Nonsmooth multigrid methods for small-strain plasticity

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Small-strain elastoplasticity

- Classic problem in numerical analysis
- Dual formulation: nonsmooth saddle point problem
- Primal formulation: minimization problem

Solvers

- Basically a solved problem
- Predictor–corrector methods

Nonsmooth multigrid

- MUCH faster than previous methods
- Globally convergent
- a.k.a. inexact predictor–corrector methods

Total strain:

$$\boldsymbol{\varepsilon}(\mathbf{u}) := \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right)$$

Additive split:

$$\boldsymbol{\varepsilon} = \mathbf{e} + \mathbf{p},$$

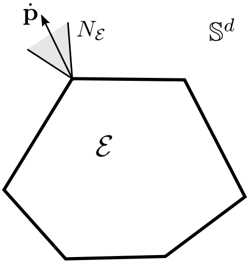
- Elastic strain $\mathbf{e} \in \mathbb{S}^d$
- Plastic strain $\mathbf{p} \in \mathbb{S}^d_0$ is symmetric and trace-free.

Only elastic strain leads to stress:

$$\boldsymbol{\sigma} = \mathbf{C} : \mathbf{e}$$

Equilibrium conditions:

$$-\operatorname{div}\boldsymbol{\sigma} = \mathbf{f}$$



Evolution of $\mathbf{p}$: Nonsmooth ODE

- Postulate existence of elastic set in space of stresses
- $\dot{\mathbf{p}} = 0$ if $\boldsymbol{\sigma} \in \operatorname{int} \mathcal{E}$
- $\dot{\mathbf{p}} \in N_{\mathcal{E}}(\boldsymbol{\sigma})$, if $\boldsymbol{\sigma} \in \partial \mathcal{E}$

Dual formulation

- Unknowns: Displacement $\mathbf{u}$ and stresses $\boldsymbol{\sigma}$

Dissipation function

- Support function of the elastic set

$$D(\mathbf{p}) := \sup_{\boldsymbol{\sigma}\in\mathcal{E}}\{\mathbf{p} : \boldsymbol{\sigma}\} \in \mathbb{R}\cup\{\infty\}.$$

- Convex, positively homogeneous, lower semicontinuous, proper
- Dual flow rule

$$\boldsymbol{\sigma} \in \partial D(\dot{\mathbf{p}})$$

Variational inequality of 2nd kind:

Find $\mathbf{w} = (\mathbf{u}, \mathbf{p}) : [0, T] \to W := H^1(\Omega, \mathbb{R}^d) \times L_2(\Omega, \mathbb{S}^d_0)$ so that

$$a(\mathbf{w}(t), \tilde{\mathbf{w}} - \dot{\mathbf{w}}(t)) + j(\tilde{\mathbf{w}}) - j(\dot{\mathbf{w}}(t)) \geq \langle l(t), \tilde{\mathbf{w}} - \dot{\mathbf{w}}(t)\rangle \quad \forall \tilde{\mathbf{w}} \in W, t \in (0, T)$$

with

$$a(\mathbf{w}, \tilde{\mathbf{w}}) := \int_\Omega [\mathbf{C}(\boldsymbol{\varepsilon}(\mathbf{u}) - \mathbf{p}) : (\boldsymbol{\varepsilon}(\tilde{\mathbf{u}}) - \tilde{\mathbf{p}}) + k_1\mathbf{p} : \tilde{\mathbf{p}}]\, dx,$$

$$j(\tilde{\mathbf{w}}) := \int_\Omega D(\tilde{\mathbf{p}}, \tilde{\eta})\, dx, \qquad \langle l(t), \tilde{\mathbf{w}}\rangle = \int_\Omega \mathbf{f}(t) \cdot \tilde{\mathbf{u}}\, dx.$$

Von Mises plasticity

$$\mathbf{p} \in \mathbb{S}_0^d, \qquad D(\mathbf{p}) = \sigma_c \|\mathbf{p}\|.$$

Tresca plasticity

$$D(\mathbf{p}) = \sigma_c \max\{|p_1|, |p_2|, |p_3|\}, \qquad p_1,\ p_2,\ p_3 \text{ the eigenvalues of } \mathbf{p}$$

Isotropic hardening

- Additional scalar variable η
- Von Mises plasticity with isotropic hardening

$$(\mathbf{p}, \eta) \in \mathbb{S}_0^d \times \mathbb{R}_+, \qquad D(\mathbf{p}, \eta) = \begin{cases} c_0 \|\mathbf{p}\| & \text{if } \|\mathbf{p}\| \leq \eta, \\ +\infty & \text{else.} \end{cases}$$

- Mutatis mutandis for Tresca plasticity with isotropic hardening

Discretization in time

- Implicit Euler method
- Increments $\Delta \mathbf{w}_n := \mathbf{w}_n - \mathbf{w}_{n-1}$ minimize the functional

$$\mathcal{L}(\mathbf{w}) = \frac{1}{2} a(\mathbf{w}, \mathbf{w}) + j(\mathbf{p}) - \langle l_n, \mathbf{w} \rangle, \qquad \mathbf{w} = (\mathbf{u}, \mathbf{p}).$$

- Functional is strictly convex and coercive.

