# Weighted Tree Automata May it be a little more? 

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Dedicated to the memory of Magnus Steinby (1941-2021)

## Preface

Weighted automata form a quantitative extension of the traditional automaton model which is fundamental for Computer Science. They can be viewed as classical nondeterministic finite automata in which the transitions are equipped with weights. These weights could model, for instance, the cost, reward or probability of executing the transition. Already in 1961, Marcel Schützenberger investigated weighted automata and described their behaviors on finite words as rational formal power series. This extended Kleene's fundamental theorem on the coincidence of the classes of regular and rational formal languages into a quantitative setting. Subsequently, several decidability problems concerning context-free languages could be solved using weighted automata techniques, and up to now no other proof methods are known for this. This developed into a flourishing field, as described in books by Samuel Eilenberg (1974), Arto Salomaa and Matti Soittola (1978), Wolfgang Wechler (1978), Werner Kuich and Arto Salomaa (1986), Jean Berstel and Christophe Reutenauer (1988), Jacques Sakarovitch (2009) and the "Handbook of Weighted Automata" (2009).

Often, the weights employed for weighted automata are taken from a field like the rational numbers, the real numbers or the complex numbers, respectively, with their usual operations. However, we might also consider just the natural numbers which, with their usual operations of addition and multiplication, form the classical example of a semiring. Moreover, in many important applications calculations of the weights combine the operations maximum and addition, or minimum and addition, where the distributivity of the second operation over the first again yields a semiring as weight structure. The weights of the transitions can very conveniently be described as a single matrix (for each letter); the weight at entry $(i, j)$ is the weight of the transition from state $i$ to $j$. Since in semirings multiplication is distributive over addition, matrix multiplication is associative. This enables us to use techniques from algebra to express and analyse the behavior of weighted automata over semirings, and this was one of the reasons for the success of the weighted automaton model. All classical automata can be recast as weighted automata by taking the Boolean semiring $\{0,1\}$ as weight structure.

With the turn of the century, further quantitative models began to be investigated as weight structure. For instance, we might be interested in average weights, or in discounting of weights. In semiring-weighted automata, the weight of runs is computed by employing the second operation of the semiring, whereas the weights of the (in case of non-deterministic automata, often several) runs are combined into a single weight by using the semiring's first operation. The required distributivity of the second operation over the first is quite a strong mathematical assumption. What happens without this assumption? Such 'semirings without requiring distributivity' were called strong bimonoids. For instance, in lattices, which arise in multi-valued logic, both operations supremum and infimum are associative but in general not distributive over each other, and large parts of lattice theory in Mathematics concern non-distributive lattices. There are many further possibilities how strong bimonoids may arise. Since for strong bimonoids matrix multiplication is in general not associative, it follows that for weighted automata, techniques from linear algebra can no longer be used. However, fortunately, direct automata constructions can very often still be used to obtain results which previously were derived only for semirings.

In Computer Science, a most important data structure arising, e.g., in programming analysis and from pushdown automata and context-free languages, is given by trees. Therefore, a large part of automata theory concerns automata over trees. Already early on, much research concerned extensions of results
from weighted automata over words to weighted tree automata, which becomes often more intricate. Moreover, weighted tree automata have found very interesting recent applications in natural language processing. In view of the above, much research has recently been devoted to weighted tree automata with strong bimonoids as weight structure.

The present book, written by two leading researchers of this area, is the first book on weighted tree automata. It presents large parts of the theory of weighted tree automata over strong bimonoids and semirings in a systematic way. As indicated above, for weighted automata over strong bimonoids two kinds of defining the behavior arise:

- an automata-theoretic oriented approach using runs of the automaton and calculating their weights, and
- a universal algebra oriented method similar to the algebraic method mentioned above for words, but adjusted to trees.
In contrast to words, trees with tree concatenation do not form a monoid; this is the reason why the algebraic method mentioned above for weighted automata on words now for weighted tree automata has been replaced by a universal algebra approach. Over semirings, the two semantics for weighted tree automata can be shown to coincide, but over strong bimonoids in general they differ. Consequently, general structure results for the possible behaviors of weighted tree automata may have two versions, using the run semantics respectively the initial algebra semantics.

The authors show a number of these structure results. These include, among others, descriptions of the behaviors of weighted tree automata by rational expressions, by weighted context-free or regular grammars, by weighted versions of monadic second order logic, by closure results under various operations, and as abstract families of weighted tree languages, as well as weighted versions of classical pumping lemmas and of determinization results. This shows that many structure results for weighted tree automata can also be derived in the case of strong bimonoids. The authors give full mathematical proofs, which are needed as weighted tree automata often have combinatorial intricacies which are easy to overlook. The reader will appreciate that the authors include numerous examples explaining the various aspects of definitions and differences in the results, often illustrating them with pictures.

The field of weighted automata over strong bimonoids is still developing. This systematic presentation of large parts of recent research is therefore very valuable and right in time. I am sure that this book will stimulate much further research in this exciting area.

## Leipzig, December 2022

Manfred Droste

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## Chapter 1

## Introduction

The purpose of this book is to present the basic definitions and some of the important results for weighted tree automata in a mathematically rigorous and coherent form with full proofs. The concept of weighted tree automata is part of Automata Theory and it touches the area of Universal Algebra. It originated from two sources: weighted string automata and finite-state tree automata.

Historical perspective. Weighted string automata were introduced in Sch61; here we recall briefly [Sch61, Def. 1']. Roughly speaking, a weighted string automaton $\mathcal{A}$ is a finite-state automaton in which each transition has a weight, which is an element of the set $\mathbb{Z}$ of integers. Together with summation and multiplication, $\mathbb{Z}$ forms the weight algebra of $\mathcal{A}$. For each input symbol $a$, the weights of the transitions are represented as a $Q \times Q$-matrix over $\mathbb{Z}$ where $Q$ is the finite set of states of $\mathcal{A}$. Due to distributivity of multiplication over summation, the set of $Q \times Q$-matrices over $\mathbb{Z}$ together with matrix multiplication and the unit matrix forms a monoid. Thus, the representation can be extended to a monoid homomorphism from the free monoid over the set of input symbols to the monoid of $Q \times Q$-matrices over $\mathbb{Z}$. Finally, for a given input string $w$, its matrix representation is multiplied from the left by an initial weight vector and from the right by a final weight vector; the result is the weight of $w$ computed by $\mathcal{A}$. In this way, $\mathcal{A}$ assigns to each input string a weight in $\mathbb{Z}$; this assignment is called the formal power series recognized by $\mathcal{A}$ or weighted language recognized by $\mathcal{A}$. We might say that this weighted language is the monoid semantics of $\mathcal{A}$. Although in Sch61 mostly the semiring of natural numbers, the ring of integers, or commutative rings were considered, the idea of [Sch61] was general enough to deal with weighted string automata over arbitrary semirings (cf. [Wec78, p. 58]).

Since 1961, weighted string automata and the set of recognizable weighted languages were studied intensively. Different kinds of semantics were investigated (run semantics, free monoid semantics, initial algebra semantics) and various classes of weight algebras were used (e.g., bounded lattices, fields, semirings, pairs of t-conorm and t-norm, valuation monoids, strong bimonoids). We note that, neither in bounded lattices nor in valuation monoids or strong bimonoids or pairs of t-conorm and t-norm, the multiplication has to be distributive over the summation.

The development of weighted string automata is witnessed by the following list of books and survey papers: Eil74, SS78, Wec78, KS86, BR88, Kui97, MM02, Sak09, DKV09, DK21. The state-of-the-art of this area was also cultivated and extended by the biennial workshops "Weighted Automata: Theory and Applications" (WATA) since 2002 (cf. DV03, DV05, DV07b, DV09, DV11a, DV14b, DÉL18, DMV22).

Finite-state tree automata were invented independently by Don65, Don70 and TW68. A finite-state tree automaton processes a given input tree over some ranked alphabet $\Sigma$ also by means of transitions. Now a transition on a $k$-ary input symbol $\sigma$ has not only one but $k$ starting states (one for the root of each of the $k$ subtrees below $\sigma$ ), and it has one ending state (at the node which is labeled by $\sigma$ ). Viewed as a

[^0]bottom-up device 2 , a finite-state tree automaton $\mathcal{A}$ over an input ranked alphabet $\Sigma$ can be considered as a finite $\Sigma$-algebra, as it is known from Universal Algebra. Additionally, $\mathcal{A}$ identifies the set of final states as a subset of the carrier set of that $\Sigma$-algebra. Then $\mathcal{A}$ recognizes the set of all $\Sigma$-trees which are interpreted in the corresponding $\Sigma$-algebra as some final state. We might say that this tree language is the initial algebra semantics of $\mathcal{A}$.

Also the concept of finite-state tree automata and the set of recognizable tree languages were investigated intensively. Here is a list of books, lecture notes, and survey papers on finite-state tree automata: Eng75b, Eng80, GS84, GS97, CDG ${ }^{+}$07, LT21. Some more developments on tree automata and tree transducers were published in the series "International Workshop on Trends in Tree Automata and Tree Transducers" Man13, Fil15, Mal16.

It is natural to combine the concepts of weighted string automata and finite-state tree automata, resulting in the concept of weighted tree automata (wta). In IF75 wta were introduced with the weight algebra being the real number interval $[0,1]$ in which summation is max and multiplication is min; such wta were called fuzzy tree automata. In [BR82], a wta is a finite-dimensional vector space $V$ over some field B equipped with a $\Sigma$-algebra. Roughly speaking, each dimension corresponds to a state and each $k$-ary input symbol is interpreted in the $\Sigma$-algebra as a $k$-ary multilinear operation over V . In the following years, wta over different weight algebras were introduced: wta over semirings AB87, wta over distributive multioperator monoids Kui98, wta over multioperator monoids FMV09, SVF09, wta over pairs of t -conorm and t -norm on the unit interval $[0,1]$ BLB10, wta over strong bimonoids Rad10], and wta over tree valuation monoids DGMM11.

Here we list some of the seminal papers and survey papers on wta: [IF75, BR82, AB87, Kui98, Kui99b, Boz99, ÉK03, ÉL07, FV09, DGMM11. We also mention some recent PhD theses: Bor05, Hög07, Mat09, May10, Tei16, Göt17, Her20b, Pau20, Jon21, Dör22.

Weighted tree automata over strong bimonoids. In this book we will investigate wta over strong bimonoids. In this paragraph we provide an overview on this automaton model. A strong bimonoid is an algebra $\mathrm{B}=(B, \oplus, \otimes, \mathbb{O}, \mathbb{1})$ such that

- $(B, \oplus, \mathbb{0})$ is a commutative monoid (and $\oplus$ is called the summation),
- $(B, \otimes, \mathbb{1})$ is a monoid (and $\otimes$ is called the multiplication), and
- $\mathbb{O}$ is annihilating with respect to $\otimes$, i.e., $b \otimes \mathbb{O}=\mathbb{O} \otimes b=\mathbb{O}$ for each $b \in B$.

For instance, each semiring (and thus each ring, semifield, and field) is a strong bimonoid; also each bounded lattice (and thus each complete lattice and residuated lattice) is a strong bimonoid; we note that a large part of the field of lattice theory confers to non-distributive lattices Bir93, Grä03. The algebra $\left(\mathbb{N}_{\infty},+, \min , 0, \infty\right)$ is a strong bimonoid, but it is not a semiring, because, e.g., $\min (a, a+a) \neq$ $\min (a, a)+\min (a, a)$ for each $a \in \mathbb{N} \backslash\{0\}$, and hence the multiplication (here: min) does not distribute over the summation (here: + ). Further examples of non-distributive strong bimonoids are the finite lattices $\mathrm{N}_{5}$ and $\mathrm{M}_{3}$ (cf. Figure 2.3) and the infinite bounded lattice $\mathrm{FL}(2+2)$ (cf. Figure 2.4).

A wta over $\Sigma$ and B , for short: $(\Sigma, \mathrm{B})$-wta, is a tuple $\mathcal{A}=(Q, \delta, F)$ which is almost like a finite-state tree automaton, i.e., $Q$ is a finite set of states, $\delta=\left(\delta_{k} \mid k \in \mathbb{N}\right)$ is a family of transition mappings, and $F: Q \rightarrow B$ is the root weight mapping. However, now each transition and each final state carries a weight, which is taken from $B$. More precisely, for each $k \in \mathbb{N}$, the mapping $\delta_{k}$ has the type

$$
\delta_{k}: Q^{k} \times \Sigma^{(k)} \times Q \rightarrow B
$$

and it maps each transition $\left(q_{1} \cdots q_{k}, \sigma, q\right)$ with starting states $q_{1}, \ldots, q_{k}$, a $k$-ary input symbol $\sigma$, and ending state $q$ to a weight, i.e., an element of $B$. Since $\Sigma$ is finite, only finitely many $\delta_{k}$ are different from the empty mapping. Finally, the mapping $F: Q \rightarrow B$ maps each state to an element of $B$; this is used as root weight (or final weight).

[^1]We can define two semantics of $\mathcal{A}$ : the run semantics $\llbracket \mathcal{A} \rrbracket^{\text {run }}$ and the initial algebra semantics $\llbracket \mathcal{A} \rrbracket^{\text {init }}$ where each of them is a weighted tree language; more precisely,

$$
\llbracket \mathcal{A} \rrbracket^{\text {run }}: \mathrm{T}_{\Sigma} \rightarrow B \quad \text { and } \quad \llbracket \mathcal{A} \rrbracket^{\text {init }}: \mathrm{T}_{\Sigma} \rightarrow B
$$

where $\mathrm{T}_{\Sigma}$ is the set of all $\Sigma$-trees.
The run semantics is based on the idea of a run of $\mathcal{A}$ on a given input tree $\xi$. This is a mapping $\rho$ which maps each position of $\xi$ to a state of $Q$. Thus $\rho$ determines, for each position $w$ of $\xi$, a transition $\left(q_{1} \cdots q_{k}, \sigma, q\right)$ where $q_{i}$ (for $\left.i \in\{1, \ldots, k\}\right)$ and $q$ are the states assigned by $\rho$ to the $i$-th successor of $w$ and to $w$ itself, respectively, and $\sigma$ is the label of $\xi$ at position $w$. By applying $\delta_{k}$ to $\left(q_{1} \cdots q_{k}, \sigma, q\right)$, we obtain the weight of this transition. Then the weight of $\rho$, denoted by $\mathrm{wt}(\xi, \rho)$, is the $\otimes$-product of the weights of the transitions for each position of $\xi$ (where the factors are ordered according to the depth-first post-order of the corresponding positions). Finally, we let

$$
\llbracket \mathcal{A} \rrbracket^{\mathrm{run}}(\xi)=\bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}}(\xi)} \mathrm{wt}(\xi, \rho) \otimes F(\rho(\varepsilon))
$$

where $\mathrm{R}_{\mathcal{A}}(\xi)$ is the set of all runs of $\mathcal{A}$ on $\xi$, and $\rho(\varepsilon)$ is the state assigned by $\rho$ to the root of $\xi$. A weighted tree language $r: \mathrm{T}_{\Sigma} \rightarrow B$ is run recognizable if there exists a $(\Sigma, \mathrm{B})$-wta $\mathcal{A}$ such that $r=\llbracket \mathcal{A} \rrbracket^{\text {run }}$. We denote by $\operatorname{Rec}^{\text {run }}(\Sigma, \mathrm{B})$ the set of weighted tree languages which are run recognizable by some $(\Sigma, \mathrm{B})$-wta.

The initial algebra semantics of $\mathcal{A}$ is based on the concept of vector algebra of $\mathcal{A}$, which is the $\Sigma$-algebra $\left(B^{Q}, \delta_{\mathcal{A}}\right)$ where $\delta_{\mathcal{A}}$ associates with each $k$-ary input symbol $\sigma \in \Sigma$ a $k$-ary operation $\delta_{\mathcal{A}}(\sigma): B^{Q} \times \cdots \times B^{Q} \rightarrow B^{Q}$. For each $q \in Q$, the $q$-component of a vector $v \in B^{Q}$ is denoted by $v_{q}$. The operation $\delta_{\mathcal{A}}(\sigma)$ is defined, for every $v_{1}, \ldots, v_{k} \in B^{Q}$ and $q \in Q$, by

$$
\delta_{\mathcal{A}}(\sigma)\left(v_{1}, \ldots, v_{k}\right)_{q}=\bigoplus_{q_{1} \cdots q_{k} \in Q^{k}}\left(\bigotimes_{i \in[k]}\left(v_{i}\right)_{q_{i}}\right) \otimes \delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right)
$$

Then, for every $\xi \in \mathrm{T}_{\Sigma}$, we let

$$
\llbracket \mathcal{A} \rrbracket^{\mathrm{init}}(\xi)=\bigoplus_{q \in Q} \mathrm{~h}_{\mathcal{A}}(\xi)_{q} \otimes F(q)
$$

where $\mathrm{h}_{\mathcal{A}}$ denotes the unique $\Sigma$-algebra homomorphism from the $\Sigma$-term algebra to the $\Sigma$-algebra $\left(B^{Q}, \delta_{\mathcal{A}}\right)$. A weighted tree language $r: \mathrm{T}_{\Sigma} \rightarrow B$ is initial algebra recognizable if there exists a $(\Sigma, \mathrm{B})$-wta $\mathcal{A}$ such that $r=\llbracket \mathcal{A} \rrbracket^{\text {init }}$. We denote by $\operatorname{Rec}^{\text {init }}(\Sigma, \mathrm{B})$ the set of weighted tree languages which are initial algebra recognizable by some ( $\Sigma, \mathrm{B}$ )-wta.

If B is a semiring, i.e., $\otimes$ distributes over $\oplus$, then $\llbracket \mathcal{A} \rrbracket^{\text {run }}=\llbracket \mathcal{A} \rrbracket^{\text {init }}$ (as in the case of weighted string automata over semirings, similar to [Eil74, Cor. 6.2]). For arbitrary strong bimonoids this equality in general does not hold (cf. DSV10, Ex. 25] and [DIV10, Ex. 3.1]). However, the equality does hold for arbitrary strong bimonoids if $\mathcal{A}$ is bottom-up deterministic.

Since the set of trees does not have a monoid structure (in contrast to the set of strings), there does not exists a semantics of wta which corresponds directly to Schützenberger's semantics of wsa described above. However, for the case that $B$ is a field, the concept of $B$ - $\Sigma$-representation BA89 can be viewed as kind of monoid-semantics of wta. In that concept it is exploited that the set of contexts over $\Sigma$ (i.e., trees with a variable at exactly one leaf) is a freely generated monoid; moreover, while switching from contexts to trees, a certain consistency condition has to be satisfied.

Topics of the book. In this book we present a part of the theory of wta over strong bimonoids. Most of the theorems which we present have previously been published in research articles; others are generalizations of the corresponding published theorems. However, some results are published for the first time in this book as, e.g., Observation 2.6.16. Theorem 5.1.1. Theorem 10.3.3 results of Subsections 10.13.6, 10.13.7 and 10.13.8 results of Section 12.4, results of Subsections 14.4.2and 14.4.3, in particular

Theorem 14.4.11 results of Section 14.5 results of Section 18.2.5 in particular Theorems 18.2 .14 and 18.2.15 Theorems 19.7.1 and 19.8.4.

In Chapter 2 we collect all the technical ingredients for our development such that the book is selfcontained and hence also accessible for, e.g., master students of Mathematics or Computer Science. So this chapter is quite long, and of course, it may be consulted on demand.

In Chapter 3 we define the concept of wta over some ranked alphabet $\Sigma$ and some strong bimonoid B, and with each such $(\Sigma, B)$-wta $\mathcal{A}$, we associate the run semantics $\llbracket \mathcal{A} \rrbracket^{\text {run }}$ and the initial algebra semantics $\llbracket \mathcal{A} \rrbracket^{\text {init }}$. We define the restricted versions bottom-up deterministic wta and crisp deterministic wta. We give several examples of a wta and both kinds of semantics. In order to connect the concept of wta with its historical predecessors, we discuss particular cases of wta:

- wta over string ranked alphabets (which are equivalent to weighted string automata),
- wta over the Boolean semiring (which are equivalent to finite-state tree automata),
- wta over the semiring of natural numbers (which reflect multiplicities in finite-state tree automata),
- wta over commutative semirings (which are equivalent to multilinear representations).

In Chapter 4 we prove basic properties of wta and of their bottom-up and crisp deterministic subconcepts. Roughly speaking, these properties are consequences of the annihilation property of $\mathbb{0}$ for the multiplication $\otimes$.

In Chapter 5 we provide two algorithms (cf. Section 5.1) which compute in a natural way the values $\llbracket \mathcal{A} \rrbracket^{\text {run }}(\xi)$ and $\llbracket \mathcal{A} \rrbracket^{\text {init }}(\xi)$ for any given wta $\mathcal{A}$ and input tree $\xi$. We analyse the complexity of these algorithms and, as is to be expected, the second algorithm is more efficient than the first one. Moreover, we compare the run semantics and the initial algebra semantics of wta and show some examples of wta for which these semantics are different. We prove that there exist a strong bimonoid B and a $(\Sigma, \mathrm{B})$-wta $\mathcal{A}$ such that $\llbracket \mathcal{A} \rrbracket^{\text {init }} \notin \operatorname{Rec}^{\text {run }}(\Sigma, \mathrm{B})$ (cf. Theorem 5.2.5). We show conditions under which, for each wta, the two semantics are equal (cf. Theorems and 5.3.1 and 5.3.2).

In Chapter 6 we prove three pumping lemmas: one for runs of wta over strong bimonoids (cf. Theorem 6.1.4), another one for supports of weighted tree languages recognizable by wta over positive strong bimonoids (cf. Corollary (6.1.6), and a third one for supports of weighted tree languages recognizable by wta over fields (cf. Theorem 6.2.9). We use these lemmas to show that certain weighted tree languages are not recognizable.

In Chapter 7 we prove under which conditions a wta can be transformed into an equivalent one which is trim, i.e., each state is "useful" (cf. Theorems 7.1.3 and 7.1.4). Moreover, we show how to normalize the root weight mapping $F$ such that there exists a unique state $q$ with $F(q)=\mathbb{1}$ and $F(p)=\mathbb{0}$ for each state $p$ different from $q$ (cf. Theorem 7.3.1). Finally, we prove that each wta can be transformed into a run equivalent wta with identity transition weights if the multiplication is locally finite 7.4 .2 ,

In Chapter 8 we recall the concept of weighted context-free grammar from CS63 and prove several normal form lemmas (cf. Section 8.2). We prove the fundamental connection between weighted tree languages that are run recognizable by wta and weighted context-free languages (cf. Theorem 8.3.1). This generalizes the well-known theorem Bra69, Don70 saying that the yield of a recognizable tree language is a context-free language, and vice versa, each context-free language can be obtained in this way.

In Chapter 9 we introduce weighted regular tree grammars as particular weighted context-free grammars. We show that, under certain conditions, weighted regular tree grammars and the run semantics of wta are equally expressive (cf. Theorem 9.2.9).

In Chapter 10 we prove a number of closure properties of the set of recognizable weighted tree languages, viz., closure under sum, scalar multiplication, Hadamard product, top-concatenations, tree concatenations, Kleene-stars, yield-intersection with weighted regular string languages, strong bimonoid homomorphisms, tree relabelings, linear and nondeleting tree homomorphisms, inverse of linear tree homomorphisms, and weighted projective bimorphisms. A brief summary of these closure results is presented in Table 10.18

In Chapter 11 we introduce two different kinds of weighted local systems and generalize another fundamental result from the theory of recognizable tree languages: each recognizable tree language is the image of a local tree language under a tree relabeling (cf. Theorem 11.2.6). Moreover, we prove a decomposition of the run semantics of a wta into the inverse of a deterministic tree relabeling, intersection with a local tree language, and a homomorphism into an evaluation algebra (cf. Theorems 11.3.1).

In Chapters 12, 13, and 14 we prove characterizations of the set of recognizable weighted tree languages in terms of rational weighted tree languages (cf. Theorem 12.1.2), in terms of representable weighted tree languages (cf. Theorem 13.1.4), and in terms of MSO-definable weighted tree languages (cf. Theorem 14.3.1). This generalizes corresponding well-known results of Kleene, Médvédjév, and Büchi-Elgot-Trakhtenbrot, respectively, from the unweighted string case to the weighted tree case.

In Chapter 15 we prove that, for commutative and $\sigma$-complete semirings, the set of recognizable weighted tree languages is the smallest principal abstract family of weighted tree languages (cf. Theorem 15.4 .5 ). This generalizes the well-known situation for the set of recognizable string languages [Gin75].

In Chapters 16 and 17 we deal with the questions under which conditions it is possible to construct, for a given wta, an equivalent crisp deterministic wta and a bottom-up deterministic wta, respectively. In general, the usual subset construction for unweighted automata leads to an infinite state set and thus cannot be employed.

In Chapter 18 we deal with the question whether the support of a recognizable weighted tree language is a recognizable tree language. We show that in general the answer is no and we give sufficient conditions under which the answer is yes.

Finally, in Chapter 19 we collect some of the results of the previous chapters for the special case of wta over bounded lattices. In particular, we prove the inclusion relationship between several sets of recognizable weighted tree languages (cf. Theorem 19.3.5) and the characterizations of $\operatorname{Rec}{ }^{\text {run }}(\Sigma, \mathrm{L})$ in terms of unrestricted representations (cf. Theorem 19.7.1) and in terms of unrestricted weighted MSO formulas (cf. Theorem 19.8.4).

In Figure 1.1 (at the end of this introduction) we show an overview of the models of automata and models of grammars which occur in this book, and we indicate their relationship.

Topics not included in this book. We have covered some of the important theorems for wta over strong bimonoids. However, this book is not meant to present the whole theory of wta over strong bimonoids, not even that of wta over semirings. The choice of the presented material is biased by our personal research in this area. In particular, we did not address the topics of the following list. For each topic, we have indicated some of the publications where the reader can start investigating the topic.

- wta on infinite trees Rah07, BP11, LP14
- wta on unranked trees HMV09, DV11b, DH15, DHV15, Göt17,
- (weighted) forest automata Str09, Dör19, Dör21, Dör22]
- wta over M-monoids Kui98, Kui99b, Mal04, Mal05b, FMV09, SVF09, FSV12, TO15, FV18,
- wta over tree valuation monoids DGMM11, DM12, TO15, DFG16, Göt17, GFD19,
- B- $\Sigma$-representation of dimension $n$ BA89, Boz94
- probabilistic tree automata MM70, Ell71, Wei15
- minimization of wta BLB83, HMM07, Mal08, Mal09 HMV09, MQ12, MQ11, JM13, MQ14, BR20
- decidability and undecidability Sei89, Sei90, Boz91, Sei94, Bor04, Mal04, Mal05b, SMK18, FKV21, Pau20, DFKV20, DFKV22 (but also cf. Lemma 9.1.2 and Theorem 16.2.14)
- varieties of weighted tree languages FS11, SJC15]
- cut sets of recognizable weighted tree languages over some ordered structure BMŠ ${ }^{+}$. STV08.
- learning of wta Mal07, DV07a, DHM11, BM15
- weighted tree transducers Kui99c, EFV02, FV03, FGV04, Mal04, Mal05b, Mal05a, MV05, Mal06b, Mal06a, Mal06c, FV09, FMV11.
- wta with storage HV15 VDH16, FHV18, FV19, HVD19, FV22b, Her20b, DFV21
- applications of wta (and weighted tree transducers) in syntactic processing of natural languages PRS94, YK02, GHKM04, GK04, KG05, GKM08, KM09, MS09, FMV10, MS10, May10, Mal11a, Mal11b, Buy13,

BVN14, Büc14, BDZ15, DGV16, Mal17, Die18, MV19b, MV19a, MV21; also cf. the workshops DK10, DK12.

## Notes to the reader.

... about prerequisites:
The reader is assumed to be familiar with the fundamental concepts, results, and constructions in the theory of tree automata Eng75b, GS84, GS97, CDG ${ }^{+} 07$ and context-free languages Har78, HU79, HMU07.
... about the notion "construction":
In general, a construction is an algorithm which takes some objects as input and produces some objects as output in an effective manner. For instance, GS84, Thm. 2.4.2] states that, for two finite-state tree automata $A_{1}$ and $A_{2}$, a finite-state tree automaton $A$ can be constructed which recognizes the intersection $\mathrm{L}\left(A_{1}\right) \cap \mathrm{L}\left(A_{2}\right)$ of the tree languages which are recognized by $A_{1}$ and $A_{2}$; then the proof of GS84, Thm. 2.4.2] (essentially) shows an algorithm which, on input $A_{1}$ and $A_{2}$, builds up effectively $A$ such that $\mathrm{L}(A)=\mathrm{L}\left(A_{1}\right) \cap \mathrm{L}\left(A_{2}\right)$.

Also in the present book we show a number of constructions. However, here the situation is more complicated for those constructions which involve some strong bimonoid B. If the algorithm involves calculations in B, it may happen that the calculations in B are not effective. This can arise, e.g., because the arguments of the calculation cannot be given effectively or because the operations $\oplus$ and $\otimes$ are not given algorithmically. In this unpleasant situation, we understand by "construction" just the definition of the output objects, and not an algorithm for their effective building up process. So, in our understanding a construction

- takes some objects as input,
- if the input objects are given effectively and the operations of the strong bimonoid are effective, then the construction gives the output objects effectively,
- otherwise the construction merely defines the output objects (in a mathematical sense).

Let us illustrate this understanding by considering the "weighted version" of GS84, Thm. 2.4.2]. In Theorem 10.4.1(1) we claim: "Let $\Sigma$ be a ranked alphabet, B be a strong bimonoid, and $\mathcal{A}_{1}$ and $\mathcal{A}_{2}$ be two ( $\Sigma, \mathrm{B})$-wta. If B is a commutative semiring, then we can construct a $(\Sigma, \mathrm{B})$-wta $\mathcal{A}$ such that $\llbracket \mathcal{A} \rrbracket=\llbracket \mathcal{A}_{1} \rrbracket \otimes \llbracket \mathcal{A}_{2} \rrbracket$." If B is, e.g., the Boolean semiring or the semiring of natural numbers, then our proof shows an algorithm which builds up $\mathcal{A}$ effectively. However, if $B$ is, e.g., the field of real numbers and some of the coefficients of $\mathcal{A}_{1}$ or $\mathcal{A}_{2}$ cannot be given effectively, then our proof just shows the definition of $\mathcal{A}$.
... about general conventions:
In order to avoid many repetitions of the statements "for each ranked alphabet $\Sigma$ " and "for each strong bimonoid B", we will use the general conventions that $\Sigma$ always stands for an arbitrary, i.e., universally quantified ranked alphabet (cf. page 40) and that $B$ denotes an arbitrary strong bimonoid (cf. page 35). Such a convention is placed inside \begin\{quote\}-\end\{quote\} in emphasized letter style. In some } chapters or sections, we develop results for a particular subset of strong bimonoids, e.g., the set of commutative semirings. Again, in order to avoid repetitions of universal quantifications over this particular subset, we will place a corresponding convention local to these chapters and sections; the scope of each local convention is explicitly indicated. Then, for each result (e.g. observation, lemma, theorem, corollary) within this scope, the general and local conventions and restrictions hold. We hope that this extraction of universal quantifications helps to focus on important argumentation.
... about theorems, lemmas, and corollaries which are in a box:
If a reader just wants to look up some result, then checking for local and global conventions and collecting them is a burden and it is better to see *all* the restrictions, requirements, and universal quantifications inside the corresponding latex-environment. To solve the contradicting wishes: avoiding repetitions of universal quantifications versus fully quantified statements, for some of the main results (lemmas, theorems, and corollaries) we show all the necessary restrictions and requirements inside the latex-environment. In
order to ease the search for such main results, we put them into a box.
... about possible mistakes:
We would like to ask (or: encourage) the readers to share with us their remarks and observations concerning the contents of the book. In particular, the indication of mistakes or of missing references to published results are very welcome. Please, write an email to fulop@inf.u-szeged.hu and heiko.vogler@tu-dresden.de.
... about an electronic version:
This book can be found on arXiv by searching for the authors or the title.

Acknowledgements. In 2017 we started to work on this book. But the seed for this work was laid much earlier. In our academic youth, we both had excellent teachers of the theory of tree automata (Ferenc Gécseg and Joost Engelfriet, respectively). After this academic qualification, we met for the first time in the year 1987, when the second author (Heiko) visited the first author (Zoltán) and his colleague Sándor Vágvölgyi in Szeged. And the result of this start was the publication [FHVV93. Since then we (the authors) jointly investigated the theory of tree automata and tree transducers. In the beginning, we communicated by hand-written letters as it was usual at that time, or we paid regular visits to each other. When in 1999 Werner Kuich visited Ferenc Gécseg, the second author happened to be also in Szeged. Then Werner showed him his fresh paper "Tree transducers and tree series" Kui99c and explained its ideas. From that point on, we (Zoltan and Heiko) worked on the theory of weighted tree automata and weighted tree transducers (starting with EFV02). During the development of that theory, we benefitted much from the cooperation with Symeon Bozapalidis, Frank Drewes, Manfred Droste, Joost Engelfriet, Andreas Maletti, Mark-Jan Nederhof, George Rahonis, and Magnus Steinby. In the years 2002-2021, the series of workshops "Weighted Automata: Theory and Applications" (WATA), co-organized by Manfred Droste, gave an excellent infrastructure for presenting our results. This book contains many ideas which were produced by these cooperations during the last 20 years; we are very grateful for this.

We would like to thank our colleagues for checking some parts of the book and contributing valuable suggestions: Johanna Björklund, Frank Drewes, Manfred Droste, Zsolt Gazdag, Luisa Herrmann, Eija Jurvanen, Andreas Maletti, George Rahonis, and Sándor Vágvölgyi. In particular, we wish to thank Manfred Droste for several fruitful discussions and suggestions, which had a very positive impact on the book. Of course, each remaining mistake is due to the authors. We are grateful to Felicita Purnama Dewi Gernhardt, Luisa Herrmann, Dávid Kószó, and, in particular, Celina Pohl for preparing the figures. In a number of situations, Celina Pohl and Richard Mörbitz helped us to find the appropriate latex-commands and settings.
"... who prologue-like your humble patience pray, gently to hear, kindly to judge, our play." W. Shakespeare, Henry V.
models of automata
model of antomata


Figure 1.1: Overview of the models of automata and models of grammars which occur in this book. Their relationship is expressed by arrows. For every pair $(X, Y)$ of models, if $X \rightarrow Y$, then $Y$ is a special case of $X$, and if $X \leftrightarrow Y$, then $X$ and $Y$ are equivalent.
wta: weighted tree automaton
wsa: weighted string automaton
fta: finite-state tree automaton
fsa: finite-state string automaton
$\Sigma, \Psi:$ ranked alphabets
$\Gamma$ : alphabet
B: strong bimonoid
rhs: right-hand side
wcfg: weighted context-free grammar
wrtg: weighted regular tree grammar
cfg: context-free grammar
wpb: weighted projective bimorphism

## List of all general and local conventions made in this book

- on page 12 .

In the rest of this book, $\Gamma$ will denote an arbitrary alphabet, if not specified otherwise.

- on page 35

In the rest of this book, B will denote an arbitrary strong bimonoid $(B, \oplus, \otimes, \mathbb{O}, \mathbb{1})$ if not specified otherwise.

- on page 40 . In the rest of this book, we assume that each ranked alphabet has at least one symbol with rank 0. Moreover, $\Sigma$ and $\Delta$ will denote arbitrary such ranked alphabets, if not specified otherwise.
- on page 41 .

If not specified otherwise, then we denote the unique $\Sigma$-algebra homomorphism from the $\Sigma$-term algebra $\mathrm{T}_{\Sigma}=\left(\mathrm{T}_{\Sigma}, \theta_{\Sigma}\right)$ to some $\Sigma$-algebra $\mathrm{A}=(A, \theta)$ by $\mathrm{h}_{\mathrm{A}}$.

- on page 48 .

In the rest of the book, $Z$ and $X$ will denote sets of variables if not specified otherwise. Moreover, we let $Z_{n}=\left\{z_{1}, \ldots, z_{n}\right\}$ and $X_{n}=\left\{x_{1}, \ldots, x_{n}\right\}$ for every $n \in \mathbb{N}$.

- on page 97 ,

In the rest of this section, we let B denote an arbitrary commutative semiring.

- on page 117

Due to Theorem 5.3.1, if $\mathcal{A}$ is bu deterministic, then we write $\llbracket \mathcal{A} \rrbracket$ instead of $\llbracket \mathcal{A} \rrbracket^{\mathrm{run}}$ and $\llbracket \mathcal{A} \rrbracket^{\mathrm{init}}$. Moreover, for a weighted tree language $r$, we say that $r$ is bu deterministically recognizable (instead of bu deterministically i-recognizable and bu deterministically r-recognizable). Also, we write bud-Rec $(\Sigma, B)$ for bud- $\operatorname{Rec}^{\text {run }}(\Sigma, B)$ (and hence, for bud-Rec ${ }^{\text {init }}(\Sigma, B)$ ). Similarly, we say that $r$ is crisp deterministically recognizable (instead of crisp deterministically i-recognizable and crisp deterministically r-recognizable). Also, we write cd-Rec $(\Sigma, \mathrm{B})$ for $\operatorname{cd}-\operatorname{Rec}^{\text {run }}(\Sigma, \mathrm{B})$ and $\operatorname{cd}-\operatorname{Rec}^{\text {init }}(\Sigma, \mathrm{B})$ ).

- on page 121

Due to Corollary 5.3.3, if B is a semiring and $\mathcal{A}$ is a $(\Sigma, \mathrm{B})$-wta, then we write $\llbracket \mathcal{A} \rrbracket$ instead of $\llbracket \mathcal{A} \rrbracket^{\text {run }}$ and $\llbracket \mathcal{A} \rrbracket^{\text {init }}$. Moreover, for an i-recognizable or r-recognizable weighted tree language $r$, we say that $r$ is recognizable. Also, we denote the set $\operatorname{Rec}^{\mathrm{run}}(\Sigma, \mathrm{B})$ (and hence, $\operatorname{Rec}^{\mathrm{init}}(\Sigma, \mathrm{B})$ ) by $\operatorname{Rec}(\Sigma, \mathrm{B})$.

- on page 122

Due to Corollary 5.3.4, if B is a semiring and $\mathcal{A}$ is a $(\Gamma, \mathrm{B})$-wsa, then we write $\llbracket \mathcal{A} \rrbracket$ instead of $\llbracket \mathcal{A} \rrbracket^{\text {run }}$ and $\llbracket \mathcal{A} \rrbracket^{\text {init }}$.

- on page 124

In this section, we let $\mathcal{A}=(Q, \delta, F)$ be an arbitrary $(\Sigma, \mathrm{B})$-wta.

- on page 130 .

In the rest of this section, we let B be a field and $(\mathrm{V}, \mu, \gamma)$ be a $(\Sigma, \mathrm{B})$-multilinear representation where V is a $\kappa$-dimensional B -vector space for some $\kappa \in \mathbb{N}_{+}$, unless specified otherwise.

- on page 174

In the rest of this book, for each wrtg $\mathcal{G}$, we will abbreviate $\llbracket \mathcal{G} \rrbracket^{\mathrm{t}}$ by $\llbracket \mathcal{G} \rrbracket$.

- on page 253

In the rest of this chapter, B denotes an arbitrary commutative semiring.

- on page 287

In the sequel we will abbreviate $\mathrm{h}_{\mathrm{M}(\Sigma, \kappa)}$ by $\mathrm{h}_{\kappa}$.

- on page 299

In this section, we assume that B is commutative.

- on page 321

In the rest of this chapter, B denotes an arbitrary commutative and $\sigma$-complete semiring.

- on page 344

In this chapter B is a semiring and $\mathcal{A}=(Q, \delta, F)$ denotes an arbitrary $(\Sigma, \mathrm{B})$-wta unless specified otherwise.

- on page 347

We recall that $\mathrm{V}(\mathcal{A})=\left(B^{Q}, \delta_{\mathcal{A}}\right)$ is the vector algebra of $\mathcal{A}$, and that $\mathbb{O}_{Q} \in B^{Q}$ is the $Q$-vector over B which contains $(1)$ in each component (cf. Section 3.1). We assume that there exists $\alpha \in \Sigma^{(0)}$ such that $\delta_{\mathcal{A}}(\alpha)() \neq \mathbb{O}_{Q}$.

- on page 370 .

In this subsection we assume that B is commutative.

- on page 381 .

In this chapter, $\mathrm{L}=(L, \vee, \wedge, \mathbb{O}, \mathbb{1})$ denotes a bounded lattice.

## Chapter 2

## Preliminaries

### 2.1 Numbers and sets

We denote the set $\{0,1,2, \ldots\}$ of natural numbers by $\mathbb{N}$ and the set $\mathbb{N} \backslash\{0\}$ by $\mathbb{N}_{+}$. Let $n, k \in \mathbb{N}$. Then we denote the set $\{i \in \mathbb{N} \mid n \leq i \leq k\}$ by $[n, k]$. We abbreviate $[1, k]$ by $[k]$. Thus $[0]=\emptyset$. We denote the sets of integers, rational numbers, and real numbers by $\mathbb{Z}, \mathbb{Q}$, and $\mathbb{R}$, respectively.

For each $a, b \in \mathbb{R}$, we denote by $\max (a, b)$ and $\min (a, b)$ the maximum and the minimum of $a$ and $b$ with respect to $\leq$, respectively. We extend max and min to each nonempty, finite subset of $\mathbb{R}$ in a natural way. Later, when $\max$ (or min) is an operation of a monoid, we will define max $\emptyset$ (and $\min \emptyset$, respectively) to be the unit element of that monoid (cf. (2.10)).

Sometimes we use the set $\mathbb{N} \cup\{\infty\}$. We abbreviate it by $\mathbb{N}_{\infty}$ and extend the operations + and min to $\mathbb{N}_{\infty}$ in the obvious way: $a+\infty=\infty$ and $\min (a, \infty)=a$ for each $a \in \mathbb{N}_{\infty}$. In a similar way we proceed with an extension by $-\infty$ and the operation $\max$ (and also with other sets like $\mathbb{Z}$ and $\mathbb{R}$ ).

For the set $\mathbb{R}$ of real numbers, we denote its subset $\{r \in \mathbb{R} \mid r \geq 0\}$ by $\mathbb{R}_{\geq 0}$. For the corresponding subsets of integers and rational numbers we use $\mathbb{Z}_{\geq 0}$ and $\mathbb{Q}_{\geq 0}$, respectively. We denote the set $\mathbb{Q}_{\geq 0} \cup\{\infty\}$ by $\mathbb{Q}_{\geq 0, \infty}$.

Let $A$ be a set. We call $A$ countable if its cardinality coincides with that of a subset of the natural numbers. If $A$ is finite and it has $n$ elements $a_{1}, \ldots, a_{n}$, then we denote this fact by $A=\left\{a_{1}, \ldots, a_{n}\right\}$. If $n=1$, then sometimes we identify $A$ with $a_{1}$ and simply write $a_{1}$ for $\left\{a_{1}\right\}$. Moreover, $|A|$ denotes the cardinality of $A$. The power-set of $A$, denoted by $\mathcal{P}(A)$, is the set of all subsets of $A$. We denote by $\mathcal{P}_{\text {fin }}(A)$ the set of all finite subsets of $A$.

Let $n \in \mathbb{N}$, and $A_{1}, \ldots, A_{n}$ be sets. The Cartesian product of $A_{1}, \ldots, A_{n}$ is the set $\left\{\left(a_{1}, \ldots, a_{n}\right) \mid a_{i} \in\right.$ $A_{i}$ for each $\left.i \in[n]\right\}$ and it is denoted by $A_{1} \times \ldots \times A_{n}$. The $n$-fold Cartesian product of a set $A$ is the set $A \times \ldots \times A$, where $A$ appears $n$ times. We abbreviate $A \times \ldots \times A$ by $A^{n}$. In particular, $A^{0}=\{()\}$.

Let $A$ and $B$ be sets. If each element of $A$ is also an element of $B$, then $A$ is a subset of $B$, denoted by $A \subseteq B$. If $A \subseteq B$ and $A \neq B$, then $A$ is a strict subset of $B$, denoted by $A \subset B$.

### 2.2 Strings and languages

We recall some notions and notations of strings and (formal) languages from Har78, HU79, HMU14, Lin12.

Let $A$ be a set. A string (or: word) over $A$ is a finite sequence $a_{1} \cdots a_{n}$ with $n \in \mathbb{N}$ and $a_{i} \in A$ for each $i \in[n]$. In particular, we denote the sequence $a_{1} \cdots a_{n}$ with $n=0$ by $\varepsilon$ and call it the empty string. We say that $a_{1} \cdots a_{n}$ has length $n$. For each $n \in \mathbb{N}$, we denote the set of strings over $A$ of length $n$ by
$A^{n}$; thus $A^{0}=\{\varepsilon\}$. Moreover, we denote by $A^{*}$ the set of all strings over $A$, i.e., $A^{*}=\bigcup_{n \in \mathbb{N}} A^{n}$ and we let $A^{+}=A^{*} \backslash\{\varepsilon\}$. For each $w \in A^{*}$, we denote by $|w|$ the length of $w$.

We note that the notation $A^{n}$ is overloaded in the sense that it denotes two sets: (a) the $n$-fold Cartesian product of $A$ and (b) the set of strings over $A$ of length $n$. Of course, formally these sets are different. But since there exists a bijection between them, we find it acceptable to use the same notation.

The concatenation of the strings $a_{1} \cdots a_{n}$ and $b_{1} \cdots b_{m}$ is the string $a_{1} \cdots a_{n} b_{1} \cdots b_{m}$. For every $v, w \in A^{*}$, we denote the concatenation of $v$ and $w$ by $v \cdot w$ or simply by $v w$.

Let $w \in A^{*}$. For each $n \in \mathbb{N}$, we define $w^{n}$ such that $w^{0}=\varepsilon$ and $w^{n}=w w^{n-1}$ for $n \in \mathbb{N}_{+}$. Moreover, we denote by postfix $(w)$ the set $\left\{v \in A^{*} \mid\left(\exists u \in A^{*}\right): w=u v\right\}$ of postfixes of $w$, and by prefix $(w)$ the set $\left\{u \in A^{*} \mid\left(\exists v \in A^{*}\right): w=u v\right\}$ of prefixes of $w$. Note that $\{\varepsilon, w\} \subseteq \operatorname{postfix}(w) \cap \operatorname{prefix}(w)$.

An alphabet is a finite and nonempty set. Let $\Gamma$ be an alphabet. Then each subset $L \subseteq \Gamma^{*}$ is called a language (or: formal language) over $\Gamma$. Let $L_{1}, L_{2} \subseteq \Gamma^{*}$ be two languages. The concatenation of $L_{1}$ and $L_{2}$, denoted by $L_{1} \cdot L_{2}$ or just $L_{1} L_{2}$, is the language $L_{1} L_{2}=\left\{w_{1} w_{2} \mid w_{1} \in L_{1}, w_{2} \in L_{2}\right\}$.

In the rest of this book, $\Gamma$ will denote an arbitrary alphabet, if not specified otherwise.

### 2.3 Binary relations and mappings

Binary relations. Let $A$ and $B$ be sets. A binary relation (on $A$ and $B$ ) is a subset $R \subseteq A \times B$. As usual, we frequently write $a R b$ for $(a, b) \in R$. For each $A^{\prime} \subseteq A$, we define $R\left(A^{\prime}\right)=\left\{b \in B \mid\left(\exists a \in A^{\prime}\right): a R b\right\}$. If $A=B$, then we call $R$ a binary relation on $A$ (or: over $A$ ). We denote by $R^{-1}$ the binary relation $\{(b, a) \mid a R b\}$, and we call it the inverse of $R$.

Let additionally $C$ be a set and $R_{1} \subseteq A \times B$ and $R_{2} \subseteq B \times C$ be binary relations. The composition of $R_{1}$ and $R_{2}$, denoted by $R_{1} ; R_{2}$, is the binary relation on $A$ and $C$ defined by $R_{1} ; R_{2}=\{(a, c) \in A \times C \mid$ $(\exists b \in B):(a, b) \in R_{1}$ and $\left.(b, c) \in R_{2}\right\}$.

Let $R$ be a binary relation on $A$. The relation $R$ is

- reflexive if $a R a$ for every $a \in A$,
- symmetric if $a R b$ implies that $b R a$ for every $a, b \in A$,
- antisymmetric if $a R b$ and $b R a$ imply that $a=b$ for every $a, b \in A$, and
- transitive if $a R b$ and $b R c$ imply that $a R c$ for every $a, b, c \in A$.

The reflexive, transitive closure of $R$, denoted by $R^{*}$, is the binary relation $R^{*}=\bigcup_{n \in \mathbb{N}} R^{n}$ where $R^{0}=$ $\{(a, a) \mid a \in A\}$ and $R^{n+1}=R^{n} ; R$ for each $n \in \mathbb{N}$. The transitive closure of $R$, denoted by $R^{+}$, is the binary relation $R^{+}=\bigcup_{n \in \mathbb{N}_{+}} R^{n}$. Then $R^{*}$ is reflexive and transitive, and $R^{+}$is transitive.

If $R$ is reflexive, symmetric, and transitive, then we call it an equivalence relation (on $A$ ). Then, for every $a \in A$, the equivalence class with representative $a$, denoted by $[a]_{R}$, is the set $\{b \in A \mid a R b\}$. The factor set of $A$ modulo $R$, denoted by $A / R$, is the set $\left\{[a]_{R} \mid a \in A\right\}$. The index of $R$ is the cardinality of $A / R$.

Let $\leq$ be a binary relation on $A$. If $\leq$ is reflexive, antisymmetric, and transitive, then we call it a partial order (on $A$ ), and we call the pair $(A, \leq)$ a partially ordered set. A partial order $\leq$ on $A$ is a linear $\operatorname{order}($ on $A$ ) if for every $a, b \in A$ we have $a \leq b$ or $b \leq a$. For $a, b \in A$, we write $a<b$ to denote that $a \leq b$ and $a \neq b$.

