Benchmarking: Deformation and Flow in Fractured Poroelastic Media

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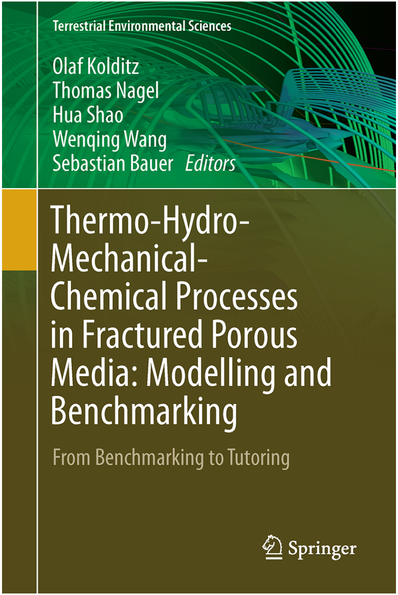
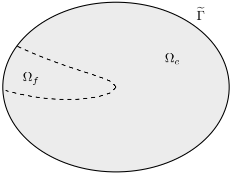
Benchmarking



Fractures in porous media



Domain contained in the medium with different material properties

- Depending on the application:
  - Filling: Only fluid or debris
  - Rough surface or smooth surface
  - ...

  $\Rightarrow$ Different approaches for the fracture modelling (Stokes, Darcy, ...)

- Small-strain Biot Equation:

$$\nabla \cdot (\sigma(\mathbf{u}) - \alpha p \mathbb{I}) = 0$$

with

$$\sigma(\mathbf{u}) = \mathbb{E} : \mathbf{e}(\mathbf{u})$$

where

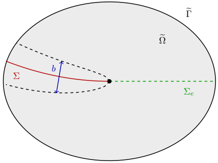
$$\mathbf{e}(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$$

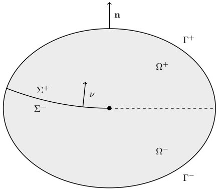
- Fluid equation (single phase):

$$S\frac{\partial p}{\partial t} - \nabla \cdot \mathbf{q} = -\alpha \frac{\partial \mathrm{tr}(\mathbf{e}(\mathbf{u}))}{\partial t}$$

with

$$\mathbf{q} = -\mathbf{K}\nabla p$$





Complexity reduction:

Two general approaches derived from equi-dimensional formulation:

- Homogenization of large fracture networks
- Dimension reduction:
  Via lubrication theory or averaging

Deformation and Flow: Darcy Averaging

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Bulk equations:

- Poroelasticity equations in $\Omega \setminus \overline{\Sigma}$,
- Darcy law in $\Omega \setminus \overline{\Sigma}$, $\alpha \in [0,1]$

$$S_\Omega \partial_t p^\Omega + \alpha \nabla \cdot (\partial_t \mathbf{u}^\Omega) + \nabla \cdot \mathbf{q}^\Omega = f^\Omega$$
$$-\mathbf{K}\left(\nabla p^\Omega - \rho^l \mathbf{g}\right) = \mathbf{q}^\Omega$$

Fracture equation:

- Averaged Darcy law on Σ
  [Martin, Jaffré, Roberts '05], with curvature terms

$$S_\Sigma b \partial_t p^\Sigma + S_\Sigma (p^\Sigma - \{p^\Omega\})\partial_t b + \alpha \partial_t b + \nabla_\tau \cdot \mathbf{q}^\Sigma + [\![\mathbf{q}^\Omega]\!] \cdot \nu = b f^\Sigma - b\kappa \{\mathbf{q}^\Omega\} \cdot \nu$$
$$-b\mathbf{K}^\tau \left(\nabla_\tau p^\Sigma - \rho^l \mathbf{g}^\tau\right) = \mathbf{q}^\Sigma$$

