Observer-Based Synthesis of Finite Horizon Linear Time-Varying Controllers

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***Abstract*—This paper proposes a computationally efficient and traceable way to synthesize finite horizon linear time-varying (LTV) output feedback controllers. It is based on a separate observer and state feedback synthesis with guaranteed performance in a mixed sensitivity setting. The approach avoids a grid-wise evaluation of coupled synthesis conditions that limits existing output feedback syntheses and instead uses two subsequent steps. However, it guarantees the same performance as the original output feedback problem. A trajectory tracking controller for an ascending space launcher in the earth's atmosphere demonstrates the feasibility of the approach.**

***Index Terms*—Observers for Linear systems, Robust control, Time-varying systems**

I. INTRODUCTION

Tracking a predefined trajectory is a fundamental control problem. Typical examples include space launchers in atmospheric ascent [1], industrial robots [2], aircraft in final approach [3], self-steering cars [4], and systems with moving loads such as telescopes or gearing wheels [5]. For a specific trajectory, the system dynamics are strictly time-varying. Hence, a linearization with respect to the predefined trajectory results in a finite horizon linear time-varying (LTV) system. Ideally, the control system should make use of the known time dependence.

Existing robust LTV output feedback synthesis approaches originate from the time-invariant finite horizon H_∞ problem ([6], [7]). Based on the min-max-principle, [8] proposes the solution of two coupled Riccati Differential Equations (RDEs), but without providing a computational approach. Another approach is provided in [9]. It proposes the solution of two coupled Riccati differential equations (RDEs) which is closely related to the classical linear quadratic Gaussian control. Here, the analysis condition only holds over an infinite horizon, limiting its applicability. In [10], a game-theoretic approach proposes the solution of two decoupled RDEs and the point-wise evaluation of a spectral radius condition. A generalization of this approach is applied to an LTV mixed sensitivity design problem in [11]. Most recently, a controller synthesis for uncertain LTV systems was proposed in [12]. It uses a classical analysis / controller synthesis ("D-K") iteration to extend the results for uncertain linear time-invariant (LTI) [13] and linear parameter-varying (LPV) [14] systems to the finite horizon LTV case. The nominal controller calculation step applies a particular case of the results in [7] and [10]. An alternative approach uses lifting methods for uncertain LTV systems [15].

The present paper proposes a novel and highly structured finite horizon LTV output feedback controller synthesis. It uses two separate synthesis steps for observer and state feedback gain that were recently proposed in [16] for linear parameter-varying control. In the LTV case, these two steps translate to solving a filter RDE and a state feedback RDE. These RDEs are unidirectionally coupled and, thus, can be solved consecutively. Section III establishes that this procedure achieves the same closed-loop performance as the original mixed sensitivity output feedback problem. This guarantee is provided using a specific input weight in the mixed sensitivity formulation based on a normalized coprime factorization [17]. Incorporating this weight in the interconnection resembles the coprime factorization approach to the parameterization of all stabilizing controllers [18]; the synthesis can be thought of as optimizing over this set. The approach also bears resemblance with partial pole placement methods in H_∞ design [19]; the poles of the observer appear in the closed-loop. Thus, the state feedback synthesis explicitly acknowledges the existence of an observer in the loop and the robustness issues associated with conventional observer-based state feedback control [20] are avoided. The consecutive synthesis provides a highly structured controller, e. g., allowing for an easy incorporation of anti-windup compensation [21]. Furthermore, the proposed approach requires only one RDE to be calculated repeatedly and another RDE to be calculated once. Hence, it is computationally more efficient than state of the art synthesis methods requiring the repeated solution of two RDEs coupled by a spectral radius condition. The latter can also only be evaluated point-wise on a grid. Section IV covers the practical implementation and evaluation of the synthesis method using an industry-inspired example of a pitch tracker for a space launcher in atmospheric ascent.

