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Coupling Deformation and Flow in Fractured Poroelastic Media

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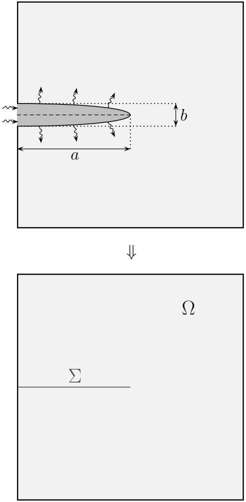

Fluid–solid coupling

- Porous material, contains thin layer with different material properties.
- Coupled processes:
  1. Fluid-fluid: The fluid diffuses from the fracture into the porous medium and vice versa.
  2. Fluid-solid: The pressure applied by the fluid onto the fracture faces induces a deformation.
  3. Solid-fluid: The flow in the fracture is affected by the fracture aperture b and thus on the deformation of the surrounding solid skeleton.
  4. Poroelasticity: The matrix fluid pressure interacts with matrix stress
- Generally $b \ll a \Rightarrow$ use dimension-reduced flow model.

Governing equations

Darcy law in $\Omega \setminus \overline{\Sigma}$

$$\nabla \cdot \mathbf{q}^{\Omega} = f^{\Omega}$$
$$-\mathbf{K}\nabla p^{\Omega} = \mathbf{q}^{\Omega}$$

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Averaged Darcy law on Σ

[Martin, Jaffré, Roberts '05]

$$\nabla_\tau \cdot \mathbf{q}^\Sigma - [\![\mathbf{q}^\Omega]\!] \cdot \nu = b f^\Sigma$$
$$-b\mathbf{K}^\tau \cdot \nabla_\tau p^\Sigma = \mathbf{q}^\Sigma$$

Coupling constraints on Σ, $\xi \in (1/2, 1]$

$$\{\mathbf{q}^\Omega\} \cdot \nu = \frac{-K^\nu}{b} [\![p^\Omega]\!]$$
$$[\![\mathbf{q}^\Omega]\!] \cdot \nu = \frac{4}{2\xi - 1} \frac{K^\nu}{b} \left(p^\Sigma - \{p^\Omega\}\right)$$

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$$-b\mathbf{K}^{\tau} \cdot \nabla_{\tau} p^{\Sigma} = \mathbf{q}^{\Sigma}$$

Poroelasticity equations in $\Omega \setminus \overline{\Sigma}$

$$\mathbf{0} = \nabla \cdot \left(\sigma(\mathbf{u}^{\Omega}) - p^{\Omega}\mathbb{I}\right)$$
$$\varepsilon(\mathbf{u}^{\Omega}) = \frac{1}{2}\left(D\mathbf{u}^{\Omega} + (D\mathbf{u}^{\Omega})^{T}\right)$$
$$\sigma(\mathbf{u}^{\Omega}) = \mathbb{E} : \varepsilon(\mathbf{u}^{\Omega})$$

Coupling constraints on Σ, $\xi \in (1/2, 1]$

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$$[\![\mathbf{q}^{\Omega}]\!] \cdot \nu = \frac{4}{2\xi - 1} \frac{K^{\nu}}{b} \left(p^{\Sigma} - \{p^{\Omega}\}\right)$$
$$-p^{\Sigma}\nu^{\pm} = \sigma\left(\mathbf{u}^{\Omega}\right) \cdot \nu^{\pm}$$
$$b = [\![\mathbf{u}^{\Omega}]\!] \cdot \nu$$

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For fixed fluid pressures p^Ω, p^Σ:

- Symmetric bilinear form:

$$a_E^\Omega(\mathbf{u}^\Omega, \mathbf{v}^\Omega) = \int_\Omega \sigma(\mathbf{u}^\Omega) : \varepsilon(\mathbf{v}^\Omega).$$

