On the Maxwell Constants in 3D

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Dedicated to Martin Costabel
on the occasion of his 65th birthday

Abstract

Using tools from functional analysis we show that for bounded and convex domains in three dimensions, the Maxwell constants are bounded from below and above by Friedrichs' and Poincaré's constants.
Key Words Maxwell inequality, Poincaré inequality, Friedrichs inequality, Maxwell's equations, Maxwell constant, second Maxwell eigenvalue, electro statics, magneto statics

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1 Introduction and Preliminaries

Throughout this paper, let us fix a bounded domain $\Omega \subset \mathbb{R}^3$ with boundary $\Gamma := \partial\Omega$, which is devided into two relatively open subsets $\Gamma_{\mathtt{t}}$ and its complement $\Gamma_{\mathtt{n}} := \Gamma \setminus \overline{\Gamma_{\mathtt{t}}}$. The letters $\mathtt{t}$ and $\mathtt{n}$ should remind on homogeneous tangential and normal boundary conditions.

It is well known that the Poincaré (or Friedrichs) inequality, i.e., for all $u \in \mathsf{H}^1_{\Gamma_{\mathtt{t}}}(\Omega)$

$$|u|_{\mathsf{L}^2(\Omega)} \leq c_{\mathtt{p},\Gamma_{\mathtt{t}},\varepsilon} |\nabla u|_{\mathsf{L}^2_\varepsilon(\Omega)}, \tag{1.1}$$

holds with some $c_{\mathtt{p},\Gamma_{\mathtt{t}},\varepsilon} > 0$, as long as Rellich's selection theorem is valid, i.e., the embedding

$$\mathsf{H}^1(\Omega) \hookrightarrow \mathsf{L}^2(\Omega) \tag{1.2}$$

is compact. Here, $\mathsf{L}^2(\Omega)$ and $\mathsf{H}^1(\Omega)$ denote the usual Lebesgue- and Sobolev (Hilbert) spaces, respectively. Moreover, $\varepsilon : \Omega \to \mathbb{R}^{3\times 3}$ denotes a symmetric and uniformly positive definite L^∞-matrix field. We introduce $\mathsf{L}^2_\varepsilon(\Omega)$ as $\mathsf{L}^2(\Omega)$ equipped with the weighted inner product $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathsf{L}^2_\varepsilon(\Omega)} := \langle \varepsilon\,\cdot\,,\,\cdot\,\rangle_{\mathsf{L}^2(\Omega)}$.[i] For $\Gamma_{\mathtt{t}} \neq \emptyset$ the Sobolev space $\mathsf{H}^1_{\Gamma_{\mathtt{t}}}(\Omega)$ is defined as the closure $\big($taken in $\mathsf{H}^1(\Omega)\big)$ of test functions

$$\mathsf{C}^\infty_{\Gamma_{\mathtt{t}}}(\Omega) := \{\varphi|_\Omega \,:\, \varphi \in \mathsf{C}^\infty(\mathbb{R}^3)\,,\, \operatorname{dist}(\operatorname{supp}\varphi, \Gamma_{\mathtt{t}}) > 0\}.$$

Otherwise we set $\mathsf{H}^1_\emptyset(\Omega) := \mathsf{H}^1(\Omega) \cap \mathbb{R}^\perp$. Let us assume that we have chosen the best constant in (1.1), this is

$$\frac{1}{c_{\mathtt{p},\Gamma_{\mathtt{t}},\varepsilon}} := \inf_{0 \neq u \in \mathsf{H}^1_{\Gamma_{\mathtt{t}}}(\Omega)} \frac{|\nabla u|_{\mathsf{L}^2_\varepsilon(\Omega)}}{|u|_{\mathsf{L}^2(\Omega)}}.$$

Analogously, it is also well known that the (let's call it) Maxwell inequality, i.e., for all $E \in \mathsf{R}_{\Gamma_{\mathtt{t}}}(\Omega) \cap \varepsilon^{-1}\mathsf{D}_{\Gamma_{\mathtt{n}}}(\Omega)$

$$|E - \pi_{\mathtt{DN}} E|_{\mathsf{L}^2_\varepsilon(\Omega)} \leq c_{\mathtt{m},\Gamma_{\mathtt{t}},\varepsilon}\big(|\operatorname{div}\varepsilon E|^2_{\mathsf{L}^2(\Omega)} + |\operatorname{rot} E|^2_{\mathsf{L}^2(\Omega)}\big)^{1/2}$$

or equivalently for all $E \in \mathsf{R}_{\Gamma_{\mathtt{t}}}(\Omega) \cap \varepsilon^{-1}\mathsf{D}_{\Gamma_{\mathtt{n}}}(\Omega) \cap \mathcal{H}_{\mathtt{DN},\varepsilon}(\Omega)^{\perp_\varepsilon}$

$$|E|_{\mathsf{L}^2_\varepsilon(\Omega)} \leq c_{\mathtt{m},\Gamma_{\mathtt{t}},\varepsilon}\big(|\operatorname{div}\varepsilon E|^2_{\mathsf{L}^2(\Omega)} + |\operatorname{rot} E|^2_{\mathsf{L}^2(\Omega)}\big)^{1/2}, \tag{1.3}$$

holds with some $c_{\mathtt{m},\Gamma_{\mathtt{t}},\varepsilon} > 0$, as long as the Maxwell selection theorem or the Maxwell compactness property is given, i.e., the embedding

$$\mathsf{R}_{\Gamma_{\mathtt{t}}}(\Omega) \cap \varepsilon^{-1}\mathsf{D}_{\Gamma_{\mathtt{n}}}(\Omega) \hookrightarrow \mathsf{L}^2(\Omega) \tag{1.4}$$

[i]Throughout this paper norms resp. scalar products will be denoted by $|\,\cdot\,|_{\mathsf{X}}$ resp. $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathsf{X}}$ if X is a normed space or a space featuring a scalar product.

is compact, see Appendix A.2.1 for details. Here, we introduce the Sobolev (Hilbert) spaces

$$\mathsf{R}(\Omega) := \{E \in \mathsf{L}^2(\Omega) : \operatorname{rot} E \in \mathsf{L}^2(\Omega)\}, \quad \mathsf{D}(\Omega) := \{E \in \mathsf{L}^2(\Omega) : \operatorname{div} E \in \mathsf{L}^2(\Omega)\}$$

in the distributional sense. As above, if $\Gamma_{\mathtt{t}} \neq \emptyset$, we define as closures $\big($taken in $\mathsf{R}(\Omega)$ resp. $\mathsf{D}(\Omega)\big)$ of test vector fields $\mathsf{C}^\infty_{\Gamma_{\mathtt{t}}}(\Omega)$ the Sobolev spaces $\mathsf{R}_{\Gamma_{\mathtt{t}}}(\Omega)$ and $\mathsf{D}_{\Gamma_{\mathtt{t}}}(\Omega)$ (and of course the same for $\Gamma_{\mathtt{n}}$). If $\Gamma_{\mathtt{t}} = \emptyset$ we set $\mathsf{R}_\emptyset(\Omega) := \mathsf{R}(\Omega)$ and $\mathsf{D}_\emptyset(\Omega) := \mathsf{D}(\Omega)$. Then, for $\Gamma_{\mathtt{t}} \neq \emptyset$ in $\mathsf{H}^1_{\Gamma_{\mathtt{t}}}(\Omega)$, $\mathsf{R}_{\Gamma_{\mathtt{t}}}(\Omega)$ and $\mathsf{D}_{\Gamma_{\mathtt{t}}}(\Omega)$ homogeneous scalar, tangential and normal traces at $\Gamma_{\mathtt{t}}$ are generalized, respectively. Moreover, we define the closed subspaces

$$\mathsf{R}_0(\Omega) := \{E \in \mathsf{L}^2(\Omega) : \operatorname{rot} E = 0\}, \quad \mathsf{D}_0(\Omega) := \{E \in \mathsf{L}^2(\Omega) : \operatorname{div} E = 0\}$$

as well as $\mathsf{R}_{\Gamma_{\mathtt{t}},0}(\Omega) := \mathsf{R}_{\Gamma_{\mathtt{t}}}(\Omega) \cap \mathsf{R}_0(\Omega)$ and $\mathsf{D}_{\Gamma_{\mathtt{t}},0}(\Omega) := \mathsf{D}_{\Gamma_{\mathtt{t}}}(\Omega) \cap \mathsf{D}_0(\Omega)$. Finally, we have the harmonic Dirichlet-Neumann fields

$$\mathcal{H}_{\mathtt{DN},\varepsilon}(\Omega) := \mathsf{R}_{\Gamma_{\mathtt{t}},0}(\Omega) \cap \varepsilon^{-1}\mathsf{D}_{\Gamma_{\mathtt{n}},0}(\Omega),$$

which are finite dimensional since by (1.4) the unit ball is compact in $\mathcal{H}_{\mathtt{DN},\varepsilon}(\Omega)$. The $\mathsf{L}^2_\varepsilon(\Omega)$-orthogonal projector onto them will be denoted by $\pi_{\mathtt{DN}} : \mathsf{L}^2_\varepsilon(\Omega) \to \mathcal{H}_{\mathtt{DN},\varepsilon}(\Omega)$ and $\perp_\varepsilon$ means orthogonality in $\mathsf{L}^2_\varepsilon(\Omega)$. If $\Gamma_{\mathtt{t}} = \Gamma$ resp. $\Gamma_{\mathtt{n}} = \Gamma$ we have the classical Dirichlet resp. Neumann fields and write $\mathcal{H}_{\mathtt{D},\varepsilon}(\Omega)$ resp. $\mathcal{H}_{\mathtt{N},\varepsilon}(\Omega)$. We also need the Neumann-Dirichlet fields $\mathcal{H}_{\mathtt{ND},\varepsilon}(\Omega) := \mathsf{R}_{\Gamma_{\mathtt{n}},0}(\Omega) \cap \varepsilon^{-1}\mathsf{D}_{\Gamma_{\mathtt{t}},0}(\Omega)$. In the case $\varepsilon = \mathrm{id}$ we usually omit ε in our notations. Again, we assume that also in (1.3) the best constant

$$\frac{1}{c_{\mathtt{m},\Gamma_{\mathtt{t}},\varepsilon}} := \inf_{0 \neq E \in \mathsf{R}_{\Gamma_{\mathtt{t}}}(\Omega) \cap \varepsilon^{-1}\mathsf{D}_{\Gamma_{\mathtt{n}}}(\Omega) \cap \mathcal{H}_{\mathtt{DN},\varepsilon}(\Omega)^{\perp_\varepsilon}} \frac{\big(|\operatorname{div} \varepsilon E|^2_{\mathsf{L}^2(\Omega)} + |\operatorname{rot} E|^2_{\mathsf{L}^2(\Omega)}\big)^{1/2}}{|E|_{\mathsf{L}^2_\varepsilon(\Omega)}}$$

is taken.

The crucial property for (1.3) to hold is the Maxwell compactness property (1.4), which holds, e.g., if Ω has a (strongly) Lipschitz continuous boundary Γ with a (strongly) Lipschitz continuous interface $\gamma := \overline{\Gamma_{\mathtt{t}}} \cap \overline{\Gamma_{\mathtt{n}}}$, see [8] for details. More precisely, the boundary Γ and the interface γ can be described locally as graphs of Lipschitz functions. From now on we assume this properties of Γ and $\Gamma_{\mathtt{t}}$, $\Gamma_{\mathtt{n}}$ as general assumption. Note that then also (1.2) and (1.1) hold. Another successful approach proving the Maxwell compactness property using a different technique from [21] has been shown in [9]. For the Maxwell compactness property in the case of full boundary conditions we refer to [21, 13, 14, 15, 20, 10, 3, 16, 17, 18, 19, 22].

With the help of the $\mathsf{L}^2_\varepsilon(\Omega)$-orthogonal Helmholtz decomposition

$$\mathsf{L}^2_\varepsilon(\Omega) = \nabla\mathsf{H}^1_{\Gamma_{\mathtt{t}}}(\Omega) \oplus_\varepsilon \mathcal{H}_{\mathtt{DN},\varepsilon}(\Omega) \oplus_\varepsilon \varepsilon^{-1} \operatorname{rot} \mathsf{R}_{\Gamma_{\mathtt{n}}}(\Omega), \tag{1.5}$$

where

$$\mathsf{R}_{\Gamma_{\mathtt{t}},0}(\Omega) = \nabla\mathsf{H}^1_{\Gamma_{\mathtt{t}}}(\Omega) \oplus_\varepsilon \mathcal{H}_{\mathtt{DN},\varepsilon}(\Omega), \quad \varepsilon^{-1}\mathsf{D}_{\Gamma_{\mathtt{n}},0}(\Omega) = \varepsilon^{-1} \operatorname{rot} \mathsf{R}_{\Gamma_{\mathtt{n}}}(\Omega) \oplus_\varepsilon \mathcal{H}_{\mathtt{DN},\varepsilon}(\Omega),$$

see Appendix A.2.2 for details, we can split the estimate (1.3) into two, namely

$$\forall\, E \in \varepsilon^{-1}\mathsf{D}_{\Gamma_{\mathtt{n}}}(\Omega) \cap \nabla\mathsf{H}^1_{\Gamma_{\mathtt{t}}}(\Omega) \qquad |E|_{\mathsf{L}^2_\varepsilon(\Omega)} \leq c_{\mathtt{m},\Gamma_{\mathtt{n}},\mathrm{div},\varepsilon} |\operatorname{div}\varepsilon E|_{\mathsf{L}^2(\Omega)}, \tag{1.6}$$

$$\forall\, E \in \mathsf{R}_{\Gamma_{\mathtt{t}}}(\Omega) \cap \varepsilon^{-1}\operatorname{rot}\mathsf{R}_{\Gamma_{\mathtt{n}}}(\Omega) \qquad |E|_{\mathsf{L}^2_\varepsilon(\Omega)} \leq c_{\mathtt{m},\Gamma_{\mathtt{t}},\mathrm{rot},\varepsilon,\mathrm{id}} |\operatorname{rot} E|_{\mathsf{L}^2(\Omega)}, \tag{1.7}$$

where we again assume to use the best constants

$$\frac{1}{c_{\mathtt{m},\Gamma_{\mathtt{n}},\mathrm{div},\varepsilon}} := \inf_{0\neq E\in\varepsilon^{-1}\mathsf{D}_{\Gamma_{\mathtt{n}}}(\Omega)\cap\nabla\mathsf{H}^1_{\Gamma_{\mathtt{t}}}(\Omega)} \frac{|\operatorname{div}\varepsilon E|_{\mathsf{L}^2(\Omega)}}{|E|_{\mathsf{L}^2_\varepsilon(\Omega)}},$$
$$\frac{1}{c_{\mathtt{m},\Gamma_{\mathtt{t}},\mathrm{rot},\varepsilon,\mathrm{id}}} := \inf_{0\neq E\in\mathsf{R}_{\Gamma_{\mathtt{t}}}(\Omega)\cap\varepsilon^{-1}\operatorname{rot}\mathsf{R}_{\Gamma_{\mathtt{n}}}(\Omega)} \frac{|\operatorname{rot} E|_{\mathsf{L}^2(\Omega)}}{|E|_{\mathsf{L}^2_\varepsilon(\Omega)}}.$$

By the assumptions on ε there exist $\underline{\varepsilon}, \overline{\varepsilon} > 0$ such that for all $E \in \mathsf{L}^2(\Omega)$

$$\frac{1}{\underline{\varepsilon}}|E|_{\mathsf{L}^2(\Omega)} \leq |E|_{\mathsf{L}^2_\varepsilon(\Omega)} \leq \overline{\varepsilon}|E|_{\mathsf{L}^2(\Omega)}.$$

We note $|E|_{\mathsf{L}^2_\varepsilon(\Omega)} = |\varepsilon^{1/2}E|_{\mathsf{L}^2(\Omega)}$ and $|\varepsilon^{1/2}E|_{\mathsf{L}^2_\varepsilon(\Omega)} = |\varepsilon E|_{\mathsf{L}^2(\Omega)}$. Thus, for all $E \in \mathsf{L}^2(\Omega)$

$$\frac{1}{\underline{\varepsilon}}|E|_{\mathsf{L}^2_\varepsilon(\Omega)} \leq |\varepsilon E|_{\mathsf{L}^2(\Omega)} \leq \overline{\varepsilon}|E|_{\mathsf{L}^2_\varepsilon(\Omega)}.$$

The inverse ε^{-1} satisfies for all $E \in \mathsf{L}^2(\Omega)$

$$\frac{1}{\overline{\varepsilon}}|E|_{\mathsf{L}^2(\Omega)} \leq |E|_{\mathsf{L}^2_{\varepsilon^{-1}}(\Omega)} \leq \underline{\varepsilon}|E|_{\mathsf{L}^2(\Omega)}, \quad \frac{1}{\overline{\varepsilon}}|E|_{\mathsf{L}^2_{\varepsilon^{-1}}(\Omega)} \leq |\varepsilon^{-1}E|_{\mathsf{L}^2(\Omega)} \leq \underline{\varepsilon}|E|_{\mathsf{L}^2_{\varepsilon^{-1}}(\Omega)},$$

which immediately follows by

$$|E|_{\mathsf{L}^2_{\varepsilon^{-1}}(\Omega)} = |\varepsilon^{-1/2}E|_{\mathsf{L}^2(\Omega)} \begin{cases} \leq \underline{\varepsilon}|\varepsilon^{-1/2}E|_{\mathsf{L}^2_\varepsilon(\Omega)} = \underline{\varepsilon}|E|_\Omega \\ \geq \overline{\varepsilon}^{-1}|\varepsilon^{-1/2}E|_{\mathsf{L}^2_\varepsilon(\Omega)} = \overline{\varepsilon}^{-1}|E|_\Omega \end{cases}.$$

For later purposes let us also define $\hat{\varepsilon} := \max\{\underline{\varepsilon}, \overline{\varepsilon}\}$.

In this contribution we will study these different constants $c_{\mathtt{p},\Gamma_{\mathtt{t}},\varepsilon}$, $c_{\mathtt{m},\Gamma_{\mathtt{t}},\varepsilon}$, $c_{\mathtt{m},\Gamma_{\mathtt{n}},\mathrm{div},\varepsilon}$, $c_{\mathtt{m},\Gamma_{\mathtt{t}},\mathrm{rot},\varepsilon,\mathrm{id}}$ and their relations to each other. It turns out that

$$c_{\mathtt{p},\Gamma_{\mathtt{t}},\varepsilon} = c_{\mathtt{m},\Gamma_{\mathtt{n}},\mathrm{div},\varepsilon}, \quad c_{\mathtt{m},\Gamma_{\mathtt{t}},\mathrm{rot},\varepsilon,\mathrm{id}} = c_{\mathtt{m},\Gamma_{\mathtt{n}},\mathrm{rot},\mathrm{id},\varepsilon}, \quad c_{\mathtt{m},\Gamma_{\mathtt{t}},\varepsilon} = \max\{c_{\mathtt{p},\Gamma_{\mathtt{t}},\varepsilon}, c_{\mathtt{m},\Gamma_{\mathtt{t}},\mathrm{rot},\varepsilon,\mathrm{id}}\}$$

hold, see Lemmas 3, 10 and 6. The main result of this paper states that in the special case of full boundary conditions, i.e., $\Gamma_{\mathtt{t}} = \Gamma$ or $\Gamma_{\mathtt{n}} = \Gamma$, and for bounded and *convex* domains we have

$$\frac{c_{\mathtt{p},\Gamma}}{\overline{\varepsilon}} \leq c_{\mathtt{m},\Gamma,\varepsilon} \leq \hat{\varepsilon}c_{\mathtt{p}}, \quad \frac{c_{\mathtt{p}}}{\overline{\varepsilon}} \leq c_{\mathtt{m},\emptyset,\varepsilon} \leq \hat{\varepsilon}c_{\mathtt{p}}$$

and especially for $\varepsilon = \mathrm{id}$

$$\max\{c_{\mathtt{p},\Gamma}, c_{\mathtt{m},\mathrm{rot}}\} = c_{\mathtt{m},\Gamma} \leq c_{\mathtt{m},\emptyset} = c_{\mathtt{p}},$$

see Theorem 17. Here, we introduce for the special case $\varepsilon = \mathrm{id}$

$$c_{\mathtt{p},\Gamma_{\mathtt{t}}} := c_{\mathtt{p},\Gamma_{\mathtt{t}},\mathrm{id}}, \quad c_{\mathtt{p}} := c_{\mathtt{p},\emptyset}, \quad c_{\mathtt{m},\Gamma_{\mathtt{t}}} := c_{\mathtt{m},\Gamma_{\mathtt{t}},\mathrm{id}}$$

and

$$c_{\mathtt{m},\Gamma_{\mathtt{t}},\mathrm{rot}} := c_{\mathtt{m},\Gamma_{\mathtt{t}},\mathrm{rot},\mathrm{id},\mathrm{id}} = c_{\mathtt{m},\Gamma_{\mathtt{n}},\mathrm{rot},\mathrm{id},\mathrm{id}} = c_{\mathtt{m},\Gamma_{\mathtt{n}},\mathrm{rot}}$$

as well as

$$c_{\mathtt{m},\mathrm{rot}} := c_{\mathtt{m},\Gamma,\mathrm{rot},\mathrm{id},\mathrm{id}} = c_{\mathtt{m},\emptyset,\mathrm{rot},\mathrm{id},\mathrm{id}}.$$

The crucial point in our analysis is that for convex domains

$$c_{\mathtt{m},\mathrm{rot}} \leq c_{\mathtt{p}}, \quad c_{\mathtt{m},\Gamma,\mathrm{rot},\varepsilon,\mathrm{id}}, c_{\mathtt{m},\emptyset,\mathrm{rot},\varepsilon,\mathrm{id}} \leq \overline{\varepsilon} c_{\mathtt{p}}$$

hold, see Lemma 16. Some of these results have also been obtained recently in [11] utilizing different and more elementary[ii] methods. We note that in the convex case we can estimate the Poincaré constant $c_{\mathtt{p}}$ by the diameter of Ω. More precisely, by the famous paper of Payne and Weinberger [12][iii] we have

$$c_{\mathtt{p}} \leq \frac{\mathrm{diam}(\Omega)}{\pi}.$$

In [12] also the optimality of this estimate has been shown. Furthermore, $c_{\mathtt{p},\Gamma} < c_{\mathtt{p}}$ is well known even for non-convex domains, see e.g. [4] and the cited literature, yielding

$$\frac{1}{\sqrt{\lambda_1}} = c_{\mathtt{p},\Gamma} < c_{\mathtt{p}} = \frac{1}{\sqrt{\mu_2}} \leq \frac{\mathrm{diam}(\Omega)}{\pi}, \tag{1.8}$$

where λ_1 resp. μ_2 is the first Dirichlet resp. second Neumann eigenvalue of the negative Laplacian.