Let $(A, \leq)$ be a partially ordered set and let $B \subseteq A$. An element $a \in A$ is an upper bound of $B$ if, for each $b \in B$, we have $b \leq a$. Moreover, if $a$ is an upper bound and, for each upper bound $a^{\prime}$ of $B$, we have $a \leq a^{\prime}$, then $a$ is the least upper bound of $B$. If the least upper bound of $B$ exists, then we call it the supremum of $B$ and denote it by $\sup _{\leq} B($ or: $\sup B)$. Similarly, we can define lower bound and the greatest lower bound, i.e., the infimum of $\bar{B}$. We denote the latter by $\inf _{\leq} B$ (or: $\left.\inf B\right)$.

Mappings. Let $f \subseteq A \times B$ be a binary relation. We say that $f$ is a mapping from $A$ to $B$, denoted by $f: A \rightarrow B$, if for each $a \in A$ there exists a unique $b \in B$ such that $(a, b) \in f$. In this case we write $f(a)=b$ as usual. The set of all mappings from $A$ to $B$ is denoted by $B^{A}$. We call $A$ and $B$ the domain and codomain of $f$, respectively. The image of $f$ is the set $\operatorname{im}(f)=\{f(a) \mid a \in A\}$. A mapping $f: A \rightarrow B$ is injective if for every $a_{1}, a_{2} \in A$ with $a_{1} \neq a_{2}$ we have that $f\left(a_{1}\right) \neq f\left(a_{2}\right)$. It is surjective if for each $b \in B$ there exists an $a \in A$ such that $f(a)=b$.

In particular, each binary relation $R \subseteq A \times B$ can be considered as a mapping $R: A \rightarrow \mathcal{P}(B)$, where $R(a)=\{b \in B \mid a R b\}$ for each $a \in A$.

Let $f: A \rightarrow B$ be a mapping and $A^{\prime}$ and $B^{\prime}$ be sets such that $A^{\prime} \subseteq A$ and $B^{\prime} \supseteq B$. Then $f$ is also a mapping from $A$ to $B^{\prime}$, i.e., $f: A \rightarrow B^{\prime}$. The restriction of $f$ to $A^{\prime}$ is the mapping $\left.f\right|_{A^{\prime}}: A^{\prime} \rightarrow B$ and defined by $\left.f\right|_{A^{\prime}}(a)=f(a)$ for each $a \in A^{\prime}$.

Let $A, B$, and $C$ be sets such that $A$ and $B$ are disjoint. Moreover, let $f: A \rightarrow C$ and $g: B \rightarrow C$. We define the union of $f$ and $g$ as the mapping $(f \cup g): A \cup B \rightarrow C$ such that, for each $x \in(A \cup B)$ we let $(f \cup g)(x)=f(x)$ if $x \in A$, and $(f \cup g)(x)=g(x)$ otherwise. Clearly, $\left.(f \cup g)\right|_{A}=f$ and $\left.(f \cup g)\right|_{B}=g$.

Let $C$ be a further set. For two mappings $f: A \rightarrow B$ and $g: B \rightarrow C$, the composition of $f$ and $g$ is the mapping $(g \circ f): A \rightarrow C$, where $(g \circ f)(a)=g(f(a))$ for every $a \in A$. If we view $f$ and $g$ as binary relations, then $g \circ f=f ; g$.

In the usual way we extend a mapping $f: A \rightarrow B$ to the mapping $f^{\prime}: \mathcal{P}(A) \rightarrow \mathcal{P}(B)$ by defining $f^{\prime}\left(A^{\prime}\right)=\left\{b \in B \mid\left(\exists a \in A^{\prime}\right): f(a)=b\right\}$ for each $A^{\prime} \subseteq A$. We denote $f^{\prime}$ also by $f$.

If we want to emphasize the listing of the values of a mapping, then we use the concept of family defined as follows. Let $I$ be a countable set and $B$ be a set. An $I$-indexed family over $B$ is a mapping $f: I \rightarrow B$. Such a family is also denoted by $\left(b_{i} \mid i \in I\right)$ where $b_{i}=f(i)$ for each $i \in I$. An $I$-indexed family over $B$ is finite (nonempty) if $I$ is finite (nonempty, respectively). We call $I$ the index set of that family.

Let $f=\left(B_{i} \mid i \in I\right)$ be an $I$-indexed family over $\mathcal{P}(B)$. We call $f$ a partitioning of $B$ (with respect to $I$ ) if $\bigcup_{i \in I} B_{i}=B$, and $B_{i} \cap B_{j}=\emptyset$ for every $i, j \in I$ with $i \neq j$.

For each $k \in \mathbb{N}$, a mapping $f: A^{k} \rightarrow A$ is called $k$-ary operation on $A$ or simply operation on $A$. As usual, we identify a nullary operation $f$ on $A$ with the element $f() \in A$. We denote the set of all $k$-ary operations on $A$ by $\mathrm{Ops}^{k}(A)$, and the set of all operations on $A$ by $\operatorname{Ops}(A)$.

We define the identity mapping on $A$ as the mapping $\operatorname{id}_{A}: A \rightarrow A$ such that $\operatorname{id}_{A}(a)=a$ for each $a \in A$.

### 2.4 Hasse diagram, Venn diagram, and Euler diagram

Let $(A \leq)$ be a partially ordered set. There are several ways to represent $(A, \leq)$ by means of a diagram in the plane.

Hasse diagram. Each element $a \in A$ is represented in the diagram by a node which is labeled by $a$. This node is called $a$-node. The nodes are placed into the diagram in such a way that whenever $a<b$ holds for $a, b \in A$, the $b$-node is drawn above the $a$-node. Moreover, for every $a, b \in A$, the $a$-node and the $b$-node are connected by an edge if $a<b$ and, for every $c \in A$, the condition $a \leq c \leq b$ implies that $a=c$ or $b=c$. From this diagram we can reconstruct $\leq$ by noting that $a \leq b$ holds if and only if $a=b$ or the $b$-node can be reached from the $a$-node via a sequence of ascending edges.

Figure 2.1] shows an example of a Hasse diagram for $A=(\mathbb{N} \times \mathbb{N})$ and the partial order $\leq$ defined by $\left(n_{1}, n_{2}\right) \leq\left(k_{1}, k_{2}\right)$ if $n_{1} \leq k_{1}$ and $n_{2} \leq k_{2}$. Of course, since $A$ is an infinite set, we can only show a finite part of the Hasse diagram.


Figure 2.1: A finite part of the Hasse diagram for $A=(\mathbb{N} \times \mathbb{N})$ and $\left(n_{1}, n_{2}\right) \leq\left(k_{1}, k_{2}\right)$ if $n_{1} \leq k_{1}$ and $n_{2} \leq k_{2}$.


Figure 2.2: The Venn diagram (left) and the Euler diagram (right) for $U=\{a, b, c\}$ and $A=$ $\{\{a\},\{a, b\},\{b, c\}\}$.

Venn diagram and Euler diagram. These diagrams can be drawn if there exists a finite set $U$ such that $A$ is a finite collection of subsets of $U$, i.e., $A=\left\{B_{1}, \ldots, B_{n}\right\}$ for some $n \in \mathbb{N}_{+}$and some subsets $B_{1}, \ldots, B_{n}$ of $U$; moreover, the partial order $\leq$ is the subset relation $\subseteq$. Then each element $a \in U$ is represented by an individual point in the diagram; this point is labeled by $a$. Each subset $B_{i}$ is represented in the diagram by an area which is delimited by an oval or some polygon; this area contains exactly the elements of $B_{i}$. We call this area the $B_{i}$-area.

In the Venn diagram, the $B_{i}$-areas are placed into the diagram in an overlapping manner such that exactly $2^{n}-1$ overlap areas are created. It is possible that an overlap area does not contain an element of $U$.

In the Euler diagram, the $B_{i}$-areas are placed into the diagram also in an overlapping manner, but now in such a way that each overlap area contains at least one element of $U$.

Figure 2.2 shows an example of a Venn diagram and an Euler diagram for $U=\{a, b, c\}$ and $A=$ $\{\{a\},\{a, b\},\{b, c\}\}$.

### 2.5 Well-founded induction

We recall the principle of proof by well-founded induction from Win93, Jec03 (cf. also [BN98, Gal03]).
Let $C$ be a set and $\prec \subseteq C \times C$ a binary relation on $C$. We say that $\prec$ is well-founded if there does not exists a family $\left(c_{i} \mid i \in \mathbb{N}\right)$ of elements of $C$ such that $\cdots \prec c_{1} \prec c_{0}$. (In BN98 such a relation is called
terminating.) It is easy to see that a reflexive relation cannot be well-founded, and that the transitive closure $\prec^{+}$of a well-founded relation $\prec$ on $C$ is also well-founded. For instance, the relation

$$
\prec_{\mathbb{N}}=\{(n, n+1) \mid n \in \mathbb{N}\}
$$

is a well-founded relation on $\mathbb{N}$. Since $<=\left(\prec_{\mathbb{N}}\right)^{+}$(where $<$is the usual "less than" relation on $\mathbb{N}$ ), the relation $<$ is also well-founded.

Let $C$ be a set and $\prec$ be a well-founded relation on $C$. We call the pair $(C, \prec)$ a well-founded set. Moreover, let $P \subseteq C$ be a subset, called property. We will abbreviate the fact that $c \in P$ by $P(c)$ and say that $c$ has the property $P$. Then the following holds:

$$
\begin{equation*}
\left((\forall c \in C):\left[\left(\forall c^{\prime} \in C\right):\left(c^{\prime} \prec c\right) \rightarrow P\left(c^{\prime}\right)\right] \rightarrow P(c)\right) \rightarrow((\forall c \in C): P(c)) \tag{2.1}
\end{equation*}
$$

The formula (2.1) is called the principle of proof by well-founded induction on $(C, \prec)$ (for short: proof by induction on $(C, \prec))$. We can use this principle to prove the claim that each $c \in C$ has property $P$.

Often, the proof of the premise of (2.1), i.e., the formula

$$
\left((\forall c \in C):\left[\left(\forall c^{\prime} \in C\right):\left(c^{\prime} \prec c\right) \rightarrow P\left(c^{\prime}\right)\right] \rightarrow P(c)\right)
$$

is split into two parts. To show them, we define the concept of minimal element. For each $c \in C$, we say that $c$ is minimal (with respect to $\prec$ ) if there does not exist $c^{\prime} \in C$ such that $c^{\prime} \prec c$. We denote by $\min _{\prec}(C)$ the set of all minimal elements of $C$. Then the first part is the proof of the induction base (for short: I.B.)

$$
\begin{equation*}
\left(\forall c \in \min _{\prec}(C)\right): P(c) \tag{2.2}
\end{equation*}
$$

and the second part is the proof of the induction step (for short: I.S.)

$$
\begin{equation*}
\left(\left(\forall c \in C \backslash \min _{\prec}(C)\right):\left[\left(\forall c^{\prime} \in C\right):\left(c^{\prime} \prec c\right) \rightarrow P\left(c^{\prime}\right)\right] \rightarrow P(c)\right) \tag{2.3}
\end{equation*}
$$

For each $c \in C$, the subformula

$$
\left[\left(\forall c^{\prime} \in C\right):\left(c^{\prime} \prec c\right) \rightarrow P\left(c^{\prime}\right)\right]
$$

is called induction hypothesis (for short: I.H.).
There are several instances of well-founded sets and well-founded induction which are relevant for us. Two instances are based on the set $\mathbb{N}$ of natural numbers and we present these instances here (later we will present two more instances which are based on the set of trees):

- proof by induction on $\left(\mathbb{N}, \prec_{\mathbb{N}}\right)$ (for short: proof by induction on $\mathbb{N}$ ): Then $\min _{\prec_{\mathbb{N}}}(\mathbb{N})=\{0\}$, and the induction base (2.2) and the induction step (2.3) read

$$
P(0) \text { and }\left(\left(\forall n \in \mathbb{N}_{+}\right): P(n-1) \rightarrow P(n)\right), \text { respectively, }
$$

- proof by induction on $(\mathbb{N},<)$ : Then $\min _{<}(\mathbb{N})=\{0\}$, and the induction base (2.2) and the induction step (2.3) read

$$
P(0) \text { and }\left(\left(\forall n \in \mathbb{N}_{+}\right):[(\forall k \in[0, n-1]): P(k)] \rightarrow P(n)\right), \text { respectively. }
$$

We also use the principle of definition of a mapping by well-founded induction, which is based on the next theorem. Its historical predecessor is Ded39, 126. Satz der Definition durch Induktion] for mappings of type $f: \mathbb{N} \rightarrow A$ for some set $A$. We follow the proof of that theorem given in Ind76 and generalize it to mappings of type $f: C \rightarrow A$ for any set $C$ and well-founded relation $\prec$ on $C$. For each $c \in C$, we define $\operatorname{pred}_{\prec}(c)=\left\{c^{\prime} \in C \mid c^{\prime} \prec c\right\}$. (We also refer to [Kla84, Thm. 1.17] and [Win93, Thm. 10.19].)

Theorem 2.5.1. (cf. Ded39, 126. Satz]) Let $A$ and $C$ be two sets, $\prec$ a well-founded relation on $C$, and $G:\left\{(c, g) \mid c \in C, g: \operatorname{pred}_{\prec}(c) \rightarrow A\right\} \rightarrow A$ a mapping. Then there exists exactly one mapping $f: C \rightarrow A$ such that, for each $c \in C$, we have

$$
\begin{equation*}
f(c)=G\left(c,\left.f\right|_{\text {pred }_{\prec}(c)}\right) \tag{2.4}
\end{equation*}
$$

Proof. We define the set

$$
\begin{equation*}
S=\left\{T \subseteq C \times A \mid(\forall c \in C)\left(\forall g: \operatorname{pred}_{\prec}(c) \rightarrow A\right):(g \subseteq T) \rightarrow((c, G(c, g)) \in T)\right\} \tag{2.5}
\end{equation*}
$$

We note that $S \neq \emptyset$ because $(C \times A) \in S$. Moreover, we define

$$
\rho=\bigcap(T \mid T \in S)
$$

Next we show the following three statements.
(1) $\rho$ is a mapping of type $\rho: C \rightarrow A$,
(2) for each $c \in C$, we have $\rho(c)=G\left(c,\left.\rho\right|_{\text {pred }_{\prec}(c)}\right)$, and
(3) for each mapping $h: C \rightarrow A$ for which $h(c)=G\left(c,\left.h\right|_{\operatorname{pred}_{\prec}(c)}\right)$ for each $c \in C$, we have $h=\rho$.

From (1)-(3) it follows that $\rho$ is the desired mapping $f$, and hence we have proved the theorem.
Proof of (1): We define

$$
M=\{c \in C \mid \text { there exists exactly one } a \in A \text { such that }(c, a) \in \rho\}
$$

By induction on $(C, \prec)$, we show that $C \subseteq M$. For this, let $c \in C$.
I.B.: Let $c \in \min _{\prec}(C)$. Since $\operatorname{pred}_{\prec}(c)=\emptyset$, we have $(c, G(c, \emptyset)) \in T$ for each $T \in S$. Hence $(c, G(c, \emptyset)) \in \rho$. We show that $c \in M$ by contradiction.

We assume that there exists an $a \in A$ such that $a \neq G(c, \emptyset)$ and $(c, a) \in \rho$. Let $\rho^{\prime}=\rho \backslash\{(c, a)\}$; hence $\rho^{\prime} \subset \rho$. We show that $\rho^{\prime} \in S$. For this, let $c^{\prime} \in C$ and let $g: \operatorname{pred}_{\prec}\left(c^{\prime}\right) \rightarrow A$ be such that $g \subseteq \rho^{\prime}$. Then for this $c^{\prime}$ and $g$ we also have $g \subseteq \rho$. Then by the definition of $\rho$ we have $\left(c^{\prime}, G\left(c^{\prime}, g\right)\right) \in \rho$. Now if $c^{\prime}=c$, then $\left(c^{\prime}, G\left(c^{\prime}, g\right)\right)=(c, G(c, \emptyset)) \neq(c, a)$, Otherwise, again $\left(c^{\prime}, G\left(c^{\prime}, g\right)\right) \neq(c, a)$. Hence in both cases $\left(c^{\prime}, G\left(c^{\prime}, g\right)\right) \in \rho^{\prime}$. Then $\rho^{\prime} \in S$ and thus $\rho \subseteq \rho^{\prime}$, which contradicts $\rho^{\prime} \subset \rho$.
I.S.: Let $c \in C \backslash \min _{\prec}(C)$. By I.H., for each $c^{\prime} \in \operatorname{pred}_{\prec}(c)$, we have $c \in M$. Let $g=\left\{\left(c^{\prime}, a\right) \in \rho \mid c^{\prime} \in\right.$ $\left.\operatorname{pred}_{\prec}(c)\right\}$. Then, by the definition of $\rho$, for each $T \in S$, we have $g \subseteq T$ and hence for each $T \in S$, we also have $(c, G(c, g)) \in T$. Consequently, $(c, G(c, g)) \in \rho$. We show that $c \in M$ by contradiction.

We assume that there is an $a \in A$ such that $a \neq G(c, g)$ and $(c, a) \in \rho$. Let $\rho^{\prime}=\rho \backslash\{(c, a)\}$; hence $\rho^{\prime} \subset \rho$. We show that $\rho^{\prime} \in S$. For this, let $\bar{c} \in C$ and let $g: \operatorname{pred}_{\prec}(\bar{c}) \rightarrow A$ be such that $g \subseteq \rho^{\prime}$. Then for this $\bar{c}$ and $g$ we also have $g \subseteq \rho$. Then by the definition of $\rho$ we have $(\bar{c}, G(\bar{c}, g)) \in \rho$. If $\bar{c}=c$, then $(\bar{c}, G(\bar{c}, g))=(c, G(c, g)) \neq(c, a)$, Otherwise, again $(\bar{c}, G(\bar{c}, g)) \neq(c, a)$. Hence in both cases $(\bar{c}, G(\bar{c}, g)) \in \rho^{\prime}$. Then $\rho^{\prime} \in S$ and thus $\rho \subseteq \rho^{\prime}$, which contradicts $\rho^{\prime} \subset \rho$.

Proof of (2): It is obvious that, for each $c \in C$, we have $\left.\rho\right|_{\text {pred }_{\prec}(c)} \subseteq \rho$. Hence, by the definition of $\rho$, for each $c \in C$, we have $\rho(c)=G\left(c,\left.\rho\right|_{\operatorname{pred}_{\prec}(c)}\right)$.

Proof of (3): Let $h: C \rightarrow A$ be a mapping for which $h(c)=G\left(c,\left.h\right|_{\operatorname{pred}_{\prec}(c)}\right)$ for each $c \in C$. By induction on $(C, \prec)$ we show that, for each $c \in C$, we have $h(c)=\rho(c)$. Let $c \in C$. By I.H. for each $c^{\prime} \in \operatorname{pred}_{\prec}(c)$, we have $h\left(c^{\prime}\right)=\rho\left(c^{\prime}\right)$. Hence, $\left.h\right|_{\operatorname{pred}_{\prec}(c)}=\left.\rho\right|_{\operatorname{pred}_{\prec}(c)}$ and thus $h(c)=G\left(c,\left.h\right|_{\operatorname{pred}_{\prec}(c)}\right)=$ $G\left(c,\left.\rho\right|_{\operatorname{pred}_{\prec}(c)}\right)=\rho(c)$.

We say that the mapping $f$ of Theorem[2.5.1]is defined by well-founded induction on $(C, \prec)$ (for short: defined by induction on $(C, \prec))$.

The mapping $G:\left\{(c, g) \mid c \in C, g: \operatorname{pred}_{\prec}(c) \rightarrow A\right\} \rightarrow A$ which determines by Theorem 2.5.1 a unique mapping $f: C \rightarrow A$, might be called a schema of primitive recursion. Such schemata have been investigated, e.g., for $C=\mathbb{N}$ in Pét57, for $C=\Gamma^{*}$ in vHIRW75, for $C$ being the set of trees over $\Sigma$ in Hup78, EV91, FHVV93, and for $C$ being a decomposition algebra in Kla84.

In many applications of this principle, we will not show the mapping $G$ explicitly, but only implicitly in the definition of $f$. Moreover, we will split (2.4) into an induction base and an induction step as follows:
I.B.: Let $c \in \min _{\prec}(C)$. Hence $\operatorname{pred}_{\prec}(c)=\emptyset$ and thus $\left.f\right|_{\emptyset}: \emptyset \rightarrow A$. Then we define $f(c)=G\left(c,\left.f\right|_{\emptyset}\right)$. That means that $f(c)$ does not depend on elements of $\operatorname{im}(f)$.
I.S.: Let $c \in C \backslash \min _{\prec}(C)$ and we assume that $f\left(c^{\prime}\right)$ is defined for each $c^{\prime} \in \operatorname{pred}_{\prec}(c)$. Then we define $f(c)=G\left(c,\left.f\right|_{\operatorname{pred}_{\prec}(c)}\right)$. That means that $f(c)$ may depend on elements of $\left\{f\left(c^{\prime}\right) \mid c^{\prime} \in \operatorname{pred}_{\prec}(c)\right\}$.
Next we demonstrate the principle of definition by well-founded induction. More precisely, given a mapping $h: D \rightarrow D$ for some set $D$, we define the family of iterated applications ( $h^{n} \mid n \in \mathbb{N}$ ) of $h$ using this principle.

For this we instantiate the objects $C, \prec, A$, and $G$ which occur in the principle as follows: we let $C=\mathbb{N}, \prec=\prec_{\mathbb{N}}$, and $A=D^{D}$. Then $G$ has the type

$$
G:\left\{(n, g) \mid n \in \mathbb{N}, g: \operatorname{pred}_{\prec \mathbb{N}}(n) \rightarrow D^{D}\right\} \rightarrow D^{D}
$$

We note that $\operatorname{pred}_{\prec_{\mathbb{N}}}(0)=\emptyset$ and, for each $n \in \mathbb{N}_{+}$, the set $\operatorname{pred}_{\prec_{\mathbb{N}}}(n)=\{n-1\}$. Now, for every $n \in \mathbb{N}$ and $g: \operatorname{pred}_{\prec_{\mathrm{N}}}(n) \rightarrow D^{D}$, we define

$$
G(n, g)= \begin{cases}\operatorname{id}_{D} & \text { if } n=0 \\ h \circ g(n-1) & \text { if } n \in \mathbb{N}_{+}\end{cases}
$$

By Theorem 2.5.1, there exists a unique mapping $f: \mathbb{N} \rightarrow D^{D}$ such that $f(n)=G\left(n,\left.f\right|_{\operatorname{pred}_{<_{\mathbb{N}}}(n)}\right)$. Finally, we define the family of iterated applications of a mapping $h: D \rightarrow D$, denoted by ( $h^{n} \mid n \in \mathbb{N}$ ), by letting $h^{n}=f(n)$.

Thus, for $n=0$, we have $h^{0}=f(0)=G\left(0,\left.f\right|_{\text {pred }_{<_{N}}(0)}\right)=\operatorname{id}_{D}$. Moreover, for each $n \in \mathbb{N}_{+}$, we have

$$
h^{n}=f(n)=G\left(n,\left.f\right|_{\operatorname{pred}_{<\mathbb{N}}(n)}\right)=G\left(n,\left.f\right|_{\{n-1\}}\right)=\left.h \circ f\right|_{\{n-1\}}(n-1)=h \circ f(n-1)=h \circ h^{n-1}
$$

Hence, the family of iterated applications of a mapping $h: D \rightarrow D$ satisfies the well known equations:

$$
h^{0}=\operatorname{id}_{D} \text { and } h^{n}=h \circ h^{n-1} \text { for each } n \in \mathbb{N}_{+} . \text {In particular, } h^{1}=h
$$

### 2.6 Algebraic structures

### 2.6.1 Ranked alphabets

A ranked alphabet is a pair $(\Sigma, \mathrm{rk})$, where

- $\Sigma$ is an alphabet and
- rk: $\Sigma \rightarrow \mathbb{N}$ is a mapping called rank mapping.

For each $k \in \mathbb{N}$, we denote the set $\mathrm{rk}^{-1}(k)$ by $\Sigma^{(k)}$. Sometimes we write $\sigma^{(k)}$ to indicate that $\sigma \in \Sigma^{(k)}$. We denote $\max \left(k \in \mathbb{N} \mid \Sigma^{(k)} \neq \emptyset\right)$ by maxrk $(\Sigma)$. Whenever the rank mapping is clear from the context or it is irrelevant, then we abbreviate the ranked alphabet $(\Sigma, \mathrm{rk})$ by $\Sigma$. If $\Sigma=\Sigma^{(0)}$, then we call $\Sigma$ trivial, and if $\Sigma=\Sigma^{(1)} \cup \Sigma^{(0)}$, then we call $\Sigma$ monadic. A monadic ranked alphabet with $\left|\Sigma^{(1)}\right| \geq 1$ and $\left|\Sigma^{(0)}\right|=1$ is called a string ranked alphabet.

### 2.6.2 $\Sigma$-algebras, subalgebras, congruences, and homomorphisms

We recall some notions from universal algebra Grä68, GTWW77, BS81, Wec92.
Let $\Sigma$ be a ranked alphabet. A $\Sigma$-algebra is a pair $\mathrm{A}=(A, \theta)$ which consists of a nonempty set $A$ and a $\Sigma$-indexed family $\theta$ over $\operatorname{Ops}(A)$ such that $\theta(\sigma): A^{k} \rightarrow A$ for every $k \in \mathbb{N}$ and $\sigma \in \Sigma^{(k)}$. We call $A$
and $\theta$ the carrier set and the $\Sigma$-interpretation (or: interpretation of $\Sigma$ ), respectively, of $A$. We denote the set $\{\theta(\sigma) \mid \sigma \in \Sigma\}$ of operations by $\theta(\Sigma)$.

In the following, let $\mathrm{A}=(A, \theta)$ be a $\Sigma$-algebra. Moreover, let $A^{\prime} \subseteq A$ and $O \subseteq \theta(\Sigma)$. We say that $A^{\prime}$ is closed under the operations in $O$, if, for every $k \in \mathbb{N}$, $k$-ary operation $\theta(\sigma) \in O$, and $a_{1}, \ldots, a_{k} \in A^{\prime}$, we have that $\theta(\sigma)\left(a_{1}, \ldots, a_{k}\right) \in A^{\prime}$. We denote by $\left\langle A^{\prime}\right\rangle_{O}$ the smallest subset of $A$ which contains $A^{\prime}$ and is closed under the operations in $O$.

A subalgebra of A is a $\Sigma$-algebra $\left(A^{\prime}, \theta^{\prime}\right)$ such that $A^{\prime} \subseteq A$ and $A^{\prime}$ is closed under the operations in $\theta(\Sigma)$ and for every $k \in \mathbb{N}$ and $\sigma \in \Sigma^{(k)}$, we have $\theta^{\prime}(\sigma)=\left.\theta(\sigma)\right|_{\left(A^{\prime}\right)^{k}}$. For the sake of convenience, we will drop the prime from $\theta^{\prime}$ in the sequel. For each $H \subseteq A$, the subalgebra of A generated by $H$ is the subalgebra $\left(\langle H\rangle_{\theta(\Sigma)}, \theta\right)$ of A . The smallest subalgebra of A is the subalgebra of A generated by $\emptyset$.

Let $\mathrm{A}=(A, \theta)$ be a $\Sigma$-algebra. We say that A is

- finite if $A$ is finite,
- locally finite if, for each finite subset $H \subseteq A$, the set $\langle H\rangle_{\theta(\Sigma)}$ is finite, and
- finitely generated if there exists a finite subset $H \subseteq A$ such that $\langle H\rangle_{\theta(\Sigma)}=A$.

Lemma 2.6.1. Let $\mathrm{A}=(A, \theta)$ be a $\Sigma$-algebra and $H \subseteq A$. Let $f: \mathcal{P}(A) \rightarrow \mathcal{P}(A)$ be defined for each $U \in \mathcal{P}(A)$ by

$$
f(U)=U \cup\left\{\theta(\sigma)\left(u_{1}, \ldots, u_{k}\right) \mid k \in \mathbb{N}, \sigma \in \Sigma^{(k)}, u_{1}, \ldots, u_{k} \in U\right\}
$$

Then $\langle H\rangle_{\theta(\Sigma)}=\bigcup\left(f^{n}(H) \mid n \in \mathbb{N}\right)$. Moreover, if $\langle H\rangle_{\theta(\Sigma)}$ is finite, then we can construct it.
Proof. Obviously, for every $U, U^{\prime} \in \mathcal{P}(A)$,
for each $n \in \mathbb{N}$, we have $f^{n}(U) \subseteq f^{n+1}(U)$, and

$$
\begin{equation*}
\text { if } U \subseteq U^{\prime} \text { then } f(U) \subseteq f\left(U^{\prime}\right) \tag{2.6}
\end{equation*}
$$

Let us abbreviate $\bigcup\left(f^{n}(H) \mid n \in \mathbb{N}\right)$ by $V$ and show that $V=\langle H\rangle_{\theta(\Sigma)}$.
First we show that $f(V) \subseteq V$ as follows:

$$
\begin{aligned}
f(V) & =f\left(\bigcup\left(f^{n}(H) \mid n \in \mathbb{N}\right)\right) \\
& =\bigcup\left(f^{n}(H) \mid n \in \mathbb{N}\right) \cup\left\{\theta(\sigma)\left(u_{1}, \ldots, u_{k}\right) \mid k \in \mathbb{N}, \sigma \in \Sigma^{(k)},(\forall i \in[k]): u_{i} \in \bigcup\left(f^{n}(H) \mid n \in \mathbb{N}\right)\right\} \\
& =\bigcup\left(f^{n}(H) \mid n \in \mathbb{N}\right) \cup\left\{\theta(\sigma)\left(u_{1}, \ldots, u_{k}\right) \mid k \in \mathbb{N}, \sigma \in \Sigma^{(k)},(\forall i \in[k])\left(\exists n_{i} \in \mathbb{N}\right): u_{i} \in f^{n_{i}}(H)\right\} \\
& =\bigcup\left(f^{n}(H) \mid n \in \mathbb{N}\right) \cup\left\{\theta(\sigma)\left(u_{1}, \ldots, u_{k}\right) \mid k \in \mathbb{N}, \sigma \in \Sigma^{(k)},(\exists n \in \mathbb{N})(\forall i \in[k]): u_{i} \in f^{n}(H)\right\} \\
& \left.=\bigcup\left(f^{n}(H) \cup\left\{\theta(\sigma)\left(u_{1}, \ldots, u_{k}\right) \mid k \in \mathbb{N}, \sigma \in \Sigma^{(k)},(\forall i \in[k]): u_{i} \in f^{n}(H)\right\} \mid n \in \mathbb{N}\right)\right) \\
& =\bigcup\left(f\left(f^{n}(H)\right) \mid n \in \mathbb{N}\right)=\bigcup\left(f^{n}(H) \mid n \in \mathbb{N}_{+}\right) \subseteq V
\end{aligned}
$$

Hence $V$ is closed under the operations of $\theta(\Sigma)$ and thus $(V, \theta)$ is a subalgebra of A such that $H \subseteq V$. Since $\left(\langle H\rangle_{\theta(\Sigma)}, \theta\right)$ is the smallest subalgebra of A with this property, we have $\langle H\rangle_{\theta(\Sigma)} \subseteq V$.

Next we prove the inclusion from right to left. By induction on $\mathbb{N}$, we show that $f^{n}(H) \subseteq\langle H\rangle_{\theta(\Sigma)}$ for each $n \in \mathbb{N}$. The statement is obvious for $n=0$. Then for each $n \in \mathbb{N}$, we have

$$
f^{n+1}(H)=f\left(f^{n}(H)\right) \subseteq f\left(\langle H\rangle_{\theta(\Sigma)}\right) \subseteq\langle H\rangle_{\theta(\Sigma)}
$$

where the first inclusion follows from the I.H. and (2.7), and the second inclusion follows from the fact that $\langle H\rangle_{\theta(\Sigma)}$, being a subalgebra, is closed under the operations of $\theta(\Sigma)$. Hence, we obtain $V \subseteq\langle H\rangle_{\theta(\Sigma)}$.

For each $n \in \mathbb{N}$, if $f^{n}(H)=f^{n+1}(H)$, then $f^{n+1}(H)=f^{n+2}(H)$ obviously. Now assume that $\langle H\rangle_{\theta(\Sigma)}$ is finite. Then we can find $N \in \mathbb{N}$ such that $f^{N}(H)=f^{N+1}(H)$ and therefore $\langle H\rangle_{\theta(\Sigma)}=\bigcup\left(f^{n}(H) \mid n \in\right.$ $[0, N])$. Hence we can construct the set $\langle H\rangle_{\theta(\Sigma)}$.

Let $\mathrm{A}=(A, \theta)$ be a $\Sigma$-algebra and $\sim \subseteq A \times A$ be an equivalence relation on $A$. We call $\sim$ a congruence relation on A if, for every $k \in \mathbb{N}, \sigma \in \Sigma^{(k)}, a_{1}, b_{1}, \ldots, a_{k}, b_{k} \in A$, we have that $a_{1} \sim b_{1}, \ldots, a_{k} \sim b_{k}$ implies $\theta(\sigma)\left(a_{1}, \ldots, a_{k}\right) \sim \theta(\sigma)\left(b_{1}, \ldots, b_{k}\right)$.

Let $\mathrm{A}=(A, \theta)$ be a $\Sigma$-algebra and $\sim$ a congruence relation on A . The quotient algebra of A modulo $\sim$ is the $\Sigma$-algebra $\mathrm{A} / \sim=(A / \sim, \theta / \sim)$, where $A / \sim$ is the factor set of $A$ modulo $\sim$ and, for every $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $a_{1}, \ldots, a_{k} \in A$ we have

$$
(\theta / \sim)(\sigma)\left(\left[a_{1}\right]_{\sim}, \ldots,\left[a_{k}\right]_{\sim}\right)=\left[\theta(\sigma)\left(a_{1}, \ldots, a_{k}\right)\right]_{\sim} .
$$

Clearly, this operation is well defined, cf. Grä68, p.36].
Let $\mathrm{A}_{1}=\left(A_{1}, \theta_{1}\right)$ and $\mathrm{A}_{2}=\left(A_{2}, \theta_{2}\right)$ be two $\Sigma$-algebras. Moreover, let $h: A_{1} \rightarrow A_{2}$ be a mapping. Then $h$ is a $\Sigma$-algebra homomorphism (from $\mathrm{A}_{1}$ to $\mathrm{A}_{2}$ ) if, for every $k \in \mathbb{N}, \sigma \in \Sigma^{(k)}$, and for every $v_{1}, \ldots, v_{k} \in A_{1}$, we have

$$
\begin{equation*}
h\left(\theta_{1}(\sigma)\left(v_{1}, \ldots, v_{k}\right)\right)=\theta_{2}(\sigma)\left(h\left(v_{1}\right), \ldots, h\left(v_{k}\right)\right) \tag{2.8}
\end{equation*}
$$

If $h$ is bijective, then $h$ is a $\Sigma$-algebra isomorphism. If there exists such an isomorphism, then we say that $\mathrm{A}_{1}$ and $\mathrm{A}_{2}$ are isomorphic and we denote this fact by $\mathrm{A}_{1} \cong \mathrm{~A}_{2}$.
Lemma 2.6.2. Grä68, $\S 7, \operatorname{Lm} .3]$ Let $\left(A_{1}, \theta_{1}\right)$ and $\left(A_{2}, \theta_{2}\right)$ be two $\Sigma$-algebras, and let $h: A_{1} \rightarrow A_{2}$ be a $\Sigma$-algebra homomorphism. Then $\operatorname{im}(h)$ is closed under the operations in $\theta_{2}(\Sigma)$. Hence $\left(\operatorname{im}(h), \theta_{2}\right)$ is a subalgebra of $\left(A_{2}, \theta_{2}\right)$.
Theorem 2.6.3. Grä68, §7, Lm. 5] Let $\left(A_{1}, \theta_{1}\right),\left(A_{2}, \theta_{2}\right)$, and $\left(A_{3}, \theta_{3}\right)$ be three $\Sigma$-algebras. Moreover, let $h: A_{1} \rightarrow A_{2}$ and $g: A_{2} \rightarrow A_{3}$ be two $\Sigma$-algebra homomorphisms. Then $g \circ h$ is a $\Sigma$-algebra homomorphism.

Let $h$ be a $\Sigma$-algebra homomorphism from $\left(A_{1}, \theta_{1}\right)$ to $\left(A_{2}, \theta_{2}\right)$. The kernel of $h$, denoted by $\operatorname{ker}(h)$, is the equivalence relation on $A_{1}$ defined by $\operatorname{ker}(h)=\left\{\left(a_{1}, a_{2}\right) \in A_{1} \times A_{1} \mid h\left(a_{1}\right)=h\left(a_{2}\right)\right\}$.
Theorem 2.6.4. Grä68, §7, Lm. 6], Grä68, §11, Thm. 1] Let $h$ be a $\Sigma$-algebra homomorphism from $\mathrm{A}_{1}=\left(A_{1}, \theta_{1}\right)$ to $\mathrm{A}_{2}=\left(A_{2}, \theta_{2}\right)$. Then $\operatorname{ker}(h)$ is a congruence relation on $\mathrm{A}_{1}$. Moreover, if $h$ is surjective, then $\mathrm{A}_{1} / \operatorname{ker}(h) \cong \mathrm{A}_{2}$.

Let $\mathcal{K}$ be a set of $\Sigma$-algebras, $\mathrm{A}=(A, \theta)$ be a $\Sigma$-algebra, and $H \subseteq A$. The algebra A is called freely generated by $H$ over $\mathcal{K}$ if the following conditions hold:
(a) $\mathrm{A} \in \mathcal{K}$,
(b) $A=\langle H\rangle_{\theta(\Sigma)}$, and
(c) for every $\Sigma$-algebra $\mathrm{A}^{\prime}=\left(A^{\prime}, \theta^{\prime}\right) \in \mathcal{K}$ and mapping $f: H \rightarrow A^{\prime}$, there exists a unique extension of $f$ to a $\Sigma$-algebra homomorphism $h: A \rightarrow A^{\prime}$ from A to $\mathrm{A}^{\prime}$.
The set $H$ is called generator set. If $H=\emptyset$, then A is called initial in $\mathcal{K}$. In this case, for each $\mathrm{A}^{\prime}=$ $\left(A^{\prime}, \theta^{\prime}\right) \in \mathcal{K}$ there exists exactly one $\Sigma$-algebra homomorphism $h: A \rightarrow A^{\prime}$ from A to $\mathrm{A}^{\prime}$.

Theorem 2.6.5. Grä68, §24, Thm. 1] Let $\mathcal{K}$ be a set of $\Sigma$-algebras. Any two algebras freely generated by the same generator set over $\mathcal{K}$ are isomorphic.

In each of the Subsections 2.6.4-2.6.6 we will consider particular ranked alphabets $\Sigma$ and consider the set of all $\Sigma$-algebras or the set of those $\Sigma$-algebras which satisfy certain algebraic laws (cf. Subsection 2.6.3). Then $\Sigma$ is clear from the context and can be neglected in both the name and the notation. In this case, we call a $\Sigma$-algebra $\mathrm{A}=(A, \theta)$ an algebra and specify it by its carrier set $A$ and by listing all its operations, i.e., we write $\mathrm{A}=\left(A, \theta\left(\sigma_{1}\right), \ldots, \theta\left(\sigma_{n}\right)\right)$, where $\Sigma=\left\{\sigma_{1}, \ldots, \sigma_{n}\right\}$.

### 2.6.3 Properties of binary operations

Let $B$ be a nonempty set. Let $\odot$ be a binary operation on $B$. The operation $\odot$ is

- associative if $(a \odot b) \odot c=a \odot(b \odot c)$ for every $a, b, c \in B$,
- commutative if $a \odot b=b \odot a$ for every $a, b \in B$,
- idempotent if $a \odot a=a$ for each $a \in B$,
- extremal if $a \odot b \in\{a, b\}$ for every $a, b \in B$.

Obviously, extremal implies idempotent. The other direction does not hold, e.g., the set union $\cup$ is idempotent but in general not extremal.

An element $e \in B$ is an identity element (of $\odot$ ) if $e \odot a=a \odot e=a$ for every $a \in B$. There is at most one identity element. Let $e \in B$ be an identity element and $a \in B$. An element $\bar{a} \in B$ is an inverse of $a$ (with respect to $\odot$ ) if $a \odot \bar{a}=\bar{a} \odot a=e$. If an inverse of $a$ exists, then we also say that $a$ has an inverse. If $\odot$ is associative, then each element has at most one inverse.

Let $\oplus$ and $\otimes$ be two binary operations on $B$. The operation $\otimes$ is

- right-distributive (with respect to $\oplus)$, if $(a \oplus b) \otimes c=(a \otimes c) \oplus(b \otimes c)$ for every $a, b, c \in B$,
- left-distributive (with respect to $\oplus)$, if $a \otimes(b \oplus c)=(a \otimes b) \oplus(a \otimes c)$ for every $a, b, c \in B$,
- distributive (with respect to $\oplus$ ) if it is both right-distributive and left-distributive (with respect to $\oplus)$.
Clearly, if $\otimes$ is commutative, then right-distributivity implies left-distributivity and vice versa. Moreover, $\oplus$ and $\otimes$ satisfy the absorption axiom if
- $a \otimes(a \oplus b)=a$ and $a \oplus(a \otimes b)=a$ for every $a, b \in B$.


### 2.6.4 Semigroups, monoids, and groups

In this subsection we recall the definitions of semigroups, monoids, and groups in the form described in the last paragraph of Subsection 2.6.2. Moreover, we define extensions of the underlying binary operations to finitely many and countably many arguments.

- A semigroup is an algebra $(B, \odot)$ where $\odot$ is an associative binary operation on $B$.
- A monoid is an algebra $(B, \odot, e)$ where $(B, \odot)$ is a semigroup and $e$ is an element of $B$ which is an identity element of $\odot$. We note that the identity element $e$ can be considered as a nullary operation on $B$.
- A group is a monoid $(B, \odot, e)$ such that each $a \in B$ has an inverse. We note that the inverses of the elements can be defined in terms of a unary operation on $B$.
The semigroup $(B, \odot)$ is commutative if $\odot$ is commutative. Similarly, we define commutative monoids and commutative groups.

Observation 2.6.6. Let $(B, \odot)$ be a semigroup such that $\odot$ is commutative and idempotent. Then $(B, \odot)$ is locally finite.

Proof. Let $H$ be a finite subset of $B$. We note that $\langle H\rangle_{\{\odot\}}=\left\{b_{1} \odot \ldots \odot b_{n} \mid n \in \mathbb{N}_{+}, b_{1}, \ldots, b_{n} \in H\right\}$. Let us abbreviate this set by $\langle H\rangle$.

By induction on $\left(\mathbb{N}_{+}, \prec_{\mathbb{N}_{+}}\right)$, where $\prec_{\mathbb{N}_{+}}=\left\{(n, n+1) \mid n \in \mathbb{N}_{+}\right\}$, we prove the following statement.

$$
\begin{equation*}
\text { For each } n \in \mathbb{N}_{+} \text {and every } b_{1}, \ldots, b_{n} \in H \text {, there exist } m \in \mathbb{N}_{+} \tag{2.9}
\end{equation*}
$$

and pairwise different $c_{1}, \ldots, c_{m} \in H$ such that $b_{1} \odot \ldots \odot b_{n}=c_{1} \odot \ldots \odot c_{m}$.
I.B.: For $n=1$ the statement is trivially true.
I.S.: Let $b=b_{1} \odot \cdots \odot b_{n+1}$ with $n \in \mathbb{N}_{+}$. By I.H. there exist $m \in \mathbb{N}_{+}$and pairwise different $c_{1}, \ldots, c_{m} \in H$ such that $b_{1} \odot \ldots \odot b_{n}=c_{1} \odot \ldots \odot c_{m}$.

If $b_{n+1} \notin\left\{c_{1}, \ldots, c_{m}\right\}$, then (2.9) holds with $m+1$ by letting $c_{m+1}=b_{n+1}$. Otherwise $b_{n+1}=c_{i}$ for some $i \in[m]$. Then, by commutativity and idempotency, we have

$$
b_{1} \odot \cdots \odot b_{n+1}=c_{1} \odot \ldots \odot c_{m} \odot b_{n+1}=c_{1} \odot \ldots \odot c_{i} \odot b_{n+1} \odot \ldots \odot c_{m}=c_{1} \odot \ldots \odot c_{m}
$$

Thus (2.9) holds with $m$. This proves (2.9). Since $H$ is finite, by (2.9) also $\langle H\rangle$ is finite.
Next we consider a monoid $(B, \odot, e)$ and formalize two ways of extending $\odot$ to finitely many arguments. Let $I$ be a finite set with $I=\left\{i_{1}, \ldots, i_{k}\right\}$ for some $k \in \mathbb{N}$. If (a) $I \subseteq \mathbb{N}$ and $i_{1}<\ldots<i_{k}$, where $<$ is the usual total order "less than" on natural numbers or $(\mathrm{b}) \odot$ is commutative, then we define the operation $\odot_{I}: B^{I} \rightarrow B$ such that, for each $I$-indexed family $\left(b_{i} \mid i \in I\right)$ of elements in $B$, we have

$$
\bigodot_{I}\left(b_{i} \mid i \in I\right)= \begin{cases}b_{i_{1}} \odot \ldots \odot b_{i_{k}} & \text { if } I \neq \emptyset  \tag{2.10}\\ e & \text { otherwise }\end{cases}
$$

We abbreviate $\bigodot_{I}\left(b_{i} \mid i \in I\right)$ by $\odot\left(b_{i} \mid i \in I\right)$ or $\bigodot_{i \in I} b_{i}$. Thus, in particular, $\bigodot_{i \in[k]} b_{i}=b_{1} \odot \ldots \odot b_{k}$ for each $k \in \mathbb{N}_{+}$, and $\bigodot_{i \in \emptyset} b_{i}=e$.
Observation 2.6.7. Let $I$ be a finite and nonempty index set such that $I \subseteq \mathbb{N}$ or $\odot$ is commutative. If $\odot$ is extremal, then for each $I$-indexed family $\left(b_{i} \mid i \in I\right)$ over $B$ there exists a $j \in I$ such that $b_{j}=\bigodot_{i \in I} b_{i}$.

Next let $(B, \odot, e)$ be commutative. We introduce a notation for the case that the index set is the Cartesian product of two sets as follows. Let $I$ and $J$ be finite sets and $\left(b_{(i, j)} \mid(i, j) \in I \times J\right)$ be an $(I \times J)$-indexed family over $B$. Then it follows from commutativity and associativity that

$$
\bigodot_{(i, j) \in I \times J} b_{(i, j)}=\bigodot_{i \in I} \bigodot_{j \in J} b_{(i, j)}
$$

Often we will abbreviate the expression $\bigodot_{i \in I} \bigodot_{j \in J} b_{(i, j)}$ by $\bigodot_{i \in I, j \in J} b_{(i, j)}$. We will use this kind of abbreviation also in the case where the index set is the Cartesian product of finitely many sets.

Finally, we consider a commutative monoid $(B, \odot, e)$ and identify algebraic laws such that $\odot$ can be extended to countably many arguments. We call $(B, \odot, e) \sigma$-complete if, for each countable index set $I$, there exists a mapping $\sum_{I}^{\odot}: B^{I} \rightarrow B$ such that for each $I$-indexed family $\left(b_{i} \mid i \in I\right)$ over $B$ the equalities (2.11) and (2.12) below are satisfied (cf. [Eil74, p. 124]) using $\sum_{i \in I}^{\odot} b_{i}$ as abbreviation for $\sum_{I}^{\odot}\left(b_{i} \mid i \in I\right):$

$$
\begin{array}{r}
\text { If } I=\{j\}, \text { then } \sum_{i \in I}^{\odot} b_{i}=b_{j}, \quad \text { and if } I=\left\{j, j^{\prime}\right\}, \text { then } \sum_{i \in I}^{\odot} b_{i}=b_{j} \odot b_{j^{\prime}} \\
\sum_{i \in I}^{\odot} b_{i}=\sum_{j \in J}^{\odot}\left(\sum_{i \in I_{j}}^{\odot} b_{i}\right) \text { for every countable set } J \text { and partitioning }\left(I_{j} \mid j \in J\right) \text { of } I . \tag{2.12}
\end{array}
$$

Observation 2.6.8. Let $(B, \odot, e)$ be a $\sigma$-complete monoid. The following statements hold.
(1) For each $\emptyset$-indexed family $\left(b_{i} \mid i \in \emptyset\right)$ over $B$, we have $\sum_{j \in \emptyset}^{\odot} b_{j}=e$.
(2) For each finite $I$-indexed family $\left(b_{i} \mid i \in I\right)$ of elements of $B$, we have $\sum_{i \in I}^{\odot} b_{i}=\bigodot_{i \in I} b_{i}$.
(3) For each countable index set $I$, we have $\sum_{j \in I}^{\odot} e=e$.

Proof. Proof of (1): Let $I=\{1\}$. We consider the $I$-indexed family $\left(b_{i} \mid i \in I\right)$ over $B$ where $b_{1}$ is an arbitrary element of $B$ (recall that $B \neq \emptyset$ ). Moreover, let $J=\{1,2\}, I_{1}=\emptyset$, and $I_{2}=I$. Then we have

$$
\begin{equation*}
b_{1}=\sum_{i \in I}^{\odot} b_{i} \tag{2.11}
\end{equation*}
$$

$$
\begin{align*}
& =\sum_{j \in J}^{\odot}\left(\sum_{i \in I_{j}}^{\odot} b_{i}\right)=\left(\sum_{i \in I_{1}}^{\odot} b_{i}\right) \odot\left(\sum_{i \in I_{2}}^{\odot} b_{i}\right) \\
& =\left(\sum_{i \in \emptyset}^{\odot} b_{i}\right) \odot b_{1} \tag{2.11}
\end{align*}
$$

Since $\odot$ is commutative, also $b_{1} \odot\left(\sum_{i \in \emptyset}^{\odot} b_{i}\right)=b_{1}$. Hence $\sum_{i \in \emptyset}^{\odot} b_{i}$ is an identity element, and hence $\sum_{i \in \emptyset}^{\odot} b_{i}=e$.

