TECHNISCHE UNIVERSITÄT DRESDEN

INSTITUT FÜR KERN- UND TEILCHENPHYSIK

Construction and properties of the NRQED Lagrangian up to order $1/M^3$

Bachelorarbeit
zur Erlangung des Hochschulgrades
Bachelor of Science
im Bachelor-Studiengang Physik

vorgelegt von

Jonas Scheibler
geboren am 09.07.1998 in Zittau

Institut für Kern- und Teilchen Physik
Fakultät Physik
Bereich Mathematik und Naturwissenschaften
Technische Universität Dresden
2022

Eingereicht am 27. 06. 2022

1. Gutachter: Prof. Dr. Dominik Stöckinger
2. Gutachter: Prof. Dr. Ralf Schützhold

Summary

Abstract

Nonrelativistic quantum electrodynamics (NRQED) is an effective field theory that describes a nonrelativistic spin $\frac{1}{2}$-particle interacting with the electromagnetic field. We present a detailed discussion of the NRQED Lagrangian up to order $1/M^3$, in which we introduce a procedure to construct all valid terms at the respective order and show the explicit calculations for each term. We motivate the construction of the NRQED Lagrangian by discussing properties of the Lagrangian of quantum electrodynamics (QED).

Abstract

Die nichtrelativistische Quantenelektrodynamik (NRQED) ist eine effektive Feldtheorie, die die Wechselwirkung eines nichtrelativistischen Spin $\frac{1}{2}$-Teilchens mit dem elektromagnetischen Feld beschreibt. Die NRQED Lagrangedichte wird bis zur Ordnung $1/M^3$ untersucht, wobei ein Verfahren präsentiert wird, welches ermöglicht, alle erlaubten Terme in der zugehörigen Ordnung zu bestimmen. Die Berechnung aller Terme wird dabei explizit angegeben. Vorüberlegungen zur Lagrangedichte der Quantenelektrodynamik (QED) motivieren die Auseinandersetzung mit der NRQED Lagrangedichte.

Contents

1 Introduction

1.1 Motivation of the QED Lagrangian $\mathcal{L}_{QED}$

Since a rigorous derivation of $\mathcal{L}_{QED}$ would be too detailed for the following considerations, $\mathcal{L}_{QED}$ is motivated by introducing the description of a free, relativistic, spin $\frac{1}{2}$-particle and the coupling to the electromagnetic field.

Description of relativistic particles

The Dirac equation describes a free, relativistic, spin $\frac{1}{2}$-particle. The associated wave-functions (Dirac spinors) ψ are 4-component complex spinors. The corresponding Lagrangian $\mathcal{L}_{D,free}$ is given as

$$\mathcal{L}_{D,free} = \bar{\psi}(i\not{\partial} - m)\psi \tag{1.1}$$

Slashed properties, such as $\not{\partial}$ are defined by a contracion with the complex, 4x4-γ^μ-matrices: $\not{a} = a_\mu \gamma^\mu$. The barred spinor $\bar{\psi}$ is defined by $\bar{\psi} = \psi^\dagger \gamma^0$. The equations of motion for the spinors ψ and $\bar{\psi}$ can be obtained by the Euler-Lagrange-equations of $\mathcal{L}_{D,free}$

$$\frac{\partial \mathcal{L}_D}{\partial \psi} = \partial_\mu \frac{\partial \mathcal{L}_D}{\partial(\partial_\mu \psi)} \implies \bar{\psi}(i\overleftarrow{\not{\partial}} + m) = 0 \tag{1.2}$$

$\bar{\psi}\overleftarrow{\not{\partial}}$ denotes, that the derivative acts on the spinor to the left.

$$\frac{\partial \mathcal{L}_D}{\partial \bar{\psi}} = \partial_\mu \frac{\partial \mathcal{L}_D}{\partial(\partial_\mu \bar{\psi})} \implies (i\not{\partial} - m)\psi = 0 \tag{1.3}$$

This reproduces the free Dirac equation $(i\not{\partial} - m)\psi = 0$.

Classical electrodynamics

Before discussing the introduction of the electromagnetic interaction, it is worthwhile to take a look into classical electrodynamics. The physical fields $\mathbf{E}$ and $\mathbf{B}$ are governed by the electromagnetic potentials ϕ and $\mathbf{A}$ in the following way: $\mathbf{E} = -\nabla\phi - \partial_t \mathbf{A}$, $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$. In 4-notation, both potentials are combined into the 4-potential $A^\mu := (\phi, \mathbf{A}) = (A^0, \mathbf{A})$. An important property of A^μ is that $\mathbf{E}$ and $\mathbf{B}$ are invariant under the following gauge transformation with an arbitrary field $\theta(\mathbf{x}, t)$: $A'^{\mu} = A^{\mu} - \partial^{\mu}\theta$. This degree of freedom of A^{μ} is essential in the following considerations.

Coupling to the electromagnetic field

$\mathcal{L}_{D,free}$ is invariant under a global gauge transformation of the spinor field $\psi \to e^{i\theta}\psi$. It is however a crucial point that the Lagrangian-density should be invariant under a local gauge transformation $\psi \to e^{ieQ\theta(x)}\psi$ as well. $\mathcal{L}_{D,free}$ requires some modification in order to achive said invariance. The minimal coupling principle presents a way to introduce a gauge field, which establishes local gauge invariance. The field A^{μ} is introduced by the replacement $i\partial_{\mu} \to i\partial_{\mu} - eQA_{\mu}$ into the free Dirac Lagrangian.

$$\mathcal{L}_{D,free} = \bar{\psi}(i\not{\partial} - m)\psi \to \mathcal{L}_{D,int} = \bar{\psi}[(i\partial^{\mu} - eQA^{\mu})\gamma_{\mu} - m]\psi \tag{1.4}$$

The obtained Lagrangian $\mathcal{L}_{D,int}$ contains a new type of interaction between the spinor field and A^{μ}, the electromagnetic interaction. In order to show local gauge invariance of $\mathcal{L}_{D,int}$, it is useful to introduce the gauge covariant derivative $D^{\mu} := \partial^{\mu} + ieQA^{\mu}$

$$\mathcal{L}_{D,int} = \bar{\psi}[(i\partial^{\mu} - eQA^{\mu})\gamma_{\mu} - m]\psi = \bar{\psi}[i\not{D} - m]\psi \tag{1.5}$$

D^{μ} transforms covariantly under a gauge transformation, hence the name.

$$D^{\mu}\psi \to D'^{\mu} = [\partial^{\mu} + ieQ\,(A^{\mu} - \partial^{\mu}\theta(x))]\, e^{ieQ\theta(x)}\psi = e^{ie\theta(x)}D^{\mu}\psi \tag{1.6}$$

With this, one can easily show the gauge invariance of $\mathcal{L}_{D,int}$

$$\mathcal{L}'_{D,int} = \bar{\psi}'[i\not{D}' - m]\psi' = \bar{\psi}[i\not{D} - m]\psi = \mathcal{L}_{D,int} \tag{1.7}$$

The Lagrangian of the photon field A^{μ}

The proposed gauge field A^{μ} can be interpreted as the photon field. The associated Lagrangian $\mathcal{L}_A$ should therefore fulfill the following demands: $\mathcal{L}_A$ should be Lorentz-invariant and the associated equations of motion should reproduce the Maxwell equations. Since the photon is a massless particle, there should be no mass term in $\mathcal{L}_A$. The kinetic term should contain Field derivatives $\partial^{\mu}A^{\nu}$. The simplest Lagrangian to fulfill these demands is

$$\mathcal{L}_A = -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{1.8}$$

with the electromagnetic field tensor $F^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}$.

The Lagrangian of QED $\mathcal{L}_{QED}$

$\mathcal{L}_{QED}$ is obtained by combining $\mathcal{L}_{D,int}$ and $\mathcal{L}_A$.

$$\begin{aligned}\mathcal{L}_{QED} &= \mathcal{L}_{D,int} + \mathcal{L}_A \\ &= \bar{\psi}(i\not{D} - m)\psi - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} \\ &= \underbrace{\bar{\psi}i\partial^\mu\gamma_\mu\psi}_{\text{kinetic term } \psi} - \underbrace{eQ\bar{\psi}\gamma_\mu\psi A^\mu}_{\text{interaction term}} - \underbrace{\bar{\psi}m\psi}_{\text{mass term}} - \underbrace{\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}}_{\text{kinetic term } A^\mu}\end{aligned} \tag{1.9}$$

The Euler-Lagrange equations for $\bar{\psi}$ and A^μ produce the coupled Dirac equation for ψ and as expected, the Maxwell equations for A^μ

$$\frac{\partial\mathcal{L}_{QED}}{\partial\bar{\psi}} = \partial_\mu\frac{\partial\mathcal{L}_{QED}}{\partial(\partial_\mu\bar{\psi})} \implies (i\not{D} - m)\psi = 0 \tag{1.10}$$

$$\frac{\partial\mathcal{L}_{QED}}{\partial A_\beta} = \partial_\alpha\frac{\partial\mathcal{L}_{QED}}{\partial(\partial_\alpha A_\beta)} \implies \partial_\alpha F^{\alpha\beta} = eQ\bar{\psi}\gamma^\beta\psi \tag{1.11}$$

The interaction term from (1.9) appears on the right side of the Maxwell equations. It is therefore reasonable to define the electron current j^β as $j^\beta := eQ\bar{\psi}\gamma^\beta\psi$, since it is the source term of the Maxwell-equations.

1.1.1 Symmetries of $\mathcal{L}_{QED}$

Lorentz-invariance

Under a Lorentz-transformation ψ, $\bar{\psi}$, ∂_μ and A_μ transform as $S(\Lambda)\psi = \psi^{'}(x^{'})$, $\bar{\psi}(x)S(\Lambda)^{-1} = \bar{\psi}^{'}(x^{'})$, $\partial_\nu = \Lambda^\mu{}_\nu\partial^{'}_\mu$ and $A_\nu = \Lambda^\mu{}_\nu A^{'}_\mu$, respectively.

$$\begin{aligned}S(\Lambda)\psi(x) = \psi^{'}(x^{'}) &\implies \psi(x) = S^{-1}(\Lambda)\psi^{'}(x^{'}) \\ \bar{\psi}(x)S(\Lambda)^{-1} = \bar{\psi}^{'}(x^{'}) &\implies \bar{\psi}(x) = \bar{\psi}^{'}(x^{'})S(\Lambda)\end{aligned} \tag{1.12}$$

Lorentz-invariance of $\mathcal{L}_{QED}$ can be shown by using $S^{-1}(\Lambda)\gamma^\mu S(\Lambda) = \Lambda^\mu{}_\nu\gamma^\nu$.

$$\begin{aligned}\mathcal{L}_{QED} &= \bar{\psi}(x)(i\partial_\nu\gamma^\nu - eQA_\nu\gamma^\nu - m)\psi(x) - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} \\ &= \bar{\psi}^{'}(x^{'})S(\Lambda)(i\Lambda^\mu{}_\nu\partial^{'}_\mu\gamma^\nu - eQ\Lambda^\mu{}_\nu A^{'}_\mu\gamma^\nu - m)S^{-1}(\Lambda)\psi^{'}(x^{'}) - \underbrace{\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}}_{\text{Lorentz-scalar}} \\ &= \bar{\psi}^{'}(x^{'})S(\Lambda)(i\partial^{'}_\mu\underbrace{\Lambda^\mu{}_\nu\gamma^\nu}_{S^{-1}\gamma^\mu S} - eQA^{'}_\mu\underbrace{\Lambda^\mu{}_\nu\gamma^\nu}_{S^{-1}\gamma^\mu S} - m)S^{-1}(\Lambda)\psi^{'}(x^{'}) - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} \\ &= \bar{\psi}^{'}(x^{'})(i\partial^{'}_\mu\gamma^\mu - eQA^{'}_\mu\gamma^\mu - m)\psi^{'}(x^{'}) - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} \\ &= \mathcal{L}^{'}{}_{QED}\end{aligned} \tag{1.13}$$

This also means that the equations of motion for ψ (1.10) and A^μ (1.11), that are obtained from $\mathcal{L}_{QED}$, are Lorentz-covariant and therefore valid in any reference frame.

