FAMILIES OF ANNIHILATING SKEW-SELFADJOINT OPERATORS AND THEIR CONNECTION TO HILBERT COMPLEXES

DIRK PAULY AND RAINER PICARD

Abstract. In this short note we show that Hilbert complexes are strongly related to what we shall call annihilating sets of skew-selfadjoint operators. This provides for a new perspective on the classical topic of Hilbert complexes viewed as families of commuting normal operators.

Contents

1. Introduction

The classical differential geometry topic of "chain complexes" has entered functional analysis as the topic of so-called "Hilbert complexes". The purpose of this note is to inspect Hilbert complexes from another functional analytical perspective linked to a four decades old construction of the skew-selfadjoint extended Maxwell operator

$$S_{\mathsf{Dir}} = \begin{pmatrix} 0 & \operatorname{div} & 0 & 0 \\ \mathring{\operatorname{grad}} & 0 & -\operatorname{curl} & 0 \\ 0 & \mathring{\operatorname{curl}} & 0 & \operatorname{grad} \\ 0 & 0 & \mathring{\operatorname{div}} & 0 \end{pmatrix},$$

see [13, 14]. For some pre-history and the scope of this construction see [15]. We merely mention here that the extended Maxwell system provides not only a deeper structural insight into the system of Maxwell's equations but also shows a deep connection to the Dirac equation. Indeed, the extended Maxwell operator has proven to be useful in important applications such as boundary integral equations in electrodynamics at low frequencies, see e.g. [5] and [16, 17, 18, 19].

Date: July 4, 2023; *Corresponding Author*: Dirk Pauly.

Key words and phrases. Hilbert complexes, de Rham complex, elasticity complex, biharmonic complex, factorisation of hyperbolic problems.

As it will turn out Hilbert complexes are intimately related to, indeed generalized by, an abstract concept, which we shall refer to as annihilating sets of skew-selfadjoint operators, which in turn is based on observations made in connection with the extended Maxwell system. This is the subject of the main part in Section 2. Our final section, Section 3 serves to illustrate the abstract setting by a number of more or less classical applications.

2. Annihilating Sets of Skew-Selfadjoint Operators and Hilbert Complexes

2.1. **Finite Sets of Annihilating Skew-Selfadjoint Operators.**

We start with particular finite sets of commuting skew-selfadjoint operators S on a Hilbert space H, i.e.,

$$S : \mathrm{dom}(S) \subset H \to H, \quad S^{*} = -S,$$

which we shall refer to as an (pair-wise) annihilating set of skew-selfadjoint operators.

Definition 1. *A finite set $\mathscr{S}$ of skew-selfadjoint operators satisfying*

$$\mathrm{ran}\,(S) \subseteq \ker\,(T)\,, \quad S \neq T,\ S, T \in \mathscr{S},$$

is called an annihilating set of skew-selfadjoint operators.

For the rest of this section, let $\mathscr{S}$ be an annihilating set of skew-selfadjoint operators.

Remark 2. *Let $S, T \in \mathscr{S}$.*

(1) *$\mathscr{S}$ is a set of commuting operators.*
(2) *It holds $TS = 0$ on $\mathrm{dom}\,(S)$ for $S \neq T$.*
(3) *The observation that $S = f_S\,(Q)$ for some suitable complex valued functions $f_S : \mathbb{R} \to \mathbb{C}$ in the sense of a function calculus associated with a single selfadjoint operator Q provides for other examples, which are not necessarily tridiagonal.*

As a consequence we have a straight-forward application of the projection theorem the following generalized (orthogonal) Helmholtz decomposition.

Theorem 3. $H = K \oplus_H \bigoplus_{S \in \mathscr{S}} \overline{\mathrm{ran}\,(S)}$ *with (generalised cohomology group)* $K \coloneqq \bigcap_{S \in \mathscr{S}} \ker\,(S)$.

2.2. **A Special Case: Tridiagonal Operator Matrices.**

We consider operator matrices of the form

$$A \coloneqq \sum_{k=1}^{N} A_k = \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 \\ a_1 & 0 & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & a_N & 0 \end{pmatrix}, \quad A_k \coloneqq \begin{pmatrix} 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & & & & \vdots \\ \vdots & \ddots & \ddots & 0 & & & & \vdots \\ \vdots & & 0 & 0 & 0 & & & \vdots \\ \vdots & & & a_k & 0 & 0 & & \vdots \\ \vdots & & & & 0 & \ddots & \ddots & \vdots \\ \vdots & & & & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 \end{pmatrix}$$

closed and densely defined on a Cartesian product $H = H_1 \times \cdots \times H_{N+1}$ of Hilbert spaces H_k. Then

$$A^* = \sum_{k=1}^{N} A_k^* = \begin{pmatrix} 0 & a_1^* & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & 0 & a_N^* \\ 0 & \cdots & \cdots & 0 & 0 \end{pmatrix}, \quad A_k^* = \begin{pmatrix} 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & & & & \vdots \\ \vdots & \ddots & \ddots & 0 & & & & \vdots \\ \vdots & & 0 & 0 & a_k^* & & & \vdots \\ \vdots & & & 0 & 0 & 0 & & \vdots \\ \vdots & & & & 0 & \ddots & \ddots & \vdots \\ \vdots & & & & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 \end{pmatrix}.$$

