Function space bases in the dune-functions module

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Discretization modules

dune-fem

- Focus on adaptivity, parallism, and efficiency

dune-pdelab

- Very flexible and powerful
- Steep learning curve

dune-fufem

- Easy to use
- Less powerful

The idea:

- Standardize on parts of the functionality

The team

- Carsten
- Christian
- Steffen
- Yours truly

History

- First meeting: Aug. 2013 in Münster (with Christoph Gersbacher and Stefan Girke)
- Further meetings every six months
- First actual users in March 2015

Functions

- Interface for functions $f : \mathbb{R}^n \to \mathbb{R}^m$, differentiable functions, grid functions, etc.
- Based on callables, concepts and type erasure
- Talk by Carsten

Function space bases

- Content of this talk

Infrastructure

- Interpolation:

  $$\text{function} + \text{basis} \Rightarrow \text{coefficient vector}$$

- VTK output of grid functions

The case for bases

- Grid function spaces are *not* the right abstraction
- More than one basis for the same space
  - E.g., P2 nodal basis vs. hierarchical basis
  - Orthogonal vs. Lagrange DG basis
- Basis + coefficients = discrete function

Functionality of a basis

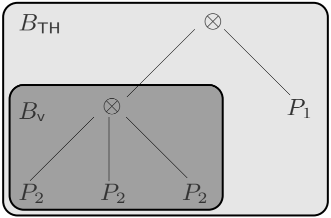
For any given grid element

- ...get restrictions of relevant basis functions to this element
  - i.e., the shape functions
  - use dune-localfunctions interfaces
- ...get local shape function numbers
- ...get global basis function numbers

Systematic construction of basis for vector-valued spaces

- Tensor products of simpler basis
- Taylor–Hood: $B_{\mathsf{TH}} = (P_2 \otimes P_2 \otimes P_2) \otimes P_1$

Tree representation



Systematic construction of

- orderings
- multi-indices

Taylor–Hood basis: lexicographic ordering

$b_{x,0}$	0	$(0,0)$	$(0,0)$	$(0,0,0)$
$b_{x,1}$	1	$(0,1)$	$(0,1)$	$(0,0,1)$
$b_{x,2}$	2	$(0,2)$	$(0,2)$	$(0,0,2)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$b_{y,0}$	n	$(0,n)$	$(1,0)$	$(0,1,0)$
$b_{y,1}$	$n+1$	$(0,n+1)$	$(1,1)$	$(0,1,1)$
$b_{y,2}$	$n+2$	$(0,n+2)$	$(1,2)$	$(0,1,2)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$b_{z,0}$	$2n$	$(0,2n)$	$(2,0)$	$(0,2,0)$
$b_{z,1}$	$2n+1$	$(0,2n+1)$	$(2,1)$	$(0,2,1)$
$b_{z,2}$	$2n+2$	$(0,2n+2)$	$(2,2)$	$(0,2,2)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
p_0	$3n$	$(1,0)$	n	$(1,0)$
p_1	$3n+1$	$(1,1)$	$n+1$	$(1,1)$
p_2	$3n+2$	$(1,2)$	$n+2$	$(1,2)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Possible index types for a Taylor–Hood basis with lexicographic ordering of the velocity basis functions

Taylor–Hood basis: interleaved ordering

$b_{x,0}$	0	$(0,0)$	$(0,0)$	$(0,0,0)$
$b_{y,0}$	1	$(0,1)$	$(0,1)$	$(0,0,1)$
$b_{z,0}$	2	$(0,2)$	$(0,2)$	$(0,0,2)$
$b_{x,1}$	3	$(0,3)$	$(1,0)$	$(0,1,0)$
$b_{y,1}$	4	$(0,4)$	$(1,1)$	$(0,1,1)$
$b_{z,1}$	5	$(0,5)$	$(1,2)$	$(0,1,2)$
$b_{x,2}$	6	$(0,6)$	$(2,0)$	$(0,2,0)$
$b_{y,2}$	7	$(0,7)$	$(2,1)$	$(0,2,1)$
$b_{z,2}$	8	$(0,8)$	$(2,2)$	$(0,2,2)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
p_0	$3n$	$(1,0)$	n	$(1,0)$
p_1	$3n+1$	$(1,1)$	$n+1$	$(1,1)$
p_2	$3n+2$	$(1,2)$	$n+2$	$(1,2)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Possible index types for a Taylor–Hood basis with interleaved ordering of the velocity basis functions