Discretization in space

- Displacements: first-order Lagrangian finite elements with values in $\mathbb{R}^d$
  - Scalar basis functions ϕ_i, $i = 1, \dots, n_1$
- Plastic strain: zero-order elements with values in $\mathbb{S}_0^d$
  - Scalar basis functions θ_i, $i = 1, \dots, n_2$

Algebraic increment functional:

$$L \; : (\mathbb{R}^d)^{n_1} \times \left(\mathbb{S}_0^d\right)^{n_2} \to \mathbb{R}$$

$$L(w) := \frac{1}{2} w^T A w - b^T w + \sum_{i=1}^{n_2} \int_\Omega \theta_i(x)\, dx \cdot D\left(\sum_{j,k=1}^{d} p_{i,jk} \right).$$

Stiffness matrix:

$$A = \begin{pmatrix} E & C \\ C^T & P \end{pmatrix}.$$

- Elasticity matrix

$$(E_{ij})_{kl} = \int_\Omega \mathbf{C}(\boldsymbol{\varepsilon}(\phi_i \mathbf{e}_k) : \boldsymbol{\varepsilon}(\phi_j \mathbf{e}_l)\, dx, \qquad 1 \leq i,j \leq n_1, \quad 1 \leq k,l \leq d$$

- “Plastic strain matrix”

$$P_{ii} = \int_\Omega \left(\mathbf{C} + k_1 \mathbf{I}\right) \theta_i(x)^2\, dx, \qquad 1 \leq i \leq n_2$$

- plus coupling terms C

Block structure

- n_1 blocks of size d, followed by n_2 blocks of size $d \times d$
- Canonical restriction operators R_i

Nonsmooth strictly convex minimization problem

Separable energy

$$L(w) = L_0(w) + \sum_{i \in \text{blocks}} \varphi_i(R_i w)$$

- $L_0(w) = \frac{1}{2} w^T A w - b^T w$ is smooth, strictly convex, coercive
- Convex, proper, lower semi-continuous functionals $\varphi_i : \mathbb{R}^{n_i} \to \mathbb{R} \cup \{\infty\}$.

$$\varphi_i(w) := \int_\Omega \theta_i(x)\, dx \cdot D\left(\sum_{j,k=1}^{d} p_{i,jk} \right).$$

The algorithm

1. Nonlinear presmoothing (Gauß–Seidel)
   - For each block i, inexactly solve a local minimization problem for
   $$L_i(w) := L_0(R_i^T w) + \varphi_i(w)$$
   in the variables of the i-th block.
2. Truncated linearization
   - Freeze all variables where L is not differentiable.
   - Linearize everywhere else
3. Linear correction
   - One linear geometric multigrid step
4. Projection onto admissible set
   - Simply in the ℓ^2-sense
5. Line search
   - 1d nonsmooth convex minimization problem: use bisection

Theorem (Gräser, S. 2016)

Let $w^0 \in \operatorname{dom} L$ and assume that the inexact solution local solution operator $\mathcal{M}_i$ satisfies the following assumptions:

- *Monotonicity:* $L(\mathcal{M}_i(w)) \leq L(w)$ *for all* $w \in \operatorname{dom} L$.
- *Continuity:* $\mathcal{M}_i$ *is continuous.*
- *Stability:* $L(\mathcal{M}_i(w)) < L(w)$ *if* $L(w)$ *is not minimal in the* i*-th block.*

Then the iterates produced by the TNNMG method converge to the unique minimizer of L.

Local minimization problems for von Mises plasticity:

$$L_i : \mathbb{S}_0^d \to \mathbb{R}, \qquad L_i(p) := \frac{1}{2} p^T P_{ii} p - p^T r_i + \lambda_i \sigma_c \|p\|$$

with

$$P_{ii} = \int_\Omega (\mathbf{C} + k_1 \mathbf{I}) \theta_i(x)^2 \, dx$$

Lemma (Folklore; Alberty, Carstensen, Zarrabi, 1999)

The exact minimizer is

$$p^* = \frac{\max\left\{(\|\operatorname{dev} r_i\| - \lambda_i \sigma_c), 0\right\}}{\lambda_i (2\mu + k_1)} \frac{\operatorname{dev} r_i}{\|\operatorname{dev} r_i\|}.$$

Alternative:

- Steepest descent methods

Linear correction problems

- Newton linearization

$$H_k = \begin{pmatrix} E & C \\ C^T & P + \nabla^2 \varphi(w_k) \end{pmatrix}.$$

- Freeze degrees of freedom where dissipation function is not differentiable
- Do one geometric multigrid iteration

Semi-definite Hesse matrices

- Freezing of variables leads to semi-definite Hesse matrices.
- Modify linear geometric multigrid to handle semi-definiteness

Alternative

- $\widetilde{P} := P + \nabla^2 \varphi(w_k)$ is block-diagonal; computing pseudo-inverse $\widetilde{P}^+$ is cheap.
- Pseudo-Schur-complement $S := E - C\widetilde{P}^+ C^T$ is positive definite.
- Do linear multigrid step for the Schur complement system.