II. BACKGROUND

A finite horizon continuous LTV system $\mathbf{P}$ is defined as

$$\left[\begin{array}{c}\dot{x}(t)\\ \hline y(t)\end{array}\right] = \left[\begin{array}{c|c}A(t) & B(t)\\ \hline C(t) & D(t)\end{array}\right]\left[\begin{array}{c}x(t)\\ \hline u(t)\end{array}\right] \qquad (1)$$

where $x(t) \in \mathbb{R}^{n_x}$ denotes the state vector, $u(t) \in \mathbb{R}^{n_u}$ the input vector, and $y(t) \in \mathbb{R}^{n_y}$ the output vector. Its system matrices are locally bounded continuous functions of time t and compatible size-wise to the corresponding vectors, e.g., $A(t) \in \mathbb{R}^{n_x \times n_x}$. The explicit time dependence will be omitted regularly to shorten the notation. The size of signals in this paper is measured by the $L_2[0,T]$ norm

$$\|u\|_{2[0,T]} = \left[\int_0^T u(t)^T u(t)\, dt \right]^{\frac{1}{2}}. \tag{2}$$

In the course of the paper, the notation $y = \mathbf{P}u$ is used to state the input-output map defined by the state space representation (1) for zero initial conditions.

The performance of such a finite horizon LTV input-output map can be quantified by its finite horizon induced $L_2[0,T]$ norm

$$\|\mathbf{P}\|_{[0,T]} := \sup_{u \in L_2[0,T], u \neq 0, x(0)=0} \frac{\|y\|_{2[0,T]}}{\|u\|_{2[0,T]}}, \tag{3}$$

where $u \in L_2[0,T]$ implies $y \in L_2[0,T]$. An upper bound on $\|\mathbf{P}\|_{[0,T]}$ is provided by a generalization of the Bounded Real Lemma (BRL) as stated in the following theorem.

Theorem 1 ([22]). *Let $\mathbf{P}$ be an LTV system defined by (1). Given $x(0)=0$, if there exists a continuous differentiable, symmetric positive semi-definite matrix function $Q(t), t \in [0,T]$ such that $Q(T) = 0$ and*

$$\begin{aligned} \dot{Q} =& - QA - A^T Q + C^T C \\ &- (QB + C^T D)(D^T D - \gamma^2 I)^{-1}(D^T C + Q^T Q), \end{aligned} \tag{4}$$

then γ is an upper bound on the induced $L_2[0,T]$ gain of $\mathbf{P}$.

Proof. The proof is given in [22]. □

A. Induced $L_2[0,T]$ Finite Horizon Synthesis

Based on Theorem 1, induced $L_2[0,T]$ controller syntheses were proposed in [7], [8], [10]. Consider an open-loop finite horizon LTV system $\mathbf{G}$ with the state space representation

$$\begin{bmatrix} \dot{x} \\ \hline z_1 \\ z_2 \\ e \end{bmatrix} = \left[\begin{array}{c|ccc} A & B_{11} & B_{12} & B_2 \\ \hline C_{11} & D_{1111} & D_{1112} & 0 \\ C_{12} & 0 & 0 & I \\ C_2 & 0 & I & 0 \end{array}\right] \begin{bmatrix} x \\ \hline w_1 \\ w_2 \\ u \end{bmatrix} \tag{5}$$

and an LTV output feedback controller $\mathbf{K}$ with the state space representation:

$$\begin{bmatrix} \dot{\xi} \\ \hline u \end{bmatrix} = \left[\begin{array}{c|c} A_K(t) & B_K(t) \\ \hline C_K(t) & D_K(t) \end{array}\right] \begin{bmatrix} \xi \\ \hline e \end{bmatrix}. \tag{6}$$

The signal $e = r - y$ denotes the measured error fed to the controller, u the control variable, and the input-output map from $\left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right]$ specifies the performance requirements. The special structure of the plant (5) is not restrictive and can be achieved through loop-shifting and scalings, see, e.g., [11] for an LTV generalization of the LTI results in [23] or [24]. The synthesis objective is to provide a controller that minimizes the induced $L_2[0,T]$-norm of the closed-loop interconnection obtained by connecting the open-loop generalized plant (5) with the controller (6). This connection is given by the lower linear fractional transformation $\mathcal{F}(\mathbf{G},\mathbf{K})$ such that the synthesis objective can be formulated as $\min_{\mathbf{K}} \|\mathcal{F}(\mathbf{G},\mathbf{K})\|_{2[0,T]}$. The solution to the induced $L_2[0,T]$-norm controller synthesis problem is stated in the next theorem.