- Coupling terms:

$$c_E^\Omega\left(p^\Omega; \mathbf{v}^\Omega\right) = -\int_\Omega \nabla p^\Omega \cdot \mathbf{v}^\Omega \cdot \nu \,\mathrm{d}\Gamma$$

$$c_E^\Sigma\left(p^\Sigma; \mathbf{v}^\Omega\right) = \int_\Sigma 2 p^\Sigma \mathbf{v}^\Omega \cdot \nu \,\mathrm{d}\Gamma$$

- Define

$$V := \left\{ v \in L^2(\widetilde{\Omega}) \,\middle|\, v|_{\Omega^\pm} \in H^1(\Omega), [\![v]\!]_{\Sigma_e} = 0 \right\}.$$

  and $\mathbf{V}_E \subset V^d$ as the space of functions satisfying the Dirichlet boundary conditions
- Weak formulation: Find $\mathbf{u}^\Omega \in \mathbf{V}_E$ such that

$$a_E^\Omega\left(\mathbf{u}^\Omega, \mathbf{v}^\Omega\right) + c_E^\Omega\left(p^\Omega; \mathbf{v}^\Omega\right) + c_E^\Sigma\left(p^\Sigma; \mathbf{v}^\Omega\right) = \mathbf{0} \qquad \forall \mathbf{v}^\Omega \in \mathbf{V}_0$$

- Linear elasticity problem with pre-stress

Poroelastic problem - Well posedness

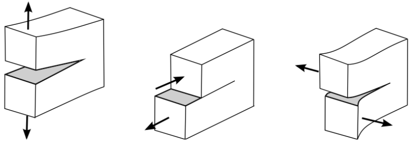
For fixed fluid pressures p^Ω, p^Σ:

Theorem ([Folklore])

Assume $\nabla p^\Omega \in L^2(\Omega)$, $p^\Sigma \in H^{\frac{1}{2}}(\Sigma)$ *and* $\boldsymbol{u}_D^\Omega \in \boldsymbol{H}^{\frac{1}{2}}(\Gamma_D^E)$. *Then there exists a unique solution* $\boldsymbol{u}^\Omega \in \boldsymbol{V}_E$ *to the linear elasticity problem with pre-stress.*
Furthermore the solution can be decomposed additively into a continuous function $\boldsymbol{u}_c$ *and a singular part* $\boldsymbol{u}_s$. *Near the crack tip the singular part can be approximated by* $\boldsymbol{u}_s = K_I \boldsymbol{u}_I + K_{II} \boldsymbol{u}_{II} + K_{III} \boldsymbol{u}_{III}$, *where*

$$u_{I,i}, u_{II,i}, u_{III,i} \in span\left\{ \sqrt{r}\sin\frac{\Theta}{2}, \sqrt{r}\cos\frac{\Theta}{2}, \sqrt{r}\sin\Theta\sin\frac{\Theta}{2}, \sqrt{r}\sin\Theta\cos\frac{\Theta}{2} \right\},$$

for $i = 1, \dots, d$ *where* (r, Θ) *are the crack tip polar coordinates.*



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For fixed fracture width b:

- Symmetric, bilinear forms:

$$a_F^\Omega\left(b;p^\Omega,r^\Omega\right)=\int_\Omega \mathbf{K}\nabla p^\Omega\cdot\nabla r^\Omega\,\mathrm{d}\Omega+\int_\Sigma \frac{K^\nu}{b}\left[\!\left[p^\Omega\right]\!\right]\left[\!\left[r^\Omega\right]\!\right]\mathrm{d}\Gamma + \frac{4}{2\xi-1}\int_\Sigma \frac{K^\nu}{b}\left\{p^\Omega\right\}\left\{r^\Omega\right\}\mathrm{d}\Gamma,$$

$$a_F^\Sigma\left(b;p^\Sigma,r^\Sigma\right)=\int_\Sigma b\mathbf{K}^\tau\nabla_\tau p^\Sigma\cdot\nabla_\tau r^\Sigma+\frac{4}{2\xi-1}\frac{K^\nu}{b}p^\Sigma r^\Sigma\,\mathrm{d}\Gamma,$$