Gauge-invariance

$\mathcal{L}_{D,int}$ is invariant under local gauge transformations by design. $\mathcal{L}_A$ is also gauge-invariant because $F_{\mu\nu}$ is invariant under the transformation of A^μ: $A^\mu \to A^\mu - \partial^\mu\theta(x)$. Therefore $\mathcal{L}_{QED}$ is gauge invariant as well.

Parity-conservation

Parity transformation $\mathcal{P}$ denotes the inversion of space: $\mathbf{x} \xrightarrow{\mathcal{P}} \textbf{-x}$. A representation of this transformation for Dirac spinors is γ^0

$$\begin{aligned} \psi'(-\mathbf{x},t) &= \gamma^0\psi(\mathbf{x},t) \\ \bar{\psi}'(-\mathbf{x},t) &= \bar{\psi}(\mathbf{x},t)\gamma^0 \end{aligned} \tag{1.14}$$

The transformation behavior of the quantities of $\mathcal{L}_{QED}$ under parity transformation is summarized in table (1.1). $\mathcal{L}_{QED}$ is invariant under Parity transformation

$$\begin{aligned} \mathcal{L}_{QED}^{\mathcal{P}} &= \bar{\psi}'(-\mathbf{x},t)(i\partial_0\gamma^0 - i\partial_j\gamma^j - eQA_0\gamma^0 + eQA_j\gamma^j - m)\psi'(-\mathbf{x},t) - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &= \bar{\psi}(\mathbf{x},t)\gamma^0(i\partial_0\gamma^0\gamma^0 - i\partial_j \underbrace{\gamma^j\gamma^0}_{-\gamma^0\gamma^j} - eQA_0\gamma^0\gamma^0 + eQA_j \underbrace{\gamma^j\gamma^0}_{-\gamma^0\gamma^j} - \gamma^0 m)\psi(\mathbf{x},t) - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &= \bar{\psi}(\mathbf{x},t)\underbrace{\gamma^0\gamma^0}_{\mathrm{I}}(i\partial_0\gamma^0 + i\partial_j\gamma^j - eQA_0\gamma^0 - eQA_j\gamma^j - m)\psi(\mathbf{x},t) - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &= \bar{\psi}(\mathbf{x},t)(i\partial_\nu\gamma^\nu - eQA_\nu\gamma^\nu - m)\psi(\mathbf{x},t) - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} \\ &= \mathcal{L}_{QED} \end{aligned} \tag{1.15}$$

This shows that $\mathcal{L}_{QED}$ is parity conserving. In other words, the combination of physical quantities of $\mathcal{L}_{QED}$ is even under parity transformation.

parity transformation $\mathcal{P}$			
quantity	∂_μ	A_μ	$F_{\mu\nu}$
$\mathcal{P}$	∂^μ	A^μ	$F^{\mu\nu}$

Table 1.1: Transformation behavior under parity transofrmation.

Time-reversal-invariance

Time reversal transformation $\mathcal{T}$ denotes the inversion of time: $t \xrightarrow{\mathcal{T}} -t$. A physical interpretation is observing a process and then observing the same process in reversed temporal order.

If a process is symmetric under this transformation, an observer could not tell the difference between the process itself and the reversed one. Dirac spinors transform like

$$\begin{aligned} \psi'(\mathbf{x},-t) &= \gamma^1\gamma^3\psi(\mathbf{x},t) \\ \bar{\psi}'(\mathbf{x},-t) &= \bar{\psi}(\mathbf{x},t)\gamma^3\gamma^1 \end{aligned} \tag{1.16}$$

The transformation behavior of the quantities of $\mathcal{L}_{QED}$ under time reversal transformation is summarized in table (1.2). As well as transforming the physical quantities, $\mathcal{T}$ also acts on i: $i \xrightarrow{\mathcal{T}} -i$. Time reversal invariance of $\mathcal{L}_{QED}$ can be shown by using

$$[\gamma^1\gamma^3,\gamma^0] = \{\gamma^1\gamma^3,\gamma^1\} = [\gamma^1\gamma^3,\gamma^2] = \{\gamma^1\gamma^3,\gamma^3\} = 0 \tag{1.17}$$

$$\begin{aligned} \mathcal{L}^{\mathcal{T}}_{QED} &= \bar{\psi}'(\mathbf{x},-t)(i\partial_0\gamma^0 - i\partial_j\gamma^j - eQA_0\gamma^0 + eQA_j\gamma^j - m)\psi'(\mathbf{x},-t) - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &= \bar{\psi}(\mathbf{x},t)\gamma^3\gamma^1(i\partial_0\gamma^0 - i\partial_1\gamma^1 + i\partial_2\gamma^2 - i\partial_3\gamma^3 - eQA_0\gamma^0 + eQA_1\gamma^1 - eQA_2\gamma^2 \\ &\quad + eQA_3\gamma^3 - m)\gamma^1\gamma^3\psi(\mathbf{x},t) - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &= \bar{\psi}(\mathbf{x},t)\gamma^3\gamma^1(i\partial_0\underbrace{\gamma^0\gamma^1\gamma^3}_{\gamma^1\gamma^3\gamma^0} - i\partial_1\underbrace{\gamma^1\gamma^1\gamma^3}_{-\gamma^1\gamma^3\gamma^1} + i\partial_2\underbrace{\gamma^2\gamma^1\gamma^3}_{\gamma^1\gamma^3\gamma^2} - i\partial_3\underbrace{\gamma^3\gamma^1\gamma^3}_{-\gamma^1\gamma^3\gamma^3} - eQA_0\underbrace{\gamma^0\gamma^1\gamma^3}_{\gamma^1\gamma^3\gamma^0} \\ &\quad + eQA_1\underbrace{\gamma^1\gamma^1\gamma^3}_{-\gamma^1\gamma^3\gamma^1} - eQA_2\underbrace{\gamma^2\gamma^1\gamma^3}_{\gamma^1\gamma^3\gamma^2} + eQA_3\underbrace{\gamma^3\gamma^1\gamma^3}_{-\gamma^1\gamma^3\gamma^3} - m)\psi(\mathbf{x},t) - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &= \bar{\psi}(\mathbf{x},t)\underbrace{\gamma^3\gamma^1\gamma^1\gamma^3}_{\mathrm{I}}(i\partial_0\gamma^0 + i\partial_j\gamma^j - eQA_0\gamma^0 - eQA_j\gamma^j - m)\psi(\mathbf{x},t) - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &= \bar{\psi}(\mathbf{x},t)(i\partial_\nu\gamma^\nu - eQA_\nu\gamma^\nu - m)\psi(\mathbf{x},t) - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} \\ &= \mathcal{L}_{QED} \end{aligned} \tag{1.18}$$

In other words, the combination of physical quantities of $\mathcal{L}_{QED}$ is even under time-reversal transformation.

time reversal transformation $\mathcal{T}$				
quantity	∂_μ	A_μ	$F_{\mu\nu}$	γ^2
$\mathcal{T}$	-∂^μ	A^μ	$F^{\mu\nu}$	-γ^2

Table 1.2: Transformation behavior under time reversal transofrmation.

Charge-conjungation-invariance

Charge conjungation $\mathcal{C}$ denotes the reversal of the sign of the electric charge: $Qe \xrightarrow{\mathcal{C}} -Qe$. This can be interpreted as changing particles into antiparticles and vice versa. Dirac spinors transform like

$$\begin{aligned}\psi^{\mathcal{C}} &= C\bar{\psi}^{\intercal} \\ \bar{\psi^{\mathcal{C}}} &= (\psi^{\mathcal{C}\dagger}\gamma^0) = \psi^{\intercal}\underbrace{\gamma^0 C^\dagger \gamma^0}_{\mathrm{C}}\end{aligned} \tag{1.19}$$

An appropriate choice for C is $C = i\gamma^2\gamma^0$. The relation used for $\psi^{\mathcal{C}\dagger}$ can be shown accordingly.

$$C^\dagger = -i\gamma^0\underbrace{\gamma^0\gamma^2\gamma^0}_{\gamma^{2\dagger}} \implies \gamma^0 C^\dagger\gamma^0 = -\underbrace{\gamma^0\gamma^0}_{\mathrm{I}}\gamma^0\gamma^2\underbrace{\gamma^0\gamma^0}_{\mathrm{I}} = -i\underbrace{\gamma^0\gamma^2}_{-\gamma^2\gamma^0} = C \tag{1.20}$$

$$\mathcal{L}^{\mathcal{C}}_{QED} = \bar{\psi^{\mathcal{C}}}\big(i\partial_\mu\gamma^\mu + eQA_\mu\gamma^\mu - m\big)\psi^{\mathcal{C}} - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{1.21}$$

Charge conjungation invariance of $\mathcal{L}_{QED}$ can be shown by analyzing $\bar{\psi^{\mathcal{C}}}\gamma^\mu\psi^{\mathcal{C}}$.

$$\bar{\psi^{\mathcal{C}}}\gamma^\mu\psi^{\mathcal{C}} = \psi^{\intercal}\underbrace{C\gamma^\mu C}_{\gamma^{\mu\intercal}}\bar{\psi}^{\intercal} = \psi^{\intercal}\gamma^{\mu\intercal}\bar{\psi}^{\intercal} = -\bar{\psi}\gamma^\mu\psi \tag{1.22}$$

The last step picks up a minus sign because ψ is an anticommuting field and the order of spinors is changed. With this result, we make up for the minus sign of the second term in $\mathcal{L}^{\mathcal{C}}_{QED}$, that was caused by the reversal of the charge. The kinetic term should not take up a minus sign, since it has not been altered.

$$\bar{\psi^{\mathcal{C}}}i\partial_\mu\gamma^\mu\psi^{\mathcal{C}} = \psi^{\intercal}i\partial_\mu\underbrace{C\gamma^\mu C}_{\gamma^{\mu\intercal}}\bar{\psi}^{\intercal} = \psi^{\intercal}i\partial_\mu\gamma^{\mu\intercal}\bar{\psi}^{\intercal} = -\bar{\psi}i\overleftarrow{\partial}_\mu\gamma^\mu\psi = \bar{\psi}i\partial_\mu\gamma^\mu\psi \tag{1.23}$$

When changing the order of the spinors, the derivative then acts on the spinor to its left. The last step uses the method of partial integration in the associated action integral in order to change the spinor on which the derivative acts. This is demonstrated in more detail in the discussion of hermicity below.