Theorem 2 (Output Feedback Synthesis [10], [11]). *Consider an LTV system (5). There exists an output feedback controller $\mathbf{K}$ as in (6) such that $\|\mathcal{F}(\mathbf{G},\mathbf{K})\|_{2[0,T]} \leq \gamma$, with $\gamma > 0$ iff the following three conditions hold.*

1) *There exists a continuously differentiable, symmetric positive semi-definite matrix function $X(t)$, $t \in [0,T]$ such that $X(T) = 0$ and*

$$\begin{aligned} \dot{X} =& - (\hat{A}^T - C_{11}^T D_{111\bullet} S B_1^T)^T X \\ &- X(\hat{A} - B_1 S D_{111\bullet} C_{11}) \\ &- X \left[\frac{1}{\gamma^2} B_1 S B_1^T + B_2 B_2^T \right] X \\ &- C_{11}^T (I - D_{111\bullet} S D_{111\bullet}^T) C_{11}, \end{aligned} \tag{7}$$

where $\hat{A} := A - B_2 C_{12}$, $D_{111\bullet} := [D_{1111} \;\; D_{1112}]$, and $S := (D_{111\bullet}^T D_{111\bullet} - \gamma^2)$.

2) *There exists a continuously differentiable, symmetric positive semi-definite matrix function Y, $t \in [0,T]$ such that $Y(0) = 0$ and*

$$\begin{aligned} \dot{Y} =& + (\tilde{A}^T - B_{11}^T D_{11\bullet 1}^T \tilde{S} \tilde{C})^T Y \\ &+ Y(\tilde{A} - \tilde{C}^T \tilde{S} D_{11\bullet 1}^T B_{11}^T) \\ &+ Y \left[\frac{1}{\gamma^2} \tilde{C}^T \tilde{S} \tilde{C} + C_2 C_2^T \right] Y \\ &+ B_{11}(I - D_{11\bullet 1}^T \tilde{S} D_{11\bullet 1}) B_{11}^T, \end{aligned} \tag{8}$$

where $\tilde{A} := A - B_{12} C_2$, $D_{11\bullet 1} := \left[\begin{smallmatrix} D_{1111} \\ 0 \end{smallmatrix}\right]$, $\tilde{C} := \left[\begin{smallmatrix} C_{11} \\ C_{12} \end{smallmatrix}\right] - \left[\begin{smallmatrix} D_{1112} \\ 0 \end{smallmatrix}\right] C_2$, and $\tilde{S} := (D_{11\bullet 1} D_{11\bullet 1}^T - \gamma^2 I)$

3) *X and Y satisfy the point-wise in time spectral radius condition*

$$\rho(X(t)Y(t)) < \gamma^2 \qquad \forall t \in [0,T]. \tag{9}$$

Proof: The proof is provided in [10] and [11]. ■

For a given positive γ, the RDEs associated with X and Y are integrated backward and forward in time, respectively. Both RDEs are coupled implicitly by the spectral radius condition (9), which must hold for all times. However, it can only be evaluated after the integration and violation renders the RDE solutions obsolete. Only if all three conditions are satisfied, a controller that achieves a closed-loop performance of γ can be constructed in closed form from the solutions and the plant matrices. The smallest feasible value of γ for which all three conditions hold is calculated via bisection. However, an unknown amount of computationally expensive RDE solutions violating (9) are calculated in the process. These do not provide a feasible controller and, thus, introduce only computational overhead.

When the complete state vector is available for feedback, the synthesis problem simplifies significantly. The following Theorem presents a special case of the synthesis condition stated in [7] and [25].

Theorem 3 (State-Feedback Synthesis). *Consider an LTV system (5) with $e = x$. There exists a controller $\mathbf{F}$ such that $\|\mathcal{F}(\mathbf{G},\mathbf{F})\| \leq \gamma$ iff there exists a (unique) symmetric positive semi-definite matrix function $X(t)$, $t \in [0,T]$ such that X(T) = 0 and*

$$\begin{aligned}\dot{X} = &- (\hat{A}^T - C_{11}^T D_{111\bullet} S B_1^T)^T X \\ &- X(\hat{A} - B_1 S D_{111\bullet} C_{11}) \\ &- X\left[\frac{1}{\gamma^2} B_1^T S B_1^T + B_2 B_2^T\right] X \\ &- C_{11}^T (I - D_{111\bullet} S D_{111\bullet}^T) C_{11},\end{aligned} \tag{10}$$

where $\hat{A} = A - B_2 C_{12}$, $D_{111\bullet} := [D_{1111}\ D_{1112}]$, and $S = (D_{111\bullet}^T D_{111\bullet} - \gamma^2 I)$. The time-dependent state feedback gain F can be calculated from the open-loop plant matrices and the solution X as

$$F = -B_2^T X - C_{12}. \tag{11}$$

Proof: The proof is provided in [7] and [25]. ∎

In contrast to the output feedback synthesis problem, the controller only depends on the solution of a single RDE. Furthermore, no undesirable auxiliary condition must be evaluated.

