

# DUALITY IN MONOIDAL CATEGORIES

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**ABSTRACT.** We compare closed and rigid monoidal categories. Closedness is defined by the tensor product having a right adjoint: the internal-hom functor. Rigidity on the other hand generalises the concept of duals in the sense of finite-dimensional vector spaces. A consequence of these axioms is that the internal-hom functor is implemented by tensoring with the respective duals. This raises the question: can one decide whether a closed monoidal category is rigid, simply by verifying that the internal-hom is tensor-representable? At the *Research School on Bicatogories, Categorification and Quantum Theory*, Heunen suggested that this is not the case. In this note, we will prove his claim by constructing an explicit counterexample.

## 1. INTRODUCTION: CLOSED AND RIGID MONOIDAL CATEGORIES

Monoidal categories are a ubiquitous tool in mathematics, physics, and computer science [BS11]. Often, they come equipped with additional structures, such as braidings or twists, see the previously cited article. In the following, we will compare two notions of duality for monoidal categories: closedness and rigidity.

We assume the reader's familiarity with standard concepts of category theory; in particular, adjunctions and monoidal categories as discussed for example in [ML98] and [EGNO15]. As rigidity and closedness are preserved, as well as reflected, by monoidal equivalences, see [Lin78], we restrict ourselves to the strict setting. As such, let  $\mathcal{C}$  be a strict monoidal category with  $- \otimes -: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$  as its *tensor product* and  $1 \in \mathcal{C}$  as its *unit*.

The category  $\mathcal{C}$  is called (*right*) *closed* if it admits a functor  $[-, -]: \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{C}$ , the (*right*) *internal-hom*, such that for all objects  $x \in \mathcal{C}$  there exists an adjunction

$$(1.1) \quad - \otimes x: \mathcal{C} \rightleftarrows \mathcal{C} : [x, -].$$

On the other hand,  $\mathcal{C}$  is said to be (*right*) *rigid* if every object  $x \in \mathcal{C}$  has a (*right*) *dual*  $x^*$  equipped with an *evaluation* and *coevaluation* morphism

$$\text{ev}_x: x^* \otimes x \rightarrow 1 \quad \text{and} \quad \text{coev}_x: 1 \rightarrow x \otimes x^*,$$

subject to the *snake identities*

$$(1.2) \quad \text{id}_x = (\text{id}_x \otimes \text{ev}_x)(\text{coev}_x \otimes \text{id}_x) \quad \text{and} \quad \text{id}_{x^*} = (\text{ev}_x \otimes \text{id}_{x^*})(\text{id}_{x^*} \otimes \text{coev}_x).$$

Rigid monoidal categories are closed, see for example Section 2.10 of [EGNO15].

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**Lemma 1.1.** *If  $\mathcal{C}$  is rigid, the internal-hom is implemented by the adjunction*

$$(1.3) \quad - \otimes x : \mathcal{C} \rightleftarrows \mathcal{C} : - \otimes x^* \quad \text{for all } x \in \mathcal{C}.$$

The main concern of this note is to show that the converse of the above result does not hold. That is, we will prove that the internal-hom being given by tensoring with the dual of an object does not imply rigidity.

In order to elucidate the underlying problem, let us assume that we are given objects  $x, y \in \mathcal{C}$  such that  $- \otimes x : \mathcal{C} \rightleftarrows \mathcal{C} : - \otimes y$ . The unit and counit of the adjunction provide us with natural candidates for the coevaluation and evaluation morphisms:

$$\text{coev}_x := \eta_1 : 1 \longrightarrow x \otimes y \quad \text{and} \quad \text{ev}_x := \varepsilon_1 : y \otimes x \longrightarrow 1.$$

The triangle identities of this adjunction evaluated at the monoidal unit state that  $\text{id}_x = \varepsilon_x \circ (\eta_1 \otimes x)$  and  $\text{id}_y = (\varepsilon_1 \otimes x) \circ \eta_y$ . However, since we a priori do not know whether  $\varepsilon_x \cong \text{id}_x \otimes \varepsilon_1$  and  $\eta_y \cong \text{id}_y \otimes \eta_1$ , the snake identities do not necessarily follow.

