Maxwell meets Korn: A New Coercive Inequality for Tensor Fields in $\mathbb{R}^{N\times N}$ with Square-Integrable Exterior Derivative

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Abstract

For a bounded domain $\Omega \subset \mathbb{R}^N$ with connected Lipschitz boundary we prove the existence of some $c > 0$, such that

$$c\,\|P\|_{\mathsf{L}^2(\Omega,\mathbb{R}^{N\times N})} \leq \|\operatorname{sym} P\|_{\mathsf{L}^2(\Omega,\mathbb{R}^{N\times N})} + \|\operatorname{Curl} P\|_{\mathsf{L}^2(\Omega,\mathbb{R}^{N\times(N-1)N/2})}$$

holds for all square-integrable tensor fields $P : \Omega \to \mathbb{R}^{N\times N}$, having square-integrable generalized 'rotation' $\operatorname{Curl} P : \Omega \to \mathbb{R}^{N\times(N-1)N/2}$ and vanishing tangential trace on $\partial\Omega$, where both operations are to be understood row-wise. Here, in each row the operator curl is the vector analytical reincarnation of the exterior derivative d in $\mathbb{R}^N$. For compatible tensor fields P, i.e., $P = \nabla v$, the latter estimate reduces to a non-standard variant of Korn's first inequality in $\mathbb{R}^N$, namely

$$c\,\|\nabla v\|_{\mathsf{L}^2(\Omega,\mathbb{R}^{N\times N})} \leq \|\operatorname{sym} \nabla v\|_{\mathsf{L}^2(\Omega,\mathbb{R}^{N\times N})}$$

for all vector fields $v \in \mathsf{H}^1(\Omega, \mathbb{R}^N)$, for which ∇v_n, $n = 1, \ldots, N$, are normal at $\partial\Omega$.
Key Words Korn's inequality, theory of Maxwell equations in $\mathbb{R}^N$, Helmholtz decomposition, Poincaré/Friedrichs type estimates

1 Introduction and Preliminaries

We extend the results from [12], which have been announced in [13], to the N-dimensional case following in close lines the arguments presented there. Let $N \in \mathbb{N}$ and Ω be a bounded domain in $\mathbb{R}^N$ with connected Lipschitz boundary $\Gamma := \partial\Omega$. We prove a Korn-type inequality in $\overset{\circ}{\mathsf{H}}(\operatorname{Curl};\Omega)$ for eventually non-symmetric tensor fields P mapping Ω to $\mathbb{R}^{N\times N}$. More precisely, there exists a positive constant c, such that

$$c\,\|P\|_{\mathsf{L}^2(\Omega)} \leq \|\operatorname{sym} P\|_{\mathsf{L}^2(\Omega)} + \|\operatorname{Curl} P\|_{\mathsf{L}^2(\Omega)}$$

holds for all tensor fields $P \in \overset{\circ}{\mathsf{H}}(\mathrm{Curl};\Omega)$, where P belongs to $\overset{\circ}{\mathsf{H}}(\mathrm{Curl};\Omega)$, if $P \in \mathsf{H}(\mathrm{Curl};\Omega)$ has vanishing tangential trace on Γ. Thereby, the generalized Curl and tangential trace are defined as row-wise operations. For compatible tensor fields $P = \nabla v$ with vector fields $v \in \mathsf{H}^1(\Omega)$, for which ∇v_n, $n = 1,\dots,N$, are normal at $\partial\Omega$, the latter estimate reduces to a non-standard variant of the well known Korn's first inequality in $\mathbb{R}^N$

$$c\,\|\nabla v\|_{\mathsf{L}^2(\Omega)} \le \|\mathrm{sym}\,\nabla v\|_{\mathsf{L}^2(\Omega)}\,.$$

Our proof relies on three essential tools, namely

1. Maxwell estimate (Poincaré-type estimate),
2. Helmholtz' decomposition,
3. Korn's first inequality.

In [12] we already pointed out the importance of the Maxwell estimate and the related question of the Maxwell compactness property*. Here, we mention the papers [2, 6, 15, 16, 17, 18, 20]. Results for the Helmholtz decomposition can be found in [3, 14, 15, 17, 20, 19, 7, 8, 9]. Nowadays, differential forms find prominent applications in numerical methods like Finite Element Exterior Calculus [1, 4] or Discrete Exterior Calculus [5].