B. Coprime Factoriztion

The normalized LTV coprime factorization [17] plays a key role for the results derived in this paper. It is defined in the following Theorem 4.

Theorem 4 (Normalized Coprime Factorization [17]). *Consider an LTV system (1). There exists a normalized left coprime factorization $\mathbf{P} = \mathbf{M}^{-1}\mathbf{N}$ if there exists a continuously differentiable, symmetric positive semi-definite matrix function $Z(t)$, $t \in [0,T]$ such that $Z(0) = 0$ and*

$$\dot{Z} = \bar{A}Z + Z\bar{A}^T - ZCS^T C^T Z + BSB^T, \tag{12}$$

with $\bar{A} = (A - BS^{-1}D^T C)$, and $S = I + D^T D$. A state space realization for $[\mathbf{M}\ \mathbf{N}]$ is:

$$\begin{bmatrix} \dot{\mu} \\ \hline \nu \end{bmatrix} = \left[\begin{array}{c|cc} A{+}LC & L & B + LD \\ \hline R^{-0.5}C & R^{-0.5} & R^{-0.5}D \end{array}\right] \begin{bmatrix} \mu \\ \hline y \\ -u \end{bmatrix}, \tag{13}$$

with

$$L = -(BD^T + ZC^T)R^{-1}. \tag{14}$$

and $R = I + DD^T$.

Proof: The proof is provided in [17]. ∎

The normalized left coprime factorization $\mathbf{P} = \mathbf{M}^{-1}\mathbf{N}$ provides a kernel representation of all stable input-output pairs of a system $\mathbf{P}$ and has the property that $\mathbf{C} = [\mathbf{M}\ \mathbf{N}]$ is co-isometric, see [17]. This property implies $\|\mathbf{GC}\|_{2[0,T]} = \|\mathbf{G}\|_{2[0,T]}$ for any finite time LTV system $\mathbf{G}$. In other words, interconnecting a dynamic system $\mathbf{G}$ in series with a co-isometric dynamic system $\mathbf{C}$ does not change its induced $L_2[0,T]$-norm. Note that this does also imply $\|\mathbf{C}\|_{2[0,T]} = 1$.

III. Observer-based Mixed Sensitivity Control

Let $[\mathbf{P_d}\,\mathbf{P_u}]$ denote a finite horizon LTV model of a plant with control input u and disturbance input d and a state space realization

$$\begin{bmatrix} \dot{x} \\ \hline y \end{bmatrix} = \left[\begin{array}{c|cc} A(t) & B_d(t) & B_u(t) \\ \hline C(t) & 0 & 0 \end{array}\right] \begin{bmatrix} x \\ d \\ u \end{bmatrix}. \tag{15}$$

The assumption that the plant model (15) is strictly proper is only made to simplify the notation and the following results can be generalized to non-strictly proper plants at the cost of more complicated expressions. An unweighted mixed sensitivity four-block formulation for the plant model (15) is pictured in Fig. 1. The performance outputs corresponding to the generalized plant (5) are thus $z_1 = e$ and $z_2 = u$. The performance inputs are $w_1 = d$ and $w_2 = r$. The

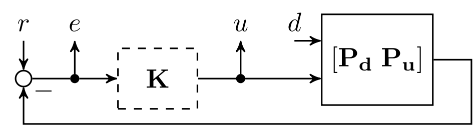


Fig. 1: four-block mixed sensitivity problem.

corresponding closed-loop input-output map is

$$\begin{bmatrix} e \\ u \end{bmatrix} = \begin{bmatrix} -\mathbf{S}\,\mathbf{P_d} & \mathbf{S} \\ -\mathbf{K}\,\mathbf{S}\,\mathbf{P_d} & \mathbf{K}\,\mathbf{S} \end{bmatrix} \begin{bmatrix} d \\ r \end{bmatrix}. \tag{16}$$

In (16), $\mathbf{S} = (I + \mathbf{P_u}\mathbf{K})^{-1}$ denotes the output sensitivity function [25]. Removing the controller $\mathbf{K}$ from the interconnection in Fig. 1 results in an open-loop generalized plant with a structure equal to (5). This structure allows for the synthesis of a dynamic output feedback LTV controller as described in Theorem 2. However, it comes with a significant computational overhead due to "unnecessary" solutions of RDE (7) and (8) which violate the coupling condition (9). The remainder of this section deals with the derivation of an equivalent synthesis procedure that avoids the spectral radius condition.