At least some of our results extend in a natural way to bounded domains $\Omega \subset \mathbb{R}^N$ or even to Riemannian manifolds with compact closure, see Remark 5 and Appendix A.1.

Our new estimates have important applications e.g. to numerical analysis, where especially an upper bound for the Maxwell constants is needed e.g. for preconditioning and for functional a posteriori error estimates in the framework of Maxwell's equations.

2 An Abstract Setting

Let X and Y be Hilbert spaces and

$$\mathrm{A} : D(\mathrm{A}) \subset \mathsf{X} \to \mathsf{Y}, \quad \mathrm{A}^* : D(\mathrm{A}^*) \subset \mathsf{Y} \to \mathsf{X}$$

be a closed and densely defined linear operator and its adjoint. Here, D denotes the domain of definition and we introduce the kernel N and the range R. Since A is closed we

[ii] In the sense that no tools from functional analysis were used.
[iii] A little mistake or inconsistency in [12] has been corrected later in [2].

have $(\mathrm{A}^*)^* = \bar{\mathrm{A}} = \mathrm{A}$ and sometimes $(\mathrm{A}, \mathrm{A}^*)$ is called a dual pair. The projection theorem yields the orthogonal 'Helmholtz' decompositions

$$\mathsf{X} = N(\mathrm{A}) \oplus \overline{R(\mathrm{A}^*)}, \quad \mathsf{Y} = N(\mathrm{A}^*) \oplus \overline{R(\mathrm{A})}. \tag{2.1}$$

Now, we collect some well known facts. For the convenience of the reader we give simple proofs of those in the Appendix A.3.

$\mathrm{A}^*\mathrm{A}$ and $\mathrm{A}\mathrm{A}^*$ are non-negative and self-adjoint and their spectra coincide if we exclude $\{0\}$, i.e.,

$$\sigma(\mathrm{A}^*\mathrm{A}) \setminus \{0\} = \sigma(\mathrm{A}\mathrm{A}^*) \setminus \{0\}, \quad \sigma_{\mathsf{p}}(\mathrm{A}^*\mathrm{A}) \setminus \{0\} = \sigma_{\mathsf{p}}(\mathrm{A}\mathrm{A}^*) \setminus \{0\}. \tag{2.2}$$

Let us assume that the embedding

$$D(\mathrm{A}) \cap \overline{R(\mathrm{A}^*)} \hookrightarrow \mathsf{X} \tag{2.3}$$

is compact.

Lemma 1 *There exist* $c_{\mathrm{A}}, c_{\mathrm{A}^*} > 0$*, such that*

$$\begin{aligned} &\forall\, x \in D(\mathrm{A}) \cap R(\mathrm{A}^*) && |x|_{\mathsf{X}} \leq c_{\mathrm{A}} |\mathrm{A}x|_{\mathsf{Y}}, \\ &\forall\, y \in D(\mathrm{A}^*) \cap R(\mathrm{A}) && |y|_{\mathsf{Y}} \leq c_{\mathrm{A}^*} |\mathrm{A}^*y|_{\mathsf{X}}. \end{aligned}$$

Moreover, $R(\mathrm{A})$ *and* $R(\mathrm{A}^*)$ *are closed and*

$$\mathsf{X} = N(\mathrm{A}) \oplus R(\mathrm{A}^*), \quad \mathsf{Y} = N(\mathrm{A}^*) \oplus R(\mathrm{A}).$$

Furthermore, $D(\mathrm{A}^*) \cap R(\mathrm{A}) \hookrightarrow \mathsf{Y}$ *is compact as well.*

We note that the same lemma can be proved assuming the compactness of the embedding of $D(\mathrm{A}^*) \cap \overline{R(\mathrm{A})} \hookrightarrow \mathsf{Y}$ instead of (2.3). By Lemma 1 the restricted operator

$$\mathcal{A} := \mathrm{A}|_{D(\mathcal{A})} : D(\mathcal{A}) \subset R(\mathrm{A}^*) \to R(\mathrm{A}), \quad D(\mathcal{A}) := D(\mathrm{A}) \cap R(\mathrm{A}^*)$$

has a bounded inverse $\mathcal{A}^{-1} : R(\mathrm{A}) \to D(\mathcal{A})$ with $|\mathcal{A}^{-1}| \leq (1 + c_{\mathrm{A}}^2)^{1/2}$, which is compact as an operator from $R(\mathrm{A})$ to $R(\mathrm{A}^*)$. Hence, $\mathrm{A}^*\mathrm{A}$ and $\mathrm{A}\mathrm{A}^*$ have pure point spectra which can only accumulate at infinity and which coincide by (2.2). Especially, the second eigenvalues equal and therefore (see Corollary 32 for details) we conclude:

Theorem 2 *For the best constants in Lemma 1 it holds* $c_{\mathrm{A}} = c_{\mathrm{A}^*}$*, this is*

$$\frac{1}{c_{\mathrm{A}}} = \min_{0 \neq x \in D(\mathrm{A}) \cap R(\mathrm{A}^*)} \frac{|\mathrm{A}x|_{\mathsf{Y}}}{|x|_{\mathsf{X}}} = \min_{0 \neq y \in D(\mathrm{A}^*) \cap R(\mathrm{A})} \frac{|\mathrm{A}^*y|_{\mathsf{X}}}{|y|_{\mathsf{Y}}} = \frac{1}{c_{\mathrm{A}^*}}.$$

Hence, $c_{\mathrm{A}}^{-2} = c_{\mathrm{A}^*}^{-2}$ is the first positive eigenvalue of $\mathrm{A}^*\mathrm{A}$ as well as of $\mathrm{A}\mathrm{A}^*$.

3 The Maxwell Estimates

We remind on Ω and its properties from the introduction.

3.1 General Lipschitz Domains

In this subsection we frequently use Lemma 1 and Theorem 2.

3.1.1 Gradient and Divergence

Let us consider A as

$$\nabla : \mathsf{H}^1_{\Gamma_t}(\Omega) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2_\varepsilon(\Omega).$$

Then A^* equals

$$-\operatorname{div}\varepsilon : \varepsilon^{-1}\mathsf{D}_{\Gamma_n}(\Omega) \subset \mathsf{L}^2_\varepsilon(\Omega) \to \mathsf{L}^2(\Omega).$$

More precisely, we have the following table:

A	$D(\mathrm{A})$	X	Y	$N(\mathrm{A})$	$R(\mathrm{A})$
∇	$\mathsf{H}^1_{\Gamma_t}(\Omega)$	$\mathsf{L}^2(\Omega)$	$\mathsf{L}^2_\varepsilon(\Omega)$	$\{0\}$	$\nabla\mathsf{H}^1_{\Gamma_t}(\Omega) = \mathsf{R}_{\Gamma_t,0}(\Omega) \cap \mathcal{H}_{\mathtt{DN}}(\Omega)^\perp$
A^*	$D(\mathrm{A}^*)$	Y	X	$N(\mathrm{A}^*)$	$R(\mathrm{A}^*)$
$-\operatorname{div}\varepsilon$	$\varepsilon^{-1}\mathsf{D}_{\Gamma_n}(\Omega)$	$\mathsf{L}^2_\varepsilon(\Omega)$	$\mathsf{L}^2(\Omega)$	$\varepsilon^{-1}\mathsf{D}_{\Gamma_n,0}(\Omega)$	$\operatorname{div}\mathsf{D}_{\Gamma_n}(\Omega)$

We note that $\operatorname{div}\mathsf{D}_{\Gamma_n}(\Omega) = \mathsf{L}^2(\Omega)$ if $\Gamma_n \neq \Gamma$ and $\operatorname{div}\mathsf{D}_\Gamma(\Omega) = \mathsf{L}^2(\Omega) \cap \mathbb{R}^\perp$. Moreover, we emphasize that indeed $D(\mathrm{A}^*) = \varepsilon^{-1}\mathsf{D}_{\Gamma_n}(\Omega)$ holds, see e.g. [8]. Note that for this one has to show the approximation property

$$\mathsf{D}_{\Gamma_n}(\Omega) = \{H \in \mathsf{D}(\Omega) \, : \, \langle \operatorname{div} H, u\rangle_{\mathsf{L}^2(\Omega)} = -\langle H, \nabla u\rangle_{\mathsf{L}^2(\Omega)} \; \forall \, u \in \mathsf{H}^1_{\Gamma_t}(\Omega)\},$$

which is not trivial at all for mixed boundary conditions. Only in the special cases of full boundary conditions this is clear. $D(\mathrm{A}^*) = \varepsilon^{-1}\mathsf{D}(\Omega)$ holds for $\Gamma_t = \Gamma$ by definition. For $\Gamma_t = \emptyset$ we see that the closed operator

$$\mathrm{B} := -\operatorname{div} : \mathsf{D}_\Gamma(\Omega) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2(\Omega)$$

has the adjoint

$$\mathrm{B}^* = \nabla : \mathsf{H}^1(\Omega) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2(\Omega)$$

by definition. Since in this case $\mathrm{A} = \mathrm{B}^*$ we have $D(\mathrm{A}^*) = D(\mathrm{B}^{**}) = D(\mathrm{B}) = \mathsf{D}_\Gamma(\Omega)$. The crucial compact embedding (2.3) reads

$$\mathsf{H}^1_{\Gamma_t}(\Omega) \cap \overline{\operatorname{div}\mathsf{D}_{\Gamma_n}(\Omega)} \hookrightarrow \mathsf{L}^2(\Omega)$$

and is just Rellich's selection theorem since

$$\mathsf{H}^1_{\Gamma_t}(\Omega) \cap \overline{\operatorname{div}\mathsf{D}_{\Gamma_n}(\Omega)} \subset \mathsf{H}^1_{\Gamma_t}(\Omega) \subset \mathsf{H}^1(\Omega) \hookrightarrow \mathsf{L}^2(\Omega).$$

Theorem 2 yields

$$0 < \frac{1}{c_{\mathtt{p},\Gamma_\mathtt{t},\varepsilon}} = \min_{0\neq u\in\mathsf{H}^1_{\Gamma_\mathtt{t}}(\Omega)} \frac{|\nabla u|_{\mathsf{L}^2_\varepsilon(\Omega)}}{|u|_{\mathsf{L}^2(\Omega)}} = \min_{0\neq E\in\varepsilon^{-1}\mathsf{D}_{\Gamma_\mathtt{n}}(\Omega)\cap\nabla\mathsf{H}^1_{\Gamma_\mathtt{t}}(\Omega)} \frac{|\operatorname{div}\varepsilon E|_{\mathsf{L}^2(\Omega)}}{|E|_{\mathsf{L}^2_\varepsilon(\Omega)}} = \frac{1}{c_{\mathtt{m},\Gamma_\mathtt{n},\operatorname{div},\varepsilon}}.$$

We note that $\lambda_{\Gamma_\mathtt{t},\varepsilon} := c^{-2}_{\mathtt{p},\Gamma_\mathtt{t},\varepsilon}$ is the first positive Dirichlet-Neumann eigenvalue of the weighted negative Laplacian $-\Delta_\varepsilon := -\operatorname{div}\varepsilon\nabla$. For $\varepsilon = \mathrm{id}$ and $\Gamma_\mathtt{t} = \Gamma$ resp. $\Gamma_\mathtt{t} = \emptyset$ we see that $\lambda_{\Gamma,\mathrm{id}} =: \lambda_1$ resp. $\lambda_{\emptyset,\mathrm{id}} =: \mu_2$ is the first Dirichlet resp. second Neumann eigenvalue of the negative Laplacian. As $\lambda_{\Gamma_\mathtt{t},\varepsilon} = c^{-2}_{\mathtt{m},\Gamma_\mathtt{n},\operatorname{div},\varepsilon}$ holds too, $\lambda_{\Gamma_\mathtt{t},\varepsilon}$ is also the first positive Neumann-Dirichlet eigenvalue of the weighted negative reduced grad-div-operator $-\nabla\operatorname{div}\varepsilon$, which can also be interpreted as the weighted negative vector Laplacian $-\vec{\Delta}_\varepsilon := -\nabla\operatorname{div}\varepsilon + \operatorname{rot}\operatorname{rot}$ on a subspace of irrotational vector fields.

Lemma 3 *The Poincaré constant in $\mathsf{H}^1_{\Gamma_\mathtt{t}}(\Omega)$ and the Maxwell divergence constant in $\varepsilon^{-1}\mathsf{D}_{\Gamma_\mathtt{n}}(\Omega)\cap\nabla\mathsf{H}^1_{\Gamma_\mathtt{t}}(\Omega)$, i.e., the best constants in the inequalities*

$$\begin{aligned} &\forall\, u \in \mathsf{H}^1_{\Gamma_\mathtt{t}}(\Omega) && |u|_{\mathsf{L}^2(\Omega)} \leq c_{\mathtt{p},\Gamma_\mathtt{t},\varepsilon}|\nabla u|_{\mathsf{L}^2_\varepsilon(\Omega)},\\ &\forall\, E \in \varepsilon^{-1}\mathsf{D}_{\Gamma_\mathtt{n}}(\Omega)\cap\nabla\mathsf{H}^1_{\Gamma_\mathtt{t}}(\Omega) && |E|_{\mathsf{L}^2_\varepsilon(\Omega)} \leq c_{\mathtt{m},\Gamma_\mathtt{n},\operatorname{div},\varepsilon}|\operatorname{div}\varepsilon E|_{\mathsf{L}^2(\Omega)}, \end{aligned}$$

coincide and correspond to the first positive Dirichlet-Neumann eigenvalue of the weighted negative Laplacian $-\Delta_\varepsilon$, more precisely $c_{\mathtt{p},\Gamma_\mathtt{t},\varepsilon} = c_{\mathtt{m},\Gamma_\mathtt{n},\operatorname{div},\varepsilon} = 1/\sqrt{\lambda_{\Gamma_\mathtt{t},\varepsilon}}$.

Lemma 4 *It holds $\overline{\varepsilon}^{-1}c_{\mathtt{p},\Gamma_\mathtt{t}} \leq c_{\mathtt{p},\Gamma_\mathtt{t},\varepsilon} \leq \underline{\varepsilon}c_{\mathtt{p},\Gamma_\mathtt{t}}$ as well as $c_{\mathtt{p},\Gamma} \leq c_{\mathtt{p},\Gamma_\mathtt{t}}$ and $c_{\mathtt{p},\Gamma,\varepsilon} \leq c_{\mathtt{p},\Gamma_\mathtt{t},\varepsilon}$.*

Proof For $u \in \mathsf{H}^1_{\Gamma_\mathtt{t}}(\Omega)$ we have

$$\begin{aligned} |u|_{\mathsf{L}^2(\Omega)} &\leq c_{\mathtt{p},\Gamma_\mathtt{t}}|\nabla u|_{\mathsf{L}^2(\Omega)} \leq \underline{\varepsilon}c_{\mathtt{p},\Gamma_\mathtt{t}}|\nabla u|_{\mathsf{L}^2_\varepsilon(\Omega)},\\ |u|_{\mathsf{L}^2(\Omega)} &\leq c_{\mathtt{p},\Gamma_\mathtt{t},\varepsilon}|\nabla u|_{\mathsf{L}^2_\varepsilon(\Omega)} \leq \overline{\varepsilon}c_{\mathtt{p},\Gamma_\mathtt{t},\varepsilon}|\nabla u|_{\mathsf{L}^2(\Omega)}, \end{aligned}$$

which gives $c_{\mathtt{p},\Gamma_\mathtt{t},\varepsilon} \leq \underline{\varepsilon}c_{\mathtt{p},\Gamma_\mathtt{t}}$ and $c_{\mathtt{p},\Gamma_\mathtt{t}} \leq \overline{\varepsilon}c_{\mathtt{p},\Gamma_\mathtt{t},\varepsilon}$. □

Remark 5 *The results of this section extend to bounded domains $\Omega \subset \mathbb{R}^N$, $N \in \mathbb{N}$, having the proper regularity of the boundary.*

3.1.2 Rotations

Now, let A be

$$\mu^{-1}\operatorname{rot} : \mathsf{R}_{\Gamma_\mathtt{t}}(\Omega) \subset \mathsf{L}^2_\varepsilon(\Omega) \to \mathsf{L}^2_\mu(\Omega).$$

Then A^* is

$$\varepsilon^{-1}\operatorname{rot} : \mathsf{R}_{\Gamma_\mathtt{n}}(\Omega) \subset \mathsf{L}^2_\mu(\Omega) \to \mathsf{L}^2_\varepsilon(\Omega),$$

where μ is another matrix field similar to ε. More precisely:

A	$D(\mathrm{A})$	X	Y	$N(\mathrm{A})$	$R(\mathrm{A})$
$\mu^{-1}\operatorname{rot}$	$\mathsf{R}_{\Gamma_{\mathtt{t}}}(\Omega)$	$\mathsf{L}^2_\varepsilon(\Omega)$	$\mathsf{L}^2_\mu(\Omega)$	$\mathsf{R}_{\Gamma_{\mathtt{t}},0}(\Omega)$	$\mu^{-1}\operatorname{rot}\mathsf{R}_{\Gamma_{\mathtt{t}}}(\Omega)$
A^*	$D(\mathrm{A}^*)$	Y	X	$N(\mathrm{A}^*)$	$R(\mathrm{A}^*)$
$\varepsilon^{-1}\operatorname{rot}$	$\mathsf{R}_{\Gamma_{\mathtt{n}}}(\Omega)$	$\mathsf{L}^2_\mu(\Omega)$	$\mathsf{L}^2_\varepsilon(\Omega)$	$\mathsf{R}_{\Gamma_{\mathtt{n}},0}(\Omega)$	$\varepsilon^{-1}\operatorname{rot}\mathsf{R}_{\Gamma_{\mathtt{n}}}(\Omega)$

We note

$$R(\mathrm{A})=\mu^{-1}\big(\mathsf{D}_{\Gamma_{\mathtt{t}},0}(\Omega)\cap\mathcal{H}_{\mathtt{ND}}(\Omega)^{\perp}\big),\quad R(\mathrm{A}^*)=\varepsilon^{-1}\big(\mathsf{D}_{\Gamma_{\mathtt{n}},0}(\Omega)\cap\mathcal{H}_{\mathtt{DN}}(\Omega)^{\perp}\big)$$

and that indeed $D(\mathrm{A}^*)=\mathsf{R}_{\Gamma_{\mathtt{n}}}(\Omega)$ holds, see again e.g. [8]. As before, for this one has to show the approximation property

$$\mathsf{R}_{\Gamma_{\mathtt{n}}}(\Omega)=\{H\in\mathsf{R}(\Omega)\,:\,\langle\operatorname{rot}H,E\rangle_{\mathsf{L}^2(\Omega)}=\langle H,\operatorname{rot}E\rangle_{\mathsf{L}^2(\Omega)}\;\forall\,E\in\mathsf{R}_{\Gamma_{\mathtt{t}}}(\Omega)\},$$

which is not trivial at all for mixed boundary conditions. Again, only in the special cases of full boundary conditions this is clear. Since $D(\mathrm{A}^*)=\mathsf{R}(\Omega)$ holds for $\Gamma_{\mathtt{t}}=\Gamma$ by definition we have also $D(\mathrm{B}^*)=D(\mathrm{A}^{**})=D(\mathrm{A})=\mathsf{R}_{\Gamma}(\Omega)$ for $\mathrm{B}=\mathrm{A}^*$, which shows the result for $\Gamma_{\mathtt{t}}=\emptyset$. The crucial compact embedding (2.3) reads

$$\mathsf{R}_{\Gamma_{\mathtt{t}}}(\Omega)\cap\varepsilon^{-1}\overline{\operatorname{rot}\mathsf{R}_{\Gamma_{\mathtt{n}}}(\Omega)}\hookrightarrow\mathsf{L}^2_\varepsilon(\Omega)$$

and is just the Maxwell compactness property (1.4) since

$$\mathsf{R}_{\Gamma_{\mathtt{t}}}(\Omega)\cap\varepsilon^{-1}\overline{\operatorname{rot}\mathsf{R}_{\Gamma_{\mathtt{n}}}(\Omega)}\subset\mathsf{R}_{\Gamma_{\mathtt{t}}}(\Omega)\cap\varepsilon^{-1}\mathsf{D}_{\Gamma_{\mathtt{n}},0}(\Omega)\subset\mathsf{R}_{\Gamma_{\mathtt{t}}}(\Omega)\cap\varepsilon^{-1}\mathsf{D}_{\Gamma_{\mathtt{n}}}(\Omega)\hookrightarrow\mathsf{L}^2(\Omega)\subset\mathsf{L}^2_\varepsilon(\Omega).$$

By Theorem 2 we have

$$\begin{aligned}0<\frac{1}{c_{\mathtt{m},\Gamma_{\mathtt{t}},\mathrm{rot},\varepsilon,\mu}}&=\min_{0\neq E\in\mathsf{R}_{\Gamma_{\mathtt{t}}}(\Omega)\cap\varepsilon^{-1}\operatorname{rot}\mathsf{R}_{\Gamma_{\mathtt{n}}}(\Omega)}\frac{|\mu^{-1}\operatorname{rot}E|_{\mathsf{L}^2_\mu(\Omega)}}{|E|_{\mathsf{L}^2_\varepsilon(\Omega)}}\\&=\min_{0\neq H\in\mathsf{R}_{\Gamma_{\mathtt{n}}}(\Omega)\cap\mu^{-1}\operatorname{rot}\mathsf{R}_{\Gamma_{\mathtt{t}}}(\Omega)}\frac{|\varepsilon^{-1}\operatorname{rot}H|_{\mathsf{L}^2_\varepsilon(\Omega)}}{|H|_{\mathsf{L}^2_\mu(\Omega)}}=\frac{1}{c_{\mathtt{m},\Gamma_{\mathtt{n}},\mathrm{rot},\mu,\varepsilon}},\end{aligned}$$

which serves also as definition for the constants $c_{\mathtt{m},\Gamma_{\mathtt{t}},\mathrm{rot},\varepsilon,\mu}$ and $c_{\mathtt{m},\Gamma_{\mathtt{n}},\mathrm{rot},\mu,\varepsilon}$. Therefore, $\kappa_{\Gamma_{\mathtt{t}},\varepsilon,\mu}:=c^{-2}_{\mathtt{m},\Gamma_{\mathtt{t}},\mathrm{rot},\varepsilon,\mu}$ is the first positive Dirichlet-Neumann eigenvalue of the weighted reduced double-rot-operator $\Box_{\varepsilon,\mu}:=\varepsilon^{-1}\operatorname{rot}\mu^{-1}\operatorname{rot}$, which can also be interpreted as the weighted negative vector Laplacian $-\vec{\Delta}_{\varepsilon,\mu}:=-\nabla\operatorname{div}\varepsilon+\varepsilon^{-1}\operatorname{rot}\mu^{-1}\operatorname{rot}$ on a subspace of ε-solenoidal vector fields. Since $\kappa_{\Gamma_{\mathtt{t}},\varepsilon,\mu}=c^{-2}_{\mathtt{m},\Gamma_{\mathtt{n}},\mathrm{rot},\mu,\varepsilon}$ holds as well, $\kappa_{\Gamma_{\mathtt{t}},\varepsilon,\mu}$ is also the first positive Neumann-Dirichlet eigenvalue of the weighted reduced double-rot-operator $\Box_{\mu,\varepsilon}=\mu^{-1}\operatorname{rot}\varepsilon^{-1}\operatorname{rot}$, which can also be interpreted as the weighted negative vector Laplacian on a subspace of μ-solenoidal vector fields, i.e., $-\vec{\Delta}_{\mu,\varepsilon}=-\nabla\operatorname{div}\mu+\mu^{-1}\operatorname{rot}\varepsilon^{-1}\operatorname{rot}$.