1.1 Differential Forms

We may look at Ω as a smooth Riemannian manifold of dimension N with compact closure and connected Lipschitz continuous boundary Γ. The alternating differential forms of rank $q \in \{0,\dots,N\}$ on Ω, briefly q-forms, with square-integrable coefficients will be denoted by $\mathsf{L}^{2,q}(\Omega)$. The exterior derivative d and the co-derivative $\delta = \pm * \mathrm{d} *$ ($*$: Hodge's star operator) are formally skew-adjoint to each other, i.e.,

$$\forall\, E \in \overset{\circ}{\mathsf{C}}{}^{\infty,q}(\Omega) \quad H \in \overset{\circ}{\mathsf{C}}{}^{\infty,q+1}(\Omega) \qquad \langle \mathrm{d}E, H\rangle_{\mathsf{L}^{2,q+1}(\Omega)} = -\,\langle E, \delta H\rangle_{\mathsf{L}^{2,q}(\Omega)}\,,$$

where the $\mathsf{L}^{2,q}(\Omega)$-scalar product is given by

$$\forall\, E, H \in \mathsf{L}^{2,q}(\Omega) \qquad \langle E, H\rangle_{\mathsf{L}^{2,q}(\Omega)} := \int_\Omega E \wedge *H.$$

Here $\overset{\circ}{\mathsf{C}}{}^{\infty,q}(\Omega)$ denotes the space of compactly supported and smooth q-forms on Ω. Using this duality, we can define weak versions of d and δ. The corresponding standard Sobolev spaces are denoted by

$$\mathsf{D}^q(\Omega) := \{E \in \mathsf{L}^{2,q}(\Omega) \,:\, \mathrm{d}E \in \mathsf{L}^{2,q+1}(\Omega)\},$$
$$\Delta^q(\Omega) := \{H \in \mathsf{L}^{2,q}(\Omega) \,:\, \delta H \in \mathsf{L}^{2,q-1}(\Omega)\}.$$

*By 'Maxwell estimate' and 'Maxwell compactness property' we mean the estimates and compact embedding results used in the theory of Maxwell's equations.

The homogeneous tangential boundary condition $\tau_\Gamma E = 0$, where τ_Γ denotes the tangential trace, is generalized in the space

$$\mathring{\mathsf{D}}^q(\Omega) := \overline{\mathring{\mathsf{C}}^{\infty,q}(\Omega)},$$

where the closure is taken in $\mathsf{D}^q(\Omega)$. In classical terms, we have for smooth q-forms $\tau_\Gamma = \iota^*$ with the canonical embedding $\iota : \Gamma \hookrightarrow \overline{\Omega}$. An index 0 at the lower right position indicates vanishing derivatives, i.e.,

$$\mathring{\mathsf{D}}^q_0(\Omega) = \{E \in \mathring{\mathsf{D}}^q(\Omega) \,:\, \mathrm{d}E = 0\}, \qquad \Delta^q_0(\Omega) = \{H \in \Delta^q(\Omega) \,:\, \delta H = 0\}.$$