- Linear forms:

$$l_F^\Omega\left(r^\Omega\right)=\int_\Omega f_F^\Omega r^\Omega\,\mathrm{d}\Omega \quad \text{and} \quad l_F^\Sigma\left(b;r^\Sigma\right)=\int_\Sigma b f_F^\Sigma r^\Sigma\,\mathrm{d}\Gamma.$$

- Coupling terms:

$$c_F\left(b;r^\Omega,p^\Sigma\right)=-\frac{4}{2\xi-1}\int_\Sigma\frac{K^\nu}{b}p^\Sigma\left\{r^\Omega\right\}\mathrm{d}\Gamma,$$

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Coupled weak problem - For fixed fracture width *b*:

Find $\left(p^{\Omega}, p^{\Sigma}\right)$, such that

$$a_F^{\Omega}\left(b; p^{\Omega}, r^{\Omega}\right) + c_F\left(b; r^{\Omega}, p^{\Sigma}\right) = l_F^{\Omega}\left(r^{\Omega}\right) \quad \forall r^{\Omega} \in H_0^1(\Omega)$$
$$a_F^{\Sigma}\left(b; p^{\Sigma}, r^{\Sigma}\right) + c_F\left(b; p^{\Omega}, r^{\Sigma}\right) = l_F^{\Sigma}\left(r^{\Sigma}\right) \quad \forall r^{\Sigma} \in H_0^1(\Sigma)$$

- Linear (if *b* is fixed!)

Existence and uniqueness?

- Simple but restrictive result: Existence and uniqueness for $0 < c \leq b(x) \leq C$ a.e. on Σ
- Crack front in domain $\Longrightarrow b(x) \to 0$ for x approaching the crack front

Existence and uniqueness with crack tips:

- Crack tip asymptotics yield: $b(x) = \mathrm{dist}(x, \gamma)^{\frac{1}{2}}$ near the crack tip and bounded away from zero otherwise
- Denote by $L^2_{b^{-1}}(\Sigma)$ and $L^2_b(\Sigma)$ the sets of all measurable functions $v : \Omega \to \mathbb{R}$ for which the norms

$$\|v\|^2_{0,b^{-1},\Sigma} := \int_\Sigma |v|^2 \, b^{-1} \, \mathrm{d}\, x < \infty,$$
$$\|v\|^2_{0,b,\Sigma} := \int_\Sigma |v|^2 \, b \, \mathrm{d}\, x < \infty,$$

  respectively. The spaces $\left(L^2_{b^{\pm 1}}(\Sigma), \|\cdot\|_{0,\pm,\Sigma}\right)$ are Hilbert spaces
- We have

$$L^2_{b^{-1}}(\Sigma) \hookrightarrow L^2(\Sigma) \hookrightarrow L^2_b(\Sigma)$$

Fluid–fluid problem: Function Spaces

Crack-weighted Sobolev spaces:

- The spaces

$$H_b^1(\Sigma) := \left\{ s \in L^2_{b^{-1}}(\Sigma) \mid \nabla_\tau s \in \mathbf{L}^2_b(\Sigma) \right\},$$

$$H^{\frac{1}{2}}_{b^{-1}}(\Sigma) := \left\{ s \in H^{\frac{1}{2}}(\Sigma) \mid s \in L^2_{b^{-1}}(\Sigma) \right\},$$

$$H_b^1(\Omega) := \left\{ v \in V \mid \gamma^\pm v \in H^{\frac{1}{2}}_{b^{-1}}(\Sigma) \right\},$$

  together with the norms

$$\|s\|^2_{1,b,\Sigma} := \|s\|^2_{0,b^{-1},\Sigma} + \|\nabla_\tau s\|^2_{0,b,\Sigma},$$