Lemma 6 *The tangential-normal and normal-tangential Maxwell rotation constants, i.e., the best constants in the inequalities*

$$\forall\,E\in\mathsf{R}_{\Gamma_{\mathtt{t}}}(\Omega)\cap\varepsilon^{-1}\operatorname{rot}\mathsf{R}_{\Gamma_{\mathtt{n}}}(\Omega)\qquad|E|_{\mathsf{L}^2_\varepsilon(\Omega)}\leq c_{\mathtt{m},\Gamma_{\mathtt{t}},\mathrm{rot},\varepsilon,\mu}|\operatorname{rot}E|_{\mathsf{L}^2_{\mu^{-1}}(\Omega)},$$

$$\forall\,H\in\mathsf{R}_{\Gamma_{\mathtt{n}}}(\Omega)\cap\mu^{-1}\operatorname{rot}\mathsf{R}_{\Gamma_{\mathtt{t}}}(\Omega)\qquad|H|_{\mathsf{L}^2_\mu(\Omega)}\leq c_{\mathtt{m},\Gamma_{\mathtt{n}},\mathrm{rot},\mu,\varepsilon}|\operatorname{rot}H|_{\mathsf{L}^2_{\varepsilon^{-1}}(\Omega)},$$

coincide and correspond to the first positive Dirichlet-Neumann eigenvalue of the weighted reduced double-rot-operator $\square_{\varepsilon,\mu}$, *more precisely* $c_{\mathtt{m},\Gamma_{\mathtt{t}},\mathrm{rot},\varepsilon,\mu} = c_{\mathtt{m},\Gamma_{\mathtt{n}},\mathrm{rot},\mu,\varepsilon} = 1/\sqrt{\kappa_{\Gamma_{\mathtt{t}},\varepsilon,\mu}}$.

Let us define for $\varepsilon = \mu$ and for $\varepsilon = \mu = \mathrm{id}$

$$c_{\mathtt{m},\Gamma_{\mathtt{t}},\mathrm{rot},\varepsilon} := c_{\mathtt{m},\Gamma_{\mathtt{t}},\mathrm{rot},\varepsilon,\varepsilon} = c_{\mathtt{m},\Gamma_{\mathtt{n}},\mathrm{rot},\varepsilon,\varepsilon}$$

and note

$$c_{\mathtt{m},\Gamma_{\mathtt{t}},\mathrm{rot},\varepsilon} = c_{\mathtt{m},\Gamma_{\mathtt{n}},\mathrm{rot},\varepsilon}, \quad c_{\mathtt{m},\Gamma_{\mathtt{t}},\mathrm{rot}} = c_{\mathtt{m},\Gamma_{\mathtt{n}},\mathrm{rot}}. \tag{3.1}$$

Corollary 7 *For all* $E \in \big(\mathsf{R}_{\Gamma_{\mathtt{t}}}(\Omega) \cap \varepsilon^{-1} \operatorname{rot} \mathsf{R}_{\Gamma_{\mathtt{n}}}(\Omega)\big) \cup \big(\mathsf{R}_{\Gamma_{\mathtt{n}}}(\Omega) \cap \varepsilon^{-1} \operatorname{rot} \mathsf{R}_{\Gamma_{\mathtt{t}}}(\Omega)\big)$

$$|E|_{\mathsf{L}^2_\varepsilon(\Omega)} \leq c_{\mathtt{m},\Gamma_{\mathtt{t}},\mathrm{rot},\varepsilon} |\operatorname{rot} E|_{\mathsf{L}^2_{\varepsilon^{-1}}(\Omega)} \leq \underline{\varepsilon} c_{\mathtt{m},\Gamma_{\mathtt{t}},\mathrm{rot},\varepsilon} |\operatorname{rot} E|_{\mathsf{L}^2(\Omega)} \tag{3.2}$$

holds with sharp constants. Moreover, the inequalities

$$\forall\, E \in \mathsf{R}_{\Gamma_{\mathtt{t}}}(\Omega) \cap \varepsilon^{-1} \operatorname{rot} \mathsf{R}_{\Gamma_{\mathtt{n}}}(\Omega) \qquad |E|_{\mathsf{L}^2_\varepsilon(\Omega)} \leq c_{\mathtt{m},\Gamma_{\mathtt{t}},\mathrm{rot},\varepsilon,\mathrm{id}} |\operatorname{rot} E|_{\mathsf{L}^2(\Omega)}, \tag{3.3}$$

$$\forall\, H \in \mathsf{R}_{\Gamma_{\mathtt{n}}}(\Omega) \cap \varepsilon^{-1} \operatorname{rot} \mathsf{R}_{\Gamma_{\mathtt{t}}}(\Omega) \qquad |H|_{\mathsf{L}^2_\varepsilon(\Omega)} \leq c_{\mathtt{m},\Gamma_{\mathtt{n}},\mathrm{rot},\varepsilon,\mathrm{id}} |\operatorname{rot} H|_{\mathsf{L}^2(\Omega)} \tag{3.4}$$

hold, where these sharp constants do not need to coincide if $\varepsilon \neq \mathrm{id}$.

Lemma 8 *It holds*

(i) $\underline{\varepsilon}^{-2} c_{\mathtt{m},\Gamma_{\mathtt{t}},\mathrm{rot}} \leq c_{\mathtt{m},\Gamma_{\mathtt{t}},\mathrm{rot},\varepsilon} \leq \overline{\varepsilon}^2 c_{\mathtt{m},\Gamma_{\mathtt{t}},\mathrm{rot}}$,

(ii) $c_{\mathtt{m},\Gamma_{\mathtt{t}},\mathrm{rot},\varepsilon,\mathrm{id}}, c_{\mathtt{m},\Gamma_{\mathtt{n}},\mathrm{rot},\varepsilon,\mathrm{id}} \begin{cases} \leq \min\{\underline{\varepsilon} c_{\mathtt{m},\Gamma_{\mathtt{t}},\mathrm{rot},\varepsilon}, \overline{\varepsilon} c_{\mathtt{m},\Gamma_{\mathtt{t}},\mathrm{rot}}\} \leq \overline{\varepsilon} c_{\mathtt{m},\Gamma_{\mathtt{t}},\mathrm{rot}}, \\ \geq \max\{\overline{\varepsilon}^{-1} c_{\mathtt{m},\Gamma_{\mathtt{t}},\mathrm{rot},\varepsilon}, \underline{\varepsilon}^{-1} c_{\mathtt{m},\Gamma_{\mathtt{t}},\mathrm{rot}}\} \geq \underline{\varepsilon}^{-1} c_{\mathtt{m},\Gamma_{\mathtt{t}},\mathrm{rot}}. \end{cases}$

Proof It is clear that $c_{\mathtt{m},\Gamma_{\mathtt{t}},\mathrm{rot},\varepsilon,\mathrm{id}}, c_{\mathtt{m},\Gamma_{\mathtt{n}},\mathrm{rot},\varepsilon,\mathrm{id}} \leq \underline{\varepsilon} c_{\mathtt{m},\Gamma_{\mathtt{t}},\mathrm{rot},\varepsilon}$ holds. To prove the other estimates, let $E \in \mathsf{R}_{\Gamma_{\mathtt{t}}}(\Omega) \cap \varepsilon^{-1} \operatorname{rot} \mathsf{R}_{\Gamma_{\mathtt{n}}}(\Omega)$. We decompose (see Appendix A.2.2)

$$E = E_0 + E_{\mathrm{rot}} \in \mathsf{R}_{\Gamma_{\mathtt{t}},0}(\Omega) \oplus \operatorname{rot} \mathsf{R}_{\Gamma_{\mathtt{n}}}(\Omega).$$

Then $E_{\mathrm{rot}} \in \mathsf{R}_{\Gamma_{\mathtt{t}}}(\Omega) \cap \operatorname{rot} \mathsf{R}_{\Gamma_{\mathtt{n}}}(\Omega)$ and $\operatorname{rot} E = \operatorname{rot} E_{\mathrm{rot}}$. Thus by orthogonality

$$|E|^2_{\mathsf{L}^2_\varepsilon(\Omega)} = \langle \varepsilon E, E_{\mathrm{rot}} \rangle_{\mathsf{L}^2(\Omega)} \leq c_{\mathtt{m},\Gamma_{\mathtt{t}},\mathrm{rot}} \underbrace{|\varepsilon E|_{\mathsf{L}^2(\Omega)}}_{\leq \overline{\varepsilon} |E|_{\mathsf{L}^2_\varepsilon(\Omega)}} |\operatorname{rot} E|_{\mathsf{L}^2(\Omega)}$$

and hence

$$|E|_{\mathsf{L}^2_\varepsilon(\Omega)} \leq \overline{\varepsilon} c_{\mathtt{m},\Gamma_{\mathtt{t}},\mathrm{rot}} |\operatorname{rot} E|_{\mathsf{L}^2(\Omega)} \leq \overline{\varepsilon}^2 c_{\mathtt{m},\Gamma_{\mathtt{t}},\mathrm{rot}} |\operatorname{rot} E|_{\mathsf{L}^2_{\varepsilon^{-1}}(\Omega)}.$$

This shows $c_{\mathtt{m},\Gamma_{\mathtt{t}},\mathrm{rot},\varepsilon,\mathrm{id}} \leq \overline{\varepsilon} c_{\mathtt{m},\Gamma_{\mathtt{t}},\mathrm{rot}}$ and $c_{\mathtt{m},\Gamma_{\mathtt{t}},\mathrm{rot},\varepsilon} \leq \overline{\varepsilon}^2 c_{\mathtt{m},\Gamma_{\mathtt{t}},\mathrm{rot}}$. Interchanging $\Gamma_{\mathtt{t}}$ and $\Gamma_{\mathtt{n}}$ proves $c_{\mathtt{m},\Gamma_{\mathtt{n}},\mathrm{rot},\varepsilon,\mathrm{id}} \leq \overline{\varepsilon} c_{\mathtt{m},\Gamma_{\mathtt{n}},\mathrm{rot}} = \overline{\varepsilon} c_{\mathtt{m},\Gamma_{\mathtt{t}},\mathrm{rot}}$. By $\underline{\varepsilon}^{-1} |E|_{\mathsf{L}^2(\Omega)} \leq |E|_{\mathsf{L}^2_\varepsilon(\Omega)}$ and (3.2) resp. (3.3) resp. (3.4) we see $c_{\mathtt{m},\Gamma_{\mathtt{t}},\mathrm{rot}} \leq \underline{\varepsilon}^2 c_{\mathtt{m},\Gamma_{\mathtt{t}},\mathrm{rot},\varepsilon}$ resp. $\underline{\varepsilon}^{-1} c_{\mathtt{m},\Gamma_{\mathtt{t}},\mathrm{rot}} \leq c_{\mathtt{m},\Gamma_{\mathtt{t}},\mathrm{rot},\varepsilon,\mathrm{id}}, c_{\mathtt{m},\Gamma_{\mathtt{n}},\mathrm{rot},\varepsilon,\mathrm{id}}$. Using $|\operatorname{rot} E|_{\mathsf{L}^2(\Omega)} \leq \overline{\varepsilon} |\operatorname{rot} E|_{\mathsf{L}^2_{\varepsilon^{-1}}(\Omega)}$ and (3.3), (3.4) we get $\overline{\varepsilon}^{-1} c_{\mathtt{m},\Gamma_{\mathtt{t}},\mathrm{rot},\varepsilon} \leq c_{\mathtt{m},\Gamma_{\mathtt{t}},\mathrm{rot},\varepsilon,\mathrm{id}}, c_{\mathtt{m},\Gamma_{\mathtt{n}},\mathrm{rot},\varepsilon,\mathrm{id}}$, which completes the proof. $\square$

3.1.3 The Full Maxwell Estimates

Theorem 9 *For all* $E \in \mathsf{R}_{\Gamma_t}(\Omega) \cap \varepsilon^{-1}\mathsf{D}_{\Gamma_n}(\Omega)$ *the tangential-normal Maxwell estimate*

$$|E - \pi_{\mathtt{DN}} E|^2_{\mathsf{L}^2_\varepsilon(\Omega)} \leq c^2_{\mathtt{p},\Gamma_t,\varepsilon} |\operatorname{div} \varepsilon E|^2_{\mathsf{L}^2(\Omega)} + c^2_{\mathtt{m},\Gamma_t,\mathrm{rot},\varepsilon,\mathrm{id}} |\operatorname{rot} E|^2_{\mathsf{L}^2(\Omega)}$$

holds with sharp constants. Moreover, $c_{\mathtt{p},\Gamma_t,\varepsilon} \leq \underline{\varepsilon} c_{\mathtt{p},\Gamma_t}$ *and* $c_{\mathtt{m},\Gamma_t,\mathrm{rot},\varepsilon,\mathrm{id}} \leq \overline{\varepsilon} c_{\mathtt{m},\Gamma_t,\mathrm{rot}}$.

Proof By the Helmholtz decomposition (see Appendix A.2.2) we have

$$\mathsf{R}_{\Gamma_t}(\Omega) \cap \varepsilon^{-1}\mathsf{D}_{\Gamma_n}(\Omega) \cap \mathcal{H}_{\mathtt{DN},\varepsilon}(\Omega)^{\perp_\varepsilon} \ni E - \pi_{\mathtt{DN}} E = E_\nabla + E_{\mathrm{rot}} \in \nabla \mathsf{H}^1_{\Gamma_t}(\Omega) \oplus_\varepsilon \varepsilon^{-1} \operatorname{rot} \mathsf{R}_{\Gamma_n}(\Omega)$$

with

$$E_\nabla \in \varepsilon^{-1}\mathsf{D}_{\Gamma_n}(\Omega) \cap \nabla \mathsf{H}^1_{\Gamma_t}(\Omega) = \mathsf{R}_{\Gamma_t,0}(\Omega) \cap \varepsilon^{-1}\mathsf{D}_{\Gamma_n}(\Omega) \cap \mathcal{H}_{\mathtt{DN},\varepsilon}(\Omega)^{\perp_\varepsilon}, \quad \operatorname{div} \varepsilon E_\nabla = \operatorname{div} \varepsilon E,$$
$$E_{\mathrm{rot}} \in \mathsf{R}_{\Gamma_t}(\Omega) \cap \varepsilon^{-1} \operatorname{rot} \mathsf{R}_{\Gamma_n}(\Omega) = \mathsf{R}_{\Gamma_t}(\Omega) \cap \varepsilon^{-1}\mathsf{D}_{\Gamma_n,0}(\Omega) \cap \mathcal{H}_{\mathtt{DN},\varepsilon}(\Omega)^{\perp_\varepsilon}, \quad \operatorname{rot} E_{\mathrm{rot}} = \operatorname{rot} E.$$

Thus, by Lemma 3 and Corollary 7 as well as orthogonality we obtain

$$|E - \pi_{\mathtt{DN}} E|^2_{\mathsf{L}^2_\varepsilon(\Omega)} = |E_\nabla|^2_{\mathsf{L}^2_\varepsilon(\Omega)} + |E_{\mathrm{rot}}|^2_{\mathsf{L}^2_\varepsilon(\Omega)} \leq c^2_{\mathtt{p},\Gamma_t,\varepsilon} |\operatorname{div} \varepsilon E|^2_{\mathsf{L}^2(\Omega)} + c^2_{\mathtt{m},\Gamma_t,\mathrm{rot},\varepsilon,\mathrm{id}} |\operatorname{rot} E|^2_{\mathsf{L}^2(\Omega)}.$$

Lemmas 4 and 8 show the two estimates for the constants, completing the proof. □

Lemma 10 *It holds*

$$c_{\mathtt{m},\Gamma_t,\varepsilon} = \max\{c_{\mathtt{p},\Gamma_t,\varepsilon}, c_{\mathtt{m},\Gamma_t,\mathrm{rot},\varepsilon,\mathrm{id}}\} \begin{cases} \leq \max\{\underline{\varepsilon} c_{\mathtt{p},\Gamma_t}, \overline{\varepsilon} c_{\mathtt{m},\Gamma_t,\mathrm{rot}}\} \leq \hat{\varepsilon} \max\{c_{\mathtt{p},\Gamma_t}, c_{\mathtt{m},\Gamma_t,\mathrm{rot}}\} \\ \geq \max\{\overline{\varepsilon}^{-1} c_{\mathtt{p},\Gamma_t}, \underline{\varepsilon}^{-1} c_{\mathtt{m},\Gamma_t,\mathrm{rot}}\} \geq \hat{\varepsilon}^{-1} \max\{c_{\mathtt{p},\Gamma_t}, c_{\mathtt{m},\Gamma_t,\mathrm{rot}}\} \end{cases}$$

and for $\varepsilon = \mathrm{id}$

$$c_{\mathtt{m},\Gamma_t} = \max\{c_{\mathtt{p},\Gamma_t}, c_{\mathtt{m},\Gamma_t,\mathrm{rot}}\}.$$

Proof We have $c_{\mathtt{m},\Gamma_t,\varepsilon} \leq \max\{c_{\mathtt{p},\Gamma_t,\varepsilon}, c_{\mathtt{m},\Gamma_t,\mathrm{rot},\varepsilon,\mathrm{id}}\}$. Inserting $E \in \varepsilon^{-1}\mathsf{D}_{\Gamma_n}(\Omega) \cap \nabla \mathsf{H}^1_{\Gamma_t}(\Omega)$ resp. $E \in \mathsf{R}_{\Gamma_t}(\Omega) \cap \varepsilon^{-1} \operatorname{rot} \mathsf{R}_{\Gamma_n}(\Omega)$ into the tangential-normal Maxwell estimate (1.3) shows $c_{\mathtt{p},\Gamma_t,\varepsilon}, c_{\mathtt{m},\Gamma_t,\mathrm{rot},\varepsilon,\mathrm{id}} \leq c_{\mathtt{m},\Gamma_t,\varepsilon}$ and the first equation follows. The other estimates are given by Lemmas 4 and 8, completing the proof. □

By the latter theorem and lemma it remains to estimate only the two constants $c_{\mathtt{p},\Gamma_t}$ and $c_{\mathtt{m},\Gamma_t,\mathrm{rot}}$ for the various Γ_t.

3.2 Full Boundary Conditions

We summarize our results for the two important extreme cases $\Gamma_t = \Gamma$ resp. $\Gamma_t = \emptyset$, i.e., the full tangential resp. the full normal case, and emphasize that in these two cases the tangential and normal Maxwell rotation constants coincide by (3.1) and hence beside the Poincaré constants we just have to estimate one constant, namely

$$c_{\mathtt{m},\mathrm{rot},\varepsilon} := c_{\mathtt{m},\Gamma,\mathrm{rot},\varepsilon} = c_{\mathtt{m},\emptyset,\mathrm{rot},\varepsilon}, \quad c_{\mathtt{m},\mathrm{rot}} = c_{\mathtt{m},\Gamma,\mathrm{rot}} = c_{\mathtt{m},\emptyset,\mathrm{rot}}. \tag{3.5}$$

For the convenience of the reader let us recall our estimates from the latter sections in these two extreme cases. Lemmas 3 and 4 read:

Corollary 11 *The Poincaré constant* $c_{\mathtt{p},\Gamma,\varepsilon}$ *in* $\mathsf{H}^1_\Gamma(\Omega)$ *resp.* $c_{\mathtt{p},\varepsilon}$ *in* $\mathsf{H}^1_\emptyset(\Omega)$ *and the Maxwell divergence constant* $c_{\mathtt{m},\emptyset,\mathrm{div},\varepsilon}$ *in* $\varepsilon^{-1}\mathsf{D}(\Omega)\cap\nabla\mathsf{H}^1_\Gamma(\Omega)$ *resp.* $c_{\mathtt{m},\Gamma,\mathrm{div},\varepsilon}$ *in* $\varepsilon^{-1}\mathsf{D}_\Gamma(\Omega)\cap\nabla\mathsf{H}^1(\Omega)$ *equal, i.e., the inequalities*

$$\begin{aligned}
&\forall\, u\in\mathsf{H}^1_\Gamma(\Omega) && |u|_{\mathsf{L}^2(\Omega)}\leq c_{\mathtt{p},\Gamma,\varepsilon}|\nabla u|_{\mathsf{L}^2_\varepsilon(\Omega)},\\
&\forall\, E\in\varepsilon^{-1}\mathsf{D}(\Omega)\cap\nabla\mathsf{H}^1_\Gamma(\Omega) && |E|_{\mathsf{L}^2_\varepsilon(\Omega)}\leq c_{\mathtt{p},\Gamma,\varepsilon}|\operatorname{div}\varepsilon E|_{\mathsf{L}^2(\Omega)}
\end{aligned}$$

resp.

$$\begin{aligned}
&\forall\, u\in\mathsf{H}^1(\Omega)\cap\mathbb{R}^\perp && |u|_{\mathsf{L}^2(\Omega)}\leq c_{\mathtt{p},\varepsilon}|\nabla u|_{\mathsf{L}^2_\varepsilon(\Omega)},\\
&\forall\, E\in\varepsilon^{-1}\mathsf{D}_\Gamma(\Omega)\cap\nabla\mathsf{H}^1(\Omega) && |E|_{\mathsf{L}^2_\varepsilon(\Omega)}\leq c_{\mathtt{p},\varepsilon}|\operatorname{div}\varepsilon E|_{\mathsf{L}^2(\Omega)}
\end{aligned}$$

hold with sharp constants. Moreover, $\overline{\varepsilon}^{-1}c_{\mathtt{p},\Gamma}\leq c_{\mathtt{p},\Gamma,\varepsilon}\leq\underline{\varepsilon}c_{\mathtt{p},\Gamma}$ *and* $\overline{\varepsilon}^{-1}c_{\mathtt{p}}\leq c_{\mathtt{p},\varepsilon}\leq\underline{\varepsilon}c_{\mathtt{p}}$.