Now let

$$S := 2\,\mathrm{skw}\,A = A - A^* = \sum_{k=1}^{N} S_k, \qquad S_k := 2\,\mathrm{skw}\,A_k = A_k - A_k^*.$$

Then

$$S = \begin{pmatrix} 0 & -a_1^* & 0 & \cdots & 0 \\ a_1 & 0 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 & -a_N^* \\ 0 & \cdots & 0 & a_N & 0 \end{pmatrix}, \quad S_k = \begin{pmatrix} 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & & & & \vdots \\ \vdots & \ddots & \ddots & 0 & & & & \vdots \\ \vdots & & 0 & 0 & -a_k^* & & & \vdots \\ \vdots & & & a_k & 0 & 0 & & \vdots \\ \vdots & & & & 0 & \ddots & \ddots & \vdots \\ \vdots & & & & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 \end{pmatrix}$$

and we have the following main result.

Theorem 4. *$\mathscr{S} := \{S_1, \dots, S_N\}$ is an annihilating set of skew-selfadjoint operators if and only if $(a_1, \dots, a_N)$ is a Hilbert complex, i.e., for all $k = 1, \dots, N$*

$$a_k : \mathrm{dom}\,(a_k) \subseteq H_k \to H_{k+1}$$

are closed and densely defined linear operators satisfying $\mathrm{ran}\,(a_k) \subseteq \ker\,(a_{k+1})$, $k = 1, \dots, N-1$.

Proof. The result follows by a straightforward calculation. See Appendix A for more details. □

Remark 5. *As $(S_k S_\ell)^* \supset S_\ell^* S_k^* = S_\ell S_k$ we see that $\mathscr{S} = \{S_1, \dots, S_N\}$ is an annihilating set of skew-selfadjoint operators if and only if $S_k S_\ell = 0$ for all $1 \leq k < \ell \leq N$if and only if $S_\ell S_k = 0$ for all $1 \leq k < \ell \leq N$.*

Remark 6. *Often a Hilbert complex $(a) := (a_1, \dots, a_N)$ is written as*

$$H_1 \xrightarrow{a_1} \cdots \xrightarrow{a_{k-1}} H_k \xrightarrow{a_k} H_{k+1} \xrightarrow{a_{k-1}} \cdots \xrightarrow{a_N} H_{N+1}.$$

It is noteworthy that the Hilbert complex (a) is equivalently turned into the property

$$\mathrm{ran}\,(A_k) \subseteq \ker\,(A_j)$$

or

$$\mathrm{ran}\,(S_k) \subseteq \ker\,(S_j)$$

for all $j, k = 1, \dots, N$, where the sequential character of Hilbert complexes seems to have disappeared. Theorem 4 suggests to consider annihilating sets of skew-selfadjoint operators as an appropriate generalization of Hilbert complexes.

Remark 7. *Note that, if preferred, the set $\mathscr{S}$ may be considered as*

(1) *a set of homomorphisms by restriction of the elements to their respective domains, i.e.,*

$$\mathscr{S}_{\mathsf{hom}} := \left\{ \widetilde{S}_1, \dots, \widetilde{S}_N \right\},$$

where $\widetilde{S}_k := S_k \iota_{\mathrm{dom}(S_k)} : \mathrm{dom}(S_k) \to H$ *are now bounded linear operators,*

(2) *a set of bounded isomorphisms by restriction of the elements to their respective domains and orthogonal complements of their kernels (and projections onto the ranges), i.e.,*

$$\mathscr{S}_{\mathsf{iso}} := \left\{ \widehat{S}_1, \dots, \widehat{S}_N \right\},$$

where $\widehat{S}_k := \iota^*_{\mathrm{ran}(S_k)} S_k \iota_{\mathrm{dom}(S_k) \cap \ker(S_k)^{\perp_H}} : \mathrm{dom}\,(S_k) \cap \ker\,(S_k)^{\perp_H} \to \mathrm{ran}(S_k)$ *are now bounded and bijective,*

(3) *a set of topological isomorphisms* $\mathscr{S}_{\mathsf{iso}}$ *if all ranges* $\mathrm{ran}(S_k)$ *are closed. Note that in this case we have* $\mathrm{ran}(S_k) = \ker\,(S_k)^{\perp_H}$ *and that* $\mathrm{ran}(S_k)$ *is closed if and only if* $\mathrm{ran}(a_k)$ *is closed.*

In the latter remark ι_X denotes the embedding of the subspace X into H. If X is closed in H the orthonormal projector onto X is given by $\pi_X := \iota_X \iota_X^* : H \to H$.