$$\|s\|^2_{\frac{1}{2},b^{-1},\Sigma} := \|s\|^2_{0,b^{-1},\Sigma} + \int_{\Sigma\times\Sigma} \frac{|s(x)-s(y)|^2}{|x-y|^d}\,\mathrm{d}x\,\mathrm{d}y,$$

$$\|v\|^2_{1,b^{-1},\Omega} := \|\gamma^+ v\|^2_{0,b^{-1},\Sigma} + \|\gamma^- v\|^2_{0,b^{-1},\Sigma} + \|u\|^2_1$$

  are Hilbert spaces.
- Define $V_F := V_F^\Sigma \times V_F^\Omega$ (where $V_F^\Sigma \subset H_b^1(\Sigma)$ and $V_F^\Omega \subset H_b^1(\Omega)$) as the spaces of functions satisfying the Dirichlet boundary data.

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For fixed crack width b:

Theorem ([Hanowski '16])

Assume that Σ is a bounded parametrized surface with smooth boundary and that the permeability tensor $\boldsymbol{K}$ is symmetric, bounded and uniform elliptic. Let $b > 0$ on Σ a.e. and $b \simeq dist(\cdot, \gamma)^{\frac{1}{2}}$ near the crack tip and $\xi \in \left(\frac{1}{2}, 1\right]$. Let $\Gamma_D^F \neq \emptyset$, $f_F^\Omega \in L^2(\Omega)$, $f_F^\Sigma \in L_b^2(\Sigma)$, $q_N^\Omega \in L^2(\Gamma_N)$ and $q_N^\Sigma \in L_b^2(\Sigma)$. Furthermore, assume that

$$p_D^\Omega \in W_D^\Omega := \{s \in H^{\frac{1}{2}}(\Gamma_D^F) \mid E_\Omega s \in V_b\}$$

and

$$p_D^\Sigma \in W_D^\Sigma := \{s \in H^{\frac{1}{2}}(\gamma_D^F) \mid E_\Sigma s \in H_b^1(\Sigma)\},$$

where $E_\Omega : H^{\frac{1}{2}}(\Gamma_D^F) \to V$ and $E_\Sigma : H^{\frac{1}{2}}(\gamma_D^F) \to H^1(\Sigma)$ are the standard extension operators. Then there exists a unique solution $(p^\Omega, p^\Sigma) \in V_F^\Omega \times V_F^\Sigma$ of the weak coupled fluid–fluid problem.

Fully Coupled Problem

Fixed-point formulation:

- Solution operator for the fluid problem :

$$S_f : H_{00}^{\frac{1}{2}}(\Sigma) \to V_F^\Omega \times V_F^\Sigma, \quad b \mapsto (p^\Omega, p^\Sigma)$$

- Solution operator for the elasticity problem:

$$S_e : V_F^\Omega \times V_F^\Sigma \to \mathbf{V}_E, \quad (p^\Omega, p^\Sigma) \mapsto \mathbf{u}^\Omega$$

- Normal jump operator

$$j : \mathbf{V}_E \to H_{00}^{1/2}(\Sigma), \quad \mathbf{u}^\Omega \mapsto b := [\![\mathbf{u}]\!] \cdot \nu$$

- Fixed Point Formulation:

$$b = (j \circ S_e \circ S_f)b$$

Existence and Uniqueness:

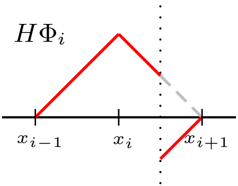
- Dependency of the solution spaces for the pressures on the fracture width b! Iterative approach leads to 'independent' solution spaces in each iteration step.