Proof of (2): Let $\left(b_{i} \mid i \in I\right)$ be a finite $I$-indexed family of elements of $B$. If $I=\emptyset$, then the statement follows from Statement (1) and (2.10). If $I \neq \emptyset$, then we can prove by induction on $\mathbb{N}$ the following statement:

$$
\begin{equation*}
\text { For every } n \in \mathbb{N} \text { and index set } I \text { with }|I|=n+1, \text { we have } \sum_{i \in I}^{\odot} b_{i}=\bigodot_{i \in I} b_{i} \tag{2.13}
\end{equation*}
$$

I.B.: For $n=0$, this follows from (2.11) and (2.10).
I.S.: Now let $n \in \mathbb{N}_{+}$and $|I|=n+1$. Moreover, let $\left(b_{i} \mid i \in I\right)$ be a finite $I$-indexed family of elements of $B$. Let $i^{\prime} \in I$ be an arbitrary index. Using $J=\{1,2\}$ and $I_{1}=\left\{i^{\prime}\right\}$ and $I_{2}=I \backslash\left\{i^{\prime}\right\}$, we have:

$$
\begin{aligned}
\sum_{i \in I}^{\odot} b_{i} & =\sum_{j \in J}^{\odot}\left(\sum_{i \in I_{j}}^{\odot} b_{i}\right)=\left(\sum_{i \in\left\{i^{\prime}\right\}}^{\odot} b_{i}\right) \odot\left(\sum_{i \in I \backslash\left\{i^{\prime}\right\}}^{\odot} b_{i}\right) \\
& =b_{i^{\prime}} \odot \bigodot_{i \in I \backslash\left\{i^{\prime}\right\}} b_{i} \\
& =\bigodot_{i \in I} b_{i}
\end{aligned}
$$

Then Statement (2) follows from (2.13).
Proof of (3): Let $I_{j}=\emptyset$ for every $j \in I$. Then $\left(I_{j} \mid j \in I\right)$ is a partitioning of $\emptyset$. Hence we have

$$
\begin{align*}
& \sum_{j \in I}^{\odot} e=\sum_{j \in I}^{\odot}\left(\bigodot_{i \in I_{j}} e\right)  \tag{2.10}\\
& =\sum_{j \in I}^{\odot}\left(\sum_{i \in I_{j}}^{\odot} e\right) \\
& =\sum_{j \in \emptyset}^{\odot} e \quad \quad \text { (because } \bigcup_{j \in I} I_{j}=\emptyset \text { and by (2.12)) } \\
& =e \\
& \text { (by Statement (2)) } \\
& \text { (by Statement (1)) }
\end{align*}
$$

Let $\left(b_{i} \mid i \in I\right)$ be an $I$-indexed family over $B$. Moreover, let $P \subseteq B$ be a property. Then we denote by ( $b_{i} \mid i \in I, b_{i} \in P$ ) the $I^{\prime}$-indexed family ( $b_{i} \mid i \in I^{\prime}$ ) over $B$ where $I^{\prime}=\left\{i \in I \mid b_{i} \in P\right\}$. Also, we abbreviate $\odot\left(b_{i} \mid i \in I, b_{i} \in P\right)$ by $\bigodot_{\substack{i \in I . \\ b_{i} \in P}} b_{i}$ and $\sum_{I}^{\odot}\left(b_{i} \mid i \in I, b_{i} \in P\right)$ by $\sum_{\substack{i \in I, b_{i} \in P}}^{\odot} b_{i}$.

### 2.6.5 Strong bimonoids, semirings, rings, and fields

A strong bimonoid DSV10, Rad10, CDIV10, DV10, DV12 is an algebra $\mathrm{B}=(B, \oplus, \otimes, 0, \mathbb{1})$ where $(B, \oplus, \mathbb{O})$ is a commutative monoid, $(B, \otimes, \mathbb{1})$ is a monoid, $\mathbb{O} \neq \mathbb{1}$, and $\mathbb{O}$ is an annihilator for $\otimes$, i.e.,
$b \otimes \mathbb{O}=\mathbb{O} \otimes b=\mathbb{O}$ holds for every $b \in B$. The operations $\oplus$ and $\otimes$ are called summation and multiplication, respectively.

Let $\mathrm{B}=(B, \oplus, \otimes, \mathbb{0}, \mathbb{1})$ be a strong bimonoid. It is

- commutative, if $\otimes$ is commutative,
- left-distributive, if $\otimes$ is left-distributive (with respect to $\oplus$ ),
- right-distributive, if $\otimes$ is right-distributive (with respect to $\oplus$ ),
- distributive, if $\otimes$ is left-distributive and right-distributive (with respect to $\oplus$ ),
- $\sigma$-complete if $(B, \oplus, \mathbb{0})$ is $\sigma$-complete, and,
- bi-locally finite if $(B, \oplus, \mathbb{O})$ and $(B, \otimes, \mathbb{1})$ are locally finite.

We declare that the precedence order of the two operations in expressions is as follows: first $\otimes$, second $\oplus$. Using this convention, we can save some parentheses in expressions over B which use both operations. For instance

$$
\bigoplus_{i \in I}\left(b_{i} \otimes b\right) \quad \text { can be written as } \bigoplus_{i \in I} b_{i} \otimes b
$$

for each finite index set $I, I$-indexed family $\left(b_{i} \mid i \in I\right)$, and $b \in B$. Similarly, if B is $\sigma$-complete, then

$$
\sum_{i \in I}^{\oplus}\left(b_{i} \otimes b\right) \text { can be written as } \sum_{i \in I}^{\oplus} b_{i} \otimes b
$$

for each countable index set $I, I$-indexed family $\left(b_{i} \mid i \in I\right)$, and $b \in B$.
A semiring HW93, Gol99 is a strong bimonoid $\mathrm{B}=(B, \oplus, \otimes, \mathbb{O}, \mathbb{1})$ which is distributive. Clearly, if B is a semiring, then for every finite and nonempty family $\left(b_{i} \mid i \in I\right)$ and $b \in B$ the following equalities hold: $\bigoplus_{i \in I} b \otimes b_{i}=b \otimes\left(\bigoplus_{i \in I} b_{i}\right)$ and $\bigoplus_{i \in I} b_{i} \otimes b=\left(\bigoplus_{i \in I} b_{i}\right) \otimes b$.

A semiring $(B, \oplus, \otimes, \mathbb{O}, \mathbb{1})$ is $\sigma$-complete if $(B, \oplus, \mathbb{O})$ is $\sigma$-complete and the following equalities hold for every countable index set $I, I$-indexed family $\left(b_{i} \mid i \in I\right)$, and $b \in B$ :

$$
\begin{equation*}
\sum_{i \in I}^{\oplus} b \otimes b_{i}=b \otimes\left(\sum_{i \in I}^{\oplus} b_{i}\right) \text { and } \sum_{i \in I}^{\oplus} b_{i} \otimes b=\left(\sum_{i \in I}^{\oplus} b_{i}\right) \otimes b \tag{2.14}
\end{equation*}
$$

We note that for semirings $\sigma$-completeness is slightly more general than completeness (cf. e.g. Eil74, p. 125] and EK03] because completeness requires the extension of $\oplus$ not only for countable but arbitrary index sets $I$.

Let $(B, \oplus, \otimes, \mathbb{0}, \mathbb{1})$ be a semiring. It is a

- $\operatorname{ring}$ if $(B, \oplus, \mathbb{O})$ is a commutative group (where we usually denote the inverse of $a \in B$ with respect to $\oplus$ by $-a)$,
- semifield (or: division semiring) if $(B \backslash\{0\}, \otimes, \mathbb{1})$ is a group (where we usually denote the inverse of $a \in B$ with respect to $\otimes$ by $a^{-1}$ ), and
- field if it is a ring and a commutative semifield.

Next we give a number of examples. We start with examples of semirings because they are more familiar than strong bimonoids.
Example 2.6.9. Here we show examples of semirings, rings, semifields, and fields. If an example shows a $\sigma$-complete semiring, i.e., there exists a mapping $\sum^{\oplus}$ which satisfies equalities (2.11), (2.12), and (2.14), then we give such a mapping.

1. The Boolean semiring Boole $=(\mathbb{B}, \vee, \wedge, 0,1)$, where $\mathbb{B}=\{0,1\}$ (the truth values) and $\vee$ and $\wedge$ denote disjunction and conjunction, respectively. The Boolean semiring Boole is $\sigma$-complete with the mapping

$$
\sum_{I}^{\vee}: \mathbb{B}^{I} \rightarrow \mathbb{B} \quad \text { with } \quad\left(b_{i} \mid i \in I\right) \mapsto \begin{cases}1 & \text { if there exists } i \in I \text { such that } b_{i}=1 \\ 0 & \text { otherwise }\end{cases}
$$

We note that Boole is a semifield.
2. The semiring $\operatorname{Nat}=(\mathbb{N},+, \cdot, 0,1)$ of natural numbers, where + and $\cdot$ are the usual addition and multiplication.
3. The semiring $\operatorname{Nat}_{\infty}=\left(\mathbb{N}_{\infty},+, \cdot, 0,1\right)$ of natural numbers, where + and $\cdot$ are extended to $\mathbb{N}_{\infty}$ in the natural way. The semiring $\mathrm{Nat}_{\infty}$ is $\sigma$-complete with the mapping

$$
\sum_{I}^{+}:\left(\mathbb{N}_{\infty}\right)^{I} \rightarrow \mathbb{N}_{\infty} \quad \text { with } \quad\left(n_{i} \mid i \in I\right) \mapsto \begin{cases}+_{i \in J} n_{j} & \text { if }\left\{n_{i} \mid i \in I\right\} \subseteq \mathbb{N} \text { and } \\ & J=\left\{i \in I \mid n_{i} \neq 0\right\} \text { is finite } \\ \infty & \text { otherwise }\end{cases}
$$

(We recall that + denotes the extension of + to finite sums in the monoid $(\mathbb{N},+, 0)$. )
4. The ring Int $=(\mathbb{Z},+, \cdot, 0,1)$ of integers.
5. The ring Intmod4 $=\left(\{0,1,2,3\},+_{4}, \cdot 4,0,1\right)$ where $+_{4}$ and $\cdot 4$ are the usual addition modulo 4 and the usual multiplication modulo 4 , respectively.
6. The field Rat $=(\mathbb{Q},+, \cdot, 0,1)$ of rational numbers and the field Real $=(\mathbb{R},+, \cdot, 0,1)$ of real numbers.
7. There exist only two semirings $(\{0,1\}, \oplus, \otimes, 0,1)$ with exactly two elements because, for every $a, b \in\{0,1\}$ and $\odot \in\{\oplus, \otimes\}$, the value of $a \odot b$ is determined by the strong bimonoid axioms, except the value of $1 \oplus 1$.
(a) If we define $1 \oplus 1=0$, then $(\{0,1\}, \oplus, \otimes, 0,1)$ is a field, denoted by $F_{2}$; we have $-0=0$ and $-1=1^{-1}=1$
(b) If we define $1 \oplus 1=1$, then $(\{0,1\}, \oplus, \otimes, 0,1)$ is the Boolean semiring Boole.
8. The arctic semiring $\mathrm{Nat}_{\max ,+}=\left(\mathbb{N}_{-\infty}, \max ,+,-\infty, 0\right)$. The arctic semiring is often called max-plus semiring.
9. The semiring $\operatorname{Nat}_{\max ,+, n}=\left([0, n]_{-\infty}, \max , \hat{+}_{n},-\infty, 0\right)$ where $n \in \mathbb{N}_{+},[0, n]_{-\infty}=[0, n] \cup\{-\infty\}$, and $b_{1} \hat{+}_{n} b_{2}=\min \left(b_{1}+b_{2}, n\right)$ for every $b_{1}, b_{2} \in[0, n]_{-\infty}$ GGol99, Ex. 1.8]. The semiring Nat ${ }_{\max ,+, n}$ is $\sigma$-complete with the mapping

$$
\sum_{I}^{\max }:\left([0, n]_{-\infty}\right)^{I} \rightarrow[0, n]_{-\infty} \quad \text { with } \quad\left(n_{i} \mid i \in I\right) \mapsto \sup \left(n_{i} \mid i \in I\right)
$$

10. The tropical semiring $\operatorname{Nat}_{\min ,+}=\left(\mathbb{N}_{\infty}, \min ,+, \infty, 0\right)$ over $\mathbb{N}$. The tropical semiring $\mathrm{Nat}_{\text {min },+}$ is $\sigma$-complete with the mapping

$$
\sum_{I}^{\min }:\left(\mathbb{N}_{\infty}\right)^{I} \rightarrow \mathbb{N}_{\infty} \quad \text { with } \quad\left(n_{i} \mid i \in I\right) \mapsto \inf \left(n_{i} \mid i \in I\right)
$$

The tropical semiring is often called the min-plus semiring.
11. The semiring $\operatorname{Nat}_{\max , \min }=\left(\mathbb{N}_{\infty}, \max , \min , 0, \infty\right)$. It is $\sigma$-complete with the mapping

$$
\sum_{I}^{\max }:\left(\mathbb{N}_{\infty}\right)^{I} \rightarrow \mathbb{N}_{\infty} \quad \text { with } \quad\left(n_{i} \mid i \in I\right) \mapsto \begin{cases}\sup \left(n_{i} \mid i \in I\right) & \text { if }\left\{n_{i} \mid i \in I\right\} \subseteq \mathbb{N} \text { and } \\ & \text { it is finite } \\ \infty & \text { otherwise }\end{cases}
$$

12. The semiring of formal languages $\operatorname{Lang}_{\Gamma}=\left(\mathcal{P}\left(\Gamma^{*}\right), \cup, \cdot, \emptyset,\{\varepsilon\}\right)$ where $\cdot$ denotes the concatenation of languages. The semiring Lang $_{\Gamma}$ is $\sigma$-complete with the mapping

$$
\sum_{I}^{\cup}: \mathcal{P}\left(\Gamma^{*}\right)^{I} \rightarrow \mathcal{P}\left(\Gamma^{*}\right) \quad \text { with } \quad\left(L_{i} \mid i \in I\right) \mapsto \bigcup_{i \in I} L_{i}
$$

13. The Viterbi semiring Viterbi $=([0,1]$, max, $\cdot, 0,1)$ where $[0,1]$ denotes the set $\{r \in \mathbb{R} \mid 0 \leq r \leq 1\}$ of real numbers $]$. The Viterbi semiring is $\sigma$-complete with the mapping

$$
\sum_{I}^{\max }:[0,1]^{I} \rightarrow[0,1] \quad \text { with } \quad\left(r_{i} \mid i \in I\right) \mapsto \sup \left(r_{i} \mid i \in I\right)
$$

The Viterbi semiring can be used for calculations with probabilities, and it is isomorphic to $\left(\mathbb{R}_{\geq 0, \infty}, \min ,+, \infty, 0\right)$ where $\mathbb{R}_{\geq 0, \infty}$ denotes the set $\{r \in \mathbb{R} \mid r \geq 0\} \cup\{\infty\}$. The isomorphism from $[0,1]$ to $[0, \infty]$ is $x \mapsto-\ln (x)$.
14. The tropical semifield $\operatorname{Int}_{\min ,+}=\left(\mathbb{Z}_{\infty}, \min ,+, \infty, 0\right)$ over $\mathbb{Z}$, and correspondingly the tropical semifield $\operatorname{Rat}_{\min ,+}=\left(\mathbb{Q}_{\infty}, \min ,+, \infty, 0\right)$ over $\mathbb{Q}$.
15. The semifields $\operatorname{Rat}_{\geq 0}=\left(\mathbb{Q}_{\geq 0},+, \cdot, 0,1\right)$ and Real $l_{\geq 0}=\left(\mathbb{R}_{\geq 0},+, \cdot, 0,1\right)$.
16. The $\log$ semifield $\left(\mathbb{R}_{-\infty}, \oplus,+,-\infty, 0\right)$ with $x \oplus y=\ln \left(e^{x}+e^{y}\right)$.
17. Let $\mathrm{B}=(B, \otimes, \mathbb{1})$ be a monoid. In the canonical way, we extend the operation $\otimes$ to finite sets of elements of $B$, i.e., for every $A_{1}, A_{2} \in \mathcal{P}_{\text {fin }}(B)$ we let $A_{1} \otimes A_{2}=\left\{a \otimes b \mid a \in A_{1}, b \in A_{2}\right\}$. Then the algebraic structure $\left(\mathcal{P}_{\text {fin }}(B), \cup, \otimes, \emptyset,\{\mathbb{1}\}\right)$ is a semiring. We denote it by $\operatorname{Sem}(\mathrm{B})$.
18. Let $A$ be a set. Then $\operatorname{PS}_{A}=(\mathcal{P}(A), \cup \cap, \emptyset, A)$ is a semiring. It is $\sigma$-complete with the mapping

$$
\sum_{I}^{\cup}: \mathcal{P}(A)^{I} \rightarrow \mathcal{P}(A) \quad \text { with } \quad\left(A_{i} \mid i \in I\right) \mapsto \bigcup_{i \in I} A_{i}
$$

Moreover, the monoid $(\mathcal{P}(A), \cap, A)$ is $\sigma$-complete with the mapping

$$
\sum_{I}^{\cap}: \mathcal{P}(A)^{I} \rightarrow \mathcal{P}(A) \quad \text { with } \quad\left(A_{i} \mid i \in I\right) \mapsto \bigcap_{i \in I} A_{i}
$$

In the literature (e.g., Grä68, p. 4]), often the expressions $\sum_{I}^{\cup}\left(A_{i} \mid i \in I\right)$ and $\sum_{i \in I}^{U} A_{i}$ are written as $\bigcup\left(A_{i} \mid i \in I\right)$. Similarly, $\sum_{I}^{\cap}\left(A_{i} \mid i \in I\right)$ and $\sum_{i \in I}^{\cap} A_{i}$ are written as $\bigcap\left(A_{i} \mid i \in I\right)$. In this book, we will also use these notations from the literature.
19. Let $(B, \oplus, \otimes, \mathbb{0}, \mathbb{1})$ be a semiring. Then $\left(B[x],+, \cdot, p_{0}, p_{1}\right)$ is a semiring called polynomial semiring, where $B[x]$ is the set of all $\mathbb{N}$-indexed families $\left(a_{i} \mid i \in \mathbb{N}\right)$ over $B$ such that the set $\left\{i \in \mathbb{N} \mid a_{i} \neq \mathbb{O}\right\}$ is finite. Moreover, for every $p=\left(a_{i} \mid i \in \mathbb{N}\right)$ and $q=\left(b_{i} \mid i \in \mathbb{N}\right)$ in $B[x]$ we define $p+q=\left(c_{i} \mid i \in \mathbb{N}\right)$ by letting $c_{i}=a_{i} \oplus b_{i}$ for each $i \in \mathbb{N}$, and we define $p \cdot q=\left(d_{i} \mid i \in \mathbb{N}\right)$ by letting $d_{i}=\bigoplus_{j \in[0, i]} a_{j} \otimes b_{i-j}$. Finally, the polynomials $p_{0}$ and $p_{1}$ are defined by $p_{0}=\left(a_{i} \mid i \in \mathbb{N}\right)$ with $a_{i}=\mathbb{O}$ for each $i \in \mathbb{N}$ and $p_{1}=\left(a_{i} \mid i \in \mathbb{N}\right)$ with $a_{0}=\mathbb{1}$ and $a_{i}=\mathbb{0}$ for each $i \geq 1$. If $B$ is commutative, then also $B[x]$ is commutative, and if $B$ is a ring, then also $B[x]$ is a ring.

Example 2.6.10. In DSV10, Ex. 1], CDIV10, Ex. 2.2], DV12, Ex. 2.1], and DFKV22, Ex. 2.2, 2.3] a number of examples of strong bimonoids are give, and in DV12, Ex. 2.1.4] a general construction principle for strong bimonoids is given, which we recall here. The first two examples are $\sigma$-complete strong bimonoids, and we show the mappings $\sum_{I}^{\oplus}$ which satisfy equalities (2.11) and (2.12).

[^2]1. The algebra TropBM $=\left(\mathbb{N}_{\infty},+, \min , 0, \infty\right)$ is a commutative strong bimonoid, called the tropical bimonoid. However, it is not bi-locally finite. Moreover, it is not a semiring, because there exist $a, b, c \in \mathbb{N}_{\infty}$ with $\min (a, b+c) \neq \min (a, b)+\min (a, c)$ (e.g., take $\left.a=b=c \neq 0\right)$. The strong bimonoid TropBM is $\sigma$-complete with the mapping

$$
\sum_{I}^{+}:\left(\mathbb{N}_{\infty}\right)^{I} \rightarrow \mathbb{N}_{\infty} \quad \text { with } \quad\left(n_{i} \mid i \in I\right) \mapsto \begin{cases}+_{i \in J} n_{j} & \text { if }\left\{n_{i} \mid i \in I\right\} \subseteq \mathbb{N} \text { and } \\ & J=\left\{i \in I \mid n_{i} \neq 0\right\} \text { is finite } \\ \infty & \text { otherwise }\end{cases}
$$

2. For each $\lambda \in \mathbb{R}$ with $0<\lambda<\frac{1}{2}$, let $\operatorname{Trunc}_{\lambda}=(B, \oplus, \odot, 0,1)$ be the algebra, where

- $B=\{0\} \cup\{b \in \mathbb{R} \mid \lambda \leq b \leq 1\}$,
- $a \oplus b=\min (a+b, 1)$, and
- $a \odot b=a \cdot b$ if $a \cdot b \geq \lambda$, and 0 otherwise,
and where + and $\cdot$ are the usual addition and multiplication of real numbers, respectively. Then Trunc ${ }_{\lambda}$ is a bi-locally finite and commutative strong bimonoid. It is $\sigma$-complete with the mapping:

$$
\sum_{I}^{\oplus}: B^{I} \rightarrow B \quad \text { with } \quad\left(n_{i} \mid i \in I\right) \mapsto \begin{cases}\min \left(\bigoplus_{i \in J} n_{j}, 1\right) & \text { if } J=\left\{i \in I \mid n_{i} \neq 0\right\} \text { is finite } \\ 1 & \text { otherwise }\end{cases}
$$

We note that Trunc ${ }_{\lambda}$ is not a semiring because $\odot$ is not right-distributive. For instance, for $a=b=$ 0.9 , and $c=\lambda$, we have $(a \oplus b) \odot c=\lambda$, while $(a \odot c) \oplus(b \odot c)=0$ because $a \odot c=b \odot c=0$.

If $\lambda=\frac{1}{4}$, then $\operatorname{Trunc}_{\lambda}$ is not locally finite, because the set $H=\left\langle\left\{\frac{1}{2}\right\}\right\rangle_{\{\oplus, \odot\}}$ is infinite. We can show this as follows Dro19. Let $\left(b_{i} \mid i \in \mathbb{N}\right)$ such that

$$
b_{i}= \begin{cases}\frac{1}{2} & \text { if } i=0 \\ \frac{1}{2} \cdot b_{i-1} & \text { if } i \text { is odd } \\ \frac{1}{2}+b_{i-1} & \text { if } i \text { is even and } i \neq 0\end{cases}
$$

Then, e.g., $b_{0}=1 / 2, b_{1}=1 / 4, b_{2}=1 / 2+1 / 4=3 / 4, b_{3}=3 / 8, b_{4}=1 / 2+3 / 8=7 / 8, b_{5}=7 / 16$, $b_{6}=1 / 2+7 / 16=15 / 16, b_{7}=15 / 32, b_{8}=1 / 2+15 / 32=31 / 32, b_{9}=31 / 64, b_{10}=1 / 2+31 / 64=$ $63 / 64$, etc. (In fact, the subsequences ( $b_{i} \mid i \in \mathbb{N}, i$ is even) and ( $b_{i} \mid i \in \mathbb{N}, i$ is odd) converge to 1 and $\frac{1}{2}$, respectively.) It is easy to see that $b_{i} \in\left\langle\left\{\frac{1}{2}\right\}\right\rangle_{\{\oplus, \odot\}}$ for each $i \in \mathbb{N}$, and that $b_{i} \neq b_{j}$ for every $i, j \in \mathbb{N}$ with $i \neq j$. Hence $\left(b_{i} \mid i \in \mathbb{N}\right)$ is an infinite family of elements in $\left\langle\left\{\frac{1}{2}\right\}\right\rangle_{\{\oplus, \odot\}}$, and thus $\operatorname{Trunc}_{\lambda}$ is not locally finite.
3. The algebra $([0,1], \oplus, \cdot, 0,1)$ with interval $[0,1]=\{r \in \mathbb{R} \mid 0 \leq r \leq 1\}$ of real numbers and the usual multiplication - of real numbers is a strong bimonoid for each of the following two definitions of $\oplus$ for every $a, b \in[0,1]$ :

- $a \oplus b=a+b-a \cdot b$ (called algebraic sum in KY95) and
- $a \oplus b=\min (a+b, 1)$ (called bounded sum in KY95).

If $\oplus$ is the algebraic sum, then we denote the algebra by Unitlnt ${ }_{\text {alg }}$; if $\oplus$ is the bounded sum, then we denote the algebra by UnitInt ${ }_{\text {bs }}$. Neither UnitInt ${ }_{\text {alg }}$ nor UnitInt ${ }_{\text {bs }}$ is a semiring.
4. Let $[0,1]=\{r \in \mathbb{R} \mid 0 \leq r \leq 1\}$ and let $i$ be a binary operation on $[0,1]$. We say that $i$ is a $t$-norm (or: fuzzy intersection, cf. [KY95, p. 62]) if $i$ satisfies the following conditions:
(a) $i$ is commutative and associative,
(b) $i(a, 1)=a$ for each $a \in[0,1]$ (boundary condition), and
(c) $a \leq b$ implies $i(c, a) \leq i(c, b)$ for every $a, b, c \in[0,1]$ (monotonicity condition).

Moreover, let $u$ be a binary operation on $[0,1]$. We say that $u$ is a $t$-conorm (or: fuzzy union, cf. KY95, p. 77]) if $u$ satisfies the the following conditions:
(a) $u$ is commutative and associative,
(b) $u(a, 0)=a$ for each $a \in[0,1]$ (boundary condition), and
(c) $a \leq b$ implies $u(c, a) \leq u(c, b)$ for every $a, b, c \in[0,1]$ (monotonicity condition).

Due to the boundary conditions, $([0,1], u, 0)$ and ( $[0,1], i, 1)$ are commutative monoids. Since $i(a, 0)=i(0, a) \leq i(0,1)=0$ for each $a \in[0,1]$ (cf. KY95, Thm. 3.10]), we have that

$$
\text { Unitlnt }_{u, i}=([0,1], u, i, 0,1)
$$

is a commutative strong bimonoid for each t-conorm $u$ and each t-norm $i$.
There are combinations of t-conorm $u$ and t-norm $i$ such that the strong bimonoid Unitlnt ${ }_{u, i}$ is not distributive. This is the case, e.g., for the t-conorm bounded sum (cf. Example 2.6.10(3)) and the t-norm bounded difference $i(a, b)=\max (0, a+b-1)$. We also refer to [KY95, Thm. 3.24] for sufficient conditions under which UnitInt ${ }_{u, i}$ is not distributive.
5. Let $(C,+, 0)$ be a commutative monoid. We consider the set $B$ of all mappings $f: C \rightarrow C$ such that $f(0)=0$. Moreover, we extend + to $B$ by a pointwise addition on elements of $B$, i.e., for every $f, g \in B$ and $c \in C$, we define $(f+g)(c)=f(c)+g(c)$. Also, we define the operation $\diamond$ on $B$ such that, for every $f, g \in B$ and $c \in C$, we have $(f \diamond g)(c)=g(f(c))$. Finally, we denote by $\tilde{0}$ the mapping $\tilde{0}: C \rightarrow C$ such that $\tilde{0}(c)=0$ for each $c \in C$. Then

$$
\operatorname{NearSem}_{C}=\left(B,+, \diamond, \tilde{0}, \mathrm{id}_{C}\right)
$$

is a strong bimonoid. Such an algebra is called a near semiring (over C) vHvR67, Kri05. We note that the condition $f(0)=0$ is needed to guarantee that $\tilde{0} \diamond f=\tilde{0}$. Except for trivial cases, the operation $\diamond$ is left-distributive over + , but not right-distributive.
As example we consider the commutative monoid $(\mathcal{P}(Q), \cup, \emptyset)$ for some set $Q$. Then we consider the near semiring $\left(B, \cup, \diamond, \widetilde{0}, \operatorname{id}_{\mathcal{P}(Q)}\right)$ as described above, i.e.,

- $B=\{f \mid f: \mathcal{P}(Q) \rightarrow \mathcal{P}(Q), f(\emptyset)=\emptyset\}$,
- $(f \cup g)(U)=f(U) \cup g(U)$ and $(f \diamond g)(U)=g(f(U))$ for every $f, g \in B$ and $U \in \mathcal{P}(Q)$, and
- $\widetilde{0}(U)=\emptyset$ and $\operatorname{id}_{\mathcal{P}(Q)}(U)=U$ for each $U \in \mathcal{P}(Q)$.

It is easy to see that the operation $\diamond$ is left-distributive over $\cup$. However, it is not right-distributive over $\cup$ if $|Q| \geq 2$. To show this, let $p, q \in Q$ with $p \neq q$ and let $f, g, h \in B$ such that for every $U \in \mathcal{P}(Q)$ :

$$
f(U)=\left\{\begin{array}{ll}
\{p\} & \text { if } U=\{p\} \\
\emptyset & \text { otherwise }
\end{array}, g(U)=\left\{\begin{array}{ll}
\{q\} & \text { if } U=\{p\} \\
\emptyset & \text { otherwise }
\end{array}, \text { and } h(U)= \begin{cases}\emptyset & \text { if }|U|=2 \\
U & \text { otherwise }\end{cases}\right.\right.
$$

Then $((f \cup g) \diamond h)(\{p\})=h(\{p, q\})=\emptyset$ and $((f \diamond h) \cup(g \diamond h))(\{p\})=h(\{p\}) \cup h(\{q\})=\{p, q\}$.
6. We consider the strong bimonoid $\left(\Gamma^{*} \cup\{\infty\}, \wedge, \cdot, \infty, \varepsilon\right)$ where

- $\wedge$ is the longest common prefix operation,
- . is the usual concatenation of strings, and
- $\infty$ is a new element such that $s \wedge \infty=\infty \wedge s=s$ and $s \cdot \infty=\infty \cdot s=\infty$ for each $s \in \Gamma^{*} \cup\{\infty\}$.

This bimonoid occurs in investigations for natural language processing, see Moh00. It is clear that $\left(\Gamma^{*} \cup\{\infty\}, \wedge, \cdot, \infty, \varepsilon\right)$ is left-distributive but not right-distributive (consider, e.g., if $\Gamma=\{a, b, c\}$, then $a b c=(a \wedge a b) \cdot b c \neq(a \cdot b c) \wedge(a b \cdot b c)=a b)$.
7. There exist only two strong bimonoids $(\{0,1\}, \oplus, \otimes, 0,1)$ with exactly two elements: the Boolean semiring Boole and the field $\mathrm{F}_{2}$; in particular, both are semirings (cf. Example 2.6.9(7)). However, there exist strong bimonoids with three elements which are not semirings, take, e.g., Three $=$ $(\{0,1,2\}, \max , \stackrel{\wedge}{\wedge}, 1)$ where $a^{\wedge} b=(a \cdot b) \bmod 3$ for every $a, b \in\{0,1,2\}$; there $\max \left(2^{\wedge} 2,2^{\wedge} 1\right) \neq$ $2 \hat{\bullet} \max (1,2)$.
8. We consider the algebra $\left(\mathbb{N}_{\mathbb{O}}, \oplus,+, \mathbb{O}, 0\right)$, where $\mathbb{N}_{\mathbb{O}}=\mathbb{N} \cup\{\mathbb{O}\}$ for some new element $\mathbb{O} \notin \mathbb{N}$. The binary operation $\oplus$, if restricted to $\mathbb{N}$, and the binary operation + , if restricted to $\mathbb{N}$, are the usual addition on natural numbers (e.g. $3+2=5$ ). Moreover, $\mathbb{O} \oplus x=x \oplus \mathbb{O}=x$ and $\mathbb{O}+x=x+\mathbb{O}=\mathbb{O}$ for each $x \in \mathbb{N}_{\mathbb{0}}$. Thus, $\left(\mathbb{N}_{\mathbb{0}}, \oplus,+, \mathbb{0}, 0\right)$ is a strong bimonoid. However, it is not a semiring (e.g., $2+(3 \oplus 4) \neq(2+3) \oplus(2+4))$. We might call this algebra the plus-plus strong bimonoid of natural numbers and denote it by $\mathrm{PP}_{\mathbb{N}}$.
9. We recall the strong bimonoid $\operatorname{Stb}=(\mathbb{N}, \oplus, \odot, 0,1)$ from DSV10, Ex. 25 ${ }^{2}$. Intuitively, both operations are commutative and consider their maximal argument, say, $b$. Then, depending on the characteristic of $b$ being even or odd, $\oplus$ delivers $b$ or $b+1$, respectively, and dually, $\odot$ delivers $b+1$ or $b$, respectively.
Formally, the two commutative operations $\oplus$ and $\odot$ on $\mathbb{N}$ are defined as follows. First, let $0 \oplus a=a$, $0 \odot a=0$, and $1 \odot a=a$ for every $a \in \mathbb{N}$. If $a, b \in \mathbb{N} \backslash\{0\}$ with $a \leq b$, we put (with + being the usual addition on $\mathbb{N}$ )

$$
a \oplus b= \begin{cases}b & \text { if } b \text { is even } \\ b+1 & \text { if } b \text { is odd }\end{cases}
$$

If $a, b \in \mathbb{N} \backslash\{0,1\}$ with $a \leq b$, let

$$
a \odot b= \begin{cases}b+1 & \text { if } b \text { is even } \\ b & \text { if } b \text { is odd }\end{cases}
$$

Then $\mathrm{Stb}=(\mathbb{N}, \oplus, \odot, 0,1)$ is a strong bimonoid. Clearly, it is not a semiring because, e.g., $2 \odot(2 \oplus$ $3)=2 \odot 4=5$ and $(2 \odot 2) \oplus(2 \odot 3)=3 \oplus 3=4$. It is easy to see that Stb is bi-locally finite. However, if we apply $\oplus$ and $\odot$ alternatingly, then the result increases arbitrarily. For instance, let ( $b_{i} \mid n \in \mathbb{N}$ ) be the family defined by $b_{0}=2$ and, for each $n \in \mathbb{N}$, by

$$
b_{n+1}= \begin{cases}b_{n} \oplus 2 & \text { if } n \text { is odd } \\ b_{n} \odot 2 & \text { otherwise }\end{cases}
$$

Then, e.g., $b_{0}=2, b_{1}=3, b_{2}=4, b_{3}=5, \ldots$. In general we have $\langle\{2\}\rangle_{\{\oplus, \odot, 0,1\}}=\mathbb{N}$, and hence Stb is not locally finite.
10. Let $(B,+)$ be a commutative semigroup and $(B, \cdot)$ be a semigroup. Combining these two semigroups, we obtain a strong bimonoid structure on $B$ by adding constants $\mathbb{O}$ and $\mathbb{1}$. Formally, let $\mathbb{0}, \mathbb{1} \notin B$ and put $B^{\prime}=B \cup\{\mathbb{0}, \mathbb{1}\}$. Then we define binary operations $\oplus$ and $\odot$ on $B^{\prime}$ by letting
$\left.\bullet \oplus\right|_{B \times B}=+$ and $\mathbb{O} \oplus b=b \oplus \mathbb{O}=b$ and $\mathbb{1} \oplus b=b \oplus \mathbb{1}=b$ if $b \neq \mathbb{O}$, and
$\left.\bullet \odot\right|_{B \times B}=\cdot$ and $\mathbb{O} \odot b=b \odot \mathbb{O}=\mathbb{O}$, and $\mathbb{1} \odot b=b \odot \mathbb{1}=b$ for each $b \in B^{\prime}$.
Then $\left(B^{\prime}, \oplus, \odot, 0, \mathbb{1}\right)$ is a strong bimonoid.
After having shown a number of examples of strong bimonoids, we define some more restrictions and prove useful relationships.

Let $\mathrm{B}=(B, \oplus, \otimes, \mathbb{0}, \mathbb{1})$ be a strong bimonoid. It is

- additively idempotent if $\oplus$ is idempotent,
- multiplicatively idempotent if $\otimes$ is idempotent,
- zero-sum free if $a \oplus b=\mathbb{0}$ implies $a=b=\mathbb{0}$ for every $a, b \in B$,
- zero-divisor free if $a \otimes b=\mathbb{O}$ implies $a=\mathbb{O}$ or $b=\mathbb{O}$ for every $a, b \in B$,
- positive if it is zero-sum free and zero-divisor free,
- zero-cancellation free if $a \otimes b \otimes c \neq \mathbb{0}$ implies $a \otimes c \neq \mathbb{O}$ for every $a, b, c \in B$,

[^3]- extremal if $\oplus$ is extremal,
- weakly locally finite if, for every finite subset $A \subseteq B$, the set $\mathrm{Cl}(A)$ is finite, where $\mathrm{Cl}(A)$ is the smallest subset $C \subseteq B$ such that $A \subseteq C$, and $b \oplus b^{\prime} \in C$ and $b \otimes a \in C$ for every $a \in A$ and $b, b^{\prime} \in C$,
We refer the reader to the proofs of Theorems 18.2 .14 and 18.2 .15 for further examples of strong bimonoids.

Let $n \in \mathbb{N}$ and $b \in B$. We define the elements $n b \in B$ and $b^{n} \in B$ by induction on $\mathbb{N}$ as follows:

$$
\begin{equation*}
0 n=\mathbb{O},(n+1) b=b \oplus n b \text { and } b^{0}=\mathbb{1}, b^{n+1}=b \otimes b^{n} \tag{2.15}
\end{equation*}
$$

Observation 2.6.11. Let $\mathrm{B}=(B, \oplus, \otimes, \mathbb{O}, \mathbb{1})$ be a strong bimonoid. Then the following statements hold.
(1) If $B$ is commutative, then it is zero-cancellation free. Moreover, there exists a zero-cancellation free semiring which is not commutative.
(2) If $B$ is zero-divisor free, then it is zero-cancellation free. Moreover, there exists a zero-cancellation free semiring which is not zero-divisor free.
(3) If $B$ is extremal, then it is additively idempotent.
(4) If $B$ is additively idempotent, then it is zero-sum free. Moreover, there exists a zero-sum free semiring which is not additively idempotent.
(5) If $B$ is locally finite, then it is bi-locally finite.
(6) If B is a bi-locally finite semiring, then it is locally finite.
(7) If B is $\sigma$-complete, then it is zero-sum free Gol99, Prop. 22.28].
(8) If B is $\sigma$-complete, then, for every countable index set $I$ and family $\left(b_{i} \mid i \in B\right)$ of elements of $B$, we have

$$
\sum_{i \in I}^{\oplus} b_{i} \neq \mathbb{0} \Longleftrightarrow(\exists i \in I): b_{i} \neq \mathbb{0}
$$

(9) If B is zero-sum free, then for every finite index set $I$ and family $\left(b_{i} \mid i \in B\right)$ of elements of $B$ we have

$$
\bigoplus_{i \in I} b_{i} \neq \mathbb{O} \Longleftrightarrow(\exists i \in I): b_{i} \neq \mathbb{0}
$$

(10) If $B$ is an additively idempotent semiring, then it is zero-sum free.

Proof. Proof of (1): For the proof of the first statement, assume that $B$ is commutative and not zerocancellation free. By the second condition, there are $a, b, c \in B$ such that $a \otimes b \otimes c \neq \mathbb{O}$ and $a \otimes c=\mathbb{0}$. Using the latter and commutativity, we obtain $a \otimes b \otimes c=a \otimes c \otimes b=0$ which is a contradiction. For the second, we can consider the semiring Lang ${ }_{\Gamma}=\left(\mathcal{P}\left(\Gamma^{*}\right), \cup, \cdot \emptyset,\{\varepsilon\}\right)$ of formal languages (cf. Example 2.6.9 (12)), which is zero-cancellation free and not commutative.

Proof of (2): Let us assume that B is zero-divisor free and not zero-cancellation free. The latter means that there exist $a, b, c \in B$ such that $a \otimes b \otimes c \neq \mathbb{O}$ and $a \otimes c=\mathbb{0}$. Since B is zero-divisor free, we have $a=\mathbb{O}$ or $c=\mathbb{O}$. Thus $a \otimes b \otimes c=\mathbb{O}$, which contradicts our assumption.

To prove the second statement, we consider the ring Intmod4 $=\left(\{0,1,2,3\},{ }_{4},{ }^{\cdot}, 0,0,1\right)$ as defined in Example 2.6.9(5). It is easy to see that Intmod4 is zero-cancellation free and not zero-divisor free.

Proof of (3): It is obvious.
Proof of (4): Let us assume that B is additively idempotent. Let $a \oplus b=\mathbb{O}$. Then $a=a \oplus \mathbb{O}=$ $a \oplus(a \oplus b)=(a \oplus a) \oplus b=a \oplus b=\mathbb{O}$. Hence B is zero-sum free. The semiring Nat of natural numbers is zero-sum free and not additively idempotent.

Proof of (5): This is obvious because for each finite subset $F \subseteq B$ we have $\langle F\rangle_{\{0, \oplus\}} \cup\langle F\rangle_{\{\mathbb{1}, \otimes\}} \subseteq$ $\langle F\rangle_{\{0, \mathbb{1}, \oplus, \otimes\}}$.

Proof of (6): Let $F \subseteq B$ be a finite subset of $B$. We show that set $\langle F\rangle_{\{0, \mathbb{1}, \oplus, \otimes\}}$ is finite. Let us abbreviate this set by $\langle F\rangle$.

Since B is bi-locally finite it is sufficient to show that $\langle F\rangle=\left\langle\langle F\rangle_{\{\mathbb{1}, \otimes\}}\right\rangle_{\{0, \oplus\}}$. The inclusion $\left\langle\langle F\rangle_{\{\mathbb{1}, \otimes\}}\right\rangle_{\{0, \oplus\}} \subseteq\langle F\rangle$ holds obviously.

We show the other inclusion. Since $F \subseteq\left\langle\langle F\rangle_{\{1, \otimes\}}\right\rangle_{\{0, \oplus\}}$, it is sufficient to show that $\left\langle\langle F\rangle_{\{1, \otimes\}}\right\rangle_{\{0, \oplus\}}$ is closed under the operations $\oplus, \otimes, 0$, and $\mathbb{1}$. The closure under $\oplus, 0$, and $\mathbb{1}$ is obvious. To show the closure under $\otimes$, let $c, d \in\left\langle\langle F\rangle_{\{\mathbb{1}, \otimes\}}\right\rangle_{\{0, \oplus\}}$. Then there exist $n_{1}, n_{2} \in \mathbb{N}, c_{1}, \ldots, c_{n_{1}}, d_{1}, \ldots, d_{n_{2}} \in\langle F\rangle_{\{1, \otimes\}}$ such that $c=\bigoplus_{i \in\left[n_{1}\right]} c_{i}$ and $d=\bigoplus_{j \in\left[n_{2}\right]} d_{j}$. By using distributivity, we obtain

$$
c \otimes d=\left(\bigoplus_{i \in\left[n_{1}\right]} c_{i}\right) \otimes\left(\bigoplus_{j \in\left[n_{2}\right]} d_{j}\right)=\bigoplus_{i \in\left[n_{1}\right]} \bigoplus_{j \in\left[n_{2}\right]} c_{i} \otimes d_{j}
$$

Since $c_{i} \otimes d_{j} \in\langle F\rangle_{\{\mathbb{1}, \otimes\}}$ for every $i \in\left[n_{1}\right]$ and $j \in\left[n_{2}\right]$, we obtain that $c \otimes d \in\left\langle\langle F\rangle_{\{\mathbb{1}, \otimes\}}\right\rangle_{\{0, \oplus\}}$.
Proof of (7): Let $a, b \in B$ such that $a \oplus b=0$ and let $c=\sum_{i \in \mathbb{N}}^{\oplus}(a \oplus b)$. Then by Observation 2.6.8 we have $c=0$. Moreover, let $d=\sum_{i \in \mathbb{N}}^{\oplus} a$ and $e=\sum_{i \in \mathbb{N}}^{\oplus} b$. By renaming indices and using axioms (2.11) and (2.12), it is easy to see that

$$
\sum_{i \in \mathbb{N}}^{\oplus}(a \oplus b)=\sum_{i \in \mathbb{N}}^{\oplus} a \oplus \sum_{i \in \mathbb{N}}^{\oplus} b,
$$

hence we have $c=d \oplus e$. Similarly, it is easy to see that

$$
\sum_{i \in \mathbb{N}}^{\oplus} a=a \oplus \sum_{i \in \mathbb{N}}^{\oplus} a,
$$

hence $d=a \oplus d$. Then we obtain $\mathbb{O}=c=d \oplus e=a \oplus d \oplus e=a$ and $\mathbb{O}=a \oplus b=b$.
Proof of (8): Let B be $\sigma$-complete. First we prove the implication $\Rightarrow$ in the equivalence. For this we assume that $\sum_{i \in I}^{\oplus} b_{i} \neq \mathbb{0}$ and $(\forall i \in I): b_{i}=\mathbb{0}$. By Observation 2.6.8 the second condition implies that $\sum_{i \in I}^{\oplus} b_{i}=\mathbb{0}$, which contradicts the first one.

Lastly we prove the implication $\Leftarrow$. Let $j \in I$ be such that $b_{j} \neq 0$. Then

$$
\sum_{i \in I}^{\oplus} b_{i}=\sum_{i \in I \backslash \backslash j\}}^{\oplus} b_{i} \oplus b_{j} \neq \mathbb{0},
$$

where at the first equality we applied the laws (2.11) and (2.12), and the second equality follows from the fact that B is zero-sum free.

Proof of (9): The proof is very similar to the proof of (7). However, instead of Observation 2.6.8 we use the fact that, for every finite index set $I$, we have $\bigoplus_{i \in I} 0=0$.

Proof of (10): Let $a, b \in B$ such that $a \oplus b=\mathbb{0}$. Then $a=\mathbb{O}$ because

$$
\mathbb{0}=a \oplus b=a \otimes \mathbb{1} \oplus b=a \otimes(\mathbb{1} \oplus \mathbb{1}) \oplus b=a \oplus a \oplus b=a .
$$

Similarly, we can show that $b=\mathbb{0}$.

### 2.6.6 Lattices and their comparison with strong bimonoids

We recall notions from lattice theory Bir93, Grä03] and BS81, Ch. 1]. A lattice is an algebra $(B, \vee, \wedge)$ in which $\vee$ (the join) and $\wedge$ (the meet) are binary operations, $(B, \vee)$ and $(B, \wedge)$ are commutative semigroups, the operations $\wedge$ and $\vee$ are idempotent and satisfy the absorption axioms $a \vee(a \wedge b)=a$ and $a \wedge(a \vee b)=a$. The lattice $(B, \vee, \wedge)$ is bounded if there exist elements $\mathbb{O}, \mathbb{1} \in B$ such that $\mathbb{O} \vee a=a$ and $\mathbb{1} \wedge a=a$ for every $a \in B$. We denote a bounded lattice also by ( $B, \vee, \wedge, \mathbb{0}, \mathbb{1}$ ).

There is an alternative order-theoretic definition of lattice. A partially ordered set ( $B, \leq$ ) is a lattice if for each $a, b \in B$ the elements $\sup _{\leq}\{a, b\}$ and $\inf _{\leq}\{a, b\}$ exist. The lattice $(B, \leq)$ is bounded if there
exist elements $\mathbb{O}, \mathbb{1} \in B$ such that $\inf _{\leq}(\mathbb{O}, a)=\mathbb{O}$ and $\sup _{\leq}(\mathbb{1}, a)=\mathbb{1}$ for every $a \in B ;$ in other words, $\mathbb{D}$ and $\mathbb{1}$ are the smallest element and greatest element, respectively, with respect to $\leq$. We denote a bounded lattice also by $(B, \leq, \mathbb{0}, \mathbb{1})$.

We recall the well known correspondence between these two definitions.
Theorem 2.6.12. (cf. BS81, Sect. I.1], Bir67, Ch. I., Thm 8], Grä78, Sect. 1.1, Thm 8], [Grä03, p.6]) Let $B$ be a set. The following two statements hold.
(1) Let the algebra $(B, \vee, \wedge)$ be a lattice. We define the partial order $\leq$ on $B$ such that

$$
a \leq b \text { if } a \wedge b=a \text { for every } a, b \in B
$$

Then the partially ordered set $(B, \leq)$ is a lattice with $\sup _{\leq}\{a, b\}=a \vee b$ and $\inf _{\leq}\{a, b\}=a \wedge b$ for every $a, b \in B$.
(2) Let the partially ordered set $(B, \leq)$ be a lattice. We define the binary operations $\vee$ and $\wedge$ on $B$ such that

$$
a \vee b=\sup _{\leq}\{a, b\} \text { and } a \wedge b=\inf _{\leq}\{a, b\} \text { for every } a, b \in B .
$$

Then the algebra $(B, \vee, \wedge)$ is a lattice.
Clearly, if we apply the transformation in Theorem $2.6 .12(1)$ to the lattice $(B, \vee, \wedge)$ and then apply to the resulting lattice $(B, \leq)$ the transformation in Theorem $[2.6 .12(2)$, then we reobtain the lattice $(B, \vee, \wedge)$. A similar invariant occurs if we apply the transformation of (2) followed by the transformation of $(1)$ to a lattice $(B, \leq)$.

Another consequence of Theorem 2.6.12 is the following. Let $(B, \vee, \wedge)$ and $(B, \leq)$ be bounded lattices such that the transformations of Theorem 2.6.12 transform the one into the other. Moreover, let $\mathbb{O}$ and $\mathbb{1}$ be two elements of $B$. Then the following equivalence holds: $(B, \vee, \wedge, \mathbb{O}, \mathbb{1})$ is a bounded lattice iff $(B, \leq, \mathbb{0}, \mathbb{1})$ is a bounded lattice.