By definition and density, we have

$$\Delta^q_0(\Omega) := (\mathrm{d}\mathring{\mathsf{D}}^{q-1}(\Omega))^\perp, \quad \Delta^q_0(\Omega)^\perp := \overline{\mathrm{d}\mathring{\mathsf{D}}^{q-1}(\Omega)},$$

where $\perp$ denotes the orthogonal complement with respect to the $\mathsf{L}^{2,q}(\Omega)$-scalar product and the closure is taken in $\mathsf{L}^{2,q}(\Omega)$. Hence, we obtain the $\mathsf{L}^{2,q}(\Omega)$-orthogonal decomposition, usually called Hodge-Helmholtz decomposition,

$$\mathsf{L}^{2,q}(\Omega) = \overline{\mathrm{d}\mathring{\mathsf{D}}^{q-1}(\Omega)} \oplus \Delta^q_0(\Omega), \tag{1.1}$$

where $\oplus$ denotes the orthogonal sum with respect to the $\mathsf{L}^{2,q}(\Omega)$-scalar product. In [20, 16] the following crucial tool has been proved:

Lemma 1 (Maxwell Compactness Property) *For all q the embeddings*

$$\mathring{\mathsf{D}}^q(\Omega) \cap \Delta^q(\Omega) \hookrightarrow \mathsf{L}^{2,q}(\Omega)$$

are compact.

As a first immediate consequence, the spaces of so called 'harmonic Dirichlet forms'

$$\mathcal{H}^q(\Omega) := \mathring{\mathsf{D}}^q_0(\Omega) \cap \Delta^q_0(\Omega)$$

are finite dimensional. In classical terms, a q-form E belongs to $\mathcal{H}^q(\Omega)$, if

$$\mathrm{d}E = 0, \quad \delta E = 0, \quad \iota^* E = 0.$$

The dimension of $\mathcal{H}^q(\Omega)$ equals the $(N-q)$th Betti number of Ω. Since we assume the boundary Γ to be connected, the $(N-1)$th Betti number of Ω vanishes and therefore there are no Dirichlet forms of rank 1 besides zero, i.e.,

$$\mathcal{H}^1(\Omega) = \{0\}. \tag{1.2}$$

This condition on the domain Ω resp. its boundary Γ is satisfied e.g. for a ball or a torus.

By a usual indirect argument, we achieve another immediate consequence:

Lemma 2 (Poincaré Estimate for Differential Forms) *For all q there exist positive constants $c_{p,q}$, such that for all $E \in \overset{\circ}{\mathsf{D}}{}^q(\Omega) \cap \Delta^q(\Omega) \cap \mathcal{H}^q(\Omega)^\perp$*

$$\|E\|_{\mathsf{L}^{2,q}(\Omega)} \leq c_{p,q}\big(\|\mathrm{d}E\|^2_{\mathsf{L}^{2,q+1}(\Omega)} + \|\delta E\|^2_{\mathsf{L}^{2,q-1}(\Omega)} \big)^{1/2}.$$

Since

$$\mathrm{d}\overset{\circ}{\mathsf{D}}{}^{q-1}(\Omega) \subset \overset{\circ}{\mathsf{D}}{}^q_0(\Omega)$$

(note that $\mathrm{dd}= 0$ and $\delta\delta= 0$ hold even in the weak sense) we get by (1.1)

$$\mathrm{d}\overset{\circ}{\mathsf{D}}{}^{q-1}(\Omega) = \mathrm{d}\big(\overset{\circ}{\mathsf{D}}{}^{q-1}(\Omega) \cap \Delta_0^{q-1}(\Omega)\big) = \mathrm{d}\big(\overset{\circ}{\mathsf{D}}{}^{q-1}(\Omega) \cap \Delta_0^{q-1}(\Omega) \cap \mathcal{H}^{q-1}(\Omega)^\perp\big).$$

Now, Lemma 2 shows that $\mathrm{d}\overset{\circ}{\mathsf{D}}{}^{q-1}(\Omega)$ is already closed. Hence, we obtain a refinement of (1.1)

Lemma 3 (Hodge-Helmholtz Decomposition for Differential Forms) *The decomposition*

$$\mathsf{L}^{2,q}(\Omega) = \mathrm{d}\overset{\circ}{\mathsf{D}}{}^{q-1}(\Omega) \oplus \Delta_0^q(\Omega)$$

holds.