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- Coupling constraints on Σ, $\xi \in (1/2, 1]$

$$\{\mathbf{q}^{\Omega}\} \cdot \nu = \frac{-K^{\nu}}{b}\left([\![p^{\Omega}]\!] - b\rho^{l}\mathbf{g} \cdot \nu\right)$$

$$[\![\mathbf{q}^{\Omega}]\!] \cdot \nu = \frac{4}{2\xi - 1}\frac{K^{\nu}}{b}\left(p^{\Sigma} - \{p^{\Omega}\}\right)$$

$$-p^{\Sigma}\nu^{\pm} = \sigma\left(\mathbf{u}^{\Omega}\right) \cdot \nu^{\pm}$$

$$b = [\![\mathbf{u}^{\Omega}]\!] \cdot \nu$$

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Bulk equations:

- Poroelasticity equations in $\Omega \setminus \overline{\Sigma}$,
- Darcy law in $\Omega \setminus \overline{\Sigma}$, $\alpha \in [0,1]$

$$S_\Omega \partial_t p^\Omega + \alpha \nabla \cdot (\partial_t \mathbf{u}^\Omega) + \nabla \cdot \mathbf{q}^\Omega = f^\Omega + Q^L$$
$$-\mathbf{K}\left(\nabla p^\Omega - \rho^l \mathbf{g}\right) = \mathbf{q}^\Omega$$

Fracture equation:

- Lubrication Equation on Σ

$$S_\Sigma b \partial_t p^\Sigma + \alpha \partial_t b + \nabla_\tau \cdot \mathbf{q}^\Sigma = f^\Sigma - Q^L$$
$$-\frac{b^3}{12\mu}\left(\nabla_\tau p^\Sigma - \rho^l \mathbf{g}^\tau\right) = \mathbf{q}^\Sigma$$

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- Coupling constraints on Σ

$$
\begin{aligned}
[\![p^\Omega]\!] &= 0 \\
p^\Sigma &= \{p^\Omega\} = p^\Omega \\
[\![\sigma(\mathbf{u}^\Omega)]\!] \cdot \tau &= 0 \\
\{\sigma(\mathbf{u}^\Omega)\} \cdot \nu &= b\mathbf{K} - \beta p^\Omega \mathbb{I} \\
b &= [\![\mathbf{u}^\Omega]\!] \cdot \nu
\end{aligned}
$$

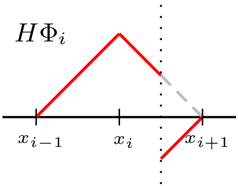
Discretization: XFEM

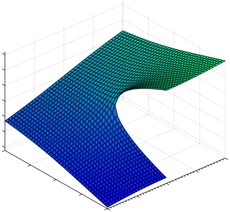
Challenge:

- Displacement $\mathbf{u}^\Omega$ discontinuous on Σ
- Fracture averaging: Pressure p^Ω discontinuous on Σ
- Singularities at the crack tip

XFEM basic idea:

- Additional bespoke FE functions near the crack
- Reproduce discontinuity: Heavyside function
- Reproduce singularity:
  Special singularity functions derived by asymptotic analysis
- Aim: Retain optimal discretization errors





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Finite element functions:

- Linear FE functions $\Phi_i^\Omega : \overline{\Omega} \to \mathbb{R}^d$ $(i = 0, \ldots, n^\Omega)$ in the bulk
- Linear FE functions $\Phi_i^\Sigma : \overline{\Sigma} \to \mathbb{R}$ $(i = 0, \ldots, n^\Sigma)$ on the fracture
- Enrichment functions H for nodes in elements fully cut by the fracture
- Enrichment functions $(F_E)_j$ for tip singularities of elasticity problem
- Enrichment functions $(F_F)_j$ for tip singularities of bulk Darcy problem

Unknowns – Darcy Averaging: p_h^Ω, p^Σ and $\mathbf{u}_h^\Omega = \left(u_{h,k}^\Omega\right)_{k=1}^d$ with

- $p_h^\Omega \in \text{span}\{\Phi_i^\Omega\}_{i=1}^{n^\Omega} \cup \text{span}\{H\Phi_i^\Omega\}_{i\in\mathscr{K}_1} \cup \text{span}\{(F_F)_j\Phi_i^\Omega \mid j = 1,2\}_{i\in\mathscr{K}_2}$
- $p_h^\Sigma \in \text{span}\{\Phi_i^\Sigma\}_{i=1}^{n^\Sigma}$,
- $u_{h,k}^\Omega \in \text{span}\{\Phi_i^\Omega\}_{i=1}^{n^\Omega} \cup \text{span}\{H\Phi_i^\Omega\}_{i\in\mathscr{K}_1} \cup \text{span}\{(F_E)_j\Phi_i^\Omega \mid j = 1,\ldots,4\}_{i\in\mathscr{K}_2}$.

