappa(\omega, \gamma_D) > 0$ and $u \in H^1_{\gamma_D}(\omega)$ with

div u = g, $\|\nabla u\|_{0,\omega} \le \kappa(\omega, \gamma_D) \|g\|_{0,\omega}$.

Then v, which is the extension by zero of u to Ω , belongs to $\mathsf{H}^{1}_{\Gamma_{D}}(\Omega)$ and $\operatorname{supp} v = \operatorname{supp} u \subset \overline{\omega}$. Moreover, $\operatorname{div} v = f$ and $\|\nabla v\|_{0,\Omega} = \|\nabla u\|_{0,\omega} \le \kappa(\omega, \gamma_{D}) \|g\|_{0,\omega} = \kappa(\omega, \gamma_{D}) \|f\|_{0,\Omega}$.

Our main result reads as follows:

Lemma 3.3 (stability lemma for exterior domains). There exists c > 0 such that for all $f \in L^2(\Omega)$ there is a vector field $v \in H^1_{-1,\Gamma_D}(\Omega)$ with

div v = f and $\|\nabla v\|_{0,\Omega} \le c \|f\|_{0,\Omega}$.

The best constant is denoted by $\kappa(\Omega, \Gamma_D)$ which equals the norm of the corresponding right inverse $f \mapsto v$. Moreover with (7)

$$\kappa(\Omega, \Gamma_D) \leq \hat{\kappa}(\Omega, \Gamma_D).$$

Note that no mean value condition is imposed on f.

Proof. We extend f by 0 to $\mathbb{R}^d \setminus \overline{\Omega}$ and identify $f \in L^2(\mathbb{R}^d)$. By Lemma 2.2 we get some $v \in H^1_{-1}(\mathbb{R}^d)$ with rot v = 0 solving div v = f in \mathbb{R}^d and

(8)
$$\|v\|_{-1,\mathbb{R}^d} \le c_d \|\nabla v\|_{0,\mathbb{R}^d}, \quad \|\nabla v\|_{0,\mathbb{R}^d} = \|\operatorname{div} v\|_{0,\mathbb{R}^d} = \|f\|_{0,\Omega}.$$

Then $\eta v \in \mathsf{H}^{1}_{-1,\Gamma}(\Omega)$ with (3) and $\operatorname{supp}(\eta v) \subset \mathbb{R}^{d} \setminus B_{r_{1}}$. We are searching for $v \in \mathsf{H}^{1}_{-1,\Gamma_{D}}(\Omega)$ solving div v = f in the form

$$v := \eta v + v_{\omega},$$

where $v_{\omega} \in \mathsf{H}^{1}_{\Gamma_{D}}(\Omega)$ with $\operatorname{supp} v_{\omega} \subset \overline{\omega}$ is the extension by zero to Ω of some vector field $u \in \mathsf{H}^{1}_{\gamma_{D}}(\omega)$. Hence, v and v_{ω} should satisfy

$$f = \operatorname{div} v = \eta f + \nabla \eta \cdot v + \operatorname{div} v_{\omega} \quad \text{in } \Omega$$

and we have to find $u \in \mathsf{H}^{1}_{\gamma_{D}}(\omega)$ with

div
$$u = g := (1 - \eta)f - \nabla \eta \cdot v \in \mathsf{L}^2(\omega)$$
 in ω .

Note that indeed $\operatorname{supp}(1-\eta) \subset \overline{B}_{r_2}$, $\operatorname{supp} \nabla \eta \subset \overline{B}_{r_2} \setminus B_{r_1}$ and hence $\operatorname{supp} g \subset \overline{\omega}$. Moreover,

$$g = (1 - \eta)f + \nabla(1 - \eta) \cdot v = \operatorname{div} ((1 - \eta)v)$$
 in \mathbb{R}^d

and therefore

$$\int_{\omega} g = \int_{\gamma} (1 - \eta) n \cdot v = \int_{\Gamma} n \cdot v = - \int_{\mathbb{R}^d \setminus \overline{\Omega}} \operatorname{div} v = - \int_{\mathbb{R}^d \setminus \overline{\Omega}} f = 0.$$

Thus, g has mean value zero independent of the particular boundary condition on Γ_D , i.e., $g \in \mathsf{L}^2_{\gamma_D}(\omega)$. Lemma 1.1 provides such a $u \in \mathsf{H}^1_{\gamma_D}(\omega)$ with $\|\nabla u\|_{0,\omega} \leq \kappa(\omega,\gamma_D)\|g\|_{0,\omega}$. We can even pick $u \in \mathsf{H}^1_{\gamma}(\omega) \subset \mathsf{H}^1_{\gamma_D}(\omega)$ with $\|\nabla u\|_{0,\omega} \leq \kappa(\omega,\gamma)\|g\|_{0,\omega}$. Hence, generally, we obtain $u \in \mathsf{H}^1_{\gamma}(\omega) \subset \mathsf{H}^1_{\gamma_D}(\omega)$ with the stability estimate

$$\|\nabla u\|_{0,\omega} \le \kappa \|g\|_{0,\omega}, \qquad \kappa = \min\left\{\kappa(\omega, \gamma_D), \kappa(\omega, \gamma)\right\},$$

see (7). Thus $v \in \mathsf{H}^{1}_{-1,\Gamma_{D}}(\Omega)$ solves div v = f. It remains to show the estimates. Using

$$\|\nabla \eta \cdot v^{\top}\|_{0,\mathbb{R}^d}, \|\nabla \eta \cdot v\|_{0,\mathbb{R}^d} \le \frac{\xi_{\infty}'}{r_2 - r_1} \|v\|_{0,B_{r_2} \setminus \overline{B}_{r_1}}$$

and by (8) we compute

$$\left\| \nabla(\eta v) \right\|_{0,\Omega} \leq \underbrace{\left\| \nabla v \right\|_{0,\mathbb{R}^d}}_{=\|f\|_{0,\Omega}} + \| \nabla \eta v^\top \|_{0,\mathbb{R}^d},$$

$$\|\nabla v_{\omega}\|_{0,\Omega} = \|\nabla u\|_{0,\omega} \le \kappa(\omega,\gamma_D) \|g\|_{0,\omega} \le \kappa(\omega,\gamma_D) \big(\|f\|_{0,\Omega} + \|\nabla \eta \cdot v\|_{0,\mathbb{R}^d}\big),$$
$$\|v\|_{0,B_{r_2}\setminus\overline{B}_{r_1}} \le \rho(r_2) \|v\|_{-1,\mathbb{R}^d} \le c_d\rho(r_2) \underbrace{\|\nabla v\|_{0,\mathbb{R}^d}}_{=\|f\|_{0,\Omega}},$$

which finally proves $\|\nabla v\|_{0,\Omega} \leq \hat{\kappa}(\Omega, \Gamma_D) \|f\|_{0,\Omega}$, finishing the proof.

4. Applications for Exterior Domains

4.1. Inf-Sup Lemma and Estimates of the Distance to Solenoidal Fields. A direct consequence of Lemma 3.3 is an estimate for the distance of vector fields to solenoidal fields:

Corollary 4.1 (distance lemma for exterior domains). For any $v \in H^1_{-1,\Gamma_D}(\Omega)$ there exists a solenoidal $v_0 \in S_{-1,\Gamma_D}(\Omega)$ such that

$$\operatorname{dist}\left(v, \mathsf{S}_{-1,\Gamma_{D}}(\Omega)\right) = \inf_{\phi \in \mathsf{S}_{-1,\Gamma_{D}}(\Omega)} \left\| \nabla(v-\phi) \right\|_{0,\Omega} \le \left\| \nabla(v-v_{0}) \right\|_{0,\Omega} \le \kappa(\Omega,\Gamma_{D}) \|\operatorname{div} v\|_{0,\Omega}.$$

Proof. For $v \in \mathsf{H}^{1}_{-1,\Gamma_{D}}(\Omega)$ solve div $\tilde{v} = \operatorname{div} v$ with $\tilde{v} \in \mathsf{H}^{1}_{-1,\Gamma_{D}}(\Omega)$ and the stability estimate $\|\nabla \tilde{v}\|_{0,\Omega} \leq \kappa(\Omega,\Gamma_{D})\|\operatorname{div} v\|_{0,\Omega}$ by Lemma 3.3. Then $v_{0} := v - \tilde{v} \in \mathsf{S}_{-1,\Gamma_{D}}(\Omega)$ and we have $\|\nabla (v - v_{0})\|_{0,\Omega} = \|\nabla \tilde{v}\|_{0,\Omega} \leq \kappa(\Omega,\Gamma_{D})\|\operatorname{div} v\|_{0,\Omega}$. \Box

Corollary 4.2 (inhomogeneous distance lemma for exterior domains). For any $v \in \mathsf{H}_{-1}^1(\Omega)$ there exists a solenoidal $v_0 \in \mathsf{S}_{-1}(\Omega)$ such that $v_0 - v \in \mathsf{H}_{-1,\Gamma_D}^1(\Omega)$, i.e., $v_0|_{\Gamma_D} = v|_{\Gamma_D}$, and $\|\nabla(v_0 - v)\|_{0,\Omega} \leq \kappa(\Omega,\Gamma_D)\| \operatorname{div} v\|_{0,\Omega}$.

Proof. For $v \in \mathsf{H}^{1}_{-1}(\Omega)$ we solve by Lemma 3.3 div $\tilde{v} = \operatorname{div} v$ with some $\tilde{v} \in \mathsf{H}^{1}_{-1,\Gamma_{D}}(\Omega)$ and $\|\nabla \tilde{v}\|_{0,\Omega} \leq \kappa(\Omega,\Gamma_{D})\|\operatorname{div} v\|_{0,\Omega}$. Then $v_{0} := v - \tilde{v} \in \mathsf{S}(\Omega)$ with $v - v_{0} = \tilde{v} \in \mathsf{H}^{1}_{\Gamma_{D}}(\Omega)$ and $\|\nabla (v_{0} - v)\|_{0,\Omega} = \|\nabla \tilde{v}\|_{0,\Omega} \leq \kappa(\Omega,\Gamma_{D})\|\operatorname{div} v\|_{0,\Omega}$.