The lattice $(B, \vee, \wedge)$ is distributive if $\wedge$ is distributive over $\vee$ and $\vee$ is distributive over $\wedge$. If fact, the first of the previous two conditions holds if and only if the second one holds [DP12, Lm. 4.3]. Hence it suffices to require one of them.

Observation 2.6.13. (1) For each lattice $(B, \vee, \wedge)$, the semigroups $(B, \vee)$ and $(B, \wedge)$ are locally finite.
(2) Each bounded lattice $(B, \vee, \wedge, \mathbb{O}, \mathbb{1})$ is a bi-locally finite and commutative strong bimonoid.
(3) Each distributive bounded lattice $(B, \vee, \wedge, \mathbb{O}, \mathbb{1})$ is a locally finite and commutative semiring.

Proof. Proof of (1): Let $(B, \vee, \wedge)$ be a lattice. The statement follows from Observation 2.6.6 because both $\vee$ and $\wedge$ are commutative and idempotent.

Proof of (2): Let $\mathrm{B}=(B, \vee, \wedge, \mathbb{O}, \mathbb{1})$ be a bounded lattice. Since we have $\mathbb{O} \wedge a=\mathbb{O} \wedge(\mathbb{O} \vee a)=\mathbb{0}$ for every $a \in B$ (by absorption law), the algebra B is a strong bimonoid. Obviously, it is commutative. Moreover, it follows from Statement (1) that both $(B, \vee, \mathbb{O})$ and $(B, \wedge, \mathbb{1})$ are locally finite.

Proof of (3): This follows from (2) and Observation 2.6.11(6).
Observation 2.6.14. Let $\mathrm{B}=(B, \oplus, \otimes, \mathbb{O}, \mathbb{1})$ be an algebra. Then the following two statements hold.
(1) B is a bounded lattice if and only if $(B, \oplus, \otimes)$ is a lattice and B is a strong bimonoid.
(2) B is a distributive bounded lattice if and only if $(B, \oplus, \otimes)$ is a lattice and B is a semiring.

We recall that a bounded lattice $(B, \vee, \wedge, \mathbb{O}, \mathbb{1})$ (which is a particular commutative strong bimonoid) is $\sigma$-complete if the monoid $(B, \vee, \mathbb{O})$ is $\sigma$-complete (cf. Section 2.6.5). We note that, for bounded lattices, our concept is more general than $\sigma$-lattice in [Bir67, Ch. 9] because in a $\sigma$-lattice both the join and the meet exist for arbitrary countable index sets. Moreover, $\sigma$-lattice is more general than complete lattice [Bir67, Bir93, Grä78, Grä03 because in a complete lattice the join and the meet exist for arbitrary index sets. Lastly, we note that if meet exists for arbitrary index sets, then also join exists for arbitrary index sets and vice versa [Bir67, Ch. V, Thm. 3] [Grä78, Sect. 1.3, Lm. 14].


Figure 2.3: The lattices $N_{5}$ and $M_{3}$ in Figure 2 of [Grä03, p.14].

## Example 2.6.15.

1. Let $A$ be a set. The $\sigma$-complete semiring $\operatorname{PS}_{A}=(\mathcal{P}(A), \cup, \cap, \emptyset, A)$ is a $\sigma$-complete distributive lattice.
2. The $\sigma$-complete semiring $\operatorname{Nat}_{\max , \min }=\left(\mathbb{N}_{\infty}, \max , \min , 0, \infty\right)$ is a $\sigma$-complete distributive lattice.
3. Grä03, Fig. 2, p.14], BS81, Fig. 5] Let $N_{5}=\{o, a, b, c, i\}$ be a set with five elements. We consider two binary operations $\vee$ and $\wedge$ such that the following requirements hold:

$$
a \wedge b=b, \quad b \vee c=i, \quad \text { and } \quad a \wedge c=o
$$

The definition of the values of $\vee$ and $\wedge$ for other combinations of arguments is given by the unique extension of the above requirements such that $\mathrm{N}_{5}=\left(N_{5}, \vee, \wedge, o, i\right)$ is a bounded lattice (cf. Figure 2.3). This lattice is not distributive.
4. Grä03, Fig. 2, p.14], BS81, Fig. 5] Let $M_{3}=\{o, a, b, c, i\}$ be a set with five elements. We consider two binary operations $\vee$ and $\wedge$ such that the following requirements hold:

$$
a \wedge b=a \wedge c=b \wedge c=o \quad \text { and } \quad a \vee b=a \vee c=b \vee c=i
$$

Similarly, the definition of the values of $\vee$ and $\wedge$ for other combinations of arguments is given by the unique extension of the above requirements such that $\mathrm{M}_{3}=\left(M_{3}, \vee, \wedge, o, i\right)$ is a bounded lattice (cf. Figure 2.3). This lattice is not distributive.
It is well known that there exist large sets of lattices that are not distributive Grä03, e.g., orthomodular lattices. In fact, a lattice is non-distributive if and only if it contains at least one of the two lattices $N_{5}$ and $M_{3}$ as a sublattice Grä03, Thm. 1 on p. 80].
5. We consider the algebra $(\mathbb{R}, \max , \min )$ with maximum and minimum based on the usual $\leq$ ordering on the set of real numbers. The algebra $(\mathbb{R}, \max , \min )$ is a lattice. It is not bounded (and hence not $\sigma$-complete). Moreover, both max and min are extremal.
6. A residuated lattice is an algebra $\mathrm{B}=(B, \vee, \wedge, \otimes, \rightarrow, \mathbb{0}, \mathbb{1})$ such that
$(\mathrm{L} 1)(B, \vee, \wedge, \mathbb{0}, \mathbb{1})$ is a bounded lattice,
(L2) $(B, \otimes, \mathbb{1})$ is a commutative monoid with the unit $\mathbb{1}$, and
(L3) $\rightarrow$ is a binary operation, and $\otimes$ and $\rightarrow$ form an adjoint pair, i.e., they satisfy the adjunction property: for all $a, b, c \in B, a \otimes b \leq c$ if and only if $a \leq(b \rightarrow c)$. (Here $\leq$ is the partial order on $B$ defined, as for lattices, by $a \leq b$ iff $a=a \wedge b$ for every $a, b \in B$.)
If, additionally, B is a $\sigma$-complete lattice, then B is called a $\sigma$-complete residuated lattice.
The operations $\otimes$ and $\rightarrow$ are called multiplication and residuum, respectively. The algebra $(B, \vee, \otimes, \mathbb{O}, \mathbb{1})$ is a semiring (cf., e.g., Gal08, Slide 11]), it is called the semiring reduct of B . Thus, in particular, the semiring reduct of each residuated lattice is a strong bimonoid.


Figure 2.4: The bounded lattice $\mathrm{FL}(2+2)$ freely generated by the two chains $a<b$ and $c<d$. The upper bound of $\mathrm{FL}(2+2)$ is $b \vee d=\mathbb{1}$, and its lower is $a \wedge c=\mathbb{0}$.
7. The $\sigma$-complete Boolean algebras of [Loo47] are $\sigma$-complete lattices.
8. A finite chain is a finite bounded lattice $(L, \leq, \mathbb{O}, \mathbb{1})$ where $\leq$ is a linear order. Trivially, each finite chain is distributive.
9. The bounded lattice $\mathrm{FL}(2+2)=(\mathrm{FL}(2+2), \vee, \wedge, \mathbb{0}, \mathbb{1})$ (cf. Rol58, Grä03, p. 361], RW09, Fig. 3a]) is the lattice freely generated by the two finite chains $a<b$ and $c<d$ (cf. Figure 2.4). The upper bound is $\mathbb{1}=b \vee d$ and the lower bound is $\mathbb{O}=a \wedge c$. This lattice is infinite [Rol58, Thm. 2], hence it is not locally finite. Thus, by Observation 2.6.11(6), it is not distributive; this also follows from the fact that $\mathrm{FL}(2+2)$ contains $\mathrm{N}_{5}$ as sublattice.

We note that semiring-reducts of semi-lattice ordered monoids and of $\sigma$-complete residuated lattices, and Brouwerian lattices are commutative semirings.

In Figure 2.5, we have placed five sets of algebras into a diagram. The sets are:

- the set $\mathcal{C}_{1}$ of all lattices,
- the set $\mathcal{C}_{2}$ of all strong bimonoids,
- the set $\mathcal{C}_{3}$ of all semirings,
- the set $\mathcal{C}_{4}$ of all bi-locally finite strong bimonoids, and
- the set $\mathcal{C}_{5}$ of all locally finite strong bimonoids.

Observation 2.6.16. Figure 2.5 shows the Euler diagram of the sets $\mathcal{C}_{1}, \mathcal{C}_{2}, \mathcal{C}_{3}, \mathcal{C}_{4}$, and $\mathcal{C}_{5}$.

Proof. By Observation 2.6.11(6), we have $\mathcal{C}_{3} \cap \mathcal{C}_{4}=\mathcal{C}_{3} \cap \mathcal{C}_{5}$. Moreover, the following subset relationships hold:

- $\mathcal{C}_{3} \subseteq \mathcal{C}_{2}$,
- $\mathcal{C}_{5} \subseteq \mathcal{C}_{4}$,
- $\mathcal{C}_{4} \subseteq \mathcal{C}_{2}$,
- $\mathcal{C}_{1} \cap \mathcal{C}_{3} \subseteq \mathcal{C}_{5}$, and
- $\mathcal{C}_{1} \cap \mathcal{C}_{2} \subseteq \mathcal{C}_{4}$.

Next, for each nonempty area of the diagram, we show a witness (i.e., an element of that area).
$\mathcal{C}_{1} \backslash \mathcal{C}_{2} \neq \emptyset$ : Let $\mathrm{B}_{1}$ be the lattice $(\mathbb{R}, \max , \min )$. Since $\mathrm{B}_{1}$ is not bounded, it is not a strong bimonoid.
$\left(\mathcal{C}_{1} \cap \mathcal{C}_{4}\right) \backslash \mathcal{C}_{5} \neq \emptyset$ Czél9: We let $\mathrm{B}_{2}$ be the bounded lattice $\mathrm{FL}(2+2)$ in Example 2.6.15(9) (cf. Figure 2.4). This lattice is infinite and it is freely generated by the two finite chains $a<b$ and $c<d$. Hence it is not locally finite.
$\left(\mathcal{C}_{1} \cap \mathcal{C}_{5}\right) \backslash \mathcal{C}_{3} \neq \emptyset$ : We let $\mathrm{B}_{3}=\mathrm{N}_{5}$ and $\mathrm{B}_{4}=\mathrm{M}_{3}$. These are finite bounded lattices; hence, they are locally finite and commutative strong bimonoids. But neither of them is distributive.
$\mathcal{C}_{1} \cap \mathcal{C}_{3} \neq \emptyset$ : We let $\mathrm{B}_{5}$ be the semiring $\mathrm{PS}_{A}=(\mathcal{P}(A), \cup, \cap, \emptyset, A)$ for some set $A \neq \emptyset$. This is a bounded lattice (cf. Example 2.6.15(1)).
$\left(\mathcal{C}_{3} \cap \mathcal{C}_{4}\right) \backslash \mathcal{C}_{1} \neq \emptyset:$ We let $\mathrm{B}_{6}$ be the ring Intmod4 $=\left(\{0,1,2,3\},{ }_{4},{ }_{4}, 0,1\right)$ defined in Example 2.6.9(5). This is a finite (hence also locally finite) semiring but not a lattice, because e.g. $+_{4}$ is not idempotent.
$\mathcal{C}_{5} \backslash\left(\mathcal{C}_{1} \cup \mathcal{C}_{3}\right) \neq \emptyset$ : We let $\mathrm{B}_{7}$ be the strong bimonoid $\left(\{0,1,2,3, \infty\},{ }_{4}, \mathrm{~min}, 0, \infty\right)$ where $+_{4}$ is defined as for $B_{6}$ (extended in the usual way to $\infty$ ). Since $B_{7}$ is finite, it is also locally finite. $B_{7}$ is neither a lattice (because $+_{4}$ is not idempotent) nor a semiring (because, e.g., $\min \left(1,1+{ }_{4} 1\right) \neq \min (1,1)+{ }_{4} \min (1,1)$ ).
$\mathcal{C}_{4} \backslash\left(\mathcal{C}_{1} \cup \mathcal{C}_{3} \cup \mathcal{C}_{5}\right) \neq \emptyset$ Dro19: Let $\mathrm{B}_{8}$ be the strong bimonoid Trunc ${ }_{\lambda}=(B, \oplus, \odot, 0,1)$ of Example 2.6.10(2) (also cf. [DV12, Ex. 2.1(2)]). Then $\mathrm{B}_{8}$ is a bi-locally finite and commutative strong bimonoid, but not a locally finite strong bimonoid. Moreover, $\mathrm{B}_{8}$ is neither a semiring nor a lattice.
$\mathcal{C}_{2} \backslash\left(\mathcal{C}_{1} \cup \mathcal{C}_{3} \cup \mathcal{C}_{4}\right) \neq \emptyset$ : We let $\mathrm{B}_{9}$ be the tropical bimonoid TropBM. This is neither a lattice (because + is not idempotent) nor a bi-locally finite strong bimonoid nor a semiring (because, e.g., $\min (1,1+1) \neq \min (1,1)+\min (1,1))$.
$\mathcal{C}_{3} \backslash\left(\mathcal{C}_{1} \cup \mathcal{C}_{4}\right)$ : We let $\mathrm{B}_{10}$ be the semiring $\mathrm{Nat}=(\mathbb{N},+, \cdot, 0,1)$ of natural numbers. This is neither bi-locally finite nor it is a lattice.

We finish this subsection with a characterization theorem. It concerns the smallest set which is closed under certain operations. Such a set can be characterized by using a combination of the fixpoint theorem of Tarski [Tar55, Thm. 1] and the first fixpoint theorem of Kleene [Kle62, p. 348] (cf. [LNS82]). We start with some preparations.

First, we recall some notions and observations from [DP12]. Let $A$ be a set. The pair $(\mathcal{P}(A), \subseteq)$ is a $\sigma$-complete lattice in the order-theoretical sense; the corresponding algebraic version is the $\sigma$-complete semiring $\mathrm{PS}_{A}=(\mathcal{P}(A), \cup, \cap, \emptyset, A)$ where $U \cup U^{\prime}$ and $U \cap U^{\prime}$ are the supremum of $U$ and infimum of $U^{\prime}$, respectively, with respect to $\subseteq$.

Let $D \subseteq \mathcal{P}(A)$ such that $D \neq \emptyset$. We call $D$ directed if, for each $U, U^{\prime} \in D$, there exists $V \in D$ such that $U \subseteq V$ and $U^{\prime} \subseteq V$.

Let $f: \mathcal{P}(A) \rightarrow \mathcal{P}(A)$ be a mapping. We say that

- $f$ is order-preserving if, for every $U, U^{\prime} \in \mathcal{P}(A)$, the inclusion $U \subseteq U^{\prime}$ implies $f(U) \subseteq f\left(U^{\prime}\right)$,
- $f$ is continuous if, for each directed set $D \subseteq \mathcal{P}(A)$ we have that $f(D)$ is a directed set and $f(\bigcup(U \mid U \in D))=\bigcup(f(U) \mid U \in D)$,
- $U \in \mathcal{P}(A)$ is a fixpoint of $f$ if $f(U)=U$, and
- $U$ is the least fixpoint of $f$ if it is a fixpoint and, for each fixpoint $U^{\prime}$, the inclusion $U \subseteq U^{\prime}$ holds. We note that each continuous mapping $f: \mathcal{P}(A) \rightarrow \mathcal{P}(A)$ is order-preserving [DP12, Lm. 8.7(ii)].

bounded lattices
distributive bounded lattices
bi-locally finite semirings $=$ locally finite semirings

Figure 2.5: The Euler diagram of the sets $\mathcal{C}_{1}$ of all lattices, $\mathcal{C}_{2}$ of all strong bimonoids, $\mathcal{C}_{3}$ of all semirings, $\mathcal{C}_{4}$ of all bi-locally finite strong bimonoids, and $\mathcal{C}_{5}$ of all locally finite strong bimonoids. Each black area denotes the empty set. The algebras $B_{1}-B_{10}$ are shown in the proof of Observation 2.6.16.

Theorem 2.6.17. (cf. least fixpoint theorems DP12, Thm. 2.35] and [DP12, Thm. 8.15]) Let $A$ be a set and $f: \mathcal{P}(A) \rightarrow \mathcal{P}(A)$ be a continuous mapping. Then:

$$
\begin{align*}
& \bigcap(U \in \mathcal{P}(A) \mid f(U) \subseteq U)=\bigcup\left(f^{n}(\emptyset) \mid n \in \mathbb{N}\right) \\
& \text { and } \bigcup\left(f^{n}(\emptyset) \mid n \in \mathbb{N}\right) \text { is the least fixpoint of } f . \tag{2.16}
\end{align*}
$$

Intuitively, the left-hand side of (2.16) reflects the definition of a set as the smallest set which satisfies a closure property (formalized by $f$ ), and the right-hand side offers a kind of slicing (or stratification) of this set; that slicing can be used in a proof by induction on $\mathbb{N}$.

In the rest of this book, B will denote an arbitrary strong bimonoid $(B, \oplus, \otimes, \mathbb{Q}, \mathbb{1})$ if not specified otherwise.

### 2.7 Matrices and vectors over a strong bimonoid

In this section $Q$ denotes a finite set.

Matrices. A $Q$-square matrix over $B$ is a mapping $M: Q \times Q \rightarrow B$ and an entry $M(p, q) \in B$ is denoted by $M_{p, q}$. We recall that the set of all $Q$-square matrices over $B$ is denoted by $B^{Q \times Q}$. In case that $Q=[\kappa]$ for some $\kappa \in \mathbb{N}$, we write $B^{\kappa \times \kappa}$ instead of $B^{[\kappa] \times[\kappa]}$.

As usual, we define the addition of $Q$-square matrices over $B$. More precisely, we define the binary operation $+: B^{Q \times Q} \times B^{Q \times Q} \rightarrow B^{Q \times Q}$ by letting

$$
\left(M_{1}+M_{2}\right)_{p, q}=\left(M_{1}\right)_{p, q} \oplus\left(M_{2}\right)_{p, q}
$$

for every $M_{1}, M_{2} \in B^{Q \times Q}$ and $p, q \in Q$. It is clear that + is associative and commutative. Moreover, the matrix $\mathrm{M}_{\mathbb{O}} \in B^{Q \times Q}$ with all entries being $\mathbb{O}$, is the identity element for + , hence $\left(B^{Q \times Q},+, \mathrm{M}_{0}\right)$ is a commutative monoid. Also we define the multiplication of $Q$-square matrices over $B$ in the usual way. We define the binary operation $\cdot: B^{Q \times Q} \times B^{Q \times Q} \rightarrow B^{Q \times Q}$ by letting for every $M_{1}, M_{2} \in B^{Q \times Q}$ and $p, q \in Q$ :

$$
\left(M_{1} \cdot M_{2}\right)_{p, q}=\bigoplus_{k \in Q}\left(M_{1}\right)_{p, k} \otimes\left(M_{2}\right)_{k, q}
$$

In general, • is not associative, because B may lack distributivity. The matrix $\mathrm{M}_{\mathbb{1}} \in B^{Q \times Q}$ with

$$
\left(\mathrm{M}_{\mathbb{1}}\right)_{p, q}= \begin{cases}\mathbb{1} & \text { if } p=q \\ \mathbb{O} & \text { otherwise }\end{cases}
$$

for every $p, q \in Q$, is the identity element for $\cdot$ Moreover, $\mathrm{M}_{\mathbb{0}} \cdot M=M \cdot \mathrm{M}_{\mathbb{0}}=\mathrm{M}_{\mathbb{0}}$ for each $M \in B^{Q \times Q}$.
Let $\mathrm{B}=(B, \oplus, \otimes, \mathbb{0}, \mathbb{1})$ be a semiring. Since the operation $\otimes$ of B is distributive over $\oplus$, the operation . on $B^{Q \times Q}$ is associative. Thus, $\left(B^{Q \times Q}, \cdot, \mathrm{M}_{\mathbb{I}}\right)$ is a monoid. Also $\cdot$ is distributive with respect to + . Hence $\mathrm{B}^{Q \times Q}=\left(B^{Q \times Q},+, \cdot, \mathrm{M}_{\mathbb{D}}, \mathrm{M}_{\mathbb{1}}\right)$ is a semiring, called the semiring of $Q$-square matrices over $B$. If B is $\sigma$-complete, then so is $\mathrm{B}^{Q \times Q}$ (cf. [DK09, Sec. 4]).

We mention that the semiring of $\{1,2\}$-square matrices over the semiring Boole (cf. Section 2.7) is not zero-cancellation free, because

$$
\left(\begin{array}{ll}
1 & 0 \\
0 & 0
\end{array}\right) \cdot\left(\begin{array}{ll}
1 & 1 \\
1 & 1
\end{array}\right) \cdot\left(\begin{array}{ll}
0 & 0 \\
0 & 1
\end{array}\right)=\left(\begin{array}{ll}
0 & 1 \\
0 & 0
\end{array}\right) \quad \text { and } \quad\left(\begin{array}{ll}
1 & 0 \\
0 & 0
\end{array}\right) \cdot\left(\begin{array}{ll}
0 & 0 \\
0 & 1
\end{array}\right)=\left(\begin{array}{ll}
0 & 0 \\
0 & 0
\end{array}\right)
$$

Vectors. A $Q$-vector $v$ over $B$ is a mapping $v: Q \rightarrow B$, and an element $v(q) \in B$ of $v$ is denoted by $v_{q}$. We recall that the set of all $Q$-vectors over $B$ is denoted by $B^{Q}$. In case that $Q=[\kappa]$ for some $\kappa \in \mathbb{N}$, we write $B^{\kappa}$ instead of $B^{[\kappa]}$.

We define the operation sum $+: B^{Q} \times B^{Q} \rightarrow B^{Q}$ and scalar multiplication $\cdot: B \times B^{Q} \rightarrow B^{Q}$. For this, let $u, v \in B^{Q}$ and $b \in B$. Then, for every $q \in Q$, we define

$$
(u+v)_{q}=u_{q} \oplus v_{q}, \quad \text { and } \quad(b \cdot u)_{q}=b \otimes u_{q}
$$

We denote by $\mathbb{O}_{Q}$ the $Q$-vector over $B$ which contains $\mathbb{D}$ in each component. Moreover, for each $q \in Q$, we denote by $\mathbb{1}_{q}$ the $q$-unit vector in $B^{Q}$, i.e., the vector in $B^{Q}$ defined by $\left(\mathbb{1}_{q}\right)_{p}=\mathbb{1}$ if $p=q$ and $\left(\mathbb{1}_{q}\right)_{p}=\mathbb{0}$ otherwise. We note that the algebra $\left(B^{Q},+, 0_{Q}\right)$ is a commutative monoid. Moreover, for each $v \in B^{Q}$, we have $v=+_{q \in Q} v_{q} \cdot \mathbb{1}_{q}$.

The scalar product of $Q$-vectors is defined for every $v_{1}, v_{2} \in B^{Q}$ by:

$$
v_{1} \cdot v_{2}=\bigoplus_{q \in Q}\left(v_{1}\right)_{q} \otimes\left(v_{2}\right)_{q}
$$

Products. Let $b \in B, v_{1}, v_{2} \in B^{Q}$, and $M \in B^{Q \times Q}$ be a matrix. We define the matrix $b \cdot M$ by

$$
(b \cdot M)_{p, q}=b \otimes M_{p, q}
$$

for every $p, q \in Q$. Moreover, we define the vector-matrix product $v_{1} \cdot M \in B^{Q}$ and the matrix-vector product $M \cdot v_{2} \in B^{Q}$ such that for every $p \in Q$ :

$$
\left(v_{1} \cdot M\right)_{p}=\bigoplus_{q \in Q}\left(v_{1}\right)_{q} \otimes M_{q, p} \quad\left(M \cdot v_{2}\right)_{p}=\bigoplus_{q \in Q} M_{p, q} \otimes\left(v_{2}\right)_{q}
$$

It is easy to calculate that, if B is left-distributive, then for every $M_{1}, M_{2} \in B^{Q \times Q}$ and $v \in B^{Q}$, we have

$$
\begin{equation*}
\left(M_{1} \cdot M_{2}\right) \cdot v=M_{1} \cdot\left(M_{2} \cdot v\right) \tag{2.17}
\end{equation*}
$$

Characteristic polynomials. Let $(B, \oplus, \otimes, \mathbb{O}, \mathbb{1})$ be a field. Moreover, let $\kappa \in \mathbb{N}_{+}$and $M \in B^{\kappa \times \kappa}$. The characteristic polynomial of $M$, denoted by $\operatorname{char}_{M}$, is defined by

$$
\operatorname{char}_{M}(x)=\operatorname{det}\left(M-x \mathrm{M}_{\mathbb{I}}\right)
$$

where $\mathrm{M}_{\mathbb{1}} \in B^{\kappa \times \kappa}$ is the identity for the matrix multiplication and $\operatorname{det}\left(M^{\prime}\right)$ is the determinant of the matrix $M^{\prime}$ (cf. MY98). Clearly, the degree of $\operatorname{char}_{M}$ is $\kappa$, and the coefficient of $x^{\kappa}$ is $(-1)^{\kappa}$. The following shows an example of $M \in \mathbb{R}^{3 \times 3}$ and $\operatorname{char}_{M}(x)$ :

$$
M=\left(\begin{array}{ccc}
0 & 0 & 1 \\
1 & 3 & 2 \\
2 & 0 & 2
\end{array}\right), \operatorname{det}\left(M-x \mathrm{M}_{\mathbb{1}}\right)=\left[\begin{array}{ccc}
-x & 0 & 1 \\
1 & 3-x & 2 \\
2 & 0 & 2-x
\end{array}\right], \operatorname{char}_{M}(x)=-x^{3}+5 x^{2}-4 x-6
$$

The following theorem is due to Cayley and Hamilton. We recall that char ${ }_{M}(M)$ is the evaluation of $\operatorname{char}_{M}(x)$ in $\left(B^{\kappa \times \kappa},+, \cdot, \mathrm{M}_{\mathbb{D}}, \mathrm{M}_{\mathbb{I}}\right)$ at $x=M$, and that $\mathrm{M}_{\mathbb{D}} \in B^{\kappa \times \kappa}$ is the identity element of + .

Theorem 2.7.1. Lan93, Ch. XIV,Thm. 3.1] Let $\kappa \in \mathbb{N}_{+}$and $M \in B^{\kappa \times \kappa}$. Then $\operatorname{char}_{M}(M)=\mathrm{M}_{\mathbb{D}}$.

### 2.8 Semimodules and vector spaces

In this section let $\mathrm{B}=(B, \oplus, \otimes, \mathbb{O}, \mathbb{1})$ be a semiring.

Semimodules and multilinear operations. Let $\mathrm{V}=(V,+, 0)$ be a commutative monoid ${ }^{3}$. Moreover, let $\cdot: B \times V \rightarrow V$ be a mapping. Then V is a (left-) B -semimodule (via $\cdot$ ) if the following laws hold:

$$
\begin{array}{r}
\left(b \otimes b^{\prime}\right) \cdot v=b \cdot\left(b^{\prime} \cdot v\right) \\
b \cdot\left(v+v^{\prime}\right)=(b \cdot v)+\left(b \cdot v^{\prime}\right) \\
\left(b \oplus b^{\prime}\right) \cdot v=(b \cdot v)+\left(b^{\prime} \cdot v\right) \\
\mathbb{1} \cdot v=v \\
b \cdot 0=\mathbb{0} \cdot v=0 \tag{2.22}
\end{array}
$$

for every $b, b^{\prime} \in B$ and $v, v^{\prime} \in V$ (cf. p. 149 of Gol99). We note that, in Equation (2.22), the element $0 \in V$ occurs twice and the element $\mathbb{D} \in B$ occurs once. In particular, $(B, \oplus, \mathbb{O})$ is a B -semimodule via $\otimes$. Furthermore, we drop • and write, e.g., $b v$ instead of $b \cdot v$.

Let B be commutative and $\mathrm{V}=(V,+, 0)$ be a B -semimodule. A linear form (over V ) is a mapping $\gamma: V \rightarrow B$ such that for every $b, b^{\prime} \in B$ and $v, v^{\prime} \in V$, we have $\gamma\left(b v+b^{\prime} v^{\prime}\right)=b \otimes \gamma(v) \oplus b^{\prime} \otimes \gamma\left(v^{\prime}\right)$.

[^4]An $m$-ary operation $\omega: V^{m} \rightarrow V$ is called multilinear (over V ) if

$$
\begin{align*}
& \omega\left(v_{1}, \ldots, v_{i-1}, b v+b^{\prime} v^{\prime}, v_{i+1}, \ldots, v_{m}\right)=  \tag{2.23}\\
& b \omega\left(v_{1}, \ldots, v_{i-1}, v, v_{i+1}, \ldots, v_{m}\right)+b^{\prime} \omega\left(v_{1}, \ldots, v_{i-1}, v^{\prime}, v_{i+1}, \ldots, v_{m}\right)
\end{align*}
$$

holds for every $i \in[m], b, b^{\prime} \in B$, and $v, v^{\prime}, v_{1}, \ldots, v_{m} \in V$. We denote by $\mathcal{L}\left(\mathrm{V}^{m}, \mathrm{~V}\right)$ the set of all $m$-ary multilinear operations over V .

We explain why commutativity of $B$ is assumed in the above. For this we consider the expression $\omega\left(b_{1} v_{1}, b_{2} v_{2}\right)$ and observe that we can evaluate it in two different ways:

$$
\begin{array}{rlll} 
& \omega\left(b_{1} v_{1}, b_{2} v_{2}\right) & & \omega\left(b_{1} v_{1}, b_{2} v_{2}\right) \\
= & (\text { by }(2.23)(i=1)) & =b_{2} \omega\left(b_{1} v_{1}, v_{2}\right) & (\text { by }(2.23)(i=2)) \\
=b_{1} \omega\left(v_{1}, b_{2} v_{2}\right) & b_{1}\left(b_{2} \omega\left(v_{1}, v_{2}\right)\right) & (\text { by }(2.23)(i=2)) & =b_{2}\left(b_{1} \omega\left(v_{1}, v_{2}\right)\right) \\
=\left(b_{1} \otimes b_{2}\right) \omega\left(v_{1}, v_{2}\right) & (\text { by }(2.18)) & (\text { by }(2.23)(i=1)) \\
= & \left(b_{2} \otimes b_{1}\right) \omega\left(v_{1}, v_{2}\right) & (\text { by }(2.18))
\end{array}
$$

Then commutativity of B guarantees the well-definedness of multilinearity.

Vector spaces and linear mappings. Let $\mathrm{B}=(B, \oplus, \otimes, \mathbb{0}, \mathbb{1})$ be a field. A B-vector space is a Bsemimodule $(V,+, 0)$ where $(V,+, 0)$ is a commutative group. In particular, $(B, \oplus, 0)$ is a B -vector space via $\otimes$.

Let $\mathrm{V}=(V,+, 0)$ be a B -vector space. Moreover, let $U \subseteq V$ be a nonempty subset such that for every $u, v \in U$ and $b \in B$, we have $u+v \in U$ and $b u \in U$. Then $\left(U,+^{\prime}, 0\right)$ is a subspace of V where $+^{\prime}=+\left.\right|_{U \times U}$. Often we will drop the prime from $+^{\prime}$. It is easy to see that any subspace U of V is a B-vector space. Two subspaces $\left(U_{1},+, 0\right)$ and $\left(U_{2},+, 0\right)$ of V are supplementary if $U_{1} \cap U_{2}=\{0\}$ and $U_{1}+U_{2}=V$, where in the latter we denote also by + the natural extension of the binary operation + on $V$ to subsets of $V$.

The elements of $V$ are called vectors. The vectors $v_{1}, \ldots, v_{m} \in V$ are linearly independent if, for every $b_{1}, \ldots, b_{m} \in B$, the equality $b_{1} v_{1}+\ldots+b_{m} v_{m}=0$ implies that $b_{1}=\ldots=b_{m}=\mathbb{0}$. Let $Q \subseteq V$. We say that

- $Q$ is linearly independent if, for every $m \in \mathbb{N}$ and pairwise different $v_{1}, \ldots, v_{m} \in Q$, the vectors $v_{1}, \ldots, v_{m}$ are linearly independent.
- $Q$ generates $V$ if, for every $v \in V$, there exists $m \in \mathbb{N}_{+}$and for every $i \in[m]$ there exist $b_{i} \in B$ and $v_{i} \in Q$ such that $v=b_{1} v_{1}+\ldots+b_{m} v_{m}$.
- $Q$ is a basis of $V$ if $Q$ is linearly independent and $Q$ generates $V$.

A vector space V that has a finite basis is called finite-dimensional. In a finite-dimensional vector space each basis has the same number of elements MY98, p. 341]. V is $\kappa$-dimensional, where $\kappa \in \mathbb{N}_{+}$, if it has a basis consisting of $\kappa$ elements.

Let $\mathrm{V}=(V,+, 0)$ and $\mathrm{V}^{\prime}=\left(V^{\prime},+^{\prime}, 0^{\prime}\right)$ be B -vector spaces via $\cdot$ and.$^{\prime}$, respectively. A mapping $f: V \rightarrow V^{\prime}$ is linear (from V to $\mathrm{V}^{\prime}$ ) if, for every $b_{1}, b_{2} \in B$ and $u, v \in V$, the equation

$$
f\left(b_{1} \cdot u+b_{2} \cdot v\right)=b_{1} \cdot^{\prime} f(u)+^{\prime} b_{2}!^{\prime} f(v)
$$

holds. The vector spaces V and $\mathrm{V}^{\prime}$ are isomorphic if there exists a bijective linear mapping from V to $\mathrm{V}^{\prime}$.
Let $f: V \rightarrow V^{\prime}$ be a linear mapping from V to $\mathrm{V}^{\prime}$. If $Q=\left\{v_{1}, \ldots, v_{m}\right\}$ is a basis of V , then $f$ is determined by the vectors $f\left(v_{1}\right), \ldots, f\left(v_{m}\right)$, because for each $v \in V$ there exist $b_{1}, \ldots, b_{m} \in B$ such that $v=b_{1} \cdot v_{1}+\ldots+b_{m} \cdot v_{m}$ and then

$$
f(v)=f\left(b_{1} \cdot v_{1}+\ldots+b_{m} \cdot v_{m}\right)=b_{1} \cdot^{\prime} f\left(v_{1}\right)+^{\prime} \ldots+^{\prime} b_{m} \cdot^{\prime} f\left(v_{m}\right)
$$

The kernel of $f$, denoted by $\operatorname{ker}(f)$, is the set $\left\{v \in V \mid f(v)=0^{\prime}\right\}$. It is easy to see that $(\operatorname{ker}(f),+, 0)$ and $\left(\operatorname{im}(f),+^{\prime}, 0^{\prime}\right)$ are subspaces of V and $\mathrm{V}^{\prime}$, respectively.

We note that each linear mapping from V to V is a unary multilinear operation over V . Thus, $\mathcal{L}(\mathrm{V}, \mathrm{V})$ is the set of all linear mappings from V to V . Each element of $\mathcal{L}(\mathrm{V}, \mathrm{V})$ is an endomorphism.

Since endomorphisms are closed under composition, $\left(\mathcal{L}(\mathrm{V}, \mathrm{V}), \circ, \mathrm{id}_{V}\right)$ is a monoid (cf. MY98, p. 384]). An endomorphism $f \in \mathcal{L}(\mathrm{~V}, \mathrm{~V})$ is pseudo-regular if $(\operatorname{ker}(f),+, 0)$ and $(\operatorname{im}(f),+, 0)$ are supplementary subspaces of V .

Representing vector spaces and endomorphisms. Let $\mathrm{B}=(B, \oplus, \otimes, \mathbb{0}, \mathbb{1})$ be a field and $\mathrm{V}=$ $\left(V,+^{\prime}, 0\right)$ be a B-vector space via $\cdot^{\prime}$ with the finite basis $Q=\left\{q_{1}, \ldots, q_{\kappa}\right\}$. Let $q_{1}, \ldots, q_{\kappa}$ be an arbitrary, but fixed order of the basis vectors. As usual, each vector of $V$ can be represented by a $[\kappa]$-vector over $B$, and vice versa (cf. MY98, Ch. 3]). The key for this representation is the fact that, for each vector $v \in V$, there exist unique $b_{1}, \ldots, b_{\kappa} \in \mathrm{B}$ such that $v=b_{1}{ }^{\prime} q_{1}+{ }^{\prime} \ldots+{ }^{\prime} b_{\kappa}{ }^{\prime} q_{\kappa}$. Then, intuitively, we will represent $v$ by $\left(b_{1}, \ldots, b_{\kappa}\right)$. Let us formalize this.

Obviously, $\left(B^{\kappa},+, \mathbb{O}_{\kappa}\right)$ is a commutative group, where + is the sum of vectors and $\mathbb{O}_{\kappa}$ is the $[\kappa]$-vector which maps each $i \in[\kappa]$ to $\mathbb{O}$. Moreover, it is easy to check that $\left(B^{\kappa},+, \mathbb{O}_{\kappa}\right)$ is a $\kappa$-dimensional B -vector space via the scalar multiplication $\cdot$. The basis of this vector space is $\left\{\mathbb{1}_{1}, \ldots, \mathbb{1}_{\kappa}\right\}$ where $\mathbb{1}_{i}$ denotes the $i$-unit vector in $B^{\kappa}$, i.e., the element in $B^{\kappa}$ of which the $i$-th component is $\mathbb{1}$ and the other components are 0.

Then the $B$-vector space $\left(V,+^{\prime}, 0\right)$ via $\cdot^{\prime}$ and the $B$-vector space $\left(B^{\kappa},+, \mathbb{O}_{\kappa}\right)$ via $\cdot$ are isomorphic via the bijective linear mapping $\psi: V \rightarrow B^{\kappa}$ defined by

$$
\psi\left(q_{i}\right)=\mathbb{1}_{i} \text { for each } i \in[\kappa]
$$

Hence, we identify $\left(V,+^{\prime}, 0\right)$ and $\left(B^{\kappa},+, \mathbb{O}_{\kappa}\right)$.
Also as usual, we can represent each endomorphism $f \in \mathcal{L}(\mathrm{~V}, \mathrm{~V})$ by a $\kappa$-square matrix over $B$. For this, we define the mapping $\psi^{\prime}: \mathcal{L}(\mathrm{V}, \mathrm{V}) \rightarrow B^{\kappa \times \kappa}$ for each $f: V \rightarrow V$ and $i, j \in[\kappa]$ by:

$$
\psi^{\prime}(f)_{i, j}=\psi\left(f\left(q_{j}\right)\right)_{i}
$$

or, in other words, the $j$ th column of $\psi^{\prime}(f)$ is the $\psi$-image of the basis vector $q_{j}$. It is easy to see that $\psi^{\prime}$ is bijective.

Moreover, $\psi^{\prime}$ is a monoid homomorphism from the monoid $\left(\mathcal{L}(\mathrm{V}, \mathrm{V}), \circ, \mathrm{id}_{V}\right)$ to the monoid $\left(B^{\kappa \times \kappa}, \cdot, \mathrm{M}_{\mathbb{1}}\right)$, which can be seen as follows. Clearly, $\psi^{\prime}\left(\mathrm{id}_{V}\right)=\mathrm{M}_{\mathbb{1}}$. Let $f_{1}, f_{2} \in \mathcal{L}(\mathrm{~V}, \mathrm{~V})$. Then, for every $i, j \in[\kappa]$, we have

$$
\begin{aligned}
\psi^{\prime}\left(f_{1} \circ f_{2}\right)_{i, j} & =\psi\left(\left(f_{1} \circ f_{2}\right)\left(q_{j}\right)\right)_{i} \\
& =\psi\left(f_{1}\left(f_{2}\left(q_{j}\right)\right)\right)_{i} \\
& =\psi\left(f_{1}\left(b_{1} \cdot^{\prime} q_{1}+^{\prime} \ldots+^{\prime} b_{\kappa} \cdot^{\prime} q_{\kappa}\right)\right)_{i}
\end{aligned}
$$

(where $b_{1}, \ldots, b_{\kappa} \in B$ are such that $f_{2}\left(q_{j}\right)=b_{1} .^{\prime} q_{1}+^{\prime} \ldots+{ }^{\prime} b_{\kappa}{ }^{\prime} q_{\kappa}$ )
$\left.=\psi\left(b_{1} \cdot{ }^{\prime} f_{1}\left(q_{1}\right)+^{\prime} \ldots+^{\prime} b_{\kappa} \cdot^{\prime} f_{1}\left(q_{\kappa}\right)\right)\right)_{i} \quad$ (since $f_{1}$ is a linear mapping)
$=\left(b_{1} \cdot \psi\left(f_{1}\left(q_{1}\right)\right)+\ldots+b_{\kappa} \cdot \psi\left(f_{1}\left(q_{\kappa}\right)\right)\right)_{i} \quad$ (since $\psi$ is a linear mapping)
$=b_{1} \otimes \psi\left(f_{1}\left(q_{1}\right)\right)_{i} \oplus \ldots \oplus b_{\kappa} \otimes \psi\left(f_{1}\left(q_{\kappa}\right)\right)_{i}$
(by definition of scalar multiplication and vector summation)
$=\psi\left(f_{2}\left(q_{j}\right)\right)_{1} \otimes \psi\left(f_{1}\left(q_{1}\right)\right)_{i} \oplus \ldots \oplus \psi\left(f_{2}\left(q_{j}\right)\right)_{\kappa} \otimes \psi\left(f_{1}\left(q_{\kappa}\right)\right)_{i} \quad$ (because $\left.b_{k}=\psi\left(f_{2}\left(q_{j}\right)\right)_{k}\right)$
$=\bigoplus_{k \in[\kappa]} \psi\left(f_{2}\left(q_{j}\right)\right)_{k} \otimes \psi\left(f_{1}\left(q_{k}\right)\right)_{i}$
$=\bigoplus_{k \in[\kappa]} \psi^{\prime}\left(f_{2}\right)_{k, j} \otimes \psi^{\prime}\left(f_{1}\right)_{i, k}$
$=\bigoplus_{k \in[\kappa]} \psi^{\prime}\left(f_{1}\right)_{i, k} \otimes \psi^{\prime}\left(f_{2}\right)_{k, j}$
$=\left(\psi^{\prime}\left(f_{1}\right) \cdot \psi^{\prime}\left(f_{2}\right)\right)_{i, j}$.

Hence, the two monoids $\left(\mathcal{L}(\mathrm{V}, \mathrm{V}), \circ, \mathrm{id}_{V}\right)$ and $\left(B^{\kappa \times \kappa}, \cdot, \mathrm{M}_{\mathbb{1}}\right)$ are isomorphic.
A matrix $M \in B^{\kappa \times \kappa}$ is pseudo-regular if the endomorphism $\left(\psi^{\prime}\right)^{-1}(M)$ is pseudo-regular. Equivalent definitions of pseudo-regularity can be found in [Reu80, Prop. 1].

### 2.9 Trees and tree languages

We recall some notions from (formal) tree languages Eng75b, GS84, GS97, CDG+07.
In the rest of this book, we assume that each ranked alphabet has at least one symbol with rank 0. Moreover, $\Sigma$ and $\Delta$ will denote arbitrary such ranked alphabets, if not specified otherwise.

Trees. Let $H$ be a set and let $\Xi$ denote the set containing the opening and closing parentheses "(" and ")", respectively, and the comma ",". We assume that $\Sigma, H$, and $\Xi$ are pairwise disjoint. Now we define the $\Sigma$-algebra $\left((\Sigma \cup H \cup \Xi)^{*}, \theta_{\Sigma}\right)$ where, for every $k \in \mathbb{N}, \sigma \in \Sigma^{(k)}$, and $w_{1}, \ldots, w_{k} \in(\Sigma \cup H \cup \Xi)^{*}$, we let

$$
\theta_{\Sigma}(\sigma)\left(w_{1}, \ldots, w_{k}\right)=\sigma\left(w_{1}, \ldots, w_{k}\right)
$$

Then the set of $\Sigma$-terms over $H$, denoted by $\mathrm{T}_{\Sigma}(H)$, is the set $\langle H\rangle_{\theta_{\Sigma}(\Sigma)}$, i.e., the smallest subset of $(\Sigma \cup H \cup \Xi)^{*}$ which contains $H$ and is closed under $\theta_{\Sigma}(\Sigma)$. We denote $\mathrm{T}_{\Sigma}(\emptyset)$ by $\mathrm{T}_{\Sigma}$ and call it the set of $\Sigma$-terms. If $H$ is finite, then we can view $\mathrm{T}_{\Sigma}(H)$ as $\mathrm{T}_{\Delta}$, where the ranked alphabet $\Delta$ is defined as follows: $\Delta^{(0)}=\Sigma^{(0)} \cup H$ and $\Delta^{(k)}=\Sigma^{(k)}$ for every $k \in \mathbb{N}_{+}$.

Since terms can be depicted in a very illustrative way as trees, i.e., particular graphs, it has become a custom to call $\Sigma$-terms also $\Sigma$-trees (or just: trees). In this book we follow this custom. Each subset $L \subseteq \mathrm{~T}_{\Sigma}$ is called a $\Sigma$-tree language (or just: tree language).

Obviously, for each $\xi \in \mathrm{T}_{\Sigma}$ there exist unique $k \in \mathbb{N}, \sigma \in \Sigma^{(k)}$, and $\xi_{1}, \ldots, \xi_{k} \in \mathrm{~T}_{\Sigma}$ such that $\xi=\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)$. (In case $k=0$ we identify $\sigma()$ and $\sigma$.) In order to avoid the repetition of all these quantifications, we henceforth only write that we consider a " $\xi \in \mathrm{T}_{\Sigma}$ of the form $\xi=\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)$ " or "for every $\xi=\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)$ ".

As an application of Lemma 2.6.1 we obtain the following characterization of the set $\mathrm{T}_{\Sigma}$.
Observation 2.9.1. Let $f_{\mathrm{T}_{\Sigma}}: \mathcal{P}\left((\Sigma \cup \Xi)^{*}\right) \rightarrow \mathcal{P}\left((\Sigma \cup \Xi)^{*}\right)$ be the mapping defined, for each $U \in$ $\mathcal{P}\left((\Sigma \cup \Xi)^{*}\right)$, by $f_{\mathrm{T}_{\Sigma}}(U)=U \cup\left\{\sigma\left(\xi_{1}, \ldots, \xi_{k}\right) \mid k \in \mathbb{N}, \sigma \in \Sigma^{(k)}, \xi_{1}, \ldots, \xi_{k} \in U\right\}$. Then, by Lemma 2.6.1, we have $\mathrm{T}_{\Sigma}=\bigcup\left(\left(f_{\mathrm{T}_{\Sigma}}\right)^{n}(\emptyset) \mid n \in \mathbb{N}\right)$.

The $\Sigma$-term algebra. Let $H$ be a set disjoint with $\Sigma$. The $\Sigma$-term algebra over $H$, denoted by $\mathrm{T}_{\Sigma}(H)$, is the subalgebra of the $\Sigma$-algebra $\left((\Sigma \cup H \cup \Xi)^{*}, \theta_{\Sigma}\right)$ which is generated by $H$. Thus, since $\langle H\rangle_{\theta_{\Sigma}(\Sigma)}=\mathrm{T}_{\Sigma}(H)$, the $\Sigma$-term algebra over $H$ is the $\Sigma$-algebra

$$
\mathrm{T}_{\Sigma}(H)=\left(\mathrm{T}_{\Sigma}(H), \theta_{\Sigma}\right)
$$

The $\Sigma$-term algebra, denoted by $\mathrm{T}_{\Sigma}$, is the $\Sigma$-term algebra over $\emptyset$, i.e., $\mathrm{T}_{\Sigma}=\mathrm{T}_{\Sigma}(\emptyset)$.
Next we prove that $\mathrm{T}_{\Sigma}(H)$ is freely generated by $H$ over the set of all $\Sigma$-algebras by using Theorem 2.5.1 For this, we use the following auxiliary definitions. We define the binary relation $\prec$ on $\mathrm{T}_{\Sigma}(H)$ by

$$
\prec=\left\{\left(\xi_{i}, \sigma\left(\xi_{1}, \ldots, \xi_{k}\right)\right) \mid k \in \mathbb{N}_{+}, \sigma \in \Sigma^{(k)}, \xi_{1}, \ldots, \xi_{k} \in \mathrm{~T}_{\Sigma}(H), i \in[k]\right\}
$$

Obviously $\prec$ is well-founded and $\min _{\prec}\left(\mathrm{T}_{\Sigma}(H)\right)=\Sigma^{(0)} \cup H$. Let $\mathrm{A}=(A, \theta)$ be a $\Sigma$-algebra and $f: H \rightarrow A$. Then we define the mapping

$$
G_{\theta, f}:\left\{(\xi, g) \mid \xi \in \mathrm{T}_{\Sigma}(H), g: \operatorname{pred}_{\prec}(\xi) \rightarrow A\right\} \rightarrow A
$$

for every $\xi \in \mathrm{T}_{\Sigma}(H)$ and $g: \operatorname{pred}_{\prec}(\xi) \rightarrow A$ as follows:

$$
G_{\theta, f}(\xi, g)= \begin{cases}\theta(\sigma)\left(g\left(\xi_{1}\right), \ldots, g\left(\xi_{k}\right)\right) & \text { if }\left(\exists k \in \mathbb{N}, \sigma \in \Sigma^{(k)}, \xi_{1}, \ldots, \xi_{k} \in \mathrm{~T}_{\Sigma}(H)\right): \xi=\sigma\left(\xi_{1}, \ldots, \xi_{k}\right) \\ f(\xi) & \text { otherwise (i.e., } \xi \in H)\end{cases}
$$

Now we prove that the unique mapping defined by $G_{\theta, f}$ is the unique $\Sigma$-algebra homomorphism from $\mathrm{T}_{\Sigma}(H)$ to A which extends $f$.
Lemma 2.9.2. Let $\mathrm{A}=(A, \theta)$ be a $\Sigma$-algebra and $f: H \rightarrow A$. Moreover, let $\phi: \mathrm{T}_{\Sigma}(H) \rightarrow A$ be a mapping. Then the following two statements are equivalent.
(A) For each $\xi \in \mathrm{T}_{\Sigma}(H)$, the mapping $\phi$ satisfies $\phi(\xi)=G_{\theta, f}\left(\xi,\left.\phi\right|_{\text {pred }_{\prec}(\xi)}\right)$.
(B) $\phi$ is a $\Sigma$-algebra homomorphism from $\mathrm{T}_{\Sigma}(H)$ to A , and $\phi$ extends $f$.