Remark 8. *For consistency we set* $a_0 := 0$ *and* $a_{N+1} := 0$. *Note that*

$$\begin{aligned}\mathrm{dom}(S) &= \bigtimes_{k=1}^{N+1} \big(\mathrm{dom}(a_k) \cap \mathrm{dom}(a^*_{k-1})\big)\\ &= \mathrm{dom}(a_1) \times (\mathrm{dom}(a_2) \cap \mathrm{dom}(a^*_1)) \times \cdots \times \big(\mathrm{dom}(a_N) \cap \mathrm{dom}(a^*_{N-1})\big) \times \mathrm{dom}(a^*_N)\end{aligned}$$

and that by the complex property

$$\begin{aligned}\ker(S) &= \bigtimes_{k=1}^{N+1} \big(\ker(a_k) \cap \ker(a^*_{k-1})\big)\\ &= \ker(a_1) \times (\ker(a_2) \cap \ker(a^*_1)) \times \cdots \times \big(\ker(a_N) \cap \ker(a^*_{N-1})\big) \times \ker(a^*_N),\\ \mathrm{ran}(S) &= \bigtimes_{k=1}^{N+1} (\mathrm{ran}(a_{k-1}) \oplus_{H_k} \mathrm{ran}(a^*_k))\\ &= \mathrm{ran}(a^*_1) \times (\mathrm{ran}(a_1) \oplus_{H_2} \mathrm{ran}(a^*_2)) \times \cdots \times (\mathrm{ran}(a_{N-1}) \oplus_{H_N} \mathrm{ran}(a^*_N)) \times \mathrm{ran}(a_N).\end{aligned}$$

In particular, the product of the cohomology groups $K_k := \ker(a_k) \cap \ker(a^*_{k-1})$ *equals the kernel of* S.

Definition 9. *Recall Remark 6. A Hilbert complex* (a) *is called*

(1) *closed if all ranges* $\mathrm{ran}(a_k)$ *are closed.*
(2) *compact if all embeddings* $\mathrm{dom}(a_k) \cap \mathrm{dom}(a^*_{k-1}) \hookrightarrow H_k$ *are compact.*

Theorem 10. *Recall Theorem 4. Let* (a) *be a Hilbert complex with associated annihilating set of skew-selfadjoint operators* $\mathscr{S}$. *Then* (a) *is*

(1) *closed if and only if* $\mathrm{ran}(S)$ *is closed.*
(2) *compact if and only if the embedding* $\mathrm{dom}(S) \hookrightarrow H$ *is compact.*

Proof. Use Remark 8 and orthogonality. □

Theorem 11. *$S^2 = \sum_{k=1}^{N} S_k^2$ is diagonal and may be considered as generalised Laplacian acting on H. More precisely,*

$$-S^2 = \begin{pmatrix} a_1^*a_1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & a_1a_1^* + a_2^*a_2 & 0 & & & & \vdots \\ \vdots & 0 & \ddots & \ddots & & & \vdots \\ \vdots & & \ddots & a_{k-1}a_{k-1}^* + a_k^*a_k & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & 0 & \vdots \\ \vdots & & & & 0 & a_{N-1}a_{N-1}^* + a_N^*a_N & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & a_Na_N^* \end{pmatrix},$$

$$-S_k^2 = \begin{pmatrix} 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & & & & \vdots \\ \vdots & \ddots & 0 & 0 & & & & \vdots \\ \vdots & & 0 & a_k^*a_k & 0 & & & \vdots \\ \vdots & & & 0 & a_ka_k^* & 0 & & \vdots \\ \vdots & & & & 0 & 0 & \ddots & \vdots \\ \vdots & & & & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 \end{pmatrix}$$

Remark 12. *By replacing the skew-selfadjoint operators with selfadjoint operators the presented theory works literally as well. The only modifications are*

$$S := 2 \operatorname{sym} A = A + A^* = \sum_{k=1}^{N} S_k, \qquad S_k := 2 \operatorname{sym} A_k = A_k + A_k^*$$

resulting in

$$S = \begin{pmatrix} 0 & a_1^* & 0 & \cdots & 0 \\ a_1 & 0 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 & a_N^* \\ 0 & \cdots & 0 & a_N & 0 \end{pmatrix}, \qquad S_k = \begin{pmatrix} 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & & & & \vdots \\ \vdots & \ddots & \ddots & 0 & & & & \vdots \\ \vdots & & 0 & 0 & a_k^* & & & \vdots \\ \vdots & & & a_k & 0 & 0 & & \vdots \\ \vdots & & & & 0 & \ddots & \ddots & \vdots \\ \vdots & & & & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 \end{pmatrix}.$$

This is in a sense a matter of taste. We prefer, however, the skew-selfadjoint setting, since it has the advantage of being closer to various applications, such as Maxwell's and Dirac's equation (written in real form). We note, in particular, that skew-selfadjointness is at the heart of energy conservation.