Here, $c_{\mathtt{p},\varepsilon}:=c_{\mathtt{p},\emptyset,\varepsilon}$. Corollary 7 and Lemma 8 read:

Corollary 12 *The tangential Maxwell rotation constant* $c_{\mathtt{m},\Gamma,\mathrm{rot},\varepsilon}$ *in* $\mathsf{R}_\Gamma(\Omega)\cap\varepsilon^{-1}\operatorname{rot}\mathsf{R}(\Omega)$ *and the normal Maxwell rotation constant* $c_{\mathtt{m},\emptyset,\mathrm{rot},\varepsilon}$ *in* $\mathsf{R}(\Omega)\cap\varepsilon^{-1}\operatorname{rot}\mathsf{R}_\Gamma(\Omega)$ *equal, i.e., for all* $E\in\big(\mathsf{R}_\Gamma(\Omega)\cap\varepsilon^{-1}\operatorname{rot}\mathsf{R}(\Omega)\big)\cup\big(\mathsf{R}(\Omega)\cap\varepsilon^{-1}\operatorname{rot}\mathsf{R}_\Gamma(\Omega)\big)$

$$|E|_{\mathsf{L}^2_\varepsilon(\Omega)}\leq c_{\mathtt{m},\mathrm{rot},\varepsilon}|\operatorname{rot}E|_{\mathsf{L}^2_{\varepsilon^{-1}}(\Omega)}\leq\underline{\varepsilon}c_{\mathtt{m},\mathrm{rot},\varepsilon}|\operatorname{rot}E|_{\mathsf{L}^2(\Omega)}$$

holds with sharp constants. Moreover, the inequalities

$$\begin{aligned}
&\forall\, E\in\mathsf{R}_\Gamma(\Omega)\cap\varepsilon^{-1}\operatorname{rot}\mathsf{R}(\Omega) && |E|_{\mathsf{L}^2_\varepsilon(\Omega)}\leq c_{\mathtt{m},\Gamma,\mathrm{rot},\varepsilon,\mathrm{id}}|\operatorname{rot}E|_{\mathsf{L}^2(\Omega)},\\
&\forall\, H\in\mathsf{R}(\Omega)\cap\varepsilon^{-1}\operatorname{rot}\mathsf{R}_\Gamma(\Omega) && |H|_{\mathsf{L}^2_\varepsilon(\Omega)}\leq c_{\mathtt{m},\emptyset,\mathrm{rot},\varepsilon,\mathrm{id}}|\operatorname{rot}H|_{\mathsf{L}^2(\Omega)}
\end{aligned}$$

hold, where these sharp constants do not need to coincide if $\varepsilon\neq\mathrm{id}$. *Moreover, it holds* $\underline{\varepsilon}^{-2}c_{\mathtt{m},\mathrm{rot}}\leq c_{\mathtt{m},\mathrm{rot},\varepsilon}\leq\overline{\varepsilon}^2c_{\mathtt{m},\mathrm{rot}}$ *and*

$$\begin{aligned}
\underline{\varepsilon}^{-1}c_{\mathtt{m},\mathrm{rot}}\leq\max\{\overline{\varepsilon}^{-1}c_{\mathtt{m},\mathrm{rot},\varepsilon},\underline{\varepsilon}^{-1}c_{\mathtt{m},\mathrm{rot}}\}&\leq c_{\mathtt{m},\Gamma,\mathrm{rot},\varepsilon,\mathrm{id}},c_{\mathtt{m},\emptyset,\mathrm{rot},\varepsilon,\mathrm{id}}\\
&\leq\min\{\underline{\varepsilon}c_{\mathtt{m},\mathrm{rot},\varepsilon},\overline{\varepsilon}c_{\mathtt{m},\mathrm{rot}}\}\leq\overline{\varepsilon}c_{\mathtt{m},\mathrm{rot}}.
\end{aligned}$$

Theorem 9 and Lemma 10 read:

Corollary 13 *For all* $E\in\mathsf{R}_\Gamma(\Omega)\cap\varepsilon^{-1}\mathsf{D}(\Omega)$ *and all* $H\in\mathsf{R}(\Omega)\cap\varepsilon^{-1}\mathsf{D}_\Gamma(\Omega)$ *the tangential and normal Maxwell estimates*

$$\begin{aligned}
|E-\pi_{\mathtt{D}}E|^2_{\mathsf{L}^2_\varepsilon(\Omega)}&\leq c^2_{\mathtt{p},\Gamma,\varepsilon}|\operatorname{div}\varepsilon E|^2_{\mathsf{L}^2(\Omega)}+c^2_{\mathtt{m},\Gamma,\mathrm{rot},\varepsilon,\mathrm{id}}|\operatorname{rot}E|^2_{\mathsf{L}^2(\Omega)},\\
|H-\pi_{\mathtt{N}}H|^2_{\mathsf{L}^2_\varepsilon(\Omega)}&\leq c^2_{\mathtt{p},\varepsilon}|\operatorname{div}\varepsilon H|^2_{\mathsf{L}^2(\Omega)}+c^2_{\mathtt{m},\emptyset,\mathrm{rot},\varepsilon,\mathrm{id}}|\operatorname{rot}H|^2_{\mathsf{L}^2(\Omega)}
\end{aligned}$$

hold with sharp constants. Furthermore, the estimates $\overline{\varepsilon}^{-1}c_{\mathtt{p},\Gamma} \leq c_{\mathtt{p},\Gamma,\varepsilon}, c_{\mathtt{p},\varepsilon} \leq \underline{\varepsilon}c_{\mathtt{p}}$ *and* $\underline{\varepsilon}^{-1}c_{\mathtt{m},\mathrm{rot}} \leq c_{\mathtt{m},\Gamma,\mathrm{rot},\varepsilon,\mathrm{id}}, c_{\mathtt{m},\emptyset,\mathrm{rot},\varepsilon,\mathrm{id}} \leq \overline{\varepsilon}c_{\mathtt{m},\mathrm{rot}}$ *as well as*

$$c_{\mathtt{m},\Gamma,\varepsilon} = \max\{c_{\mathtt{p},\Gamma,\varepsilon}, c_{\mathtt{m},\Gamma,\mathrm{rot},\varepsilon,\mathrm{id}}\} \begin{cases} \leq \max\{\underline{\varepsilon}c_{\mathtt{p},\Gamma}, \overline{\varepsilon}c_{\mathtt{m},\mathrm{rot}}\} \leq \hat{\varepsilon}\max\{c_{\mathtt{p},\Gamma}, c_{\mathtt{m},\mathrm{rot}}\}, \\ \geq \max\{\overline{\varepsilon}^{-1}c_{\mathtt{p},\Gamma}, \underline{\varepsilon}^{-1}c_{\mathtt{m},\mathrm{rot}}\} \geq \hat{\varepsilon}^{-1}\max\{c_{\mathtt{p},\Gamma}, c_{\mathtt{m},\mathrm{rot}}\}, \end{cases}$$

$$c_{\mathtt{m},\emptyset,\varepsilon} = \max\{c_{\mathtt{p},\varepsilon}, c_{\mathtt{m},\emptyset,\mathrm{rot},\varepsilon,\mathrm{id}}\} \begin{cases} \leq \max\{\underline{\varepsilon}c_{\mathtt{p}}, \overline{\varepsilon}c_{\mathtt{m},\mathrm{rot}}\} \leq \hat{\varepsilon}\max\{c_{\mathtt{p}}, c_{\mathtt{m},\mathrm{rot}}\}, \\ \geq \max\{\overline{\varepsilon}^{-1}c_{\mathtt{p}}, \underline{\varepsilon}^{-1}c_{\mathtt{m},\mathrm{rot}}\} \geq \hat{\varepsilon}^{-1}\max\{c_{\mathtt{p}}, c_{\mathtt{m},\mathrm{rot}}\} \end{cases}$$

hold. Therefore, in both cases

$$\begin{aligned} \hat{\varepsilon}^{-1}\max\{c_{\mathtt{p},\Gamma}, c_{\mathtt{m},\mathrm{rot}}\} \leq \max\{\overline{\varepsilon}^{-1}c_{\mathtt{p},\Gamma}, \underline{\varepsilon}^{-1}c_{\mathtt{m},\mathrm{rot}}\} &\leq c_{\mathtt{m},\Gamma,\varepsilon}, c_{\mathtt{m},\emptyset,\varepsilon} \\ &\leq \max\{\underline{\varepsilon}c_{\mathtt{p}}, \overline{\varepsilon}c_{\mathtt{m},\mathrm{rot}}\} \leq \hat{\varepsilon}\max\{c_{\mathtt{p}}, c_{\mathtt{m},\mathrm{rot}}\}. \end{aligned}$$

For $\varepsilon = \mathrm{id}$ *it holds*

$$c_{\mathtt{m},\Gamma} = \max\{c_{\mathtt{p},\Gamma}, c_{\mathtt{m},\mathrm{rot}}\}, \quad c_{\mathtt{m},\emptyset} = \max\{c_{\mathtt{p}}, c_{\mathtt{m},\mathrm{rot}}\}.$$

As the two Poincaré constants $c_{\mathtt{p},\Gamma} < c_{\mathtt{p}}$ are more or less well known, by the latter corollaries it remains only to estimate the Maxwell constant $c_{\mathtt{m},\mathrm{rot}}$.

3.2.1 Convex Domains

Now, let $\Omega \subset \mathbb{R}^3$ be a bounded and convex domain. Then Ω is strongly Lipschitz, see e.g. [6, Corollary 1.2.2.3]. Moreover, there are no Dirichlet or Neumann fields since Ω is simply connected and has a connected boundary. As noted before in (1.8), in the convex case we can estimate the Poincaré constant $c_{\mathtt{p}}$ by the diameter of Ω, i.e.,

$$c_{\mathtt{p},\Gamma} < c_{\mathtt{p}} \leq \frac{\mathrm{diam}(\Omega)}{\pi}.$$

We show that we can also estimate the Maxwell constant $c_{\mathtt{m},\mathrm{rot}}$ in the two extreme cases $\Gamma_{\mathtt{t}} = \Gamma$ resp. $\Gamma_{\mathtt{t}} = \emptyset$ by $c_{\mathtt{p}}$. In [1, Theorem 2.17] the following crucial lemma has been proved, which is the key point in our investigations for convex domains.

Lemma 14 *Let* E *belong to* $\mathsf{R}_\Gamma(\Omega) \cap \mathsf{D}(\Omega)$ *or* $\mathsf{R}(\Omega) \cap \mathsf{D}_\Gamma(\Omega)$*. Then* $E \in \mathsf{H}^1(\Omega)$ *and*

$$|\nabla E|^2_{\mathsf{L}^2(\Omega)} \leq |\operatorname{rot} E|^2_{\mathsf{L}^2(\Omega)} + |\operatorname{div} E|^2_{\mathsf{L}^2(\Omega)}. \tag{3.6}$$

We note that the latter lemma has already been proved in [19] in the case $\mathsf{R}_\Gamma(\Omega) \cap \mathsf{D}(\Omega)$.

Remark 15 *For* $E \in \mathsf{H}^1_\Gamma(\Omega)$ *it is clear that for any domain* $\Omega \subset \mathbb{R}^3$ *(or even in* $\mathbb{R}^N$*)*

$$|\nabla E|^2_{\mathsf{L}^2(\Omega)} = |\operatorname{rot} E|^2_{\mathsf{L}^2(\Omega)} + |\operatorname{div} E|^2_{\mathsf{L}^2(\Omega)}$$

holds since $-\Delta = \operatorname{rot}\operatorname{rot} - \nabla\operatorname{div}$*. In general, this formula is no longer valid if* E *has just the tangential or normal boundary condition.*

With the help of Lemma 14 we can now estimate $c_{\mathtt{m},\mathrm{rot}}$.

Lemma 16 $c_{\mathtt{m},\mathrm{rot}} \leq c_{\mathtt{p}}$. *More precisely, for all E in $\mathsf{R}_\Gamma(\Omega) \cap \operatorname{rot} \mathsf{R}(\Omega)$ or $\mathsf{R}(\Omega) \cap \operatorname{rot} \mathsf{R}_\Gamma(\Omega)$*

$$|E|_{\mathsf{L}^2(\Omega)} \leq c_{\mathtt{p}} |\operatorname{rot} E|_{\mathsf{L}^2(\Omega)}.$$

Furthermore, $c_{\mathtt{m},\Gamma,\mathrm{rot},\varepsilon,\mathrm{id}}, c_{\mathtt{m},\emptyset,\mathrm{rot},\varepsilon,\mathrm{id}} \leq \overline{\varepsilon} c_{\mathtt{p}}$.

Proof By (3.5) the boundary condition does not matter. So, let

$$E \in \mathsf{R}(\Omega) \cap \operatorname{rot} \mathsf{R}_\Gamma(\Omega) = \mathsf{R}(\Omega) \cap \mathsf{D}_{\Gamma,0}(\Omega)$$

with $E = \operatorname{rot} H$ for some $H \in \mathsf{R}_\Gamma(\Omega)$. Then, for any constant vector $a \in \mathbb{R}^3$

$$\langle E, a \rangle_{\mathsf{L}^2(\Omega)} = \langle \operatorname{rot} H, a \rangle_{\mathsf{L}^2(\Omega)} = 0 \tag{3.7}$$

holds. Thus, by Poincaré's estimate and Lemma 14 we get $E \in \mathsf{H}^1(\Omega) \cap (\mathbb{R}^3)^\perp$ and

$$|E|_{\mathsf{L}^2(\Omega)} \leq c_{\mathtt{p}} |\nabla E|_{\mathsf{L}^2(\Omega)} \leq c_{\mathtt{p}} |\operatorname{rot} E|_{\mathsf{L}^2(\Omega)},$$

which shows $c_{\mathtt{m},\mathrm{rot}} = c_{\mathtt{m},\emptyset,\mathrm{rot}} \leq c_{\mathtt{p}}$. □

We can now formulate the main result for convex domains, which follows immediately from Corollary 13 and Lemma 16.

Theorem 17 *For all $E \in \mathsf{R}_\Gamma(\Omega) \cap \varepsilon^{-1}\mathsf{D}(\Omega)$ and all $H \in \mathsf{R}(\Omega) \cap \varepsilon^{-1}\mathsf{D}_\Gamma(\Omega)$ the tangential and normal Maxwell estimates*

$$|E|^2_{\mathsf{L}^2_\varepsilon(\Omega)} \leq \underline{\varepsilon}^2 c^2_{\mathtt{p},\Gamma} |\operatorname{div} \varepsilon E|^2_{\mathsf{L}^2(\Omega)} + \overline{\varepsilon}^2 c^2_{\mathtt{p}} |\operatorname{rot} E|^2_{\mathsf{L}^2(\Omega)},$$
$$|H|^2_{\mathsf{L}^2_\varepsilon(\Omega)} \leq \underline{\varepsilon}^2 c^2_{\mathtt{p}} |\operatorname{div} \varepsilon H|^2_{\mathsf{L}^2(\Omega)} + \overline{\varepsilon}^2 c^2_{\mathtt{p}} |\operatorname{rot} H|^2_{\mathsf{L}^2(\Omega)}$$

hold. Moreover,

$$\frac{c_{\mathtt{p},\Gamma}}{\overline{\varepsilon}} \leq c_{\mathtt{m},\Gamma,\varepsilon} \leq \hat{\varepsilon} c_{\mathtt{p}}, \qquad \frac{c_{\mathtt{p}}}{\overline{\varepsilon}} \leq c_{\mathtt{m},\emptyset,\varepsilon} \leq \hat{\varepsilon} c_{\mathtt{p}}.$$

Especially, for $\varepsilon = \mathrm{id}$

$$\max\{c_{\mathtt{p},\Gamma}, c_{\mathtt{m},\mathrm{rot}}\} = c_{\mathtt{m},\Gamma} \leq c_{\mathtt{m},\emptyset} = c_{\mathtt{p}}.$$

Theorem 18 *For all $E \in \big(\mathsf{R}_\Gamma(\Omega) \cap \varepsilon^{-1}\mathsf{D}(\Omega)\big) \cup \big(\mathsf{R}(\Omega) \cap \varepsilon^{-1}\mathsf{D}_\Gamma(\Omega)\big)$*

$$|E|_{\mathsf{L}^2_\varepsilon(\Omega)} \leq \hat{\varepsilon} c_{\mathtt{p}} \big(|\operatorname{div} \varepsilon E|^2_{\mathsf{L}^2(\Omega)} + |\operatorname{rot} E|^2_{\mathsf{L}^2(\Omega)}\big)^{1/2}.$$

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A Appendix

A.1 More General Operators

There are obvious generalizations to differential forms. Let Ω be a smooth Riemannian manifold of dimension $N \geq 2$ with boundary Γ and compact closure. We assume that the boundary manifold Γ is divided into two $(N-1)$-dimensional Riemannian sub-manifolds $\Gamma_{\mathtt{t}}$ and $\Gamma_{\mathtt{n}}$ with boundaries. Let us denote by $\mathsf{L}^{2,q}(\Omega)$ the usual Lebesgue (Hilbert) space of q-forms. For the exterior derivative and co-derivative we define the well known Sobolev spaces

$$\mathsf{D}^q(\Omega) := \{E \in \mathsf{L}^{2,q}(\Omega) \,:\, \mathrm{d}\, E \in \mathsf{L}^{2,q+1}(\Omega)\}, \quad \Delta^q(\Omega) := \{E \in \mathsf{L}^{2,q}(\Omega) \,:\, \delta\, E \in \mathsf{L}^{2,q-1}(\Omega)\}.$$

As before, we introduce weak homogeneous boundary conditions by closures of respective test forms, yielding the Sobolev spaces

$$\mathsf{D}^q_{\Gamma_{\mathtt{t}}}(\Omega), \quad \Delta^q_{\Gamma_{\mathtt{n}}}(\Omega).$$

Let A be

$$\mu^{-1}\,\mathrm{d} : \mathsf{D}^q_{\Gamma_\mathtt{t}}(\Omega) \subset \mathsf{L}^{2,q}_\varepsilon(\Omega) \to \mathsf{L}^{2,q+1}_\mu(\Omega).$$

Then A^* is

$$-\varepsilon^{-1}\,\delta : \Delta^{q+1}_{\Gamma_\mathtt{n}}(\Omega) \subset \mathsf{L}^{2,q+1}_\mu(\Omega) \to \mathsf{L}^{2,q}_\varepsilon(\Omega),$$

where ε resp. μ are bounded, symmetric, real and uniformly positive definite linear transformations on q- resp. $(q+1)$-forms. More precisely:

A	$D(\mathrm{A})$	X	Y	$N(\mathrm{A})$	$R(\mathrm{A})$
$\mu^{-1}\,\mathrm{d}$	$\mathsf{D}^q_{\Gamma_\mathtt{t}}(\Omega)$	$\mathsf{L}^{2,q}_\varepsilon(\Omega)$	$\mathsf{L}^{2,q+1}_\mu(\Omega)$	$\mathsf{D}^q_{\Gamma_\mathtt{t},0}(\Omega)$	$\mu^{-1}\,\mathrm{d}\,\mathsf{D}^q_{\Gamma_\mathtt{t}}(\Omega)$
A^*	$D(\mathrm{A}^*)$	Y	X	$N(\mathrm{A}^*)$	$R(\mathrm{A}^*)$
$-\varepsilon^{-1}\,\delta$	$\Delta^{q+1}_{\Gamma_\mathtt{n}}(\Omega)$	$\mathsf{L}^{2,q+1}_\mu(\Omega)$	$\mathsf{L}^{2,q}_\varepsilon(\Omega)$	$\Delta^{q+1}_{\Gamma_\mathtt{n},0}(\Omega)$	$\varepsilon^{-1}\,\delta\,\Delta^{q+1}_{\Gamma_\mathtt{n}}(\Omega)$

Here,

$$\mathsf{D}^q_{\Gamma_\mathtt{t},0}(\Omega) := \{E \in \mathsf{D}^q_{\Gamma_\mathtt{t}}(\Omega) \,:\, \mathrm{d}\,E = 0\}, \quad \Delta^q_{\Gamma_\mathtt{n},0}(\Omega) := \{E \in \Delta^q_{\Gamma_\mathtt{n}}(\Omega) \,:\, \delta\,E = 0\}$$

and we note

$$R(\mathrm{A}) = \mu^{-1}\big(\mathsf{D}^{q+1}_{\Gamma_\mathtt{t},0}(\Omega) \cap \mathcal{H}^{q+1}_{\mathtt{DN}}(\Omega)^\perp\big), \quad R(\mathrm{A}^*) = \varepsilon^{-1}\big(\Delta^q_{\Gamma_\mathtt{n},0}(\Omega) \cap \mathcal{H}^q_{\mathtt{DN}}(\Omega)^\perp\big),$$

where $\mathcal{H}^q_{\mathtt{DN}}(\Omega) := \mathsf{D}^q_{\Gamma_\mathtt{t},0}(\Omega) \cap \Delta^q_{\Gamma_\mathtt{n},0}(\Omega)$. Indeed $D(\mathrm{A}^*) = \Delta^{q+1}_{\Gamma_\mathtt{n}}(\Omega)$ holds. We have the same remarks as in Section 3.1.2. Again, for this one has to show the approximation property

$$\Delta^{q+1}_{\Gamma_\mathtt{n}}(\Omega) = \{H \in \Delta^{q+1}(\Omega) \,:\, \langle \delta\,H, E\rangle_{\mathsf{L}^{2,q}(\Omega)} = -\langle H, \mathrm{d}\,E\rangle_{\mathsf{L}^{2,q+1}(\Omega)}\ \forall\, E \in \mathsf{D}^q_{\Gamma_\mathtt{t}}(\Omega)\},$$

which is not trivial at all for mixed boundary conditions. And again, only in the special cases of full boundary conditions this is clear. Since $D(\mathrm{A}^*) = \Delta^{q+1}(\Omega)$ holds for $\Gamma_\mathtt{t} = \Gamma$ by definition we have also $D(\mathrm{B}^*) = D(\mathrm{A}^{**}) = D(\mathrm{A}) = \mathsf{D}^q_\Gamma(\Omega)$ for $\mathrm{B} = \mathrm{A}^*$, which shows the result for $\Gamma_\mathtt{t} = \emptyset$. The crucial compact embedding (2.3) is

$$\mathsf{D}^q_{\Gamma_\mathtt{t}}(\Omega) \cap \varepsilon^{-1}\overline{\delta\,\Delta^{q+1}_{\Gamma_\mathtt{n}}(\Omega)} \hookrightarrow \mathsf{L}^{2,q}_\varepsilon(\Omega).$$