Proof. Proof of $(\mathrm{A}) \Rightarrow(\mathrm{B})$ : Let $\xi=\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)$. Then we can compute as follows:

$$
\begin{align*}
\phi\left(\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)\right) & =G_{\theta, f}\left(\xi,\left.\phi\right|_{\operatorname{pred}_{\prec}(\xi)}\right)  \tag{A}\\
& =\theta(\sigma)\left(\left.\phi\right|_{\operatorname{pred}_{\prec}(\xi)}\left(\xi_{1}\right), \ldots, \phi\right. \\
& =\theta(\sigma)\left(\phi\left(\xi_{1}\right), \ldots, \phi\left(\xi_{k}\right)\right) .
\end{align*}
$$

$$
\left.=\theta(\sigma)\left(\left.\phi\right|_{\operatorname{pred}_{\prec}(\xi)}\left(\xi_{1}\right), \ldots,\left.\phi\right|_{\operatorname{pred}_{\prec}(\xi)}\left(\xi_{k}\right)\right) \quad \quad \text { (by definition of } G_{\theta, f}\right)
$$

Hence $\phi$ is a $\Sigma$-algebra homomorphism from $\mathrm{T}_{\Sigma}(H)$ to A. By definition of $G_{\theta, f}$, the homomorphism $\phi$ extends $f$.

Proof of $(\mathrm{B}) \Rightarrow(\mathrm{A})$ : We prove by case analysis. Let $\xi \in H$. Then $\phi(\xi)=f(\xi)=G_{\theta, f}\left(\xi,\left.\phi\right|_{\text {pred }_{\prec}(\xi)}\right)$. Let $\xi=\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)$. Then we can compute as follows:

$$
\begin{align*}
G_{\theta, f}\left(\xi,\left.\phi\right|_{\operatorname{pred}_{\prec}(\xi)}\right) & \left.=\theta(\sigma)\left(\left.\phi\right|_{\operatorname{pred}_{\prec}(\xi)}\left(\xi_{1}\right), \ldots,\left.\phi\right|_{\operatorname{pred}_{\prec}(\xi)}\left(\xi_{k}\right)\right) \quad \text { (by definition of } G_{\theta, f}\right) \\
& =\theta(\sigma)\left(\phi\left(\xi_{1}\right), \ldots, \phi\left(\xi_{k}\right)\right) \\
& =\phi(\xi) \tag{B}
\end{align*}
$$

Theorem 2.9.3. GTWW77, Prop. 2.3] and [GS84, Thm. 1.3.11] The $\Sigma$-term algebra $\mathrm{T}_{\Sigma}(H)$ over $H$ is freely generated by $H$ over the set of all $\Sigma$-algebras. Moreover, the $\Sigma$-term algebra $\mathrm{T}_{\Sigma}$ is initial in the set of all $\Sigma$-algebras.

Proof. For the proof of the first statement of the lemma, we proceed along the items (a), (b), and (c) of the definition of freely generated algebra.
(a) As we saw, the $\Sigma$-term algebra $\mathrm{T}_{\Sigma}(H)$ over $H$ is a $\Sigma$-algebra.
(b) By definition, we have $\mathrm{T}_{\Sigma}(H)=\langle H\rangle_{\theta_{\Sigma}(\Sigma)}$.
(c) Let $\mathrm{A}=(A, \theta)$ be a $\Sigma$-algebra and $f: H \rightarrow A$ a mapping. By Theorem 2.5.1 there exists a unique mapping $h: \mathrm{T}_{\Sigma}(H) \rightarrow A$ such that, for each $\xi \in \mathrm{T}_{\Sigma}(H)$, we have $h(\xi)=G_{\theta, f}\left(\xi,\left.h\right|_{\operatorname{pred}_{\swarrow}(\xi)}\right)$. Thus, due to Lemma 2.9.2, we obtain that $h$ is the unique $\Sigma$-algebra homomorphism from $\mathrm{T}_{\Sigma}(H)$ to A which extends $f$.

This proves the first statement of the lemma. The second statement follows from the first one with $H=\emptyset$.

If not specified otherwise, then we denote the unique $\Sigma$-algebra homomorphism from the $\Sigma$ term algebra $\mathrm{T}_{\Sigma}=\left(\mathrm{T}_{\Sigma}, \theta_{\Sigma}\right)$ to some $\Sigma$-algebra A by $\mathrm{h}_{\mathrm{A}}$.
Observation 2.9.4. Let $\mathrm{A}=(A, \theta)$ be a $\Sigma$-algebra and $\left(A^{\prime}, \theta\right)$ be the smallest subalgebra of A . Then $A^{\prime}=\operatorname{im}\left(\mathrm{h}_{\mathrm{A}}\right)$.

Proof. By Lemma 2.6.2, $\left(\mathrm{im}\left(\mathrm{h}_{\mathrm{A}}\right), \theta\right)$ is a subalgebra of A. Moreover, we can show by induction on $\mathrm{T}_{\Sigma}$ that, for each $\xi \in \mathrm{T}_{\Sigma}$, we have $\mathrm{h}_{\mathrm{A}}(\xi) \in A^{\prime}$. Hence $\operatorname{im}\left(\mathrm{h}_{\mathrm{A}}\right) \subseteq A^{\prime}$, and since $\left(A^{\prime}, \theta\right)$ is the smallest subalgebra of A , we have $\operatorname{im}\left(\mathrm{h}_{\mathrm{A}}\right)=A^{\prime}$.

String-like terms. We can view strings over an alphabet $\Gamma$ as trees. For this, we first define an appropriate ranked alphabet.

Let $e \notin \Gamma$ be a symbol. Then $\Gamma$ and $e$ determine the string ranked alphabet $\Gamma_{e}=\left\{a^{(1)} \mid a \in \Gamma\right\} \cup\left\{e^{(0)}\right\}$. Vice versa, each string ranked alphabet $\Sigma$ can be written in the form $\Gamma_{e}$, where $\Gamma=\Sigma^{(1)}$ and $e$ is the only element of $\Sigma^{(0)}$. In the following, sometimes we denote a string ranked alphabet by $\Gamma_{e}$ without mentioning what $\Gamma$ and $e$ mean.

Now we define the $\Gamma_{e}$-algebrat $\left(\Gamma^{*}, \widehat{\Gamma_{e}}\right)$ where $\widehat{\Gamma_{e}}=\left(\widehat{b} \mid b \in \Gamma_{e}\right)$ is the $\Gamma_{e}$-indexed family over $\operatorname{Ops}\left(\Gamma^{*}\right)$ such that

$$
\widehat{e}=\varepsilon \text { and } \widehat{a}(w)=w a \text { for every } w \in \Gamma^{*} \text { and } a \in \Gamma .
$$

Then we consider the $\Gamma_{e}$-term algebra $\left(\mathrm{T}_{\Gamma_{e}}, \theta_{\Gamma_{e}}\right)$ and show that

$$
\begin{equation*}
\left(\Gamma^{*}, \widehat{\Gamma_{e}}\right) \cong\left(\mathrm{T}_{\Gamma_{e}}, \theta_{\Gamma_{e}}\right) \tag{2.24}
\end{equation*}
$$

i.e., the two $\Gamma_{e}$-algebras $\left(\Gamma^{*}, \widehat{\Gamma_{e}}\right)$ and $\left(\mathrm{T}_{\Gamma_{e}}, \theta_{\Gamma_{e}}\right)$ are isomorphic. For this, we define the mapping

$$
\operatorname{tree}_{e}: \Gamma^{*} \rightarrow \mathrm{~T}_{\Gamma_{e}}
$$

as follows. We consider the well-founded set $\left(\Gamma^{*}, \prec\right)$ where $\prec \subseteq \Gamma^{*} \times \Gamma^{*}$ is defined by $\prec=\{(w, w a) \mid w \in$ $\left.\Gamma^{*}, a \in \Gamma\right\}$. Clearly, $\prec$ is well-founded and $\min _{\prec}\left(\Gamma^{*}\right)=\{\varepsilon\}$. Then we define tree $e_{e}$ by induction on ( $\Gamma^{*}, \prec$ ) by
I.B.: $\operatorname{tree}_{e}(\varepsilon)=e$ and
I.S.: $\operatorname{tree}_{e}(w a)=a\left(\operatorname{tree}_{e}(w)\right)$ for every $w \in \Gamma^{*}$ and $a \in \Gamma$.

It is obvious that tree $e$ is a bijection. Lastly, we show that tree $e$ is a $\Gamma_{e}$-algebra homomorphism from $\left(\Gamma^{*}, \widehat{\Gamma_{e}}\right)$ to $\left(\mathrm{T}_{\Gamma_{e}}, \theta_{\Gamma_{e}}\right)$ :
(i) $\operatorname{tree}_{e}(\widehat{e})=\operatorname{tree}_{e}(\varepsilon)=e=\theta_{\Gamma_{e}}(e)()$, and
(ii) for every $w \in \Gamma^{*}$ and $a \in \Gamma$, we have $\operatorname{tree}_{e}(\widehat{a}(w))=\operatorname{tree}_{e}(w a)=a\left(\operatorname{tree}_{e}(w)\right)=\theta_{\Gamma_{e}}(a)\left(\operatorname{tree}_{e}(w)\right)$.

Hence (2.24) holds.

Proof by induction on $\mathrm{T}_{\Sigma}(H)$ and proof by induction on $\mathrm{T}_{\Sigma}$. Let $H$ be a set disjoint with $\Sigma$. Many times we will use the following two instances of well-founded induction to prove that each element of $\mathrm{T}_{\Sigma}(H)$ has a property:

- proof by induction on $\left(\mathrm{T}_{\Sigma}(H), \prec_{\Sigma, H}\right)$ (for short: proof by induction on $\left.\mathrm{T}_{\Sigma}(H)\right)$ : we define the binary relation $\sqrt[5]{ } \prec_{\Sigma, H}$ on $\mathrm{T}_{\Sigma}(H)$ by

$$
\prec \Sigma, H=\left\{\left(\xi_{i}, \sigma\left(\xi_{1}, \ldots, \xi_{k}\right)\right) \mid k \in \mathbb{N}_{+}, \sigma \in \Sigma^{(k)}, \xi_{1}, \ldots, \xi_{k} \in \mathrm{~T}_{\Sigma}(H), i \in[k]\right\} .
$$

Obviously, $\prec_{\Sigma, H}$ is well-founded and $\min _{\prec_{\Sigma, H}}\left(\mathrm{~T}_{\Sigma}(H)\right)=\Sigma^{(0)} \cup H$. In this case, the induction base (2.2) and induction step (2.3) read

$$
\begin{aligned}
& \text { I.B.: }\left(\forall \alpha \in \Sigma^{(0)} \cup H\right): P(\alpha) \\
& \text { I.S.: }\left(\left(\forall k \in \mathbb{N}_{+}, \sigma \in \Sigma^{(k)}, \xi_{1}, \ldots, \xi_{k} \in \mathrm{~T}_{\Sigma}(H)\right):\left[P\left(\xi_{1}\right) \wedge \ldots \wedge P\left(\xi_{k}\right)\right] \rightarrow P\left(\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)\right)\right) .
\end{aligned}
$$

- proof by induction on $\left(\mathrm{T}_{\Sigma}(H), \prec_{\Sigma, H}^{+}\right)$, where $\prec_{\Sigma, H}^{+}=\left(\prec_{\Sigma, H}\right)^{+}$, i.e., $\prec_{\Sigma, H}^{+}$is the transitive closure of $\prec_{\Sigma, H}$. Again we have $\min _{\prec_{\Sigma, H}^{+}}\left(\mathrm{T}_{\Sigma}(H)\right)=\Sigma^{(0)} \cup H$. In this case, the induction base (2.2) and the induction step (2.3) read

$$
\begin{aligned}
& \text { I.B.: }\left(\forall \alpha \in \Sigma^{(0)} \cup H\right): P(\alpha) \\
& \text { I.S.: }\left(\left(\forall \xi \in \mathrm{T}_{\Sigma}(H) \backslash\left(\Sigma^{(0)} \cup H\right)\right):\left[\left(\forall \xi^{\prime} \in \mathrm{T}_{\Sigma}(H) \text { with } \xi^{\prime} \prec_{\Sigma, H}^{+} \xi\right): P\left(\xi^{\prime}\right)\right] \rightarrow P(\xi)\right) .
\end{aligned}
$$

[^5]Often we will use induction on $\mathrm{T}_{\Sigma}(H)$ for the case that $H=\emptyset$. We will abbreviate $\prec_{\Sigma, \emptyset}$ and $\prec_{\Sigma, \emptyset}^{+}$by $\prec_{\Sigma}$ and $\prec_{\Sigma}^{+}$, respectively; then $\min _{\prec_{\Sigma}}\left(\mathrm{T}_{\Sigma}\right)=\min _{\prec_{\Sigma}^{+}}\left(\mathrm{T}_{\Sigma}\right)=\Sigma^{(0)}$. We will call a proof by induction on $\mathrm{T}_{\Sigma}(\emptyset)$ (i.e., on $\left(\mathrm{T}_{\Sigma}, \prec_{\Sigma}\right)$ ) a proof by induction on $\mathrm{T}_{\Sigma}$. As in the general case, we will use the idioms "we prove $P$ by induction on $\mathrm{T}_{\Sigma}(H)$ " and "we prove $P$ by induction on $\mathrm{T}_{\Sigma}$ " with their natural meanings.

In some proofs by induction on $\mathrm{T}_{\Sigma}$, we combine the induction base (2.2) and induction step (2.3) by proving

$$
\left(\forall k \in \mathbb{N}, \sigma \in \Sigma^{(k)}, \xi_{1}, \ldots, \xi_{k} \in \mathrm{~T}_{\Sigma}\right):\left[P\left(\xi_{1}\right) \wedge \ldots \wedge P\left(\xi_{k}\right)\right] \rightarrow P\left(\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)\right)
$$

We will define several mappings on $\mathrm{T}_{\Sigma}(H)$ by induction on $\left(\mathrm{T}_{\Sigma}(H), \prec_{\Sigma, H}\right)$ (cf. (2.4)). Then we will use the idiom "we define a mapping by induction on $\mathrm{T}_{\Sigma}(H)$ " or, if $H=\emptyset$, "we define a mapping by induction on $\mathrm{T}_{\Sigma}$ ".

Mappings on trees. Let $H$ be a set disjoint with $\Sigma$. First, we define the three mappings

$$
\text { height : } \mathrm{T}_{\Sigma}(H) \rightarrow \mathbb{N}, \quad \text { size }: \mathrm{T}_{\Sigma}(H) \rightarrow \mathbb{N}, \quad \text { and } \quad \operatorname{pos}: \mathrm{T}_{\Sigma}(H) \rightarrow \mathcal{P}\left(\mathbb{N}_{+}{ }^{*}\right)
$$

which, intuitively, for each tree $\xi \in \mathrm{T}_{\Sigma}(H)$ (viewed as graph), deliver the maximal number of edges from the root of $\xi$ to some leaf, the number of nodes of $\xi$, and the set of Gorn-addresses of $\xi$, respectively. For instance, let $\Sigma=\left\{\sigma^{(2)}, \gamma^{(1)}, \alpha^{(0)}, \beta^{(0)}\right\}, H=\{a\}$, and $\xi=\sigma(\gamma(\alpha), \sigma(\gamma(a), \beta))$; then

$$
\operatorname{height}(\xi)=3, \quad \operatorname{size}(\xi)=7, \quad \operatorname{pos}(\xi)=\{\varepsilon, 1,11,2,21,211,22\}
$$

Formally, we use Theorem 2.9.3 for the definition of height, size, and pos as follows.

1. Let $\left(\mathbb{N}, \theta_{1}\right)$ be the $\Sigma$-algebra where $\theta_{1}$ is defined as follows:

- for each $\alpha \in \Sigma^{(0)}$, we let $\theta_{1}(\alpha)()=0$, and
- for each $k \in \mathbb{N}_{+}, \sigma \in \Sigma^{(k)}$, and $n_{1}, \ldots, n_{k} \in \mathbb{N}$, we let $\theta_{1}(\sigma)\left(n_{1}, \ldots, n_{k}\right)=1+\max \left(n_{1}, \ldots, n_{k}\right)$.

Moreover, let $f: H \rightarrow \mathbb{N}$ be defined such that, for each $a \in H$, we let $f(a)=0$. Then

$$
\text { height : } \mathrm{T}_{\Sigma}(H) \rightarrow \mathbb{N}
$$

is the unique extension of $f$ to a $\Sigma$-algebra homomorphism from the $\Sigma$-term algebra $\mathrm{T}_{\Sigma}(H)$ over $H$ to the $\Sigma$-algebra $\left(\mathbb{N}, \theta_{1}\right)$.
2. Let $\left(\mathbb{N}, \theta_{2}\right)$ be the $\Sigma$-algebra where $\theta_{2}$ is defined as follows: for each $k \in \mathbb{N}, \sigma \in \Sigma^{(k)}$, and $n_{1}, \ldots, n_{k} \in$ $\mathbb{N}$, we let $\theta_{2}(\sigma)\left(n_{1}, \ldots, n_{k}\right)=1++_{i \in[k]} n_{i}$. Moreover, let $f: H \rightarrow \mathbb{N}$ be defined such that, for each $a \in H$, we let $f(a)=1$. Then

$$
\text { size }: \mathrm{T}_{\Sigma}(H) \rightarrow \mathbb{N}
$$

is the unique extension of $f$ to a $\Sigma$-algebra homomorphism from the $\Sigma$-term algebra $\mathrm{T}_{\Sigma}(H)$ over $H$ to the $\Sigma$-algebra $\left(\mathbb{N}, \theta_{2}\right)$.
3. Let $\left(\mathcal{P}\left(\mathbb{N}_{+}{ }^{*}\right), \theta_{3}\right)$ be the $\Sigma$-algebra where $\theta_{3}$ is defined as follows: for each $k \in \mathbb{N}, \sigma \in \Sigma^{(k)}$, and $U_{1}, \ldots, U_{k} \in \mathcal{P}\left(\mathbb{N}_{+}{ }^{*}\right)$, we let $\theta_{3}(\sigma)\left(U_{1}, \ldots, U_{k}\right)=\{\varepsilon\} \cup\left\{i v \mid i \in[k], v \in U_{i}\right\}$. Moreover, let $f: H \rightarrow \mathcal{P}\left(\mathbb{N}_{+}{ }^{*}\right)$ be defined such that, for each $a \in H$, we let $f(a)=\{\varepsilon\}$. Then

$$
\operatorname{pos}: \mathrm{T}_{\Sigma}(H) \rightarrow \mathcal{P}\left(\mathbb{N}_{+}{ }^{*}\right)
$$

is the unique extension of $f$ to a $\Sigma$-algebra homomorphism from the $\Sigma$-term algebra $\mathrm{T}_{\Sigma}(H)$ over $H$ to the $\Sigma$-algebra $\left(\mathcal{P}\left(\mathbb{N}_{+}{ }^{*}\right), \theta_{3}\right)$.

It is easy to verify that the following equalities hold.


Figure 2.6: For a given tree $\xi=\sigma(\gamma(\alpha), \sigma(\gamma(\beta), \beta))$ and position $w=21$, the label of $\xi$ at position $w$, the subtree of $\xi$ at position $w$, and the replacement of the subtree of $\xi$ at position $w$ by $\zeta$.
(i) For each $\alpha \in \Sigma^{(0)} \cup H$, we have height $(\alpha)=0$, $\operatorname{size}(\alpha)=1$, and $\operatorname{pos}(\alpha)=\{\varepsilon\}$.
(ii) For each $\xi=\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)$ with $k \in \mathbb{N}_{+}$, we have:
$-\operatorname{height}(\xi)=1+\max \left(\operatorname{height}\left(\xi_{i}\right) \mid i \in[k]\right)$,
$-\operatorname{size}(\xi)=1++_{i \in[k]} \operatorname{size}\left(\xi_{i}\right)$, and
$-\operatorname{pos}(\xi)=\{\varepsilon\} \cup\left\{i v \mid i \in[k], v \in \operatorname{pos}\left(\xi_{i}\right)\right\}$.
Observation 2.9.5. For the mapping $f_{\mathrm{T}_{\Sigma}}: \mathcal{P}\left((\Sigma \cup \Xi)^{*}\right) \rightarrow \mathcal{P}\left((\Sigma \cup \Xi)^{*}\right)$ defined in Observation 2.9.1, there is a connection between the set $f_{\mathrm{T}_{\Sigma}}^{n+1}(\emptyset)$ and the set of $\Sigma$-trees of height at most $n$. Indeed, we can prove by induction on $\mathbb{N}$ that, for each $n \in \mathbb{N}$, we have $f_{\mathrm{T}_{\Sigma}}^{n+1}(\emptyset)=\left\{\xi \in \mathrm{T}_{\Sigma} \mid\right.$ height $\left.(\xi) \leq n\right\}$.

Next we define three mappings $f, g$, and $h$ which show the label at a position of a tree, the subtree at a position of a tree, and the replacement of the subtree at a position of a tree, respectively (cf. Figure 2.6). We define these mappings by well-founded induction. For this, we define the set $\mathrm{TP}=\{(\xi, w) \mid \xi \in$ $\left.\mathrm{T}_{\Sigma}(H), w \in \operatorname{pos}(\xi)\right\}$ and the binary relation $\prec$ on TP by

$$
\prec=\left\{\left(\left(\xi_{i}, v\right),\left(\sigma\left(\xi_{1}, \ldots, \xi_{k}\right), i v\right)\right) \mid k \in \mathbb{N}_{+}, \sigma \in \Sigma^{(k)}, \xi_{1}, \ldots, \xi_{k} \in \mathrm{~T}_{\Sigma}(H), i \in[k], v \in \operatorname{pos}\left(\xi_{i}\right)\right\}
$$

Obviously, $\prec$ is well-founded and $\min _{\prec}(\mathrm{TP})=\left\{(\xi, \varepsilon) \mid \xi \in \mathrm{T}_{\Sigma}(H)\right\}$. We define the mappings

$$
f: \mathrm{TP} \rightarrow \Sigma, \quad g: \mathrm{TP} \rightarrow \mathrm{~T}_{\Sigma}(H), \quad \text { and } \quad h: \mathrm{TP} \rightarrow \mathrm{~T}_{\Sigma}(H)^{\mathrm{T}_{\Sigma}(H)}
$$

by induction on ( $\mathrm{TP}, \prec$ ) as follows:
I.B.: Let $\xi \in \mathrm{T}_{\Sigma}(H)$. Moreover, let $\zeta \in \mathrm{T}_{\Sigma}(H)$. Then we define $f(\xi, \varepsilon), g(\xi, \varepsilon)$, and $h(\xi, \varepsilon)(\zeta)$ as follows.

- If $\xi \in H$, then we define $f(\xi, \varepsilon)=\xi$. If there exist $k \in \mathbb{N}, \sigma \in \Sigma^{(k)}$, and $\xi_{1}, \ldots, \xi_{k} \in \mathrm{~T}_{\Sigma}(H)$ such that $\xi=\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)$, then we define $f(\xi, \varepsilon)=\sigma$,
- $g(\xi, \varepsilon)=\xi$, and
- $h(\xi, \varepsilon)(\zeta)=\zeta$.
I.S.: Let $\xi \in \mathrm{T}_{\Sigma}(H)$ and $i v \in \operatorname{pos}(\xi)$ with $i \in \mathbb{N}$ and $v \in \mathbb{N}^{*}$. Thus, there exists $k \in \mathbb{N}_{+}$with $k \geq i$ and there exist $\sigma \in \Sigma^{(k)}$ and $\xi_{1}, \ldots, \xi_{k} \in \mathrm{~T}_{\Sigma}(H)$ such that $\xi=\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)$. Moreover, let $\zeta \in \mathrm{T}_{\Sigma}(H)$. Then we define $f(\xi, i v), g(\xi, i v)$, and $h(\xi, i v)(\zeta)$ as follows.
- $f(\xi, i v)=f\left(\xi_{i}, v\right)$,
- $g(\xi, i v)=g\left(\xi_{i}, v\right)$, and
- $h(\xi, i v)(\zeta)=\sigma\left(\xi_{1}, \ldots, \xi_{i-1}, h\left(\xi_{i}, v\right)(\zeta), \xi_{i+1}, \ldots, \xi_{k}\right)$.

For every $\xi, \zeta \in \mathrm{T}_{\Sigma}(H)$ and $w \in \operatorname{pos}(\xi)$ we call

- $f(\xi, w)$ the label of $\xi$ at $w$, and we denote it by $\xi(w)$,
- $g(\xi, w)$ the subtree of $\xi$ at $w$, and we denote it by $\left.\xi\right|_{w}$, and
- $h(\xi, w)(\zeta)$ the replacement of the subtree of $\xi$ at $w$ by $\zeta$, and we denote it by $\xi[\zeta]_{w}$.

For each subset $Q \subseteq \Sigma \cup H$, we define $\operatorname{pos}_{Q}: \mathrm{T}_{\Sigma}(H) \rightarrow \mathcal{P}\left(\mathbb{N}_{+}{ }^{*}\right)$ by

$$
\operatorname{pos}_{Q}(\xi)=\{w \in \operatorname{pos}(\xi) \mid \xi(w) \in Q\}
$$

and, for each $\xi \in \mathrm{T}_{\Sigma}(H)$, we define the set of subtrees of $\xi$, denoted by $\operatorname{sub}(\xi)$, as the set

$$
\operatorname{sub}(\xi)=\left\{\left.\xi\right|_{w} \mid w \in \operatorname{pos}(\xi)\right\}
$$

In fact, $\operatorname{sub}(\xi)=\{\xi\} \cup\left\{\xi^{\prime} \in \mathrm{T}_{\Sigma}(H) \mid \xi^{\prime} \prec_{\Sigma, H}^{+} \xi\right\}$.
Let $\Gamma \subseteq \Sigma^{(0)}$. We define the mapping yield ${ }_{\Gamma}: \mathrm{T}_{\Sigma} \rightarrow \Gamma^{*}$ by induction on $\mathrm{T}_{\Sigma}$ as follows.
I.B.: Let $\xi \in \Sigma^{(0)}$. If $\xi \in \Gamma$, then let $\operatorname{yield}_{\Gamma}(\xi)=\xi$, otherwise let yield ${ }_{\Gamma}(\xi)=\varepsilon$.
I.S.: Let $\xi=\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)$ with $k \in \mathbb{N}_{+}$. Then we define yield ${ }_{\Gamma}(\xi)=\operatorname{yield}_{\Gamma}\left(\xi_{1}\right) \cdots \operatorname{yield}_{\Gamma}\left(\xi_{k}\right)$.

We abbreviate yield ${ }_{\Sigma^{(0)}}$ by yield.

Comparison. Now we wish to compare two methods for defining mappings $h: \mathrm{T}_{\Sigma}(H) \rightarrow A$, where $H$ and $A$ are two sets.
(1) We define the mapping $h$ by induction on $\mathrm{T}_{\Sigma}(H)$, i.e., by using the well-founded relation $\prec_{\Sigma, H}$ and Theorem 2.5.1.
(2) As additional information, we know that $A$ has a $\Sigma$-algebra structure $(A, \theta)$. Then we define $h$ to be the unique extension of some mapping $f: H \rightarrow A$ to a $\Sigma$-algebra homomorphism, i.e., by using algebraic methods and, in particular, Theorem 2.9.3, this theorem guarantees that $h$ is a $\Sigma$-algebra homomorphism form $\mathrm{T}_{\Sigma}(H)$ to $(A, \theta)$.
Obviously, Method (1) is more general than Method (2) because the first can be used to define not only $\Sigma$-algebra homomorphisms. For instance, let $\Sigma=\left\{\sigma^{(2)}, \alpha^{(0)}, \beta^{(0)}\right\}, H=\emptyset$, and $A=\mathrm{T}_{\Sigma}$. Moreover, we define the mapping $G:\left\{(\xi, g) \mid \xi \in \mathrm{T}_{\Sigma}, g: \operatorname{pred}_{\prec_{\Sigma}}(\xi) \rightarrow \mathrm{T}_{\Sigma}\right\} \rightarrow \mathrm{T}_{\Sigma}$ such that for each $(\xi, g)$ we let

$$
G(\xi, g)= \begin{cases}\sigma\left(\alpha, g\left(\xi^{\prime}\right)\right) & \text { if there exists } \xi^{\prime} \in \mathrm{T}_{\Sigma} \text { such that } \xi=\sigma\left(\alpha, \xi^{\prime}\right) \\ \alpha & \text { otherwise }\end{cases}
$$

According to Theorem 2.5.1 there exists a unique mapping $h: \mathrm{T}_{\Sigma} \rightarrow \mathrm{T}_{\Sigma}$ such that $h(\xi)=$ $G\left(\xi,\left.h\right|_{\operatorname{pred}_{\prec_{\Sigma}}(\xi)}\right)$ for each $\xi \in \mathrm{T}_{\Sigma}$.

We show that $h$ is not a $\Sigma$-algebra homomorphism from the $\Sigma$-term algebra $\mathrm{T}_{\Sigma}=\left(\mathrm{T}_{\Sigma}, \theta_{\Sigma}\right)$ to itself. (We note that $G \neq G_{\theta_{\Sigma}, \emptyset}$.) We prove this claim by contradiction. For this, we assume that $h$ is a $\Sigma$-algebra homomorphism from $\mathrm{T}_{\Sigma}$ to itself. Then we can calculate as follows:

$$
\alpha=G\left(\sigma(\beta, \alpha),\left.h\right|_{\{\alpha, \beta\}}\right)
$$

$$
=h(\sigma(\beta, \alpha)) \quad(\text { by definition of } h)
$$

$$
=\theta_{\Sigma}(\sigma)(h(\beta), h(\alpha)) \quad\left(\text { because } h \text { is a } \Sigma \text {-algebra homomorphism from } \mathrm{T}_{\Sigma} \text { to } \mathrm{T}_{\Sigma}\right)
$$

$$
=\sigma(h(\beta), h(\alpha)) \quad \quad \quad \text { by definition of } \theta_{\Sigma}(\sigma)
$$

$$
=\sigma(\alpha, \alpha) . \quad(\text { by definition of } h \text { and by definition of } G)
$$

This is a contradiction. Hence $h$ is not a $\Sigma$-algebra homomorphism from $\mathrm{T}_{\Sigma}$ to $\mathrm{T}_{\Sigma}$.
Nevertheless, given that $\mathrm{A}=(A, \theta)$ is some $\Sigma$-algebra, Method (1) can also be used to define a $\Sigma$ algebra homomorphism from $\mathrm{T}_{\Sigma}(H)$ to A . However, in this case we have to prove explicitly that $h$ is a homomorphism. By means of an example, we compare the two methods in the case that $h$ is a $\Sigma$-algebra homomorphism.

By using Method (1): We define the mapping size ${ }_{1}$ by well-founded induction on $\left(\mathrm{T}_{\Sigma}(H), \prec_{\Sigma, H}\right)$ with the following induction base and induction step:
I.B.: For each $a \in \Sigma^{(0)} \cup H$, we define $\operatorname{size}_{1}(a)=1$.
I.S.: For every $k \in \mathbb{N}_{+}, \sigma \in \Sigma^{(k)}$, and $\xi_{1}, \ldots, \xi_{k} \in \mathrm{~T}_{\Sigma}(H)$, we define $\operatorname{size}_{1}\left(\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)\right)=1+$ $+_{i \in[k]} \operatorname{size}_{1}\left(\xi_{i}\right)$.
By Theorem 2.5.1, there exists exactly one mapping which satisfies these conditions.

Now we consider the $\Sigma$-algebra $(\mathbb{N}, \theta)$ with $\theta(\sigma)\left(n_{1}, \ldots, n_{k}\right)=1+\sum_{i \in[k]} n_{i}$ for every $k \in \mathbb{N}, \sigma \in \Sigma^{(k)}$, and $n_{1}, \ldots, n_{k} \in \mathbb{N}$. Then we prove that size ${ }_{1}$ is a $\Sigma$-algebra homomorphism from $\mathrm{T}_{\Sigma}(H)$ to $(\mathbb{N}, \theta)$. For every $k \in \mathbb{N}, \sigma \in \Sigma^{(k)}$, and $\xi_{1}, \ldots, \xi_{k} \in \mathrm{~T}_{\Sigma}(H)$, we have

$$
\operatorname{size}_{1}\left(\theta_{\Sigma}(\sigma)\left(\xi_{1}, \ldots, \xi_{k}\right)\right)=\operatorname{size}_{1}\left(\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)\right)=1+\underset{i \in[k]}{\nmid} \operatorname{size}_{1}\left(\xi_{i}\right)=\theta(\sigma)\left(\operatorname{size}_{1}\left(\xi_{1}\right), \ldots, \operatorname{size}_{1}\left(\xi_{k}\right)\right)
$$

Hence size $_{1}$ is a $\Sigma$-algebra homomorphism,
By using Method (2): We consider the $\Sigma$-algebra $(\mathbb{N}, \theta)$ as above and define the mapping $f: H \rightarrow \mathbb{N}$ by $f(a)=1$ for each $a \in H$. Then Theorem 2.9.3 guarantees that there exists a unique extension of $f$, say, size $_{2}$, into a $\Sigma$-algebra homomorphism from $\mathrm{T}_{\Sigma}(H)$ to $(\mathbb{N}, \theta)$.

Since size ${ }_{1}$ is a $\Sigma$-algebra homomorphism and $\operatorname{size}_{1}(a)=\operatorname{size}_{2}(a)$ for each $a \in H$, we know that $\operatorname{size}_{1}=\operatorname{size}_{2}$, due to the uniqueness property.

In comparison, for $\Sigma$-algebra homomorphisms, Method (1) seems to be slightly less efficient than Method (2), because in the first we have to prove explicitly that the mapping $h$ is a $\Sigma$-algebra homomorphism; while in Method (2) this is guaranteed by Theorem 2.9.3.

Representing trees by tree domains. At the beginning of this section, we have defined the set $\mathrm{T}_{\Sigma}$ of $\Sigma$-trees as the smallest set of strings which contains the nullary symbols of $\Sigma$ and which is closed under top-concatenations. Here we recall an alternative way, which is equivalent.

A tree domain is a finite and nonempty set $W \in\left(\mathbb{N}_{+}\right)^{*}$ such that

- $W$ is prefix-closed, i.e., prefix $(W) \subseteq W$, and
- $W$ is left-closed, i.e., for every $u \in\left(\mathbb{N}_{+}\right)^{*}$ and $i \in \mathbb{N}_{+}$, if $u i \in W$ and $i \geq 2$, then $u(i-1) \in W$.

Thus $\varepsilon$ is an element of each tree domain. Obviously, for each $\xi \in \mathrm{T}_{\Sigma}$, the set $\operatorname{pos}(\xi)$ is a tree domain.
A $\Sigma$-tree mapping is a mapping $t: W \rightarrow \Sigma$ such that

- $W$ is a tree domain and
- $t$ is rank preserving, i.e., for each $w \in W$, we have $\left|\left\{j \in \mathbb{N}_{+} \mid w j \in W\right\}\right|=\operatorname{rk}_{\Sigma}(t(w))$.

Let us denote the set of $\Sigma$-tree mappings by $\mathrm{TF}_{\Sigma}$.
We define the mapping $\varphi: \mathrm{T}_{\Sigma} \rightarrow \mathrm{TF}_{\Sigma}$ for each $\xi \in \mathrm{T}_{\Sigma}$ by $\varphi(\xi)=t$ such that $t: \operatorname{pos}(\xi) \rightarrow \Sigma$ and for each $w \in \operatorname{pos}(\xi)$ we let $t(w)=\xi(w)$. For instance, let $\xi$ be the tree in Figure 2.6. Then $\varphi(\xi)=t$ where $t: W \rightarrow \Sigma$ with tree domain $W=\{\varepsilon, 1,11,2,21,211,22\}$ and $t(\varepsilon)=t(2)=\sigma, t(1)=t(21)=\gamma$, $t(11)=\alpha$, and $t(211)=t(22)=\beta$.

We show that $\varphi$ is bijective. To show that $\varphi$ is injective, let $\xi_{1}, \xi_{2} \in \mathrm{~T}_{\Sigma}$ be such that $\xi_{1} \neq \xi_{2}$ and let $\varphi\left(\xi_{i}\right)=t_{i}$ for $i \in\{1,2\}$. Then there exists $w \in \operatorname{pos}\left(\xi_{1}\right) \cap \operatorname{pos}\left(\xi_{2}\right)$ such that $\xi_{1}(w) \neq \xi_{2}(w)$. By definition of $\varphi$, we have $t_{1}(w) \neq t_{2}(w)$, i.e. $t_{1} \neq t_{2}$. Hence $\varphi$ is injective.

For the proof of surjectivity of $\varphi$, we define the well-founded set (TD, $\prec)$ where TD is the set of all tree domains and $W^{\prime} \prec W$ if there exists $i \in \mathbb{N}_{+}$such that $W^{\prime}=\left\{w \in\left(\mathbb{N}_{+}\right)^{*} \mid i w \in W\right\}$. Then $\min _{\prec}(\mathrm{TD})=\{\{\varepsilon\}\}$. By induction on (TD,$\left.\prec\right)$, we prove that the following statement holds:

$$
\begin{equation*}
\text { For every } W \in \mathrm{TD} \text { and } \Sigma \text {-tree mapping } t: W \rightarrow \Sigma, \tag{2.25}
\end{equation*}
$$

there exists a $\xi \in \mathrm{T}_{\Sigma}$ such that $W=\operatorname{pos}(\xi)$ and $\varphi(\xi)=t$.
For the proof, let $W \in \mathrm{TD}$ and $t: W \rightarrow \Sigma$ be a $\Sigma$-tree mapping. Then there exist $k \in \mathbb{N}$ and $\sigma \in \Sigma$ such that $t(\varepsilon)=\sigma$. Moreover, for each $i \in[k]$, we define $W_{i}=\left\{w \in\left(\mathbb{N}_{+}\right)^{*} \mid i w \in W\right\}$ and the $\Sigma$-tree mapping $t_{i}: W_{i} \rightarrow \Sigma$ by $t_{i}(w)=t(i w)$ for each $w \in W_{i}$. Then $W=\{\varepsilon\} \cup\left\{i w \mid i \in[k], w \in W_{i}\right\}$.

Since $\left\{W^{\prime} \in \mathrm{TD} \mid W^{\prime} \prec W\right\}=\left\{W_{1}, \ldots, W_{k}\right\}$, the I.H. implies that, for each $i \in[k]$, there exists $\xi_{i} \in \mathrm{~T}_{\Sigma}$ such that $W_{i}=\operatorname{pos}\left(\xi_{i}\right)$ and $\varphi\left(\xi_{i}\right)=t_{i}$.

Then let $\xi=\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)$. We claim that $\varphi(\xi)=t$. First, by definition we have that $\varphi(\xi): \operatorname{pos}(\xi) \rightarrow \Sigma$. Moreover,

$$
\operatorname{pos}(\xi)=\{\varepsilon\} \cup\left\{i w \mid i \in[k], w \in \operatorname{pos}\left(\xi_{i}\right)\right\}=\{\varepsilon\} \cup\left\{i w \mid i \in[k], w \in W_{i}\right\}=W
$$

where the second equality follows from I.H. Second, for the proof of $\varphi(\xi)=t$ we let $w \in W$. If $w=\varepsilon$, then $\xi(w)=\sigma=t(w)$. If $w=i w^{\prime}$ for some $i \in \mathbb{N}_{+}$and $w^{\prime} \in W_{i}$, then

$$
\xi\left(i w^{\prime}\right)=\xi_{i}\left(w^{\prime}\right)=t_{i}\left(w^{\prime}\right)=t\left(i w^{\prime}\right)
$$

where the second equality follows from I.H. Hence $\varphi(\xi)=t$. This proves (2.25), and hence $\varphi$ is surjective.
Since $\varphi$ is bijective we can specify a tree $\xi \in \mathrm{T}_{\Sigma}$ also by means of a $\Sigma$-tree mapping (cf., e.g., the proofs of Lemma 8.3.3, of Theorem 10.6.1 and of Theorem 11.3.1).

Orders on positions of trees. The prefix order on $\operatorname{pos}(\xi)$, denoted by $\leq_{\text {pref }}$, is the partial order defined for every $w, v \in \operatorname{pos}(\xi)$ by

$$
w \leq_{\text {pref }} v \text { iff } w \in \operatorname{prefix}(v)
$$

We let $w<_{\text {pref }} v$ if $\left(w \leq_{\text {pref }} v\right) \wedge(w \neq v)$.
The lexicographic order on $\operatorname{pos}(\xi)$, denoted by $\leq_{l e x}$, is the linear order defined for every $w, v \in \operatorname{pos}(\xi)$ by

$$
\begin{aligned}
w \leq_{\text {lex }} v \operatorname{iff}(w \in \operatorname{prefix}(v)) \vee( & (\exists u \in \operatorname{prefix}(w) \cap \operatorname{prefix}(v))\left(\exists i, j \in \mathbb{N}_{+}\right): \\
& (u i \in \operatorname{prefix}(w)) \wedge(u j \in \operatorname{prefix}(v)) \wedge(i<j)
\end{aligned}
$$

The lexicographic order is also called depth-first pre-order. We let $w<_{\operatorname{lex}} v$ if $\left(w \leq_{\operatorname{lex}} v\right) \wedge(w \neq v)$.
The depth-first post-order on $\operatorname{pos}(\xi)$, denoted by $\leq_{d p}$, is the linear order defined for every $w, v \in \operatorname{pos}(\xi)$ by

$$
\begin{aligned}
& w \leq_{\mathrm{dp}} v \text { iff }(v \in \operatorname{prefix}(w)) \vee(\exists u \in \operatorname{prefix}(w) \cap \operatorname{prefix}(v))\left(\exists i, j \in \mathbb{N}_{+}\right): \\
&(u i \in \operatorname{prefix}(w)) \wedge(u j \in \operatorname{prefix}(v)) \wedge(i<j) .
\end{aligned}
$$

We let $w<_{\mathrm{dp}} v$ if $\left(w \leq_{\mathrm{dp}} v\right) \wedge(w \neq v)$.

Tree relabelings. A $(\Sigma, \Delta)$-tree relabeling (or simply: tree relabeling) [Eng75b, Def. 3.1] is an $\mathbb{N}$ indexed family $\tau=\left(\tau_{k} \mid k \in \mathbb{N}\right)$ of mappings $\tau_{k}: \Sigma^{(k)} \rightarrow \mathcal{P}\left(\Delta^{(k)}\right)$. We call a tree relabeling $\tau$

- non-overlapping if $\tau_{k}(\sigma) \cap \tau_{k}\left(\sigma^{\prime}\right)=\emptyset$ for every $k \in \mathbb{N}$ and $\sigma, \sigma^{\prime} \in \Sigma^{(k)}$ with $\sigma \neq \sigma^{\prime}$.
- deterministic if for every $k \in \mathbb{N}$ and $\sigma \in \Sigma^{(k)}$, the set $\tau_{k}(\sigma)$ contains exactly one element. Then we specify the tree relabeling as an $\mathbb{N}$-indexed family of mappings $\tau_{k}: \Sigma^{(k)} \rightarrow \Delta^{(k)}$.
Now let $\tau=\left(\tau_{k} \mid k \in \mathbb{N}\right)$ be a $(\Sigma, \Delta)$-tree relabeling. We extend $\tau$ to the mapping

$$
\tau^{\prime}: \mathrm{T}_{\Sigma} \rightarrow \mathcal{P}_{\mathrm{fin}}\left(\mathrm{~T}_{\Delta}\right)
$$

by using Theorem 2.9 .3 as follows. We define the $\Sigma$-algebra $\left(\mathcal{P}_{\text {fin }}\left(\mathrm{T}_{\Delta}\right), \theta_{\tau}\right)$ such that, for every $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $U_{1}, \ldots, U_{k} \in \mathcal{P}_{\text {fin }}\left(\mathrm{T}_{\Delta}\right)$, we let

$$
\theta_{\tau}(\sigma)\left(U_{1}, \ldots, U_{k}\right)=\left\{\gamma\left(\zeta_{1}, \ldots, \zeta_{k}\right) \mid \gamma \in \tau_{k}(\sigma), \zeta_{1} \in U_{1}, \ldots, \zeta_{k} \in U_{k}\right\}
$$

Then $\tau^{\prime}: \mathrm{T}_{\Sigma} \rightarrow \mathcal{P}_{\text {fin }}\left(\mathrm{T}_{\Delta}\right)$ is the unique $\Sigma$-homomorphism $h$ from the $\Sigma$-term algebra $\mathrm{T}_{\Sigma}$ to the $\Sigma$-algebra $\left(\mathcal{P}_{\text {fin }}\left(\mathrm{T}_{\Delta}\right), \theta_{\tau}\right)$ (cf. Theorem 2.9.3). We also call the mapping $\tau^{\prime}: \mathrm{T}_{\Sigma} \rightarrow \mathcal{P}_{\text {fin }}\left(\mathrm{T}_{\Delta}\right)$ a $(\Sigma, \Delta)$-tree relabeling (or simply: tree relabeling) and we abbreviate $\tau^{\prime}$ by $\tau$. The mapping $\tau$ can also be considered as a binary relation $\tau \subseteq \mathrm{T}_{\Sigma} \times \mathrm{T}_{\Delta}$, and thus, in particular, $\tau^{-1}$ is defined.

We note that $\operatorname{pos}(\zeta)=\operatorname{pos}(\xi)$ for each $\zeta \in \tau(\xi)$. Thus, for every $\zeta \in \mathrm{T}_{\Delta}$, the set $\tau^{-1}(\zeta)$ is finite and, if $\tau$ is non-overlapping, then $\left|\tau^{-1}(\zeta)\right| \leq 1$. If $\tau$ is deterministic, then for every $\xi \in \mathrm{T}_{\Sigma}$ we have $|\tau(\xi)|=1$.

Let $\Omega$ be a ranked alphabet, $\omega=\left(\omega_{k} \mid k \in \mathbb{N}\right)$ be a $(\Sigma, \Omega)$-tree relabeling, and $\tau=\left(\tau_{k} \mid k \in \mathbb{N}\right)$ be a $(\Omega, \Delta)$-tree relabeling. The syntactic composition of $\omega$ and $\tau$, denoted by $\tau \hat{o} \omega$, is the $(\Sigma, \Delta)$-tree relabeling $\left((\tau \hat{o} \omega)_{k} \mid k \in \mathbb{N}\right)$ such that $(\tau \hat{o} \omega)_{k}=\tau_{k} \circ \omega_{k}$, i.e., $(\tau \hat{o} \omega)_{k}(\sigma)=\bigcup_{\delta \in \omega_{k}(\sigma)} \tau_{k}(\delta)$ for every $k \in \mathbb{N}$ and $\sigma \in \Sigma^{(k)}$. It follows that if $\omega$ and $\tau$ are deterministic, then $\tau \hat{o} \omega$ is also deterministic.

Theorem 2.9.6. (cf. Eng75a, Lm. 3.4]) Let $\Omega$ be a ranked alphabet, $\omega$ be a $(\Sigma, \Omega)$-tree relabeling, and $\tau$ be a $(\Omega, \Delta)$-tree relabeling. Then $\tau \hat{o} \omega=\tau \circ \omega$, i.e., the syntactic composition of $\omega$ and $\tau$ determines the mapping $\tau \circ \omega$.

Contexts and substitution. Next we define contexts. Let $Z=\left\{z_{1}, z_{2}, \ldots\right\}$ be a set of variables, disjoint with $\Sigma$, and let $Z_{n}=\left\{z_{1}, \ldots, z_{n}\right\}$ for every $n \in \mathbb{N}$. Sometimes we write $z$ for $z_{1}$. Moreover, let $\xi \in \mathrm{T}_{\Sigma}(Z)$ and $V \subseteq Z$. We say that $\xi$ is linear in $V(\xi$ is nondeleting in $V$ ) if for each $z \in V$ we have $\left|\operatorname{pos}_{z}(\xi)\right| \leq 1$ (and $\left|\operatorname{pos}_{z}(\xi)\right| \geq 1$, respectively). We denote by $\mathrm{C}_{\Sigma}\left(Z_{n}\right)$ the set of all trees $\xi \in \mathrm{T}_{\Sigma}\left(Z_{n}\right)$ which are both linear and nondeleting in $Z_{n}$. Since $Z_{0}=\emptyset$, we have $\mathrm{C}_{\Sigma}\left(Z_{0}\right)=\mathrm{T}_{\Sigma}$. We call the elements of $\mathrm{C}_{\Sigma}\left(Z_{n}\right)$ contexts over $\Sigma$ and $Z_{n}$ (or: $n$-contexts if $\Sigma$ is clear). We abbreviate $\mathrm{C}_{\Sigma}(\{z\})$ by $\mathrm{C}_{\Sigma}$ and call its elements contexts.