3. Applications

In this final section we give several examples of annihilating sets of skew-selfadjoint operators, i.e., of Hilbert complexes, cf. Theorem 4. All operators will be considered as closures of unbounded linear operators densely defined on smooth and compactly supported test fields. For example, $\mathring{\operatorname{grad}}$,

$\operatorname{sym}\mathring{\operatorname{Curl}}_{\mathbb{T}}$, and $\operatorname{div}\mathring{\operatorname{Div}}_{\mathbb{S}}$ – where the tiny circle on top of an operator indicates the full Dirichlet boundary condition associated to the respective differential operator – are the closures of

$$\begin{aligned}
\mathring{\operatorname{grad}}^{\infty} &: \mathring{C}^{\infty}(\Omega) \subset L^2(\Omega) \to L^2(\Omega); && u \mapsto \operatorname{grad} u,\\
\operatorname{sym}\mathring{\operatorname{Curl}}_{\mathbb{T}}^{\infty} &: \mathring{C}_{\mathbb{T}}^{\infty}(\Omega) \subset L^2_{\mathbb{T}}(\Omega) \to L^2_{\mathbb{S}}(\Omega); && M \mapsto \operatorname{sym}\operatorname{Curl} M,\\
\operatorname{div}\mathring{\operatorname{Div}}_{\mathbb{S}}^{\infty} &: \mathring{C}_{\mathbb{S}}^{\infty}(\Omega) \subset L^2_{\mathbb{S}}(\Omega) \to L^2(\Omega); && M \mapsto \operatorname{div}\operatorname{Div} M,
\end{aligned}$$

where $\Omega \subset \mathbb{R}^3$ is an open set, $\mathring{C}^{\infty}(\Omega)$ denotes the space of smooth and compactly supported fields in Ω, and $\mathbb{S}$ and $\mathbb{T}$ indicate symmetric and deviatoric tensor fields, respectively. The corresponding adjoints $-\operatorname{div}$, $\operatorname{Curl}_{\mathbb{S}}$, and $\operatorname{Grad}\operatorname{grad}$ are then given by

$$\begin{aligned}
-\operatorname{div} &: H(\operatorname{div},\Omega) \subset L^2(\Omega) \to L^2(\Omega); && E \mapsto -\operatorname{div} E,\\
\operatorname{Curl}_{\mathbb{S}} &: H_{\mathbb{S}}(\operatorname{Curl},\Omega) \subset L^2_{\mathbb{S}}(\Omega) \to L^2_{\mathbb{T}}(\Omega); && M \mapsto \operatorname{Curl} M,\\
\operatorname{Grad}\operatorname{grad} &: H^2(\Omega) \subset L^2(\Omega) \to L^2_{\mathbb{S}}(\Omega); && u \mapsto \operatorname{Grad}\operatorname{grad} u.
\end{aligned}$$

3.1. **The Classical de Rham Complexes.**

3.1.1. *De Rham Complex of Vector Fields.* Let Ω be an open set in $\mathbb{R}^3$ with boundary $\Gamma := \partial\Omega$. The most prominent example is the classical de Rahm complex of vector fields involving the classical operators of vector calculus grad, curl, and div with full Dirichlet or Neumann boundary conditions:

$$S_{\mathsf{Dir}} = \begin{pmatrix} 0 & \operatorname{div} & 0 & 0\\ \mathring{\operatorname{grad}} & 0 & -\operatorname{curl} & 0\\ 0 & \mathring{\operatorname{curl}} & 0 & \operatorname{grad}\\ 0 & 0 & \mathring{\operatorname{div}} & 0 \end{pmatrix}, \qquad S_{\mathsf{Neu}} = \begin{pmatrix} 0 & \mathring{\operatorname{div}} & 0 & 0\\ \operatorname{grad} & 0 & -\mathring{\operatorname{curl}} & 0\\ 0 & \operatorname{curl} & 0 & \mathring{\operatorname{grad}}\\ 0 & 0 & \operatorname{div} & 0 \end{pmatrix}$$

Inhomogeneous and anisotropic coefficients and mixed boundary conditions can also be considered:

$$S_{\mathsf{mix}} = \begin{pmatrix} 0 & \nu^{-1}\mathring{\operatorname{div}}_{\Gamma_1}\varepsilon & 0 & 0\\ \mathring{\operatorname{grad}}_{\Gamma_0} & 0 & -\varepsilon^{-1}\mathring{\operatorname{curl}}_{\Gamma_1} & 0\\ 0 & \mu^{-1}\mathring{\operatorname{curl}}_{\Gamma_0} & 0 & \mathring{\operatorname{grad}}_{\Gamma_1}\\ 0 & 0 & \kappa^{-1}\mathring{\operatorname{div}}_{\Gamma_0}\mu & 0 \end{pmatrix} \tag{3.1}$$

Here the boundary Γ is decomposed into two parts Γ_0 and Γ_1 where the Dirichlet and Neumann boundary condition is imposed, respectively. Note that the de Rham off-diagonals are skew-adjoint to each other.