Both latter properties of Ω, i.e., the approximation and the compactness property, hold, e.g., if the boundary manifolds Γ, $\Gamma_\mathtt{t}$, $\Gamma_\mathtt{n}$ are Lipschitz and the boundary manifolds $\Gamma_\mathtt{t}$, $\Gamma_\mathtt{n}$ are separated by a $(N-2)$-dimensional Riemannian and Lipschitz sub-manifold, the interface $\gamma := \overline{\Gamma_\mathtt{t}} \cap \overline{\Gamma_\mathtt{n}}$, see [5, 7] for details and proofs. We note that

$$\mathsf{D}^q_{\Gamma_\mathtt{t}}(\Omega) \cap \varepsilon^{-1}\overline{\delta\,\Delta^{q+1}_{\Gamma_\mathtt{n}}(\Omega)} \subset \mathsf{D}^q_{\Gamma_\mathtt{t}}(\Omega) \cap \varepsilon^{-1}\Delta^q_{\Gamma_\mathtt{n},0}(\Omega) \subset \mathsf{D}^q_{\Gamma_\mathtt{t}}(\Omega) \cap \varepsilon^{-1}\Delta^q_{\Gamma_\mathtt{n}}(\Omega)$$

holds and that even the compact embedding of the latter space into $\mathsf{L}^{2,q}(\Omega)$, this is

$$\mathsf{D}^q_{\Gamma_\mathtt{t}}(\Omega) \cap \varepsilon^{-1}\Delta^q_{\Gamma_\mathtt{n}}(\Omega) \hookrightarrow \mathsf{L}^{2,q}(\Omega) \subset \mathsf{L}^{2,q}_\varepsilon(\Omega),$$

has been shown in [7][iv]. By Theorem 2 we have

$$\kappa := \min_{0\neq E\in\mathsf{D}^q_{\Gamma_t}(\Omega)\cap\varepsilon^{-1}\,\delta\,\Delta^{q+1}_{\Gamma_n}(\Omega)} \frac{|\mu^{-1}\,\mathrm{d}\,E|_{\mathsf{L}^{2,q+1}_\mu(\Omega)}}{|E|_{\mathsf{L}^{2,q}_\varepsilon(\Omega)}} = \min_{0\neq H\in\Delta^{q+1}_{\Gamma_n}(\Omega)\cap\mu^{-1}\,\mathrm{d}\,\mathsf{D}^q_{\Gamma_t}(\Omega)} \frac{|\varepsilon^{-1}\,\delta\,H|_{\mathsf{L}^{2,q}_\varepsilon(\Omega)}}{|H|_{\mathsf{L}^{2,q+1}_\mu(\Omega)}}$$

and κ^2 is the first positive Dirichlet-Neumann eigenvalue of the weighted reduced δ-d-operator $-\varepsilon^{-1}\,\delta\,\mu^{-1}\,\mathrm{d}$. Analogously κ^2 is also the first positive Neumann-Dirichlet eigenvalue of the weighted reduced d-δ-operator $-\mu^{-1}\,\mathrm{d}\,\varepsilon^{-1}\,\delta$.

Lemma 19 *The tangential-normal and normal-tangential generalized Maxwell constants, i.e., the best constants in the inequalities*

$$\forall\, E\in\mathsf{D}^q_{\Gamma_t}(\Omega)\cap\varepsilon^{-1}\,\delta\,\Delta^{q+1}_{\Gamma_n}(\Omega) \qquad |E|_{\mathsf{L}^{2,q}_\varepsilon(\Omega)}\leq c_{\mathtt{gm},\Gamma_t,\mathrm{d},\varepsilon,\mu}|\,\mathrm{d}\,E|_{\mathsf{L}^{2,q+1}_{\mu^{-1}}(\Omega)},$$

$$\forall\, H\in\Delta^{q+1}_{\Gamma_n}(\Omega)\cap\mu^{-1}\,\mathrm{d}\,\mathsf{D}^q_{\Gamma_t}(\Omega) \qquad |H|_{\mathsf{L}^{2,q+1}_\mu(\Omega)}\leq c_{\mathtt{gm},\Gamma_n,\delta,\mu,\varepsilon}|\,\delta\,H|_{\mathsf{L}^{2,q}_{\varepsilon^{-1}}(\Omega)},$$

coincide and equal to $1/\kappa$, *i.e.,* $c_{\mathtt{gm},\Gamma_t,\mathrm{d},\varepsilon,\mu}=c_{\mathtt{gm},\Gamma_n,\delta,\mu,\varepsilon}=\kappa^{-1}$.

Remark 20 *It is clear that more results of this contribution can be generalized to the differential form setting.*

A.2 Maxwell Tools

Let the general assumptions from the introduction be satisfied.

A.2.1 The Maxwell Estimates

By the Maxwell compactness property we get immediately the Maxwell estimate.

Lemma 21 *There exists* $c_{\mathtt{m},\Gamma_t,\varepsilon}>0$, *such that for all* E *in* $\mathsf{R}_{\Gamma_t}(\Omega)\cap\varepsilon^{-1}\mathsf{D}_{\Gamma_n}(\Omega)\cap\mathcal{H}_{\mathtt{DN},\varepsilon}(\Omega)^{\perp_\varepsilon}$

$$|E|_{\mathsf{L}^2_\varepsilon(\Omega)}\leq c_{\mathtt{m},\Gamma_t,\varepsilon}\big(|\operatorname{rot}E|^2_{\mathsf{L}^2(\Omega)}+|\operatorname{div}\varepsilon E|^2_{\mathsf{L}^2(\Omega)}\big)^{1/2}.$$

Proof If the estimate would not hold, there would exist a sequence of vector fields $(E_n)\subset\mathsf{R}_{\Gamma_t}(\Omega)\cap\varepsilon^{-1}\mathsf{D}_{\Gamma_n}(\Omega)\cap\mathcal{H}_{\mathtt{DN},\varepsilon}(\Omega)^{\perp_\varepsilon}$ with $|E_n|_{\mathsf{L}^2_\varepsilon(\Omega)}=1$ and

$$|\operatorname{rot}E_n|_{\mathsf{L}^2(\Omega)}+|\operatorname{div}\varepsilon E_n|_{\mathsf{L}^2(\Omega)}<\frac{1}{n}.$$

By the Maxwell compactness property we can assume w.l.o.g. that (E_n) converges in $\mathsf{L}^2_\varepsilon(\Omega)$ to some $E\in\mathsf{L}^2_\varepsilon(\Omega)$. By testing, E belongs to $\mathsf{R}_0(\Omega)\cap\varepsilon^{-1}\mathsf{D}_0(\Omega)\cap\mathcal{H}_{\mathtt{DN},\varepsilon}(\Omega)^{\perp_\varepsilon}$ and (E_n) converges to E also in $\mathsf{R}(\Omega)\cap\varepsilon^{-1}\mathsf{D}(\Omega)$. As $\mathsf{R}_{\Gamma_t}(\Omega)$ resp. $\mathsf{D}_{\Gamma_n}(\Omega)$ is a closed subspace of $\mathsf{R}(\Omega)$ resp. $\mathsf{D}(\Omega)$, E belongs even to $\mathsf{R}_{\Gamma_t,0}(\Omega)\cap\varepsilon^{-1}\mathsf{D}_{\Gamma_n,0}(\Omega)=\mathcal{H}_{\mathtt{DN},\varepsilon}(\Omega)$. Hence, $E=0$, which contradicts $1=|E_n|_{\mathsf{L}^2(\Omega)}\to 0$. □

[iv] In [7] it is proved that $\mathsf{D}^q_{\Gamma_t}(\Omega)\cap\Delta^q_{\Gamma_n}(\Omega)$ even embeds continuously to $\mathsf{H}^{1/2,q}(\Omega)$ and hence compactly to $\mathsf{L}^{2,q}(\Omega)$. We note that the compactness property is independent of ε, see e.g. [9].

Corollary 22 *For all E in* $\mathsf{R}_{\Gamma_t}(\Omega) \cap \varepsilon^{-1}\mathsf{D}_{\Gamma_n}(\Omega)$

$$|(1-\pi_{\mathtt{DN}})E|_{\mathsf{L}^2_\varepsilon(\Omega)} \leq c_{\mathtt{m},\Gamma_t,\varepsilon}\big(|\operatorname{rot} E|^2_{\mathsf{L}^2(\Omega)} + |\operatorname{div}\varepsilon E|^2_{\mathsf{L}^2(\Omega)}\big)^{1/2}.$$

Proof As $H := (1-\pi_{\mathtt{DN}})E \in \mathsf{R}_{\Gamma_t}(\Omega) \cap \varepsilon^{-1}\mathsf{D}_{\Gamma_n}(\Omega) \cap \mathcal{H}_{\mathtt{DN},\varepsilon}(\Omega)^{\perp_\varepsilon}$ with $\operatorname{div}\varepsilon H = \operatorname{div}\varepsilon E$ and $\operatorname{rot} H = \operatorname{rot} E$, Lemma 21 completes the proof. □

The same arguments show that the Maxwell estimate remains valid in any dimension and even for compact Riemannian manifolds, as long as the crucial Maxwell compactness property holds.

A.2.2 Helmholtz-Weyl Decompositions

By the projection theorem we have for the operator ∇

$$\mathsf{L}^2_\varepsilon(\Omega) = \overline{\nabla\mathsf{H}^1_{\Gamma_t}(\Omega)} \oplus_\varepsilon \varepsilon^{-1}\mathsf{D}_{\Gamma_n,0}(\Omega),$$

where indeed $\big(\nabla\mathsf{H}^1_{\Gamma_t}(\Omega)\big)^\perp = \mathsf{D}_{\Gamma_n,0}(\Omega)$ holds by [8]. Note that $\nabla\mathsf{H}^1_{\Gamma_t}(\Omega)$ is already closed by Rellich's selection theorem. Analogously, we obtain for the operator rot

$$\mathsf{L}^2_\varepsilon(\Omega) = \mathsf{R}_{\Gamma_t,0}(\Omega) \oplus_\varepsilon \varepsilon^{-1}\overline{\operatorname{rot}\mathsf{R}_{\Gamma_n}(\Omega)}, \tag{A.1}$$

where again and indeed $\big(\operatorname{rot}\mathsf{R}_{\Gamma_n}(\Omega)\big)^\perp = \mathsf{R}_{\Gamma_t,0}(\Omega)$ holds by [8]. For $\varepsilon = \mathrm{id}$ we get by (A.1)

$$\mathsf{R}_{\Gamma_t}(\Omega) = \mathsf{R}_{\Gamma_t,0}(\Omega) \oplus \big(\mathsf{R}_{\Gamma_t}(\Omega) \cap \overline{\operatorname{rot}\mathsf{R}_{\Gamma_n}(\Omega)}\big)$$

and therefore

$$\operatorname{rot}\mathsf{R}_{\Gamma_t}(\Omega) = \operatorname{rot}\big(\mathsf{R}_{\Gamma_t}(\Omega) \cap \overline{\operatorname{rot}\mathsf{R}_{\Gamma_n}(\Omega)}\big).$$

As $\overline{\operatorname{rot}\mathsf{R}_{\Gamma_n}(\Omega)} \subset \mathsf{D}_{\Gamma_n,0}(\Omega) \cap \mathcal{H}_{\mathtt{DN}}(\Omega)^\perp$, the Maxwell estimate Lemma 21 implies that also $\operatorname{rot}\mathsf{R}_{\Gamma_t}(\Omega)$ is already closed. Moreover,

$$\operatorname{rot}\mathsf{R}_{\Gamma_t}(\Omega) = \operatorname{rot}\mathcal{R}_{\Gamma_t}(\Omega), \quad \mathcal{R}_{\Gamma_t}(\Omega) := \mathsf{R}_{\Gamma_t}(\Omega) \cap \operatorname{rot}\mathsf{R}_{\Gamma_n}(\Omega) = \mathsf{R}_{\Gamma_t}(\Omega) \cap \operatorname{rot}\mathcal{R}_{\Gamma_n}(\Omega).$$

Since $\nabla\mathsf{H}^1_{\Gamma_t}(\Omega) \subset \mathsf{R}_{\Gamma_t,0}(\Omega)$ and $\operatorname{rot}\mathsf{R}_{\Gamma_n}(\Omega) \subset \mathsf{D}_{\Gamma_n,0}(\Omega)$ we obtain

$$\mathsf{R}_{\Gamma_t,0}(\Omega) = \nabla\mathsf{H}^1_{\Gamma_t}(\Omega) \oplus_\varepsilon \big(\underbrace{\mathsf{R}_{\Gamma_t,0}(\Omega) \cap \varepsilon^{-1}\mathsf{D}_{\Gamma_n,0}(\Omega)}_{=\mathcal{H}_{\mathtt{DN},\varepsilon}(\Omega)}\big),$$

$$\varepsilon^{-1}\mathsf{D}_{\Gamma_n,0}(\Omega) = \varepsilon^{-1}\operatorname{rot}\mathsf{R}_{\Gamma_n}(\Omega) \oplus_\varepsilon \big(\overbrace{\mathsf{R}_{\Gamma_t,0}(\Omega) \cap \varepsilon^{-1}\mathsf{D}_{\Gamma_n,0}(\Omega)}\big).$$

Finally, we have the well known Helmholtz decompositions:

Lemma 23 *It holds*

$$\begin{aligned}\mathsf{L}^2_\varepsilon(\Omega) &= \nabla\mathsf{H}^1_{\Gamma_t}(\Omega)\oplus_\varepsilon \varepsilon^{-1}\mathsf{D}_{\Gamma_n,0}(\Omega) = \mathsf{R}_{\Gamma_t,0}(\Omega)\oplus_\varepsilon \varepsilon^{-1}\operatorname{rot}\mathcal{R}_{\Gamma_n}(\Omega)\\ &= \nabla\mathsf{H}^1_{\Gamma_t}(\Omega)\oplus_\varepsilon \mathcal{H}_{\mathtt{DN},\varepsilon}(\Omega)\oplus_\varepsilon \varepsilon^{-1}\operatorname{rot}\mathcal{R}_{\Gamma_n}(\Omega)\end{aligned}$$

as well as

$$\nabla\mathsf{H}^1_{\Gamma_t}(\Omega) = \mathsf{R}_{\Gamma_t,0}(\Omega)\cap\mathcal{H}_{\mathtt{DN},\varepsilon}(\Omega)^{\perp_\varepsilon},\quad \varepsilon^{-1}\operatorname{rot}\mathcal{R}_{\Gamma_n}(\Omega) = \varepsilon^{-1}\mathsf{D}_{\Gamma_n,0}(\Omega)\cap\mathcal{H}_{\mathtt{DN},\varepsilon}(\Omega)^{\perp_\varepsilon}$$

and $\mathcal{R}_{\Gamma_t}(\Omega) = \mathsf{R}_{\Gamma_t}(\Omega)\cap\mathsf{D}_{\Gamma_n,0}(\Omega)\cap\mathcal{H}_{\mathtt{DN}}(\Omega)^{\perp}$.

A.3 Functional Analytical Tools

Let us recall that for a self-adjoint operator $\mathrm{T}: D(\mathrm{T})\subset\mathsf{H}\to\mathsf{H}$, where H denotes some Hilbert space,

$$\mathbb{C}\setminus\mathbb{R}\subset\rho(\mathrm{T}),\quad \sigma(\mathrm{T}) = \sigma_{\mathtt{p}}(\mathrm{T})\cup\sigma_{\mathtt{c}}(\mathrm{T})\subset\mathbb{R},\quad \sigma_{\mathtt{r}}(\mathrm{T}) = \emptyset$$

hold. Here, $\rho(\mathrm{T})$, $\sigma(\mathrm{T})$, $\sigma_{\mathtt{p}}(\mathrm{T})$, $\sigma_{\mathtt{c}}(\mathrm{T})$, $\sigma_{\mathtt{r}}(\mathrm{T})$ denote the resolvent set, the spectrum, the point spectrum, the continuous spectrum and the residual spectrum, respectively. Moreover, we have the 'Helmholtz' decompositions

$$\mathsf{H} = N(\mathrm{T}-\bar\lambda)\oplus\overline{R(\mathrm{T}-\lambda)}.$$

For $\lambda\in\rho(\mathrm{T})$ the continuity of $(\mathrm{T}-\lambda)^{-1}$ is equivalent to

$$\exists\, c>0\quad\forall\, u\in D(\mathrm{T})\quad |u|_{\mathsf{H}}\leq c|(\mathrm{T}-\lambda)u|_{\mathsf{H}}.$$

Hence, as T is closed, $R(\mathrm{T}-\lambda)=\mathsf{H}$ holds for $\lambda\in\rho(\mathrm{T})$, see e.g. [23, VIII.1, Theorem]. Thus the resolvent set $\rho(\mathrm{T})$, i.e., the set of all $\lambda\in\mathbb{C}$ with $N(\mathrm{T}-\lambda)=\{0\}$, $\overline{R(\mathrm{T}-\lambda)}=\mathsf{H}$ and $(\mathrm{T}-\lambda)^{-1}: R(\mathrm{T}-\lambda)\to D(\mathrm{T}-\lambda)$ bounded, is just given by

$$\rho(\mathrm{T}) = \{\lambda\in\mathbb{C}\,:\,(\mathrm{T}-\lambda)^{-1}:\mathsf{H}\to D(\mathrm{T})\text{ bounded}\}.$$

We note that for all $\lambda\in\mathbb{C}$ the norms in $D(\mathrm{T}-\lambda)$ and $D(\mathrm{T})$ are equivalent.

We give simple proofs of the results of section 2. For this, we recall the Hilbert spaces X and Y and the closed and densely defined linear operator $\mathrm{A}: D(\mathrm{A})\subset\mathsf{X}\mapsto\mathsf{Y}$ with adjoint $\mathrm{A}^*: D(\mathrm{A}^*)\subset\mathsf{Y}\mapsto\mathsf{X}$. $\mathrm{A}^*\mathrm{A}: D(\mathrm{A}^*\mathrm{A})\subset\mathsf{X}\mapsto\mathsf{X}$ and $\mathrm{A}\mathrm{A}^*: D(\mathrm{A}\mathrm{A}^*)\subset\mathsf{Y}\mapsto\mathsf{Y}$ are self-adjoint and non-negative. Furthermore, we introduce the Maxwell-type operator

$$\mathrm{M}: D(\mathrm{M})\subset\mathsf{Z}\to\mathsf{Z},\quad D(\mathrm{M}) := D(\mathrm{A})\times D(\mathrm{A}^*),\quad \mathsf{Z}:=\mathsf{X}\times\mathsf{Y}$$

by $\mathrm{M}(x,y) = (\mathrm{A}^*y,\mathrm{A}x)$ and note that

$$\mathrm{M} = \begin{bmatrix}0 & \mathrm{A}^*\\ \mathrm{A} & 0\end{bmatrix},\quad \mathrm{M}^2 = \begin{bmatrix}\mathrm{A}^*\mathrm{A} & 0\\ 0 & \mathrm{A}\mathrm{A}^*\end{bmatrix}$$

are self-adjoint as well and M^2 is non-negative. Moreover, we introduce two projections $\pi_{\mathsf{X}} : \mathsf{Z} \to \mathsf{X}$ and $\pi_{\mathsf{Y}} : \mathsf{Z} \to \mathsf{Y}$ by $\pi_{\mathsf{X}} z := x$ and $\pi_{\mathsf{Y}} z := y$ for $z = (x, y)$ and two embeddings $\iota_{\mathsf{X}} : \mathsf{X} \to \mathsf{Z}$ and $\iota_{\mathsf{Y}} : \mathsf{Y} \to \mathsf{Z}$ by $\iota_{\mathsf{X}} x := (x, 0)$ and $\iota_{\mathsf{Y}} y := (0, y)$.

First we show a stronger version of (2.2).

Lemma 24 *It holds*

- **(i)** $0 \in \sigma(\mathrm{M}) \Leftrightarrow 0 \in \sigma(\mathrm{A}^*\mathrm{A}) \cup \sigma(\mathrm{A}\mathrm{A}^*)$,
- **(i')** $0 \in \sigma_{\mathsf{c}}(\mathrm{M}) \Leftrightarrow 0 \in \sigma_{\mathsf{c}}(\mathrm{A}^*\mathrm{A}) \Leftrightarrow 0 \in \sigma_{\mathsf{c}}(\mathrm{A}\mathrm{A}^*)$,
- **(i")** $0 \in \sigma_{\mathsf{p}}(\mathrm{M}) \Leftrightarrow 0 \in \sigma_{\mathsf{p}}(\mathrm{A}^*\mathrm{A}) \cup \sigma_{\mathsf{p}}(\mathrm{A}\mathrm{A}^*)$

and for $\lambda \in \mathbb{R} \setminus \{0\}$

- **(ii)** $\lambda \in \rho(\mathrm{M}) \Leftrightarrow \lambda^2 \in \rho(\mathrm{A}^*\mathrm{A}) \Leftrightarrow \lambda^2 \in \rho(\mathrm{A}\mathrm{A}^*)$,
- **(ii')** $\lambda \in \sigma(\mathrm{M}) \Leftrightarrow \lambda^2 \in \sigma(\mathrm{A}^*\mathrm{A}) \Leftrightarrow \lambda^2 \in \sigma(\mathrm{A}\mathrm{A}^*)$,
- **(iii)** $\lambda \in \sigma_{\mathsf{c}}(\mathrm{M}) \Leftrightarrow \lambda^2 \in \sigma_{\mathsf{c}}(\mathrm{A}^*\mathrm{A}) \Leftrightarrow \lambda^2 \in \sigma_{\mathsf{c}}(\mathrm{A}\mathrm{A}^*)$,
- **(iv)** $\lambda \in \sigma_{\mathsf{p}}(\mathrm{M}) \Leftrightarrow \lambda^2 \in \sigma_{\mathsf{p}}(\mathrm{A}^*\mathrm{A}) \Leftrightarrow \lambda^2 \in \sigma_{\mathsf{p}}(\mathrm{A}\mathrm{A}^*)$. *More precisely: If* $z := (x, y)$ *is an eigenvector to the eigenvalue* λ *of* M, *then* x *is an eigenvector to the eigenvalue* λ^2 *of* $\mathrm{A}^*\mathrm{A}$ *and* y *is an eigenvector to the eigenvalue* λ^2 *of* $\mathrm{A}\mathrm{A}^*$. *If* x *is an eigenvector to the eigenvalue* λ^2 *of* $\mathrm{A}^*\mathrm{A}$, *then* $z_\pm := (x, \pm\lambda^{-1}\mathrm{A}x)$ *is an eigenvector to the eigenvalue* $\pm\lambda$ *of* M, *respectively. If* y *is an eigenvector to the eigenvalue* λ^2 *of* $\mathrm{A}\mathrm{A}^*$, *then* $z_\pm := (\pm\lambda^{-1}\mathrm{A}^*y, y)$ *is an eigenvector to the eigenvalue* $\pm\lambda$ *of* M, *respectively.*

Therefore,

- **(v)** $\rho(\mathrm{M})$ *and* $\sigma(\mathrm{M})$, $\sigma_{\mathsf{c}}(\mathrm{M})$, $\sigma_{\mathsf{p}}(\mathrm{M})$ *are point symmetric to the origin.*

Proof As (i')∧(i")⇒(i), (ii)⇒(ii') and (ii')∧(iv)⇒(iii), we only have to show (i'), (i"), (ii) and (iv). Then (v) is clear.