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Finite element functions:

- Linear FE functions $\Phi_i^\Omega : \overline{\Omega} \to \mathbb{R}^d$ $(i = 0, \ldots, n^\Omega)$ in the bulk
- Linear FE functions $\Phi_i^\Sigma : \overline{\Sigma} \to \mathbb{R}$ $(i = 0, \ldots, n^\Sigma)$ on the fracture
- Enrichment functions H for nodes in elements fully cut by the fracture
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- Enrichment functions $(F_F)_j$ for tip singularities of bulk Darcy problem

Unknowns – Lubrication: p_h^Ω and $\mathbf{u}_h^\Omega = \left(u_{h,k}^\Omega\right)_{k=1}^d$ with

- $p_h^\Omega \in \text{span}\{\Phi_i^\Omega\}_{i=1}^{n^\Omega} \cup \text{span}\{(F_F)_j \Phi_i^\Omega \mid j = 1,2\}_{i \in \mathscr{K}_2}$
- $u_{h,k}^\Omega \in \text{span}\{\Phi_i^\Omega\}_{i=1}^{n^\Omega} \cup \text{span}\{H\Phi_i^\Omega\}_{i \in \mathscr{K}_1} \cup \text{span}\{(F_E)_j \Phi_i^\Omega \mid j = 1, \ldots, 4\}_{i \in \mathscr{K}_2}$.

Darcy Averaging:

- (r, Θ) front polar coordinates
- Enrichment functions for the displacement

$$(F_E)_1(r,\Theta) = \sqrt{r}\sin\left(\frac{\Theta}{2}\right)$$

$$(F_E)_2(r,\Theta) = \sqrt{r}\cos\left(\frac{\Theta}{2}\right)$$

$$(F_E)_3(r,\Theta) = \sqrt{r}\sin\left(\frac{\Theta}{2}\right)\sin(\Theta)$$

$$(F_E)_4(r,\Theta) = \sqrt{r}\cos\left(\frac{\Theta}{2}\right)\sin(\Theta)$$

- Laplace enrichment for the matrix pressure:

$$(F_F)_1(r,\Theta) = \sqrt{r}\sin\left(\frac{\Theta}{2}\right)$$

$$(F_F)_2(r,\Theta) = \sqrt{r}\cos\left(\frac{\Theta}{2}\right)$$

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Lubrication

- (r, Θ) front polar coordinates
- Enrichment functions for the displacement (Kovalyshen, Detournay, 2010)

$$(F_E)_1(r,\Theta) = r^{\frac{1}{3}} \sin\left(\frac{\Theta}{2}\right)$$

$$(F_E)_2(r,\Theta) = r^{\frac{1}{3}} \cos\left(\frac{\Theta}{2}\right)$$

$$(F_E)_3(r,\Theta) = r^{\frac{1}{3}} \sin\left(\frac{\Theta}{2}\right) \sin(\Theta)$$

$$(F_E)_4(r,\Theta) = r^{\frac{1}{3}} \cos\left(\frac{\Theta}{2}\right) \sin(\Theta)$$

- Distance enrichment for the fracture

$$(F_F)_1(r,\Theta) = \text{unclear } (r?)$$

- Coefficient vectors p for the pressure, u for the displacement and fracture width

$$b = [\![u_h]\!] \cdot \nu$$

- **Static Problem**

$$\begin{aligned} L(b)p &= F^F \\ Bu + Cp &= F^E \end{aligned}$$

Linear for fixed fracture width b

- **Dynamic Problem**

$$\begin{aligned} M\frac{\partial p}{\partial t} + C\frac{\partial b}{\partial t} + L(b)p &= F^F \\ Bu + Cp &= F^E \end{aligned}$$