3.1.2. *De Rham Complex of Differential Forms.* Let Ω be an N-dinensional Riemannian manifold, e.g., an open set in $\mathbb{R}^N$. Another prominent example is the classical de Rahm complex of differential forms involving the exterior derivative d and its formal skew-adjoint the co-derivative $\delta = -\mathring{\mathrm{d}}^*$, $\mathrm{d} = -\mathring{\delta}^*$ with full Dirichlet or Neumann boundary conditions:

$$S_{\mathsf{Dir}} = \begin{pmatrix} 0 & \delta & 0 & \cdots & 0\\ \mathring{\mathrm{d}} & \ddots & \ddots & \ddots & \vdots\\ 0 & \ddots & \ddots & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & \delta\\ 0 & \cdots & 0 & \mathring{\mathrm{d}} & 0 \end{pmatrix}, \qquad S_{\mathsf{Neu}} = \begin{pmatrix} 0 & \mathring{\delta} & 0 & \cdots & 0\\ \mathrm{d} & \ddots & \ddots & \ddots & \vdots\\ 0 & \ddots & \ddots & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & \mathring{\delta}\\ 0 & \cdots & 0 & \mathrm{d} & 0 \end{pmatrix}$$

Again, the de Rham off-diagonals are skew-adjoint to each other. Note that S_{Dir} and S_{Neu} are unitarily congruent via transposition, permutation, sign change and Hodge-$*$-isomorphism.

3.2. **Other Complexes in Three Dimensions.**

There are plenty of extensions and restrictions of the de Rham complex. A nice overview and list of complexes is given in [1], from which we extract the following discussion. Their construction is based on the BGG-resolution using copies of the de Rham complex. For this section let Ω be an open set in $\mathbb{R}^3$.

3.2.1. *More De Rham Complexes.*

- Grad curl complex:

$$S_{\mathsf{Dir}} = \begin{pmatrix} 0 & \operatorname{div} & 0 & 0 & 0 \\ \mathring{\operatorname{grad}} & 0 & \operatorname{curl}\operatorname{Div}_{\mathbb{T}} & 0 & 0 \\ 0 & \mathring{\operatorname{Grad}\operatorname{curl}} & 0 & -\operatorname{dev}\operatorname{Curl} & 0 \\ 0 & 0 & \mathring{\operatorname{Curl}}_{\mathbb{T}} & 0 & \operatorname{Grad} \\ 0 & 0 & 0 & \mathring{\operatorname{Div}} & 0 \end{pmatrix}$$

- curl Div complex (formal dual of the Grad curl complex):

$$S_{\mathsf{Dir}} = \begin{pmatrix} 0 & \operatorname{Div} & 0 & 0 & 0 \\ \mathring{\operatorname{Grad}} & 0 & -\operatorname{Curl}_{\mathbb{T}} & 0 & 0 \\ 0 & \mathring{\operatorname{dev}\operatorname{Curl}} & 0 & \operatorname{Grad}\operatorname{curl} & 0 \\ 0 & 0 & \mathring{\operatorname{curl}\operatorname{Div}}_{\mathbb{T}} & 0 & \operatorname{grad} \\ 0 & 0 & 0 & \mathring{\operatorname{div}} & 0 \end{pmatrix}$$

- grad div complex (formally self-dual):

$$S_{\mathsf{Dir}} = \begin{pmatrix} 0 & \operatorname{div} & 0 & 0 & 0 & 0 \\ \mathring{\operatorname{grad}} & 0 & -\operatorname{curl} & 0 & 0 & 0 \\ 0 & \mathring{\operatorname{curl}} & 0 & -\operatorname{grad}\operatorname{div} & 0 & 0 \\ 0 & 0 & \mathring{\operatorname{grad}\operatorname{div}} & 0 & -\operatorname{curl} & 0 \\ 0 & 0 & 0 & \mathring{\operatorname{curl}} & 0 & \operatorname{grad} \\ 0 & 0 & 0 & 0 & \mathring{\operatorname{div}} & 0 \end{pmatrix}$$

3.2.2. *Elasticity Complexes.*

- Kröner complex (formally self-dual):

$$S_{\mathsf{Dir}} = \begin{pmatrix} 0 & \operatorname{Div}_{\mathbb{S}} & 0 & 0 \\ \mathring{\operatorname{sym}\operatorname{Grad}} & 0 & -\operatorname{Curl}\top\operatorname{Curl}_{\mathbb{S}} & 0 \\ 0 & \mathring{\operatorname{Curl}\top\operatorname{Curl}_{\mathbb{S}}} & 0 & \operatorname{sym}\operatorname{Grad} \\ 0 & 0 & \mathring{\operatorname{Div}}_{\mathbb{S}} & 0 \end{pmatrix}$$

  Here $\top$ denotes the formal transpose.