<u>(ii):</u> We just show the assertions for $\mathrm{A}^*\mathrm{A}$. The corresponding results for $\mathrm{A}\mathrm{A}^*$ can be proven analogously.

⇒: Let $\lambda \in \rho(\mathrm{M})$, i.e., $N(\mathrm{M} - \lambda) = \{0\}$ and $(\mathrm{M} - \lambda)^{-1} : \mathsf{Z} \to D(\mathrm{M})$ is continuous.

• First we show $N(\mathrm{A}^*\mathrm{A} - \lambda^2) = \{0\}$. Let $x \in N(\mathrm{A}^*\mathrm{A} - \lambda^2)$. Then $z := (x, y) \in D(\mathrm{M})$ with $y := \lambda^{-1}\mathrm{A}x \in D(\mathrm{A}^*)$ belongs to $N(\mathrm{M}-\lambda)$ since $(\mathrm{M}-\lambda)z = (\mathrm{A}^*y-\lambda x, \mathrm{A}x-\lambda y) = 0$. Hence $z = 0$, especially $x = 0$.

• Let $f \in \mathsf{X}$. We want to solve $(\mathrm{A}^*\mathrm{A} - \lambda^2)x = f$ with $x \in D(\mathrm{A}^*\mathrm{A})$. Defining the 'dual variable' $y := \lambda^{-1}\mathrm{A}x \in D(\mathrm{A}^*)$ and $z := (x, y) \in D(\mathrm{M})$, the mixed formulation of this problem is

$$\lambda(\mathrm{A}^*y - \lambda x) = f, \quad \mathrm{A}x = \lambda y \quad \Leftrightarrow \quad (\mathrm{M} - \lambda)z = (\mathrm{A}^*y - \lambda x, \mathrm{A}x - \lambda y) = \lambda^{-1}(f, 0).$$

These heuristic considerations suggest to set $x := \pi_{\mathsf{X}} z \in D(\mathrm{A})$ and $y := \pi_{\mathsf{Y}} z \in D(\mathrm{A}^*)$ with $z := \lambda^{-1}(\mathrm{M} - \lambda)^{-1}\iota_{\mathsf{X}} f \in D(\mathrm{M})$. Then $(\mathrm{A}^*y - \lambda x, \mathrm{A}x - \lambda y) = (\mathrm{M} - \lambda)z = \lambda^{-1}(f, 0)$, i.e., $\mathrm{A}x = \lambda y \in D(\mathrm{A}^*)$ and $(\mathrm{A}^*\mathrm{A} - \lambda^2)x = f$. Moreover, x depends continuously on f since

$$|x|_{D(\mathrm{A}^*\mathrm{A})} \leq \underbrace{|x|_{\mathsf{X}}}_{\leq |z|_{\mathsf{Z}}} + \underbrace{|\mathrm{A}^*\mathrm{A}x|_{\mathsf{X}}}_{=|f|_{\mathsf{X}}+\lambda^2|x|_{\mathsf{X}}} \leq c\big(|z|_{\mathsf{Z}} + |f|_{\mathsf{X}}\big) \leq c|f|_{\mathsf{X}}.$$

Therefore, $(\mathrm{A}^*\mathrm{A} - \lambda^2)^{-1} = \lambda^{-1}\pi_{\mathsf{X}}(\mathrm{M} - \lambda)^{-1}\iota_{\mathsf{X}} : \mathsf{X} \to D(\mathrm{A}^*\mathrm{A})$ is continuous and thus $\lambda^2 \in \rho(\mathrm{A}^*\mathrm{A})$.

$\Leftarrow$: Let $\lambda^2 \in \rho(\mathrm{A}^*\mathrm{A})$, i.e., $N(\mathrm{A}^*\mathrm{A} - \lambda^2) = \{0\}$ and $(\mathrm{A}^*\mathrm{A} - \lambda^2)^{-1} : \mathsf{X} \to D(\mathrm{A}^*\mathrm{A})$ is continuous.

• First we show $N(\mathrm{M} - \lambda) = \{0\}$. Let $z = (x, y) \in N(\mathrm{M} - \lambda)$. As

$$(\mathrm{A}^*y - \lambda x, \mathrm{A}x - \lambda y) = (\mathrm{M} - \lambda)z = 0,$$

$\mathrm{A}x = \lambda y \in D(\mathrm{A}^*)$ with $\mathrm{A}^*\mathrm{A}x = \lambda\mathrm{A}^*y = \lambda^2 x$. Hence, $x \in N(\mathrm{A}^*\mathrm{A} - \lambda^2)$ yields $x = 0$ and $y = 0$, i.e., $z = 0$.

• Let $h = (f, g) \in \mathsf{Z}$. We want to solve $(\mathrm{M} - \lambda)z = h$ with $(x, y) = z \in D(\mathrm{M})$. As $(\mathrm{A}^*y - \lambda x, \mathrm{A}x - \lambda y) = (f, g)$, $y \in D(\mathrm{A}^*)$ is already given by the second equation $\lambda y = \mathrm{A}x - g$, if x is known. Hence, rewriting everything in terms of x, this is

$$(f, g) = \big(\lambda^{-1}\mathrm{A}^*(\mathrm{A}x - g) - \lambda x, \mathrm{A}x - (\mathrm{A}x - g)\big) = \big(\lambda^{-1}\mathrm{A}^*(\mathrm{A}x - g) - \lambda x, g\big),$$

we see that we need to solve $\mathrm{A}^*(\mathrm{A}x - g) - \lambda^2 x = \lambda f$. Since g does not belong to $D(\mathrm{A}^*)$ in general, we cannot apply $(\mathrm{A}^*\mathrm{A} - \lambda^2)^{-1}$ directly. The ansatz $x = \tilde{x} + \hat{x} \in D(\mathrm{A})$ with $\hat{x} \in D(\mathrm{A}^*\mathrm{A})$ leads to

$$\mathrm{A}^*(\mathrm{A}\tilde{x} - g) - \lambda^2\tilde{x} + (\mathrm{A}^*\mathrm{A} - \lambda^2)\hat{x} = \lambda f. \tag{A.2}$$

By the Lax-Milgram lemma we can solve, e.g., $\mathrm{A}^*(\mathrm{A}\tilde{x} - g) + \tilde{x} = \lambda f$. More precisely, there exists a unique $\tilde{x} \in D(\mathrm{A})$ with

$$\forall\, \varphi \in D(\mathrm{A}) \quad \langle \mathrm{A}\tilde{x}, \mathrm{A}\varphi\rangle_{\mathsf{Y}} + \langle \tilde{x}, \varphi\rangle_{\mathsf{X}} = \lambda\langle f, \varphi\rangle_{\mathsf{X}} + \langle g, \mathrm{A}\varphi\rangle_{\mathsf{Y}} \tag{A.3}$$

depending continuously on f and g and hence on h, i.e., $|\tilde{x}|_{D(\mathrm{A})} \leq |\lambda||f|_{\mathsf{X}} + |g|_{\mathsf{Y}} \leq c|h|_{\mathsf{Z}}$. Let us denote this bounded linear operator mapping h to $\tilde{x}$ by $\mathrm{Ł} : \mathsf{Z} \to D(\mathrm{A})$. Now, (A.2) turns to

$$(\mathrm{A}^*\mathrm{A} - \lambda^2)\hat{x} = (1 + \lambda^2)\tilde{x}.$$

The latter heuristic computations suggest to define $z := (x, y)$ by

$$x := \tilde{x} + (1 + \lambda^2)(\mathrm{A}^*\mathrm{A} - \lambda^2)^{-1}\tilde{x} \in D(\mathrm{A}), \quad y := \lambda^{-1}(\mathrm{A}x - g)$$

with $\tilde{x}$ from (A.3). $\tilde{x} \in D(\mathrm{A})$ is uniquely defined and depends continuously on h, i.e., $|\tilde{x}|_{D(\mathrm{A})} \leq c|h|_{\mathsf{Z}}$. Moreover, $\mathrm{A}\tilde{x} - g \in D(\mathrm{A}^*)$ and $\mathrm{A}^*(\mathrm{A}\tilde{x} - g) = \lambda f - \tilde{x}$ by (A.3). As $x - \tilde{x} \in D(\mathrm{A}^*\mathrm{A})$, we get $y = \lambda^{-1}\big(\mathrm{A}(x - \tilde{x}) + \mathrm{A}\tilde{x} - g\big) \in D(\mathrm{A}^*)$. Thus, z belongs to $D(\mathrm{M})$. Since

$$\lambda\mathrm{A}^*y = \mathrm{A}^*\mathrm{A}(x - \tilde{x}) + \mathrm{A}^*(\mathrm{A}\tilde{x} - g) = (1 + \lambda^2)\tilde{x} + \lambda^2(x - \tilde{x}) + \lambda f - \tilde{x} = \lambda^2 x + \lambda f$$

we obtain

$$(\mathrm{M}-\lambda)z = (\mathrm{A}^*y - \lambda x, \mathrm{A}x - \lambda y) = (f, g) = h.$$

Furthermore, z depends continuously on h, i.e., using

$$\forall\, \varphi \in D(\mathrm{A}^*\mathrm{A}) \quad |\mathrm{A}\varphi|_{\mathsf{Y}}^2 = \langle \mathrm{A}^*\mathrm{A}\varphi, \varphi\rangle_{\mathsf{X}} \leq |\mathrm{A}^*\mathrm{A}\varphi|_{\mathsf{X}} |\varphi|_{\mathsf{X}} \leq |\varphi|_{\mathsf{X}}^2 + |\mathrm{A}^*\mathrm{A}\varphi|_{\mathsf{X}}^2$$

we have

$$\begin{aligned} |z|_{D(\mathrm{M})} &\leq |x|_{D(\mathrm{A})} + |y|_{D(\mathrm{A}^*)} \leq c\big(|x|_{D(\mathrm{A})} + |f|_{\mathsf{X}} + |g|_{\mathsf{Y}}\big) \leq c\big(|x - \tilde{x}|_{D(\mathrm{A})} + |\tilde{x}|_{D(\mathrm{A})} + |h|_{\mathsf{Z}}\big) \\ &\leq c\big(|x - \tilde{x}|_{D(\mathrm{A}^*\mathrm{A})} + |\tilde{x}|_{D(\mathrm{A})} + |h|_{\mathsf{Z}}\big) \leq c\big(|\tilde{x}|_{D(\mathrm{A})} + |h|_{\mathsf{Z}}\big) \leq c|h|_{\mathsf{Z}}. \end{aligned}$$

Therefore, with $\chi : D(\mathrm{A}) \to \mathsf{Z}$ defined by $\chi(x) := (x, \lambda^{-1}\mathrm{A}x)$ we finally obtain that

$$(\mathrm{M}-\lambda)^{-1} = \chi\big(1 + (1+\lambda^2)(\mathrm{A}^*\mathrm{A} - \lambda^2)^{-1}\big)\text{Ł} - \lambda^{-1}\iota_{\mathsf{Y}}\pi_{\mathsf{Y}} : \mathsf{Z} \to D(\mathrm{M})$$

is bounded and hence $\lambda \in \rho(\mathrm{M})$.

(iv): $\Rightarrow$: Let $\lambda \in \sigma_{\mathsf{p}}(\mathrm{M})$ and $z := (x, y)$ be an eigenvector to λ, i.e., $0 \neq z \in N(\mathrm{M}-\lambda)$. As $0 = (\mathrm{M}-\lambda)z = (\mathrm{A}^*y - \lambda x, \mathrm{A}x - \lambda y)$, neither x nor y can be zero. Moreover, since $\mathrm{M}z = \lambda z \in D(\mathrm{M})$, $z \in N\big((\mathrm{M}+\lambda)(\mathrm{M}-\lambda)\big)$ holds, this is

$$0 = (\mathrm{M}+\lambda)(\mathrm{M}-\lambda)z = (\mathrm{M}^2 - \lambda^2)z = \big((\mathrm{A}^*\mathrm{A} - \lambda^2)x, (\mathrm{A}\mathrm{A}^* - \lambda^2)y\big).$$

Thus, $0 \neq x \in N(\mathrm{A}^*\mathrm{A} - \lambda^2)$ and $0 \neq y \in N(\mathrm{A}\mathrm{A}^* - \lambda^2)$ yielding $\lambda^2 \in \sigma_{\mathsf{p}}(\mathrm{A}^*\mathrm{A}) \cap \sigma_{\mathsf{p}}(\mathrm{A}\mathrm{A}^*)$.

$\Leftarrow$: Let $\lambda^2 \in \sigma_{\mathsf{p}}(\mathrm{A}^*\mathrm{A})$ and x be an eigenvector to λ^2, i.e., $0 \neq x \in N(\mathrm{A}^*\mathrm{A} - \lambda^2)$. Then $z_\pm := (x, \pm\lambda^{-1}\mathrm{A}x) \in D(\mathrm{M})$ and

$$(\mathrm{M} \mp \lambda)z_\pm = (\pm\lambda^{-1}\mathrm{A}^*\mathrm{A}x \mp \lambda x, \mathrm{A}x - \lambda\lambda^{-1}\mathrm{A}x) = \pm\lambda^{-1}(\mathrm{A}^*\mathrm{A}x - \lambda^2 x, 0) = 0.$$

Hence, $0 \neq z_\pm \in N(\mathrm{M} \mp \lambda)$, i.e., $\pm\lambda \in \sigma_{\mathsf{p}}(\mathrm{M})$. Similar arguments apply to the case $\lambda^2 \in \sigma_{\mathsf{p}}(\mathrm{A}\mathrm{A}^*)$.

(i'): It holds with (ii')

$$\begin{aligned} 0 \in \sigma_{\mathsf{c}}(\mathrm{M}) \quad &\Leftrightarrow \quad \exists \quad (\lambda_n) \subset \sigma(\mathrm{M}) \setminus \{0\} \quad \lambda_n \to 0 \\ &\Leftrightarrow \quad \exists \quad (\lambda_n^2) \subset \sigma(\mathrm{A}^*\mathrm{A}) \setminus \{0\} \quad \lambda_n^2 \to 0 \\ &\Leftrightarrow \quad 0 \in \sigma_{\mathsf{c}}(\mathrm{A}^*\mathrm{A}) \end{aligned}$$

and the same is valid for $\mathrm{A}\mathrm{A}^*$.

(i"): If $0 \in \sigma_{\mathsf{p}}(\mathrm{M})$, then there exists $0 \neq z = (x, y) \in N(\mathrm{M})$, i.e., $0 = \mathrm{M}z = (\mathrm{A}^*y, \mathrm{A}x)$. But then $0 \neq z \in N(\mathrm{M}^2)$, i.e., $0 = \mathrm{M}^2 z = (\mathrm{A}^*\mathrm{A}x, \mathrm{A}\mathrm{A}^*y)$. As either $x \neq 0$ or $y \neq 0$, we get $0 \in \sigma_{\mathsf{p}}(\mathrm{A}^*\mathrm{A}) \cup \sigma_{\mathsf{p}}(\mathrm{A}\mathrm{A}^*)$. Now, let e.g. $0 \in \sigma_{\mathsf{p}}(\mathrm{A}^*\mathrm{A})$. Then, there exists $0 \neq x \in N(\mathrm{A}^*\mathrm{A})$, i.e., $\mathrm{A}^*\mathrm{A}x = 0$. This implies $\mathrm{A}x = 0$ since

$$0 = \langle \mathrm{A}^*\mathrm{A}x, x\rangle_{\mathsf{X}} = \langle \mathrm{A}x, \mathrm{A}x\rangle_{\mathsf{Y}} = |\mathrm{A}x|_{\mathsf{Y}}^2.$$

Thus $0 \neq z := (x, 0) \in N(\mathrm{M})$ because $\mathrm{M}z = (\mathrm{A}^*0, \mathrm{A}x) = 0$. Therefore, $0 \in \sigma_{\mathsf{p}}(\mathrm{M})$. □

We recall the 'Helmholtz' decompositions

$$\mathsf{X} = N(\mathrm{A}) \oplus \overline{R(\mathrm{A}^*)}, \quad D(\mathrm{A}) = N(\mathrm{A}) \oplus \big(D(\mathrm{A}) \cap \overline{R(\mathrm{A}^*)}\big)$$

and define the restricted operator

$$\mathcal{A} := \mathrm{A}|_{D(\mathcal{A})} : D(\mathcal{A}) \subset \overline{R(\mathrm{A}^*)} \to \overline{R(\mathrm{A})}, \quad \mathcal{A}x := \mathrm{A}x, \quad x \in D(\mathcal{A}) := D(\mathrm{A}) \cap \overline{R(\mathrm{A}^*)}.$$

Let us compute the adjoint $\mathcal{A}^* : D(\mathcal{A}^*) \subset \overline{R(\mathrm{A})} \to \overline{R(\mathrm{A}^*)}$. For $y \in D(\mathcal{A}^*)$ we have for all $\varphi \in D(\mathcal{A})$

$$\langle \mathcal{A}\varphi, y\rangle_{\mathsf{Y}} = \langle \varphi, \mathcal{A}^* y\rangle_{\mathsf{X}}.$$

Hence, for all $\psi = \psi_0 + \varphi \in D(\mathrm{A}) = N(\mathrm{A}) \oplus D(\mathcal{A})$ we get with $\mathcal{A}\varphi = \mathrm{A}\varphi = \mathrm{A}\psi$ and by $\mathcal{A}^* y \in \overline{R(\mathrm{A}^*)} \perp N(\mathrm{A})$

$$\langle \mathrm{A}\psi, y\rangle_{\mathsf{Y}} = \langle \mathcal{A}\varphi, y\rangle_{\mathsf{Y}} = \langle \varphi, \mathcal{A}^* y\rangle_{\mathsf{X}} = \langle \psi, \mathcal{A}^* y\rangle_{\mathsf{X}}.$$

Thus, $y \in D(\mathrm{A}^*)$ and $\mathrm{A}^* y = \mathcal{A}^* y$. This shows $D(\mathcal{A}^*) = D(\mathrm{A}^*) \cap \overline{R(\mathrm{A})}$ and $\mathcal{A}^* := \mathrm{A}^*|_{D(\mathcal{A}^*)}$, i.e.,

$$\mathcal{A}^* = \mathrm{A}^*|_{D(\mathcal{A}^*)} : D(\mathcal{A}^*) \subset \overline{R(\mathrm{A})} \to \overline{R(\mathrm{A}^*)}, \quad \mathcal{A}^* y = \mathrm{A}^* y, \quad y \in D(\mathcal{A}^*) = D(\mathrm{A}^*) \cap \overline{R(\mathrm{A})}.$$

Moreover, we have $(\mathcal{A}^*)^* = \mathcal{A}$ and the operators $\mathcal{A}^*\mathcal{A} : D(\mathcal{A}^*\mathcal{A}) \subset \overline{R(\mathrm{A}^*)} \to \overline{R(\mathrm{A}^*)}$ and $\mathcal{A}\mathcal{A}^* : D(\mathcal{A}\mathcal{A}^*) \subset \overline{R(\mathrm{A})} \to \overline{R(\mathrm{A})}$ are self-adjoint and non-negative. Finally, also the restriction

$$\mathcal{M} := \mathrm{M}|_{D(\mathcal{M})} : D(\mathcal{M}) \subset \overline{R(\mathrm{M})} \to \overline{R(\mathrm{M})}, \quad \mathcal{M}z := \mathrm{M}z, \quad z \in D(\mathcal{M}) := D(\mathrm{M}) \cap \overline{R(\mathrm{M})}$$

is self-adjoint and we have

$$\mathcal{M} = \begin{bmatrix} 0 & \mathcal{A}^* \\ \mathcal{A} & 0 \end{bmatrix}, \quad \mathcal{M}^2 = \begin{bmatrix} \mathcal{A}^*\mathcal{A} & 0 \\ 0 & \mathcal{A}\mathcal{A}^* \end{bmatrix}.$$

Remark 25 *Let us emphasize once more the 'Helmholtz' decompositions*

$$\begin{aligned} \mathsf{X} &= N(\mathrm{A}) \oplus \overline{R(\mathrm{A}^*)}, & D(\mathrm{A}) &= N(\mathrm{A}) \oplus D(\mathcal{A}), \\ \mathsf{Y} &= N(\mathrm{A}^*) \oplus \overline{R(\mathrm{A})}, & D(\mathrm{A}^*) &= N(\mathrm{A}^*) \oplus D(\mathcal{A}^*), \\ \mathsf{Z} &= N(\mathrm{M}) \oplus \overline{R(\mathrm{M})}, & D(\mathrm{M}) &= N(\mathrm{M}) \oplus D(\mathcal{M}). \end{aligned}$$

We introduce the orthogonal projectors

$$\pi_0 : \mathsf{Z} \to N(\mathrm{M}), \quad \pi : \mathsf{Z} \to \overline{R(\mathrm{M})}$$

and note $\pi|_{D(\mathrm{M})} : D(\mathrm{M}) \to D(\mathcal{M})$.