Solving the Static Problem

Algebraic problem:

- Linear coupled fluid–fluid problem
- Linear elasticity problem with pre-stress
- Nonlinear coupling through changing crack width b

Decouple by fixed-point iteration

- Iteration variable: fracture width function b_k

Init: b_0 ;
while $(err > acc)$ **do**
  $p_{k+1} \leftarrow$ Solve fluid problem with $b = b_k$;
  $u_{k+1} \leftarrow$ Solve the elasticity problem with $p = p_{k+1}$;
  $b_{k+1} \leftarrow b_k + \beta\left(\llbracket u_{k+1} \rrbracket \cdot \nu - b_k\right)$;
end

- Damping parameter $\beta \in (0, 1]$
- Solve two linear, symmetric systems in each iteration step
- Direct subdomain solvers

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Dynamic Problem

$$M\frac{\partial p}{\partial t} + C\frac{\partial b}{\partial t} + L(b)p = F^F$$
$$Bu + Cp = F^E$$

Use first order finite Difference approximations

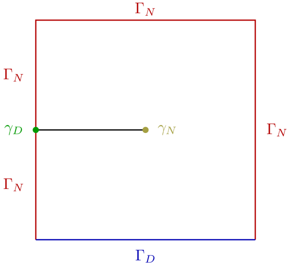
$$M\frac{p^{(m+1)} - p^{(m)}}{\tau} + C\frac{b^{(m+1)} - b^{(m)}}{\tau} + L(b^{(m+1)})p^{(m+1)} = F^F$$
$$Bu^{(m+1)} + Cp^{(m+1)} = F^E$$

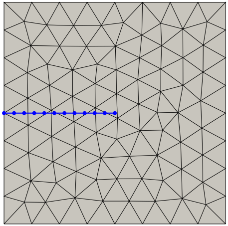
and reorder

$$\big(\frac{1}{\tau}M + L(b^{(m+1)})\big)p^{(m+1)} = F^F - C\frac{b^{(m+1)} - b^{(m)}}{\tau} - \frac{1}{\tau}Mp^{(m)}$$
$$Bu^{(m+1)} + Cp^{(m+1)} = F^E$$

$\Longrightarrow$ Looks like static problem!

2D Darcy Averaging: Single Fracture, with Tip





- Domain $\Omega = [0,1]^2$ (in km), $\Sigma = [0, 0.5] \times \{0.5\}$
- $E = 10\,\mathrm{GPa}$, $\nu = 0.3$, $\alpha = 1$, $\rho^s = 0$,
- $\mu = 1$, $K = 0.1\,\mathrm{mD}$, $K^\nu = 100\,\mathrm{D}$, $K^\tau = 100\,\mathrm{D}$, $\rho^l = 0$,
- $p_0^\Sigma = 10\,\mathrm{MPa}$, $b_0 = 10^{-6}\,\mathrm{m}$
- Zero boundary conditions on Γ_N, Γ_D and γ_N
- No damping: $\beta = 1$, $\xi = 0.75$

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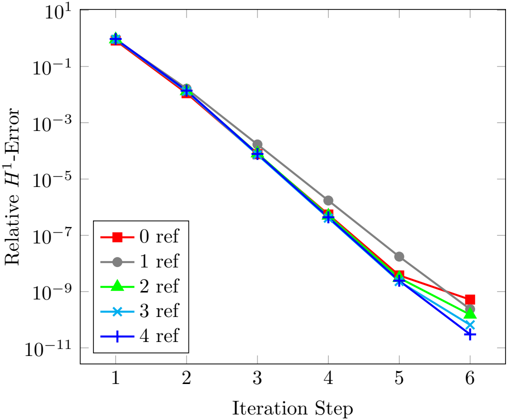
Algebraic error