Predictor–corrector with consistent tangent

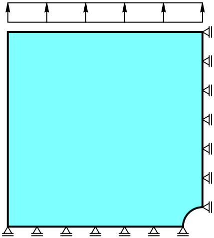
1. Predictor
   - Freeze all variables where L is not differentiable.
   - Linearize everywhere else
   - Solve exactly!
2. Corrector
   - For each plastic strain block i, solve a local minimization problem for

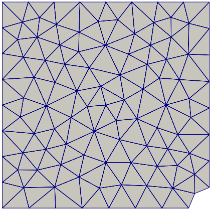
     $$L_k(w) := L_0(R_i^T w) + \varphi_i(w)$$

     in the variables of the i-th block.
3. Line search
   - 1d nonsmooth convex minimization problem

Benchmark: Square with a hole – 2d

Geometry and boundary conditions

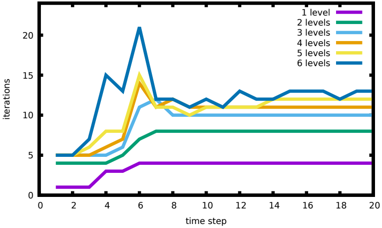




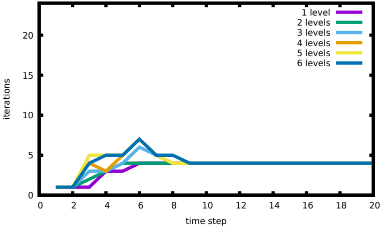
levels	cells	vertices
1	176	105
2	704	385
3	2 816	1 473
4	11 264	5 761
5	45 056	22 785
6	180 224	90 625

Evolution of the plastic zone

Iteration numbers



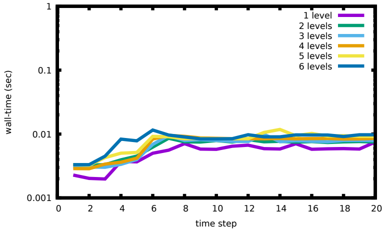
TNNMG



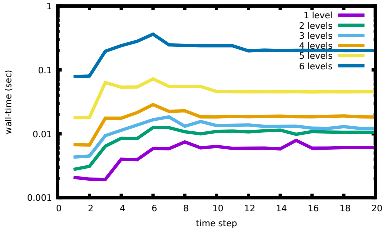
predictor–corrector

- One nonlinear smoothing step per iteration
- Using exact local solver
- TNNMG: one $V(3,3)$-cycle for the linear correction
- Predictor–corrector: Solve linear correction with UMFPack

Wall-time normalized with problem size



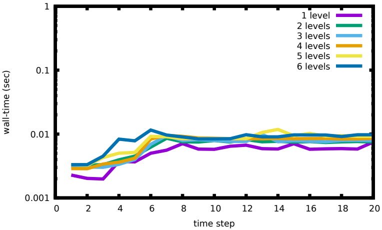
TNNMG



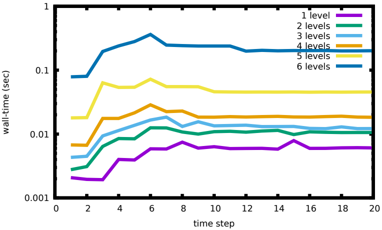
predictor–corrector

- This is a logarithmic plot!

Wall-time normalized with problem size



TNNMG

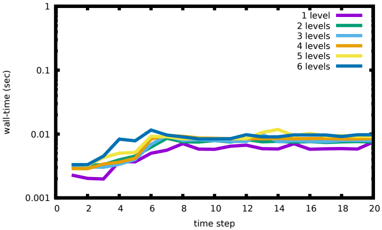


predictor–corrector

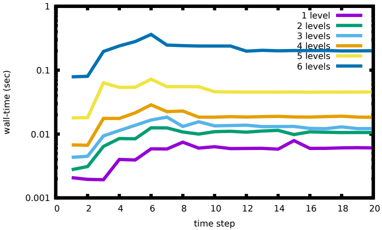
- This is a logarithmic plot!
- On finest grid, TNNMG is about 40 times faster than predictor–corrector!

Time to solution

Wall-time normalized with problem size



TNNMG



predictor–corrector

- This is a logarithmic plot!
- On finest grid, TNNMG is about 40 times faster than predictor–corrector!
- Unfair comparison: UMFPack uses my second CPU core

Conclusion

- Direct multigrid method for small-strain von Mises plasticity
- Much faster than state-of-the-art algorithms
- Nevertheless provably convergent
- Interpretation as inexact predictor–corrector methods

Outlook

Works also for

- Tresca flow rules and others
- Isotropic hardening and others
- Certain gradient plasticity models
- Certain crystal plasticity models