Now we define tree substitution (cf. derived operator in GTWW77, p. 73]). Let $H$ be a set disjoint with $\Sigma$; moreover, let $n \in \mathbb{N}, \xi \in \mathrm{~T}_{\Sigma}\left(Z_{n}\right)$, and $\xi_{1}, \ldots, \xi_{n} \in \mathrm{~T}_{\Sigma}(H)$. Intuitively, we want to define the tree in $\mathrm{T}_{\Sigma}(H)$ which is obtained from $\xi$ by replacing, for each $z_{i} \in Z_{n}$, each occurrence of $z_{i}$ by $\xi_{i}$. Formally, let $v: Z_{n} \rightarrow \mathrm{~T}_{\Sigma}(H)$ be the mapping defined by $v\left(z_{i}\right)=\xi_{i}$ for each $i \in[n]$. Then we denote by $\bar{v}$ the unique extension of $v$ to a $\Sigma$-algebra homomorphism from the $\Sigma$-term algebra ( $\left.\mathrm{T}_{\Sigma}\left(Z_{n}\right), \theta_{\Sigma}\right)$ over $Z_{n}$ to the $\Sigma$-term algebra $\left(\mathrm{T}_{\Sigma}(H), \theta_{\Sigma}\right)$ over $H$. It is easy to see that $\bar{v}(\xi)$ is the desired tree from our informal discussion. In the sequel, we abbreviate $\bar{v}(\xi)$ by $\xi\left[\xi_{1}, \ldots, \xi_{n}\right]$. For $n=1$ and $H=\{z\}$, we also denote $\xi\left[\xi_{1}\right]$ by $\xi \circ_{z} \xi_{1}$. Hence, $\circ_{z}$ can be thought of as a binary operation on $\mathrm{T}_{\Sigma}(\{z\})$.

In the rest of the book, $Z$ and $X$ will denote sets of variables if not specified otherwise. Moreover, we let $Z_{n}=\left\{z_{1}, \ldots, z_{n}\right\}$ and $X_{n}=\left\{x_{1}, \ldots, x_{n}\right\}$ for every $n \in \mathbb{N}$.

Tree homomorphisms. A $(\Sigma, \Delta)$-tree homomorphism (or just tree homomorphism) Eng75b, Def. 3.62] is a family $h=\left(h_{k} \mid k \in \mathbb{N}\right)$ of mappings $h_{k}: \Sigma^{(k)} \rightarrow \mathrm{T}_{\Delta}\left(Z_{k}\right)$. We call a tree homomorphism $h$

- linear if, for every $k \in \mathbb{N}$ and $\sigma \in \Sigma^{(k)}$, the tree $h_{k}(\sigma)$ is linear in $Z_{k}$;
- nondeleting if, for every $k \in \mathbb{N}$ and $\sigma \in \Sigma^{(k)}$, the tree $h_{k}(\sigma)$ is nondeleting in $Z_{k}$;
- alphabetic if, for every $k \in \mathbb{N}$ and $\sigma \in \Sigma^{(k)}$, we have $\left|\operatorname{pos}_{\Delta}\left(h_{k}(\sigma)\right)\right| \leq 1$;
- productive if, for every $k \in \mathbb{N}$ and $\sigma \in \Sigma^{(k)}$, we have $h_{k}(\sigma) \notin Z_{k}$ (this notion is borrowed from [FMV10] where it was called output-productive);
- ordered if $h$ is linear and nondeleting and, for every $k \in \mathbb{N}$ and $\sigma \in \Sigma^{(k)}$, the sequence of variables occurring in $h_{k}(\sigma)$ is ordered by increasing indices from left to right, i.e., it is $z_{1}, \ldots, z_{k}$;
- simple if $h$ is alphabetic and ordered (and hence also linear and nondeleting).

Now let $H$ be a set disjoint with $\Sigma$ and $\Delta$. Moreover, let $h=\left(h_{k} \mid k \in \mathbb{N}\right)$ be a $(\Sigma, \Delta)$-tree homomorphism. We extend $h$ to the mapping

$$
h: \mathrm{T}_{\Sigma}(H) \rightarrow \mathrm{T}_{\Delta}(H)
$$

by using Theorem 2.9 .3 as follows. We define the $\Sigma$-algebra $\left(\mathrm{T}_{\Delta}(H), \theta_{h}\right)$ such that, for every $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $\xi_{1}, \ldots, \xi_{k} \in \mathrm{~T}_{\Delta}(H)$, we let

$$
\theta_{h}(\sigma)\left(\xi_{1}, \ldots, \xi_{k}\right)=h_{k}(\sigma)\left[\xi_{1}, \ldots, \xi_{k}\right]
$$

Then $h: \mathrm{T}_{\Sigma}(H) \rightarrow \mathrm{T}_{\Delta}(H)$ is the unique extension of $\mathrm{id}_{H}$ to a $\Sigma$-algebra homomorphism from the $\Sigma$-term algebra $\left(\mathrm{T}_{\Sigma}(H), \theta_{\Sigma}\right)$ over $H$ to the $\Sigma$-algebra $\left(\mathrm{T}_{\Delta}(H), \theta_{h}\right)$ (cf. Theorem 2.9.3).

We also call the mapping $h: \mathrm{T}_{\Sigma}(H) \rightarrow \mathrm{T}_{\Delta}(H)$ a $(\Sigma, \Delta)$-tree homomorphism (or simply: tree homomorphism).

We note that each deterministic tree relabeling can be thought of as a particular tree homomorphism. Indeed, let $\left(\tau_{k} \mid k \in \mathbb{N}\right)$ be a deterministic $(\Sigma, \Delta)$-tree relabeling. Then $\tau$ determines the $(\Sigma, \Delta)$-tree
homomorphism $\left(\tau_{k}^{\prime} \mid k \in \mathbb{N}\right)$ where, for every $k \in \mathbb{N}$ and $\sigma \in \Sigma^{(k)}$, we have $\tau_{k}^{\prime}(\sigma)=\tau_{k}(\sigma)\left(z_{1}, \ldots, z_{k}\right)$. It is obvious that $\tau=\tau^{\prime}$.

Let $\Omega$ be a ranked alphabet, $g=\left(g_{k} \mid k \in \mathbb{N}\right)$ be a $(\Sigma, \Omega)$-tree homomorphism, and $h=\left(h_{k} \mid k \in \mathbb{N}\right)$ be an $(\Omega, \Delta)$-tree homomorphism. The syntactic composition of $g$ and $h$, denoted by $h \hat{o} g$, is the $(\Sigma, \Delta)$-tree homomorphism $\left((h \hat{o} g)_{k} \mid k \in \mathbb{N}\right)$ such that $(h \hat{o} g)_{k}(\sigma)=h\left(g_{k}(\sigma)\right)$ for every $k \in \mathbb{N}$ and $\sigma \in \Sigma^{(k)}$.

Theorem 2.9.7. (cf. Eng75a, Lm. 3.4], GS84, Thm. 4.3.7]) Let $\Omega$ be a ranked alphabet, $g$ be a $(\Sigma, \Omega)$-tree homomorphism, and $h$ be an $(\Omega, \Delta)$-tree homomorphism. Then $h \hat{o} g=h \circ g$, i.e., the syntactic composition of $g$ and $h$ determines the mapping $h \circ g$.

### 2.10 Weighted tree languages and transformations

### 2.10.1 Weighted sets and weighted languages

Let $\mathrm{B}=(B, \oplus, \otimes, \mathbb{0}, \mathbb{1})$ be a strong bimonoid, $A$ be a set, and $f: A \rightarrow B$ be a mapping. We will define several notions for and characteristics of $f$. We call $f$ a B -weighted set. We note that if $\mathrm{B}_{1}$ and $\mathrm{B}_{2}$ are different strong bimonoids with the same carrier set $B$, then a mapping $f: A \rightarrow B$ is both a $\mathrm{B}_{1}$-weighted set and a $B_{2}$-weighted set.

Let $f: A \rightarrow B$ be a B -weighted set. The support of $f$, denoted by $\operatorname{supp}_{\mathrm{B}}(f)$, is defined by $\operatorname{supp}_{\mathrm{B}}(f)=$ $\{a \in A \mid f(a) \neq \mathbb{O}\}$. If B is clear from the context, then we abbreviate $\operatorname{supp}_{\mathrm{B}}(f)$ by $\operatorname{supp}(f)$.

We call $f$

- polynomial if $\operatorname{supp}(f)$ is finite.
- monomial if $\operatorname{supp}(f) \subseteq\{a\}$ for some $a \in A$; then we denote $f$ also by $f(a) . a$.
- constant if there exists a $b \in B$ such that for every $a \in A$, we have $f(a)=b$; then we denote $f$ by $\widetilde{b}$. We note that $\widetilde{\mathbb{O}}$ is a monomial and $\widetilde{\mathbb{O}}=\mathbb{O}$. $a$ for each $a \in A$.

Let $L \subseteq A$. The characteristic mapping of $L$ with respect to B , denoted by $\chi_{\mathrm{B}}(L)$, is the B -weighted set $\chi_{\mathrm{B}}(L): A \rightarrow B$ defined, for each $a \in A$, by $\chi_{\mathrm{B}}(L)(a)=\mathbb{1}$ if $a \in L$, and $\chi_{\mathrm{B}}(L)(a)=\mathbb{0}$ otherwise. In particular, $\chi_{\mathrm{B}}(\emptyset)=\widetilde{\mathbb{0}}$ and $\chi_{\mathrm{B}}(\{a\})=\mathbb{1} . a$. Certainly, $\operatorname{supp}_{\mathrm{B}}\left(\chi_{\mathrm{B}}(L)\right)=L$ and, if $|B|>2$, then there exists a $B$-weighted set $f: A \rightarrow B$ such that $\chi_{\mathrm{B}}\left(\operatorname{supp}_{\mathrm{B}}(f)\right) \neq f$. However, for each Boole-weighted set $f: A \rightarrow \mathbb{B}$, we have $\chi_{\text {Boole }}\left(\operatorname{supp}_{\text {Boole }}(f)\right)=f$. If B is clear from the context, then we abbreviate $\chi_{\mathrm{B}}$ by $\chi_{\text {. }}$.

A weighted language over $\Gamma$ and B (for short: $(\Gamma, \mathrm{B})$-weighted language) is a B-weighted set $r: \Gamma^{*} \rightarrow B$.

### 2.10.2 Weighted tree languages and operations

Let $H$ be a set with $\Sigma \cap H=\emptyset$. A weighted tree language over $\Sigma, H$, and B is a weighted set $r: \mathrm{T}_{\Sigma}(H) \rightarrow B$. If $H=\emptyset$, then we just say weighted tree language over $\Sigma$ and B or $(\Sigma, \mathrm{B})$-weighted tree language. A Bweighted tree language is a $(\Sigma, \mathrm{B})$-weighted tree language for some ranked alphabet $\Sigma$. A weighted $\Sigma$-tree language is a $(\Sigma, \mathrm{B})$-weighted tree language for some strong bimonoid B . Finally, a weighted tree language is a $(\Sigma, \mathrm{B})$-weighted tree language for some ranked alphabet $\Sigma$ and some strong bimonoid B .

A $(\Sigma, \mathrm{B})$-weighted tree language $r$ has the preimage property if, for each $b \in B$, the $\Sigma$-tree language $r^{-1}(b)$ is recognizable (in the sense of Section 2.13).

We denote the set of polynomial $(\Sigma, \mathrm{B})$-weighted tree languages by $\operatorname{Pol}(\Sigma, \mathrm{B})$. We call each element of $\operatorname{Pol}(\Sigma, \mathrm{B})$ a ( $\Sigma, \mathrm{B})$-polynomial.

We extend the concept of support to sets of weighted tree languages in the natural way: for each set $\mathcal{C}$ of weighted tree languages, we define $\operatorname{supp}(\mathcal{C})=\{\operatorname{supp}(r) \mid r \in \mathcal{C}\}$.

Let $r: \mathrm{T}_{\Sigma} \rightarrow B$ and $b \in B$. The scalar multiplication of $r$ with $b$ from the left (with respect to B), denoted by $b \otimes r$, is the weighted tree language $(b \otimes r): \mathrm{T}_{\Sigma} \rightarrow B$ defined for each $\xi \in \mathrm{T}_{\Sigma}$ by $(b \otimes r)(\xi)=b \otimes r(\xi)$. In a similar way, we define the scalar multiplication of $r$ with $b$ from the right (with
respect to B ) and denote it by $r \otimes b$.
Let $r_{1}: \mathrm{T}_{\Sigma} \rightarrow B$ and $r_{2}: \mathrm{T}_{\Sigma} \rightarrow B$. The sum of $r_{1}$ and $r_{2}$ (with respect to B ), denoted by $\left(r_{1} \oplus r_{2}\right)$, is the weighted tree language $\left(r_{1} \oplus r_{2}\right): \mathrm{T}_{\Sigma} \rightarrow B$ defined for each $\xi \in \mathrm{T}_{\Sigma}$ by $\left(r_{1} \oplus r_{2}\right)(\xi)=r_{1}(\xi) \oplus r_{2}(\xi)$. The algebra $\left(B^{\mathrm{T}_{\Sigma}}, \oplus, \widetilde{\mathbb{0}}\right)$ is a commutative monoid. If B is $\sigma$-complete, then also this monoid is $\sigma$-complete.

Moreover, the Hadamard product of $r_{1}$ and $r_{2}$ (with respect to B ), denoted by $\left(r_{1} \otimes r_{2}\right.$ ), is the weighted tree language $\left(r_{1} \otimes r_{2}\right): \mathrm{T}_{\Sigma} \rightarrow B$ defined for each $\xi \in \mathrm{T}_{\Sigma}$ by $\left(r_{1} \otimes r_{2}\right)(\xi)=r_{1}(\xi) \otimes r_{2}(\xi)$. In general, the Hadamard product is not commutative. However, for each $L \subseteq \mathrm{~T}_{\Sigma}$, the equality $r_{1} \otimes \chi(L)=\chi(L) \otimes r_{1}$ holds.

The algebra $\left(B^{\mathrm{T}_{\Sigma}}, \otimes, \widetilde{\mathbb{1}}\right)$ is a monoid. Since $\widetilde{\mathbb{D}} \otimes r=r \otimes \widetilde{\mathbb{D}}=\widetilde{\mathbb{D}}$ for each $r \underset{\sim}{ } \in B^{\mathrm{T}_{\Sigma}}$, the algebra $\left(B^{\mathrm{T}_{\Sigma}}, \oplus, \otimes, \widetilde{\mathbb{0}}, \widetilde{1}\right)$ is a strong bimonoid. If B is a semiring, then so is $\left(B^{\mathrm{T}_{\Sigma}}, \oplus, \otimes, \widetilde{\mathbb{0}}, \widetilde{\mathbb{1}}\right)$. In particular, the semiring $\left(\mathbb{B}^{T_{\Sigma}}, \oplus, \otimes, \widetilde{\mathbb{0}}, \widetilde{\mathbb{1}}\right)$ is isomorphic to the semiring $\left(\mathcal{P}\left(\mathrm{T}_{\Sigma}\right), \cup, \cap, \emptyset, \mathrm{T}_{\Sigma}\right)$.

Let $r: \mathrm{T}_{\Sigma} \rightarrow B$ and $L \subseteq \mathrm{~T}_{\Sigma}$ be finite. Then we define $r(L) \in B$ by $r(L)=\bigoplus_{\xi \in L} r(\xi)$.
Evaluation algebras. We define particular $(\Sigma, \mathrm{B})$-weighted tree languages; they are the unique $\Sigma$ algebra homomorphisms from $T_{\Sigma}$ to $\Sigma$-algebras which are based on the evaluation of $\Sigma$-symbols in B.

Formally, let $\kappa=\left(\kappa_{k} \mid k \in \mathbb{N}\right)$ be an $\mathbb{N}$-indexed family of mappings $\kappa_{k}: \Sigma^{(k)} \rightarrow B$. The $(\Sigma, \kappa)$ evaluation algebra, denoted by $\mathrm{M}(\Sigma, \kappa)$, is the $\Sigma$-algebra $(B, \bar{\kappa})$, where

$$
\bar{\kappa}(\sigma)\left(b_{1}, \ldots, b_{k}\right)=b_{1} \otimes \cdots \otimes b_{k} \otimes \kappa_{k}(\sigma)
$$

for every $k \in \mathbb{N}, \sigma \in \Sigma^{(k)}$, and $b_{1}, \ldots, b_{k} \in B$. We recall that $\mathrm{h}_{\mathrm{M}(\Sigma, \kappa)}$ denotes the unique $\Sigma$-algebra homomorphism from $\mathrm{T}_{\Sigma}$ to $\mathrm{M}(\Sigma, \kappa)$. Then, for each $\xi=\sigma\left(\xi, \ldots, \xi_{k}\right)$ in $\mathrm{T}_{\Sigma}$, we have

$$
\begin{equation*}
\mathrm{h}_{\mathrm{M}(\Sigma, \kappa)}(\xi)=\bar{\kappa}(\sigma)\left(\mathrm{h}_{\mathrm{M}(\Sigma, \kappa)}\left(\xi_{1}\right), \ldots, \mathrm{h}_{\mathrm{M}(\Sigma, \kappa)}\left(\xi_{k}\right)\right)=\left(\bigotimes_{i \in[k]} \mathrm{h}_{\mathrm{M}(\Sigma, \kappa)}\left(\xi_{i}\right)\right) \otimes \kappa_{k}(\sigma) . \tag{2.26}
\end{equation*}
$$

Obviously, we have

$$
\begin{equation*}
\mathrm{h}_{\mathrm{M}(\Sigma, \kappa)}(\xi)=\bigotimes_{\substack{w \in \operatorname{pos}(\xi) \\ \text { in } \leq \operatorname{dp} \text { order }}} \kappa_{\mathrm{rk}(\xi(w))}(\xi(w)) \tag{2.27}
\end{equation*}
$$

In particular, $\mathrm{h}_{\mathrm{M}(\Sigma, \kappa)}: \mathrm{T}_{\Sigma} \rightarrow B$ is a $(\Sigma, \mathrm{B})$-weighted tree language.

### 2.10.3 Weighted tree transformations

Let $A$ be a set and $\tau: \mathrm{T}_{\Sigma} \times A \rightarrow B$ a B -weighted set. We say that

- $\tau$ is supp-i-finite if, for every $a \in A$, the set $\left\{\xi \in \mathrm{T}_{\Sigma} \mid \tau(\xi, a) \neq \mathbb{O}\right\}$ is finite;
- $\tau$ is supp-o-finite if, for every $\xi \in \mathrm{T}_{\Sigma}$, the set $\{a \in A \mid \tau(\xi, a) \neq \mathbb{O}\}$ is finite.

Here supp-i-finite and supp-o-finite abbreviate "support input finite" and "support output finite", respectively. If $A=\mathrm{T}_{\Delta}$, then we call $\tau$ a $(\Sigma, \Delta, \mathrm{B})$-weighted tree transformation or simply weighted tree transformation.

Let $r: \mathrm{T}_{\Sigma} \rightarrow B$ be a weighted tree language and $\tau: \mathrm{T}_{\Sigma} \times A \rightarrow B$ be a B -weighted set. We say that $r$ is $\tau$-summable if $\tau$ is supp-i-finite or $r$ has finite support. If $r$ is $\tau$-summable or B is $\sigma$-complete, then the application of $\tau$ to $r$, denoted by $\tau(r)$, is the B-weighted set $\tau(r): A \rightarrow B$ defined, for each $a \in A$, by

$$
\begin{equation*}
\tau(r)(a)=\sum_{\xi \in \mathrm{T}_{\Sigma}}^{\oplus} r(\xi) \otimes \tau(\xi, a) . \tag{2.28}
\end{equation*}
$$

Obviously, if $r$ is $\tau$-summable, then

$$
\begin{equation*}
\sum_{\xi \in \mathrm{T}_{\Sigma}}^{\oplus} r(\xi) \otimes \tau(\xi, a)=\bigoplus_{\xi \in \mathrm{T}_{\Sigma}} r(\xi) \otimes \tau(\xi, a) \tag{2.29}
\end{equation*}
$$

We will use this application in three different scenarios.

Scenario 1. In this scenario, $\tau$ is the characteristic mapping of a binary relation. Formally, let $r$ : $\mathrm{T}_{\Sigma} \rightarrow B$ and $g \subseteq \mathrm{~T}_{\Sigma} \times A$. If $r$ is $\chi(g)$-summable or B is $\sigma$-complete, then for each $a \in A$ we have

$$
\begin{equation*}
\chi(g)(r)(a)=\sum_{\xi \in \mathrm{T}_{\Sigma}}^{\oplus} r(\xi) \otimes \chi(g)(\xi, a)=\sum_{\xi \in g^{-1}(a)}^{\oplus} r(\xi) \tag{2.30}
\end{equation*}
$$

We will use the following five cases of this scenario. In Case (a) we have $A=\Gamma^{*}$, and in Cases (b)-(e) we have $A=\mathrm{T}_{\Delta}$. Moreover, $\tau$ is the characteristic mapping of
(a) a yield mapping $g: \mathrm{T}_{\Sigma} \rightarrow \Gamma^{*}$, where $\Gamma \subseteq \Sigma^{(0)}$ is an alphabet (Section 8.3),
(b) a tree relabeling $g \subseteq \mathrm{~T}_{\Sigma} \times \mathrm{T}_{\Delta}$ (Section 10.10),
(c) a tree homomorphism $g: \mathrm{T}_{\Sigma} \rightarrow \mathrm{T}_{\Delta}$ (Section 10.11),
(d) the inverse of a tree homomorphism $g \subseteq \mathrm{~T}_{\Delta} \times \mathrm{T}_{\Sigma}$ (Section 10.12), and
(e) a tree relabeling $g \subseteq \mathrm{~T}_{\Sigma} \times \mathrm{T}_{\Delta}$ (Subsection 14.5.1).

As a consequence of (2.30), we obtain the following inclusion and equality:

$$
\begin{equation*}
\operatorname{supp}(\chi(g)(r)) \subseteq g(\operatorname{supp}(r)) \quad \text { and } \tag{2.31}
\end{equation*}
$$

if B is zero-sum free, then $\operatorname{supp}(\chi(g)(r))=g(\operatorname{supp}(r))$.
First we prove the inclusion. For this, let $a \in A$. Then we have

$$
\begin{align*}
a \in \operatorname{supp}(\chi(g)(r)) & \Longleftrightarrow\left(\sum_{\xi \in g^{-1}(a)}^{\oplus} r(\xi)\right) \neq 0  \tag{2.30}\\
& \Longleftrightarrow\left(\exists \xi \in g^{-1}(a)\right): r(\xi) \neq 0 \quad \text { (by (2.30)) } \\
& \Longleftrightarrow a \in g(\operatorname{supp}(r))
\end{align*}
$$

If $B$ is zero-sum free, then the above implication can be turned into an equivalence by Observation 2.6.11(8) or (9), respectively.

The next observation will be useful later.
Observation 2.10.1. Let $g \subseteq \mathrm{~T}_{\Sigma} \times \mathrm{T}_{\Delta}$ be a $(\Sigma, \Delta)$-tree relabeling, $r: \mathrm{T}_{\Sigma} \rightarrow B, L \subseteq \mathrm{~T}_{\Sigma}$, and $\xi \in \mathrm{T}_{\Delta}$. Then $\chi(g)(\chi(L) \otimes r)(\xi)=r\left(g^{-1}(\xi) \cap L\right)$.

Proof.

$$
\begin{aligned}
r\left(g^{-1}(\xi) \cap L\right) & =\bigoplus_{\zeta \in g^{-1}(\xi) \cap L} r(\zeta) \quad \text { (we note that } g^{-1}(\xi) \cap L \text { is finite) } \\
& =\bigoplus_{\zeta \in g^{-1}(\xi)} \chi(L)(\zeta) \otimes r(\zeta)=\bigoplus_{\zeta \in g^{-1}(\xi)}(\chi(L) \otimes r)(\zeta) \\
& =\chi(g)(\chi(L) \otimes r)(\xi)
\end{aligned}
$$

Scenario 2. In this scenario, $A=\mathrm{T}_{\Sigma}$ and $\tau$ is the diagonalization of a $(\Sigma, \mathrm{B})$-tree language $r^{\prime}: \mathrm{T}_{\Sigma} \rightarrow B$. We define the diagonalization of $r^{\prime}$ [FMV11, Sect. 2.6] to be the $(\Sigma, \Sigma, \mathrm{B})$-weighted tree transformation $\overline{r^{\prime}}: \mathrm{T}_{\Sigma} \times \mathrm{T}_{\Sigma} \rightarrow B$ such that, for every $\xi, \zeta \in \mathrm{T}_{\Sigma}$, we let

$$
\overline{r^{\prime}}(\xi, \zeta)= \begin{cases}r^{\prime}(\xi) & \text { if } \xi=\zeta  \tag{2.32}\\ \mathbb{O} & \text { otherwise }\end{cases}
$$

Obviously, $\overline{r^{\prime}}$ is supp-i-finite and hence, for each $r: \mathrm{T}_{\Sigma} \rightarrow B$, the application $\overline{r^{\prime}}(r)$ is defined. Then the application of the diagonalization of $r^{\prime}$ to $r$ is the Hadamard product of $r$ and $r^{\prime}$, i.e., $\overline{r^{\prime}}(r)=r \otimes r^{\prime}$, because for each $\zeta \in \mathrm{T}_{\Sigma}$ we have

$$
\begin{equation*}
\left(\overline{r^{\prime}}(r)\right)(\zeta)=\sum_{\xi \in \mathrm{T}_{\Sigma}}^{\oplus} r(\xi) \otimes \overline{r^{\prime}}(\xi, \zeta)=r(\zeta) \otimes r^{\prime}(\zeta)=\left(r \otimes r^{\prime}\right)(\zeta) \tag{2.33}
\end{equation*}
$$

In the sequel we will sometimes drop the bar from $\overline{r^{\prime}}$. Then it will be clear from the context whether $r^{\prime}$ denotes a $(\Sigma, \mathrm{B})$-weighted tree language or its diagonalization. We will use this type of application, e.g., in the alternative proof of closure of the set of recognizable weighted tree languages under Hadamard product (cf. Subsection 10.13.6).

Scenario 3. Finally, in this scenario, we have $A=\mathrm{T}_{\Delta}$, i.e., $\tau$ is a $(\Sigma, \Delta, \mathrm{B})$-weighted tree transformation. Let $r: \mathrm{T}_{\Sigma} \rightarrow B$ and $\tau: \mathrm{T}_{\Sigma} \times \mathrm{T}_{\Delta} \rightarrow B$. If $r$ is $\tau$-summable or B is $\sigma$-complete, then for each $\zeta \in \mathrm{T}_{\Delta}$ we have

$$
\begin{equation*}
\tau(r)(\zeta)=\sum_{\xi \in \mathrm{T}_{\Sigma}}^{\oplus} r(\xi) \otimes \tau(\xi, \zeta) \tag{2.34}
\end{equation*}
$$

We will use this type of application, e.g., in Section 10.13 and in Chapter 15 ,

Next we define the composition of weighted tree transformations. Let $\Omega$ be a ranked alphabet and $\tau: \mathrm{T}_{\Sigma} \times \mathrm{T}_{\Delta} \rightarrow B$ and $\tau^{\prime}: \mathrm{T}_{\Delta} \times \mathrm{T}_{\Omega} \rightarrow B$ be weighted tree transformations. If $\tau$ is supp-o-finite, $\tau^{\prime}$ is supp-i-finite, or B is $\sigma$-complete, then we define the composition of $\tau$ and $\tau^{\prime}$ to be the weighted tree transformation $\left(\tau ; \tau^{\prime}\right): \mathrm{T}_{\Sigma} \times \mathrm{T}_{\Omega} \rightarrow B$ defined by

$$
\begin{equation*}
\left(\tau ; \tau^{\prime}\right)(\xi, \zeta)=\sum_{\eta \in \mathrm{T}_{\Delta}}^{\oplus} \tau(\xi, \eta) \otimes \tau^{\prime}(\eta, \zeta) \tag{2.35}
\end{equation*}
$$

for every $\xi \in \mathrm{T}_{\Sigma}$ and $\zeta \in \mathrm{T}_{\Omega}$. Clearly, if $\tau$ is supp-o-finite or $\tau^{\prime}$ is supp-i-finite, then

$$
\sum_{\eta \in \mathrm{T}_{\Delta}}^{\oplus} \tau(\xi, \eta) \otimes \tau^{\prime}(\eta, \zeta)=\bigoplus_{\eta \in \mathrm{T}_{\Delta}} \tau(\xi, \eta) \otimes \tau^{\prime}(\eta, \zeta)
$$

We show that, roughly speaking, if $B$ is a semiring, then the composition of weighted tree transformations is associative. Intuitively, the conditions $P_{1}$ and $P_{2}$ of the next observation guarantee that the expressions $\left(\tau_{1} ; \tau_{2}\right) ; \tau_{3}$ and $\tau_{1} ;\left(\tau_{2} ; \tau_{3}\right)$, respectively, are well defined.

Observation 2.10.2. Let B be a semiring. Moreover, let $\Omega$ and $\Psi$ be ranked alphabets and $\tau_{1}: \mathrm{T}_{\Sigma} \times$ $\mathrm{T}_{\Delta} \rightarrow B, \tau_{2}: \mathrm{T}_{\Delta} \times \mathrm{T}_{\Omega} \rightarrow B$, and $\tau_{3}: \mathrm{T}_{\Omega} \times \mathrm{T}_{\Psi} \rightarrow B$ be weighted tree transformations such that the condition $P_{1} \wedge P_{2}$ holds or B is $\sigma$-complete, where

- $P_{1}$ : [ $\tau_{1}$ is supp-o-finite or $\tau_{2}$ is supp-i-finite $]$ and $\left[\tau_{1} ; \tau_{2}\right.$ is supp-o-finite or $\tau_{3}$ is supp-i-finite] and
- $P_{2}:\left[\tau_{2}\right.$ is supp-o-finite or $\tau_{3}$ is supp-i-finite] and [ $\tau_{1}$ is supp-o-finite or $\tau_{2} ; \tau_{3}$ is supp-i-finite].

Then $\left(\tau_{1} ; \tau_{2}\right) ; \tau_{3}=\tau_{1} ;\left(\tau_{2} ; \tau_{3}\right)$.
Proof. Due to the first conjunct (and the second conjunct) of $P_{1}$ the compositions $\tau_{1} ; \tau_{2}$ (and $\left(\tau_{1} ; \tau_{2}\right) ; \tau_{3}$, respectively) are defined. Moreover, due to the first conjunct (and the second conjunct) of $P_{2}$ the compositions $\tau_{2} ; \tau_{3}$ (and $\tau_{1} ;\left(\tau_{2} ; \tau_{3}\right)$, respectively) are defined.

Let $\xi \in \mathrm{T}_{\Sigma}$ and $\zeta \in \mathrm{T}_{\Psi}$. Then

$$
\left(\left(\tau_{1} ; \tau_{2}\right) ; \tau_{3}\right)(\xi, \zeta)=\sum_{\eta \in \mathrm{T}_{\Omega}}^{\oplus}\left(\tau_{1} ; \tau_{2}\right)(\xi, \eta) \otimes \tau_{3}(\eta, \zeta)=\sum_{\eta \in \mathrm{T}_{\Omega}}^{\oplus}\left(\sum_{\theta \in \mathrm{T}_{\Delta}}^{\oplus} \tau_{1}(\xi, \theta) \otimes \tau_{2}(\theta, \eta)\right) \otimes \tau_{3}(\eta, \zeta)
$$

$$
\begin{aligned}
& =\sum_{\eta \in \mathrm{T}_{\Omega}}^{\oplus}\left(\sum_{\theta \in \mathrm{T}_{\Delta}}^{\oplus} \tau_{1}(\xi, \theta) \otimes \tau_{2}(\theta, \eta) \otimes \tau_{3}(\eta, \zeta)\right) \quad \text { (by right-distributivity) } \\
& =\sum_{\theta \in \mathrm{T}_{\Delta}}^{\oplus} \tau_{1}(\xi, \theta) \otimes\left(\sum_{\eta \in \mathrm{T}_{\Omega}}^{\oplus} \tau_{2}(\theta, \eta) \otimes \tau_{3}(\eta, \zeta)\right) \\
& =\sum_{\theta \in \mathrm{T}_{\Delta}}^{\oplus} \tau_{1}(\xi, \theta) \otimes\left(\tau_{2} ; \tau_{3}\right)(\theta, \zeta)=\left(\tau_{1} ;\left(\tau_{2} ; \tau_{3}\right)\right)(\xi, \zeta) .
\end{aligned}
$$

Next we show that, roughly speaking, if B is a semiring, then the application of the composition of weighted tree transformations $\tau_{1}$ and $\tau_{2}$ to a weighted tree language $r$ can be expressed as the consecutive applications of $\tau_{1}$ and $\tau_{2}$ to $r$. Intuitively, the conditions $P_{1}$ and $P_{2}$ of the next observation guarantee that the expressions $\left(\tau ; \tau^{\prime}\right)(r)$ and $\tau^{\prime}(\tau(r))$, respectively, are well defined.

Observation 2.10.3. Let B be a semiring. Moreover, let $\Omega$ be a ranked alphabet, $\tau: \mathrm{T}_{\Sigma} \times \mathrm{T}_{\Delta} \rightarrow B$ and $\tau^{\prime}: \mathrm{T}_{\Delta} \times \mathrm{T}_{\Omega} \rightarrow B$ be weighted tree transformations, and let $r: \mathrm{T}_{\Sigma} \rightarrow B$ be a weighted tree language such that the condition $P_{1} \wedge P_{2}$ holds or B is $\sigma$-complete, where

- $P_{1}:\left[\tau\right.$ is supp-o-finite or $\tau^{\prime}$ is supp-i-finite $]$ and $\left[\tau ; \tau^{\prime}\right.$ is supp-i-finite or $r$ has finite support $]$ and
- $P_{2}$ : [ $\tau$ is supp-i-finite or $r$ has finite support $]$ and $\left[\tau^{\prime}\right.$ is supp-i-finite or $\tau(r)$ has finite support].

Then we have $\left(\tau ; \tau^{\prime}\right)(r)=\tau^{\prime}(\tau(r))$.
Proof. Due to the first conjunct (and the second conjunct) of $P_{1}$ the composition $\tau ; \tau^{\prime}$ (and the application $\left(\tau ; \tau^{\prime}\right)(r)$, respectively) are defined. Moreover, due to the first conjunct (and the second conjunct) of $P_{2}$ the applications $\tau(r)$ (and $\tau^{\prime}(\tau(r))$, respectively) are defined.

Let $\theta \in \mathrm{T}_{\Omega}$. Then

$$
\begin{array}{rlr}
\left(\tau ; \tau^{\prime}\right)(r)(\theta) & =\sum_{\xi \in \mathrm{T}_{\Sigma}}^{\oplus} r(\xi) \otimes\left(\tau ; \tau^{\prime}\right)(\xi, \theta)=\sum_{\xi \in \mathrm{T}_{\Sigma}}^{\oplus} r(\xi) \otimes \sum_{\zeta \in \mathrm{T}_{\Delta}}^{\oplus} \tau(\xi, \zeta) \otimes \tau^{\prime}(\zeta, \theta) \\
& =\sum_{\xi \in \mathrm{T}_{\Sigma}}^{\oplus} \sum_{\zeta \in \mathrm{T}_{\Delta}}^{\oplus} r(\xi) \otimes \tau(\xi, \zeta) \otimes \tau^{\prime}(\zeta, \theta) & \text { (by left-distributivity) } \\
& =\sum_{\zeta \in \mathrm{T}_{\Delta}}^{\oplus} \sum_{\xi \in \mathrm{T}_{\Sigma}}^{\oplus} r(\xi) \otimes \tau(\xi, \zeta) \otimes \tau^{\prime}(\zeta, \theta) & \\
& =\sum_{\zeta \in \mathrm{T}_{\Delta}}^{\oplus}\left(\sum_{\xi \in \mathrm{T}_{\Sigma}}^{\oplus} r(\xi) \otimes \tau(\xi, \zeta)\right) \otimes \tau^{\prime}(\zeta, \theta) & \text { (by right-distributivity) } \\
& =\sum_{\zeta \in \mathrm{T}_{\Delta}}^{\oplus} \tau(r)(\zeta) \otimes \tau^{\prime}(\zeta, \theta)=\tau^{\prime}(\tau(r))(\theta) .
\end{array}
$$

### 2.11 Finite-state string automata

We recall the definition of $\Gamma$-automaton from Eil74, p. 12]. A finite-state string automaton over $\Gamma$ (for short: $\Gamma$ - $f s a)$ is a quadruple $A=(Q, I, \delta, F)$ where $Q$ is a finite nonempty set of states such that $Q \cap \Gamma=\emptyset$, $I \subseteq Q$ (initial states), $\delta \subseteq Q \times \Gamma \times Q$ (transitions), and $F \subseteq Q$ (final states).

Let $w=a_{1} \cdots a_{n}$ be a string in $\Gamma^{*}$ with $n \in \mathbb{N}$ and $a_{1}, \ldots, a_{n} \in \Gamma$. A run of $A$ on $w$ is a string $\rho=q_{0} \cdots q_{n}$ in $Q^{n+1}$ such that, for each $i \in[0, n-1]$ we have $\left(q_{i}, a_{i+1}, q_{i+1}\right) \in \delta$. We denote $q_{0}$ and $q_{n}$ by first $(\rho)$ and last $(\rho)$, respectively. The language recognized by $A$, denoted by $\mathrm{L}(A)$, is the set

$$
\mathrm{L}(A)=\left\{w \in \Gamma^{*} \mid \text { there exists a run } \rho \text { of } A \text { on } w \text { such that } \operatorname{first}(\rho) \in I \text { and last }(\rho) \in F\right\} .
$$

Let $L \subseteq \Gamma^{*}$. We call $L$ recognizable if there exists an fsa $A$ such that $\mathrm{L}(A)=L$.

### 2.12 Context-free grammars

The theory of context-free grammars is well established; for more details, we refer the reader to, e.g., Har78, HU79, HMU07. Here we only recall the most basic definitions.

A context-free grammar over $\Gamma$ (for short: $\Gamma$-cfg) is a triple $G=(N, S, R)$ where $N$ is a finite set (nonterminals) with $N \cap \Gamma=\emptyset, S \subseteq N$ with $S \neq \emptyset$ (initial nonterminals), and $R$ is a finite set (rules); each rule has the form $A \rightarrow \alpha$ where $A \in N$ and $\alpha \in(N \cup \Gamma)^{*}$.

Let $r=(A \rightarrow \alpha)$ be a rule. We call $r$ a chain rule (an $\varepsilon$-rule, a terminal rule) if $\alpha \in N$ (and if $\alpha=\varepsilon$, and if $\alpha \in \Gamma^{*}$, respectively). The left-hand side of $r$ is the nonterminal $A$, denoted by lhs $(r)$. Moreover, the $i$-th occurrence of a nonterminal in $\alpha$ (counted from left to right) is denoted by $\operatorname{rhs}_{N, i}(r)$. We sometimes want to show the occurrences of elements of $N$ in the right-hand side of rule $r$ more explicitly. Then we will also write $r$ in the form

$$
A \rightarrow u_{0} A_{1} u_{1} \cdots A_{k} u_{k}
$$

where $k \in \mathbb{N}, u_{0}, u_{1}, \ldots, u_{k} \in \Gamma^{*}$, and $A_{1}, \ldots, A_{k} \in N$. If $S$ contains only one element, say $S_{0}$, then we denote $G$ by $\left(N, S_{0}, R\right)$.

The derivation relation (of $G$ ) is the binary relation $\Rightarrow_{G}$ on $(N \cup \Gamma)^{*}$ defined as follows. For every $\gamma, \delta \in(N \cup \Gamma)^{*}$ and rule $A \rightarrow \alpha$ in $R$, we have $\gamma A \delta \Rightarrow_{G} \gamma \alpha \delta$. If $G$ is clear from the context, then we denote $\Rightarrow_{G}$ by $\Rightarrow$. We recall that the reflexive and transitive closure of $\Rightarrow$ is denoted by $\Rightarrow^{*}$.

A derivation (of $G$ ) is a sequence $d=\left(\gamma_{1}, \ldots, \gamma_{n}\right)$ with $n \in \mathbb{N}_{+}, \gamma_{i} \in(N \cup \Gamma)^{*}$, and $\gamma_{i} \Rightarrow \gamma_{i+1}$ for each $i \in[n-1]$. We denote $d$ also by $\gamma_{1} \Rightarrow{ }_{G}^{*} \gamma_{n}$. Let $L_{1}, L_{2} \subseteq(N \cup \Gamma)^{*}$. We denote by $\mathrm{D}_{G}\left(L_{1}, L_{2}\right)$ the set of all derivations $\gamma_{1} \Rightarrow_{G}^{*} \gamma_{n}$ such that $\gamma_{1} \in L_{1}$ and $\gamma_{n} \in L_{2}$.

The language generated by $G$ is the set $\mathrm{L}(G)=\left\{w \in \Gamma^{*} \mid S_{0} \in S, S_{0} \Rightarrow_{G}^{*} w\right\}$ or, equivalently, $\mathrm{L}(G)=\left\{w \in \Gamma^{*} \mid \mathrm{D}_{G}(S, w) \neq \emptyset\right\}$. Let $L \subseteq \Gamma^{*}$. We call $L$ a context-free language if there exists a $\Gamma$-cfg $G$ such that $\mathrm{L}(G)=L$.

The leftmost derivation relation $\Rightarrow_{G, 1}$ is defined such that, for every $u \in \Gamma^{*}, \gamma \in(N \cup \Gamma)^{*}$, and rule $A \rightarrow \alpha$ in $R$, we have $u A \gamma \Rightarrow_{G, 1} u \alpha \gamma$. If $G$ is clear from the context, then we denote $\Rightarrow_{G, 1}$ by $\Rightarrow_{1}$. The concept of leftmost derivation is defined analogously to the concept of derivation. For every $L_{1}, L_{2} \subseteq(N \cup \Gamma)^{*}$, we denote by $\mathrm{D}_{G, 1}\left(L_{1}, L_{2}\right)$ the set of all leftmost derivations $\gamma_{1} \Rightarrow_{G, 1}^{*} \gamma_{n}$ such that $\gamma_{1} \in L_{1}$ and $\gamma_{n} \in L_{2}$. It is well known that $\mathrm{L}(G)=\left\{w \in \Gamma^{*} \mid S_{0} \in S, S_{0} \Rightarrow_{G, 1}^{*} w\right\}$.

Let $G=(N, S, R)$ be a context-free grammar over $\Gamma$ and $A \in N$ be a nonterminal. We say that

- $A$ is generating if there exists $w \in \Gamma^{*}$ such that $A \Rightarrow^{*} w$,
- $A$ is reachable if there exist $S_{0} \in S$ and $\gamma, \delta \in(N \cup \Gamma)^{*}$ such that $S_{0} \Rightarrow^{*} \gamma A \delta$, and
- $A$ is useful if it is generating and reachable.

We say that $G$ is reduced if $R=\emptyset$ or each nonterminal $A \in N$ is useful.
Theorem 2.12.1. Har78, Thm. 3.2.3] For each $\Gamma$-cfg $G$ with a single initial nonterminal $S_{0}$, we can construct a $\Gamma$-cfg $G^{\prime}$ such that $G^{\prime}$ has the single initial nonterminal $S_{0}, \mathrm{~L}\left(G^{\prime}\right)=\mathrm{L}(G)$, and $G^{\prime}$ is reduced.

Finally, we give a characterization of the languages generated by context-free grammars which have a terminal rule. The characterization uses the concept of rule trees (cf. [Eng75b, Def. 3.54] and [GS84, Def. 3.2.8]). Intuitively, each leftmost derivation corresponds to exactly one rule tree, and vice versa. Because of our general assumption made at the beginning of Section 2.9, that each ranked alphabet contains a symbol of rank zero, such a characterization is only possible for cfg which have a terminal rule. Clearly, we can transform each cfg into an equivalent one which has a terminal rule.

Formally, let $G=(N, S, R)$ be a $\Gamma$-cfg which has a terminal rule. We consider $R$ as ranked alphabet by defining the rank of each rule $r$ to be the number of nonterminals in the right-hand side of $r$. Hence, each terminal rule has rank 0 and, due to our assumption, $R^{(0)} \neq \emptyset$.

We define the projection of $G$, denoted by $\pi_{G}$, to be the mapping

$$
\pi_{G}: \mathrm{T}_{R} \rightarrow \Gamma^{*}
$$



Figure 2.7: The $A$-rule tree $d \in \mathrm{RT}_{G}(A, a b c b c)$ of $G$ for $a b c b c$.
defined by induction on $\mathrm{T}_{R}$ as follows. For every $k \in \mathbb{N}, r \in R^{(k)}$ of the form $r=\left(A \rightarrow u_{0} A_{1} u_{1} \cdots A_{k} u_{k}\right)$, and $d_{1}, \ldots, d_{k} \in \mathrm{~T}_{R}$, we define

$$
\pi_{G}\left(r\left(d_{1}, \ldots, d_{k}\right)\right)=u_{0} \pi_{G}\left(d_{1}\right) u_{1} \ldots \pi_{G}\left(d_{k}\right) u_{k}
$$

If there is no confusion, then we drop the index $G$ from $\pi_{G}$ and just write $\pi$.
Now let $d \in \mathrm{~T}_{R}$. We say that $d$ is a rule tree of $G$ if, for every $w \in \operatorname{pos}_{R}(d)$ and $i \in\left[\mathrm{rk}_{R}(d(w))\right]$, we have $\operatorname{rhs}_{N, i}(d(w))=\operatorname{lhs}(d(w i))$. Let $A \in N, u \in \Gamma^{*}$, and $d \in \mathrm{~T}_{R}$ be a rule tree of $G$. We say that $d$ is

- an $A$-rule tree of $G$ if $\operatorname{lhs}(d(\varepsilon))=A$.
- a rule tree of $G$ for $u$ if $\pi_{G}(d)=u$.

We denote the set of all $A$-rule trees of $\mathcal{G}$ for $u$ by $\operatorname{RT}_{G}(A, u)$. For every $N^{\prime} \subseteq N$ and $L \subseteq \Gamma^{*}$, we define $\operatorname{RT}_{G}\left(N^{\prime}, L\right)=\bigcup_{A \in N^{\prime}, u \in L} \operatorname{RT}_{G}(A, u)$, and we abbreviate $\mathrm{RT}_{G}(S, L)$ by $\mathrm{RT}_{G}(L)$. Obviously, for each $u \in \Gamma^{*}$, we have $\pi_{G}^{-1}(u) \cap \mathrm{RT}_{G}\left(\Gamma^{*}\right)=\mathrm{RT}_{G}(u)$. Finally, we abbreviate $\mathrm{RT}_{G}\left(\Gamma^{*}\right)$ by $\mathrm{RT}_{G}$. Then the following is easy to see.

Observation 2.12.2. For each $\Gamma$-cfg $G$ which has a terminal rule we have that $\mathrm{L}(G)=\pi_{G}\left(\mathrm{RT}_{G}\right)$.
We note that $\mathrm{L}(G)=\emptyset$ for each $\Gamma$-cfg $G$ which does not have a terminal rule. Therefore such grammars are not relevant. On the other hand, $\mathrm{L}(G)=\emptyset$ does not imply that $G$ does not have a terminal rule.

Example 2.12.3. We consider the alphabet $\Gamma=\{a, b, c\}$ and the $\Gamma-\operatorname{cfg} G=(N, S, R)$ with $N=\{S, A, B\}$ and the rules

$$
S \rightarrow a A B, A \rightarrow A a B A, A \rightarrow B, B \rightarrow b c, A \rightarrow \varepsilon
$$

Figure 2.7 shows the $A$-rule tree $d \in \operatorname{RT}_{G}(A, a b c b c)$ of $G$ for $a b c b c$.

### 2.13 Finite-state tree automata

We recall the notions of finite-state tree automaton and recognizable tree language from GS84, Eng75b, GS97, $\mathrm{CDG}^{+} 07$.

A finite-state tree automaton over $\Sigma$ (for short: $\Sigma$-fta, or just fta) is a triple $A=(Q, \delta, F)$ where

- $Q$ is a finite nonempty set (states) such that $Q \cap \Sigma=\emptyset$,
- $\delta=\left(\delta_{k} \mid k \in \mathbb{N}\right)$ is a family of relations $\delta_{k} \subseteq Q^{k} \times \Sigma^{(k)} \times Q^{6}$ (transition relations) where we consider $Q^{k}$ as a set of strings over $Q$ of length $k$, and
- $F \subseteq Q$ (set of root states).

[^6]We say that $A$ is bottom-up deterministic (for short: bu deterministic) if for every $k \in \mathbb{N}, w \in Q^{k}$, and $\sigma \in \Sigma^{(k)}$, there exists at most one $q \in Q$ such that $(w, \sigma, q) \in \delta_{k}$. And we say that $A$ is total if for every $k \in \mathbb{N}, w \in Q^{k}$, and $\sigma \in \Sigma^{(k)}$, there exists at least one $q \in Q$ such that $(w, \sigma, q) \in \delta_{k}$.

We can associate two semantics with $A$ : the initial algebra semantics and the run semantics. Both lead to the same tree language. Since each of the semantics has its benefits we present them both.

Initial algebra semantics. We define the $\Sigma$-algebra associated with $A$ to be the $\Sigma$-algebra $\left(\mathcal{P}(Q), \delta_{A}\right)$ where, for every $k \in \mathbb{N}$ and $\sigma \in \Sigma^{(k)}$, the mapping $\delta_{A}(\sigma): \mathcal{P}(Q)^{k} \rightarrow \mathcal{P}(Q)$ is defined by

$$
\delta_{A}(\sigma)\left(P_{1}, \ldots, P_{k}\right)=\left\{q \in Q \mid\left(\exists q_{1} \in P_{1}\right) \ldots\left(\exists q_{k} \in P_{k}\right):\left(q_{1} \cdots q_{k}, \sigma, q\right) \in \delta_{k}\right\}
$$

for every $P_{1}, \ldots, P_{k} \in \mathcal{P}(Q)$. We denote the unique $\Sigma$-algebra homomorphism from the term algebra $\mathrm{T}_{\Sigma}$ to $\left(\mathcal{P}(Q), \delta_{A}\right)$ by $\mathrm{h}_{A}$. The tree language $i$-recognized by $A$, denoted by $\mathrm{L}_{\mathrm{i}}(A)$, is defined by

$$
\mathrm{L}_{\mathrm{i}}(A)=\left\{\xi \in \mathrm{T}_{\Sigma} \mid \mathrm{h}_{A}(\xi) \cap F \neq \emptyset\right\}
$$

Run semantics. Let $\xi \in \mathrm{T}_{\Sigma}$. A run of $A$ on $\xi$ is a mapping $\rho: \operatorname{pos}(\xi) \rightarrow Q$. Let $q \in Q$. Then $\rho$ is called

- $q$-run if $\rho(\varepsilon)=q$,
- valid if for every $w \in \operatorname{pos}(\xi)$ it holds that $(\rho(w 1) \cdots \rho(w k), \xi(w), \rho(w)) \in \delta_{k}$ where $\xi(w) \in \Sigma^{(k)}$, and
- accepting if $\rho$ is valid and $\rho(\varepsilon) \in F$.