- deviatoric Kröner complex (formally self-dual):

$$S_{\mathsf{Dir}} = \begin{pmatrix} 0 & \operatorname{Div}_{\mathbb{ST}} & 0 & 0 \\ \mathring{\operatorname{dev}\operatorname{sym}\operatorname{Grad}} & 0 & -\operatorname{Curl}\widetilde{\top}\operatorname{Curl}\widetilde{\top}\operatorname{Curl}_{\mathbb{ST}} & 0 \\ 0 & \mathring{\operatorname{Curl}\widetilde{\top}\operatorname{Curl}\widetilde{\top}\operatorname{Curl}_{\mathbb{ST}}} & 0 & \operatorname{dev}\operatorname{sym}\operatorname{Grad} \\ 0 & 0 & \mathring{\operatorname{Div}}_{\mathbb{ST}} & 0 \end{pmatrix}$$

  Here[1] $\widetilde{\top}M := M^{\top} - \frac{1}{2}(\operatorname{tr}M)\operatorname{id}$. Note that $\widetilde{\top}\operatorname{Curl}_{\mathbb{S}} = \top\operatorname{Curl}_{\mathbb{S}}$ as $\operatorname{tr}\operatorname{Curl}_{\mathbb{S}} = 0$.

3.2.3. *Biharmonic Complexes.*

- first Hessian complex:

$$S_{\mathsf{Dir}} = \begin{pmatrix} 0 & -\operatorname{div}\operatorname{Div}_{\mathbb{S}} & 0 & 0 \\ \mathring{\operatorname{Grad}\operatorname{grad}} & 0 & -\operatorname{sym}\operatorname{Curl}_{\mathbb{T}} & 0 \\ 0 & \mathring{\operatorname{Curl}}_{\mathbb{S}} & 0 & \operatorname{dev}\operatorname{Grad} \\ 0 & 0 & \mathring{\operatorname{Div}}_{\mathbb{T}} & 0 \end{pmatrix}$$

- second Hessian complex (formal dual of the first Hessian complex):

$$S_{\mathsf{Dir}} = \begin{pmatrix} 0 & \operatorname{Div}_{\mathbb{T}} & 0 & 0 \\ \mathring{\operatorname{dev}\operatorname{Grad}} & 0 & -\operatorname{Curl}_{\mathbb{S}} & 0 \\ 0 & \mathring{\operatorname{sym}\operatorname{Curl}_{\mathbb{T}}} & 0 & -\operatorname{Grad}\operatorname{grad} \\ 0 & 0 & \mathring{\operatorname{div}\operatorname{Div}_{\mathbb{S}}} & 0 \end{pmatrix}$$

[1] In $\mathbb{R}^N$ we have $\widetilde{\top}M := M^{\top} - \frac{1}{N-1}(\operatorname{tr}M)\operatorname{id}$.

- conformal Hessian complex (formally self-dual):

$$S_{\mathsf{Dir}} = \begin{pmatrix} 0 & -\operatorname{div}\operatorname{Div}_{\mathbb{ST}} & 0 & 0 \\ \operatorname{dev}\mathring{\operatorname{Grad}}\operatorname{grad} & 0 & -\operatorname{sym}\operatorname{Curl}_{\mathbb{ST}} & 0 \\ 0 & \operatorname{sym}\mathring{\operatorname{Curl}}_{\mathbb{ST}} & 0 & -\operatorname{dev}\operatorname{Grad}\operatorname{grad} \\ 0 & 0 & \operatorname{div}\mathring{\operatorname{Div}}_{\mathbb{ST}} & 0 \end{pmatrix}$$

3.3. **Some Remarks.**

Remark 13. *Theorem 4 shows that all operators $S_{\dots} = S_{\mathsf{Dir/Neu/mix}}$ are sums of annihilating skew-selfadjoint operators $S_1, \dots, S_N$, i.e.,*

$$S_{\dots} = \sum_{k=1}^{N} S_k.$$

In particular, we have for the Dirichlet de Rham complex

$$\begin{aligned} S_{\mathsf{Dir}} &= \begin{pmatrix} 0 & \operatorname{div} & 0 & 0 \\ \mathring{\operatorname{grad}} & 0 & -\operatorname{curl} & 0 \\ 0 & \mathring{\operatorname{curl}} & 0 & \operatorname{grad} \\ 0 & 0 & \mathring{\operatorname{div}} & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \operatorname{div} & 0 & 0 \\ \mathring{\operatorname{grad}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\operatorname{curl} & 0 \\ 0 & \mathring{\operatorname{curl}} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \operatorname{grad} \\ 0 & 0 & \mathring{\operatorname{div}} & 0 \end{pmatrix}, \\ -S_{\mathsf{Dir}}^2 &= \begin{pmatrix} -\operatorname{div}\mathring{\operatorname{grad}} & 0 & 0 & 0 \\ 0 & -\mathring{\operatorname{grad}}\operatorname{div} + \operatorname{curl}\mathring{\operatorname{curl}} & 0 & 0 \\ 0 & 0 & \mathring{\operatorname{curl}}\operatorname{curl} - \operatorname{grad}\mathring{\operatorname{div}} & 0 \\ 0 & 0 & 0 & -\mathring{\operatorname{div}}\operatorname{grad} \end{pmatrix} \\ &= \begin{pmatrix} -\operatorname{div}\mathring{\operatorname{grad}} & 0 & 0 & 0 \\ 0 & -\mathring{\operatorname{grad}}\operatorname{div} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \operatorname{curl}\mathring{\operatorname{curl}} & 0 & 0 \\ 0 & 0 & \mathring{\operatorname{curl}}\operatorname{curl} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &\quad + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\operatorname{grad}\mathring{\operatorname{div}} & 0 \\ 0 & 0 & 0 & -\mathring{\operatorname{div}}\operatorname{grad} \end{pmatrix}. \end{aligned}$$