## 2. A COUNTEREXAMPLE

First, we define a strict monoidal category  $(\mathcal{D}, \oplus, \mathbb{0})$  in terms of generators and relations. For details of this type of construction we refer the reader to [Kas98, Chapter XII]. The objects of  $\mathcal{D}$  are the natural numbers  $\mathbb{N}_0$  with addition as the tensor product and  $\mathbb{0} \in \mathbb{N}_0$  as monoidal unit.<sup>1</sup> Its arrows are tensor products and compositions of identities, and the *generating morphisms*

$$(2.1) \quad \eta_{m,n} : m \longrightarrow m \oplus n \oplus n, \quad \varepsilon_{m,n} : m \oplus n \oplus n \longrightarrow m, \quad n, m \in \mathbb{N}_0, n \geq 1.$$

These are for all  $i, j, k, l, n \in \mathbb{N}_0$  with  $n, k \geq 1$  subject to the relations

$$(2.2) \quad \eta_{i+j+2k+l,n}(\text{id}_i \oplus \eta_{j,k} \oplus \text{id}_l) = ((\text{id}_i \oplus \eta_{j,k} \oplus \text{id}_l) \oplus \text{id}_{2n})\eta_{i+j+l,n},$$

$$(2.3) \quad \eta_{i+j+l,n}(\text{id}_i \oplus \varepsilon_{j,k} \oplus \text{id}_l) = ((\text{id}_i \oplus \varepsilon_{j,k} \oplus \text{id}_l) \oplus \text{id}_{2n})\eta_{i+j+2k+l,n},$$

$$(2.4) \quad \varepsilon_{i+j+2k+l,n}((\text{id}_i \oplus \eta_{j,k} \oplus \text{id}_l) \oplus \text{id}_{2n}) = (\text{id}_i \oplus \eta_{j,k} \oplus \text{id}_l)\varepsilon_{i+j+l,n},$$

$$(2.5) \quad \varepsilon_{i+j+l,n}((\text{id}_i \oplus \varepsilon_{j,k} \oplus \text{id}_l) \oplus \text{id}_{2n}) = (\text{id}_i \oplus \varepsilon_{j,k} \oplus \text{id}_l)\varepsilon_{i+j+2k+l,n}.$$

These relations are tailored to implement for any  $n \in \mathbb{N}$  natural transformations

$$\eta_{x,n} : x \longrightarrow x \oplus (n \oplus n), \quad \varepsilon_{x,n} : x \oplus (n \oplus n) \longrightarrow x, \quad \text{for all } x \in \mathcal{D}.$$

For example, let  $i, j, k, l, n$  be as above. Further, define  $x := i \oplus j \oplus l$ ,  $y := i \oplus j \oplus 2k \oplus j$ , and  $f := \text{id}_i \oplus \eta_{j,k} \oplus \text{id}_j : x \longrightarrow y$ . In this setting, Equation (2.2) translates to the usual naturality condition, expressed by the commutativity of the following diagram:

$$\begin{array}{ccc} x & \xrightarrow{f} & y \\ \eta_{x,n} \downarrow & & \downarrow \eta_{y,n} \\ x \oplus (n \oplus n) & \xrightarrow{f \oplus (\text{id}_n \oplus \text{id}_n)} & y \oplus (n \oplus n) \end{array}$$

By quotienting out the triangle identities, we obtain a category  $\mathcal{C}$  in which tensoring with any fixed object gives rise to a self-adjoint functor. Explained in more detail,

<sup>1</sup>A strict monoidal category whose monoid of objects is (isomorphic to) the natural numbers is also called a *PRO*.

the monoidal category  $(\mathcal{C}, \oplus, \mathbb{0})$  has the same objects and generating morphisms as  $\mathcal{D}$  and the same identities hold. In addition, for any  $i, n \in \mathbb{N}_0$  with  $n \geq 1$  we require

$$(2.6) \quad \varepsilon_{i+n,n}(\eta_{i,n} \oplus \text{id}_n) = \text{id}_{i+n}, \quad \text{and} \quad (\varepsilon_{i,n} \oplus \text{id}_n)(\eta_{i+n,n}) = \text{id}_{i+n}.$$

The next result succinctly summarises the observations made so far concerning the internal-hom of  $\mathcal{C}$ .