The set of all $q$-runs (all valid $q$-runs, all accepting $q$-runs) of $A$ on $\xi$ is denoted by $\mathrm{R}_{A}(q, \xi)$ (respectively, $\mathrm{R}_{A}^{\mathrm{v}}(q, \xi)$ and $\left.\mathrm{R}_{A}^{\mathrm{a}}(q, \xi)\right)$. We let

$$
\mathrm{R}_{A}(\xi)=\bigcup_{q \in Q} \mathrm{R}_{A}(q, \xi) \text { and } \mathrm{R}_{A}^{\mathrm{v}}(\xi)=\bigcup_{q \in Q} \mathrm{R}_{A}^{\mathrm{v}}(q, \xi) \text { and } \mathrm{R}_{A}^{\mathrm{a}}(\xi)=\bigcup_{q \in F} \mathrm{R}_{A}^{\mathrm{a}}(q, \xi)
$$

The tree language $r$-recognized by $A$, denoted by $\mathrm{L}_{\mathrm{r}}(A)$, is defined by

$$
\mathrm{L}_{\mathrm{r}}(A)=\left\{\xi \in \mathrm{T}_{\Sigma} \mid \mathrm{R}_{A}^{\mathrm{a}}(\xi) \neq \emptyset\right\}
$$

Lemma 2.13.1. Let $A=(Q, \delta, F)$ be a $\Sigma$-fta. Then $\mathrm{L}_{\mathrm{i}}(A)=\mathrm{L}_{\mathrm{r}}(A)$.
Proof. Let $\xi \in \mathrm{T}_{\Sigma}$. We have

$$
\xi \in \mathrm{L}_{\mathrm{i}}(A) \operatorname{iff} \mathrm{h}_{A}(\xi) \cap F \neq \emptyset \text { iff }(\exists q \in F): q \in \mathrm{~h}_{A}(\xi) \operatorname{iff}^{(*)}(\exists q \in F): \mathrm{R}_{A}^{\mathrm{v}}(q, \xi) \neq \emptyset \text { iff } \xi \in \mathrm{L}_{\mathrm{r}}(A)
$$

It remains to prove $\left(^{*}\right)$. By induction on $\mathrm{T}_{\Sigma}$, we prove that the following more general statement holds:

For every $\xi \in \mathrm{T}_{\Sigma}$ and $q \in Q$, we have: $q \in \mathrm{~h}_{A}(\xi) \operatorname{iff}\left(\left(\exists \rho \in \mathrm{R}_{A}(q, \xi)\right): \rho\right.$ is valid $)$.
Let $\xi=\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)$. Then
$q \in \mathrm{~h}_{A}\left(\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)\right)$
iff $\left(\exists q_{1} \cdots q_{k} \in Q^{k}\right):\left(q_{1} \in \mathrm{~h}_{A}\left(\xi_{1}\right)\right) \wedge \ldots \wedge\left(q_{k} \in \mathrm{~h}_{A}\left(\xi_{k}\right)\right) \wedge\left(\left(q_{1} \cdots q_{k}, \sigma, q\right) \in \delta_{k}\right)$
iff $\quad\left(\exists q_{1} \cdots q_{k} \in Q^{k}\right)$ :
$\left(\left(\exists \rho_{1} \in \mathrm{R}_{A}\left(q_{1}, \xi\right)\right): \rho_{1}\right.$ is valid $) \wedge \ldots \wedge\left(\left(\exists \rho_{k} \in \mathrm{R}_{A}\left(q_{k}, \xi\right)\right): \rho_{k}\right.$ is valid $) \wedge\left(\left(q_{1} \cdots q_{k}, \sigma, q\right) \in \delta_{k}\right)$
iff $\quad\left(\exists q_{1} \cdots q_{k} \in Q^{k}\right)\left(\exists \rho_{1} \in \mathrm{R}_{A}\left(q_{1}, \xi\right)\right) \ldots\left(\exists \rho_{k} \in \mathrm{R}_{A}\left(q_{k}, \xi\right)\right)$ :
$\left(\rho_{1}\right.$ is valid $) \wedge \ldots \wedge\left(\rho_{k}\right.$ is valid $) \wedge\left(\left(q_{1} \cdots q_{k}, \sigma, q\right) \in \delta_{k}\right)$
(by distributivity of $\wedge$ over $\exists$ )
iff $\quad\left(\exists \rho \in \mathrm{R}_{A}(q, \xi)\right): \rho$ is valid.

Since $\mathrm{L}_{\mathrm{i}}(A)=\mathrm{L}_{\mathrm{r}}(A)$ for each $\Sigma$-fta $A$, we will sometimes drop the indices i and r from $\mathrm{L}_{\mathrm{i}}(A)$ and $\mathrm{L}_{\mathrm{r}}(A)$, respectively, and we call the language $\mathrm{L}(A)$ the tree language recognized by $A$. Two $\Sigma$-fta $A$ and $A^{\prime}$ are equivalent if $\mathrm{L}(A)=\mathrm{L}\left(A^{\prime}\right)$. A tree language $L \subseteq \mathrm{~T}_{\Sigma}$ is recognizable if there exists a $\Sigma$-fta $A$ such that $\mathrm{L}(A)=L$. The set of all recognizable $\Sigma$-tree languages is denoted by $\operatorname{Rec}(\Sigma)$.

Theorem 2.13.2. (cf. GS84, Thm. 2.2.6 $7^{7}$ and Eng75b, Thm. 3.8] ${ }^{8}$ ) For each $\Sigma$-fta $A$ we can construct a total and bu deterministic $\Sigma$-fta $B$ such that $\mathrm{L}(A)=\mathrm{L}(B)$.

Theorem 2.13.3. (cf. Eng75b, Thm. 3.25 and 3.32] and [GS84, Thm. 2.4.2] Let $A_{1}, A_{2}$ be two $\Sigma$-fta.
(1) We can construct a $\Sigma$-fta $A$ such that $\mathrm{L}(A)=\mathrm{L}\left(A_{1}\right) \cup \mathrm{L}\left(A_{2}\right)$.
(2) We can construct a $\Sigma$-fta $A$ such that $\mathrm{L}(A)=\mathrm{L}\left(A_{1}\right) \cap \mathrm{L}\left(A_{2}\right)$.
(3) We can construct a $\Sigma$-fta $A$ such that $\mathrm{L}(A)=\mathrm{L}\left(A_{1}\right) \backslash \mathrm{L}\left(A_{2}\right)$.

Thus, the set $\operatorname{Rec}(\Sigma)$ is closed under union, intersection, set subtraction, and complement.
For the theory of recognizable tree languages we refer to Eng75b, GS84, GS97, [CDG ${ }^{+}$07].

### 2.14 Recognizable step mappings

We recall the concept of recognizable step mapping from [DV06, (also cf. DG05, DG07, DG09]). Roughly speaking, these are ( $\Sigma, \mathrm{B}$ )-weighted tree languages which can be obtained from characteristic mappings of finitely many recognizable $\Sigma$-tree language by multiplying with elements of $B$ and summing up.

Formally, a weighted tree language $r: \mathrm{T}_{\Sigma} \rightarrow B$ is a ( $\Sigma, \mathrm{B}$ )-recognizable step mapping (or just: recognizable step mapping) if there exist $n \in \mathbb{N}_{+}, b_{1}, \ldots, b_{n} \in B$, and recognizable $\Sigma$-tree languages $L_{1}, \ldots, L_{n}$ such that we have

$$
\begin{equation*}
r=\bigoplus_{i \in[n]} b_{i} \otimes \chi\left(L_{i}\right) \tag{2.37}
\end{equation*}
$$

(where we have extended the sum of two weighted tree languages to the sum of finitely many weighted tree languages, cf. page 21). In other words, for each $\xi \in \mathrm{T}_{\Sigma}$, we have

$$
r(\xi)=\bigoplus_{\substack{i \in[n]: \\ \xi \in L_{i}}} b_{i}
$$

In particular, if $\xi \notin L_{1} \cup \ldots \cup L_{n}$, then $r(\xi)=\mathbb{O}$. Moreover, if $n=1$ and $L_{1}=\emptyset$, then $r=\widetilde{0}$. Also, each polynomial $(\Sigma, B)$-weighted tree language is a recognizable step mapping, because singleton $\Sigma$-tree languages are recognizable. The tree languages $L_{i}$ are called step languages. If $n=1$, then $r$ is called ( $\Sigma, \mathrm{B})$-recognizable one-step mapping. We denote by $\operatorname{RecStep}(\Sigma, \mathrm{B})$ the set of $(\Sigma, \mathrm{B})$-recognizable step mappings.

In general, the step languages of a recognizable step mapping need not be disjoint. However, for each recognizable step mapping we can find a characterization in terms of pairwise disjoint step languages.

Observation 2.14.1. [DV06, Lm. 3.1] Let $r: \mathrm{T}_{\Sigma} \rightarrow B$ be a recognizable step mapping. There exist a finite set $F$ and for each $f \in F$, there exist a recognizable $\Sigma$-tree language $U_{f}$ and an element $a_{f} \in B$ such that $\left(U_{f} \mid f \in F\right)$ is a partitioning of $\mathrm{T}_{\Sigma}$ and $r=\bigoplus_{f \in F} a_{f} \otimes \chi\left(U_{f}\right)$.

Proof. Let $n \in \mathbb{N}_{+}, b_{1}, \ldots, b_{n} \in B$, and recognizable $\Sigma$-tree languages $L_{1}, \ldots, L_{n}$ be such that, for each $\xi \in \mathrm{T}_{\Sigma}$, Equation (2.37) holds.

Let $F$ be the set of all mappings of type $\{1, \ldots, n\} \rightarrow\{1, c\}$ (where 1 and $c$ are just viewed as two distinct symbols). For each mapping $f \in F$, we define the $\Sigma$-tree language $U_{f}=\bigcap_{i=1}^{n} L_{i}^{f(i)}$ where $L_{i}^{1}=L_{i}$

[^7]and $L_{i}^{c}=\mathrm{T}_{\Sigma} \backslash L_{i}$. We note that the family $\left(U_{f} \mid f \in F\right)$ forms a partitioning of $\mathrm{T}_{\Sigma}$. Moreover, since the set of recognizable $\Sigma$-tree languages is closed under intersection and complementation GS84, we have that $U_{f}$ is a recognizable $\Sigma$-tree language.

For each $f \in F$, we define the element $a_{f}=\bigoplus_{i \in f^{-1}(1)} b_{i}$ in $B$. Then clearly, for each $\xi \in \mathrm{T}_{\Sigma}$ we have $r(\xi)=a_{f}$ if $\xi \in U_{f}$, where the $f$ is uniquely determined because ( $U_{f} \mid f \in F$ ) forms a partitioning of $\mathrm{T}_{\Sigma}$. Thus $r=\bigoplus_{f \in F} a_{f} \otimes \chi\left(U_{f}\right)$.

We mention that recognizable step mappings play an important role in the characterization of recognizable weighted languages by weighted MSO-logic (cf. DG05, DG07, DG09] for strings, and DV06, DV11b, FSV12, DHV15 for trees; also cf. Section 14.4). In fact, the semantics of the weighted MSO-formula $\forall x . \varphi$ is a recognizable weighted language if the semantics of $\varphi$ is a recognizable step mapping ([DG05, Lm. 4.2] and [DG09, Lm. 5.4], also cf. [DV06, Lm. 5.5] and Lemma 14.4.16] for the tree case); moreover, there exists a weighted MSO-formula $\varphi$ of which the semantics is a recognizable weighted language and the semantics $\forall x . \varphi$ is not recognizable DG09, Ex. 3.6].

We note that in GZ12, Def. 3.4], a ( $\Sigma, \mathrm{L})$-recognizable step mapping for some $\sigma$-complete residuated lattice L is called an L -valued regular tree language.

### 2.15 Fta-hypergraphs

The concept of hypergraph BC87, HK87, CE12 generalizes that of graphs. In this book we consider particular hypergraphs which we call fta-hypergraphs. Let $Q$ be a finite set. A $(Q, \Sigma)$-hypergraph is a pair $g=(Q, E)$, where $E \subseteq \bigcup_{k \in \mathbb{N}} Q^{k} \times \Sigma^{(k)} \times Q$ (set of hyperedges); the elements of $Q$ are called nodes. An fta-hypergraph is a $(Q, \Sigma)$-hypergraph for some finite set $Q$ and some ranked alphabet $\Sigma$.

We can represent a $(Q, \Sigma)$-hypergraph as a picture as follows. Each node $q \in Q$ is represented as a circle with $q$ in its center. Each hyperedge $\left(q_{1} \cdots q_{k}, \sigma, q\right)$ is represented as a box with $\sigma$ in its center and with incoming and outgoing arrows. More specifically, this box has exactly one outgoing arrow, which leads to the representation of the node $q$. Moreover, it has $k$ incoming arrows, which come from the representations of the nodes $q_{1} \ldots, q_{k}$, respectively. The order among $q_{1}, \ldots, q_{k}$ which is determined by the string $q_{1} \cdots q_{k}$, is represented in the picture as follows: starting from the unique outgoing arrow and moving counter-clockwise around the box, the $i$-th incoming arrow comes from the representation of the $i$-th component of the string $q_{1} \cdots q_{k}$.

Figure 2.8 illustrates the $(Q, \Sigma)$-hypergraph $(Q, E)$ with $Q=\{$ h, 0$\}, \Sigma=\left\{\sigma^{(2)}, \alpha^{(0)}\right\}$, and

$$
E=\{(\varepsilon, \alpha, \mathrm{h}),(\mathrm{h} 0, \sigma, \mathrm{~h}),(0 \mathrm{~h}, \sigma, \mathrm{~h}),(\varepsilon, \alpha, 0),(00, \sigma, 0)\}
$$

For instance, the hyperedge $(0 h, \sigma, h)$ is represented by the left-lower box. In the sequel, we will not distinguish between an fta-hypergraph and its representation as picture.


Figure 2.8: A $(Q, \Sigma)$-hypergraph.

## Chapter 3

## The model of weighted tree automata

In this chapter we present the basic model of weighted tree automata. Intuitively, it results from the concept of fta by applying two steps.

In the first step, for a given $\Sigma$-fta $\mathcal{A}=(Q, \delta, F)$, we replace each transition relation

$$
\delta_{k} \subseteq Q^{k} \times \Sigma^{(k)} \times Q \quad \text { by its characteristic mapping } \quad \chi_{\text {Boole }}\left(\delta_{k}\right): Q^{k} \times \Sigma^{(k)} \times Q \rightarrow \mathbb{B}
$$

with respect to the Boolean semiring Boole $=(\mathbb{B}, \vee, \wedge, 0,1)$ (where $\mathbb{B}=\{0,1\}$ ), and similarly, we replace

$$
F \subseteq Q \quad \text { by its characteristic mapping } \quad \chi_{\text {Boole }}(F): Q \rightarrow \mathbb{B}
$$

For the sake of simplicity, we denote $\chi_{\text {Boole }}\left(\delta_{k}\right)$ and $\chi_{\text {Boole }}(F)$ also by $\delta_{k}$ and $F$, respectively.
Since there is a bijection between the set of subsets of $Q^{k} \times \Sigma^{(k)} \times Q$ and set of mappings of type $Q^{k} \times \Sigma^{(k)} \times Q \rightarrow \mathbb{B}$ (and the same holds for subsets of $Q$ and mappings of type $Q \rightarrow \mathbb{B}$ ), this step does not change the concept of an fta; it is simply another way to specify an fta.

In the second step, we replace the Boolean semiring Boole by an arbitrary strong bimonoid B, i.e., we let

$$
\delta_{k}: Q^{k} \times \Sigma^{(k)} \times Q \rightarrow B \quad \text { and } \quad F: Q \rightarrow B
$$

In contrast to the first step, the second one generalizes the concept of fta considerably, because
(a) a strong bimonoid B need not be finite and
(b) a number of algebraic laws which hold for the summation $\vee$ and multiplication $\wedge$ of Boole need not hold anymore in B, e.g., idempotency, commutativity of $\wedge$, zero-sum freeness, zero-divisor freeness, absorption axiom, and distributivity.
Hence, in the proofs of lemmas and theorems on weighted tree automata over some arbitrary strong bimonoid $B$ we have to be careful not to use the latter properties out of habit.

In Section 3.1 we recall the concept of weighted tree automata and define the run semantics, the initial algebra semantics, and the corresponding notions of recognizable weighted tree language. In Section 3.2 we illustrate these definitions by a number of examples. In Sections 3.3 3.6 we connect the concept of weighted tree automata with its historical predecessors in a formal way: weighted string automata, finite-state tree automata, finite-state tree automata with multiplicities, and multilinear representations, respectively. Finally, in Section 3.7 we briefly mention a consequence of extending the weight algebra and we define the concept of Fatou extension.

### 3.1 Basic definitions

A weighted tree automaton over $\Sigma$ and B (for short: $(\Sigma, \mathrm{B})$-wta, or: wta) is a tuple $\mathcal{A}=(Q, \delta, F)$ where

- $Q$ is a finite nonempty set (states) such that $Q \cap \Sigma=\emptyset$,
- $\delta=\left(\delta_{k} \mid k \in \mathbb{N}\right)$ is a family of mappings $\delta_{k}: Q^{k} \times \Sigma^{(k)} \times Q \rightarrow B$ (transition mappings) $\sqrt{1}$ where we consider $Q^{k}$ as set of strings over $Q$ of length $k$, and
- $F: Q \rightarrow B$ is a mapping (root weight vector).

For each $k \in \mathbb{N}$, we call each element in $Q^{k} \times \Sigma^{(k)} \times Q$ a transition. For each transition $(w, \sigma, q)$, the element $\delta_{k}(w, \sigma, q)$ of $B$ is its weight, and for each $q \in Q$, the element $F_{q}$ is its root weight. (We recall that $F_{q}$ denotes $F(q)$.) We denote the set of all transition weights and root weights occurring in $\mathcal{A}$ by $\mathrm{wts}^{(\mathcal{A})}$, i.e.,

$$
\operatorname{wts}(\mathcal{A})=\operatorname{im}(\delta) \cup \operatorname{im}(F) \text { where } \operatorname{im}(\delta)=\bigcup_{k \in \mathbb{N}} \operatorname{im}\left(\delta_{k}\right)
$$

Let $\mathcal{A}=(Q, \delta, F)$ be a $(\Sigma, \mathrm{B})$-wta.

- $\mathcal{A}$ is bottom-up deterministic (for short: bu deterministic) if for every $k \in \mathbb{N}, \sigma \in \Sigma^{(k)}$, and $w \in Q^{k}$ there exists at most one $q \in Q$ such that $\delta_{k}(w, \sigma, q) \neq \mathbb{0}$.
- $\mathcal{A}$ is total if for every $k \in \mathbb{N}, \sigma \in \Sigma^{(k)}$, and $w \in Q^{k}$ there exists at least one state $q$ such that $\delta_{k}(w, \sigma, q) \neq \mathbb{0}$.
- $\mathcal{A}$ has identity transition weights if $\operatorname{im}(\delta) \subseteq\{0, \mathbb{1}\}$.
- $\mathcal{A}$ is crisp deterministic if $\mathcal{A}$ is bu deterministic, total, and $\mathcal{A}$ has identity transition weights (cf. CDIV10, Sec. 5]).
- $\mathcal{A}$ has identity root weights if $\operatorname{im}(F) \subseteq\{\mathbb{0}, \mathbb{1}\}$.
- $\mathcal{A}$ is root weight normalized if there exists a state $q \in Q$ such that $\operatorname{supp}(F)=\{q\}$ and $F_{q}=\mathbb{1}$.

We note that, for each $(\Sigma, \mathrm{B})$-wta $\mathcal{A}=(Q, \delta, F)$, we consider the strong bimonoid B as integral part of the specification of $\mathcal{A}$. Thus, if $\mathrm{B}^{\prime}$ is another strong bimonoid which has the same carrier set as B (but different operations and/or unit elements), then we consider $\mathcal{A}$ to be different from the ( $\Sigma, \mathrm{B}^{\prime}$ )-wta $\mathcal{A}^{\prime}=(Q, \delta, F)$. In other words, instead of saying that $\mathcal{A}=(Q, \delta, F)$ is a ( $\left.\Sigma, \mathrm{B}\right)$-wta, we could equivalently say that $\mathcal{A}=(Q, \Sigma, \mathrm{~B}, \delta, F)$ is a wta. And clearly, $(Q, \Sigma, \mathrm{~B}, \delta, F) \neq\left(Q, \Sigma, \mathrm{~B}^{\prime}, \delta, F\right)$.

Representation of wta by fta-hypergraphs. We can represent each ( $\Sigma, \mathrm{B}$ )-wta $\mathcal{A}=(Q, \delta, F)$ as an fta-hypergraph with extra annotations. For this we first consider the ( $Q, \Sigma$ )-hypergraph

$$
g_{\mathcal{A}}=\left(Q, \bigcup_{k \in \mathbb{N}} \operatorname{supp}\left(\delta_{k}\right)\right)
$$

Then we add to $g_{\mathcal{A}}$ the weights of transitions and the root weights of $\mathcal{A}$ as follows. For each $q \in Q$ such that $F_{q} \neq \mathbb{O}$, we add $F_{q}$ to the node which represents $q$. If $F_{q}=\mathbb{O}$, then we do not illustrate $F_{q}$ in the picture. Moreover, for each transition of $\mathcal{A}$ with non- 0 -weight, i.e., element in $\bigcup_{k \in \mathbb{N}} \operatorname{supp}\left(\delta_{k}\right)$, we add its weight to its representing hyperedge.

For instance, let us consider the $\left(\Sigma\right.$, Nat $\left._{\text {max },+}\right)$-wta $\mathcal{A}=(Q, \delta, F)$, where

- $\Sigma=\left\{\sigma^{(2)}, \alpha^{(0)}\right\}$,
- $\operatorname{Nat}_{\max ,+}=\left(\mathbb{N}_{-\infty}, \max ,+,-\infty, 0\right)$, the arctic semiring,
- $Q=\{\mathrm{h}, 0\}$,
- $\delta_{0}(\varepsilon, \alpha, \mathrm{~h})=\delta_{0}(\varepsilon, \alpha, 0)=0$ and for every $q_{1}, q_{2}, q \in Q$,

$$
\delta_{2}\left(q_{1} q_{2}, \sigma, q\right)= \begin{cases}1 & \text { if } q_{1} q_{2} q \in\{\mathrm{~h} 0 \mathrm{~h}, 0 \mathrm{hh}\} \\ 0 & \text { if } q_{1} q_{2} q=000 \\ -\infty & \text { otherwise }\end{cases}
$$

[^8]

Figure 3.1: The fta-hypergraph for the $\left(\Sigma, \operatorname{Nat}_{\max ,+}\right)$-wta $\mathcal{A}$.

- $F_{\mathrm{h}}=0$ and $F_{0}=-\infty$.

In Figure 3.1 we show how $\mathcal{A}$ is represented as fta-hypergraph with extra annotations. We note that, e.g., the transition (hh, $\sigma, 0$ ) is not represented in the fta-hypergraph, because $\delta_{2}(\mathrm{hh}, \sigma, 0)=-\infty$ and $-\infty$ is the identity for the summation of the arctic semiring $\mathrm{Nat}_{\text {max },+}$. Also, the value $F_{0}$ is not represented because $F_{0}=-\infty$.

Run semantics. The run semantics of a ( $\Sigma, \mathrm{B}$ )-wta can be viewed as a generalization of the algebraic path problem (cf. Car71, Zim81, Ch. 8], Mah81, [Rot90, CLR90, Sec. 26.4], and [HW93, Ch. IV.6]). Roughly speaking, given a finite, directed graph such that each edge is labeled by an element of some $\sigma$-complete strong bimonoid B , the algebraic path problem asks the following: for each pair $q, q^{\prime}$ of nodes, how can the value

$$
l_{q, q^{\prime}}=\sum_{\rho \text { path from } q \text { to } q^{\prime}}^{\oplus} \lambda(\rho)
$$

in $B$ be computed where, for each path $\rho=\left(q_{0}, q_{1}\right)\left(q_{1}, q_{2}\right) \cdots\left(q_{n-1}, q_{n}\right)$ from $q_{0}$ to $q_{n}$, we let

$$
\lambda(\rho)=\bigotimes_{i \in[n]} w\left(q_{i-1}, q_{i}\right)
$$

and $w\left(q_{i-1}, q_{i}\right) \in B$ is the label of the edge $\left(q_{i-1}, q_{i}\right)$. For instance, if B is the formal language semiring $\operatorname{Lang}_{\Gamma}=\left(\mathcal{P}\left(\Gamma^{*}\right), \cup, \cdot, \emptyset,\{\varepsilon\}\right)$ and, for every two vertices $q, q^{\prime}$, we have $w\left(\left(q, q^{\prime}\right)\right) \subseteq \Gamma$, then the graph can be considered as a $\Gamma$-fsa $A$. Then $l_{q, q^{\prime}} \in \mathcal{P}\left(\Gamma^{*}\right)$ is the formal language recognized by $A$ starting from $q$ and ending in $q^{\prime}$. Another example is that B is the tropical semiring $\mathrm{Nat}_{\min ,+}=\left(\mathbb{N}_{\infty}, \min ,+, \infty, 0\right)$ and, for every two vertices $q, q^{\prime}$, the value $w\left(\left(q, q^{\prime}\right)\right)$ is the distance between $q$ and $q^{\prime}$ (which may be $\infty$ ). Then the value $l_{q, q^{\prime}} \in \mathbb{N}_{\infty}$ is the length of a shortest path from $q$ to $q^{\prime}$.

Since a wta $\mathcal{A}$ corresponds to an fta-hypergraph, the algebraic path problem can be generalized to an "algebraic fta-hyperpath problem" in order to capture the recursive structure of $\mathcal{A}$ (cf. [Knu77] where the fta-hypergraph is called superior context-free grammar, and each hyperedge is labeled by a superior function). Given an fta-hypergraph $\mathcal{A}$ and a node $q$, a $q$-hyperpath is an unfolding of the fta-hypergraph, starting in node $q$ and moving in the direction opposite to the direction of the hyperedges; this unfolding results in a $\Sigma$-tree $\xi$ (constituted by protocoling the labels of the visited hyperedges) together with a decoration of each position of $\xi$ by some node of the fta-hypergraph $\mathcal{A}$; we call this decoration a " $q$-run of $\mathcal{A}$ on $\xi "$. We note that the first node of a path in a graph (see above) disappears when generalizing to hyperpaths and fta-hypergraphs, because at each leaf of an fta-hyperpath a transition in $\delta_{0}$ of $\mathcal{A}$ is applied, which does not have source states.

Formally, let $\xi \in \mathrm{T}_{\Sigma}$. A run of $\mathcal{A}$ on $\xi$ is a mapping $\rho: \operatorname{pos}(\xi) \rightarrow Q$. If $\rho(\varepsilon)=q$ for some $q \in Q$, then $\rho$ is also called a $q$-run. The set of all runs of $\mathcal{A}$ on $\xi$ and the set of all $q$-runs of $\mathcal{A}$ on $\xi$ are denoted


Figure 3.2: An example element $\left(\left(\left.\xi\right|_{1},\left.\rho\right|_{1}\right),(\xi, \rho)\right)$ of $\prec$.
by $\mathrm{R}_{\mathcal{A}}(\xi)$ and $\mathrm{R}_{\mathcal{A}}(q, \xi)$, respectively. Each run $\rho \in \mathrm{R}_{\mathcal{A}}(\xi)$ determines, for each $w \in \operatorname{pos}(\xi)$, a unique transition, viz., if $k=\operatorname{rk}(\xi(w))$ and $\sigma=\xi(w)$, then $\rho$ determines the transition $(\rho(w 1) \cdots \rho(w k), \sigma, \rho(w))$. We call this the transition induced by $\rho$ on $\xi$ at $w$. For every $\rho \in \mathrm{R}_{\mathcal{A}}(\xi)$ and $w \in \operatorname{pos}(\xi)$, the run induced by $\rho$ at position $w$, denoted by $\left.\rho\right|_{w}$, is the run in $\mathrm{R}_{\mathcal{A}}\left(\left.\xi\right|_{w}\right)$ defined for every $w^{\prime} \in \operatorname{pos}\left(\left.\xi\right|_{w}\right)$ by $\left.\rho\right|_{w}\left(w^{\prime}\right)=\rho\left(w w^{\prime}\right)$.

Next we define the weight of a run $\rho \in \mathrm{R}_{\mathcal{A}}(\xi)$. For this, we will define a mapping $\mathrm{wt}_{\mathcal{A}}: C \rightarrow B$ by well-founded induction on $(C, \prec)$ (using Theorem 2.5.1 and (2.4)). One might be tempted to choose $C=\mathrm{R}_{\mathcal{A}}(\xi)$. However, this leads to an underspecification, because one can easily imagine a ranked alphabet $\Sigma$ and two trees $\xi_{1}, \xi_{2} \in \mathrm{~T}_{\Sigma}$ such that $\xi_{1} \neq \xi_{2}$ and $\operatorname{pos}\left(\xi_{1}\right)=\operatorname{pos}\left(\xi_{2}\right)$; then $\mathrm{R}_{\mathcal{A}}\left(\xi_{1}\right)=\mathrm{R}_{\mathcal{A}}\left(\xi_{2}\right)$ and the mapping $\mathrm{wt}_{\mathcal{A}}$ is not able to "see" the $\Sigma$-labels at the positions; but, of course, the weight of a run depends on these labels ${ }_{2}$ Thus, we let $C$ be the set of pairs where each pair consists of a tree $\xi \in \mathrm{T}_{\Sigma}$ and a run $\rho \in \mathrm{R}_{\mathcal{A}}(\xi)$; due to this origin, we denote $C$ by TR (tree- $r$ un); and we define an appropriate well-founded relation on this set of pairs.

Formally, we let $\mathrm{TR}=\left\{(\xi, \rho) \mid \xi \in \mathrm{T}_{\Sigma}, \rho \in \mathrm{R}_{\mathcal{A}}(\xi)\right\}$ and we define the binary relation $\prec$ on TR by

$$
\prec=\left\{\left(\left(\left.\xi\right|_{i},\left.\rho\right|_{i}\right),(\xi, \rho)\right) \mid(\xi, \rho) \in \mathrm{TR}, i \in[\operatorname{rk}(\xi(\varepsilon))]\right\} .
$$

Obviously, $\prec$ is well-founded and $\min _{\prec}(\mathrm{TR})=\left\{(\alpha, \rho) \mid \alpha \in \Sigma^{(0)}, \rho:\{\varepsilon\} \rightarrow Q\right\}$. The well-founded relation $\prec$ is illustrated in Figure 3.2,

We define the mapping

$$
\mathrm{wt}_{\mathcal{A}}: \mathrm{TR} \rightarrow B
$$

by induction on $(\mathrm{TR}, \prec)$ for every $\xi \in \mathrm{T}_{\Sigma}$ and $\rho \in \mathrm{R}_{\mathcal{A}}(\xi)$ by

$$
\begin{equation*}
\mathrm{wt}_{\mathcal{A}}(\xi, \rho)=\left(\bigotimes_{i \in[k]} \mathrm{wt}_{\mathcal{A}}\left(\left.\xi\right|_{i},\left.\rho\right|_{i}\right)\right) \otimes \delta_{k}(\rho(1) \cdots \rho(k), \sigma, \rho(\varepsilon)) \tag{3.1}
\end{equation*}
$$

where $k$ and $\sigma$ abbreviate $\operatorname{rk}(\xi(\varepsilon))$ and $\xi(\varepsilon)$, respectively. We call $\mathrm{wt}_{\mathcal{A}}(\xi, \rho)$ the weight of $\rho$ by $\mathcal{A}$ on $\xi$. If $\xi$ is uniquely determined by the context, then the phrase weight of $\rho$ by $\mathcal{A}$ means the value $\mathrm{wt}_{\mathcal{A}}(\xi, \rho)$. If the wta $\mathcal{A}$ is clear from the context, then we drop "by $\mathcal{A}$ " and the index $\mathcal{A}$ from the phrase "weight of $\rho$ by $\mathcal{A}$ on $\xi$ " and from the denotation $\mathrm{wt}_{\mathcal{A}}(\xi, \rho)$, respectively.

Intuitively, the run semantics of $\mathcal{A}$ on a tree $\xi \in \mathrm{T}_{\Sigma}$ is the finite summation of the weights of each run $\rho$ on $\xi$, which is additionally multiplied by the root weight $F_{\rho(\varepsilon)}$. Here we use the fact that the binary summation $\oplus$ of $B$ is extended in a unique way to the summation over the finite $\mathrm{R}_{\mathcal{A}}(\xi)$-family $\left(\operatorname{wt}(\xi, \rho) \otimes F_{\rho(\varepsilon)} \mid \rho \in \mathrm{R}_{\mathcal{A}}(\xi)\right)$ over $B$ (cf. page 21). Formally, the run semantics of $\mathcal{A}$, denoted by $\llbracket \mathcal{A} \rrbracket^{\mathrm{run}}$, is the weighted tree language $\llbracket \mathcal{A} \rrbracket^{\mathrm{run}}: \mathrm{T}_{\Sigma} \rightarrow B$ such that, for each $\xi \in \mathrm{T}_{\Sigma}$, we let

$$
\llbracket \mathcal{A} \rrbracket^{\mathrm{run}}(\xi)=\bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}}(\xi)} \operatorname{wt}(\xi, \rho) \otimes F_{\rho(\varepsilon)}
$$

[^9]Obviously, we have

$$
\llbracket \mathcal{A} \rrbracket^{\mathrm{run}}(\xi)=\bigoplus_{q \in Q} \bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}}(q, \xi)} \mathrm{wt}(\xi, \rho) \otimes F_{q}
$$

because the $Q$-indexed family $\left(\mathrm{R}_{\mathcal{A}}(q, \xi) \mid q \in Q\right)$ is a partitioning of $\mathrm{R}_{\mathcal{A}}(\xi)$.
Next we prove that the weight $\operatorname{wt}_{\mathcal{A}}(\xi, \rho)$ of a run $\rho$ is the product of the weights of the transitions induced by $\rho$ on $\xi$.

Observation 3.1.1. For every $\xi \in \mathrm{T}_{\Sigma}$ and $\rho \in \mathrm{R}_{\mathcal{A}}(\xi)$, we have

$$
\mathrm{wt}_{\mathcal{A}}(\xi, \rho)=\bigotimes_{\substack{u \in \operatorname{pos}(\xi) \\ \text { in } \leq \operatorname{dp} \text { order }}} \delta_{\operatorname{rk}(\xi(u))}(\rho(u 1) \cdots \rho(u \operatorname{rk}(\xi(u))), \xi(u), \rho(u))
$$

Proof. We prove the statement by induction on $\mathrm{T}_{\Sigma}$. Let $\xi=\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)$ and $\rho \in \mathrm{R}_{\mathcal{A}}(\xi)$. Then we can calculate as follows:

$$
\begin{aligned}
& \mathrm{wt}_{\mathcal{A}}(\xi, \rho)=\left(\bigotimes_{i \in[k]} \mathrm{wt}_{\mathcal{A}}\left(\xi_{i},\left.\rho\right|_{i}\right)\right) \otimes \delta_{k}(\rho(1) \cdots \rho(k), \sigma, \rho(\varepsilon)) \\
& \begin{array}{l}
=\bigotimes_{\substack{u \in \operatorname{pos}\left(\xi_{1}\right) \\
\text { in } \leq \operatorname{dp} \text { order }}} \delta_{\operatorname{rk}\left(\xi_{1}(u)\right)}\left(\left.\left.\rho\right|_{1}(u 1) \cdots \rho\right|_{1}\left(u \operatorname{rk}\left(\xi_{1}(u)\right)\right), \xi_{1}(u),\left.\rho\right|_{1}(u)\right) \\
\otimes \ldots \otimes \bigotimes_{\operatorname{rk}\left(\xi_{k}(u)\right)}\left(\left.\left.\rho\right|_{k}(u 1) \cdots \rho\right|_{k}\left(u \operatorname{rk}\left(\xi_{k}(u)\right)\right), \xi_{k}(u),\left.\rho\right|_{k}(u)\right)
\end{array} \\
& \begin{array}{c}
u \in \operatorname{pos}\left(\xi_{k}\right) \\
\text { in } \leq \mathrm{dp} \\
\text { order }
\end{array} \\
& \otimes \delta_{k}(\rho(1) \cdots \rho(k), \sigma, \rho(\varepsilon)) \\
& =\bigotimes_{u \in \operatorname{pos}(\xi)} \delta_{\operatorname{rk}(\xi(u))}(\rho(u 1) \cdots \rho(u \operatorname{rk}(\xi(u))), \xi(u), \rho(u)) \\
& \text { (by associativity of } \otimes \text { and definition of } \leq_{\mathrm{dp}} \text { ) }
\end{aligned}
$$

A weighted tree language $r: \mathrm{T}_{\Sigma} \rightarrow B$ is run recognizable over B (for short: run recognizable or r-recognizable) if there exists a $(\Sigma, \mathrm{B})$-wta $\mathcal{A}$ such that $r=\llbracket \mathcal{A} \rrbracket^{\text {run }}$. The set of all $(\Sigma, \mathrm{B})$-weighted tree languages which are run recognizable over $B$, is denoted by $\operatorname{Rec}^{r u n}(\Sigma, B)$. In the obvious way, we can define the notions of bu deterministically r-recognizable over B and crisp deterministically r-recognizable over B . We denote the corresponding sets of all such weighted tree languages by $\operatorname{bud}-\operatorname{Rec}^{\text {run }}(\Sigma, \mathrm{B})$ and $\operatorname{cd}-\operatorname{Rec}^{\text {run }}(\Sigma, \mathrm{B})$, respectively.

Two ( $\Sigma, \mathrm{B}$ )-wta $\mathcal{A}_{1}$ and $\mathcal{A}_{2}$ are run equivalent (r-equivalent) if $\llbracket \mathcal{A}_{1} \rrbracket^{\text {run }}=\llbracket \mathcal{A}_{2} \rrbracket^{\text {run }}$.
The following result is an obvious consequence of the corresponding definitions.
Lemma 3.1.2. Let B be bi-locally finite and $\mathcal{A}$ be a $(\Sigma, \mathrm{B})$-wta $\mathcal{A}$. Then $\operatorname{im}\left(\llbracket \mathcal{A} \rrbracket^{\mathrm{run}}\right)$ is finite.

Proof. Let $\mathcal{A}=(Q, \delta, F)$. For each $\xi \in \mathrm{T}_{\Sigma}$, we have $\mathrm{wt}(\xi, \rho) \otimes F_{\rho(\varepsilon)} \in\langle\operatorname{wts}(\mathcal{A})\rangle_{\{\otimes\}}$. Thus $\llbracket \mathcal{A} \rrbracket^{\text {run }}(\xi)=$ $\bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}}(\xi)} \mathrm{wt}(\xi, \rho) \otimes F_{\rho(\varepsilon)} \subseteq\left\langle\langle\operatorname{wts}(\mathcal{A})\rangle_{\{\otimes\}}\right\rangle_{\{\oplus\}}$. This latter set is finite because B is bi-locally finite. Hence $\operatorname{im}\left(\llbracket \mathcal{A} \rrbracket^{\text {run }}\right)$ is also finite.

In Theorem 16.2 .7 we will prove the converse result: if for each $(\Sigma, \mathrm{B})$-wta $\mathcal{A}$ the set $\operatorname{im}\left(\llbracket \mathcal{A} \rrbracket^{\text {run }}\right)$ is finite, then $B$ is bi-locally finite.

Initial algebra semantics. The heart of the initial algebra semantics GTWW77 of a ( $\Sigma, \mathrm{B}$ )-wta $\mathcal{A}$ is a particular $\Sigma$-algebra, called the vector algebra of $\mathcal{A}$; it is tailormade for $\mathcal{A}$. Due to the uniqueness of the $\Sigma$-algebra homomorphism from the $\Sigma$-term algebra $\mathrm{T}_{\Sigma}=\left(\mathrm{T}_{\Sigma}, \theta_{\Sigma}\right)$ to the vector algebra of $\mathcal{A}$, each $\Sigma$-tree $\xi$ can be interpreted (or: evaluated) in a unique way in the vector algebra. Eventually, the interpretation of $\xi$ is modified with root weights. On first glance, the definition of the vector algebra of a wta below might look a bit artificial. However, conceptually, this definition results from the straightforward application of the two steps described at the beginning of this chapter to an fta and its initial algebra semantics.

Formally, the vector algebra of $\mathcal{A}$ is the $\Sigma$-algebra $\bigvee(\mathcal{A})=\left(B^{Q}, \delta_{\mathcal{A}}\right)$ where, for every $k \in \mathbb{N}, \sigma \in \Sigma^{(k)}$, the $k$-ary operation $\delta_{\mathcal{A}}(\sigma): B^{Q} \times \cdots \times B^{Q} \rightarrow B^{Q}$ is defined by

$$
\begin{equation*}
\delta_{\mathcal{A}}(\sigma)\left(v_{1}, \ldots, v_{k}\right)_{q}=\bigoplus_{q_{1} \cdots q_{k} \in Q^{k}}\left(\bigotimes_{i \in[k]}\left(v_{i}\right)_{q_{i}}\right) \otimes \delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right) \tag{3.2}
\end{equation*}
$$

for every $v_{1}, \ldots, v_{k} \in B^{Q}$ and $q \in Q$.
We abbreviate $\mathrm{h}_{\mathrm{V}_{(\mathcal{A})}}$ by $\mathrm{h}_{\mathcal{A}}$, i.e., we denote the unique $\Sigma$-algebra homomorphism from the $\Sigma$-term algebra $\mathrm{T}_{\Sigma}$ to the vector algebra $\mathrm{V}(\mathcal{A})$ by $\mathrm{h}_{\mathcal{A}}$. Then, for every $\xi=\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)$ in $\mathrm{T}_{\Sigma}$ and $q \in Q$, we have

$$
\begin{aligned}
\mathrm{h}_{\mathcal{A}}\left(\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)\right)_{q} & =\mathrm{h}_{\mathcal{A}}\left(\theta_{\Sigma}(\sigma)\left(\xi_{1}, \ldots, \xi_{k}\right)\right)_{q}=\delta_{\mathcal{A}}(\sigma)\left(\mathrm{h}_{\mathcal{A}}\left(\xi_{1}\right), \ldots, \mathrm{h}_{\mathcal{A}}\left(\xi_{k}\right)\right)_{q} \\
& =\bigoplus_{q_{1} \cdots q_{k} \in Q^{k}}\left(\bigotimes_{i \in[k]} \mathrm{h}_{\mathcal{A}}\left(\xi_{i}\right)_{q_{i}}\right) \otimes \delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right),
\end{aligned}
$$

where $\theta_{\Sigma}(\sigma)$ is the operation of the $\Sigma$-term algebra associated to $\sigma$; the second equality holds, because $\mathrm{h}_{\mathcal{A}}$ is a $\Sigma$-algebra homomorphism. In the subsequent proofs, we will show only the first and the last expressions of the above calculation. For the particular case that $k=0$, we obtain the following:

$$
\mathrm{h}_{\mathcal{A}}(\sigma)_{q}=\mathrm{h}_{\mathcal{A}}\left(\theta_{\Sigma}(\sigma)()\right)_{q}=\delta_{\mathcal{A}}(\sigma)()_{q}=\bigoplus_{\varepsilon \in Q^{0}} \mathbb{1} \otimes \delta_{0}(\varepsilon, \sigma, q)=\delta_{0}(\varepsilon, \sigma, q)
$$

and thus we will write $\mathrm{h}_{\mathcal{A}}(\sigma)_{q}=\delta_{0}(\varepsilon, \sigma, q)$ directly.
The initial algebra semantics of $\mathcal{A}$, denoted by $\llbracket \mathcal{A} \rrbracket^{\text {init }}$, is the weighted tree language $\llbracket \mathcal{A} \rrbracket^{\text {init }}: \mathrm{T}_{\Sigma} \rightarrow B$ defined for every $\xi \in \mathrm{T}_{\Sigma}$ by

$$
\llbracket \mathcal{A} \rrbracket^{\text {init }}(\xi)=\bigoplus_{q \in Q} \mathrm{~h}_{\mathcal{A}}(\xi)_{q} \otimes F_{q}
$$

The vector algebra $\mathrm{V}(\mathcal{A})$ may contain $Q$-vectors which are not accessible by $\mathcal{A}$, i.e., in general, we have $B^{Q} \backslash \operatorname{im}\left(\mathrm{~h}_{\mathcal{A}}\right) \neq \emptyset$. Next we define a $\Sigma$-algebra which is a subalgebra of $\mathrm{V}(\mathcal{A})$ and has exactly $\operatorname{im}\left(\mathrm{h}_{\mathcal{A}}\right)$ as carrier set. We define the accessible subalgebra of $\mathrm{V}(\mathcal{A})$, denoted by $\operatorname{aV}(\mathcal{A})$, to be the $\Sigma$-algebra

$$
\mathrm{a} \mathrm{~V}(\mathcal{A})=\left(\operatorname{im}\left(\mathrm{h}_{\mathcal{A}}\right), \delta_{\mathrm{aV}(\mathcal{A})}\right)
$$

where for each $k \in \mathbb{N}$ and $\sigma \in \Sigma^{(k)}$, the operation $\delta_{\mathrm{aV}(\mathcal{A})}(\sigma)$ is the restriction of $\delta_{\mathcal{A}}(\sigma)$ to $\operatorname{im}\left(\mathrm{h}_{\mathcal{A}}\right)^{k}$. By Observation 2.9.4, $\mathrm{a} \mathrm{V}(\mathcal{A})$ is the smallest subalgebra of $\mathrm{V}(\mathcal{A})$.

We denote the unique $\Sigma$-algebra homomorphism from $\mathrm{T}_{\Sigma}$ to $\mathrm{aV}(\mathcal{A})$ by $\mathrm{h}_{\mathrm{aV}}(\mathcal{A})$. Since $\mathrm{aV}(\mathcal{A})$ is a subalgebra of $\mathrm{V}(\mathcal{A})$, the homomorphism $\mathrm{h}_{\mathrm{a} \mathrm{V}(\mathcal{A})}$ is a $\Sigma$-algebra homomorphism also from $\mathrm{T}_{\Sigma}$ to $\mathrm{V}(\mathcal{A})$, and since $T_{\Sigma}$ is initial, we have

$$
\begin{equation*}
\mathrm{h}_{\mathrm{a}(\mathcal{A})}(\xi)=\mathrm{h}_{\mathcal{A}}(\xi) \text { for every } \xi \in \mathrm{T}_{\Sigma} \tag{3.3}
\end{equation*}
$$

Then we can express the initial algebra semantics $\llbracket \mathcal{A} \rrbracket^{\text {init }}$ also in terms of the homomorphism $\mathrm{h}_{\mathrm{a}}(\mathcal{A})$ as follows:

$$
\begin{equation*}
\llbracket \mathcal{A} \rrbracket^{\text {init }}=F^{\prime} \circ \mathrm{h}_{\mathrm{aV}(\mathcal{A})} \tag{3.4}
\end{equation*}
$$

where $F^{\prime}: \operatorname{im}\left(\mathrm{h}_{\mathcal{A}}\right) \rightarrow B$ is the mapping defined by $F^{\prime}(u)=\bigoplus_{q \in Q} u_{q} \otimes F_{q}$ for every $u \in \operatorname{im}\left(\mathrm{~h}_{\mathcal{A}}\right)$. Indeed, for each $\xi \in \mathrm{T}_{\Sigma}$, we have

$$
\llbracket \mathcal{A} \rrbracket^{\text {init }}(\xi)=\bigoplus_{q \in Q} \mathrm{~h}_{\mathcal{A}}(\xi)_{q} \otimes F_{q} \stackrel{\boxed{(3.3)}}{=} \bigoplus_{q \in Q} \mathrm{~h}_{\mathrm{aV}(\mathcal{A})}(\xi)_{q} \otimes F_{q}=\left(F^{\prime} \circ \mathrm{h}_{\mathrm{aV}(\mathcal{A})}\right)(\xi)
$$

By Theorem 2.6.4, the kernel $\operatorname{ker}\left(\mathrm{h}_{\mathcal{A}}\right)$ is a congruence relation on the $\Sigma$-term algebra $\left(\mathrm{T}_{\Sigma}, \theta_{\Sigma}\right) 3^{3}$
Lemma 3.1.3. $\operatorname{aV}(\mathcal{A}) \cong \mathrm{T}_{\Sigma} / \operatorname{ker}\left(\mathrm{h}_{\mathcal{A}}\right)$, i.e., the accessible subalgebra of $\mathrm{V}(\mathcal{A})$ is isomorphic to the quotient algebra of the $\Sigma$-term algebra modulo $\operatorname{ker}\left(\mathrm{h}_{\mathcal{A}}\right)$.

Proof. Since $\mathrm{h}_{\mathrm{aV}(\mathcal{A})}$ is a surjective $\Sigma$-algebra homomorphism from the $\Sigma$-term algebra to $\mathrm{aV}(\mathcal{A})$, Theorem 2.6.4 implies $\operatorname{aV}(\mathcal{A}) \cong \mathrm{T}_{\Sigma} / \operatorname{ker}\left(\mathrm{h}_{\mathrm{aV}(\mathcal{A})}\right)$. Since (3.3) implies $\operatorname{ker}\left(\mathrm{h}_{\mathcal{A}}\right)=\operatorname{ker}\left(\mathrm{h}_{\mathrm{aV}}(\mathcal{A})\right)$, we obtain $\mathrm{aV}(\mathcal{A}) \cong$ $\mathrm{T}_{\Sigma} / \operatorname{ker}\left(\mathrm{h}_{\mathcal{A}}\right)$.