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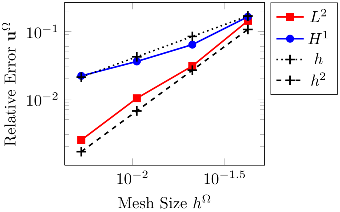
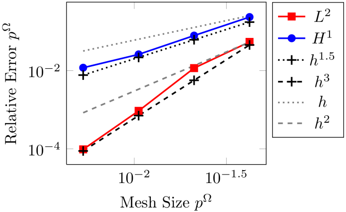
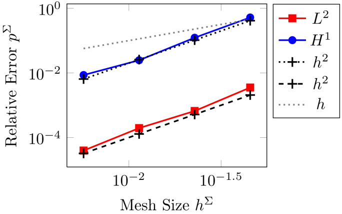
Measure discretization error

- Compare with solution on fine grid
- Optimal error rates!

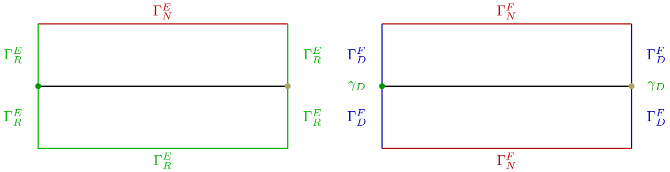






Existing benchmark

- Appeared originally in Wijesinghe, 1986
- Single fracture, no tip
- Fluid-injection from the left
- Semi-analytical similarity solution



Our implementation:

- Still hunting for bugs...

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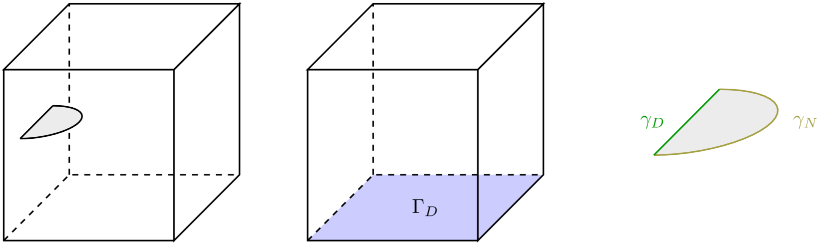
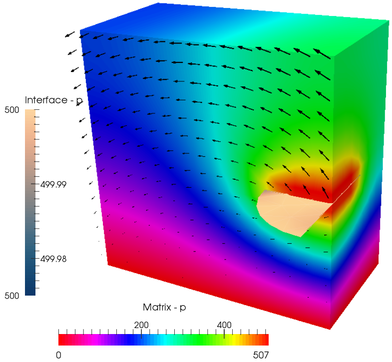
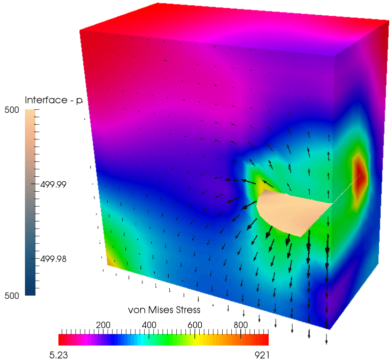
3d matrix + 2d fracture:

- Unit cube, tetrahedal mesh
- Half-penny crack
- Material parameters as previously




Interface - p
500
499.99
499.98
500
Matrix - p
200
400
0
507


Interface - p
500
499.99
499.98
500
von Mises Stress
200
400
600
800
5.23
921

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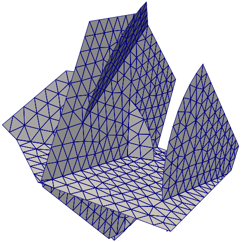
Features:

- Pure DUNE code
- Extension modules for XFEM methods
- 2d and 3d
- Separate grid objects for fracture and bulk
- Coupled by dune-grid-glue
- Fracture grid supports networks
- Python bindings



Open Problems

Hopes and dreams of a numerical analyst

Analysis

- Existence of solutions to the coupled problems
- Rigorous asymptotics at the crack tip

Discretization

- A priori error bounds

Solvers

- Robust & efficient multigrid methods for XFEM spaces
- Show fixed-point solver convergence