As in the case of a bounded domain, Corollary 4.2 can be seen as a lifting lemma, lifting the boundary datum $v|_{\Gamma_D}$ to the domain Ω , in this case with a solenoidal representative. By solving $g = \operatorname{div} v$ Lemma 3.3 yields immediately also the following inf-sup result:

Corollary 4.3 (inf-sup lemma for exterior domains). It holds

$$\inf_{f \in \mathsf{L}^2(\Omega)} \sup_{v \in \mathsf{H}^1_{-1,\Gamma_{\mathcal{D}}}(\Omega)} \frac{\langle f, \operatorname{div} v \rangle_{0,\Omega}}{\|f\|_{0,\Omega} \|\nabla v\|_{0,\Omega}} \ge \frac{1}{\kappa(\Omega,\Gamma_D)}.$$

4.2. Solution Theory for the Stationary Stokes System. For $\nu > 0$, $F \in L^2_1(\Omega)$, $v_D \in S_{-1}(\Omega)$ a solution theory for the stationary Stokes problem follows. The equations or first resp. second order systems are the same as in the case of a bounded domain, e.g.,

$$\begin{aligned} -\operatorname{Div} \sigma &= F & \text{in } \Omega, \\ \sigma &= \nu \, \nabla \, v - p \, \mathbb{I} & \text{in } \Omega, \\ -\operatorname{div} v &= 0 & \text{in } \Omega, \\ v &= v_D & \text{on } \Gamma_D \\ \sigma n &= 0 & \text{on } \Gamma_N \end{aligned}$$

(for simplicity we assume again $\sigma_N = 0$) resp.

$$\begin{aligned} -\nu\Delta v + \nabla p &= F & \text{in } \Omega, \\ -\operatorname{div} v &= 0 & \text{in } \Omega, \\ v &= v_D & \text{on } \Gamma_D, \\ (\nu \nabla v - p)n &= 0 & \text{on } \Gamma_N \end{aligned}$$

with additional proper decay conditions at infinity $v \in L^{2}_{-1}(\Omega)$ and $\nabla v \in L^{2}(\Omega)$ which read in classical point wise terms (more vaguely) as

$$v(x) \xrightarrow{|x| \to \infty} 0$$

Note that (1) is no longer a necessary condition in the case of an exterior domain due to the possible lack of integrability. Therefore, our lifting lemma Corollary 4.2 does not need an additional assumption on div v as in Corollary 1.6. Let us assume a slightly more general viscosity $\nu \in \mathsf{L}^{\infty}(\Omega)$, bounded from below and above by two positive constants ν_{-} and ν_{+} , respectively. A possible variational formulation (see, e.g., [13, 9]) is given by the following: Find $v \in v_D + \mathsf{S}_{-1,\Gamma_D}(\Omega)$ such that for all $\phi \in \mathsf{S}_{-1,\Gamma_D}(\Omega)$

$$\langle \nu \nabla v, \nabla \phi \rangle_{0,\Omega} = \langle F, \phi \rangle_{0,\Omega}.$$

Note that by $\langle F, \phi \rangle_{0,\Omega} = \langle \rho F, \rho^{-1} \phi \rangle_{0,\Omega}$ the right hand side is well defined. Using the ansatz $v = v_D + \hat{v}$ with $\hat{v} \in \mathsf{S}_{-1,\Gamma_D}(\Omega)$ we reduce this formulation to find $\hat{v} \in \mathsf{S}_{-1,\Gamma_D}(\Omega)$ such that for all $\phi \in \mathsf{S}_{-1,\Gamma_D}(\Omega)$

$$\langle \nu \nabla \hat{v}, \nabla \phi \rangle_{0,\Omega} = \langle F, \phi \rangle_{0,\Omega} - \langle \nu \nabla v_D, \nabla \phi \rangle_{0,\Omega}.$$

Again, another formulation taking the pressure into account and removing the unpleasant solenoidal condition from the Hilbert space is the following saddle point formulation: Find $(\hat{v}, p) \in \mathsf{H}^{1}_{-1,\Gamma_{D}}(\Omega) \times \mathsf{L}^{2}(\Omega)$ such that for all $(\phi, \varphi) \in \mathsf{H}^{1}_{-1,\Gamma_{D}}(\Omega) \times \mathsf{L}^{2}(\Omega)$

(9)
$$\langle \nu \nabla \hat{v}, \nabla \phi \rangle_{0,\Omega} - \langle p, \operatorname{div} \phi \rangle_{0,\Omega} = \langle F, \phi \rangle_{0,\Omega} - \langle \nu \nabla v_D, \nabla \phi \rangle_{0,\Omega}, \\ - \langle \operatorname{div} \hat{v}, \varphi \rangle_{0,\Omega} = 0.$$

Corollary 4.4 (Stokes lemma for exterior domains). For ν , $F \in L^2_1(\Omega)$, $v_D \in S_{-1}(\Omega)$ the Stokes system is uniquely solvable with $v = v_D + \hat{v} \in v_D + \mathsf{S}_{-1,\Gamma_D}(\Omega) \subset \mathsf{S}_{-1}(\Omega)$ and $p \in L^2(\Omega)$. Moreover,

$$\begin{split} \nu \| \nabla \hat{v} \|_{0,\Omega} &\leq c_{fp}(\Omega, \Gamma_D) \| F \|_{1,\Omega} + \nu \| \nabla v_D \|_{0,\Omega}, \\ \nu \| \nabla v \|_{0,\Omega} &\leq c_{fp}(\Omega, \Gamma_D) \| F \|_{1,\Omega} + 2\nu \| \nabla v_D \|_{0,\Omega}, \\ \| p \|_{0,\Omega} &\leq 2\kappa(\Omega, \Gamma_D) \big(c_{fp}(\Omega, \Gamma_D) \| F \|_{1,\Omega} + \nu \| \nabla v_D \|_{0,\Omega} \big). \end{split}$$

Proof. Standard saddle point theory and the inf-sup lemma, Corollary 4.3, shows existence and the estimates follow by standard arguments, which provide also uniqueness. Note that by the Friedrichs/Poincaré estimates in Lemma 2.1 the principal part of the bilinear form is positive over $\mathsf{H}^1_{-1,\Gamma_{\mathcal{D}}}(\Omega)$, and that we solve $p = \operatorname{div} \phi$ by Lemma 3.3 to get the estimates for the pressure p. \square

4.3. A Posteriori Error Estimates for Stationary Stokes Equations. Before proceeding, we need one more polynomial weighted Sobolev space. For this, we recall Div acting as usual row wise on $\mathbb{R}^{d \times d}$ -tensor fields and define

$$\mathsf{D}(\Omega) := \big\{ \Theta \in \mathsf{L}^2(\Omega) \, : \, \operatorname{Div} \Theta \in \mathsf{L}^2_1(\Omega) \big\}, \qquad \mathsf{D}_{\Gamma_N}(\Omega),$$

where $\tilde{\mathsf{D}}_{\Gamma_N}(\Omega)$ is the closure of $\mathsf{C}^{\infty}_{\Gamma_N}(\Omega)$ -tensor fields in the norm of the Sobolev space $\tilde{\mathsf{D}}(\Omega)$. Then we observe for all $\phi \in \mathsf{H}^{1}_{-1,\Gamma_{D}}(\Omega)$ and all $\tau \in \tilde{\mathsf{D}}_{\Gamma_{N}}(\Omega)$

(10)
$$\langle \tau, \nabla \phi \rangle_{0,\Omega} = -\langle \operatorname{Div} \tau, \phi \rangle_{0,\Omega}.$$

Note that the right hand side is well defined since $(\text{Div }\tau, \phi)_{0,\Omega} = \langle \rho \text{Div }\tau, \rho^{-1}\phi \rangle_{0,\Omega}$.

From now on we assume that we have approximations

$$\tilde{v} \in \mathsf{L}^2_{-1}(\Omega), \qquad \tilde{p} \in \mathsf{L}^2(\Omega), \qquad \tilde{T} \in \mathsf{L}^2(\Omega), \qquad \tilde{\sigma} \in \mathsf{L}^2(\Omega)$$

of our exact solutions from (9) and Corollary 4.4

$$\begin{aligned} v &= v_D + \hat{v} \in v_D + \mathsf{S}_{-1,\Gamma_D}(\Omega) \subset \mathsf{S}_{-1}(\Omega), \qquad & T := \nabla v \in \mathsf{L}^2(\Omega), \\ p \in \mathsf{L}^2(\Omega), \qquad & \sigma = \nu \, \nabla v - p \, \mathbb{I} \in \mathsf{L}^2(\Omega), \end{aligned}$$

respectively, for given data $\nu, F \in L^2(\Omega)$, and $v_D \in S_{-1}(\Omega)$. We recall from (9) that (v, p)solves for all $\phi \in \mathsf{H}^{1}_{-1,\Gamma_{\mathcal{D}}}(\Omega)$

(11)
$$\langle \nu \, \nabla \, v, \nabla \, \phi \rangle_{0,\Omega} - \langle p, \operatorname{div} \phi \rangle_{0,\Omega} = \langle F, \phi \rangle_{0,\Omega}.$$

^vThe viscosity ν can even be assumed to be a bounded, positive definite, symmetric tensor field. Moreover, we note that $\nu_{-}|T|^{2} \leq |\nu^{1/2}T|^{2} = \nu T : T \leq \nu_{+}|T|^{2}$ and thus also $\nu_{-}|\nu^{-1/2}T|^{2} \leq |T|^{2} \leq \nu_{+}|\nu^{-1/2}T|^{2}$.