Lemma 26 *We have* $0 \notin \sigma_{\mathtt{p}}(\mathcal{M}) \cup \sigma_{\mathtt{p}}(\mathcal{A}^*\mathcal{A}) \cup \sigma_{\mathtt{p}}(\mathcal{A}\mathcal{A}^*)$. *Moreover:*

(i) *The inverse operators* $\mathcal{A}^{-1}$, $(\mathcal{A}^*)^{-1}$ *and* $\mathcal{M}^{-1}$ *exist.*

(ii) $R(\mathrm{A}) = R(\mathcal{A}), \quad R(\mathrm{A}^*) = R(\mathcal{A}^*), \quad R(\mathrm{M}) = R(\mathcal{M})$

(iii) *Lemma 24 holds for* $\mathcal{A}$, $\mathcal{A}^*$ *and* $\mathcal{M}$ *as well, which follows immediately by replacing* X *by* $\overline{R(\mathrm{A}^*)}$ *and* Y *by* $\overline{R(\mathrm{A})}$ *as well as* A *by* $\mathcal{A}$ *and* A^* *by* $\mathcal{A}^*$.

Lemma 27 *It holds*

(i) $\sigma(\mathrm{M}) \setminus \{0\} = \sigma(\mathcal{M}) \setminus \{0\}$, *more precisely even* $\sigma_{\mathtt{c}}(\mathrm{M}) \setminus \{0\} = \sigma_{\mathtt{c}}(\mathcal{M}) \setminus \{0\}$ *and* $\sigma_{\mathtt{p}}(\mathrm{M}) \setminus \{0\} = \sigma_{\mathtt{p}}(\mathcal{M}) \setminus \{0\}$,

(ii) $\rho(\mathrm{M}) \setminus \{0\} = \rho(\mathcal{M}) \setminus \{0\}$,

(iii) $\sigma_{\mathtt{p}}(\mathcal{M}^{-1}) \setminus \{0\} = \dfrac{1}{\sigma_{\mathtt{p}}(\mathcal{M}) \setminus \{0\}}$, *more precisely* $N(\mathcal{M} - \lambda) = N(\mathcal{M}^{-1} - \lambda^{-1})$ *for* $\lambda \neq 0$.

Proof We start with proving (ii).

$\Rightarrow$: Let $0 \neq \lambda \in \rho(\mathrm{M})$. We note that $R(\mathrm{M}-\lambda) = \mathsf{Z}$. For $h \in \overline{R(\mathrm{M})} \subset \mathsf{Z}$ we want solve $(\mathcal{M}-\lambda)z = h$. $z := (\mathrm{M}-\lambda)^{-1}h \in D(\mathrm{M})$ with $(\mathrm{M}-\lambda)z = h$ satisfies $\lambda z = \mathrm{M}z - h \in \overline{R(\mathrm{M})}$ and thus $z \in D(\mathcal{M})$. As $|z|_{D(\mathcal{M})} = |z|_{D(\mathrm{M})} \leq c|h|_{\mathsf{Z}} = c|h|_{\overline{R(\mathrm{M})}}$, z depends continuously on h. Hence $\lambda \in \rho(\mathcal{M})$.

$\Leftarrow$: Let $0 \neq \lambda \in \rho(\mathcal{M})$. We note that $R(\mathcal{M} - \lambda) = \overline{R(\mathrm{M})}$. For $h \in \mathsf{Z}$ we want solve $(\mathrm{M} - \lambda)z = h$. Decomposing

$$h = h_0 + \tilde{h} \in \mathsf{Z} = N(\mathrm{M}) \oplus \overline{R(\mathrm{M})}, \quad z = z_0 + \tilde{z} \in D(\mathrm{M}) = N(\mathrm{M}) \oplus D(\mathcal{M})$$

shows with $\mathrm{M}\tilde{z} \in R(\mathrm{M})$

$$-\lambda z_0 + (\mathrm{M} - \lambda)\tilde{z} = h_0 + \tilde{h} \quad \Leftrightarrow \quad -\lambda z_0 = h_0 \wedge (\mathrm{M} - \lambda)\tilde{z} = \tilde{h}.$$

This gives rise to define $z \in D(\mathrm{M})$ by

$$z := z_0 + \tilde{z}, \quad \tilde{z} := (\mathcal{M} - \lambda)^{-1}\tilde{h} \in D(\mathcal{M}), \quad z_0 := -\lambda^{-1}h_0 \in N(\mathrm{M}).$$

Then $(\mathrm{M} - \lambda)z = h_0 + \tilde{h} = h$ and z depends continuously on h, i.e.,

$$|z|_{D(\mathrm{M})} \leq |z_0|_{D(\mathrm{M})} + |\tilde{z}|_{D(\mathrm{M})} = |z_0|_{\mathsf{Z}} + |\tilde{z}|_{D(\mathcal{M})} \leq c\big(|h_0|_{\mathsf{Z}} + |\tilde{h}|_{\mathsf{Z}}\big) \leq c|h|_{\mathsf{Z}}.$$

Therefore, $\lambda \in \rho(\mathrm{M})$. We note that the inverse $(\mathrm{M} - \lambda)^{-1} : \mathsf{Z} \to D(\mathrm{M})$ is given by

$$(\mathcal{M} - \lambda)^{-1}\pi - \lambda^{-1}\pi_0.$$

(i): Since (ii) implies $\sigma(\mathrm{M}) \setminus \{0\} = \sigma(\mathcal{M}) \setminus \{0\}$ we just have to show the assertion for the point spectrum.

$\Rightarrow$: Let $0 \neq \lambda \in \sigma_{\mathtt{p}}(\mathrm{M})$. For $0 \neq z \in N(\mathrm{M} - \lambda)$ we have $\lambda z = \mathrm{M}z \in R(\mathrm{M})$. Hence, $z \in D(\mathcal{M})$ and thus $z \in N(\mathcal{M} - \lambda)$, i.e., $\lambda \in \sigma_{\mathtt{p}}(\mathcal{M})$.

$\Leftarrow$: Of course $N(\mathcal{M}-\lambda) \subset N(\mathrm{M}-\lambda)$. Thus, $\lambda \in \sigma_{\mathtt{p}}(\mathcal{M})$ implies $\lambda \in \sigma_{\mathtt{p}}(\mathrm{M})$.
(iii): For $\lambda \neq 0$ we have

$$\begin{aligned}
\lambda \in \sigma_{\mathtt{p}}(\mathcal{M}) \quad &\Leftrightarrow \quad \exists\, 0 \neq z \in N(\mathcal{M}-\lambda) \\
&\Leftrightarrow \quad \exists\, 0 \neq z \in D(\mathcal{M}) \quad \mathrm{M}z = \lambda z \in R(\mathrm{M}) \\
&\Leftrightarrow \quad \exists\, 0 \neq z \in R(\mathrm{M}) \quad \mathcal{M}^{-1}z = \lambda^{-1}\mathcal{M}^{-1}\mathcal{M}z = \lambda^{-1}z \in D(\mathcal{M}) \\
&\Leftrightarrow \quad \exists\, 0 \neq z \in N(\mathcal{M}^{-1}-\lambda^{-1}) \\
&\Leftrightarrow \quad \lambda^{-1} \in \sigma_{\mathtt{p}}(\mathcal{M}^{-1}).
\end{aligned}$$

The proof is complete. □

The latter lemma holds true for $\mathcal{A}^*\mathcal{A}$ and $\mathcal{A}\mathcal{A}^*$ we well. More precisely:

Lemma 28 *It holds*

- **(i)** $\sigma(\mathrm{A}^*\mathrm{A}) \setminus \{0\} = \sigma(\mathcal{A}^*\mathcal{A}) \setminus \{0\}$, *more precisely even* $\sigma_{\mathtt{c}}(\mathrm{A}^*\mathrm{A}) \setminus \{0\} = \sigma_{\mathtt{c}}(\mathcal{A}^*\mathcal{A}) \setminus \{0\}$ *and* $\sigma_{\mathtt{p}}(\mathrm{A}^*\mathrm{A}) \setminus \{0\} = \sigma_{\mathtt{p}}(\mathcal{A}^*\mathcal{A}) \setminus \{0\}$,
- **(ii)** $\rho(\mathrm{A}^*\mathrm{A}) \setminus \{0\} = \rho(\mathcal{A}^*\mathcal{A}) \setminus \{0\}$,
- **(iii)** $\sigma_{\mathtt{p}}\big((\mathcal{A}^*\mathcal{A})^{-1}\big) \setminus \{0\} = \dfrac{1}{\sigma_{\mathtt{p}}(\mathcal{A}^*\mathcal{A}) \setminus \{0\}}$ *and* $N(\mathcal{A}^*\mathcal{A}-\lambda^2) = N\big((\mathcal{A}^*\mathcal{A})^{-1}-\lambda^{-2}\big)$ *for* $\lambda \neq 0$.

The corresponding assertions are valid for $\mathrm{A}\mathrm{A}^*$ *and* $\mathcal{A}\mathcal{A}^*$ *as well.*

Proof With Lemma 24 (ii'), Lemma 27 (i) and Lemma 26 we have for $\lambda \neq 0$

$$\lambda^2 \in \sigma(\mathrm{A}^*\mathrm{A}) \quad \Leftrightarrow \quad \lambda \in \sigma(\mathrm{M}) \quad \Leftrightarrow \quad \lambda \in \sigma(\mathcal{M}) \quad \Leftrightarrow \quad \lambda^2 \in \sigma(\mathcal{A}^*\mathcal{A}).$$

and the corresponding results hold for $\sigma_{\mathtt{p}}$, $\sigma_{\mathtt{c}}$ and ρ as well. This shows (i) and (ii). To prove (iii) we can follow the proof of Lemma 27 (iii) and see for $\lambda \neq 0$

$$\begin{aligned}
\lambda^2 \in \sigma_{\mathtt{p}}(\mathcal{A}^*\mathcal{A}) \quad &\Leftrightarrow \quad \exists\, 0 \neq x \in N(\mathcal{A}^*\mathcal{A}-\lambda^2) \\
&\Leftrightarrow \quad \exists\, 0 \neq x \in D(\mathcal{A}^*\mathcal{A}) \quad \mathrm{A}^*\mathrm{A}x = \lambda^2 x \in R(\mathrm{A}^*) \\
&\Leftrightarrow \quad \exists\, 0 \neq x \in D(\mathcal{A}^*\mathcal{A}) \quad \mathrm{A}x = \lambda^2(\mathcal{A}^*)^{-1}x \in R(\mathrm{A}) \\
&\Leftrightarrow \quad \exists\, 0 \neq x \in D(\mathcal{A}^*\mathcal{A}) \quad x = \lambda^2(\mathcal{A})^{-1}(\mathcal{A}^*)^{-1}x \in R(\mathrm{A}^*) \\
&\Leftrightarrow \quad \exists\, 0 \neq x \in R(\mathrm{A}^*) \quad (\mathcal{A}^*\mathcal{A})^{-1}x = \lambda^{-2}x \in D(\mathcal{A}^*\mathcal{A}) \\
&\Leftrightarrow \quad \exists\, 0 \neq x \in N\big((\mathcal{A}^*\mathcal{A})^{-1}-\lambda^{-2}\big) \\
&\Leftrightarrow \quad \lambda^{-2} \in \sigma_{\mathtt{p}}\big((\mathcal{A}^*\mathcal{A})^{-1}\big),
\end{aligned}$$

which completes the proof. □

A.3.1 Results for Compact Resolvents

From now on we assume generally that the embedding

$$D(\mathcal{A}) \hookrightarrow \mathsf{X} \tag{A.4}$$

is compact.

Lemma 29 *The following assertions hold:*

- **(i)** $\exists\, c_{\mathrm{A}} > 0 \quad \forall\, x \in D(\mathcal{A}) \quad |x|_{\mathsf{X}} \leq c_{\mathrm{A}} |\mathrm{A}x|_{\mathsf{Y}}$
- **(i')** $\exists\, c_{\mathrm{A}^*} > 0 \quad \forall\, y \in D(\mathcal{A}^*) \quad |y|_{\mathsf{Y}} \leq c_{\mathrm{A}^*} |\mathrm{A}^* y|_{\mathsf{X}}$
- **(i")** $\exists\, c_{\mathrm{M}} > 0 \quad \forall\, z \in D(\mathcal{M}) \quad |z|_{\mathsf{Z}} \leq c_{\mathrm{M}} |\mathrm{M}z|_{\mathsf{Z}}$
- **(ii)** $R(\mathrm{A})$, $R(\mathrm{A}^*)$ *and* $R(\mathrm{M})$ *are closed.*
- **(iii)** $\mathsf{X} = N(\mathrm{A}) \oplus R(\mathrm{A}^*)$, $\mathsf{Y} = N(\mathrm{A}^*) \oplus R(\mathrm{A})$ *and* $\mathsf{Z} = N(\mathrm{M}) \oplus R(\mathrm{M})$.
- **(iv)** $\mathcal{A}^{-1} : R(\mathrm{A}) \to D(\mathcal{A})$ *is continuous and* $\mathcal{A}^{-1} : R(\mathrm{A}) \to R(\mathrm{A}^*)$ *is compact.*
- **(iv')** $(\mathcal{A}^*)^{-1} : R(\mathrm{A}^*) \to D(\mathcal{A}^*)$ *is continuous and* $(\mathcal{A}^*)^{-1} : R(\mathrm{A}^*) \to R(\mathrm{A})$ *is compact.*
- **(iv")** $\mathcal{M}^{-1} : R(\mathrm{M}) \to D(\mathcal{M})$ *is continuous and* $\mathcal{M}^{-1} : R(\mathrm{M}) \to R(\mathrm{M})$ *is compact.*
- **(v)** $D(\mathcal{A}^*) \hookrightarrow \mathsf{Y}$ *is compact.*
- **(v')** $D(\mathcal{M}) \hookrightarrow \mathsf{Z}$ *is compact.*

Proof (i): Let us assume that the estimate is wrong. Then there exists a sequence $(x_n) \subset D(\mathcal{A})$ with $|x_n|_{\mathsf{X}} = 1$ and $|\mathrm{A}x_n|_{\mathsf{Y}} \to 0$. As (x_n) is bounded in $D(\mathcal{A})$, by the general assumption (A.4) we can extract a subsequence, again denoted by (x_n), with $x_n \to x \in \mathsf{X}$. Since A and $\overline{R(\mathrm{A}^*)}$ are closed, we have $x \in N(\mathrm{A}) \cap N(\mathrm{A})^{\perp} = \{0\}$, in contradiction to $1 = |x_n|_{\mathsf{X}} \to |x|_{\mathsf{X}} = 0$.

(ii): For $y \in \overline{R(\mathrm{A})} = \overline{R(\mathcal{A})}$ there exists a sequence $(x_n) \subset D(\mathcal{A})$ with $\mathrm{A}x_n \to y$. By (i') (x_n) is a Cauchy sequence in X. Hence, (x_n) converges to some $x \in \mathsf{X}$. Since A is closed, we obtain $x \in D(\mathrm{A})$ and $\mathrm{A}x = y$, showing that $R(\mathrm{A})$ is closed. By the closed range theorem, see e.g. [23, VII, 5, Theorem], $R(\mathrm{A}^*)$ is closed as well. Hence, also $R(\mathrm{M}) = R(\mathrm{A}^*) \times R(\mathrm{A})$ is closed.

(iii) follows immediately by (ii).

(iv) follows directly by (i) and (A.4). Indeed, (i) is equivalent to the continuity of $\mathcal{A}^{-1}$.

(v): Let (y_n) be a bounded sequence in $D(\mathcal{A}^*)$. By (ii), $(y_n) \in R(\mathrm{A}) = R(\mathcal{A})$ and hence there exists a sequence $(x_n) \subset D(\mathcal{A})$ with $\mathrm{A}x_n = y_n$. By (i), (x_n) is bounded in $D(\mathcal{A})$. By (A.4), we can extract a subsequence, again denoted by (x_n), such that (x_n) converges in X. Then, for $x_{n,m} := x_n - x_m$ and $y_{n,m} := y_n - y_m$ we have

$$|y_{n,m}|_{\mathsf{Y}}^2 = \langle \mathrm{A}x_{n,m}, y_{n,m} \rangle_{\mathsf{Y}} = \langle x_{n,m}, \mathrm{A}^* y_{n,m} \rangle_{\mathsf{X}} \leq c |x_{n,m}|_{\mathsf{X}}.$$

Thus, (y_n) is a Cauchy sequence in Y.

(v') is clear by (A.4) and (v).

(i')[v] follows by (v) analogously to (i).

(i") follows by (i) and (i').

(iv')[vi] follows by (i') and (v).

(iv) and (iv') imply (iv"). $\square$

Let us recall some facts: By Lemma 29 (v') for all $\lambda \in \mathbb{C}$

$$D(\mathcal{M} - \lambda) \hookrightarrow \mathsf{Z} \tag{A.5}$$

is compact. For $\lambda \in \rho(\mathrm{M}) \supset \mathbb{C} \setminus \mathbb{R}$ we have

$$N(\mathrm{M} - \lambda) = \{0\}, \quad R(\mathrm{M} - \lambda) = \mathsf{Z}, \quad N(\mathcal{M} - \lambda) = \{0\}, \quad R(\mathcal{M} - \lambda) = R(\mathrm{M})$$

and the boundedness of $(\mathrm{M} - \lambda)^{-1} : \mathsf{Z} \to D(\mathrm{M})$ is equivalent to

$$\exists\, c_{\mathrm{M},\lambda} > 0 \quad \forall\, z \in D(\mathrm{M}) \quad |z|_{\mathsf{Z}} \leq c_{\mathrm{M},\lambda} |(\mathrm{M} - \lambda) z|_{\mathsf{Z}},$$

which holds for $\mathcal{M}$ as well. For $0 \neq \lambda \in \sigma(\mathrm{M}) \subset \mathbb{R}$ we have

$$\mathsf{Z} = N(\mathrm{M} - \lambda) \oplus \overline{R(\mathrm{M} - \lambda)}, \qquad R(\mathrm{M}) = N(\mathcal{M} - \lambda) \oplus \overline{R(\mathcal{M} - \lambda)}.$$

Lemma 30 *For $\lambda \in \mathbb{R} \setminus \{0\}$ the following assertions hold:*

(i) $N(\mathrm{M} - \lambda) \subset R(\mathrm{M})$ *and* $N(\mathrm{M} - \lambda) = N(\mathcal{M} - \lambda)$ *has finite dimension.*

(ii) $\exists\, c_{\mathrm{M},\lambda} > 0 \quad \forall\, z \in D(\mathcal{M}) \cap N(\mathrm{M} - \lambda)^{\perp} \quad |z|_{\mathsf{Z}} \leq c_{\mathrm{M},\lambda} |(\mathrm{M} - \lambda) z|_{\mathsf{Z}}$

(iii) $R(\mathcal{M} - \lambda)$ *is closed.*

(iii') $R(\mathrm{M} - \lambda)$ *is closed.*

(iii") $R(\mathcal{M} - \lambda) = R(\mathrm{M} - \lambda) \cap R(\mathrm{M})$

(iv) $\mathsf{Z} = N(\mathrm{M} - \lambda) \oplus R(\mathrm{M} - \lambda)$ *and* $R(\mathrm{M}) = N(\mathcal{M} - \lambda) \oplus R(\mathcal{M} - \lambda)$.

(v) *Let* $N(\mathcal{M} - \lambda) = \{0\}$. *Then* $(\mathcal{M} - \lambda)^{-1} : R(\mathrm{M}) \to D(\mathcal{M})$ *is continuous and* $(\mathcal{M} - \lambda)^{-1} : R(\mathrm{M}) \to R(\mathrm{M})$ *is compact. Especially* $\lambda \in \rho(\mathcal{M})$.

(v') *Let* $N(\mathrm{M} - \lambda) = \{0\}$. *Then* $(\mathrm{M} - \lambda)^{-1} : \mathsf{Z} \to D(\mathrm{M})$ *is continuous. Especially* $\lambda \in \rho(\mathrm{M})$.

Corresponding results hold for $\mathrm{A}^*\mathrm{A}$, $\mathrm{A}\mathrm{A}^*$ *resp.* $\mathcal{A}^*\mathcal{A}$, $\mathcal{A}\mathcal{A}^*$ *we well.*

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[v] (i') follows also by (iv'), since (i') is equivalent to the continuity of $(\mathcal{A}^*)^{-1}$.

[vi] Another proof of (iv') is the following: As $\mathcal{A}^{-1} : R(\mathrm{A}) \to R(\mathrm{A}^*)$ is compact by (iv), so is the adjoint $(\mathcal{A}^*)^{-1} : R(\mathrm{A}^*) \to R(\mathrm{A})$ by Schauder's theorem, see e.g. [23, X, 4, Theorem]. Especially $(\mathcal{A}^*)^{-1}$ is bounded and hence also $(\mathcal{A}^*)^{-1} : R(\mathrm{A}^*) \to D(\mathrm{A}^*)$.

Proof It is enough to consider $0 \neq \lambda \in \sigma(\mathrm{M}) \subset \mathbb{R}$.

(i): Of course, $N(\mathcal{M}-\lambda) \subset N(\mathrm{M}-\lambda)$. For $z \in N(\mathrm{M}-\lambda)$ we have $\mathrm{M}z = \lambda z$. Thus $z \in R(\mathrm{M})$, i.e., $z \in D(\mathcal{M})$. Hence $z \in N(\mathcal{M}-\lambda)$. By (A.5) the unit ball in $N(\mathcal{M}-\lambda)$ is compact, i.e., $\dim N(\mathcal{M}-\lambda) < \infty$.

(ii): If the estimate is wrong, then there exists a sequence $(z_n) \subset D(\mathcal{M}) \cap N(\mathcal{M}-\lambda)^{\perp}$ with $|z_n|_{\mathsf{Z}} = 1$ and $|(\mathrm{M}-\lambda)z_n|_{\mathsf{Z}} \to 0$. By (A.5) we can extract a subsequence, again denoted by (z_n), with $z_n \to z \in \mathsf{Z}$. Moreover, $\mathrm{M}z_n = (\mathrm{M}-\lambda)z_n + \lambda z_n \to \lambda z$. As M and $N(\mathcal{M}-\lambda)^{\perp}$ are closed, z belongs to $N(\mathcal{M}-\lambda) \cap N(\mathcal{M}-\lambda)^{\perp} = \{0\}$, in contradiction to $1 = |z_n|_{\mathsf{Z}} \to |z|_{\mathsf{Z}} = 0$.