Hermicity

In order to investigate the hermicity of (1.9), the terms are discussed seperately. The only term that needs further consideration is the kinetic term. The hermitian conjungate of the kinetic term is

$$(\bar{\psi}i\partial_\mu\gamma^\mu\psi)^\dagger = -i\psi^\dagger\gamma^0\gamma^\mu\gamma^0\overleftarrow{\partial}_\mu\underbrace{\psi^\dagger\gamma^0}_{\gamma^0\psi} = -i\bar{\psi}\overleftarrow{\not\partial}\underbrace{\gamma^0\gamma^0}_{\mathrm{I}}\psi \tag{1.24}$$

Hermicity of the kinetic term can be shown by integration by parts in the associated action integral.

$$\begin{aligned} S_{kin} &= \int d^4x\, \bar{\psi} i \not{\partial} \psi \\ &\overset{p.i.}{=} \int d^4x\, \text{-}i \bar{\psi} \overleftarrow{\not{\partial}} \psi \\ &= \int d^4x\, (\bar{\psi} i \not{\partial} \psi)^\dagger \end{aligned} \tag{1.25}$$

Locality

The Lagrangian obeys the mathematical definition of locality: It is a functional of the Dirac spinor field, A^μ and their respecive derivatives, wich are all evaluated at the same spacetime-point.

2 The NRQED Lagrangian

2.1 Preliminary considerations

This section is composed of information gathered from [3],[4],[5] and [9]. In the previous chapter, we have presented the fundamental theory for a spin $\frac{1}{2}$-particle, interacting with an electromagnetic field. Despite being generally valid, the relativistic equations, which arise from this theory, are rather difficult to apply when describing non-relativistic problems like bound states and low energy scattering processes. Non-relativistic atoms like muonium (μ^+e^-) are described very precisely by non-relativistic quantum mechanics. However, corrections need to be accounted for. This is done by introducing pertubation theory into non-relativistic quantum mechanics, in order to discuss for example the hydrogen fine-strucutre. More accurate results can only be produced by abandoning non-relativistic quantum mechanics and applying the covariant pertubation theory of QED. This was accomplished by Bethe and Salpeter [2], but the approach turned out to be very difficult to apply.

In 1986, Caswell and Lepage [4] presented a fundamentally different approach. Instead of applying a relativistic quantum field theory (QED) to non-relativistic problems, they introduced a new, nonrelativistic field theory called non-relativistic quantum electrodynamics (NRQED). This effective field theory approach is applicable to any non-relativistic problem and can achive any desired accuracy. Although being traditionally applied to point-like particles, NRQED can also be used to analyse the electromagnetic interactions of a spin $\frac{1}{2}$-particle that is not elementary (cf.[3]).

Effective field theory approach to non-relativistic QED

A very remarkable fact about nature is that different physical phenomena arise at different scales. In order to describe the physics at certain regions of distance, time or energy relatively easy and with sufficent precision, it is useful to focus on a separate parameter space. Isolating certain physical quantities and comparing parameters of the same dimension to it, is the first step of getting a reasonable approximation around the desired quantity. Setting relatively large parameters to infinity and neglectable ones to zero leaves a range of parameters with finite effects, that can be treated as perturbations (cf.[5]). An effective theory simplifies calculations because of its limitations to a well defined range.

When describing a non-relativistic spin $\frac{1}{2}$-particle interacting with an electromagnetic field, the predominant quantity is the particles rest mass M. We can make the following approximations

$$\begin{aligned} E \approx M \implies |\mathbf{p}| \ll M \\ |\partial_t A^\mu| \ll |MA^\mu| \end{aligned} \tag{2.1}$$

This means that only particles with small momenta relative to the rest mass are allowed and that the particles cannot obtain relativistic properties by interacting with the electromagnetic field. Phenomena like pair production are therefore not possible.
The NRQED Lagrangian for the spin $\frac{1}{2}$-particle field is constructed in the folloing way

$$\mathcal{L}_\psi = \sum_n \psi^\dagger \frac{O_n}{M^n} \psi \qquad n = 0, 1, 2, ... \tag{2.2}$$

The effective theory approach is applied by comparing the finite effects of the Operators O_n with mass dimension $n+1$ to the particles rest mass at power n. The spinors ψ and $\psi^\dagger$ have mass dimension $\frac{3}{2}$. The Lagrangian has mass dimension 4. In this way, this expansion does not diverge and can be continued to an arbitrary order to reach desired precision. (2.2) is however not the full NRQED Lagrangian $\mathcal{L}_{NRQED}$. We can write $\mathcal{L}_{NRQED}$ as

$$\mathcal{L}_{NRQED} = \mathcal{L}_\psi + \mathcal{L}_{\psi\chi} + \mathcal{L}_A \tag{2.3}$$

The other terms beside $\mathcal{L}_\psi$ are discussed below and in chapter 2.4.

Symmetries of $\mathcal{L}_\psi$

So far, no constraints, besides the structure of (2.2) have been imposed on $\mathcal{L}_\psi$. In (1.1.1), symmetries of the QED Lagrangian $\mathcal{L}_{QED}$ have been discussed. It is well worth to discuss how these change in the non-relativistic case.
Gauge-invariance and hermicity are necessary demands for a physical description and therefore cannot change. Parity and Time invariance also remain vaild. Since a non relativistic theory does not produce covariant equations, Lorentz-invariance cannot be achieved. Rotational invariance is however a reasonable demand. NRQED disregards the description of antiparticles. Since charge-conjungation-invariance relies on the exchange of particles and antiparticles, NRQED is not invariant under charge conjungation.

Building blocks of $\mathcal{L}_\psi$

Combining (2.2) with the symmetries, we have obtained a set of powerful demands for constructing the Lagrangian in terms of mass dimension of the Operators $\hat{O}_n$ and behavior under said symmetry-transformations.
According to (1.9), $\mathcal{L}_{QED}$ is $\mathcal{L}_{QED} = \mathcal{L}_{D,int} + \mathcal{L}_A$. Since NRQED describes the interaction of

a non-relativistic $\frac{1}{2}$-particle field with an electromagnetic field, electromagnetism according to $\mathcal{L}_A$ has to be described as well. $\mathcal{L}_A$ has to be included in the full NRQED Lagrangian in an unchanged way, beacause there is no non-relativistic description for photons. The components of $\mathcal{L}_{D,int}$ change in the non-relativistic case in the following way. Since there is no demand for Lorentz-invariance, the temporal and spacial components of 4-vectors can be treated seperately. D_μ splits into $D_t := \partial_t + iQeA_0$ and $\mathbf{D} := \boldsymbol{\nabla} - iQe\mathbf{A}$. Both iD_t and $i\mathbf{D}$ are hermitian operators. This can be easily shown by relating to the energy and momentum operators, which are both hermitian.

Since there is no longer a need to represent 4-dimensional Lorentz-transformations, the 4-dimensional Dirac spinors change into 2-dimensional Pauli spinors, which are sufficient in order to represent 3-dimensional rotations of a spin $\frac{1}{2}$ system. Therefore, γ matrices change into the hermitian, complex 2x2-Pauli matrices σ^i ($\boldsymbol{\sigma} = (\sigma^1, \sigma^2, \sigma^3)^\intercal$).

The properties of the discussed quantities are listed in table (2.1). At higher orders, combina-

quantity	iD_t	$i\mathbf{D}$	$\boldsymbol{\sigma}$
mass dim.	1	1	0
$\mathcal{P}$	+	-	+
$\mathcal{T}$	+	-	-
hermicity	+	+	+
gauge cov.	✓	✓	✓

Table 2.1: Properties of NRQED building blocks.

tions involving powers of $i\mathbf{D}$, iD_t and σ appear. $(i\mathbf{D})^n$ and $(iD_t)^n$ are gauge covariant for any given $n \in \mathbb{N}$

$$\begin{aligned} &\psi \to \psi' = e^{iQe\theta(\mathbf{x},\mathbf{t})}\psi \qquad D_t\psi \to e^{iQe\theta(\mathbf{x},\mathbf{t})}D_t\psi \\ (iD_t)^n\psi = \underbrace{iD_t iD_t ... iD_t}_{\text{n-times}}\psi \to \underbrace{iD_t iD_t ... iD_t}_{\text{n-1-times}} e^{iQe\theta(\mathbf{x},\mathbf{t})} \underbrace{iD_t\psi}_{\chi} &\to e^{iQe\theta(\mathbf{x},\mathbf{t})} \underbrace{iD_t iD_t ... iD_t}_{\text{n-times}}\psi \\ &= e^{iQe\theta(\mathbf{x},\mathbf{t})}(iD_t)^n\psi \end{aligned} \tag{2.4}$$

$$\begin{aligned} &\psi \to \psi' = e^{iQe\theta(\mathbf{x},\mathbf{t})}\psi \qquad i\mathbf{D}\psi \to e^{iQe\theta(\mathbf{x},\mathbf{t})}i\mathbf{D}\psi \\ (i\mathbf{D})^n\psi = \underbrace{i\mathbf{D}i\mathbf{D}...i\mathbf{D}}_{\text{n-times}}\psi \to \underbrace{i\mathbf{D}i\mathbf{D}...i\mathbf{D}}_{\text{n-1-times}} e^{iQe\theta(\mathbf{x},\mathbf{t})} \underbrace{i\mathbf{D}\psi}_{\chi} &\to e^{iQe\theta(\mathbf{x},\mathbf{t})} \underbrace{i\mathbf{D}i\mathbf{D}...i\mathbf{D}}_{\text{n-times}}\psi \\ &= e^{iQe\theta(\mathbf{x},\mathbf{t})}(i\mathbf{D})^n\psi \end{aligned} \tag{2.5}$$

$$(iD_t)^n\psi = \underbrace{iD_t^\dagger iD_t^\dagger ... iD_t^\dagger}_{\text{n-times}}\psi = \underbrace{(iD_t iD_t ... iD_t)^\dagger}_{\text{n-times}}\psi = (iD_t)^{n\dagger}\psi \tag{2.6}$$

$$(i\mathbf{D})^n\psi = \underbrace{i\mathbf{D}^\dagger i\mathbf{D}^\dagger ... i\mathbf{D}^\dagger}_{\text{n-times}}\psi = \underbrace{(i\mathbf{D}i\mathbf{D}...i\mathbf{D})^\dagger}_{\text{n-times}}\psi = (i\mathbf{D})^{n\dagger}\psi \tag{2.7}$$