4.3.1. A Posteriori Estimates for the Velocity Field: Solenoidal Approximations. First, we assume the simplest case that

$$\tilde{T} = \nabla \tilde{v}, \qquad \tilde{v} \in v_D + \mathsf{S}_{-1,\Gamma_D}(\Omega) \subset \mathsf{S}_{-1}(\Omega),$$

i.e., $\tilde{v} - v_D \in \mathsf{S}_{-1,\Gamma_D}(\Omega)$. Then by (11) we have for all solenoidal $\phi \in \mathsf{S}_{-1,\Gamma_D}(\Omega)$

$$\langle \nu \nabla (v - \tilde{v}), \nabla \phi \rangle_{0,\Omega} = \langle F, \phi \rangle_{0,\Omega} - \langle \nu \nabla \tilde{v}, \nabla \phi \rangle_{0,\Omega}.$$

Let $\tau \in \tilde{\mathsf{D}}_{\Gamma_N}(\Omega)$ and $q \in \mathsf{L}^2(\Omega)$. Using (10) and $\langle q \mathbb{I}, \nabla \phi \rangle_{0,\Omega} = 0$ (actually this holds for all $\phi \in \mathsf{S}_{-1}(\Omega)$ since $\mathbb{I} : \nabla \phi = \operatorname{div} \phi$) as well as the Friedrichs/Poincaré estimate from Lemma 2.1 we compute

$$\langle \nu \,\nabla(v-\tilde{v}), \nabla \,\phi \rangle_{0,\Omega}$$

$$= \langle \operatorname{Div} \tau + F, \phi \rangle_{0,\Omega} + \langle \tau + q \,\mathbb{I} - \nu \,\nabla \,\tilde{v}, \nabla \,\phi \rangle_{0,\Omega}$$

$$\leq \| \operatorname{Div} \tau + F \|_{1,\Omega} \|\phi\|_{-1,\Omega} + \|\nu^{-1/2} (\tau + q \,\mathbb{I} - \nu \,\nabla \,\tilde{v})\|_{0,\Omega} \|\nu^{1/2} \,\nabla \,\phi\|_{0,\Omega}$$

$$\leq \left(\nu_{-}^{-1/2} c_{fp}(\Omega, \Gamma_{D}) \| \operatorname{Div} \tau + F \|_{1,\Omega} + \|\nu^{-1/2} (\tau + q \,\mathbb{I} - \nu \,\nabla \,\tilde{v})\|_{0,\Omega} \right) \|\nu^{1/2} \,\nabla \,\phi\|_{0,\Omega}$$

Choosing $\phi_v := v - \tilde{v} = \hat{v} + v_D - \tilde{v} \in S_{-1,\Gamma_D}(\Omega)$ shows a first a posteriori estimate:

Theorem 4.5 (a posteriori error estimate I for exterior domains). Let $\tilde{v} \in v_D + S_{-1,\Gamma_D}(\Omega)$. Then for all $\tau \in \tilde{\mathsf{D}}_{\Gamma_N}(\Omega)$ and all $q \in \mathsf{L}^2(\Omega)$ it holds

$$\|\nu^{1/2} \nabla (v - \tilde{v})\|_{0,\Omega} \le \nu_{-}^{-1/2} c_{fp}(\Omega, \Gamma_D) \|\operatorname{Div} \tau + F\|_{1,\Omega} + \|\nu^{-1/2} (\tau + q \mathbb{I} - \nu \nabla \tilde{v})\|_{0,\Omega}.$$

The upper bound coincides with the norm of the error on the left hand side, if $\tau = \sigma$ (i.e., τ coincides with the exact stress tensor) and q = p (i.e., q represents the exact pressure p), i.e., we have

$$\|\nu^{1/2} \nabla(v - \tilde{v})\|_{0,\Omega} = \min_{\substack{\tau \in \tilde{\mathsf{D}}_{\Gamma_N}(\Omega), \\ q \in \mathsf{L}^2(\Omega)}} \left(\nu_{-}^{-1/2} c_{fp}(\Omega, \Gamma_D) \|\operatorname{Div} \tau + F\|_{1,\Omega} + \|\nu^{-1/2} (\tau + q \,\mathbb{I} - \nu \,\nabla \,\tilde{v})\|_{0,\Omega} \right)$$

and the minimum is attained at $(\tau, q) = (\sigma, p)$. However, Theorem 4.5 has a drawback: The estimate is valid only for those approximate vector fields \tilde{v} , which exactly satisfy the solenoidal condition and the boundary condition. In practice, the solenoidal requirement is difficult to fulfill and approximations arising in 'real life' computations often satisfy the solenoidal condition only approximately. Therefore, our next goal is to extend the estimate to a wider class of non-solenoidal vector fields.

4.3.2. A Posteriori Estimates for the Velocity Field: Non-Solenoidal Approximations. Now we assume only

$$\tilde{T} = \nabla \, \tilde{v}, \qquad \tilde{v} \in v_D + \mathsf{H}^1_{-1,\Gamma_D}(\Omega) \subset \mathsf{H}^1_{-1}(\Omega),$$

i.e., $\tilde{v} - v_D \in \mathsf{H}^1_{-1,\Gamma_D}(\Omega)$, this is \tilde{v} is not solenoidal but satisfies the boundary condition exactly. Utilizing the stability lemma, Lemma 3.3, there exists $w \in \mathsf{H}^1_{-1,\Gamma_D}(\Omega)$ such that div $w = -\operatorname{div} \tilde{v}$ and $\|\nabla w\|_{0,\Omega} \leq \kappa(\Omega,\Gamma_D)\|\operatorname{div} \tilde{v}\|_{0,\Omega}$. Then $\tilde{v}_0 := \tilde{v} + w \in v_D + \mathsf{S}_{-1,\Gamma_D}(\Omega)$ and by Theorem 4.5

Theorem 4.6 (a posteriori error estimate II for exterior domains). Let $\tilde{v} \in v_D + \mathsf{H}^1_{-1,\Gamma_D}(\Omega)$. Then for all $\tau \in \tilde{\mathsf{D}}_{\Gamma_N}(\Omega)$ and all $q \in \mathsf{L}^2(\Omega)$ it holds

$$\|\nu^{1/2} \nabla (v - \tilde{v})\|_{0,\Omega} \le \nu_{-}^{-1/2} c_{fp}(\Omega, \Gamma_D) \|\operatorname{Div} \tau + F\|_{1,\Omega} + \|\nu^{-1/2} (\tau + q \mathbb{I} - \nu \nabla \tilde{v})\|_{0,\Omega} + 2\nu_{+}^{1/2} \kappa(\Omega, \Gamma_D) \|\operatorname{div} \tilde{v}\|_{0,\Omega}.$$

If the approximation \tilde{v} is solenoidal we get back Theorem 4.5 and, again, the upper bound coincides with the norm of the error on the left hand side if $\tau = \sigma$, q = p. If the approximation \tilde{v} is solenoidal just in, e.g., $\mathbb{R}^d \setminus \overline{B}_{r_2}$ then we get trivially an estimate by Theorem 4.6, replacing the term $\| \operatorname{div} \tilde{v} \|_{0,\Omega}$ by $\| \operatorname{div} \tilde{v} \|_{0,\omega}$. But with a moderate additional assumption on the decay of the approximation we can even do better in this case, replacing the stability constant $\kappa(\Omega, \Gamma_D)$ by a stability constant of the bounded domain ω . For this let $\tilde{v} = v_D + w \in v_D + \operatorname{H}^1_{-1,\Gamma_D}(\Omega)$ with div $\tilde{v} = \operatorname{div} w = 0$ in $\mathbb{R}^d \setminus \overline{B}_{r_2}$ and additionally, if $\gamma_D = \gamma$, i.e., $\Gamma_D = \Gamma$,

(14)
$$|w| \le c r^{-m}, \quad m > d-1$$

for $r \to \infty$ with some c > 0 independent of r. Note that for $r^{-m} \in L^2_{-1}(\mathbb{R}^d \setminus \overline{B}_1)$ it is sufficient that m > d/2 - 1. We consider the ansatz

$$\tilde{v}_0 := \tilde{v} + \begin{cases} u & \text{in } \omega, \\ 0 & \text{in } \mathbb{R}^d \setminus \overline{B}_{r_2}, \end{cases}$$

with $u \in \mathsf{H}^{1}_{\gamma_{D}}(\omega)$ and div $u = -\operatorname{div} \tilde{v}$ in ω . Utilizing Lemma 1.1 we find such a u together with the stability estimate $\|\nabla u\|_{0,\omega} \leq \kappa(\omega, \gamma_{D})\|\operatorname{div} \tilde{v}\|_{0,\omega}$, provided that in the case $\gamma_{D} = \gamma$, i.e., $\Gamma_{D} = \Gamma$, additionally div $\tilde{v} \in \mathsf{L}^{2}_{\perp}(\omega)$ holds. For this we notice (for $\Gamma_{D} = \Gamma$) that for any $r > r_{2}$

$$\left|\int_{\omega} \operatorname{div} \tilde{v}\right| = \left|\int_{\omega} \operatorname{div} w\right| = \left|\int_{S_r} n \cdot w\right| \le c r^{d-1-m} \xrightarrow{r \to \infty} 0.$$

Therefore, $\tilde{v}_0 \in v_D + \mathsf{S}_{-1,\Gamma_D}(\Omega)$ is an admissible vector field for (13) showing

$$\begin{aligned} \|\nu^{1/2} \,\nabla(v - \tilde{v})\|_{0,\Omega} &\leq \nu_{-}^{-1/2} c_{fp}(\Omega, \Gamma_D) \|\operatorname{Div} \tau + F\|_{1,\Omega} + \|\nu^{-1/2} (\tau + q \,\mathbb{I} - \nu \,\nabla \,\tilde{v})\|_{0,\Omega} \\ &+ 2\nu_{+}^{1/2} \underbrace{\|\nabla(\tilde{v}_0 - \tilde{v})\|_{0,\Omega}}_{=\|\nabla \,u\|_{0,\omega}}. \end{aligned}$$

Hence we have the following:

Corollary 4.7 (a posteriori error estimate III for exterior domains). Let $\tilde{v} \in v_D + \mathsf{H}^1_{-1,\Gamma_D}(\Omega)$ with div $\tilde{v} = 0$ in $\mathbb{R}^d \setminus \overline{B}_{r_2}$ and, if $\Gamma_D = \Gamma$, (14). Then for all $\tau \in \tilde{\mathsf{D}}_{\Gamma_N}(\Omega)$ and all $q \in \mathsf{L}^2(\Omega)$

$$\|\nu^{1/2} \nabla(\nu - \tilde{v})\|_{0,\Omega} \le \nu_{-}^{-1/2} c_{fp}(\Omega, \Gamma_D) \|\operatorname{Div} \tau + F\|_{1,\Omega} + \|\nu^{-1/2} (\tau + q \mathbb{I} - \nu \nabla \tilde{v})\|_{0,\Omega} + 2\nu_{+}^{1/2} \kappa(\omega, \gamma_D) \|\operatorname{div} \tilde{v}\|_{0,\omega}.$$

Here the last term on the right hand side is a penalty for possible violation of the solenoidal condition in ω .