Consider an observer-based controller $\mathbf{K}$ with state space representation

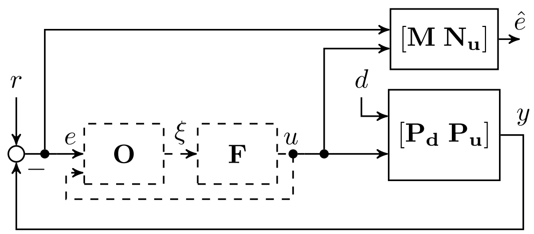
$$\begin{bmatrix} \dot{\xi} \\ \hline \xi \end{bmatrix} = \left[\begin{array}{c|cc} A + LC & L & B_u \\ \hline I & 0 & 0 \end{array}\right] \begin{bmatrix} \xi \\ \hline e \\ u \end{bmatrix} \tag{17a}$$

$$u = F(t)\,\xi. \tag{17b}$$

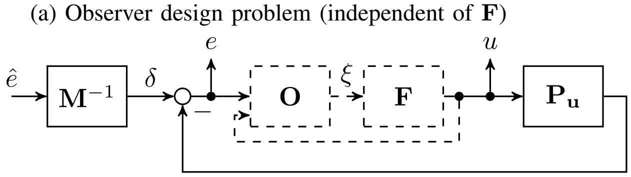
Equation (17a) represents the observer $\mathbf{O}$ with the output injection gain $L(t)$. Equation (17b) represents the state feedback controller $\mathbf{F}$ with the state feedback gain $F(t)$. For this controller structure, [16] provides an equivalent representation of the mixed sensitivity problem (16) based on the identity

$$\underbrace{\begin{bmatrix} -\mathbf{S}\mathbf{P_d} & \mathbf{S} \\ -\mathbf{K}\,\mathbf{S}\mathbf{P_d} & \mathbf{K}\,\mathbf{S} \end{bmatrix}}_{\text{Fig. 1}} = \underbrace{\begin{bmatrix} \mathbf{S} \\ \mathbf{K}\,\mathbf{S} \end{bmatrix} \mathbf{M}^{-1}}_{\text{Fig. 2b}} \underbrace{\begin{bmatrix} -\mathbf{N_d} & \mathbf{M} \end{bmatrix}}_{\text{Fig. 2a}} \tag{18}$$

where $[\mathbf{M}\ \ \mathbf{N_d}]$ is a left coprime factorization of $\mathbf{P_d}$. The identity (18) divides the mixed sensitivity problem from Fig. 1 into separate observer and state feedback synthesis problems as pictured in Fig. 2.



(a) Observer design problem (independent of $\mathbf{F}$)



(b) State feedback design problem (with given observer $\mathbf{O}$)

Fig. 2: Rearranged mixed sensitivity four-block problem with observer / state feedback separation.

Specifically, [16] shows that the observer design problem of Fig. 2a is independent of the choice of $\mathbf{F}$. In fact, obtaining a controller only requires the calculation of a normalized coprime factorization in accordance with Theorem 4. While the observer design affects the achievable closed-loop performance γ, obtaining the observer is independent of γ. As such, the observer synthesis only requires solving a single RDE once. Reference [16] further shows that there is a one-on-one correspondence between the normalized coprime factorization and the observer as they are both completely parameterized by the same output injection gain L. Consequently, $\mathbf{O}$ is known in Fig. 2b and the available feedback signal is the state vector of the observer. The observer state is further shown to be identical to the complete state vector of the generalized plant, see [16]. As such, the design problem pictured in Fig. 2b can be solved as a state feedback synthesis in accordance with Theorem 3. Doing so requires the solution of a second RDE which depends on the solution of the first RDE. Thus, both problems are only coupled in one direction and can be solved in sequence. Therefore, the bisection over γ only requires the iterative solution of one RDE. Furthermore, each fully solved RDE provides a feasible controller. This renders the proposed output feedback synthesis numerically more efficient than state-of-the-art finite horizon synthesis approaches relying on derivatives of Theorem 2 ([4], [7], [10]).