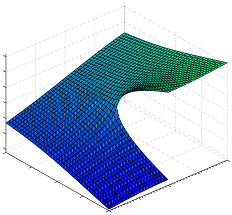
Challenge:

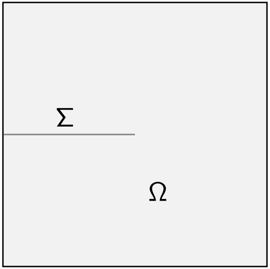
- Pressure p^Ω and displacement $\mathbf{u}^\Omega$ discontinuous on Σ
- Derivatives of p^Ω and $\mathbf{u}^\Omega$ singular at crack front

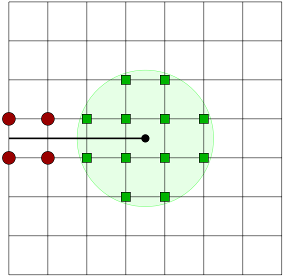
XFEM basic idea:

- Additional bespoke FE functions near the crack
- Reproduce discontinuity: Heavyside function
- Reproduce singularity:
  Special singularity functions derived by asymptotic analysis
- Aim: Retain optimal discretization errors









Given:

- Grids for $\overline{\Omega}$ and $\overline{\Sigma}$ and/or implicit crack representation
- Element subsets:
  - $\mathscr{K}_1$: elements fully cut by the fracture
  - $\mathscr{K}_2$: elements in the vicinity of the crack tip

Crack Front Enrichment Functions

Crack front enrichment functions:

- (r, Θ) front polar coordinates
- Standard enrichment functions for the displacement

$$(F_E)_1(r,\Theta) = \sqrt{r}\sin\left(\frac{\Theta}{2}\right)$$

$$(F_E)_2(r,\Theta) = \sqrt{r}\cos\left(\frac{\Theta}{2}\right)$$

$$(F_E)_3(r,\Theta) = \sqrt{r}\sin\left(\frac{\Theta}{2}\right)\sin(\Theta)$$

$$(F_E)_4(r,\Theta) = \sqrt{r}\cos\left(\frac{\Theta}{2}\right)\sin(\Theta)$$

- Laplace enrichment for the matrix pressure:

$$(F_F)_1(r,\Theta) = \sqrt{r}\sin\left(\frac{\Theta}{2}\right)$$

$$(F_F)_2(r,\Theta) = \sqrt{r}\cos\left(\frac{\Theta}{2}\right)$$

Decoupling the system: Substructuring method

Algebraic problem:

- Linear coupled fluid–fluid problem
- Linear elasticity problem with pre-stress
- Nonlinear coupling through changing crack width b

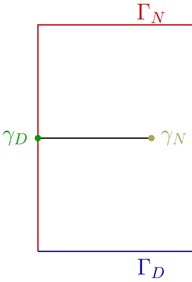
Decouple by fixed-point iteration

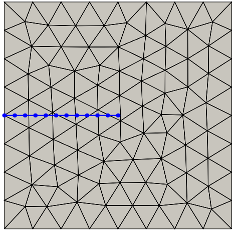
- Iteration variable: fracture width function b_k

Init: b_0 ;
while $(err > acc)$ **do**
  $\left(p^{\Omega}_{k+1}, p^{\Sigma}_{k+1}\right) \leftarrow$ Solve fluid problem with $b_{\mathbf{u}} = b_k$;
  $\mathbf{u}^{\Omega}_{k+1} \leftarrow$ Solve the elasticity problem with $p^{\Omega} = p^{\Omega}_{k+1}$ and $p^{\Sigma} = p^{\Sigma}_{k+1}$;
  $b_{k+1} \leftarrow b_k + \beta \left([\![\mathbf{u}^{\Omega}_{k+1}]\!] \cdot \nu - b_k \right)$;
end

- Damping parameter $\beta \in (0, 1]$
- Solve two linear, symmetric systems in each iteration step
- Direct subdomain solvers

2D Example - Grid and Boundary conditions





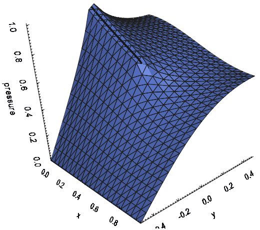
Grid

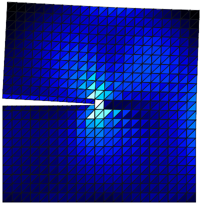
- Unstructured triangle grid for the bulk, uniform grid for the fracture