4.3.3. A Posteriori Estimates for the Pressure Function. By Lemma 3.3 there exists a vector field $\phi_p \in \mathsf{H}^1_{-1,\Gamma_D}(\Omega)$ with div $\phi_p = p - \tilde{p}$ and $\|\nabla \phi_p\|_{0,\Omega} \leq \kappa(\Omega,\Gamma_D) \|p - \tilde{p}\|_{0,\Omega}$. (11) implies for all $\psi \in v_D + \mathsf{H}^1_{-1,\Gamma_D}(\Omega)$ and all $\tau \in \tilde{\mathsf{D}}_{\Gamma_N}(\Omega)$

$$\begin{split} \|p - \tilde{p}\|_{0,\Omega}^2 &= \langle p - \tilde{p}, \operatorname{div} \phi_p \rangle_{0,\Omega} \\ &= \langle \nu \, \nabla(v - \psi), \nabla \, \phi_p \rangle_{0,\Omega} - \langle \operatorname{Div} \tau + F, \phi_p \rangle_{0,\Omega} + \langle \nu \, \nabla \, \psi - \tilde{p} \, \mathbb{I} - \tau, \nabla \, \phi_p \rangle_{0,\Omega} \\ &\leq \left(\|\nu \, \nabla(v - \psi)\|_{0,\Omega} + c_{fp}(\Omega, \Gamma_D) \| \operatorname{Div} \tau + F\|_{1,\Omega} + \|\nu \, \nabla \, \psi - \tilde{p} \, \mathbb{I} - \tau\|_{0,\Omega} \right) \| \, \nabla \, \phi_p \|_{0,\Omega}, \end{split}$$

where we have used Lemma 2.1 for ϕ_p and the equation div $\phi_p = \mathbb{I} : \nabla \phi_p$. Therefore, we obtain

$$\|p - \tilde{p}\|_{0,\Omega} \le \kappa(\Omega, \Gamma_D) \Big(\nu_+^{1/2} \|\nu^{1/2} \nabla (v - \psi)\|_{0,\Omega} + c_{fp}(\Omega, \Gamma_D) \|\operatorname{Div} \tau + F\|_{1,\Omega} \Big)$$

12

$$+ \nu_{+}^{1/2} \| \nu^{-1/2} (\tau + \tilde{p} \mathbb{I} - \nu \nabla \psi) \|_{0,\Omega} \Big)$$

and by Theorem 4.6 with $\tilde{v} := \psi$, $q := \tilde{p}$ we get:

Theorem 4.8 (a posteriori error estimate IV for exterior domains). Let $\tilde{p} \in L^2(\Omega)$. Then for all $\tau \in \tilde{D}_{\Gamma_N}(\Omega)$ and all $\psi \in v_D + H^1_{-1,\Gamma_D}(\Omega)$ it holds

$$\begin{split} \|p - \tilde{p}\|_{0,\Omega} &\leq \kappa(\Omega, \Gamma_D) \Big((\nu_{-}^{-1/2} \nu_{+}^{1/2} + 1) c_{fp}(\Omega, \Gamma_D) \| \operatorname{Div} \tau + F \|_{1,\Omega} \\ &+ 2\nu_{+}^{1/2} \| \nu^{-1/2} (\tau + \tilde{p} \,\mathbb{I} - \nu \,\nabla \,\psi) \|_{0,\Omega} + 2\nu_{+} \kappa(\Omega, \Gamma_D) \| \operatorname{div} \psi \|_{0,\Omega} \Big). \end{split}$$

The upper bound consists of the same terms as the upper bound of Theorem 4.6 and vanishes if $\psi = v$, $\tau = \sigma$, $\tilde{p} = p$. However, in this case, the quantity (error measure) on the left hand side depends, e.g., on the stability constant $\kappa(\Omega, \Gamma_D)$.

4.3.4. A Posteriori Estimates for Non-Conforming Approximations. Let us now assume that we have a very non-conforming approximation of the strain tensor field

$$T := \nabla v, \qquad v = v_D + \hat{v} \in v_D + \mathsf{S}_{-1,\Gamma_D}(\Omega) \subset \mathsf{S}_{-1}(\Omega),$$

given just by some $\tilde{T} \in L^2(\Omega)$. An example could be a broken gradient tensor field as output of some discontinuous Galerkin method. By the triangle inequality, i.e.,

$$\|\nu^{1/2}(T-\tilde{T})\|_{0,\Omega} \le \|\nu^{1/2} \nabla(v-\psi)\|_{0,\Omega} + \|\nu^{1/2}(\nabla\psi-\tilde{T})\|_{0,\Omega},$$

and Theorem 4.6 ($\tilde{v} = \psi$), and Theorem 4.8 (and again triangle inequality) we obtain the following result:

Theorem 4.9 (a posteriori error estimate V for exterior domains). Let $\tilde{T} \in L^2(\Omega)$ and $\tilde{p} \in L^2(\Omega)$. Then for all $\psi \in v_D + H^1_{-1,\Gamma_D}(\Omega)$, all $\tau \in \tilde{D}_{\Gamma_N}(\Omega)$, and all $q \in L^2(\Omega)$ it holds

$$\begin{split} \|\nu^{1/2}(T-\tilde{T})\|_{0,\Omega} &\leq \nu_{-}^{-1/2} c_{fp}(\Omega,\Gamma_{D}) \|\operatorname{Div}\tau + F\|_{1,\Omega} + \|\nu^{-1/2}(\tau+q\,\mathbb{I}-\nu\,\nabla\,\psi)\|_{0,\Omega} \\ &\quad + 2\nu_{+}^{1/2}\kappa(\Omega,\Gamma_{D}) \|\operatorname{div}\psi\|_{0,\Omega} + \|\nu^{1/2}(\nabla\,\psi-\tilde{T})\|_{0,\Omega} \\ &\leq \nu_{-}^{-1/2} c_{fp}(\Omega,\Gamma_{D}) \|\operatorname{Div}\tau + F\|_{1,\Omega} + \|\nu^{-1/2}(\tau+q\,\mathbb{I}-\nu\tilde{T})\|_{0,\Omega} \\ &\quad + 2\nu_{+}^{1/2}\kappa(\Omega,\Gamma_{D}) \|\operatorname{div}\psi\|_{0,\Omega} + 2\|\nu^{1/2}(\nabla\,\psi-\tilde{T})\|_{0,\Omega}, \\ \|p-\tilde{p}\|_{0,\Omega} &\leq \kappa(\Omega,\Gamma_{D}) \Big((\nu_{-}^{-1/2}\nu_{+}^{1/2}+1)c_{fp}(\Omega,\Gamma_{D}) \|\operatorname{Div}\tau + F\|_{1,\Omega} \\ &\quad + 2\nu_{+}^{1/2} \|\nu^{-1/2}(\tau+\tilde{p}\,\mathbb{I}-\nu\tilde{T})\|_{0,\Omega} + 2\nu_{+}\kappa(\Omega,\Gamma_{D}) \|\operatorname{div}\psi\|_{0,\Omega} \\ &\quad + 2\nu_{+}^{1/2} \|\nu^{1/2}(\nabla\,\psi-\tilde{T})\|_{0,\Omega} \Big). \end{split}$$

For $\tilde{T} = \nabla \tilde{v}$, $\psi = \tilde{v} \in v_D + \mathsf{H}^1_{-1,\Gamma_D}(\Omega)$ we get back Theorem 4.6 and Theorem 4.8. Let us investigate the latter summands a bit closer and identify them in terms of parts of the error. For this, we use the well known (row wise) Helmholtz decomposition, see (??) of the Appendix, and decompose^{vi} the error according to

$$T - \tilde{T} = \nabla w + \tilde{T}_0 \in \nabla \mathsf{H}^1_{-1,\Gamma_D}(\Omega) \oplus_{0,\nu} \nu^{-1}{}_0\mathsf{D}_{\Gamma_N}(\Omega).$$

Note that due to orthogonality

(15)
$$\left\|\nu^{1/2}(T-\tilde{T})\right\|_{0,\Omega}^2 = \left\|\nu^{1/2}\nabla w\right\|_{0,\Omega}^2 + \left\|\nu^{1/2}\tilde{T}_0\right\|_{0,\Omega}^2$$

and that $T - \nabla v_D = \nabla \hat{v} \in \nabla \mathsf{S}_{-1,\Gamma_D}(\Omega) \subset \nabla \mathsf{H}^1_{-1,\Gamma_D}(\Omega)$ already belongs to the first space. Hence the latter decomposition is actually a decomposition of $\nabla v_D - \tilde{T}$, more precisely $\nabla v_D - \tilde{T} = T - \nabla \hat{v} - \tilde{T} = \nabla (w - \hat{v}) + \tilde{T}_0$. For the second error part \tilde{T}_0 we observe for all $\phi \in \mathsf{H}^1_{-1,\Gamma_D}(\Omega)$ by orthogonality

$$\|\nu^{1/2}\tilde{T}_0\|_{0,\Omega}^2 = \langle T - \tilde{T}, \nu \tilde{T}_0 \rangle_{0,\Omega} = \langle \nabla(v_D + \phi) - \tilde{T}, \nu \tilde{T}_0 \rangle_{0,\Omega}$$