From the co-isometric property of the LTV left coprime factorization $[\mathbf{N_d}\,\mathbf{M}]$ it further follows that

$$\left\| \begin{bmatrix} \mathbf{S} \\ \mathbf{K}\,\mathbf{S} \end{bmatrix} \mathbf{M}^{-1} \begin{bmatrix} -\mathbf{N_d} & \mathbf{M} \end{bmatrix} \right\| = \left\| \begin{bmatrix} \mathbf{S} \\ \mathbf{K}\,\mathbf{S} \end{bmatrix} \mathbf{M}^{-1} \right\|. \tag{19}$$

Hence, the $L_2[0,T]$ norm of the state feedback problem with the given observer (Fig. 2b) is equal to the $L_2[0,T]$ norm of the original four block problem (Fig. 1). In other words, the sequential synthesis achieves the exact same guaranteed mixed sensitivity performance as the conventional output feedback synthesis. The following Theorem 5 formalizes this result.

Theorem 5 (Observer-Based Controller Synthesis)**.** *Consider an LTV system (5). There exists an observer-based controller* $\mathbf{K}$ *defined by (17a) and (17b) such that* $\|\mathcal{F}(\mathbf{G},\mathbf{K})\| \leq \gamma$ *iff the following two conditions hold.*

1) *There exists a continuously differentiable, symmetric positive semi-definite matrix function* $Z(t)$, $t \in [0,T]$ *such that* $Z(0) = 0$ *and*

$$\dot{Z} = AZ + ZA^T - ZC^TCZ + B_dB_d^T. \tag{20}$$

2) *There exists a continuously differentiable, symmetric positive semi-definite matrix function* $X(t)$, $t \in [0,T]$ *such that* $X(T) = 0$ *and*

$$\dot{X} = -\bar{A}^TX - X\bar{A} + X\bar{T}X + C^T\bar{U}C, \tag{21}$$

with $\bar{A} = A - \frac{1}{1-\gamma^2}ZC^T$, $\bar{T} = \frac{1}{1-\gamma^2}ZC^TCZ + B_uB_u^T$, *and* $\bar{U} = \frac{\gamma^2}{1-\gamma^2}C^TC$.

Proof. Limited space only allows a sketch of the proof. In essence, solvability of the RDE (20) establishes the existence of a normalized left coprime factorization as given in Theorem 4 and yields the output injection gain $L = -Z\,C^T$. Similarly, solvability of the RDE (21) establishes the existence of a state feedback gain $F = -B_u^T\,X$ for the generalized plant shown in Fig. 2b. As previously stated, this generalized plant includes the observer and represents the exact same performance specifications as Fig. 1. Finally, the co-isometric property of the LTV coprime factorization guarantees that both synthesis problems yield the exact same induced $L_2[0,T]$-norms. □

The line of argument is identical to the proof in [16] for LPV systems and the reader is referred there for a discussion of the general case, including weights. The noteworthy difference is that no normalized coprime factorization exists for LPV systems. Instead, a contractive coprime factorization needs to be used and only an upper bound on the mixed sensitivity problem can be guaranteed. On the contrary, the equality condition in (19) establishes a stronger result for LTV systems than the previous results for LPV systems.

IV. Control Design Example: Space Launcher

The control design for a space launcher in atmospheric ascent demonstrates the effectiveness of the approach. A representative LTV model for this scenario is taken from [26].

It represents the first stage rigid body pitch dynamics of the Vanguard space launcher along a gravity turn trajectory. To include the effects of external wind disturbances, the model is extended with an additional input $\delta_{\dot{\alpha}}$ representing the additional wind-induced angle of attack. The corresponding LTV model is

$$\begin{bmatrix}\dot{\alpha}\\ \dot{\theta}\\ \dot{q}\end{bmatrix} = \begin{bmatrix}\frac{Z_\alpha}{mV} & \frac{-g\sin\Theta}{V} & 1\\ 0 & 0 & 1\\ \frac{M_\alpha}{J_{yy}} & 0 & \frac{M_q}{J_{yy}}\end{bmatrix}\begin{bmatrix}\alpha\\ \theta\\ q\end{bmatrix} + \begin{bmatrix}\frac{T}{mV} & 1\\ 0 & 0\\ \frac{T\xi}{J_{yy}} & 0\end{bmatrix}\begin{bmatrix}\delta_\mu\\ \delta_{\dot{\alpha}}\end{bmatrix} \tag{22}$$