1.2 Functions and Vector Fields

Let us turn to the special case $q = 1$. In this case, we choose (e.g.) the identity as single global chart for Ω and use the canonical identification isomorphism for 1-forms (i.e., Riesz' representation theorem) with vector fields $\mathrm{d}x_n \cong e^n$, namely

$$\sum_{n=1}^N v_n(x)\,\mathrm{d}x_n \cong v(x) = \begin{bmatrix} v_1(x) \\ \vdots \\ v_N(x) \end{bmatrix}, \quad x \in \Omega.$$

0-forms will be isomorphically identified with functions on Ω. Then, $\mathrm{d} \cong \operatorname{grad} = \nabla$ for 0-forms (functions) and $\delta \cong \operatorname{div} = \nabla\cdot$ for 1-forms (vector fields). Hence, the well known first order differential operators from vector analysis occur. Moreover, on 1-forms we define a new operator $\operatorname{curl} :\cong \mathrm{d}$, which turns into the usual curl if $N = 3$ or $N = 2$. $\mathsf{L}^{2,q}(\Omega)$ equals the usual Lebesgue spaces of square integrable functions or vector fields on Ω with values in $\mathbb{R}^n$, $n := n_{N,q} := \binom{N}{q}$, which will be denoted by $\mathsf{L}^2(\Omega) := \mathsf{L}^2(\Omega, \mathbb{R}^n)$. $\mathsf{D}^0(\Omega)$ and $\Delta^1(\Omega)$ are identified with the standard Sobolev spaces

$$\mathsf{H}(\operatorname{grad};\Omega) := \{u \in \mathsf{L}^2(\Omega,\mathbb{R}) \,:\, \operatorname{grad} u \in \mathsf{L}^2(\Omega,\mathbb{R}^N)\} = \mathsf{H}^1(\Omega),$$
$$\mathsf{H}(\operatorname{div};\Omega) := \{v \in \mathsf{L}^2(\Omega,\mathbb{R}^N) \,:\, \operatorname{div} v \in \mathsf{L}^2(\Omega,\mathbb{R})\},$$

respectively. Moreover, we may now identify $\mathsf{D}^1(\Omega)$ with

$$\mathsf{H}(\mathrm{curl};\Omega) := \{v \in \mathsf{L}^2(\Omega,\mathbb{R}^N) \,:\, \mathrm{curl}\, v \in \mathsf{L}^2(\Omega,\mathbb{R}^{(N-1)N/2})\},$$

which is the well known $\mathsf{H}(\mathrm{curl};\Omega)$ for $N = 2,3$. E.g., for $N = 4$ we have

$$\mathrm{curl}\, v = \begin{bmatrix} \partial_1 v_2 - \partial_2 v_1 \\ \partial_1 v_3 - \partial_3 v_1 \\ \partial_1 v_4 - \partial_4 v_1 \\ \partial_2 v_3 - \partial_3 v_2 \\ \partial_2 v_4 - \partial_4 v_2 \\ \partial_3 v_4 - \partial_4 v_3 \end{bmatrix} \in \mathbb{R}^6$$

and for $N = 5$ we get $\mathrm{curl}\, v \in \mathbb{R}^{10}$. In general, the entries of the $(N-1)N/2$-vector $\mathrm{curl}\, v$ consist of all possible combinations of

$$\partial_n v_m - \partial_m v_n, \quad 1 \leq n < m \leq N.$$

Similarly, we obtain the closed subspaces

$$\overset{\circ}{\mathsf{H}}(\mathrm{grad};\Omega) = \overset{\circ}{\mathsf{H}}{}^1(\Omega), \quad \overset{\circ}{\mathsf{H}}(\mathrm{curl};\Omega)$$