^{vi}Note that the decomposition is orthogonal with respect to the weighted $\langle \nu \cdot, \cdot \rangle_{0,\Omega}$ -inner product.

and thus $\|\nu^{1/2}\tilde{T}_0\|_{0,\Omega} \leq \|\nu^{1/2}(\nabla(v_D+\phi)-\tilde{T})\|_{0,\Omega}$. In other words,

(16)
$$\begin{aligned} \|\nu^{1/2}\tilde{T}_{0}\|_{0,\Omega} &= \min_{\phi \in \mathsf{H}^{1}_{-1,\Gamma_{D}}(\Omega)} \|\nu^{1/2}(\nabla(v_{D}+\phi)-\tilde{T})\|_{0,\Omega} \\ &= \min_{\psi \in v_{D}+\mathsf{H}^{1}_{-1,\Gamma_{D}}(\Omega)} \|\nu^{1/2}(\nabla\psi-\tilde{T})\|_{0,\Omega} \end{aligned}$$

and the minima are attained at

$$\hat{\phi} = \hat{v} - w \in \mathsf{H}^{1}_{-1,\Gamma_{D}}(\Omega), \qquad \hat{\psi} = v_{D} + \hat{\phi} = v - w \in v_{D} + \mathsf{H}^{1}_{-1,\Gamma_{D}}(\Omega),$$

as $\nabla \hat{\psi} - \tilde{T} = T - \nabla w - \tilde{T} = \tilde{T}_0$. Therefore, the minima of the last terms on the right hand sides in Theorem 4.9 equal the error part $\|\nu^{1/2}\tilde{T}_0\|_{0,\Omega}$.

4.3.5. A Posteriori Estimates for the Stress Tensor Field. Error estimates for the stress tensor field follow immediately from the above derived estimates for the velocity vector field and the pressure function. Indeed, let $\tilde{\sigma} \in L^2(\Omega)$ be an approximation of the exact stress tensor $\sigma = \nu \nabla v - p \mathbb{I} = \nu T - p \mathbb{I}$. Moreover, let $\tilde{T} \in L^2(\Omega)$ and $\tilde{p} \in L^2(\Omega)$. Then, the respective error is simply subject to the triangle inequality

$$\|\tilde{\sigma} - \sigma\|_{0,\Omega} \le \|\tilde{\sigma} - \nu \tilde{T} + \tilde{p}\mathbb{I}\|_{0,\Omega} + \nu_+^{1/2} \|\nu^{1/2}(T - \tilde{T})\|_{0,\Omega} + d^{1/2} \|p - \tilde{p}\|_{0,\Omega},$$

where we can also put $T = \nabla \tilde{v}$, $\tilde{v} \in v_D + \mathsf{H}^1_{-1,\Gamma_D}(\Omega)$. The first term on the right hand side contains only known tensor fields and the second and third ones are estimated by, e.g., Theorem 4.6, Theorem 4.8, and Theorem 4.9.

4.3.6. Lower Bounds for the Velocity Field. Let $\tilde{v} \in v_D + \mathsf{H}^1_{-1,\Gamma_D}(\Omega)$, i.e., $v - \tilde{v} \in \mathsf{H}^1_{-1,\Gamma_D}(\Omega)$. Obviously, (as the subsequent max-property holds for any Hilbert^{vii} space) we have by (11)

$$\begin{split} \|\nu^{1/2} \nabla(v-\tilde{v})\|_{0,\Omega}^2 &= \max_{\phi \in \mathsf{H}_{-1,\Gamma_D}^1(\Omega)} \left(2 \langle \nu \, \nabla(v-\tilde{v}), \nabla \, \phi \rangle_{0,\Omega} - \|\nu^{1/2} \, \nabla \, \phi\|_{0,\Omega}^2 \right) \\ &\geq 2 \langle \nu \, \nabla \, v, \nabla \, \phi \rangle_{0,\Omega} - 2 \langle \nu \, \nabla \, \tilde{v}, \nabla \, \phi \rangle_{0,\Omega} - \|\nu^{1/2} \, \nabla \, \phi\|_{0,\Omega}^2 \\ &= 2 \langle F, \phi \rangle_{0,\Omega} + 2 \langle q, \operatorname{div} \phi \rangle_{0,\Omega} - \langle \nu \, \nabla(2\tilde{v} + \phi), \nabla \, \phi \rangle_{0,\Omega} \\ &+ 2 \langle p - q, \operatorname{div} \phi \rangle_{0,\Omega} \end{split}$$

and the maximum is attained at $\phi = v - \tilde{v} \in H^1_{-1,\Gamma_D}(\Omega)$. The last term can simply and roughly be estimated by Theorem 4.8 ($\tilde{p} = q$) showing the following result.

Theorem 4.10 (a posteriori error estimate VI for exterior domains). Let $\tilde{v} \in v_D + \mathsf{H}^1_{-1,\Gamma_D}(\Omega)$. Then for all $\phi \in \mathsf{H}^1_{-1,\Gamma_D}(\Omega)$, all $\tau \in \tilde{\mathsf{D}}_{\Gamma_N}(\Omega)$, all $\psi \in v_D + \mathsf{H}^1_{-1,\Gamma_D}(\Omega)$, and all $q \in \mathsf{L}^2(\Omega)$

$$\begin{aligned} \|\nu^{1/2} \nabla(v-\tilde{v})\|_{0,\Omega}^{2} \\ &\geq 2\langle F, \phi \rangle_{0,\Omega} + 2\langle q, \operatorname{div} \phi \rangle_{0,\Omega} - \langle \nu \nabla(2\tilde{v}+\phi), \nabla \phi \rangle_{0,\Omega} \\ &\quad - 2\kappa(\Omega, \Gamma_{D}) \|\operatorname{div} \phi\|_{0,\Omega} \Big((\nu_{-}^{-1/2}\nu_{+}^{1/2}+1)c_{fp}(\Omega, \Gamma_{D}) \|\operatorname{Div} \tau + F\|_{1,\Omega} \\ &\quad + 2\nu_{+}^{1/2} \|\nu^{-1/2}(\tau+q \mathbb{I}-\nu \nabla \psi)\|_{0,\Omega} + 2\nu_{+}\kappa(\Omega, \Gamma_{D}) \|\operatorname{div} \psi\|_{0,\Omega} \Big). \end{aligned}$$

In particular, $\psi = \tilde{v}$ is possible.

For solenoidal ϕ , i.e., $\phi \in \mathsf{S}_{-1,\Gamma_D}(\Omega)$ we simply get

$$\|\nu^{1/2} \nabla (v - \tilde{v})\|_{0,\Omega}^2 \ge 2\langle F, \phi \rangle_{0,\Omega} - \left\langle \nu \nabla (2\tilde{v} + \phi), \nabla \phi \right\rangle_{0,\Omega}$$

and equality holds for $\phi = v - \tilde{v}$, provided that the approximation \tilde{v} is also solenoidal, i.e., $\tilde{v} \in v_D + S_{-1,\Gamma_D}(\Omega)$. To handle a very non-conforming approximation $\tilde{T} \in L^2(\Omega)$ we can simply utilize for all $\varphi \in v_D + H^1_{-1,\Gamma_D}(\Omega)$ the triangle inequality

$$\|\nu^{1/2}(\nabla v - \tilde{T})\|_{0,\Omega} \ge \|\nu^{1/2} \nabla (v - \varphi)\|_{0,\Omega} - \|\nu^{1/2}(\nabla \varphi - \tilde{T})\|_{0,\Omega}$$

in combination with Theorem 4.10 ($\tilde{v} = \varphi$). More precisely, we note the following result:

^{vii}In any Hilbert space H it holds $|x|^2 = \max_{y \in H} (2\langle x, y \rangle - |y|^2)$. Here $H = \nabla H^1_{-1, \Gamma_D}(\Omega)$.

Theorem 4.11 (a posteriori error estimate VII for exterior domains). Let $\tilde{T} \in L^2(\Omega)$. Then for all $\phi \in H^1_{-1,\Gamma_D}(\Omega)$, all $\tau \in \tilde{\mathsf{D}}_{\Gamma_N}(\Omega)$, all $\varphi, \psi \in v_D + \mathsf{H}^1_{-1,\Gamma_D}(\Omega)$, and all $q \in \mathsf{L}^2(\Omega)$

$$\begin{split} \|\nu^{1/2} (\nabla v - \tilde{T})\|_{0,\Omega} &\geq \|\nu^{1/2} \,\nabla (v - \varphi)\|_{0,\Omega} - \|\nu^{1/2} (\nabla \varphi - \tilde{T})\|_{0,\Omega}, \\ \|\nu^{1/2} \,\nabla (v - \varphi)\|_{0,\Omega}^2 &\geq 2\langle F, \phi \rangle_{0,\Omega} + 2\langle q, \operatorname{div} \phi \rangle_{0,\Omega} - \langle \nu \,\nabla (2\varphi + \phi), \nabla \phi \rangle_{0,\Omega} \\ &\quad - 2\kappa(\Omega, \Gamma_D) \|\operatorname{div} \phi\|_{0,\Omega} \Big((\nu_-^{-1/2} \nu_+^{1/2} + 1) c_{fp}(\Omega, \Gamma_D) \|\operatorname{Div} \tau + F\|_{1,\Omega} \\ &\quad + 2\nu_+^{1/2} \|\nu^{-1/2} (\tau + q \,\mathbb{I} - \nu \,\nabla \,\psi)\|_{0,\Omega} + 2\nu_+ \kappa(\Omega, \Gamma_D) \|\operatorname{div} \psi\|_{0,\Omega} \Big). \end{split}$$