Note that the $\dagger$ operation reverses the order of operators in (2.6) and (2.7). When applied to any product of hermitian operators $O_k^\dagger = O_k$, there are in general two hermitian product-combinations

$$(O_1O_2...O_n + O_nO_{n-1}...O_1) \qquad \& \qquad i(O_1O_2...O_n - O_nO_{n-1}...O_1) \tag{2.8}$$

(2.4) and (2.5) are also valid for any multiplicative combination of $i\mathbf{D}$ and iD_t.
The electromagnetic field $\mathbf{E}$ and $\mathbf{B}$ can be obtained by the electromagnetic field tensor $F^{\mu\nu}$ via $E^i = F^{0i}$ and $B^i = \frac{1}{2}\epsilon^{ijk}F_{jk}$. These expressions can also be formed using the covariant derivatives

$$\begin{aligned}
E^j &= -\tfrac{i}{Qe}\left[D_t, D^j\right] = -\tfrac{i}{Qe}(\left[\partial_t + iQeA^0, \partial^j - iQeA^j\right]) \\
&= -\tfrac{i}{Qe}(\underbrace{\left[\partial_t, \partial^j\right]}_{0} - \underbrace{\left[\partial_t, iQeA^j\right]}_{iQe\partial_t A^j} + \underbrace{\left[iQeA^0, \partial^j\right]}_{-iQe\partial_j A^0} - \underbrace{\left[iQeA^0, iQeA^j\right]}_{0}) \\
&= -\tfrac{i}{Qe}(-iQe\underbrace{\partial_t}_{\partial^t}A^j - iQe\underbrace{\partial_j}_{-\partial^j}A^0) \\
&= F^{0j}
\end{aligned} \tag{2.9}$$

$$\begin{aligned}
B^i &= \epsilon^{ijk}\tfrac{i}{2Qe}\left[D^j, D^k\right] = \epsilon^{ijk}\tfrac{i}{2Qe}(\left[\partial_j - iQeA^j, \partial_k - iQeA^k\right]) \\
&= \epsilon^{ijk}\tfrac{i}{2Qe}(\underbrace{\left[\partial_j, \partial_k\right]}_{0} - \underbrace{\left[\partial_j, iQeA^k\right]}_{iQe\partial_j A^k} - \underbrace{\left[iQeA^j, \partial_k\right]}_{-iQe\partial_k A^j} + \underbrace{\left[iQeA^j, iQeA^k\right]}_{0}) \\
&= \epsilon^{ijk}\tfrac{i}{2Qe}(-iQe\underbrace{\partial_j}_{-\partial^j}A^k + iQe\underbrace{\partial_k}_{-\partial^k}A^j) \\
&= \epsilon^{ijk}\tfrac{1}{2}F_{jk}
\end{aligned} \tag{2.10}$$

We use $\mathbf{E}$ and $\mathbf{B}$ to simplify terms and calculations at higher orders.
Because $(\sigma^i)^2 = \mathrm{I}$, powers of σ^i can be traced back to σ^1 or σ^2. In order to achieve rotational invariance, all occuring indices of vector-like objects, such as $i\mathbf{D}$ and $\boldsymbol{\sigma}$ need to be contracted by either δ^{ij} or ϵ^{ijk}.
The construction of the Lagrangian according to (2.2) is described in [9] up to $1/M^2$-order. [8] and [6] present the NRQED/NRQCD Lagrangian at $1/M^3$- and $1/M^4$-order, respectively. In the following, we present a systematic approach to constructing each term of $\mathcal{L}_\psi$ up to any desired order and explicitly show the calculations for each term up to $1/M^3$-order.

2.2 Derivation of the leading power term

Beginning the construction of $\mathcal{L}_\psi$, we consider the first order of (2.2) $\propto 1/M^0$. The Lagrangian consists of all valid combinations of $i\mathbf{D}$, iD_t and $\boldsymbol{\sigma}$ according to the symmetries discussed above. By design, O_0 has mass dimension 1, therefore it can only be constructed using one covariant derivative. iD_t meets all criteria[1] and is therefore a valid option. The introduction of a mass term $\psi^\dagger M\psi$, though not permitted, needs further consideration. Preliminarily we achieve the following result

$$\mathcal{L}_\psi = \psi^\dagger M\psi + \psi^\dagger c_1 iD_t\psi + \mathcal{O}(1/M) \tag{2.11}$$

Elimination of the mass term

The implications of field redefinitions are explained in the next chapter. For convenience, the elimination of the mass term is demonstrated here since it belongs to this order. We only demonstrate the calculation.

The elimination of the mass term can be interpreted as shifting the zero point energy. This is useful because according to (2.1), only finite perturbations relative to the rest mass are discussed and these are not affected by this shift. The mass term is eliminated by the field redefinition $\psi \to e^{iMt}\psi$.

$$\begin{aligned}
\psi^\dagger M\psi + \psi^\dagger c_1 iD_t\psi \to & \psi^\dagger e^{-iMt}Me^{iMt}\psi + \psi^\dagger e^{-iMt}c_1 iD_t e^{iMt}\psi \\
&= \psi^\dagger M\psi + \underbrace{\psi^\dagger e^{-iMt}c_1 iD_t e^{iMt}\psi}_{-\psi^\dagger M\psi + \psi^\dagger c_1 iD_t\psi} \\
&= \psi^\dagger c_1 iD_t\psi
\end{aligned} \tag{2.12}$$

This operation is only possible in the non relativistic realm because it relies on an absolute time t instead of a proper time τ. Without the mass term, power counting in $1/M$ is well defined.

Because $\psi^\dagger c_1 iD_t\psi$ is the unique leading power term, c_1 is normalized to 1. We arrive at

$$\mathcal{L}_\psi = \psi^\dagger iD_t\psi + \mathcal{O}(1/M) \tag{2.13}$$

2.3 $1/M$-order construction

At the next order, $1/M$, O_1 has mass dimension 2. The following procedure proposes a general approach to find all valid terms for the Lagrangian and can be applied to any order.

Mass dimension and parity conservation

At mass dimension 2, two covariant derivatives are required. Both, two $i\mathbf{D}$ and $(iD_t)^2$, are parity conserving. Combinations containing one of each are ruled out by parity, which cannot be restored. The conservation of time reversal invariance can be postponed to the next step because it is intertwined with hermicity. $(iD_t)^2$ will be discussed in detail below.

[1] $i\mathbf{D}$ violates parity and time reversal and $\boldsymbol{\sigma}\cdot i\mathbf{D}$ is also ruled out because of parity.

Intermezzo - Field redefinitions

Field redefinitions of the form $\psi \to \psi + a\epsilon\psi$ are allowed to apply, since they do not change the observables (cf.[1],[7]). By appropriately choosing the redefinition, this allows to drop certain terms from the Lagrangian. We will demonstrate, that applying a field redefinition is equivalent to using the classical equation of motion for ψ and thus legitimating the application of said equation of motion.
Suppose that the Lagrangian at order $1/M$ is

$$\mathcal{L}_\psi = \psi^\dagger iD_t\psi + \frac{c}{M}\psi^\dagger(iD_t)^2\psi + ... \tag{2.14}$$

We can apply the following field redefinitions

$$\psi \to \psi + aiD_t\psi \quad \& \quad \psi^\dagger \to \big(\psi + aiD_t\psi\big)^\dagger \tag{2.15}$$

Note, that $a \propto 1/M$. By inserting these into the Lagrangian, we get

$$\begin{aligned}
\mathcal{L}'_\psi =&(\psi + aiD_t\psi)^\dagger iD_t(\psi + aiD_t\psi) + \frac{c}{M}(\psi + aiD_t\psi)^\dagger(iD_t)^2(\psi + aiD_t\psi) \\
=&\psi^\dagger iD_t\psi + \psi^\dagger a(iD_t)^2\psi + \psi^\dagger a(iD_t)^2\psi + a^2\psi^\dagger(iD_t)^2\psi + \frac{c}{M}\psi^\dagger(iD_t)^2\psi \\
&+ \frac{ac^2}{M}\psi^\dagger(iD_t)^3\psi + \frac{ac}{M}\psi^\dagger(iD_t)^3\psi + \frac{a^2c}{M}\psi^\dagger(iD_t)^4\psi \\
=&\mathcal{L}_\psi + 2a\psi^\dagger(iD_t)^2\psi + \mathcal{O}(1/M^2)
\end{aligned} \tag{2.16}$$

We generated a term that is proportional to $\psi^\dagger(iD_t)^2\psi$ and some contributions that can be neglected, since they do not belong to this order. The real prefactor a is yet to be determined. When applying the Euler-Lagrange-equations to (2.14), we get

$$\begin{aligned}
\frac{\partial\mathcal{L}_\psi}{\partial\psi^\dagger} = \partial_\mu\frac{\partial\mathcal{L}_\psi}{\partial(\partial_\mu\psi^\dagger)} \implies & iD_t\psi + \frac{c}{M}(iD_t)^2\psi = 0 \\
\Leftrightarrow & iD_t\psi = -\frac{c}{M}(iD_t)^2\psi
\end{aligned} \tag{2.17}$$

Since the term on the right side of the equation of motion is also proportional $\psi^\dagger(iD_t)^2\psi$, we can set the terms equal and determine a.

$$-\frac{c}{M}(iD_t)^2\psi \stackrel{!}{=} 2a\psi^\dagger(iD_t)^2\psi \implies a = -\frac{c}{2M} \tag{2.18}$$

We insert the determined prefactor back into (2.16).

$$\begin{aligned}
\mathcal{L}'_\psi &= \mathcal{L}_\psi - \frac{c}{M}\psi^\dagger(iD_t)^2\psi + \mathcal{O}(1/M^2) \\
&= \psi^\dagger iD_t\psi + \mathcal{O}(1/M^2)
\end{aligned} \tag{2.19}$$

With this we have shown, that a field redefinition of the form $\psi \to \psi + aiD_t\psi$ is equivalent to applying the equation of motion $iD_t\psi$ and it is therefore allowed to use the equation of motion for every term that has an iD_t acting on ψ. We can also express the elimination of $\frac{c}{M}\psi^\dagger(iD_t)^2\psi$ by using the equation of motion in a more general form

$$\begin{aligned}\frac{\partial\mathcal{L}_\psi}{\partial\psi^\dagger} = \partial_\mu\frac{\partial\mathcal{L}_\psi}{\partial(\partial_\mu\psi^\dagger)} \implies iD_t\psi + \frac{1}{M}\frac{\partial}{\partial\psi^\dagger}\psi^\dagger\sum_i O_1^i\psi + \mathcal{O}(1/M^2) = 0 \\ \Leftrightarrow iD_t\psi = -\frac{1}{M}\frac{\partial}{\partial\psi^\dagger}\psi^\dagger\sum_i O_1^i\psi + \mathcal{O}(1/M^2)\end{aligned} \tag{2.20}$$

$$\implies \frac{c}{M}\psi^\dagger(iD_t)^2\psi \overset{(2.20)}{=} \frac{c}{M}\psi^\dagger iD_t\left(-\frac{1}{M}\frac{\partial}{\partial\psi^\dagger}\psi^\dagger\sum_i O_1^i\psi + \mathcal{O}(1/M^2)\right)\psi \propto \frac{1}{M^2} \tag{2.21}$$

The same argumentation can be made for $\psi^\dagger iD_t$ by using the equation of motion for $\psi^\dagger$. All terms of this form can therefore be removed from the Lagrangian without changing the physical content.