Remark 14. *Recalling Theorem 10 it has been shown in* [2, 3, 7] *that in case of the de Rham complexes the embeddings*

$$\operatorname{dom}(S_{\dots}) \hookrightarrow H$$

are compact, provided that (Ω, Γ_0) is a bounded weak Lipschitz pair, see [21, 20, 12, 22] *and* [6, 4] *for the first results about the respective compact embeddings. For corresponding results in case of elasticity and biharmonic complexes and bounded strong Lipschitz pairs (Ω, Γ_0) see* [8, 9] *and* [10, 11].

Remark 15. *There is no doubt that all the latter complexes may be generalised to inhomogeneous and anisotropic coefficients and to mixed boundary conditions, cf.* (3.1)*. Moreover, the techniques of* [7, 8, 9] *(for the de Rham, Kröner, and Hessian complexes) can be extended to show that all the embeddings $\operatorname{dom}(S_{\dots}) \hookrightarrow H$ are compact for bounded strong Lipschitz pairs (Ω, Γ_0).*

3.4. **A Factorization Result.** An interesting consequence for annihilating sets of skew-selfadjoint operators (not necessarily tridiagonal operator matrices as in the special case) is the following factorization result: Let T be a strictly m-accretive operator commuting with $\mathscr{S} = \{S_1, \dots, S_N\}$. Then with $S = \sum_{k=1}^{N} S_k$ we have

$$(T+S) = T^{1-N} T^{N-1} \Big(T + \sum_{k=1}^{N} S_k \Big) = T^{1-N} \prod_{k=1}^{N} (T + S_k)$$

and conversely,

$$T + S_\ell = T^{N-1} \prod_{\ell \neq k=1}^{N} (T+S_k)^{-1} (T+S) = \prod_{\ell \neq k=1}^{N} \big(1 + T^{-1} S_k\big)^{-1} (T+S) \,.$$

This is, in terms of inverses (solution operators)

$$(T+S)^{-1} = T^{N-1} \prod_{k=1}^{N} (T+S_k)^{-1} \,, \qquad (T+S_\ell)^{-1} = \prod_{\ell \neq k=1}^{N} \big(1 + T^{-1} S_k\big) (T+S)^{-1} \,.$$

An example of particular interest is given by $T = \partial_t$, the case of evolutionary systems, which in a suitable setting, see e.g. [15], leads to

$$(\partial_t + S) = \partial_t^{1-N} \prod_{k=1}^{N} (\partial_t + S_k), \qquad \partial_t + S_\ell = \prod_{\ell \neq k=1}^{N} (1 + \partial_t^{-1} S_k)^{-1} (\partial_t + S),$$

$$(\partial_t + S)^{-1} = \partial_t^{N-1} \prod_{k=1}^{N} (\partial_t + S_k)^{-1}, \qquad (\partial_t + S_\ell)^{-1} = \prod_{\ell \neq k=1}^{N} (1 + \partial_t^{-1} S_k)(\partial_t + S)^{-1}.$$

References

[1] D.N. Arnold and K. Hu. Complexes from complexes. *Found. Comput. Math.*, 21(6):1739–1774, 2021.

[2] S. Bauer, D. Pauly, and M. Schomburg. The Maxwell compactness property in bounded weak Lipschitz domains with mixed boundary conditions. *SIAM J. Math. Anal.*, 48(4):2912–2943, 2016.

[3] S. Bauer, D. Pauly, and M. Schomburg. Weck's selection theorem: The Maxwell compactness property for bounded weak Lipschitz domains with mixed boundary conditions in arbitrary dimensions. *Maxwell's Equations: Analysis and Numerics, Radon Ser. Comput. Appl. Math., De Gruyter*, 24:77–104, 2019.

[4] P. Fernandes and G. Gilardi. Magnetostatic and electrostatic problems in inhomogeneous anisotropic media with irregular boundary and mixed boundary conditions. *Math. Models Methods Appl. Sci.*, 7(7):957–991, 1997.

[5] R. Hiptmair and E. Schulz. First-kind boundary integral equations for the Dirac operator in 3-dimensional Lipschitz domains. *SIAM J. Math. Anal.*, 54(1):616–648, 2022.

[6] F. Jochmann. A compactness result for vector fields with divergence and curl in $L^q(\Omega)$ involving mixed boundary conditions. *Appl. Anal.*, 66:189–203, 1997.