as reincarnations of $\overset{\circ}{\mathsf{D}}{}^0(\Omega)$ and $\overset{\circ}{\mathsf{D}}{}^1(\Omega)$, respectively. We note

$$\overset{\circ}{\mathsf{H}}(\mathrm{grad};\Omega) = \overline{\overset{\circ}{\mathsf{C}}{}^\infty(\Omega)}, \quad \overset{\circ}{\mathsf{H}}(\mathrm{curl};\Omega) = \overline{\overset{\circ}{\mathsf{C}}{}^\infty(\Omega)},$$

where the closures are taken in the respective graph norms, and that in these Sobolev spaces the classical homogeneous scalar and tangential (compare to $N = 3$) boundary conditions

$$u|_\Gamma = 0, \quad \nu \times v|_\Gamma = 0$$

are generalized. Here, ν denotes the outward unit normal for Γ. Furthermore, we have the spaces of irrotational or solenoidal vector fields

$$\begin{aligned} \mathsf{H}(\mathrm{curl}_0;\Omega) &= \{v \in \mathsf{H}(\mathrm{curl};\Omega) \,:\, \mathrm{curl}\, v = 0\}, \\ \overset{\circ}{\mathsf{H}}(\mathrm{curl}_0;\Omega) &= \{v \in \overset{\circ}{\mathsf{H}}(\mathrm{curl};\Omega) \,:\, \mathrm{curl}\, v = 0\}, \\ \mathsf{H}(\mathrm{div}_0;\Omega) &= \{v \in \mathsf{H}(\mathrm{div};\Omega) \,:\, \mathrm{div}\, v = 0\}. \end{aligned}$$

Again, all these spaces are Hilbert spaces. Now, we have two compact embeddings

$$\overset{\circ}{\mathsf{H}}(\mathrm{grad};\Omega) \hookrightarrow \mathsf{L}^2(\Omega), \quad \overset{\circ}{\mathsf{H}}(\mathrm{curl};\Omega) \cap \mathsf{H}(\mathrm{div};\Omega) \hookrightarrow \mathsf{L}^2(\Omega),$$

i.e., Rellich's selection theorem and the Maxwell compactness property. Moreover, the following Poincaré and Maxwell estimates hold:

Corollary 4 (Poincaré Estimate for Functions) *Let* $c_p := c_{p,0}$. *Then, for all functions* $u \in \overset{\circ}{\mathsf{H}}(\mathrm{grad};\Omega)$

$$\|u\|_{\mathsf{L}^2(\Omega)} \leq c_p \, \|\mathrm{grad}\, u\|_{\mathsf{L}^2(\Omega)} .$$

Corollary 5 (Maxwell Estimate for Vector Fields) *Let* $c_m := c_{p,1}$. *Then, for all vector fields* $v \in \overset{\circ}{\mathsf{H}}(\mathrm{curl};\Omega) \cap \mathsf{H}(\mathrm{div};\Omega)$

$$\|v\|_{\mathsf{L}^2(\Omega)} \leq c_m \big(\|\mathrm{curl}\, v\|^2_{\mathsf{L}^2(\Omega)} + \|\mathrm{div}\, v\|^2_{\mathsf{L}^2(\Omega)} \big)^{1/2}.$$

We note that generally $\mathcal{H}^0(\Omega) = \{0\}$ and by (1.2) also $\mathcal{H}^1(\Omega) = \{0\}$. The appropriate Helmholtz decomposition for our needs is

Corollary 6 (Helmholtz Decomposition for Vector Fiels)

$$\mathsf{L}^2(\Omega) = \mathrm{grad}\, \overset{\circ}{\mathsf{H}}(\mathrm{grad};\Omega) \oplus \mathsf{H}(\mathrm{div}_0;\Omega)$$