(iii): Let $h \in \overline{R(\mathcal{M}-\lambda)}$. Then there exists a sequence $(z_n) \subset D(\mathcal{M})$ such that $(\mathrm{M}-\lambda)z_n =: h_n \to h$. Decomposing $z_n = z_{n,0} + \tilde{z}_n \in N(\mathrm{M}-\lambda) \oplus \overline{R(\mathrm{M}-\lambda)}$ shows $(\mathrm{M}-\lambda)\tilde{z}_n = h_n$ and $\tilde{z}_n \in D(\mathcal{M}) \cap N(\mathrm{M}-\lambda)^{\perp}$. By (ii) $(\tilde{z}_n)$ is a Cauchy sequence in Z converging to some $z \in \mathsf{Z}$. Moreover, $\mathrm{M}\tilde{z}_n = (\mathrm{M}-\lambda)\tilde{z}_n + \lambda\tilde{z}_n \to h + \lambda z$. As $\mathcal{M}$ is closed, we obtain $z \in D(\mathcal{M})$ and $(\mathrm{M}-\lambda)z = h$, i.e., $h \in R(\mathcal{M}-\lambda)$.

(iii'): Let $h \in \overline{R(\mathrm{M}-\lambda)}$. By (i) we have $R(\mathrm{M}) = N(\mathrm{M}-\lambda) \oplus \big(R(\mathrm{M}) \cap \overline{R(\mathrm{M}-\lambda)}\big)$ and hence it holds

$$R(\mathrm{M}) \cap \overline{R(\mathrm{M}-\lambda)} = R(\mathcal{M}-\lambda) \tag{A.6}$$

by (iii). Let us decompose $h = h_0 + \tilde{h} \in N(\mathrm{M}) \oplus R(\mathrm{M})$. As $(\mathrm{M}-\lambda)h_0 = -\lambda h_0 \in R(\mathrm{M}-\lambda)$, we get $\tilde{h} \in R(\mathrm{M}) \cap \overline{R(\mathrm{M}-\lambda)}$. Hence $\tilde{h} \in R(\mathcal{M}-\lambda) \subset R(\mathrm{M}-\lambda)$ and thus $h \in R(\mathrm{M}-\lambda)$.

(iii") follows by (iii') and (A.6).

(iv) follows by (iii) and (iii').

(v): If $N(\mathrm{M}-\lambda) = \{0\}$, then $R(\mathrm{M}-\lambda) = \mathsf{Z}$ and $R(\mathcal{M}-\lambda) = R(\mathrm{M})$. By (ii) $(\mathcal{M}-\lambda)^{-1} : R(\mathrm{M}) \to D(\mathcal{M})$ is continuous, more precisely, for $h \in R(\mathrm{M})$ we have $z := (\mathcal{M}-\lambda)^{-1}h \in D(\mathcal{M})$ and hence $|z|_{\mathsf{Z}} \leq c_{\mathrm{M},\lambda}|h|_{\mathsf{Z}}$.

(v'): By (i), (v) and Lemma 27 (ii) we get $\lambda \in \rho(\mathcal{M}) \setminus \{0\} = \rho(\mathrm{M}) \setminus \{0\}$. Hence, $(\mathrm{M}-\lambda)^{-1} : \mathsf{Z} \to D(\mathrm{M})$ is continuous. □

Theorem 31 *$\mathcal{M}$ has a pure point spectrum, which is contained in $\mathbb{R} \setminus \{0\}$ and point symmetric to the origin. More precisely,*

$$-\sigma_{\mathrm{p}}(\mathcal{M}) = \sigma_{\mathrm{p}}(\mathcal{M}) = \sigma(\mathcal{M}) = \sigma(\mathrm{M}) \setminus \{0\} = \sigma_{\mathrm{p}}(\mathrm{M}) \setminus \{0\}$$

and

$$\begin{aligned}\sigma(\mathcal{M})^2 = \sigma_{\mathrm{p}}(\mathcal{A}^*\mathcal{A}) &= \sigma(\mathcal{A}^*\mathcal{A}) = \sigma(\mathrm{A}^*\mathrm{A}) \setminus \{0\} = \sigma_{\mathrm{p}}(\mathrm{A}^*\mathrm{A}) \setminus \{0\} \\ = \sigma_{\mathrm{p}}(\mathcal{A}\mathcal{A}^*) &= \sigma(\mathcal{A}\mathcal{A}^*) = \sigma(\mathrm{A}\mathrm{A}^*) \setminus \{0\} = \sigma_{\mathrm{p}}(\mathrm{A}\mathrm{A}^*) \setminus \{0\}\end{aligned}$$

as well as

$$\rho(\mathcal{M}) \ni 0 \in \begin{cases} \sigma_{\mathrm{p}}(\mathrm{M}) & , \textit{ if } N(\mathrm{M}) \neq \{0\}, \\ \rho(\mathrm{M}) & , \textit{ if } N(\mathrm{M}) = \{0\} \end{cases}$$

hold. Moreover, there exist sequences of eigenvalues and eigenvectors

$$(\lambda_n)_{n\in\mathbb{N}} \subset (0,\infty), \quad (z_n^\pm)_{n\in\mathbb{N}} = \big((x_n, y_n^\pm)\big)_{n\in\mathbb{N}} \subset D(\mathcal{M}),$$

which might be finite or empty if (e.g.) A *is bounded, such that the following holds:*

(i) $\sigma(\mathcal{M}) = (\lambda_n) \cup (-\lambda_n)$ *and* $\sigma(\mathcal{M})^2 = \sigma(\mathcal{A}^*\mathcal{A}) = \sigma(\mathcal{A}\mathcal{A}^*) = (\lambda_n^2)$.

(ii) (λ_n) *is monotone increasing with* $\lambda_n \to \infty$, *if* (λ_n) *is not finite.*

(iii) $(\mathrm{M} \mp \lambda_n) z_n^\pm = 0$ *holds for all* n, *i.e.,* $\mathrm{A}x_n = \pm\lambda_n y_n^\pm$ *and* $\mathrm{A}^* y_n^\pm = \pm\lambda_n x_n$ *and thus* $z_n^\pm = (x_n, \pm\lambda_n^{-1}\mathrm{A}x_n) = (\pm\lambda_n^{-1}\mathrm{A}^* y_n^\pm, y_n^\pm)$.

(iii') $(\mathrm{M}^2 - \lambda_n^2) z_n^\pm = 0$ *holds for all* n, *i.e.,* $\mathrm{A}^*\mathrm{A}x_n = \lambda_n^2 x_n$ *and* $\mathrm{A}\mathrm{A}^* y_n^\pm = \lambda_n^2 y_n^\pm$.

(iv) (x_n) *is a complete orthonormal system in* $R(\mathrm{A}^*)$, *i.e.,*

$$\forall\, x \in R(\mathrm{A}^*) \qquad x = \sum_{n=1}^{\infty} \xi_n x_n,$$

and furthermore

$$\forall\, \tilde{x} = x_0 + x \in \mathsf{X} = N(\mathrm{A}) \oplus R(\mathrm{A}^*) \qquad x = \sum_{n=1}^{\infty} \xi_n x_n,$$

$$\forall\, \tilde{x} = x_0 + x \in D(\mathrm{A}) = N(\mathrm{A}) \oplus D(\mathcal{A}) \qquad \mathrm{A}\tilde{x} = \mathrm{A}x = \pm \sum_{n=1}^{\infty} \lambda_n \xi_n y_n^\pm,$$

$$\forall\, x \in D(\mathrm{A}^*\mathrm{A}) \qquad \mathrm{A}^*\mathrm{A}x = \sum_{n=1}^{\infty} \lambda_n^2 \xi_n x_n,$$

where $\xi_n = \langle x, x_n\rangle_{\mathsf{X}} = \langle \tilde{x}, x_n\rangle_{\mathsf{X}}$. *Moreover,* $|\tilde{x}|_{\mathsf{X}}^2 = |x_0|_{\mathsf{X}}^2 + |x|_{\mathsf{X}}^2$ *and*

$$|x|_{\mathsf{X}}^2 = \sum_{n=1}^{\infty} \xi_n^2, \quad |\mathrm{A}x|_{\mathsf{Y}}^2 = \sum_{n=1}^{\infty} \lambda_n^2 \xi_n^2, \quad |\mathrm{A}^*\mathrm{A}x|_{\mathsf{X}}^2 = \sum_{n=1}^{\infty} \lambda_n^4 \xi_n^2.$$

(iv') $(y_n^\pm)$ *is a complete orthonormal system in* $R(\mathrm{A})$, *i.e.,*

$$\forall\, y \in R(\mathrm{A}) \qquad y = \sum_{n=1}^{\infty} \zeta_n^\pm y_n^\pm,$$

and furthermore

$$\forall\, \tilde{y} = y_0 + y \in \mathsf{Y} = N(\mathrm{A}^*) \oplus R(\mathrm{A}) \qquad y = \sum_{n=1}^{\infty} \zeta_n^\pm y_n^\pm,$$

$$\forall\, \tilde{y} = y_0 + y \in D(\mathrm{A}^*) = N(\mathrm{A}^*) \oplus D(\mathcal{A}^*) \qquad \mathrm{A}^*\tilde{y} = \mathrm{A}^*y = \pm \sum_{n=1}^{\infty} \lambda_n \zeta_n^\pm x_n,$$

$$\forall\, y \in D(\mathrm{A}\mathrm{A}^*) \qquad \mathrm{A}\mathrm{A}^*y = \sum_{n=1}^{\infty} \lambda_n^2 \zeta_n^\pm y_n^\pm,$$

where $\zeta_n^{\pm} = \langle y, y_n^{\pm}\rangle_{\mathsf{Y}} = \langle \tilde{y}, y_n^{\pm}\rangle_{\mathsf{Y}}$. *Moreover,* $|\tilde{y}|_{\mathsf{Y}}^2 = |y_0|_{\mathsf{Y}}^2 + |y|_{\mathsf{Y}}^2$ *and*

$$|y|_{\mathsf{Y}}^2 = \sum_{n=1}^{\infty} (\zeta_n^{\pm})^2, \quad |\mathrm{A}^* y|_{\mathsf{X}}^2 = \sum_{n=1}^{\infty} \lambda_n^2 (\zeta_n^{\pm})^2, \quad |\mathrm{A}\mathrm{A}^* y|_{\mathsf{Y}}^2 = \sum_{n=1}^{\infty} \lambda_n^4 (\zeta_n^{\pm})^2.$$

Proof By Lemma 29 (iv") we have $0 \in \rho(\mathcal{M})$. $\mathrm{M} = \mathcal{M}$ holds if $N(\mathrm{M}) = \{0\}$. By Lemma 30 (v) $\mathcal{M}$ has a pure point spectrum and by Lemma 30 (v') $\sigma(\mathrm{M}) \setminus \{0\} = \sigma_{\mathtt{p}}(\mathrm{M}) \setminus \{0\}$. By Lemma 27 (i) we have $\sigma_{\mathtt{p}}(\mathcal{M}) = \sigma_{\mathtt{p}}(\mathcal{M}) \setminus \{0\} = \sigma_{\mathtt{p}}(\mathrm{M}) \setminus \{0\}$. By Lemma 24 (v) the spectra are point symmetric to the origin. The other assertions about the spectra follow immediately by Lemmas 24, 27, 28 and Lemma 26.

As $\mathcal{A}^{-1} : R(\mathrm{A}) \to R(\mathrm{A}^*)$ or $(\mathcal{A}^*)^{-1} : R(\mathrm{A}^*) \to R(\mathrm{A})$ are compact by Lemma 29 (iv) or (iv'), so is e.g. $(\mathcal{A}^*\mathcal{A})^{-1} : R(\mathrm{A}^*) \to R(\mathrm{A}^*)$. Moreover, $(\mathcal{A}^*\mathcal{A})^{-1}$ is self-adjoint and positive. Let us assume that A is unbounded[vii]. By the spectral theorem for self-adjoint, compact and non-negative operators there exists a monotone decreasing sequence $(\lambda_n^{-1})_{n\in\mathbb{N}} \subset (0, \infty)$ converging to zero and a sequence $(x_n)_{n\in\mathbb{N}} \subset R(\mathrm{A}^*)$, such that λ_n^{-2} is an eigenvalue to the eigenvector x_n of $(\mathcal{A}^*\mathcal{A})^{-1}$, i.e., $(\mathcal{A}^*\mathcal{A})^{-1} x_n = \lambda_n^{-2} x_n$. Moreover, (x_n) is a complete orthonormal system in $R(\mathrm{A}^*)$, i.e., for all $x \in R(\mathrm{A}^*)$ we have

$$x = \sum_{n=1}^{\infty} \xi_n(x) x_n, \quad \xi_n(x) = \langle x, x_n\rangle_{\mathsf{X}}.$$

$(x_n) \subset D(\mathcal{A}^*\mathcal{A})$ is also a complete orthonormal system of eigenvectors of $\mathcal{A}^*\mathcal{A}$ since $\mathrm{A}^*\mathrm{A}x_n = \lambda_n^2 x_n$. Defining

$$y_n^{\pm} := \pm\lambda_n^{-1} \mathrm{A} x_n \in D(\mathcal{A}^*)$$

we see $\mathrm{A}^* y_n^{\pm} = \pm\lambda_n x_n \in D(\mathcal{A})$. Hence, $y_n^{\pm} \in D(\mathcal{A}\mathcal{A}^*)$ with $\mathrm{A}\mathrm{A}^* y_n^{\pm} = \pm\lambda_n \mathrm{A} x_n = \lambda_n^2 y_n^{\pm}$, i.e., $y_n^{\pm}$ is an eigenvector of $\mathcal{A}\mathcal{A}^*$ to the eigenvalue λ_n^2. For all $y \in R(\mathrm{A})$ with $y = \mathrm{A}x$ for some $x \in D(\mathrm{A})$ we have

$$\langle y, y_n^{\pm}\rangle_{\mathsf{Y}} = \langle x, \mathrm{A}^* y_n^{\pm}\rangle_{\mathsf{X}} = \pm\lambda_n \langle x, x_n\rangle_{\mathsf{X}}. \tag{A.7}$$

This shows two things. First, putting $y := y_m^{\pm} = \mathrm{A}(\pm\lambda_m^{-1} x_m)$ we get

$$\langle y_m^{\pm}, y_n^{\pm}\rangle_{\mathsf{Y}} = \frac{\lambda_n}{\lambda_m} \langle x_m, x_n\rangle_{\mathsf{X}},$$

which shows that (y_n^+) and (y_n^-) are both orthonormal systems in $R(\mathrm{A})$, and second, that they are even complete in $R(\mathrm{A})$. Thus, for all $y \in R(\mathrm{A})$ we obtain

$$y = \sum_{n=1}^{\infty} \zeta_n^{\pm}(y) y_n^{\pm}, \quad \zeta_n^{\pm}(y) = \langle y, y_n^{\pm}\rangle_{\mathsf{Y}}.$$

[vii]If A is bounded, the sequences (λ_n) and $(z_n^{\pm})$ are finite.

A little more careful inspection shows the following: For all $y = \mathrm{A}x \in R(\mathrm{A})$ with $x \in D(\mathrm{A})$ we have again with (A.7)

$$y = \sum_{n=1}^{\infty} \zeta_n^{\pm}(y) y_n^{\pm} = \pm \sum_{n=1}^{\infty} \lambda_n \xi_n(x) y_n^{\pm} = \sum_{n=1}^{\infty} \xi_n(x) \mathrm{A} x_n,$$
$$\zeta_n^{\pm}(y) = \langle y, y_n^{\pm} \rangle_{\mathsf{Y}} = \pm\lambda_n \langle x, x_n \rangle_{\mathsf{X}} = \pm\lambda_n \xi_n(x).$$

If even $y = \mathrm{AA}^* \tilde{y} \in R(\mathrm{AA}^*)$ with $\tilde{y} \in D(\mathrm{AA}^*)$ we see

$$y = \sum_{n=1}^{\infty} \zeta_n^{\pm}(y) y_n^{\pm} = \sum_{n=1}^{\infty} \lambda_n^2 \zeta_n^{\pm}(\tilde{y}) y_n^{\pm} = \sum_{n=1}^{\infty} \zeta_n^{\pm}(\tilde{y}) \mathrm{AA}^* y_n^{\pm},$$
$$\zeta_n^{\pm}(y) = \langle \tilde{y}, \mathrm{AA}^* y_n^{\pm} \rangle_{\mathsf{Y}} = \lambda_n^2 \langle \tilde{y}, y_n^{\pm} \rangle_{\mathsf{Y}} = \lambda_n^2 \zeta_n^{\pm}(\tilde{y}).$$

Analogously for some $x = \mathrm{A}^* y \in R(\mathrm{A}^*)$ with $y \in D(\mathrm{A}^*)$ it holds

$$x = \sum_{n=1}^{\infty} \xi_n(x) x_n = \pm \sum_{n=1}^{\infty} \lambda_n \zeta_n^{\pm}(y) x_n = \sum_{n=1}^{\infty} \zeta_n^{\pm}(y) \mathrm{A}^* y_n^{\pm},$$
$$\xi_n(x) = \langle x, x_n \rangle_{\mathsf{X}} = \langle y, \mathrm{A} x_n \rangle_{\mathsf{Y}} = \pm\lambda_n \langle y, y_n^{\pm} \rangle_{\mathsf{Y}} = \pm\lambda_n \zeta_n^{\pm}(y).$$

If even $x = \mathrm{A}^*\mathrm{A}\tilde{x} \in R(\mathrm{A}^*\mathrm{A})$ with $\tilde{x} \in D(\mathrm{A}^*\mathrm{A})$ we have

$$x = \sum_{n=1}^{\infty} \xi_n(x) x_n = \sum_{n=1}^{\infty} \lambda_n^2 \xi_n(\tilde{x}) x_n = \sum_{n=1}^{\infty} \xi_n(\tilde{x}) \mathrm{A}^*\mathrm{A} x_n,$$
$$\xi_n(x) = \langle \tilde{x}, \mathrm{A}^*\mathrm{A} x_n \rangle_{\mathsf{X}} = \lambda_n^2 \langle \tilde{x}, x_n \rangle_{\mathsf{X}} = \lambda_n^2 \xi_n(\tilde{x}).$$

For $z_n^{\pm} := (x_n, y_n^{\pm}) \in D(\mathcal{M})$ we have

$$\mathrm{M} z_n^{\pm} = (\mathrm{A}^* y_n^{\pm}, \mathrm{A} x_n) = \pm\lambda_n (x_n, y_n^{\pm}) = \pm\lambda_n z_n^{\pm}.$$

Hence, $z_n^{\pm}$ is an eigenvector to the eigenvalue $\pm\lambda_n$ of M, i.e., $z_n^{\pm} \in N(\mathrm{M} \mp \lambda_n)$. Of course, $z_n^{\pm}$ is also an eigenvector to the eigenvalue λ_n^2 of M^2 since

$$\begin{bmatrix} \mathrm{A}^*\mathrm{A} - \lambda_n^2 & 0 \\ 0 & \mathrm{AA}^* - \lambda_n^2 \end{bmatrix} = \mathrm{M}^2 - \lambda_n^2 = (\mathrm{M} \pm \lambda_n)(\mathrm{M} \mp \lambda_n).$$

The assertions about the norms follow immediately by orthogonality and the continuity of the norms, concluding the proof. □

Corollary 32 *It holds*

$$\lambda_\ell^2 = |\mathrm{A} x_\ell|_{\mathsf{Y}}^2 = \min_{\substack{0 \neq x \in D(\mathcal{A}) \\ x \perp_X \{x_1, \dots, x_{\ell-1}\}}} \frac{|\mathrm{A}x|_{\mathsf{Y}}^2}{|x|_{\mathsf{X}}^2} = \min_{\substack{0 \neq y \in D(\mathcal{A}^*) \\ y \perp_Y \{y_1^{\pm}, \dots, y_{\ell-1}^{\pm}\}}} \frac{|\mathrm{A}^* y|_{\mathsf{X}}^2}{|y|_{\mathsf{Y}}^2} = |\mathrm{A}^* y_\ell^{\pm}|_{\mathsf{X}}^2,$$

especially

$$\lambda_1^2 = \min_{0 \neq x \in D(\mathcal{A})} \frac{|\mathrm{A}x|_{\mathsf{Y}}^2}{|x|_{\mathsf{X}}^2} = \min_{0 \neq y \in D(\mathcal{A}^*)} \frac{|\mathrm{A}^* y|_{\mathsf{X}}^2}{|y|_{\mathsf{Y}}^2}.$$

Proof First, we emphasize that the dimensions of the eigenspaces $N(\mathrm{A}^*\mathrm{A} - \lambda_n^2)$ and $N(\mathrm{A}\mathrm{A}^* - \lambda_n^2)$ equal. Using the latter theorem we can represent $x \in D(\mathcal{A})$ and $\mathrm{A}x$ by

$$x = \sum_{n=1}^{\infty} \xi_n x_n, \quad \mathrm{A}x = \pm \sum_{n=1}^{\infty} \lambda_n \xi_n y_n^{\pm}.$$

If additionally $x \bot_X \{x_1, \ldots, x_{\ell-1}\}$ we see $\xi_1 = \cdots = \xi_{\ell-1} = 0$ and thus

$$|x|_{\mathsf{X}}^2 = \sum_{n=\ell}^{\infty} \xi_n^2, \quad |\mathrm{A}x|_{\mathsf{Y}}^2 = \sum_{n=\ell}^{\infty} \lambda_n^2 \xi_n^2 \geq \lambda_\ell^2 \sum_{n=\ell}^{\infty} \xi_n^2 = \lambda_\ell^2 |x|_{\mathsf{X}}^2.$$

Therefore, $\dfrac{|\mathrm{A}x|_{\mathsf{Y}}^2}{|x|_{\mathsf{X}}^2} \geq \lambda_\ell^2$ holds for all $0 \neq x \in D(\mathcal{A})$ with $x \bot_X \{x_1, \ldots, x_{\ell-1}\}$. On the other hand $|\mathrm{A}x_\ell|_{\mathsf{Y}}^2 = \langle x_\ell, \mathrm{A}^*\mathrm{A}x_\ell \rangle_{\mathsf{X}} = \lambda_\ell^2 |x_\ell|_{\mathsf{X}}^2$ and $0 \neq x_\ell \in D(\mathcal{A})$ with $x_\ell \bot_X \{x_1, \ldots, x_{\ell-1}\}$. Thus,

$$\lambda_\ell^2 = |\mathrm{A}x_\ell|_{\mathsf{Y}}^2 = \min_{\substack{0 \neq x \in D(\mathcal{A}) \\ x \bot_X \{x_1, \ldots, x_{\ell-1}\}}} \frac{|\mathrm{A}x|_{\mathsf{Y}}^2}{|x|_{\mathsf{X}}^2}.$$

The other assertion about y and A^*y follows analogously. □