**Lemma 2.1.** *The category  $\mathcal{C}$  is closed monoidal; its internal-hom functor is given by*

$$(2.7) \quad - \otimes n: \mathcal{C} \rightleftarrows \mathcal{C} : - \otimes n, \quad \text{for all } n \in \mathcal{C}.$$

In order to analyse the morphisms in  $\mathcal{C}$  and show that it is not rigid monoidal, we will rely on two tools. The first is the *length* of an arrow  $f \in \mathcal{C}(n, m)$ . It is defined as the minimal number of generating morphisms needed to present  $f$ . The second tool will be given by invariants for morphisms in  $\mathcal{C}$  arising from functors into the category  $\mathbf{vect}_k$  of finite-dimensional vector spaces over a field  $k$ . Note that for any such vector space  $V$  there exists an isomorphism  $\phi: V \rightarrow V^*$  to its dual  $V^*$ . The morphisms

$$\overline{\text{coev}}_V := (\text{id}_V \otimes \phi^{-1}) \text{coev}_V: k \rightarrow V \otimes V, \quad \overline{\text{ev}}_V := (\phi \otimes \text{id}_V) \text{ev}_V: V \otimes V \rightarrow k$$

satisfy the snake identities, turning  $V$  into its own dual. The next theorem is an application of [Kas98, Proposition XII.1.4].

**Theorem 2.2.** *For any  $V \in \mathbf{vect}_k$  and isomorphism  $\phi: V \rightarrow V^*$  there exists a strong monoidal functor  $F_{(V,\phi)}: \mathcal{C} \rightarrow \mathbf{vect}_k$  such that for all  $n, m \in \mathbb{N}_0$  with  $n \geq 1$*

$$F_{(V,\phi)}(\eta_{m,n}) = \text{id}_m \otimes \overline{\text{coev}}_{V^{\otimes n}} \quad \text{and} \quad F_{(V,\phi)}(\varepsilon_{m,n}) = \text{id}_m \otimes \overline{\text{ev}}_{V^{\otimes n}}.$$

To prove the statement, one has to show that relations in  $\mathcal{C}$  are mapped to relations in  $\mathbf{vect}_k$ . This amounts to verifying that  $V$  is its own right dual, in the rigid sense.

**Corollary 2.3.** *The category  $\mathcal{C}$  is skeletal. Furthermore, for any  $g \in \mathcal{C}(m, n)$  the following arrows cannot be isomorphisms:*

$$(2.8) \quad (\text{id}_{j_1} \otimes \eta_{l,m} \otimes \text{id}_{j_2})g, \quad g(\text{id}_{i_1} \otimes \varepsilon_{j,k} \otimes \text{id}_{i_2}).$$

*Proof.* Let  $V \in \mathbf{vect}_k$  of dimension at least 2 and fix an isomorphism  $\phi: V \rightarrow V^*$ . For any  $n, m \in \mathcal{C}$  we have  $F_{(V,\phi)}(n) = V^{\otimes n} = V^{\otimes m} = F_{(V,\phi)}(m)$  if and only if  $n = m$ . Thus,  $\mathcal{C}$  must be skeletal.

Now suppose that  $g \in \mathcal{C}(m, n)$  and consider the morphism  $f := g(\text{id}_{i_1} \otimes \varepsilon_{j,k} \otimes \text{id}_{i_2})$ . Applying  $F_{(V,\phi)}$  to  $f$ , we get  $F_{(V,\phi)}(f) = F_{(V,\phi)}(g)F_{(V,\phi)}(\text{id}_{i_1} \otimes \varepsilon_{j,k} \otimes \text{id}_{i_2})$ . However, due to the difference in the dimensions of its source and target,  $F_{(V,\phi)}(\text{id}_{i_1} \otimes \varepsilon_{j,k} \otimes \text{id}_{i_2})$  must have a non-trivial kernel and thus  $f$  cannot be an isomorphism.