The states, which are also the system's output, are the angle of attack α, the pitch angle θ and the pitch rate q. They represent deviation values from the time-varying reference trajectory. The input δ_μ is the corrective gimbal input rotating the thrust vector for attitude control. The functions Z_α, M_α and M_q denote the aerodynamic stability derivatives. Their values along the trajectory are provided in [26] together with explicit expressions for mass m and pitch inertia J_{yy} in dependence on t. The variables V and Θ represent the reference velocity and pitch angle of the launcher along the trajectory. The values are given in [26] for the time interval $t \in [11.35, 146.35]$ with a step size of $2.7\,$s. The thrust T, the distance ξ of the center of gravity from the gimbal, as well as the gravitational acceleration g are constants. However, all coefficients in (22) are time-varying. The control design considers a time horizon from $15\,$s to $100\,$s after lift-off; the start and end point of the gravity turn maneuver. Fig. 3 shows the reference pitch angle value Θ along this trajectory segment as well as the value $\mu_\alpha = \frac{M_\alpha}{J_{yy}}$. The magnitude of μ_α is a measure for pitch stability. Larger values of μ indicate faster unstable dynamics and, thus, a more difficult control problem.

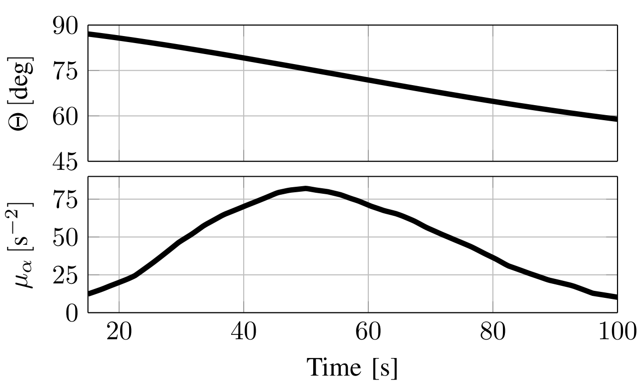


Fig. 3: Reference pitch angle Θ and pitch stiffness μ_α along the trajectory.

For the mixed sensitivity control design, a weighting structure is required to impose meaningful closed loop requirements, such as tracking and limited control authority. A suitable weighting structure is provided in [27]. It is depicted in Fig. 4 and uses weights W_e, W_u, V_e, V_u, and V_d. It represents the weighted closed loop

$$\begin{bmatrix}z_1\\ z_2\end{bmatrix} = \begin{bmatrix}\mathbf{W_e}V_e^{-1} & 0\\ 0 & \mathbf{W_u}V_u^{-1}\end{bmatrix}\begin{bmatrix}-\mathbf{SP_d} & \mathbf{S}\\ -\mathbf{KSP_d} & \mathbf{KS}\end{bmatrix}\begin{bmatrix}V_d & 0\\ 0 & V_e\end{bmatrix}\begin{bmatrix}w_1\\ w_2\end{bmatrix} \tag{23}$$

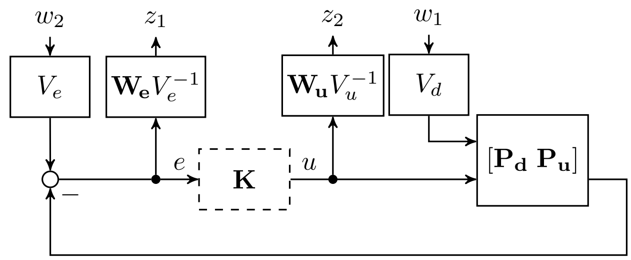


Fig. 4: Weighted four-block mixed sensitivity problem.

or equivalently in a separated form (18) becomes

$$\begin{bmatrix}z_1\\ z_2\end{bmatrix} = \underbrace{\begin{bmatrix}\mathbf{W_e}V_e^{-1} & 0\\ 0 & \mathbf{W_u}V_u^{-1}\end{bmatrix}\begin{bmatrix}\mathbf{S}\\ \mathbf{KS}\end{bmatrix}V_e\mathbf{M}^{-1}}_{\text{state feedback}}\underbrace{\begin{bmatrix}-\mathbf{N_d} & \mathbf{M}\end{bmatrix}}_{\text{observer}}\begin{bmatrix}w_1\\ w_2\end{bmatrix}, \tag{24}$$

where $\mathbf{M}^{-1}\mathbf{N_d} = V_e^{-1}\mathbf{P_d}V_d$. A detailed derivation is provided in [16].