[7] D. Pauly and M. Schomburg. Hilbert complexes with mixed boundary conditions – Part 1: De Rham complex. *Math. Methods Appl. Sci.*, 45(5):2465–2507, 2022.

[8] D. Pauly and M. Schomburg. Hilbert complexes with mixed boundary conditions – Part 2: Elasticity complex. *Math. Methods Appl. Sci.*, 45(16):8971–9005, 2022.

[9] D. Pauly and M. Schomburg. Hilbert complexes with mixed boundary conditions – Part 3: Biharmonic complex. *Math. Methods Appl. Sci.*, https://arxiv.org/abs/2207.11778, 2023.

[10] D. Pauly and W. Zulehner. The divDiv-complex and applications to biharmonic equations. *Appl. Anal.*, 99(9):1579–1630, 2020.

[11] D. Pauly and W. Zulehner. The elasticity complex: Compact embeddings and regular decompositions. *Appl. Anal.*, https://doi.org/10.1080/00036811.2022.2117497, 2022.

[12] R. Picard. An elementary proof for a compact imbedding result in generalized electromagnetic theory. *Math. Z.*, 187:151–164, 1984.

[13] R. Picard. On the low frequency asymptotics in electromagnetic theory. *J. Reine Angew. Math.*, 354:50–73, 1984.

[14] R. Picard. On a structural observation in generalized electromagnetic theory. *J. Math. Anal. Appl.*, 110:247–264, 1985.

[15] R. Picard, S. Trostorff, and M. Waurick. On a connection between the Maxwell system, the extended Maxwell system, the Dirac operator and gravito-electromagnetism. *Math. Methods Appl. Sci.*, 40(2):415–434, 2017.

[16] M. Taskinen and S. Vänskä. Current and charge integral equation formulations and Picard's extended Maxwell system. *IEEE Trans. Antennas and Propagation*, 55(12):3495–3503, 2007.

[17] M. Taskinen and S. Vänskä. Picard's extended maxwell system and frequency stable surface integral equations. *2007 Computational Electromagnetics Workshop*, pages 49–53, 2007.
[18] M. Taskinen and S. Vänskä. Surface integral equations of the picard's extended maxwell system. *IEEE Antennas and Propagation Society, AP-S International Symposium (Digest)*, 06 2007.
[19] M. Taskinen, S. Vänskä, and P. Yla-Oijala. Frequency domain surface integral equations of picard's extended maxwell system. *IET Seminar Digest*, 2007:1–6, 12 2007.
[20] C. Weber. A local compactness theorem for Maxwell's equations. *Math. Methods Appl. Sci.*, 2:12–25, 1980.
[21] N. Weck. Maxwell's boundary value problems on Riemannian manifolds with nonsmooth boundaries. *J. Math. Anal. Appl.*, 46:410–437, 1974.
[22] K.-J. Witsch. A remark on a compactness result in electromagnetic theory. *Math. Methods Appl. Sci.*, 16:123–129, 1993.

Appendix A. Sketch of a Proof of Theorem 4

For simplicity and readability we look at the special case $N = 3$ and consider (a_1, a_2, a_3). Then

$$S_1S_1 = \begin{pmatrix} 0 & -a_1^* & 0 & 0 \\ a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -a_1^* & 0 & 0 \\ a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -a_1^*a_1 & 0 & 0 & 0 \\ 0 & -a_1a_1^* & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$S_1S_2 = \begin{pmatrix} 0 & -a_1^* & 0 & 0 \\ a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -a_2^* & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a_1^*a_2^* & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$S_1S_3 = \begin{pmatrix} 0 & -a_1^* & 0 & 0 \\ a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_3^* \\ 0 & 0 & a_3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$S_2S_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -a_2^* & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -a_1^* & 0 & 0 \\ a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_2a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$S_2S_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -a_2^* & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -a_2^* & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -a_2^*a_2 & 0 & 0 \\ 0 & 0 & -a_2a_2^* & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$S_2S_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -a_2^* & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_3^* \\ 0 & 0 & a_3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_2^*a_3^* \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$S_3S_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_3^* \\ 0 & 0 & a_3 & 0 \end{pmatrix} \begin{pmatrix} 0 & -a_1^* & 0 & 0 \\ a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$S_3S_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_3^* \\ 0 & 0 & a_3 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -a_2^* & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & a_3a_2 & 0 & 0 \end{pmatrix},$$

$$S_3S_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_3^* \\ 0 & 0 & a_3 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_3^* \\ 0 & 0 & a_3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -a_3^*a_3 & 0 \\ 0 & 0 & 0 & -a_3a_3^* \end{pmatrix}.$$

We read off that (a_1, a_2, a_3) is a Hilbert complex if and only if $S_k S_\ell = 0$ for all $k > \ell$.

Institut für Analysis, Technische Universität Dresden, Germany
Email address, Dirk Pauly: dirk.pauly@tu-dresden.de
Email address, Rainer Picard: rainer.picard@tu-dresden.de