A similar argument involving the cokernel proves that  $(\text{id}_{j_1} \otimes \eta_{l,m} \otimes \text{id}_{j_2})g$  is not invertible.  $\square$

We can now state and prove our main theorem.

**Theorem 2.4.** *The category  $\mathcal{C}$  is not rigid.*

*Proof.* We assume that  $1 \in \mathcal{C}$  admits a right dual. Due to the uniqueness of adjoints, there exist isomorphisms  $\vartheta: 2n \rightarrow 2n$  and  $\theta: n \rightarrow n$  such that the evaluation and coevaluation morphisms are given by

$$\text{coev}_1 := \vartheta \eta_{0,1}: 0 \rightarrow 2, \quad \text{ev}_1 := \varepsilon_{0,1}(\theta \otimes \text{id}_n): 2 \rightarrow 0.$$

We now want to consider the following subset of homomorphisms of  $\mathcal{D}$ :

$$S := \left\{ (\text{id}_1 \otimes \varepsilon_{0,1}) \phi (\eta_{0,1} \otimes \text{id}_1) \in \mathcal{D}(1, 1) \mid \phi \in \mathcal{D}(3, 3) \text{ such that } \pi(\phi) \text{ is invertible} \right\},$$

where  $\pi: \mathcal{D} \rightarrow \mathcal{C}$  is the ‘projection’ functor. By construction, the morphism  $s = (\text{id}_1 \otimes \text{ev}_1)(\text{coev}_1 \otimes \text{id}_1)$  corresponding to one of the two snake-identities is an element of  $S$ . Furthermore, every element of  $S$  has length at least two.<sup>2</sup> Thus, by proving that  $S$  is closed under the relations arising from Equation (2.6), it follows that  $\pi(s) \neq \text{id}_1$ , which concludes the proof.

To that end, let us consider an element  $x = (\text{id}_1 \otimes \varepsilon_{0,1}) \phi (\eta_{0,1} \otimes \text{id}_1) \in S$ . There are two types of ‘moves’ we have to study. First, suppose we expand an identity into one of the triangle-morphisms. This equates to either pre- or postcomposing  $\phi$  with an arrow  $\psi \in \mathcal{D}(3, 3)$  which projects onto an isomorphism in  $\mathcal{C}$ , leading to another element in  $S$ . Second, a triangle-morphism might be contracted to an identity. A priori, there are three ways in which this might occur

$$(2.9) \quad x = (\text{id}_1 \otimes \varepsilon_{0,1}) \varepsilon_{1,1} (\eta_{0,1} \otimes \text{id}_1), \quad \text{where } \phi = \phi' \varepsilon_{1,1}, \text{ or}$$

$$(2.10) \quad x = (\text{id}_1 \otimes \varepsilon_{0,1}) \eta_{1,1} \phi'' (\eta_{0,1} \otimes \text{id}_1), \quad \text{with } \phi = \eta_{1,1} \phi'', \text{ or}$$

$$(2.11) \quad x = (\text{id}_1 \otimes \varepsilon_{0,1}) \phi_2 t \phi_1 (\eta_{0,1} \otimes \text{id}_1) \quad \text{with } \phi = \phi_2 t \phi_1 \text{ and } \pi(t) = \text{id}.$$