1.3 Tensor Fields

We extend our calculus to $(N \times N)$-tensor (matrix) fields. For vector fields v with components in $\mathsf{H}(\mathrm{grad};\Omega)$ and tensor fields P with rows in $\mathsf{H}(\mathrm{curl};\Omega)$ resp. $\mathsf{H}(\mathrm{div};\Omega)$, i.e.,

$$v = \begin{bmatrix} v_1 \\ \vdots \\ v_N \end{bmatrix}, \quad v_n \in \mathsf{H}(\mathrm{grad};\Omega), \quad P = \begin{bmatrix} {P_1}^T \\ \vdots \\ {P_N}^T \end{bmatrix}, \quad P_n \in \mathsf{H}(\mathrm{curl};\Omega) \text{ resp. } \mathsf{H}(\mathrm{div};\Omega)$$

for $n = 1, \ldots, N$, we define

$$\mathrm{Grad}\, v := \begin{bmatrix} \mathrm{grad}^T v_1 \\ \vdots \\ \mathrm{grad}^T v_N \end{bmatrix} = J_v = \nabla v, \quad \mathrm{Curl}\, P := \begin{bmatrix} \mathrm{curl}^T P_1 \\ \vdots \\ \mathrm{curl}^T P_N \end{bmatrix}, \quad \mathrm{Div}\, P := \begin{bmatrix} \mathrm{div}\, P_1 \\ \vdots \\ \mathrm{div}\, P_N \end{bmatrix},$$

where J_v denotes the Jacobian of v and ^{T} the transpose. We note that v and $\mathrm{Div}\, P$ are N-vector fields, P and $\mathrm{Grad}\, v$ are $(N \times N)$-tensor fields, whereas $\mathrm{Curl}\, P$ is a $(N \times (N-1)N/2)$-tensor field which may also be viewed as a totally anti-symmetric third order tensor field with entries

$$(\mathrm{Curl}\, P)_{ijk} = \partial_j P_{ik} - \partial_k P_{ij}.$$

The corresponding Sobolev spaces will be denoted by

$$\begin{array}{llll} \mathsf{H}(\mathrm{Grad};\Omega), & \overset{\circ}{\mathsf{H}}(\mathrm{Grad};\Omega), & \mathsf{H}(\mathrm{Div};\Omega), & \mathsf{H}(\mathrm{Div}_0;\Omega), \\ \mathsf{H}(\mathrm{Curl};\Omega), & \overset{\circ}{\mathsf{H}}(\mathrm{Curl};\Omega), & \mathsf{H}(\mathrm{Curl}_0;\Omega), & \overset{\circ}{\mathsf{H}}(\mathrm{Curl}_0;\Omega). \end{array}$$

There are three crucial tools to prove our estimate. First, we have obvious consequences from Corollaries 4, 5 and 6:

Corollary 7 (Poincaré Estimate for Vector Fields) *For all* $v \in \overset{\circ}{\mathsf{H}}(\mathrm{Grad};\Omega)$

$$\|v\|_{\mathsf{L}^2(\Omega)} \leq c_p \|\mathrm{Grad}\, v\|_{\mathsf{L}^2(\Omega)} .$$

Corollary 8 (Maxwell Estimate for Tensor Fields) *The estimate*

$$\|P\|_{\mathsf{L}^2(\Omega)} \leq c_m \big(\|\mathrm{Curl}\, P\|^2_{\mathsf{L}^2(\Omega)} + \|\mathrm{Div}\, P\|^2_{\mathsf{L}^2(\Omega)} \big)^{1/2}$$

holds for all tensor fields $P \in \overset{\circ}{\mathsf{H}}(\mathrm{Curl};\Omega) \cap \mathsf{H}(\mathrm{Div};\Omega)$.

Corollary 9 (Helmholtz Decomposition for Tensor Fields)

$$\mathsf{L}^2(\Omega) = \mathrm{Grad}\, \overset{\circ}{\mathsf{H}}(\mathrm{Grad};\Omega) \oplus \mathsf{H}(\mathrm{Div}_0;\Omega)$$

The last important tool is Korn's first inequality.