4.4. Applications for 2D Exterior Domains. For a Lipschitz domain $\mathcal{D} \subset \mathbb{R}^2$ we introduce modified polynomially weighted spaces using logarithms by

$$\begin{split} \mathsf{L}^{2}_{\pm 1,\ln}(\mathcal{D}) &:= \big\{ \phi \in \mathsf{L}^{2}_{\mathsf{loc}}(\mathcal{D}) \,:\, \big(\rho \ln(e+\rho)\big)^{\pm 1} \phi \in \mathsf{L}^{2}(\mathcal{D}) \big\}, \qquad e : \text{Euler's number}, \\ \mathsf{H}^{1}_{-1,\ln}(\mathcal{D}) &:= \big\{ \phi \in \mathsf{L}^{2}_{-1,\ln}(\mathcal{D}) \,:\, \nabla \, \phi \in \mathsf{L}^{2}(\mathcal{D}) \big\}. \end{split}$$

Note that at infinity $(\rho \ln(e+\rho))^{\pm 1}$ behaves like $(r \ln r)^{\pm 1}$. The Inner product in $L^2_{\pm 1,\ln}(\mathcal{D})$ is defined and denoted by $\langle \cdot, \cdot \rangle_{\pm 1,\ln,\mathcal{D}} := \langle (\rho \ln(e+\rho))^{\pm 2} \cdot, \cdot \rangle_{0,\mathcal{D}}$. All other weighted spaces and norms etc. are modified and defined in the same way.

Let $\Omega \subset \mathbb{R}^2$ and $\omega \subset \mathbb{R}^2$ be defined as in Section 2, i.e., $\Omega \subset \mathbb{R}^2$ is an exterior Lipschitz domain. The situation is now different from the case $d \geq 3$ as the constants will be integrable in our weighted spaces. More precisely, for $0 < \epsilon < 1$

$$(r\ln r)^{-1} \in \mathsf{L}^{2}(B_{\epsilon}), \qquad (r\ln r)^{-1} \notin \mathsf{L}^{2}(B_{1+\epsilon} \setminus \overline{B}_{1-\epsilon}), (r\ln r)^{-1} \in \mathsf{L}^{2}(\mathbb{R}^{2} \setminus \overline{B}_{e}).$$

Introducing

$$\mathsf{H}^{1}_{-1,\ln,\emptyset}(\Omega) := \mathsf{H}^{1}_{-1,\ln}(\Omega) \cap \mathbb{R}^{\perp_{-1,\ln,\Omega}}$$

we have the following Friedrichs/Poincare estimate:

Lemma 4.12 (Friedrichs/Poincaré estimate for 2D exterior domains). There exists c > 0such that $||v||_{-1,\ln,\Omega} \leq c ||\nabla v||_{0,\Omega}$ for all $v \in H^1_{-1,\ln,\Gamma_D}(\Omega)$. The best constant c will be denoted by $c_{fp}(\Omega,\Gamma_D)$. In the special case $B_e \subset \mathbb{R}^2 \setminus \Omega$ and $\Gamma_D = \Gamma$ it holds $c_{fp}(\Omega,\Gamma_D) \leq 2$.

Note that we need boundary or mean value conditions as in the case of a bounded domain.

Proof. From [24, Appendix 4.2, Lemma 4.1, Corollary 4.2, Remark 4.3], see also [38, Lemma 4.1], we have for all $v \in \mathsf{C}^{\infty}_{\Gamma}(\Omega)$

$$\|v\|_{-1,\ln,\Omega} \le \|(r\ln r)^{-1}v\|_{0,\Omega} \le 2\|\nabla v\|_{0,\Omega},$$

provided that, e.g., $B_e \subset \mathbb{R}^2 \setminus \Omega$, which extends by density and continuity to $\mathsf{H}^1_{-1,\ln,\Gamma}(\Omega)$, i.e., all $v \in \mathsf{H}^1_{-1,\ln,\Gamma}(\Omega)$,

(17)
$$\|v\|_{-1,\ln\Omega} \le 2\|\nabla v\|_{0,\Omega}.$$

Let $v \in \mathsf{H}^{1}_{-1,\ln}(\Omega)$ and let us assume w.l.o.g. $r_{1} > e, r_{2} := r_{1} + 1$ and $\xi'_{\infty} \leq 1$. Then $\eta v \in \mathsf{H}^{1}_{-1,\ln,\Gamma}(\operatorname{supp} \eta)$ and $\|\eta v\|_{-1,\ln,\Omega} \leq 2\|\nabla(\eta v)\|_{0,\Omega}$ by (17) for $\Omega = \operatorname{supp} \eta$. Hence

$$||v||_{-1,\ln,\Omega} \le 2||\nabla v||_{0,\Omega} + 2||v\nabla \eta||_{0,\Omega} + ||(1-\eta)v||_{-1,\ln,\Omega}$$

$$\le 2||\nabla v||_{0,\Omega} + 2||v||_{0,\omega} + ||v||_{0,\omega},$$

showing for all $v \in \mathsf{H}^{1}_{-1,\ln}(\Omega)$

(18)
$$\|v\|_{-1,\ln,\Omega} \le 2\|\nabla v\|_{0,\Omega} + 3\|v\|_{0,\omega}.$$

Now, if the assertion of Lemma 4.12 is false, there is a sequence $(v_n) \subset \mathsf{H}^1_{-1,\ln,\Gamma_D}(\Omega)$ with $||v_n||_{-1,\ln,\Omega} = 1$ and $||\nabla v_n||_{0,\Omega} < 1/n$. Hence, (v_n) is bounded in $\mathsf{H}^1(\omega)$ as well. By Rellich's selection theorem we can assume w.l.o.g. that (v_n) already converges in $\mathsf{L}^2(\omega)$. Thus, by (18) (v_n) is a Cauchy sequence in $\mathsf{L}^2_{-1,\ln}(\Omega)$ and hence also in $\mathsf{H}^1_{-1,\ln,\Gamma_D}(\Omega)$. Therefore, (v_n)

converges in $\mathsf{H}^{1}_{-1,\ln,\Gamma_{D}}(\Omega)$ to some $v \in \mathsf{H}^{1}_{-1,\ln,\Gamma_{D}}(\Omega)$ with $\nabla v = 0$. We conclude that v is constant and hence v = 0, which implies a contradiction by $1 = ||v_{n}||_{-1,\ln,\Omega} \to 0$.

Now, all results from the sections for $d \ge 3$ follow with the obvious modifications, where we just present the most relevant ones.

Lemma 4.13 (stability lemma for 2D exterior domains). There exists c > 0 such that for all $f \in L^2(\Omega)$ there is a vector field $v \in H^1_{-1,\ln,\Gamma_D}(\Omega)$ with

$$\operatorname{div} v = f \quad and \quad \|\nabla v\|_{0,\Omega} \le c \|f\|_{0,\Omega}.$$

The best constant is denoted by $\kappa(\Omega, \Gamma_D)$ which equals the norm of the corresponding right inverse $f \mapsto v$. Moreover, with κ from (7)

$$\kappa(\Omega, \Gamma_D) \leq \hat{\kappa}(\Omega, \Gamma_D) := (1+\kappa) \big(1 + c_{fp}(\mathbb{R}^2) \frac{\xi'_{\infty} \rho(r_2) \ln \left(e + \rho(r_2)\right)}{r_2 - r_1} \big).$$

In particular, it holds $\hat{\kappa}(\Omega, \Gamma_D) \leq (1+\kappa)(1+c_{fp}(\mathbb{R}^2)\rho(r_2)\ln(e+\rho(r_2)))$ for $r_2 = r_1+1$ and $\xi'_{\infty} \leq 1$. If f has compact support in $\overline{\omega}$ and if additionally $\int_{\omega} f = \int_{\Omega} f = 0$ in the case $\Gamma_D = \Gamma$, then v can be chosen with compact support in $\overline{\omega}$, especially $v \in \mathsf{H}^1_{\gamma_D}(\omega) \subset \mathsf{H}^1_{\Gamma_D}(\Omega)$. In this case, $\kappa(\Omega, \Gamma_D) \leq \kappa(\omega, \gamma_D)$.

Corollary 4.14 (distance lemma for 2D exterior domains). For any $v \in H^1_{-1,\ln,\Gamma_D}(\Omega)$ there exists a solenoidal $v_0 \in S_{-1,\ln,\Gamma_D}(\Omega)$ such that

$$\operatorname{dist}\left(v,\mathsf{S}_{-1,\operatorname{ln},\Gamma_{D}}(\Omega)\right) = \inf_{\phi\in\mathsf{S}_{-1,\operatorname{ln},\Gamma_{D}}(\Omega)} \left\|\nabla(v-\phi)\right\|_{0,\Omega} \le \left\|\nabla(v-v_{0})\right\|_{0,\Omega} \le \kappa(\Omega,\Gamma_{D}) \|\operatorname{div} v\|_{0,\Omega}.$$

Corollary 4.15 (inhomogeneous distance lemma for 2D exterior domains). For any vector field $v \in \mathsf{H}^{1}_{-1,\ln}(\Omega)$ there exists a solenoidal $v_{0} \in \mathsf{S}_{-1,\ln}(\Omega)$ such that $v_{0} - v \in \mathsf{H}^{1}_{-1,\ln,\Gamma_{D}}(\Omega)$, i.e., $v_{0}|_{\Gamma_{D}} = v|_{\Gamma_{D}}$, and $\|\nabla(v_{0} - v)\|_{0,\Omega} \leq \kappa(\Omega,\Gamma_{D})\|\operatorname{div} v\|_{0,\Omega}$.