Guidelines for the selection of the weights are formulated in [27]. The weight $\mathbf{W}_\mathrm{e}$ determines the requirements on the sensitivity $\mathbf{S}$ and disturbance sensitivity $\mathbf{SP_d}$. To achieve tracking in θ at a closed-loop bandwidth sufficiently above the frequency of the unstable dynamics ($1.14\,$rad/s at $50\,$s into the ascent), $\mathbf{W}_\mathrm{e}$ is selected with integral behavior in the θ channel up to $10\,$rad/s. Further, a magnitude of 0.5 is selected at high frequencies in the θ channel and at all frequencies in the α and q channels to limit the peak sensitivity for good robustness. The weighting filter $\mathbf{W}_\mathrm{u}$ determines requirements on the control sensitivity $\mathbf{KS}$, corresponding to actuator limitations, noise rejection, and robustness. The actuator for the launcher is a gimbal whose dynamics can be approximated by a first order lag with corner frequency $50\,$rad/s. To keep actuator activity sufficiently below this frequency, $\mathbf{W}_\mathrm{u}$ is selected with unit magnitude up to $25\,$rad/s and differentiating behavior above $25\,$rad/s. The static weight V_e balances the three output errors. It is chosen as $0.5\,$deg for θ, $0.5\,$deg/s for q, and $1\,$deg for α. To specify the available control effort for the previously selected errors, the weight V_u is used. Based on typical deflection limits of thrust vector control systems, its value is chosen as $5\,$deg. The ratio V_e/V_d determines the trade-off between tracking performance and disturbance rejection. A value of $V_d = 0.2\,$deg/s is chosen for the current example.

Using these weights, the observer synthesis and calculation of the output injection gain L can be performed. It requires to solve a scaled version of the RDE (20), which can be readily derived following the explanations in [16]. The RDE is solved using the Matlab solver `ODE15s` [28], suitable for stiff differential equations. The solution takes $0.09\,$s on a standard desktop PC.

Next, the state feedback synthesis is conducted, where L from the second step defines the input weight $\mathbf{M}^{-1}$. The calculated feedback gain minimizes the $L_2[0,T]$ norm of the state feedback problem in (23) and, thus, for the output feedback problem as a whole. Again, `ODE15s` in Matlab is used to solve the RDE. A bisection calculates the minimal feasible γ as $\gamma = 2.77$ and takes $4.31\,\mathrm{s}$. Thus, the complete controller synthesis requires approximately $4.4\,\mathrm{s}$.

Finally, the finite horizon LTV controller is formed from the feasible solutions of the RDEs, the plant state space matrices and the weighting filters as described in [16].

An LTV simulation of the resulting closed loop model including the gimbal dynamics is conducted in Matlab. The closed loop is exited by an external wind disturbance. The wind disturbance consists of a white Gaussian noise signal, with a sampling time of $0.5\,\mathrm{s}$ and variance 1, and a static mean value. It induces an α disturbance on the launcher shown in Fig. 6. The control signal counteracts the disturbance keeping the deviation θ close to zero for the whole ascent. The required gimbal deflection remains below the saturation limit. Thus, the controller displays excellent disturbance rejection keeping the θ disturbance close to zero along the complete trajectory. It is particularly noteworthy that no variation in performance is visible, although the unstable dynamics of the launcher vary significantly over time (see Fig. 3). In conclusion, the proposed control design approach proofs suitable for a highly time-varying problem.

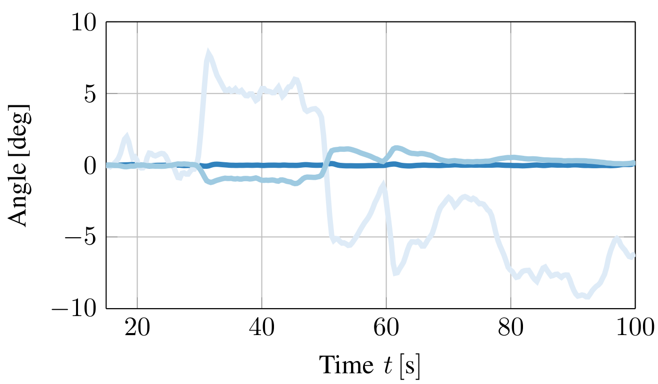


Fig. 5: Results of the LTV simulation: α disturbance (—), θ deviation (—), control signal (—).

V. Conclusion

The paper presents a novel observer-based output feedback synthesis approach for finite horizon linear time-varying systems. It separates the synthesis into a sequential observer synthesis and state feedback synthesis step. Thus,the additional coupling conidition present in state-of-the-art approaches is avoided, reducing the computational overhead. Furthermore, only one of the corresponding RDEs depends on the performance index γ which significantly reduces RDE evaluations and saves computational effort. The feasibility of the synthesis machinery is demonstrated using a pitch tracker design for a space launcher.

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