Lemma 10 (Korn's First Inequality) *For all vector fields* $v \in \overset{\circ}{\mathsf{H}}(\mathrm{Grad};\Omega)$

$$\|\mathrm{Grad}\, v\|_{\mathsf{L}^2(\Omega)} \leq \sqrt{2}\, \|\mathrm{sym}\, \mathrm{Grad}\, v\|_{\mathsf{L}^2(\Omega)} .$$

Here, we introduce the symmetric and skew-symmetric parts

$$\mathrm{sym}\, P := \frac{1}{2}(P + P^T), \quad \mathrm{skew}\, P := \frac{1}{2}(P - P^T)$$

of a $(N \times N)$-tensor $P = \mathrm{sym}\, P + \mathrm{skew}\, P$.

Remark 11 *We note that the proof including the value of the constant is simple. By density we may assume* $v \in \overset{\circ}{\mathsf{C}}{}^\infty(\Omega)$. *Twofold partial integration yields*

$$\langle \partial_n v_m, \partial_m v_n \rangle_{\mathsf{L}^2(\Omega)} = \langle \partial_m v_m, \partial_n v_n \rangle_{\mathsf{L}^2(\Omega)}$$

and hence

$$\begin{aligned} 2\, \|\mathrm{sym}\, \mathrm{Grad}\, v\|^2_{\mathsf{L}^2(\Omega)} &= \frac{1}{2} \sum_{n,m=1}^{N} \|\partial_n v_m + \partial_m v_n\|^2_{\mathsf{L}^2(\Omega)} \\ &= \sum_{n,m=1}^{N} \big(\|\partial_n v_m\|^2_{\mathsf{L}^2(\Omega)} + \langle \partial_n v_m, \partial_m v_n \rangle_{\mathsf{L}^2(\Omega)} \big) \\ &= \|\mathrm{Grad}\, v\|^2_{\mathsf{L}^2(\Omega)} + \|\mathrm{div}\, v\|^2_{\mathsf{L}^2(\Omega)} \geq \|\mathrm{Grad}\, v\|^2_{\mathsf{L}^2(\Omega)} . \end{aligned}$$

More on Korn's first inequality can be found, e.g., in [10].

2 Results

For tensor fields $P \in \mathsf{H}(\mathrm{Curl};\Omega)$ we define the semi-norm

$$|||P||| := \big(\|\mathrm{sym}\,P\|^2_{\mathsf{L}^2(\Omega)} + \|\mathrm{Curl}\,P\|^2_{\mathsf{L}^2(\Omega)} \big)^{1/2}.$$

The main step is to prove the following

Lemma 12 *Let $\hat{c} := \max\{2, \sqrt{5}c_m\}$. Then, for all $P \in \overset{\circ}{\mathsf{H}}(\mathrm{Curl};\Omega)$*

$$\|P\|_{\mathsf{L}^2(\Omega)} \leq \hat{c}\, |||P||| \, .$$

Proof Let $P \in \overset{\circ}{\mathsf{H}}(\mathrm{Curl};\Omega)$. According to Corollary 9 we orthogonally decompose

$$P = \mathrm{Grad}\, v + S \in \mathrm{Grad}\, \overset{\circ}{\mathsf{H}}(\mathrm{Grad};\Omega) \oplus \mathsf{H}(\mathrm{Div}_0;\Omega).$$

Then, $\mathrm{Curl}\, P = \mathrm{Curl}\, S$ and we observe $S \in \overset{\circ}{\mathsf{H}}(\mathrm{Curl};\Omega) \cap \mathsf{H}(\mathrm{Div}_0;\Omega)$ since

$$\mathrm{Grad}\, \overset{\circ}{\mathsf{H}}(\mathrm{Grad};\Omega) \subset \overset{\circ}{\mathsf{H}}(\mathrm{Curl}_0;\Omega). \tag{2.1}$$