Hermitian combinations

According to (2.8), two hermitian combinations can be formed with iD^i and iD^j.

$$① \quad (iD^i)(iD^j) + (iD^j)(iD^i) \qquad \& \qquad ② \quad i((iD^i)(iD^j) - (iD^j)(iD^i)) \tag{2.22}$$

Note, that the introduction of i in ② causes time-reversal violation. This can be fixed by introducting σ^k.

$$② \quad i((iD^i)(iD^j) - (iD^j) \to i\sigma^k((iD^i)(iD^j) - (iD^j)(iD^i)) \tag{2.23}$$

Rotational invariance

All spacial indices need to be contracted in order to gain rotational-invariant combinations of the vector-like objects. For ①, the choice is contraction by δ^{ij} (dot-product). ② is contracted via ϵ^{ijk} (triple product).

$$① \quad \delta^{ij}((iD^i)(iD^j) + (iD^j)(iD^i)) = 2(i\mathbf{D})^2 \tag{2.24}$$

$$\begin{aligned}② \quad i\epsilon^{ijk}\sigma^k(\underbrace{(iD^i)(iD^j) - (iD^j)(iD^i)}_{[iD^i,iD^j]}) &= -i\epsilon^{ijk}\sigma^k\left[D^i, D^j\right] \\ &\overset{(2.10)}{=} -2Qe\boldsymbol{\sigma}\cdot\mathbf{B}\end{aligned} \tag{2.25}$$

Results

Since $\frac{c}{M}\psi^\dagger(iD_t)^2\psi$ can be eliminated, we arrive at

$$\mathcal{L}_\psi = \psi^\dagger\left(iD_t + c_2\frac{\mathbf{D}^2}{M} + c_3Qe\frac{\boldsymbol{\sigma}\cdot\mathbf{B}}{M}\right)\psi + \mathcal{O}(1/M^2) \tag{2.26}$$

Physical Interpretation of $\mathcal{L}_\psi$

The equation of motion for ψ at this order produces very familiar results.

$$iD_t\psi \overset{(2.20)}{=} -c_2\frac{\mathbf{D}^2\psi}{M} - c_3Qe\frac{(\boldsymbol{\sigma}\cdot\mathbf{B})\psi}{M} + \mathcal{O}(1/M^2) \tag{2.27}$$

Which is the Pauli equation. Without an external electromagnetic field, this equation reduces to the Schrödinger equation.

$$i\partial_t\psi + c_2\frac{\boldsymbol{\nabla}^2\psi}{M} = 0 \tag{2.28}$$

2.4 $1/M^2$-order construction

At this order, $1/M^2$, O_2 has mass dimension 3. Following the procedure proposed above, we find all valid terms.

Mass dimension and parity conservation

In order to achieve mass dimension 3, three covariant derivatives are needed. The two valid options are: either three iD_t and one iD_t combined with two $i\mathbf{D}$. Three $i\mathbf{D}$, one iD_t combined with two $i\mathbf{D}$ and one $i\mathbf{D}$ combined with two iD_t are at least ruled out because of parity. The elimination of $(iD_t)^n$ has been discussed in the construction of the previous order. Considering the case of one iD_t combined with two iD^i and iD^j we have three unique operators. This leads to $3! = 6$ occurring combinations.

$$\begin{array}{llll} ① & (iD_t)(iD^i)(iD^j) & ④ & (iD^j)(iD_t)(iD^i) \\ ② & (iD_t)(iD^j)(iD^i) & ⑤ & (iD^i)(iD^j)(iD_t) \\ ③ & (iD^i)(iD_t)(iD^j) & ⑥ & (iD^j)(iD^i)(iD_t) \end{array} \tag{2.29}$$

At first, we evaluate the terms, in which (iD_t) appears either on the left or right, denoted as the first case. We can then use these results in order to analyse the second case, in which (iD_t) appears in the middle.

2.4.1 Evaluation of the 1st case

Hermitian combinations

①&⑥ and ②&⑤ form two hermitian conjungate pairs that can each be combined into two hermitian combinations according to (2.8).

$$\begin{array}{ccc} ①+⑥ & \& & i\sigma^k(①-⑥) \\ ②+⑤ & \& & i\sigma^k(②-⑤) \end{array} \tag{2.30}$$

Again, σ^k is introduced to restore time-reversal invariance as seen above.

Rotational invariance

$$\delta^{ij}(①+⑥) \quad \Leftrightarrow \quad \delta^{ij}(②+⑤) \quad \Leftrightarrow \quad \delta^{ij}((iD_t)(iD^i)(iD^j) + (iD^j)(iD^i)(iD_t)) \tag{2.31}$$

$$\begin{aligned} \delta^{ij}\big((iD_t)(iD^i)(iD^j) + (iD^j)(iD^i)(iD_t)\big) &= (iD_t)(i\mathbf{D})^2 + (i\mathbf{D})^2(iD_t) \\ &= -\big((iD_t)(\mathbf{D})^2 + (\mathbf{D})^2(iD_t)\big) \end{aligned} \tag{2.32}$$

After contraction with δ^{ij}, combinations ①+⑥ and ②+⑤ produce equal results. The same applies for the contraction with ϵ^{ijk}, because two indices are equally paired with covariant derivatives.

$$i\epsilon^{ijk}\sigma^k(①-⑥) \quad \Leftrightarrow \quad i\epsilon^{ijk}\sigma^k(②-⑤) \quad \Leftrightarrow \quad i\epsilon^{ijk}\sigma^k((iD_t)(iD^i)(iD^j) - (iD^j)(iD^i)(iD_t)) \tag{2.33}$$

$$\begin{aligned} i\epsilon^{ijk}\sigma^k\big((iD_t)(iD^i)(iD^j) - (iD^j)(iD^i)(iD_t)\big) &= i\epsilon^{ijk}\sigma^k\big((iD_t)(iD^i)(iD^j) + \underbrace{(iD^j)(iD^i)(iD_t)}_{\epsilon^{ijk}=-\epsilon^{jik}}\big) \\ &= \frac{i}{2}\epsilon^{ijk}\sigma^k\big((iD_t)(iD^i)(iD^j) + (iD_t)(iD^i)(iD^j) \\ &\quad + (iD^j)(iD^i)(iD_t) + (iD^j)(iD^i)(iD_t)\big) \\ &= \frac{i}{2}\epsilon^{ijk}\sigma^k\big((iD_t)(iD^i)(iD^j) - (iD_t)(iD^j)(iD^i) \\ &\quad + (iD^j)(iD^i)(iD_t) - (iD^i)(iD^j)(iD_t)\big) \\ &= \frac{i}{2}\epsilon^{ijk}\sigma^k\big((iD_t)\big[(iD^i),(iD^j)\big] + \big[(iD^i),(iD^j)\big](iD_t)\big) \\ &\overset{(2.10)}{=} -Qe\sigma^k\big((iD_t)B^k + B^k(iD_t)\big) \\ &= -Qe\big((iD_t)\boldsymbol{\sigma}\cdot\mathbf{B} + \boldsymbol{\sigma}\cdot\mathbf{B}(iD_t)\big) \end{aligned} \tag{2.34}$$

Both, (2.33) and (2.34) can be eliminated from the Lagrangian as discussed in the previous order.

2.4.2 Evaluation of 2nd case

We now turn to the analysis of terms with (iD_t) in the middle. Considering ③, we can use the commutator to commute iD_t to the left or right side.

$$\begin{aligned}(iD^i)(iD_t)(iD^j) &= \underbrace{\big[(iD^i),(iD_t)\big]}_{=[D_t,D^i]}(iD^j) + (iD_t)(iD^i)(iD^j) \\ &= iQeE^i(iD^j) + (iD_t)(iD^i)(iD^j)\end{aligned} \tag{2.35}$$

$$\begin{aligned}(iD^i)(iD_t)(iD^j) &= (iD^i)\underbrace{\big[(iD_t),(iD^j)\big]}_{=-[D_t,D^j]} + (iD^i)(iD^j)(iD_t) \\ &= -iQe(iD^i)E^j + (iD^i)(iD^j)(iD_t)\end{aligned} \tag{2.36}$$

The same can be applied to ④.

$$\begin{aligned}(iD^j)(iD_t)(iD^i) &= \underbrace{\big[(iD^j),(iD_t)\big]}_{=[D_t,D^j]}(iD^i) + (iD_t)(iD^j)(iD^i) \\ &= iQeE^j(iD^i) + (iD_t)(iD^j)(iD^i)\end{aligned} \tag{2.37}$$

$$\begin{aligned}(iD^j)(iD_t)(iD^i) &= (iD^j)\underbrace{\big[(iD_t),(iD^i)\big]}_{=-[D_t,D^i]} + (iD^j)(iD^i)(iD_t) \\ &= -iQe(iD^j)E^i + (iD^j)(iD^i)(iD_t)\end{aligned} \tag{2.38}$$

This means, that we can construct terms of the form $(iD^i)(iD_t)(iD^j)$ with $E^i(iD^j)$ and $(iD_t)(iD^i)(iD^j)$, of which we have already analysed the latter. We can thus use all combinations of the new basis elements $E^i(iD^j)$ and apply the same procedure as shown above. There are two hermitian and time-reversal invariant combinations.

$$\sigma^k\big((iD^i)E^j + E^j(iD^i)\big) \qquad \& \qquad i\big((iD^i)E^j - E^j(iD^i)\big) \tag{2.39}$$

Rotational invariance is achieved via contraction of all indices.

$$\begin{aligned}\epsilon^{ijk}\sigma^k\big((iD^i)E^j + E^j(iD^i)\big) &= \epsilon^{ijk}\sigma^k\big((iD^i)E^j - \underbrace{E^i(iD^j)}_{\epsilon^{ijk}=-\epsilon^{jik}}\big) \\ &= i\boldsymbol{\sigma}\cdot\big(\mathbf{D}\times\mathbf{E} - \mathbf{E}\times\mathbf{D}\big)\end{aligned} \tag{2.40}$$

$$\begin{aligned}\delta^{ij}i\big((iD^i)E^j - E^j(iD^i)\big) &= -\delta^{ij}\big[(D^i),E^j\big] \\ &= -\delta^{ij}\big(\underbrace{\big[\partial^i,E^j\big]}_{\partial^i E^j} - \underbrace{\big[eQA^i,E^j\big]}_{0}\big) = -\big[\boldsymbol{\partial}\cdot\mathbf{E}\big]\end{aligned} \tag{2.41}$$

Note, that the derivative acts only on $\mathbf{E}$ and not on the spinor ψ.