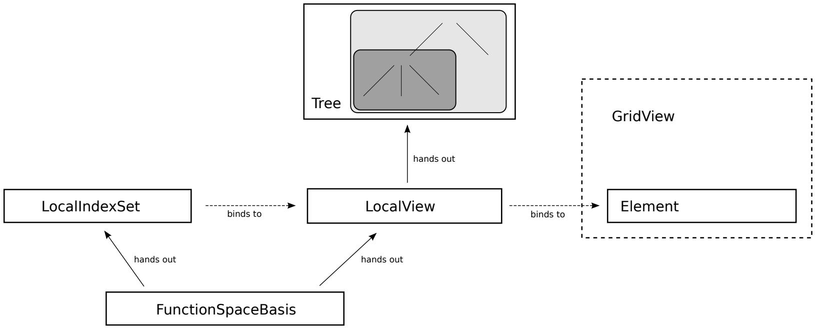


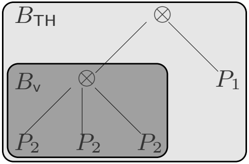
Figure: Overview of the classes making up the interface to finite element space bases

Interface

- `size_type size() const`
  Total number of basis functions
- `size_type size(const SizePrefix& prefix) const`
  Number of basis functions with a given multi-index prefix
- `LocalView localView() const`
  Get a local view object
- `LocalIndexSet localIndexSet() const`
  Get a local index object

Interface

- `void bind(const Element& e)`
  Bind the view to grid element `e`
- `const Tree& tree() const`
  Get the shape function tree for the current element
- `size_type size() const`
  Total number of shape functions on the current element
- `size_type maxSize() const`
  Maximum number of shape functions over *all* elements



Leaf nodes

- `const FiniteElement& finiteElement() const`
- `size_type localIndex(size_type i) const`

Inner nodes

- `PowerNode`: Combines identical subtrees
- `CompositeNode`: Combines differing subtrees

Node access

- `tree.child(a,b,c,...)`,
  with `a,b,c,...` either `int` or `std::integral_constant<size_type,.>`
- Example: `tree.child(_0,0)`: first component of velocity basis

Interface

- `void bind(const LocalView& localView)`
  Bind to `localView` object
- `size_type size() const`
  Total number of shape functions for the current element
- `MultiIndex index(size_type i) const`
  Get global (multi-)index for the `i`-th shape function

Open question:

- How to request *different* orderings / index types?

Setting

- Models a viscous incompressible fluid in a d-dimensional domain Ω.
- Unknowns: fluid velocity field $\mathbf{u} : \Omega \to \mathbb{R}^d$, pressure $p : \Omega \to \mathbb{R}$.
- The pressure is therefore usually normalized such that $\int_\Omega p \, dx = 0$.

Weak form

- Spaces

$$\mathbf{H}^1_D(\Omega) := \left\{ \mathbf{v} \in \mathbf{H}^1(\Omega) \ : \ \operatorname{tr} \mathbf{v} = \mathbf{u}_D \right\},$$
$$L_{2,0}(\Omega) := \left\{ q \in L_2(\Omega) \ : \ \int_\Omega q \, dx = 0 \right\},$$

- Bilinear forms

$$a(\mathbf{u}, \mathbf{v}) := \int_\Omega \nabla \mathbf{u} \nabla \mathbf{v} \, dx, \qquad \text{and} \qquad b(\mathbf{v}, q) := \int_\Omega \div \mathbf{v} \cdot q \, dx.$$

- Saddle-point problem: Find $(\mathbf{u}, p) \in \mathbf{H}^1_D(\Omega) \times L_{2,0}(\Omega)$ such that

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) &= 0 \qquad \text{for all } \mathbf{v} \in \mathbf{H}^1_0(\Omega) \\ b(\mathbf{u}, q) \qquad\qquad &= 0 \qquad \text{for all } q \in L_{2,0}(\Omega). \end{aligned}$$

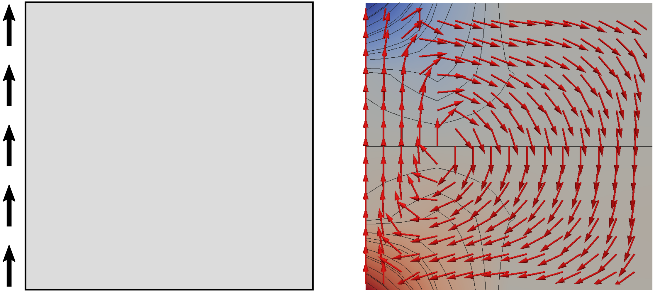


Figure: Left: setting, right: simulation result. The arrows show the *normalized* velocity.

Current status

Technology preview

- Most work is done
- Details of the API may still change(!)
- Go use it!

Basis implementations

- PQkNodalBasis
- LagrangeDGBasis
- TaylorHoodBasis
- BSplineBasis
- ... more to come

Further information

- `www.dune-project.org/modules/dune-functions`