By Corollary 8, we have

$$\|S\|_{\mathsf{L}^2(\Omega)} \leq c_m \|\mathrm{Curl}\, P\|_{\mathsf{L}^2(\Omega)} \, . \tag{2.2}$$

Then, by Lemma 10 and (2.2) we obtain

$$\begin{aligned} \|P\|^2_{\mathsf{L}^2(\Omega)} &= \|\mathrm{Grad}\, v\|^2_{\mathsf{L}^2(\Omega)} + \|S\|^2_{\mathsf{L}^2(\Omega)} \\ &\leq 2\, \|\mathrm{sym}\,\mathrm{Grad}\, v\|^2_{\mathsf{L}^2(\Omega)} + \|S\|^2_{\mathsf{L}^2(\Omega)} \leq 4\, \|\mathrm{sym}\, P\|^2_{\mathsf{L}^2(\Omega)} + 5\, \|S\|^2_{\mathsf{L}^2(\Omega)} \, , \end{aligned}$$

which completes the proof. □

The immediate consequence is our main result

Theorem 13 *On $\overset{\circ}{\mathsf{H}}(\mathrm{Curl};\Omega)$ the norms $\|\cdot\|_{\mathsf{H}(\mathrm{Curl};\Omega)}$ and $|||\cdot|||$ are equivalent. In particular, $|||\cdot|||$ is a norm on $\overset{\circ}{\mathsf{H}}(\mathrm{Curl};\Omega)$ and there exists a positive constant c, such that*

$$c\, \|P\|^2_{\mathsf{H}(\mathrm{Curl};\Omega)} \leq |||P|||^2 = \|\mathrm{sym}\, P\|^2_{\mathsf{L}^2(\Omega)} + \|\mathrm{Curl}\, P\|^2_{\mathsf{L}^2(\Omega)}$$

holds for all $P \in \overset{\circ}{\mathsf{H}}(\mathrm{Curl};\Omega)$.

Remark 14 *For a skew-symmetric tensor field $P : \Omega \to \mathfrak{so}(N)$ our estimate reduces to a Poincaré inequality in disguise, since* $\mathrm{Curl}\, P$ *controls all partial derivatives of P (compare to [11]) and the homogeneous tangential boundary condition for P is implied by $P|_\Gamma = 0$.*

Setting $P := \operatorname{Grad} v$ we obtain

Remark 15 (Korn's First Inequality: Tangential-Variant) *For all* $v \in \overset{\circ}{\mathsf{H}}(\operatorname{Grad};\Omega)$

$$\|\operatorname{Grad} v\|_{\mathsf{L}^2(\Omega)} \leq \hat{c}\,\|\operatorname{sym}\operatorname{Grad} v\|_{\mathsf{L}^2(\Omega)} \tag{2.3}$$

holds by Lemma 12 and (2.1). *This is just Korn's first inequality from Lemma 10 with a larger constant* $\hat{c}$. *Since* Γ *is connected, i.e.,* $\mathcal{H}^1(\Omega) = \{0\}$, *we even have*

$$\operatorname{Grad}\overset{\circ}{\mathsf{H}}(\operatorname{Grad};\Omega) = \overset{\circ}{\mathsf{H}}(\operatorname{Curl}_0;\Omega).$$

Thus, (2.3) *holds for all* $v \in \mathsf{H}(\operatorname{Grad};\Omega)$ *with* $\operatorname{Grad} v \in \overset{\circ}{\mathsf{H}}(\operatorname{Curl}_0;\Omega)$, *i.e., with* $\operatorname{Grad} v_n$, $n = 1,\dots,N$, *normal at* Γ, *which then extends Lemma 10 through the (apparently) weaker boundary condition.*

The elementary arguments above apply certainly to much more general situations, e.g., to not necessarily connected boundaries Γ and to tangential boundary conditions which are imposed only on parts of Γ. These discussions are left to forthcoming papers.

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