Results

We arrive at

$$\mathcal{L}_\psi = \psi^\dagger \bigg(iD_t + c_2 \frac{\mathbf{D}^2}{M} + c_3 Qe \frac{\boldsymbol{\sigma}\cdot\mathbf{B}}{M} + c_4 Qe \frac{i\boldsymbol{\sigma}\cdot\big(\mathbf{D}\times\mathbf{E} - \mathbf{E}\times\mathbf{D}\big)}{M^2} + c_5 Qe \frac{\big[\boldsymbol{\partial}\cdot\mathbf{E}\big]}{M^2} \bigg)\psi + \mathcal{O}(1/M^3) \tag{2.42}$$

Intermezzo - $\mathcal{L}_{\psi\chi}$

Mass dimension 3 can also be achieved by the introduction of another spinor field χ, that *can* be different from ψ. The interaction terms between the two spinor fields are included in $\mathcal{L}_{\psi\chi}$.

$$\mathcal{L}_{\psi\chi} = d_1 \frac{\psi^\dagger\psi\chi^\dagger\chi}{M^2} + d_2 \frac{\psi^\dagger\sigma^i\psi\chi^\dagger\sigma^i\chi}{M^2} + \mathcal{O}(1/M^4) \tag{2.43}$$

To show that the terms in $\mathcal{L}_{\psi\chi}$ are the only linear independent ones, we use the Fierz identities. In order to use the Fierz identities, some requirements have to be met. See [7] for a detailed discussion. The following set Γ spans the space of complex 2×2 matrices.

$$\Gamma = \big\{ I, \sigma^1, \sigma^2, \sigma^3 \big\} \tag{2.44}$$

Since $\sigma^i\sigma^j = \delta^{ij} I + i\epsilon^{ijk}\sigma^k$, the following orthogonality relation is fulfilled

$$Tr(\Gamma_A\Gamma_B) = n_\Gamma \delta^{AB} \tag{2.45}$$

With $n_\Gamma = 2$, since $(\sigma^i)^2 = I$ and $Tr(I) = 2$. We now apply the Fierz rearrangement formula in order to investigate the relations between products of four spinors with different ordering.

$${}_a\langle\Gamma^A\rangle_b\, {}_c\langle\Gamma^B\rangle_d = \sum_{C,D} \tfrac{1}{n_\Gamma^2} Tr(\Gamma^A\Gamma^C\Gamma^B\Gamma^D)\, {}_a\langle\Gamma^C\rangle_d\, {}_c\langle\Gamma^D\rangle_b \tag{2.46}$$

The following notation can be adapted to the case at hand

$${}_a\langle\Gamma^A\rangle_b\, {}_c\langle\Gamma^B\rangle_d \Leftrightarrow \big(\psi^\dagger\Gamma^A\psi\big)\big(\chi^\dagger\Gamma^B\chi\big) \tag{2.47}$$

Since rotational invariance is demanded, we only consider (2.46) with $\Gamma_A = \Gamma_B$. (2.46) then deliveres the following results

$$\begin{aligned} \underbrace{\psi^\dagger I\psi\chi^\dagger I\chi}_{\psi^\dagger\psi\chi^\dagger\chi} &= \tfrac{1}{2} \underbrace{\psi^\dagger I\chi\chi^\dagger I\psi}_{\psi^\dagger\chi\chi^\dagger\psi} + \tfrac{1}{2}\psi^\dagger\sigma^i\chi\chi^\dagger\sigma^i\psi \\ \psi^\dagger\sigma^j\psi\chi^\dagger\sigma^j\chi &= \tfrac{1}{2}\psi^\dagger\chi\chi^\dagger\psi + \tfrac{1}{2}\psi^\dagger\sigma^j\chi\chi^\dagger\sigma^j\psi - \tfrac{1}{2}\sum_{i\neq j}\psi^\dagger\sigma^i\chi\chi^\dagger\sigma^i\psi \end{aligned} \tag{2.48}$$

In the second equation, the implicit sum notation is violated, since there is no sum over j but a sum over i, this is only the case in this equation.
With (2.48) we have demonstrated, that (2.43) consists of linear independent terms and all other combinations can be related to with Fierz identities.
New contributions to $\mathcal{L}_{\psi\chi}$ arise at order $1/M^4$, since terms at order ${}^1/M^3$ introduce only iD_t that act on the spinors and can therefore be dropped from the Lagrangian.

Physical Interpretation of $\mathcal{L}_\psi$

When discussing pertubations to the hydrogen spectrum, the term $c_5 Qe\big[\boldsymbol{\partial}\cdot\mathbf{E}\big]/M^2$ corresponds to the known Darwin correction term (cf. [9]). In literature, c_5 is often denoted as c_D.

2.5 $1/M^3$-order construction

General selection

At this order, 4 covariant derivatives are required. Parity is conserved by either $(iD_t)^4$, four $(i\mathbf{D})$ or two $(i\mathbf{D})$ combined with two (iD_t). $(iD_t)^4$ can be treated analogously as in the previous orders. The other two cases are discussed separately.

2.5.1 Evaluation of the 1st case

All possible combinations of four (iD^i) operators correspond to $4! = 24$ permutations.

$$
\begin{array}{llll}
① & (iD^i)(iD^j)(iD^k)(iD^l) & ⑬ & (iD^l)(iD^k)(iD^j)(iD^i) \\
② & (iD^i)(iD^j)(iD^l)(iD^k) & ⑭ & (iD^k)(iD^l)(iD^j)(iD^i) \\
③ & (iD^i)(iD^k)(iD^j)(iD^l) & ⑮ & (iD^l)(iD^j)(iD^k)(iD^i) \\
④ & (iD^i)(iD^k)(iD^l)(iD^j) & ⑯ & (iD^j)(iD^l)(iD^k)(iD^i) \\
⑤ & (iD^i)(iD^l)(iD^j)(iD^k) & ⑰ & (iD^k)(iD^j)(iD^l)(iD^i) \\
⑥ & (iD^i)(iD^l)(iD^k)(iD^j) & ⑱ & (iD^j)(iD^k)(iD^l)(iD^i) \\
⑦ & (iD^j)(iD^i)(iD^k)(iD^l) & ⑲ & (iD^l)(iD^k)(iD^i)(iD^j) \\
⑧ & (iD^j)(iD^i)(iD^l)(iD^k) & ⑳ & (iD^k)(iD^l)(iD^i)(iD^j) \\
⑨ & (iD^j)(iD^k)(iD^i)(iD^l) & ㉑ & (iD^l)(iD^i)(iD^k)(iD^j) \\
⑩ & (iD^j)(iD^l)(iD^i)(iD^k) & ㉒ & (iD^k)(iD^i)(iD^l)(iD^j) \\
⑪ & (iD^k)(iD^i)(iD^j)(iD^l) & ㉓ & (iD^l)(iD^j)(iD^i)(iD^k) \\
⑫ & (iD^k)(iD^j)(iD^i)(iD^l) & ㉔ & (iD^l)(iD^i)(iD^j)(iD^k)
\end{array}
\tag{2.49}
$$

The inclusion of all permutations means, that for each combination its hermitian conjugate is also contained. Hermitian conjungate pairs are listed next to each other above. Only one pair needs to be thoroughly analysed, since they all act equally after contraction. We analyse the following two hermitian combinations.

$$\begin{aligned} &(iD^i)(iD^j)(iD^k)(iD^l) + (iD^l)(iD^k)(iD^j)(iD^i) \\ i\sigma^m\big(&(iD^i)(iD^j)(iD^k)(iD^l) - (iD^l)(iD^k)(iD^j)(iD^i)\big) \end{aligned} \tag{2.50}$$

Since we need to contract all four spacial indices, we now have several options available.

$$\begin{aligned}
&\delta^{ij}\delta^{kl}\big((iD^i)(iD^j)(iD^k)(iD^l) + (iD^l)(iD^k)(iD^j)(iD^i)\big) \\
&\quad =(i\mathbf{D})^2(i\mathbf{D})^2 + (i\mathbf{D})^2(i\mathbf{D})^2 = 2\mathbf{D}^4 \\
\\
&\epsilon^{klh}\epsilon^{hij}\big((iD^i)(iD^j)(iD^k)(iD^l) + (iD^l)(iD^k)(iD^j)(iD^i)\big) \\
&\quad = \epsilon^{klh}\epsilon^{hij}\big((iD^i)(iD^j)(iD^k)(iD^l) - (iD^l)(iD^k)(iD^i)(iD^j)\big) \\
&\quad = -\tfrac{1}{2}\epsilon^{klh}\epsilon^{hij}\big(\left[(D^i),(D^j)\right](iD^k)(iD^l) - (iD^l)(iD^k)\left[(D^i),(D^j)\right]\big) \\
&\quad = iQe\epsilon^{klh}\big(B^h(iD^k)(iD^l) - (iD^l)(iD^k)B^h\big) \\
&\quad = \epsilon^{klh} iQe\tfrac{1}{2}\big(B^h\left[(D^k),(D^l)\right] + \left[(D^k),(D^l)\right]B^h\big) \\
&\quad = -(Qe)^2\big(B^hB^h + B^hB^h\big) \\
&\quad = -2(Qe)^2\mathbf{B}^2 \\
\\
&\epsilon^{ihl}\epsilon^{hjk}\big((iD^i)(iD^j)(iD^k)(iD^l) + (iD^l)(iD^k)(iD^j)(iD^i)\big) \\
&\quad = \epsilon^{ihl}\epsilon^{hjk}\big((iD^i)(iD^j)(iD^k)(iD^l) - (iD^l)(iD^j)(iD^k)(iD^i)\big) \\
&\quad = -\tfrac{1}{2}\epsilon^{ihl}\epsilon^{hjk}\big((iD^i)\left[(D^j),(D^k)\right](iD^l) - (iD^l)\left[(D^j),(D^k)\right](iD^i)\big) \\
&\quad = iQe\epsilon^{ihl}\big((iD^i)B^h(iD^l) - (iD^l)B^h(iD^i)\big) \\
&\quad = iQe\epsilon^{ihl}\big((iD^i)B^h(iD^l) + (iD^i)B^h(iD^l)\big) \\
&\quad = -2iQe\mathbf{D}\cdot(\mathbf{B}\times\mathbf{D})
\end{aligned} \tag{2.51}$$

Note, that the difference between the last two contractions is the choice of indices for the ϵ-contractions and therefore the different placements of the commutators. For the other hermitian combination, we also need to pay close attention to all possible contractions.