Corollary 4.16 (inf-sup lemma for 2D exterior domains). It holds

$$\inf_{f \in \mathsf{L}^{2}(\Omega)} \sup_{v \in \mathsf{H}^{1}_{-1, \ln, \Gamma_{D}}(\Omega)} \frac{\langle f, \operatorname{div} v \rangle_{0, \Omega}}{\|f\|_{0, \Omega} \|\nabla v\|_{0, \Omega}} \geq \frac{1}{\kappa(\Omega, \Gamma_{D})}.$$

Corollary 4.17 (Stokes lemma for 2D exterior domains). For ν , $F \in L^2_{1,\ln,\Gamma_N}(\Omega)$, and $v_D \in S_{-1,\ln}(\Omega)$ the 2D Stokes system is uniquely solvable with a solenoidal vector field $v = v_D + \tilde{v} \in v_D + S_{-1,\ln,\Gamma_D}(\Omega) \subset S_{-1,\ln}(\Omega)$ and $p \in L^2(\Omega)$. Moreover,

$$\nu \| \nabla \tilde{v} \|_{0,\Omega} \le c_{fp}(\Omega, \Gamma_D) \| F \|_{1,\ln,\Omega} + \nu \| \nabla v_D \|_{0,\Omega},$$

$$\nu \| \nabla v \|_{0,\Omega} \le c_{fp}(\Omega, \Gamma_D) \| F \|_{1,\ln,\Omega} + 2\nu \| \nabla v_D \|_{0,\Omega},$$

$$\|p\|_{0,\Omega} \le 2\kappa(\Omega,\Gamma_D) \big(c_{fp}(\Omega,\Gamma_D) \|F\|_{1,\ln,\Omega} + \nu \|\nabla v_D\|_{0,\Omega} \big).$$

Here we have introduced

$$\begin{aligned} \mathsf{L}^{2}_{1,\ln,\Gamma_{N}}(\Omega) &:= \begin{cases} \mathsf{L}^{2}_{1,\ln}(\Omega) &, \text{ if } \Gamma_{D} \neq \emptyset, \\ \mathsf{L}^{2}_{1,\ln,\perp}(\Omega) &, \text{ if } \Gamma_{D} = \emptyset, \end{cases} \\ \mathsf{L}^{2}_{1,\ln,\perp}(\Omega) &:= \mathsf{L}^{2}_{1,\ln}(\Omega) \cap (\mathbb{R}^{2})^{\perp_{0,\Omega}} = \left\{ \phi \in \mathsf{L}^{2}_{1,\ln}(\Omega) : \int_{\Omega} \phi_{i} = 0 \right\}. \end{aligned}$$

4.4.1. A Posteriori Error Estimates for Stationary Stokes Equations in 2D. We introduce

$$\tilde{\mathsf{D}}(\Omega) := \left\{ \Theta \in \mathsf{L}^2(\Omega) \, : \, \operatorname{Div} \Theta \in \mathsf{L}^2_{1,\ln}(\Omega) \right\}$$

and $\tilde{\mathsf{D}}_{\Gamma_N}(\Omega)$ as closure of $\mathsf{C}^{\infty}_{\Gamma_N}(\Omega)$ -tensor fields in the norm of $\tilde{\mathsf{D}}(\Omega)$.

For the approximation of the velocity field we have the following result:

Theorem 4.18 (a posteriori error estimate I for 2D exterior domains). Let the approximation \tilde{v} belong to $v_D + \mathsf{H}^1_{-1,\ln,\Gamma_D}(\Omega)$. Then for all $\tau \in \tilde{\mathsf{D}}_{\Gamma_N}(\Omega)$ and all $q \in \mathsf{L}^2(\Omega)$

$$\begin{aligned} \|\nu^{1/2} \,\nabla(v - \tilde{v})\|_{0,\Omega} &\leq \nu_{-}^{-1/2} c_{fp}(\Omega, \Gamma_D) \|\operatorname{Div} \tau + F\|_{1,\ln,\Omega} + \left\|\nu^{-1/2} (\tau + q \,\mathbb{I} - \nu \,\nabla \,\tilde{v})\right\|_{0,\Omega} \\ &+ 2\nu_{+}^{1/2} \kappa(\Omega, \Gamma_D) \|\operatorname{div} \tilde{v}\|_{0,\Omega}. \end{aligned}$$

If additionally div $\tilde{v} = 0$ in $\mathbb{R}^2 \setminus \overline{B}_{r_2}$ and, if $\Gamma_D = \Gamma$, (14) holds, then $\kappa(\Omega, \Gamma_D)$ can be replaced by $\kappa(\omega, \gamma_D)$ in Theorem 4.18. If the approximation \tilde{v} is solenoidal, i.e., div $\tilde{v} = 0$ in Ω , the upper bound coincides with the norm of the error on the left hand side if $(\tau, q) = (\sigma, p)$. For the approximation of the pressure function we get:

Theorem 4.19 (a posteriori error estimate II for 2D exterior domains). Let $\tilde{p} \in L^2(\Omega)$. Then for all $\tau \in \tilde{\mathsf{D}}_{\Gamma_N}(\Omega)$ and all $\psi \in v_D + \mathsf{H}^1_{-1,\ln,\Gamma_D}(\Omega)$ it holds

$$\begin{split} \|p - \tilde{p}\|_{0,\Omega} &\leq \kappa(\Omega, \Gamma_D) \Big((\nu_{-}^{-1/2} \nu_{+}^{1/2} + 1) c_{fp}(\Omega, \Gamma_D) \| \operatorname{Div} \tau + F\|_{1,\ln,\Omega} \\ &+ 2\nu_{+}^{1/2} \|\nu^{-1/2} (\tau + \tilde{p} \,\mathbb{I} - \nu \,\nabla \,\psi) \|_{0,\Omega} + 2\nu_{+} \kappa(\Omega, \Gamma_D) \| \operatorname{div} \psi \|_{0,\Omega} \Big). \end{split}$$

For non-conforming approximations of the velocity field we see:

Theorem 4.20 (a posteriori error estimate III for 2D exterior domains). Let $\tilde{T} \in L^2(\Omega)$ and $\tilde{p} \in L^2(\Omega)$. Then for all $\psi \in v_D + H^1_{-1,\ln,\Gamma_D}(\Omega)$, all $\tau \in \tilde{\mathsf{D}}_{\Gamma_N}(\Omega)$, and all $q \in \mathsf{L}^2(\Omega)$ it holds

$$\begin{split} \|\nu^{1/2}(T-\tilde{T})\|_{0,\Omega} &\leq \nu_{-}^{-1/2} c_{fp}(\Omega,\Gamma_{D}) \|\operatorname{Div}\tau + F\|_{1,\ln,\Omega} + \|\nu^{-1/2}(\tau+q\,\mathbb{I}-\nu\,\nabla\,\psi)\|_{0,\Omega} \\ &\quad + 2\nu_{+}^{1/2}\kappa(\Omega,\Gamma_{D}) \|\operatorname{div}\psi\|_{0,\Omega} + \|\nu^{1/2}(\nabla\,\psi-\tilde{T})\|_{0,\Omega} \\ &\leq \nu_{-}^{-1/2} c_{fp}(\Omega,\Gamma_{D}) \|\operatorname{Div}\tau + F\|_{1,\ln,\Omega} + \|\nu^{-1/2}(\tau+q\,\mathbb{I}-\nu\tilde{T})\|_{0,\Omega} \\ &\quad + 2\nu_{+}^{1/2}\kappa(\Omega,\Gamma_{D}) \|\operatorname{div}\psi\|_{0,\Omega} + 2\|\nu^{1/2}(\nabla\,\psi-\tilde{T})\|_{0,\Omega}, \\ &\quad \|p-\tilde{p}\|_{0,\Omega} \leq \kappa(\Omega,\Gamma_{D}) \Big((\nu_{-}^{-1/2}\nu_{+}^{1/2}+1)c_{fp}(\Omega,\Gamma_{D}) \|\operatorname{Div}\tau + F\|_{1,\ln,\Omega} \\ &\quad + 2\nu_{+}^{1/2} \|\nu^{-1/2}(\tau+\tilde{p}\,\mathbb{I}-\nu\tilde{T})\|_{0,\Omega} + 2\nu_{+}\kappa(\Omega,\Gamma_{D}) \|\operatorname{div}\psi\|_{0,\Omega} \\ &\quad + 2\nu_{+}^{1/2} \|\nu^{1/2}(\nabla\,\psi-\tilde{T})\|_{0,\Omega} \Big). \end{split}$$

For $\tilde{T} = \nabla \tilde{v}$, $\psi = \tilde{v} \in v_D + \mathsf{H}^1_{-1,\ln,\Gamma_D}(\Omega)$ we get back Theorem 4.18 and Theorem 4.19. Moreover, using the Helmholtz decomposition

$$T - \tilde{T} = \nabla w + \tilde{T}_0 \in \nabla \mathsf{H}^1_{-1,\ln,\Gamma_D}(\Omega) \oplus_{0,\nu} \nu^{-1}{}_0\mathsf{D}_{\Gamma_N}(\Omega),$$

we observe

$$\|\nu^{1/2}\tilde{T}_0\|_{0,\Omega} = \min_{\psi \in v_D + \mathsf{H}^1_{-1,\ln,\Gamma_D}(\Omega)} \|\nu^{1/2} (\nabla \psi - \tilde{T})\|_{0,\Omega}.$$

As before, error estimates for the stress tensor field σ follow immediately by the triangle inequality. For a lower bound we have the following result:

Theorem 4.21 (a posteriori error estimate IV for 2D exterior domains). Let the approximation \tilde{v} belong to $v_D + \mathsf{H}^1_{-1,\ln,\Gamma_D}(\Omega)$. Then for all $\phi \in \mathsf{H}^1_{-1,\ln,\Gamma_D}(\Omega)$, all $\tau \in \tilde{\mathsf{D}}_{\Gamma_N}(\Omega)$, all $\psi \in v_D + \mathsf{H}^1_{-1,\ln,\Gamma_D}(\Omega)$, and all $q \in \mathsf{L}^2(\Omega)$ it holds

$$\begin{split} \|\nu^{1/2} \nabla(v-\tilde{v})\|_{0,\Omega}^{2} \\ &\geq 2\langle F, \phi \rangle_{0,\Omega} + 2\langle q, \operatorname{div} \phi \rangle_{0,\Omega} - \langle \nu \nabla(2\tilde{v}+\phi), \nabla \phi \rangle_{0,\Omega} \\ &\quad - 2\kappa(\Omega, \Gamma_{D}) \|\operatorname{div} \phi\|_{0,\Omega} \Big((\nu_{-}^{-1/2}\nu_{+}^{1/2}+1)c_{fp}(\Omega, \Gamma_{D}) \|\operatorname{Div} \tau + F\|_{1,\ln,\Omega} \\ &\quad + 2\nu_{+}^{1/2} \|\nu^{-1/2}(\tau+q \mathbb{I}-\nu \nabla \psi)\|_{0,\Omega} + 2\nu_{+}\kappa(\Omega, \Gamma_{D}) \|\operatorname{div} \psi\|_{0,\Omega} \Big). \end{split}$$

In particular, $\psi = \tilde{v}$ is possible.