Boundary Conditions

- Solid: Zero Dirichlet and Neumann boundary conditions; Fluid: $p_0^\Sigma = 0.5$ MP, zero Dirichlet and Neumann boundary conditions otherwise

2D Example - Setting





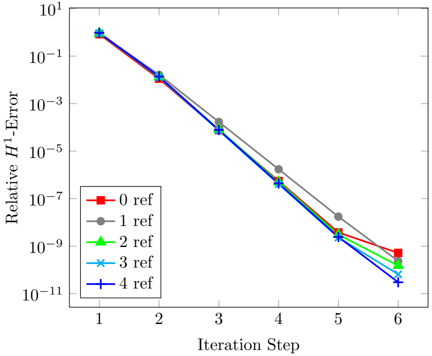
Material Properties:

- Bulk Domain: $\Omega = [0,1] \times \left[-\frac{1}{2}, \frac{1}{2}\right]$ km^2
- Fracture: $\Sigma = \left[0, \frac{1}{2}\right] \times \{0\}$ km
- Fluid: Homogeneous and isotropic permeability tensors, with $K = 0.1$ mD for the bulk and $K^\nu = K^\tau = 100$D
- Solid: St.Venant–Kirchhoff material law with Young's modulus $E = 1$ GP and Poisson ratio $\nu = 0.3$

Parameters

- Damping parameter: $\beta = 1$
- Discretization parameter $\xi = 0.75$

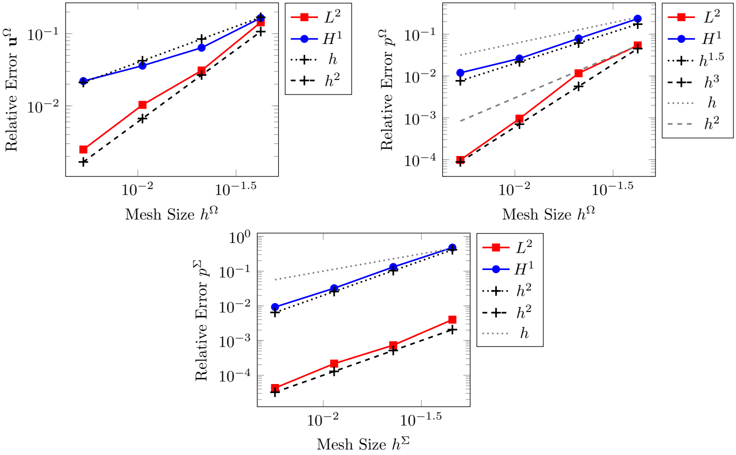
Discretization Error Rates



Measuring the Solver Convergence:

- Constant rates, fast convergence, nearly independent of the mesh size!

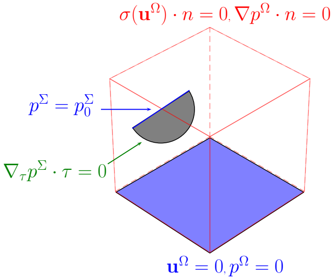
Discretization Error Rates

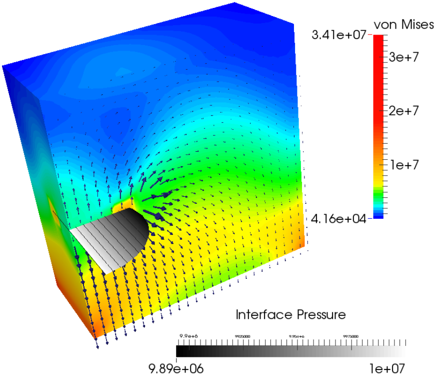


Measuring the Discretization Error

- Compare with solution on fine grid
- Optimal error rates!

3D Example





3d matrix + 2d fracture:

- Unit cube
- Half-penny crack