$$\begin{aligned}
&\delta^{ij}\epsilon^{klm}i\sigma^m\big((iD^i)(iD^j)(iD^k)(iD^l)-(iD^l)(iD^k)(iD^j)(iD^i)\big)\\
&=\delta^{ij}\epsilon^{klm}i\sigma^m\big((iD^i)(iD^j)(iD^k)(iD^l)+(iD^k)(iD^l)(iD^j)(iD^i)\big)\\
&=-i\tfrac{1}{2}\delta^{ij}\epsilon^{klm}\sigma^m\big((iD^i)(iD^j)\left[(D^k),(D^l)\right]+\left[(D^k),(D^l)\right](iD^j)(iD^i)\big)\\
&=Qe\big\{\mathbf{D}^2,\boldsymbol{\sigma}\cdot\mathbf{B}\big\}
\end{aligned}$$

$$\begin{aligned}
&\delta^{il}\epsilon^{jkm}i\sigma^m\big((iD^i)(iD^j)(iD^k)(iD^l)-(iD^l)(iD^k)(iD^j)(iD^i)\big)\\
&=\delta^{il}\epsilon^{klm}i\sigma^m\big((iD^i)(iD^j)(iD^k)(iD^l)+(iD^l)(iD^j)(iD^k)(iD^i)\big)\\
&=-i\tfrac{1}{2}\delta^{il}\epsilon^{klm}\sigma^m\big((iD^i)\left[(D^j),(D^k)\right](iD^l)+(iD^l)\left[(D^j),(D^k)\right](iD^i)\big)\\
&=Qe\big\{\mathbf{D}^2,\boldsymbol{\sigma}\cdot\mathbf{B}\big\}\\
&=Qe\delta^{ij}\big((D^i)\boldsymbol{\sigma}\cdot\mathbf{B}(D^l)+(D^l)\boldsymbol{\sigma}\cdot\mathbf{B}(D^i)\big)\\
&=2Qe\big((D^i)\boldsymbol{\sigma}\cdot\mathbf{B}(D^i)\big)
\end{aligned} \tag{2.52}$$

$$\begin{aligned}
&\delta^{ml}\epsilon^{ijk}i\sigma^m\big((iD^i)(iD^j)(iD^k)(iD^l)-(iD^l)(iD^k)(iD^j)(iD^i)\big)\\
&=\delta^{ml}\epsilon^{ijk}i\sigma^m\big((iD^i)(iD^j)(iD^k)(iD^l)+(iD^l)(iD^j)(iD^k)(iD^i)\big)\\
&=-i\tfrac{1}{2}\delta^{ml}\epsilon^{ijk}\sigma^m\big((iD^i)\left[(D^j),(D^k)\right](iD^l)+(iD^l)\left[(D^j),(D^k)\right](iD^i)\big)\\
&=Qe\big((\mathbf{D}\cdot\mathbf{B})(\boldsymbol{\sigma}\cdot\mathbf{D})+(\boldsymbol{\sigma}\cdot\mathbf{D})(\mathbf{D}\cdot\mathbf{B})\big)
\end{aligned}$$

2.5.2 Evaluation of the 2nd case

When combining two (iD^i) and two (iD_t), the number of combinations is $\frac{4!}{2}=12$, because the two (iD_t) are identical.

$$\begin{array}{llll}
① & (iD_t)(iD_t)(iD^i)(iD^j) & ⑦ & (iD^i)(iD_t)(iD_t)(iD^j)\\
② & (iD_t)(iD_t)(iD^j)(iD^i) & ⑧ & (iD^j)(iD_t)(iD_t)(iD^i)\\
③ & (iD_t)(iD^i)(iD_t)(iD^j) & ⑨ & (iD^i)(iD_t)(iD^j)(iD_t)\\
④ & (iD_t)(iD^j)(iD_t)(iD^i) & ⑩ & (iD^j)(iD_t)(iD^i)(iD_t)\\
⑤ & (iD_t)(iD^i)(iD^j)(iD_t) & ⑪ & (iD^i)(iD^j)(iD_t)(iD_t)\\
⑥ & (iD_t)(iD^j)(iD^i)(iD_t) & ⑫ & (iD^j)(iD^i)(iD_t)(iD_t)
\end{array} \tag{2.53}$$

The analysis of these operators requires a more systematic approach. It is useful to regroup the permutations into different subcases.

$$Ⓐ\begin{cases}(iD_t)(iD_t)(iD^i)(iD^j) & (iD^j)(iD^i)(iD_t)(iD_t)\\ (iD_t)(iD_t)(iD^j)(iD^i) & (iD^i)(iD^j)(iD_t)(iD_t)\end{cases} \tag{2.54}$$

$$
\begin{array}{llll}
\text{Ⓑ} & (iD_t)(iD^i)(iD^j)(iD_t) & (iD_t)(iD^j)(iD^i)(iD_t) & (2.55)\\
\text{Ⓒ}\left\{\begin{array}{l}\\ \\ \end{array}\right. & \begin{array}{l}(iD_t)(iD^i)(iD_t)(iD^j)\\ (iD_t)(iD^j)(iD_t)(iD^i)\end{array} & \begin{array}{l}(iD^j)(iD_t)(iD^i)(iD_t)\\ (iD^i)(iD_t)(iD^j)(iD_t)\end{array} & (2.56)\\
\text{Ⓓ} & (iD^i)(iD_t)(iD_t)(iD^j) & (iD^j)(iD_t)(iD_t)(iD^i) & (2.57)
\end{array}
$$

Subcase Ⓐ

At first, we analyse the combinations, in which two (iD_t) are at the left or at the right. Following the steps practiced above, we get

$$
\begin{aligned}
&\delta^{ij}\big((iD_t)(iD_t)(iD^i)(iD^j) + (iD^j)(iD^i)(iD_t)(iD_t)\big)\\
=&(iD_t)^2(i\mathbf{D})^2 + (i\mathbf{D})^2(iD_t)^2\\
\\
&\epsilon^{ijk}i\sigma^k\big((iD_t)(iD_t)(iD^i)(iD^j) - (iD^j)(iD^i)(iD_t)(iD_t)\big)\\
=&\epsilon^{ijk}i\sigma^k\big((iD_t)(iD_t)(iD^i)(iD^j) + (iD^i)(iD^j)(iD_t)(iD_t)\big)\\
=&-\tfrac{1}{2}\epsilon^{ijk}i\sigma^k\big((iD_t)(iD_t)\left[(D^i),(D^j)\right] + \left[(D^i),(D^j)\right](iD_t)(iD_t)\big)\\
=&Qe\big((iD_t)^2\boldsymbol{\sigma}\cdot\mathbf{B} + \mathbf{B}\cdot\boldsymbol{\sigma}(iD_t)^2\big)
\end{aligned}
\tag{2.58}
$$

As discussed in order 1/M, these terms can be removed from the Lagrangian.

Subcase Ⓑ

Next, we consider the combinations that have one iD_t at the left and the right. We find the following results

$$
\begin{aligned}
&\delta^{ij}\big((iD_t)(iD^i)(iD^j)(iD_t) + (iD_t)(iD^j)(iD^i)(iD_t)\big)\\
=&2(iD_t)(i\mathbf{D})^2(iD_t)\\
\\
&\epsilon^{ijk}i\sigma^k\big((iD_t)(iD^i)(iD^j)(iD_t) - (iD_t)(iD^j)(iD^i)(iD_t)\big)\\
=&\epsilon^{ijk}i\sigma^k\big((iD_t)(iD^i)(iD^j)(iD_t) + (iD_t)(iD^i)(iD^j)(iD_t)\big)\\
=&-\tfrac{1}{2}\epsilon^{ijk}i\sigma^k\big((iD_t)\left[(D^i),(D^j)\right](iD_t) + (iD_t)\left[(D^i),(D^j)\right](iD_t)\big)\\
=&Qe\big((iD_t)\boldsymbol{\sigma}\cdot\mathbf{B}(iD_t) + (iD_t)\mathbf{B}\cdot\boldsymbol{\sigma}(iD_t)\big)
\end{aligned}
\tag{2.59}
$$

These terms can also be removed from the Lagrangian.

Subcase Ⓒ

The combinations with one iD_t in between two iD^i, can be rewritten into terms containing a commutator and residual terms in which iD_t is commuted to the outside, analogous to chapter

(2.4.2).

$$\begin{aligned}(iD_t)(iD^i)(iD_t)(iD^j) &= (iD_t)(iD^i)\left[(iD_t),(iD^j)\right] + (iD_t)(iD^i)(iD^j)(iD_t) \\ &= \tfrac{Qe}{i}(iD_t)(iD^i)E^j + (iD_t)(iD^i)(iD^j)(iD_t)\end{aligned} \tag{2.60}$$

$$\begin{aligned}(iD_t)(iD^j)(iD_t)(iD^i) &= (iD_t)(iD^j)\left[(iD_t),(iD^i)\right] + (iD_t)(iD^j)(iD^i)(iD_t) \\ &= \tfrac{Qe}{i}(iD_t)(iD^j)E^i + (iD_t)(iD^j)(iD^i)(iD_t)\end{aligned} \tag{2.61}$$

$$\begin{aligned}(iD^j)(iD_t)(iD^i)(iD_t) &= \left[(iD^j),(iD_t)\right](iD^i)(iD_t) + (iD_t)(iD^j)(iD^i)(iD_t) \\ &= \tfrac{Qe}{-i}E^j(iD^i)(iD_t) + (iD_t)(iD^j)(iD^i)(iD_t)\end{aligned} \tag{2.62}$$

$$\begin{aligned}(iD^i)(iD_t)(iD^j)(iD_t) &= \left[(iD^i),(iD_t)\right](iD^j)(iD_t) + (iD_t)(iD^i)(iD^j)(iD_t) \\ &= \tfrac{Qe}{-i}E^i(iD^j)(iD_t) + (iD_t)(iD^i)(iD^j)(iD_t)\end{aligned} \tag{2.63}$$

Since the residual terms are of the form which has already been discussed in the previous subcase, they can be disregarded. This leaves the terms containing electric field.

$$\begin{matrix}\tfrac{Qe}{i}(iD_t)(iD^i)E^j & \qquad \tfrac{Qe}{-i}E^j(iD^i)(iD_t) \\ \tfrac{Qe}{i}(iD_t)(iD^j)E^i & \qquad \tfrac{Qe}{-i}E^i(iD^j)(iD_t)\end{matrix} \tag{2.64}$$

When considering time reversal invariance in the following combinations, it is crucial to take into account, that the electric field picks up a minus sign. The first combination below is therefore time reversal invariant since E^i comes with a prefactor containing i. In the second combination σ^k has to be introduced to ensure said invariance.

$$\begin{aligned}&\delta^{ij}\left(\tfrac{Qe}{i}(iD_t)(iD^i)E^j + \tfrac{Qe}{-i}E^j(iD^i)(iD_t)\right) \\ &=\delta^{ij} iQe\big(E^j(iD^i)(iD_t) - (iD_t)(iD^i)E^j\big) \\ &=Qe\big((iD_t)(\mathbf{D}\cdot\mathbf{E}) - (\mathbf{E}\cdot\mathbf{D})(iD_t)\big) \\ \\ &\epsilon^{ijk}\sigma^k i\left(\tfrac{Qe}{i}(iD_t)(iD^i)E^j + \tfrac{Qe}{i}E^j(iD^i)(iD_t)\right) \\ &=Qe\epsilon^{ijk}\sigma^k((iD_t)(iD^i)E^j + E^j(iD^i)(iD_t)) \\ &=-Qe\big((iD_t)\boldsymbol{\sigma}\cdot(\mathbf{D}\times\mathbf{E}) + \boldsymbol{\sigma}\cdot(\mathbf{E}\times\mathbf{D})(iD_t)\big)\end{aligned} \tag{2.65}$$

Once more, these terms can be removed from the Lagrangian.