Again, for solenoidal ϕ , i.e., $\phi \in \mathsf{S}_{-1,\ln,\Gamma_D}(\Omega)$ we simply get

$$\|\nu^{1/2} \nabla(v - \tilde{v})\|_{0,\Omega}^2 \ge 2\langle F, \phi \rangle_{0,\Omega} - \left\langle \nu \nabla(2\tilde{v} + \phi), \nabla \phi \right\rangle_{0,\Omega}$$

and equality holds for $\phi = v - \tilde{v}$, provided that the approximation \tilde{v} is also solenoidal, i.e., $\tilde{v} \in v_D + S_{-1,\ln,\Gamma_D}(\Omega)$. Finally, to handle also very non-conforming approximations $\tilde{T} \in \mathsf{L}^2(\Omega)$ we can simply utilize for all $\varphi \in v_D + \mathsf{H}^1_{-1,\ln,\Gamma_D}(\Omega)$ the triangle inequality

$$\|\nu^{1/2}(\nabla v - \tilde{T})\|_{0,\Omega} \ge \|\nu^{1/2} \nabla (v - \varphi)\|_{0,\Omega} - \|\nu^{1/2} (\nabla \varphi - \tilde{T})\|_{0,\Omega}$$

in combination with Theorem 4.21 ($\tilde{v} = \varphi$).

References

- I. Babuška, A. K. Aziz, Survey Lectures on the Mathematical Foundations of the Finite Element Method. Academic Press, New York, 1972.
- [2] S. Bauer, P. Neff, D. Pauly, and G. Starke. New Poincaré-type inequalities. C. R. Math. Acad. Sci. Paris, 352(2):163–166, 2014.
- [3] S. Bauer, P. Neff, D. Pauly, and G. Starke. Dev-Div- and DevSym-DevCurl-Inequalities for Incompatible Square Tensor Fields with Mixed Boundary Conditions. ESAIM Control Optim. Calc. Var., 22:112–133, 2016.
- [4] S. Bauer, D. Pauly, and M. Schomburg. The Maxwell compactness property in bounded weak Lipschitz domains with mixed boundary conditions. SIAM J. Math. Anal., 48(4):2912–2943, 2016.
- [5] F. Brezzi, On the existence, uniqueness and approximation of saddle-point problems arising from Lagrange multipliers, R.A.I.R.O., Anal. Numer. R2(1974) 129–151.
- [6] M. Costabel. A remark on the regularity of solutions of Maxwell's equations on Lipschitz domains. Math. Methods Appl. Sci., 12(4):365–368, 1990.
- [7] M. Costabel and M. Dauge. On the inequalities of Babuskä–Aziz, Friedrichs and Horgan–Payne, arXiv 1303.6141v1, 2013.
- [8] G. Galdi. An introduction to the mathematical theory of the Navier–Stokes equations. Volume 1. Springer, New York, 1992
- [9] V. Girault and A. Sequeira. A well-posed problem for the exterior Stokes equations in two and three dimensions. Archive for Rational Mechanics and Analysis December 1991, Volume 114, Issue 4, pp 313–333.
- [10] C. Horgan and L. Payne. On inequalities of Korn, Friedrichs and BabuskaAziz. Arch. Ration. Mech. Anal., 82(1983), 165–179.
- [11] M. Kessler. Die Ladyzhenskaya-Konstante in der numerischen Behand- lung von Stromungsproblemen Dissertation zur Erlangung des natur- wissenschaftlichen. Doktorgrades der Bayerischen Julius-Maximilians- Universität Würzburg, 2000.
- [12] P. Kuhn and D. Pauly. Regularity results for generalized electro-magnetic problems. Analysis (Munich), 30(3):225–252, 2010.
- [13] O. A. Ladyzhenskaya, Mathematical Problems in the Dynamics of a Viscous Incompressible Flow, second ed., Gordon and Breach, New York, 1969.
- [14] O. A. Ladyzenskaja, V. A. Solonnikov, Some problems of vector analysis, and generalized formulations of boundary value problems for the Navier-Stokes equation, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov., 59 (1976), 81–116 (Russian).
- [15] R. Leis. Initial Boundary Value Problems in Mathematical Physics. Teubner, Stuttgart, 1986.
- [16] S. G. Mikhlin, Variational Methods in Mathematical Physics, Pergamon Press, Oxford, 1964.
- [17] S. G. Mikhlin, Constants in some inequalities of analysis, John Wiley & Sons, Ltd., Chichester, 1986.
- [18] J. Nečas, Les Méthodes Directes en Théorie des Équations Elliptiques, Masson et Cie, Éditeurs, Paris; Academia, Éditeurs, Prague 1967.
- [19] M.A. Olshanskii, E.V. Chizhonkov, On the best constant in the *inf sup* condition for prolonged rectangular domains, Mat. Zametki 67 (2000) 387–396 (Russian).
- [20] D. Pauly. Low frequency asymptotics for time-harmonic generalized Maxwell equations in nonsmooth exterior domains. Adv. Math. Sci. Appl., 16(2):591–622, 2006.
- [21] D. Pauly. Generalized electro-magneto statics in nonsmooth exterior domains. Analysis (Munich), 27(4):425–464, 2007.
- [22] D. Pauly. Hodge-Helmholtz decompositions of weighted Sobolev spaces in irregular exterior domains with inhomogeneous and anisotropic media. *Math. Methods Appl. Sci.*, 31:1509–1543, 2008.
- [23] D. Pauly. On polynomial and exponential decay of eigen-solutions to exterior boundary value problems for the generalized time-harmonic Maxwell system. Asymptot. Anal., 79(1-2):133–160, 2012.
- [24] D. Pauly and S. Repin. Functional a posteriori error estimates for elliptic problems in exterior domains. J. Math. Sci. (N.Y.), 162(3):393–406, 2009.
- [25] L. E. Payne, A bound for the optimal constant in an inequality of Ladyzhenskaya and Solonnikov, IMA J. Appl. Math. 72 (2007), 563–569.
- [26] R. Picard. Zur Theorie der harmonischen Differentialformen. Manuscripta Math., 1:31–45, 1979.
- [27] R. Picard. Randwertaufgaben der verallgemeinerten Potentialtheorie. Math. Methods Appl. Sci., 3:218– 228, 1981.
- [28] R. Picard. On the boundary value problems of electro- and magnetostatics. Proc. Roy. Soc. Edinburgh Sect. A, 92:165–174, 1982.
- [29] R. Picard. An elementary proof for a compact imbedding result in generalized electromagnetic theory. Math. Z., 187:151–164, 1984.

- [30] R. Picard. Some decomposition theorems and their applications to non-linear potential theory and Hodge theory. Math. Methods Appl. Sci., 12:35–53, 1990.
- [31] R. Picard, N. Weck, and K.-J. Witsch. Time-harmonic Maxwell equations in the exterior of perfectly conducting, irregular obstacles. *Analysis (Munich)*, 21:231–263, 2001.
- [32] S. Repin, A posteriori estimates for the Stokes problem, J. Math. Sci. (New York), 109 (2002) 1950– 1964.
- [33] S. Repin, Estimates of deviations from exact solutions for some boundary-value problems with incompressibility condition, St. Petersburg Math. J., 16 (2004) 124–161.
- [34] S. Repin, A Posteriori Estimates for Partial Differential Equations, Walter de Gruyter, Berlin, 2008.
- [35] S. Repin, Estimates of the distance to the set of divergence free fields, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov., 425 (2014) 99–116.
- [36] S. Repin, Estimates of the distance to the set of divergence free fields, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 425 (2014) 99–116, J. Math. Sci. (N.Y.) 210 (2015), no. 6, 822?834.
- [37] S. Repin, Estimates of the distance to the set of solenoidal vector fields and applications to a posteriori error control. (English summary) Comput. Methods Appl. Math. 15 (2015), no. 4, 515?530.
- [38] J. Saranen and K.-J. Witsch. Exterior boundary value problems for elliptic equations. Ann. Acad. Sci. Fenn. Math., 8(1):3–42, 1983.
- [39] G. Stoyan, Towards discrete Velte decompositions and narrow bounds for Inf-Sup constants, Comput. Math. Appl., 38 (1999) 243-261.
- [40] C. Weber. A local compactness theorem for Maxwell's equations. Math. Methods Appl. Sci., 2:12–25, 1980.
- [41] N. Weck. Maxwell's boundary value problems on Riemannian manifolds with nonsmooth boundaries. J. Math. Anal. Appl., 46:410–437, 1974.
- [42] K.-J. Witsch. A remark on a compactness result in electromagnetic theory. Math. Methods Appl. Sci., 16:123–129, 1993.

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