Due to Corollary 2.3, neither  $\pi(\phi')\pi(\varepsilon_{1,1})$  nor  $\pi(\eta_{1,1})\pi(\phi'')$  are isomorphisms, contradicting Cases (2.9) and (2.10). Now assume  $x = (\text{id}_1 \otimes \varepsilon_{0,1}) \phi_2 t \phi_1 (\eta_{0,1} \otimes \text{id}_1)$  and  $\phi = \phi_2 t \phi_1$ . Using the functoriality of  $\pi: \mathcal{D} \rightarrow \mathcal{C}$ , we get

$$\pi(\phi) = \pi(\phi_2 t \phi_1) = \pi(\phi_2) \pi(t) \pi(\phi_1) = \pi(\phi_2) \pi(\phi_1) = \pi(\phi_2 \phi_1).$$

Since  $\pi(\phi_2 \phi_1)$  is an isomorphism,  $(\text{id}_1 \otimes \varepsilon_{0,1}) \phi_2 \phi_1 (\eta_{0,1} \otimes \text{id}_1)$  is an element of  $S$ .  $\square$

### 3. TENSOR-REPRESENTABILITY AND GROTHENDIECK–VERDIER CATEGORIES

Although the internal-hom of a closed monoidal category  $\mathcal{C}$  being tensor-representable does not imply rigidity,  $\mathcal{C}$  often admits additional structure.

**Definition 3.1** ([BD13, Section 1.1]). A *Grothendieck–Verdier category* is a pair  $(\mathcal{C}, d)$  of a monoidal category  $\mathcal{C}$  and an object  $d \in \mathcal{C}$ , such that there exists an antiequivalence  $D: \mathcal{C} \rightarrow \mathcal{C}^{\text{op}}$  and for all  $x \in \mathcal{C}$  the functor  $\mathcal{C}(- \otimes x, d)$  is representable by  $D(x)$ .

If  $d = 1$  is the monoidal unit, one speaks of an *r-category*.

Symmetric Grothendieck–Verdier categories are also called  $\star$ -autonomous categories, see [Bar95]. Any rigid monoidal category is an instance of an *r-category*. The converse does not hold, as shown by the counterexamples [BD13, Example 1.9] and [BD13, Example 3.3].

We conclude this note by showing that any monoidal category where tensoring with an object has tensor-representable left and right adjoints is an *r-category*. To this end, we fix a monoidal category  $\mathcal{C}$  such that for any  $x \in \mathcal{C}$  there exist objects  $L(x)$  and  $R(x)$  such that

$$- \otimes L(x) \dashv - \otimes x \dashv - \otimes R(x).$$

**Theorem 3.2.** *If  $\mathcal{C}$  is as described above, it is an r-category.*

*Proof.* By the parameter theorem, see for example [ML98, Theorem IV.7.3], the object maps  $L, R: \text{Ob}(\mathcal{C}) \rightarrow \text{Ob}(\mathcal{C})$  can be promoted to functors

$$R: \mathcal{C} \rightarrow \mathcal{C}^{\text{op}} \quad \text{and} \quad L: \mathcal{C}^{\text{op}} \rightarrow \mathcal{C}.$$

<sup>2</sup>Note that the relations of  $\mathcal{D}$  leave the number of generating morphisms in any presentation of a given arrow invariant.

We verify that  $L$  and  $R$  are quasi-inverses of each other. By assumption, for all  $y, z \in \mathcal{C}$  we have

$$\mathcal{C}(y \otimes LR(x), z) \cong \mathcal{C}(y, z \otimes R(x)) \cong \mathcal{C}(y \otimes x, z).$$

Setting  $y = 1$ , the Yoneda embedding gives rise to a natural isomorphism  $LR \rightarrow \text{Id}_{\mathcal{C}}$ . A similar argument gives  $RL \cong \text{Id}_{\mathcal{C}^{\text{op}}}$ .

In order to show that  $\mathcal{C}(- \otimes x, 1)$  is representable by  $R(x)$ , we have to prove that for all  $y \in \mathcal{C}$  there exists a natural isomorphism

$$\mathcal{C}(y \otimes x, 1) \cong \mathcal{C}(y, R(x)).$$

By assumption, we have  $\mathcal{C}(y \otimes x, z) \cong \mathcal{C}(y, z \otimes Rx)$ . The claim follows by setting  $z = 1$ .  $\square$

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