Subcase Ⓓ

For these combinations, we can introduce commutators in the same way as we did above.

$$\begin{aligned}(iD^i)(iD_t)(iD_t)(iD^j) &= \left[(iD^i),(iD_t)\right](iD_t)(iD^j) + (iD_t)(iD^i)(iD_t)(iD^j) \\ &= \tfrac{Qe}{-i}E^i(iD_t)(iD^j) + (iD_t)(iD^i)(iD_t)(iD^j)\end{aligned} \tag{2.66}$$

$$\begin{aligned}(iD^i)(iD_t)(iD_t)(iD^j) &= (iD^i)(iD_t)\left[(iD_t),(iD^j)\right] + (iD^i)(iD_t)(iD^j)(iD_t)\\ &= \tfrac{Qe}{i}(iD^i)(iD_t)E^j + (iD^i)(iD_t)(iD^j)(iD_t)\end{aligned} \tag{2.67}$$

$$\begin{aligned}(iD^j)(iD_t)(iD_t)(iD^i) &= \left[(iD^j),(iD_t)\right](iD_t)(iD^i) + (iD_t)(iD^j)(iD_t)(iD^i)\\ &= \tfrac{Qe}{-i}E^j(iD_t)(iD^i) + (iD_t)(iD^j)(iD_t)(iD^i)\end{aligned} \tag{2.68}$$

$$\begin{aligned}(iD^j)(iD_t)(iD_t)(iD^i) &= (iD^j)(iD_t)\left[(iD_t),(iD^i)\right] + (iD^j)(iD_t)(iD^i)(iD_t)\\ &= \tfrac{Qe}{i}(iD^j)(iD_t)E^i + (iD^j)(iD_t)(iD^i)(iD_t)\end{aligned} \tag{2.69}$$

The residual combinations can be disregarded again, since they match the combinations analysed in the previous subcase.

$$\begin{matrix}\tfrac{Qe}{-i}E^i(iD_t)(iD^j) & \qquad \tfrac{Qe}{i}(iD^j)(iD_t)E^i\\ \tfrac{Qe}{i}(iD^i)(iD_t)E^j & \qquad \tfrac{Qe}{-i}E^j(iD_t)(iD^i)\end{matrix} \tag{2.70}$$

The following valid terms arise

$$\begin{aligned}&\delta^{ij}\big(\tfrac{Qe}{-i}E^i(iD_t)(iD^j) + \tfrac{Qe}{i}(iD^j)(iD_t)E^i\big)\\ &= iQe\delta^{ij}\big(E^i(iD_t)(iD^j) + (iD^j)(iD_t)E^i\big)\\ &= iQe\delta^{ij}\big(E^i\left[(iD_t),(iD^j)\right] + E^i(iD^j)(iD_t) - \left[(iD^j),(iD_t)\right]E^i - (iD_t)(iD^j)E^i\big)\\ &= iQe\delta^{ij}\big(E^i\tfrac{Qe}{i}E^j + E^i(iD^j)(iD_t) - (\tfrac{Qe}{-i}E^j)E^i - (iD_t)(iD^j)E^i\big)\\ &= iQe\delta^{ij}\big(\tfrac{Qe}{i}E^iE^j + \tfrac{Qe}{i}E^jE^i + E^i(iD^j)(iD_t) - (iD_t)(iD^j)E^i\big)\\ &= 2(Qe)^2\mathbf{E}^2 - \big((\mathbf{E}\cdot\mathbf{D})(iD_t) - (iD_t)(\mathbf{D}\cdot\mathbf{E})\big)\end{aligned}$$

$$\tag{2.71}$$

$$\begin{aligned}&\epsilon^{ijk}\sigma^k i\big(\tfrac{Qe}{-i}E^i(iD_t)(iD^j) - \tfrac{Qe}{i}(iD^j)(iD_t)E^i\big)\\ &= -Qe\epsilon^{ijk}\sigma^k\big(E^i(iD_t)(iD^j) + (iD^j)(iD_t)E^i\big)\\ &= -Qe\epsilon^{ijk}\sigma^k\big(E^i\left[(iD_t),(iD^j)\right] + E^i(iD^j)(iD_t)\left[(iD^j),(iD_t)\right]E^i + (iD_t)(iD^j)E^i\big)\\ &= -Qe\epsilon^{ijk}\sigma^k\big(E^i\tfrac{Qe}{i}E^j + E^i(iD^j)(iD_t)\tfrac{Qe}{-i}E^jE^i + (iD_t)(iD^j)E^i\big)\\ &= -2\tfrac{(Qe)^2}{i}\boldsymbol{\sigma}\cdot\underbrace{(\mathbf{E}\times\mathbf{E})}_{0} + Qe\big((iD_t)\boldsymbol{\sigma}\cdot(\mathbf{D}\times\mathbf{E}) + \boldsymbol{\sigma}(\mathbf{E}\times\mathbf{D})(iD_t)\big)\end{aligned}$$

The term $\propto \mathbf{E}^2$ is the only one that cannot be removed from the Lagrangian.

2.5.3 Results

For the final Lagrangian up to order $1/M^3$, we have

$$
\begin{aligned}
\mathcal{L}_\psi = \psi^\dagger \Bigg(& iD_t + c_2 \frac{\mathbf{D}^2}{M} + c_3 Qe \frac{\boldsymbol{\sigma}\cdot\mathbf{B}}{M} + c_4 Qe \frac{i\boldsymbol{\sigma}\cdot\big(\mathbf{D}\times\mathbf{E}-\mathbf{E}\times\mathbf{D}\big)}{M^2} + c_5 Qe \frac{\big[\boldsymbol{\partial}\cdot\mathbf{E}\big]}{M^2} \\
& + c_6 \frac{\mathbf{D}^4}{M^3} + c_7 (Qe)^2 \frac{\mathbf{B}^2}{M^3} + c_8 iQe \frac{\big(\mathbf{D}\cdot(\mathbf{B}\times\mathbf{D})\big)}{M^3} + c_9 Qe \frac{\{\mathbf{D}^2, \boldsymbol{\sigma}\cdot\mathbf{B}\}}{M^3} \\
& + c_{10} Qe \frac{\big((D^i)\boldsymbol{\sigma}\cdot\mathbf{B}(D^i)\big)}{M^3} + c_{11} Qe \frac{\big((\mathbf{D}\cdot\mathbf{B})(\boldsymbol{\sigma}\cdot\mathbf{D}) + (\boldsymbol{\sigma}\cdot\mathbf{D})(\mathbf{D}\cdot\mathbf{B})\big)}{M^3} \\
& + c_{12} (Qe)^2 \frac{\mathbf{E}^2}{M^3} \Bigg) \psi + \mathcal{O}(1/M^4)
\end{aligned}
\tag{2.72}
$$

This result is confirmes the Lagrangian determined in [6].

Physical Interpretation of $\mathcal{L}_\psi$

The term $c_6\mathbf{D}^4/M^3$ is also known from the discussion of the hydrogen spectrum. It corresponds to the relativistic correction $(\mathbf{p})^4$ (cf.[9]).

3 Conclusions

We have investigated the symmetries of $\mathcal{L}_{QED}$ and described the effective theory approach to NRQED. In order to construct the NRQED Lagrangian we have introduced a procedure, which relies on the conservation of demanded symmetries and explicitly shown the calculations of all terms. The obtained result is backed by [6].
This work leaves out some very interesting aspects of NRQED. The coefficients for all terms in the Lagrangian are yet to be determined. This is done by comparing results of high precision QED calculations with NRQED calculations applied to the same processes (matching calculations) (cf.[3]). Interesting connections between NRQED and other EFT's like HQET (heavy quark effective field theory) have also not been discussed (cf.[9],[6]).

Acknowledgements

I thank both, my supervisor Prof. Dominik Stöckinger and Dr. Hyejung Stöckinger-Kim, for devoting generous ammounts of time into consultations and helpful discussions. I am also grateful to the IKTP-theory group, in particular I thank Sophie Kollatzsch and Javier Vargas for their support.

4 References

[1] Christopher Arzt. "Reduced effective Lagrangians". In: *Phys. Lett. B* 342 (1995), pp. 189–195. DOI: 10.1016/0370-2693(94)01419-D. arXiv: hep-ph/9304230.

[2] Hans A. Bethe and Edwin E. Salpeter. *Quantum Mechanics of One- and Two-Electron Atoms*. 1957. DOI: 10.1007/978-3-662-12869-5.

[3] C.P. Burgess. "An Introduction to Effective Field Theory". In: *Annual Review of Nuclear and Particle Science* 57.1 (Nov. 2007), pp. 329–362. DOI: 10.1146/annurev.nucl.56.080805.140508. URL: https://doi.org/10.1146%2Fannurev.nucl.56.080805.140508.

[4] W. E. Caswell and G. P. Lepage. "Effective Lagrangians for Bound State Problems in QED, QCD, and Other Field Theories". In: *Phys. Lett. B* 167 (1986), pp. 437–442. DOI: 10.1016/0370-2693(86)91297-9.

[5] H. Georgi. "Effective field theory". In: *Ann. Rev. Nucl. Part. Sci.* 43 (1993), pp. 209–252. DOI: 10.1146/annurev.ns.43.120193.001233.

[6] Richard J. Hill et al. "NRQED Lagrangian at order $1/M^4$". In: *Phys. Rev. D* 87 (2013), p. 053017. DOI: 10.1103/PhysRevD.87.053017. arXiv: 1212.4508 [hep-ph].

[7] Sophie Kollatzsch. "Fun with Fierz (master thesis)". In: (2022 (in preparation)).

[8] Aneesh V. Manohar. "The HQET / NRQCD Lagrangian to order alpha / m-3". In: *Phys. Rev. D* 56 (1997), pp. 230–237. DOI: 10.1103/PhysRevD.56.230. arXiv: hep-ph/9701294.

[9] Gil Paz. "An Introduction to NRQED". In: *Mod. Phys. Lett. A* 30.26 (2015), p. 1550128. DOI: 10.1142/S021773231550128X. arXiv: 1503.07216 [hep-ph].

Erklärung

Hiermit erkläre ich, dass ich diese Arbeit im Rahmen der Betreuung am Institut für Kern- und Teilchenphysik ohne unzulässige Hilfe Dritter verfasst und alle Quellen als solche gekennzeichnet habe.

Declaration

I hereby declare that I have written this thesis within the framework of my supervision at the Institut for Nuclear and Particle Physics without unauthorized help of third parties and that I have marked all sources as such.

Jonas Scheibler
Dresden, 27.06.2022