A weighted tree language $r: \mathrm{T}_{\Sigma} \rightarrow B$ is initial algebra recognizable over B (for short: initial algebra recognizable or i-recognizable) if there exists a ( $\Sigma, \mathrm{B}$ )-wta $\mathcal{A}$ such that $r=\llbracket \mathcal{A} \rrbracket^{\text {init }}$. The set of all weighted tree languages over $\Sigma$ and B which are i-recognizable, is denoted by $\operatorname{Rec}^{i \text { init }}(\Sigma, B)$. In an obvious way, we can define the notion of bu deterministically i-recognizable weighted tree language and crisp deterministically i-recognizable weighted tree language. We denote the corresponding sets of all such weighted tree languages by bud- $\operatorname{Rec}^{\text {init }}(\Sigma, \mathrm{B})$ and $\operatorname{cd}-\operatorname{Rec}^{\text {init }}(\Sigma, \mathrm{B})$, respectively.

Let $\mathcal{A}_{1}$ and $\mathcal{A}_{2}$ be ( $\Sigma, \mathrm{B}$ )-wta. We say that $\mathcal{A}_{1}$ and $\mathcal{A}_{2}$ are initial algebra equivalent (i-equivalent) if $\llbracket \mathcal{A}_{1} \rrbracket^{\text {init }}=\llbracket \mathcal{A}_{2} \rrbracket^{\text {init }}$. Moreover, $\mathcal{A}_{1}$ and $\mathcal{A}_{2}$ are equivalent if they are both r-equivalent and i-equivalent.

The following result is an obvious consequence of the corresponding definitions.
Lemma 3.1.4. Let B be locally finite and $\mathcal{A}$ be a $(\Sigma, \mathrm{B})$-wta $\mathcal{A}$. Then $\operatorname{im}\left(\llbracket \mathcal{A} \rrbracket^{\mathrm{init}}\right)$ is finite.
Proof. Let $\mathcal{A}=(Q, \delta, F)$. For each $\xi \in \mathrm{T}_{\Sigma}$ and $q \in Q$, we have that $\llbracket \mathcal{A} \rrbracket^{\mathrm{init}}(\xi) \in\langle\operatorname{wts}(\mathcal{A})\rangle_{\{\oplus, \otimes\}}$. This latter set is finite because B is locally finite. Hence $\operatorname{im}\left(\llbracket \mathcal{A} \rrbracket^{\text {init }}\right)$ is also finite.

In Theorem 16.1 .6 we will prove the converse result: if for each $(\Sigma, \mathrm{B})$-wta $\mathcal{A}$ the set $\operatorname{im}\left(\llbracket \mathcal{A} \rrbracket^{\text {init }}\right)$ is finite, then $B$ is locally finite.

We finish this section with an interesting property of the initial algebra semantics. This result will be applied in several proofs.

Theorem 3.1.5. Rad10, Lm. 6.1] Let $A \subseteq B$ be a finite subset. If $\left|\Sigma^{(0)}\right| \geq|A \cup\{\mathbb{0}, \mathbb{1}\}|$ and $\left|\Sigma^{(2)}\right| \geq 2$, then we can construct a $(\Sigma, \mathrm{B})$-wta $\mathcal{A}$ such that $\operatorname{im}\left(\llbracket \mathcal{A} \rrbracket^{\text {init }}\right)=\langle A\rangle_{\{\oplus, \otimes, 0, \mathbb{1}\}}$. In particular, if B is generated by $A$, then we obtain $\operatorname{im}\left(\llbracket \mathcal{A} \rrbracket^{\text {init }}\right)=B$.

Proof. Let us abbreviate $\langle A\rangle_{\{\oplus, \otimes, 0, \mathbb{1}\}}$ by $\langle A\rangle$. By our assumption on $\Sigma$, we may also assume that $A \cup$ $\{\mathbb{O}, \mathbb{1}\} \subseteq \Sigma^{(0)}$ and $\left\{+^{(2)}, \times^{(2)}\right\} \subseteq \Sigma^{(2)}$.

Now we construct the $(\Sigma, \mathrm{B})$-wta $\mathcal{A}=(Q, \delta, F)$ as follows. We let $Q=\{v, 1\}, F_{v}=\mathbb{1}$, and $F_{1}=\mathbb{0}$. Moreover, we define $\delta_{0}(\varepsilon, a, v)=a$ and $\delta_{0}(\varepsilon, a, 1)=\mathbb{1}$ for each $a \in A \cup\{\mathbb{0}, \mathbb{1}\}$, and for every $p, q, r \in Q$ we define

$$
\delta_{2}(p q,+, r)= \begin{cases}\mathbb{1} & \text { if }(p q, r) \in\{(11,1),(v 1, v),(1 v, v)\} \\ \mathbb{0} & \text { otherwise }\end{cases}
$$

and

$$
\delta_{2}(p q, \times, r)= \begin{cases}\mathbb{1} & \text { if }(p q, r) \in\{(11,1),(v v, v)\} \\ \mathbb{0} & \text { otherwise }\end{cases}
$$

[^10]Lastly, for every $k \in \mathbb{N}, \sigma \in \Sigma^{(k)}$ such that $\sigma \notin A \cup\{\mathbb{0}, \mathbb{1},+, \times\}$, and $q_{1}, \ldots, q_{k}, q \in Q$, we let $\delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right)=a$ for an arbitrary $a \in A \cup\{\mathbb{O}, \mathbb{1}\}$.

Since $\operatorname{wts}(\mathcal{A}) \subseteq A \cup\{\mathbb{0}, \mathbb{1}\}$, we have $\operatorname{im}\left(\llbracket \mathcal{A} \rrbracket^{\text {init }}\right) \subseteq\langle A\rangle$.
As preparation for the proof of the other inclusion, we define the relation $\prec$ on $\langle A\rangle$ as follows. For each $a \in\langle A\rangle$, let $i(a) \in \mathbb{N}$ be the minimal number of operations in $\{\oplus, \otimes\}$ which must be applied to elements of $A \cup\{\mathbb{0}, \mathbb{1}\}$ in order to produce $a$. Then, for each $a, b \in\langle A\rangle$, we define $a \prec b$ if $i(a)<i(b)$. It is easy to see that $\prec$ is a well-founded relation on $\langle A\rangle$ with $\min _{\prec}(\langle A\rangle)=A \cup\{\mathbb{D}, \mathbb{1}\}$. By induction on $(\langle A\rangle, \prec)$, we prove that the following statement holds:

$$
\begin{equation*}
\text { For each } a \in\langle A\rangle \text {, there exists } \xi \in \mathrm{T}_{\Sigma} \text { such that } \mathrm{h}_{\mathcal{A}}(\xi)_{v}=a \text { and } \mathrm{h}_{\mathcal{A}}(\xi)_{1}=\mathbb{1} . \tag{3.5}
\end{equation*}
$$

I.B.: Let $a \in A \cup\{\mathbb{O}, \mathbb{1}\}$. Then with $\xi=a$ we have $\mathrm{h}_{\mathcal{A}}(a)_{v}=\delta_{0}(\varepsilon, a, v)=a$ and $\mathrm{h}_{\mathcal{A}}(\xi)_{1}=\delta_{0}(\varepsilon, a, 1)=$ 1.
I.S.: Now let $a=a_{1} \oplus a_{2}$. By the I.H., for each $i \in\{1,2\}$, there exists $\xi_{i} \in \mathrm{~T}_{\Sigma}$ such that $\mathrm{h}_{\mathcal{A}}\left(\xi_{i}\right)_{v}=a_{i}$ and $\mathrm{h}_{\mathcal{A}}\left(\xi_{i}\right)_{1}=\mathbb{1}$. We let $\xi=+\left(\xi_{1}, \xi_{2}\right)$. Then

$$
\begin{aligned}
\mathrm{h}_{\mathcal{A}}\left(+\left(\xi_{1}, \xi_{2}\right)\right)_{v} & =\bigoplus_{p, q \in Q} \mathrm{~h}_{\mathcal{A}}\left(\xi_{1}\right)_{p} \otimes \mathrm{~h}_{\mathcal{A}}\left(\xi_{2}\right)_{q} \otimes \delta_{2}(p q,+, v) \\
& =\left(\mathrm{h}_{\mathcal{A}}\left(\xi_{1}\right)_{v} \otimes \mathrm{~h}_{\mathcal{A}}\left(\xi_{2}\right)_{1} \otimes \delta_{2}(v 1,+, v)\right) \oplus\left(\mathrm{h}_{\mathcal{A}}\left(\xi_{1}\right)_{1} \otimes \mathrm{~h}_{\mathcal{A}}\left(\xi_{2}\right)_{v} \otimes \delta_{2}(1 v,+, v)\right) \\
& =\left(a_{1} \otimes \mathbb{1} \otimes \mathbb{1}\right) \oplus\left(\mathbb{1} \otimes a_{2} \otimes \mathbb{1}\right)=a_{1} \oplus a_{2}=a
\end{aligned}
$$

and $\mathrm{h}_{\mathcal{A}}\left(+\left(\xi_{1}, \xi_{2}\right)\right)_{1}=\mathrm{h}_{\mathcal{A}}\left(\xi_{1}\right)_{1} \otimes \mathrm{~h}_{\mathcal{A}}\left(\xi_{2}\right)_{1} \otimes \delta_{2}(11,+, 1)=\mathbb{1}$.
Next let $a=a_{1} \otimes a_{2}$. As before, for each $i \in\{1,2\}$, there exists $\xi_{i} \in \mathrm{~T}_{\Sigma}$ such that $\mathrm{h}_{\mathcal{A}}\left(\xi_{i}\right)_{v}=a_{i}$ and $\mathrm{h}_{\mathcal{A}}\left(\xi_{i}\right)_{1}=\mathbb{1}$. We let $\xi=\times\left(\xi_{1}, \xi_{2}\right)$. Then

$$
\begin{aligned}
\mathrm{h}_{\mathcal{A}}\left(\times\left(\xi_{1}, \xi_{2}\right)\right)_{v} & =\bigoplus_{p, q \in Q} \mathrm{~h}_{\mathcal{A}}\left(\xi_{1}\right)_{p} \otimes \mathrm{~h}_{\mathcal{A}}\left(\xi_{2}\right)_{q} \otimes \delta_{2}(p q, \times, v) \\
& =\mathrm{h}_{\mathcal{A}}\left(\xi_{1}\right)_{v} \otimes \mathrm{~h}_{\mathcal{A}}\left(\xi_{2}\right)_{v} \otimes \delta_{2}(v v, \times, v) \\
& =a_{1} \otimes a_{2} \otimes \mathbb{1}=a
\end{aligned}
$$

and $\mathrm{h}_{\mathcal{A}}\left(\times\left(\xi_{1}, \xi_{2}\right)\right)_{1}=\mathbb{1}$ as above. This proves (3.5).
Finally, we note that, for each $\xi \in \mathrm{T}_{\Sigma}$, we have

$$
\llbracket \mathcal{A} \rrbracket^{\text {init }}(\xi)=\left(\mathrm{h}_{\mathcal{A}}(\xi)_{v} \otimes F_{v}\right) \oplus\left(\mathrm{h}_{\mathcal{A}}(\xi)_{1} \otimes F_{1}\right)=\mathrm{h}_{\mathcal{A}}(\xi)_{v}
$$

Hence, by (3.5), for each $a \in\langle A\rangle$, there exists $\xi \in \mathrm{T}_{\Sigma}$ such that $\llbracket \mathcal{A} \rrbracket^{\text {init }}(\xi)=a$. This proves $\langle A\rangle \subseteq$ $\operatorname{im}\left(\llbracket \mathcal{A} \rrbracket^{\text {init }}\right)$, i.e., that $\langle A\rangle=\operatorname{im}\left(\llbracket \mathcal{A} \rrbracket^{\text {init }}\right)$.

A consequence of Lemma 3.1.2 and Theorem 3.1.5 is the following result.
Theorem 3.1.6. Let $\mathrm{B}=(B, \oplus, \otimes, \mathbb{O}, \mathbb{1})$ be bi-locally finite and not locally finite. Moreover, let $A \subseteq B$ be a finite subset such that $\langle A\rangle_{\{\oplus, \otimes, \mathbb{0}, \mathbb{1}\}}$ is an infinite set. If $\left|\Sigma^{(0)}\right| \geq|A \cup\{0, \mathbb{1}\}|$ and $\left|\Sigma^{(2)}\right| \geq 2$, then we can construct a $(\Sigma, \mathrm{B})$-wta $\mathcal{A}$ such that $\llbracket \mathcal{A} \rrbracket^{\text {init }} \notin \operatorname{Rec}^{\text {run }}(\Sigma, \mathrm{B})$. Hence, for such a $\Sigma$, we have $\operatorname{Rec}^{\text {init }}(\Sigma, \mathrm{B}) \backslash \operatorname{Rec}^{\text {run }}(\Sigma, \mathrm{B}) \neq \emptyset$ 。

Proof. By Theorem 3.1.5 we can construct a $(\Sigma, \mathrm{B})$-wta $\mathcal{A}$ such that $\left.\operatorname{im}(\llbracket \mathcal{A}]^{\text {init }}\right)=\langle A\rangle_{\{\oplus, \odot, 0,1\}}$. Thus $\operatorname{im}\left(\llbracket \mathcal{A} \rrbracket^{\text {init }}\right)$ is an infinite set. On the other hand, by Lemma 3.1.2 $\operatorname{im}(r)$ is finite for each $r \in \operatorname{Rec}^{\text {run }}(\Sigma, \mathrm{B})$. Hence $\operatorname{im}\left(\llbracket \mathcal{A} \rrbracket^{\text {init }}\right) \notin \operatorname{Rec}^{\text {run }}(\Sigma, \mathrm{B})$.

We show three applications of Theorem 3.1.6. First, we consider the strong bimonoid Trunc ${ }_{\lambda}$ from Example 2.6.10(2) which is not a semiring. We recall that $\operatorname{Trunc}_{\lambda}=(B, \oplus, \odot, 0,1)$ where $\lambda \in \mathbb{R}$ and $B=\{0\} \cup\{b \in \mathbb{R} \mid \lambda \leq b \leq 1\} ;$ Trunc $_{\lambda}$ is bi-locally finite and not locally finite. In fact, for $\lambda=\frac{1}{4}$, in Example 2.6.10(2) an infinite family $\left(b_{i} \mid \in i \in \mathbb{N}\right)$ of elements of $B$ is given which is generated by $A=\left\{\frac{1}{2}\right\}$. Thus, for each ranked alphabet $\Sigma$ with $\left|\Sigma^{(0)}\right| \geq 3$ and $\left|\Sigma^{(2)}\right| \geq 2$, Theorem 3.1.6 implies that $\operatorname{Rec}^{\text {init }}\left(\Sigma, \operatorname{Trunc}_{\frac{1}{4}}\right) \backslash \operatorname{Rec}^{\text {run }}\left(\Sigma, \operatorname{Trunc}_{\frac{1}{4}}\right) \neq \emptyset$.

Second, we consider the bounded lattice $\mathrm{FL}(2+2)$ in Example 2.6.15(9). This lattice is infinite and freely generated by the two chains $a<b$ and $c<d$, i.e., by four elements (cf. Figure 2.4). By Observation 2.6.13, the bounded lattice $\mathrm{FL}(2+2)$ is a bi-locally finite strong bimonoid. Thus, for each ranked alphabet $\Sigma$ with $\left|\Sigma^{(0)}\right| \geq 6$ and $\left|\Sigma^{(2)}\right| \geq 2$, Theorem 3.1.6implies that $\operatorname{Rec}^{\text {init }}(\Sigma, \operatorname{FL}(2+2)) \backslash \operatorname{Rec}^{\text {run }}(\Sigma, \operatorname{FL}(2+2)) \neq \emptyset$.

Third, we consider the strong bimonoid $\operatorname{Stb}=(\mathbb{N}, \oplus, \odot, 0,1)$ from Example 2.6.10(9). We saw that Stb is bi-locally finite and not locally finite. Moreover, $\langle\{2\}\rangle_{\{\oplus, \odot, 0,1\}}=\mathbb{N}$. Thus, by Theorem 3.1.6, for each ranked alphabet $\Sigma$ with $\left|\Sigma^{(0)}\right| \geq 3$ and $\left|\Sigma^{(2)}\right| \geq 2$, we have that $\operatorname{Rec}^{\text {init }}(\Sigma, \operatorname{Stb}) \backslash \operatorname{Rec}^{\text {run }}(\Sigma, \operatorname{Stb}) \neq \emptyset$. Later a stronger version of this statement will be shown (cf. Theorem 5.2.5 in which we make a weaker assumption on $\Sigma$ ).

### 3.2 Examples

Here we list a number of weighted tree languages and show how they can be recognized by wta.
Example 3.2.1. (Number of accepting runs of an fta on a tree.) Let $A=(Q, \delta, F)$ be a $\Sigma$-fta. We recall that $\mathrm{R}_{A}(\xi), \mathrm{R}_{A}^{\mathrm{v}}(\xi)$, and $\mathrm{R}_{A}^{\mathrm{a}}(\xi)$ denote the set of runs, valid runs, and accepting runs of $A$ on a tree $\xi$, respectively. We consider the mapping

$$
\#_{\mathrm{R}_{A}^{\mathrm{a}}}: \mathrm{T}_{\Sigma} \rightarrow \mathbb{N} \quad \text { with } \quad \#_{\mathrm{R}_{A}^{\mathrm{a}}}(\xi)=\left|\mathrm{R}_{A}^{\mathrm{a}}(\xi)\right| \quad \text { for each } \xi \in \mathrm{T}_{\Sigma}
$$

We call $\#_{\mathrm{R}_{A}^{\mathrm{a}}}$ the multiplicity mapping of $A$ (cf. Eil74, Sect. VI.1]).
As weight algebra we use the semiring $N a t=(\mathbb{N},+, \cdot, 0,1)$ of natural numbers. We construct a $(\Sigma$, Nat $)$-wta $\mathcal{A}=\left(Q, \delta^{\prime}, F^{\prime}\right)$ which r-recognizes and also i-recognizes $\#_{\mathrm{R}_{A}^{\mathrm{a}}}$, as follows:

- for each $k \in \mathbb{N}$, we have $\left(\delta^{\prime}\right)_{k}=\chi\left(\delta_{k}\right)$ and
- $F^{\prime}=\chi(F)$.

Obviously, $\mathcal{A}$ has identity transition weights and identity root weights.
Next we show that $\mathcal{A}$ r-recognizes $\#_{\mathrm{R}_{A}^{\mathrm{a}}}$, i.e., $\llbracket \mathcal{A} \rrbracket^{\text {run }}=\#_{\mathrm{R}_{A}^{\mathrm{a}}}$. We observe that, for each $\xi \in \mathrm{T}_{\Sigma}$, we have $\mathrm{R}_{\mathcal{A}}(\xi)=\mathrm{R}_{A}(\xi)$. By induction on (TR, $\left.\prec\right)$ (defined on page 64), we prove that, for each $(\xi, \rho) \in \mathrm{TR}$, we have

$$
\mathrm{wt}(\xi, \rho)= \begin{cases}1 & \text { if } \rho \in \mathrm{R}_{A}^{\mathrm{v}}(\xi)  \tag{3.6}\\ 0 & \text { otherwise }\end{cases}
$$

I.B.: Let $\xi=\alpha$ for some $\alpha \in \Sigma^{(0)}$, and let $\rho:\{\varepsilon\} \rightarrow Q$. Then $\mathrm{wt}(\xi, \rho)=\left(\delta^{\prime}\right)_{0}(\varepsilon, \alpha, \rho(\varepsilon))$ by definition of wt. We proceed by case analysis.

Case (a): Let $\rho \in \mathrm{R}_{A}^{\mathrm{v}}(\alpha)$. Then $(\varepsilon, \alpha, \rho(\varepsilon)) \in \delta_{0}$ and by the definition of $\left(\delta^{\prime}\right)_{0}$ we have $\left.\left(\delta^{\prime}\right) \overline{0_{0}(\varepsilon, \alpha, \rho(\varepsilon)}\right)=1$. Thus wt $(\xi, \rho)=1$.

Case (b): Let $\rho \notin \mathrm{R}_{A}^{\mathrm{v}}(\alpha)$. Then $(\varepsilon, \alpha, \rho(\varepsilon)) \notin \delta_{0}$ and we have $\left(\delta^{\prime}\right)_{0}(\varepsilon, \alpha, \rho(\varepsilon))=0$, i.e., wt $(\xi, \rho)=0$.
I.S.: Let $(\xi, \rho) \in \mathrm{TR}$ such that $\xi=\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)$ for some $k \in \mathbb{N}_{+}$. Then

$$
\operatorname{wt}(\xi, \rho)=\left(\operatorname{wt}\left(\xi_{1},\left.\rho\right|_{1}\right) \cdot \ldots \cdot \operatorname{wt}\left(\xi_{k},\left.\rho\right|_{k}\right)\right) \cdot\left(\delta^{\prime}\right)_{k}(\rho(1) \cdots \rho(k), \sigma, \rho(\varepsilon))
$$

by definition of wt. Again, we proceed by case analysis.

Case (a): Let $\rho \in \mathrm{R}_{A}^{\mathrm{v}}(\xi)$. Then, for each $i \in[k]$, we have $\left.\rho\right|_{i} \in \mathrm{R}_{A}^{\mathrm{v}}\left(\xi_{i}\right)$, and $(\rho(1) \cdots \rho(k), \sigma, \rho(\varepsilon)) \in$ $\delta_{k}$. Hence, by I.H., we have $\operatorname{wt}\left(\xi_{i},\left.\rho\right|_{i}\right)=1$ for each $i \in[k]$ and, by construction, we have $\left(\delta^{\prime}\right)_{k}(\rho(1) \cdots \rho(k), \sigma, \rho(\varepsilon))=1$. Thus wt $(\xi, \rho)=1$.

Case (b1): Let $\rho \notin \mathrm{R}_{A}^{\mathrm{v}}(\xi)$ and $(\forall i \in[k]):\left.\rho\right|_{i} \in \mathrm{R}_{A}^{\mathrm{v}}\left(\xi_{i}\right)$. Then $(\rho(1) \cdots \rho(k), \sigma, \rho(\varepsilon)) \notin \delta_{k}$, i.e., $\left.\left(\delta^{\prime}\right) \overline{k(\rho(1) \cdots \rho}(k), \sigma, \rho(\varepsilon)\right)=0$. Hence $\operatorname{wt}(\xi, \rho)=0$.
Case (b2): $\rho \notin \mathrm{R}_{A}^{\mathrm{v}}(\xi)$ and $(\exists i \in[k]):\left.\rho\right|_{i} \notin \mathrm{R}_{A}^{\mathrm{v}}\left(\xi_{i}\right)$. Then, by I.H., $\mathrm{wt}\left(\xi_{i},\left.\rho\right|_{i}\right)=0$ and hence $\mathrm{wt}_{\mathcal{A}} \overline{\mathcal{A}}(\xi, \rho)=0$. This finishes the proof of (3.6).

Hence, for each $\xi \in \mathrm{T}_{\Sigma}$,
i.e., $\llbracket \mathcal{A} \rrbracket^{\text {run }}(\xi)$ is the number of accepting runs of $A$ on $\xi$. Hence $\#_{\mathrm{R}_{A}^{\mathrm{a}}} \in \operatorname{Rec}^{\mathrm{run}}(\Sigma, \mathrm{Nat})$.

Next we show that $\mathcal{A}$ also i-recognizes $\#_{\mathrm{R}_{A}^{\mathrm{a}}}$, i.e., $\llbracket \mathcal{A} \rrbracket^{\text {init }}=\#_{\mathrm{R}_{A}^{\mathrm{a}}}$. By induction on $\mathrm{T}_{\Sigma}$, we prove that the following statement holds:

$$
\begin{equation*}
\text { For every } \xi \in \mathrm{T}_{\Sigma} \text { and } q \in Q \text { we have: } \mathrm{h}_{\mathcal{A}}(\xi)_{q}=\left|\mathrm{R}_{A}^{\mathrm{v}}(q, \xi)\right| \tag{3.7}
\end{equation*}
$$

Let $\xi=\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)$. Then we can calculate as follows.

$$
\begin{align*}
\mathrm{h}_{\mathcal{A}}\left(\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)\right)_{q} & =\prod_{q_{1}, \ldots, q_{k} \in Q} \mathrm{~h}_{\mathcal{A}}\left(\xi_{1}\right)_{q_{1}} \cdot \ldots \cdot \mathrm{~h}_{\mathcal{A}}\left(\xi_{k}\right)_{q_{k}} \cdot\left(\delta^{\prime}\right)_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right) \\
& =\prod_{q_{1}, \ldots, q_{k} \in Q}\left|\mathrm{R}_{A}^{\mathrm{v}}\left(q_{1}, \xi_{1}\right)\right| \cdot \ldots \cdot\left|\mathrm{R}_{A}^{\mathrm{v}}\left(q_{k}, \xi_{k}\right)\right| \cdot\left(\delta^{\prime}\right)_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right)  \tag{byI.H.}\\
& \left.=\left|\mathrm{R}_{A}^{\mathrm{v}}\left(q, \sigma\left(\xi_{1}, \ldots, \xi_{k}\right)\right)\right| . \quad \text { (by definitions of }\left(\delta^{\prime}\right)_{k} \text { and of } \mathrm{R}_{A}^{\mathrm{v}}\left(q, \sigma\left(\xi_{1}, \ldots, \xi_{k}\right)\right)\right)
\end{align*}
$$

Then, for each $\xi \in \mathrm{T}_{\Sigma}$, we have

$$
\begin{align*}
& \llbracket \mathcal{A} \rrbracket^{\text {init }}(\xi)={\underset{q \in Q}{ }} \mathrm{~h}_{\mathcal{A}}(\xi)_{q} \cdot F_{q}^{\prime} \tag{3.7}
\end{align*}
$$

$$
\begin{aligned}
& =\left|\bigcup_{q \in Q} \mathrm{R}_{A}^{\mathrm{a}}(q, \xi)\right| \quad \quad\left(\text { because } \mathrm{R}_{A}^{\mathrm{a}}\left(q_{1}, \xi\right) \cap \mathrm{R}_{A}^{\mathrm{a}}\left(q_{2}, \xi\right)=\emptyset \text { for every } q_{1}, q_{2} \in Q\right) \\
& \left.=\left|\mathrm{R}_{A}^{\mathrm{a}}(\xi)\right| \quad \quad \text { (because } \mathrm{R}_{A}^{\mathrm{a}}(\xi)=\bigcup_{q \in Q} \mathrm{R}_{A}^{\mathrm{a}}(q, \xi)\right) \\
& =\#_{\mathrm{R}_{A}^{\mathrm{a}}}(\xi) \text {. }
\end{aligned}
$$

Hence $\# R_{A}^{\mathrm{a}} \in \operatorname{Rec}^{\mathrm{init}}(\Sigma, N a t)$.
There will be a general result which says that, if B is a semiring, then $\llbracket \mathcal{A} \rrbracket^{\text {init }}=\llbracket \mathcal{A} \rrbracket^{\text {run }}$ for each ( $\Sigma, \mathrm{B}$ )-wta $\mathcal{A}$ (cf. Theorem 5.3.2).

Example 3.2.2. Dro21 (Number of occurrences of transitions of accepting computations of an fta on a tree.) Let $A=(Q, \delta, F)$ be a $\Sigma$-fta. We consider the mapping

$$
\#_{A}: \mathrm{T}_{\Sigma} \rightarrow \mathbb{N} \quad \text { with } \quad \#_{A}(\xi)=\left|\mathrm{R}_{A}^{\mathrm{a}}(\xi)\right| \cdot \operatorname{size}(\xi) \text { for each } \xi \in \mathrm{T}_{\Sigma}
$$

As weight algebra we use the plus-plus strong bimonoid $\mathrm{PP}_{\mathbb{N}}=\left(\mathbb{N}_{\mathbb{0}}, \oplus,+, 0,0\right)($ cf. Example 2.6.10 (8) $)$.


Figure 3.3: The fta-hypergraph for the $\left(\Sigma, \operatorname{Nat}_{\text {min },+}\right)$-wta $\mathcal{A}=(Q, \delta, F)$ which r-recognizes size.

We construct the $\left(\Sigma, \mathrm{PP}_{\mathbb{N}}\right)$-wta $\mathcal{A}=\left(Q, \delta^{\prime}, F^{\prime}\right)$ which r-recognizes $\#_{A}$, as follows. For every $k \in \mathbb{N}$, $q_{1}, \ldots, q_{k}, q \in Q$, and $\sigma \in \Sigma^{(k)}$, we let:

$$
\delta_{k}^{\prime}\left(q_{1} \cdots q_{k}, \sigma, q\right)=\left\{\begin{array}{ll}
1 & \text { if }\left(q_{1} \cdots q_{k}, \sigma, q\right) \in \delta_{k} \\
\mathbb{0} & \text { otherwise }
\end{array} \quad \text { and } \quad F_{q}^{\prime}= \begin{cases}0 & \text { if } q \in F \\
0 & \text { otherwise }\end{cases}\right.
$$

Obviously, for each $\xi \in \mathrm{T}_{\Sigma}$, we have $\mathrm{R}_{\mathcal{A}}(\xi)=\mathrm{R}_{A}(\xi)$, and for each $\rho \in \mathrm{R}_{\mathcal{A}}(\xi)$, we have

$$
\mathrm{wt}(\xi, \rho)= \begin{cases}\operatorname{size}(\xi) & \text { if } \rho \in \mathrm{R}_{A}^{\mathrm{v}}(\xi)  \tag{3.8}\\ 0 & \text { otherwise }\end{cases}
$$

Hence, for each $\xi \in \mathrm{T}_{\Sigma}$,

$$
\begin{array}{rlr}
\llbracket \mathcal{A} \rrbracket^{\mathrm{run}}(\xi) & =\bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}}(\xi)} \operatorname{wt}(\xi, \rho)+F_{\rho(\xi)}^{\prime} \\
& =\bigoplus_{\rho \in \mathrm{R}_{A}^{\mathrm{v}}(\xi)} \operatorname{size}(\xi)+F_{\rho(\varepsilon)}^{\prime} \\
& =\bigoplus_{\rho \in \mathrm{R}_{A}^{\mathrm{a}}(\xi)} \operatorname{size}(\xi)+0 \quad \quad \text { (because } \mathbb{0} \text { is the identity element of } \oplus \text { ) } \\
& \left.=\bigoplus_{\rho \in \mathrm{R}_{A}^{\mathrm{a}}(\xi)} \operatorname{siz}(\xi)\right) \\
& =\left|\mathrm{R}_{A}^{\mathrm{a}}(\xi)\right| \cdot \operatorname{size}(\xi)=\#_{A}(\xi) . & \quad \text { (because } \operatorname{size}(\xi) \in \mathbb{N})
\end{array}
$$

Hence $\#_{A} \in \operatorname{Rec}^{\text {run }}\left(\Sigma, \mathrm{PP}_{\mathbb{N}}\right)$.

Example 3.2.3. (Size of trees.) Let $\Sigma=\left\{\sigma^{(2)}, \alpha^{(0)}\right\}$. We consider the mapping

$$
\text { size }: \mathrm{T}_{\Sigma} \rightarrow \mathbb{N}
$$

defined on page 43, As weight algebra we use the tropical semiring $\operatorname{Nat}_{\min ,+}=\left(\mathbb{N}_{\infty}, \min ,+, \infty, 0\right)$. Since $\mathbb{N} \subseteq \mathbb{N}_{\infty}$, by our convention in Section [2.3, size is also a mapping of type size : $\mathrm{T}_{\Sigma} \rightarrow \mathbb{N}_{\infty}$, i.e., a $\left(\Sigma, \mathrm{Nat}_{\mathrm{min},+}\right)$-weighted tree language.

We construct the $\left(\Sigma\right.$, Nat $\left._{\text {min },+}\right)$-wta $\mathcal{A}=(Q, \delta, F)$ which r-recognizes size, as follows.

- $Q=\{s\}$, (intuitively, the state $s$ computes the size of the tree),
- $\delta_{0}(\varepsilon, \alpha, s)=\delta_{2}(s s, \sigma, s)=1$, and
- $F_{s}=0$.

In Figure 3.3 we represent $\mathcal{A}$ as an fta-hypergraph. Clearly, $\mathcal{A}$ is bu deterministic, total, and root weight normalized; $\mathcal{A}$ is not crisp deterministic, because $1 \notin\{\infty, 0\}$.

For each $\xi \in \mathrm{T}_{\Sigma}$, there is exactly one run on $\xi$. We denote it by $\rho^{\xi}$, hence $\rho^{\xi}(w)=s$ for each $w \in \operatorname{pos}(\xi)$. Obviously, for each $\xi \in \mathrm{T}_{\Sigma}$ :

$$
\mathrm{wt}\left(\xi, \rho^{\xi}\right)=|\operatorname{pos}(\xi)|
$$



Figure 3.4: The fta-hypergraph for the $\left(\Sigma, \operatorname{Nat}_{\max ,+}\right)$-wta $\mathcal{A}$ which i-recognizes height.

Then, for each $\xi \in \mathrm{T}_{\Sigma}$, we have

$$
\llbracket \mathcal{A} \rrbracket^{\mathrm{run}}(\xi)=\min \left(\mathrm{wt}(\xi, \rho)+F_{\rho(\varepsilon)} \mid \rho \in \mathrm{R}_{\mathcal{A}}(\xi)\right)=\mathrm{wt}\left(\xi, \rho^{\xi}\right)+0=|\operatorname{pos}(\xi)|=\operatorname{size}(\xi)
$$

Hence $\llbracket \mathcal{A} \rrbracket^{\text {run }}=$ size and thus size $\in \operatorname{bud}-\operatorname{Rec}^{\mathrm{run}}\left(\Sigma, \operatorname{Nat}_{\mathrm{min},+}\right)$. The construction can easily be generalized to an arbitrary ranked alphabet $\Sigma$.

Example 3.2.4. (Height of trees.) Let $\Sigma=\left\{\sigma^{(2)}, \alpha^{(0)}\right\}$. We consider the mapping

$$
\text { height : } \mathrm{T}_{\Sigma} \rightarrow \mathbb{N}
$$

defined on page 43. As weight algebra we use the arctic semiring $\operatorname{Nat}_{\text {max },+}=\left(\mathbb{N}_{-\infty}, \max ,+,-\infty, 0\right)$. Thus, the mapping height is a $\left(\Sigma, N_{\text {at }}{ }_{\text {max },+}\right)$-weighted tree language.

We construct the $\left(\Sigma, \operatorname{Nat}_{\max ,+}\right)$-wta $\mathcal{A}=(Q, \delta, F)$ which i-recognizes height, as follows.

- $Q=\{\mathrm{h}, 0\}$, (intuitively, $h$ and 0 should calculate the height and the natural number 0 , respectively)
- $\delta_{0}(\varepsilon, \alpha, \mathrm{~h})=\delta_{0}(\varepsilon, \alpha, 0)=0$ and for every $q_{1}, q_{2}, q \in Q$,

$$
\delta_{2}\left(q_{1} q_{2}, \sigma, q\right)= \begin{cases}1 & \text { if } q_{1} q_{2} q \in\{\mathrm{~h} 0 \mathrm{~h}, 0 \mathrm{hh}\} \\ 0 & \text { if } q_{1} q_{2} q=000 \\ -\infty & \text { otherwise }\end{cases}
$$

- $F_{\mathrm{h}}=0$ and $F_{0}=-\infty$.

In Figure 3.4 we represent $\mathcal{A}$ as an fta-hypergraph. Clearly, $\mathcal{A}$ is root weight normalized; $\mathcal{A}$ is not total, because there does not exist a state $q$ such that $\delta_{2}(\mathrm{hh}, \sigma, q) \neq-\infty ; \mathcal{A}$ is not bu deterministic, because $\delta_{0}(\varepsilon, \alpha, \mathrm{~h})=\delta_{0}(\varepsilon, \alpha, 0)=0 \neq-\infty$.

By induction on $\mathrm{T}_{\Sigma}$, we prove that the following statement holds:

$$
\begin{equation*}
\text { For each } \xi \in \mathrm{T}_{\Sigma} \text {, we have } \mathrm{h}_{\mathcal{A}}(\xi)_{\mathrm{h}}=\operatorname{height}(\xi) \text { and } \mathrm{h}_{\mathcal{A}}(\xi)_{0}=0 \tag{3.9}
\end{equation*}
$$

I.B.: Let $\xi=\alpha$. Then $\mathrm{h}_{\mathcal{A}}(\xi)_{\mathrm{h}}=\delta(\varepsilon, \alpha, \mathrm{h})=0$ (and similarly for $\left.\mathrm{h}_{\mathcal{A}}(\xi)_{0}\right)$.
I.S.: Now let $\xi=\sigma\left(\xi_{1}, \xi_{2}\right)$ for some trees $\xi_{1}, \xi_{2} \in \mathrm{~T}_{\Sigma}$. Then

$$
\begin{aligned}
& \mathrm{h}_{\mathcal{A}}\left(\sigma\left(\xi_{1}, \xi_{2}\right)\right)_{\mathrm{h}} \\
= & \max \left(\mathrm{h}_{\mathcal{A}}\left(\xi_{1}\right)_{q_{1}}+\mathrm{h}_{\mathcal{A}}\left(\xi_{2}\right)_{q_{2}}+\delta_{2}\left(q_{1} q_{2}, \sigma, \mathrm{~h}\right) \mid q_{1}, q_{2} \in Q\right)
\end{aligned}
$$

$$
\begin{aligned}
& =\max \left(\mathrm{h}_{\mathcal{A}}\left(\xi_{1}\right)_{\mathrm{h}}+\mathrm{h}_{\mathcal{A}}\left(\xi_{2}\right)_{0}+\delta_{2}(\mathrm{~h} 0, \sigma, \mathrm{~h}), \mathrm{h}_{\mathcal{A}}\left(\xi_{1}\right)_{0}+\mathrm{h}_{\mathcal{A}}\left(\xi_{2}\right)_{\mathrm{h}}+\delta_{2}(0 \mathrm{~h}, \sigma, \mathrm{~h})\right) \\
& \quad \text { (using that } \delta_{2}(\mathrm{hh}, \sigma, \mathrm{~h})=\delta_{2}(00, \sigma, \mathrm{~h})=-\infty \text { and }-\infty \text { is neutral for max) } \\
& \left.=\max \left(\operatorname{height}\left(\xi_{1}\right)+0+1,0+\operatorname{height}\left(\xi_{2}\right)+1\right) \quad \text { (by I.H. and definition of } \delta\right) \\
& =1+\max \left(\operatorname{height}\left(\xi_{1}\right), \operatorname{height}\left(\xi_{2}\right)\right)=\operatorname{height}(\xi) .
\end{aligned}
$$

Moreover, using similar arguments, we can calculate:

$$
\mathrm{h}_{\mathcal{A}}\left(\sigma\left(\xi_{1}, \xi_{2}\right)\right)_{0}=\mathrm{h}_{\mathcal{A}}\left(\xi_{1}\right)_{0}+\mathrm{h}_{\mathcal{A}}\left(\xi_{2}\right)_{0}+\delta_{2}(00, \sigma, 0)=0+0+0=0
$$

This finishes the proof of Statement (3.9). Then, for each $\xi \in \mathrm{T}_{\Sigma}$, we have

$$
\llbracket \mathcal{A} \rrbracket^{\mathrm{init}}(\xi)=\max \left(\mathrm{h}_{\mathcal{A}}(\xi)_{q}+F_{q} \mid q \in Q\right)=\max \left(\mathrm{h}_{\mathcal{A}}(\xi)_{\mathrm{h}}+0,0+(-\infty)\right)=\mathrm{h}_{\mathcal{A}}(\xi)_{\mathrm{h}}=\operatorname{height}(\xi)
$$

Hence $\llbracket \mathcal{A} \rrbracket^{\text {init }}=$ height, and thus height $\in \operatorname{Rec}^{\text {init }}\left(\Sigma\right.$, Nat $\left._{\text {max },+}\right)$.
Next we show that the $\left(\Sigma, \operatorname{Nat}_{\mathrm{max},+}\right)$-wta $\mathcal{A}$ also r-recognizes the weighted tree language height. First, we observe that for each $\xi \in \mathrm{T}_{\Sigma}$,

$$
\begin{equation*}
\operatorname{height}(\xi)=\max \left(|w| \mid w \in \operatorname{pos}_{\alpha}(\xi)\right) \tag{3.10}
\end{equation*}
$$

Let $\xi \in \mathrm{T}_{\Sigma}$ be an arbitrary $\Sigma$-tree. We define the run $\rho_{0}: \operatorname{pos}(\xi) \rightarrow Q$ by $\rho_{0}(w)=0$ for each $w \in \operatorname{pos}(\xi)$. It is obvious that $\operatorname{wt}\left(\xi, \rho_{0}\right)=0$.

Moreover, for every $w \in \operatorname{pos}_{\alpha}(\xi)$, we define the run $\rho_{w}: \operatorname{pos}(\xi) \rightarrow Q$ such that for each $v \in \operatorname{pos}(\xi)$ we let $\rho_{w}(v)=\mathrm{h}$ if $v$ is a prefix of $w$, and 0 otherwise. By induction on $\mathrm{T}_{\Sigma}$, we prove that the following statement holds:

$$
\begin{equation*}
\text { For every } \xi \in \mathrm{T}_{\Sigma} \text { and } w \in \operatorname{pos}_{\alpha}(\xi), \text { we have } \mathrm{wt}\left(\xi, \rho_{w}\right)=|w| \tag{3.11}
\end{equation*}
$$

I.B.: Let $\xi=\alpha$. Then $w=\varepsilon$ and $\operatorname{wt}\left(\xi, \rho_{\varepsilon}\right)=\delta_{0}(\varepsilon, \alpha, \mathrm{~h})=0$.
I.S.: Now let $\xi=\sigma\left(\xi_{1}, \xi_{2}\right)$ and assume that $w=1 v$ for some $v \in \operatorname{pos}\left(\xi_{1}\right)$. Then

$$
\left.\begin{array}{rl}
\mathrm{wt}\left(\sigma\left(\xi_{1}, \xi_{2}\right), \rho_{w}\right) & =\mathrm{wt}\left(\xi_{1},\left.\rho_{w}\right|_{1}\right)+\mathrm{wt}\left(\xi_{2},\left.\rho_{w}\right|_{2}\right)+\delta_{2}\left(\rho_{w}(1) \rho_{w}(2), \sigma, \rho_{w}(\varepsilon)\right) \\
& =\operatorname{wt}\left(\xi_{1}, \rho_{v}\right)+\operatorname{wt}\left(\xi_{2}, \rho_{0}\right)+\delta_{2}(\mathrm{~h} 0, \sigma, \mathrm{~h}) \\
& =|v|+0+1 \\
& =|w|
\end{array} \quad \text { (by I.H. and definition of } \delta_{2}\right) \text { ) } \quad \text {. }
$$

In a similar way we prove $\operatorname{wt}\left(\xi, \rho_{w}\right)=|w|$ if $w=2 v$ for some $v$. This proves (3.11). Also it is obvious that

$$
\begin{equation*}
\text { for every } \rho \in \mathrm{R}_{\mathcal{A}}(\xi) \text { and } w \in \operatorname{pos}_{\alpha}(\xi) \text {, if } \rho \notin\left\{\rho_{0}, \rho_{w}\right\}, \text { then } \mathrm{wt}(\xi, \rho)=-\infty \tag{3.12}
\end{equation*}
$$

Then we can compute the run semantics of $\mathcal{A}$ on $\xi$ as follows:

$$
\begin{array}{rlr}
\llbracket \mathcal{A} \rrbracket^{\mathrm{run}}(\xi) & =\max _{\rho \in \mathrm{R}_{\mathcal{A}}(\xi)} \operatorname{wt}(\xi, \rho)+F_{\rho(\varepsilon)}=\max \left(\operatorname{wt}(\xi, \rho)+F_{\rho(\varepsilon)} \mid \rho \in \mathrm{R}_{\mathcal{A}}(\xi)\right) \\
& =\max \left(\operatorname{wt}(\xi, \rho)+F_{\mathrm{h}} \mid \rho \in \mathrm{R}_{\mathcal{A}}(\mathrm{h}, \xi)\right) \\
& =\max \left(\operatorname{wt}\left(\xi, \rho_{w}\right)+0 \mid w \in \operatorname{pos}_{\alpha}(\xi)\right) \\
& =\max \left(|w| \mid w \in \operatorname{pos}_{\alpha}(\xi)\right) \\
& =\operatorname{height}(\xi) . & \left(\text { because } F_{0}=-\infty\right) \\
& \text { (by (3.12) (3.11)) } \\
\text { (3.10) })
\end{array}
$$

Hence $\llbracket \mathcal{A} \rrbracket^{\text {run }}=$ height, and thus height $\in \operatorname{Rec}^{\text {run }}\left(\Sigma, \operatorname{Nat}_{\text {max }},+\right)$. The construction can easily be generalized to an arbitrary ranked alphabet $\Sigma$.


Figure 3.5: The ( $\Sigma$, Pos)-wta $\mathcal{A}=(Q, \delta, F)$ which r-recognizes pos.

Example 3.2.5. (Set of positions.) Let $\Sigma=\left\{\sigma^{(2)}, \gamma^{(1)}, \alpha^{(0)}\right\}$. We consider the mapping

$$
\operatorname{pos}: \mathrm{T}_{\Sigma} \rightarrow \mathcal{P}\left(\mathbb{N}_{+}^{*}\right)
$$

defined on page 43 ,
As weight algebra we use the semiring Pos $=\left(\mathcal{P}\left(\mathbb{N}_{+}{ }^{*}\right), \cup, \circ^{R}, \emptyset,\{\varepsilon\}\right)$ where $U \circ^{R} V=V U$ for every $U, V \in \mathcal{P}\left(\mathbb{N}_{+}{ }^{*}\right)$. Thus, the mapping pos is a ( $\Sigma$, Pos)-weighted tree language.

We construct the ( $\Sigma$, Pos)-wta $\mathcal{A}=(Q, \delta, F)$ which r-recognizes pos, as follows.

- $Q=\{e, p\}$ (intuitively, $e$ and $p$ calculate $\varepsilon \in \mathbb{N}_{+}{ }^{*}$ and a path in $\mathbb{N}_{+}{ }^{*}$, respectively),
- for every $q_{1}, q_{2}, q \in Q$ we define $\delta_{0}(\varepsilon, \alpha, q)=\{\varepsilon\}$ and

$$
\delta_{1}\left(q_{1}, \gamma, q\right)=\left\{\begin{array}{ll}
\{\varepsilon\} & \text { if } q_{1}=e \\
\{1\} & \text { if } q_{1}=q=p \\
\emptyset & \text { otherwise }
\end{array} \quad \text { and } \quad \delta_{2}\left(q_{1} q_{2}, \sigma, q\right)= \begin{cases}\{\varepsilon\} & \text { if } q_{1} q_{2}=e e \\
\{1\} & \text { if } q_{1} q_{2} q=p e p \\
\{2\} & \text { if } q_{1} q_{2} q=e p p \\
\emptyset & \text { otherwise }\end{cases}\right.
$$

- $F_{e}=\emptyset$ and $F_{p}=\{\varepsilon\}$.

In Figure 3.5 we represent $\mathcal{A}$ as fta-hypergraph. Clearly, $\mathcal{A}$ is root weight normalized; $\mathcal{A}$ is not bu deterministic, because e.g. $\delta_{1}(e, \gamma, e)=\delta_{1}(e, \gamma, p)=\{\varepsilon\}$, which is different from $\emptyset ; \mathcal{A}$ is not total, because there does not exist a state $q$ such that $\delta_{2}(p p, \sigma, q) \neq \emptyset$.

Let $\xi \in \mathrm{T}_{\Sigma}$. First we define particular runs on $\xi$ and show their weights. For each $u \in \operatorname{pos}(\xi)$, we define the run $\rho_{u}^{\xi}: \operatorname{pos}(\xi) \rightarrow Q$ for each $w \in \operatorname{pos}(\xi)$ by

$$
\rho_{u}^{\xi}(w)= \begin{cases}p & \text { if } w \text { is a prefix of } u \\ e & \text { otherwise }\end{cases}
$$

(We recall that $\varepsilon$ and $u$ are prefixes of $u$.) Also, we define the run $\rho_{e}^{\xi}: \operatorname{pos}(\xi) \rightarrow Q$ by $\rho_{e}^{\xi}(w)=e$ for each $w \in \operatorname{pos}(\xi)$.

Then we claim that the following three statements hold for each $\xi \in \mathrm{T}_{\Sigma}$ :

$$
\begin{equation*}
\mathrm{wt}\left(\xi, \rho_{e}^{\xi}\right)=\{\varepsilon\} \tag{3.13}
\end{equation*}
$$

for each $\rho \in \mathrm{R}_{\mathcal{A}}(\xi) \backslash\left(\left\{\rho_{u}^{\xi} \mid u \in \operatorname{pos}(\xi)\right\} \cup\left\{\rho_{e}^{\xi}\right\}\right)$, we have wt $(\xi, \rho)=\emptyset$

$$
\begin{equation*}
\text { for each } u \in \operatorname{pos}(\xi), \text { we have } \operatorname{wt}\left(\xi, \rho_{u}^{\xi}\right)=\{u\} \tag{3.14}
\end{equation*}
$$

By direct inspection of the corresponding definitions we get the proofs of Statements (3.13) and (3.14).
Next we prove Statement (3.15). For this we fix $\xi \in \mathrm{T}_{\Sigma}$ and $u \in \operatorname{pos}(\xi)$. We define the binary relation $\prec_{\mathrm{d}}$ (direct postfix) on postfix $(u)$ by letting $v_{1} \prec_{\mathrm{d}} v_{2}$ if there exists an $i \in \mathbb{N}_{+}$such that $v_{2}=i v_{1}$. It is obvious that $\prec_{d}$ is well-founded on $\operatorname{postfix}(u)$ and $\min _{\prec_{d}}(\operatorname{postfix}(u))=\{\varepsilon\}$. By induction on (postfix $(u), \prec_{d}$ ), we prove that the following statement holds (where $w$ is determined by $w v=u$.)

$$
\begin{equation*}
\text { For each } v \in \operatorname{postfix}(u) \text {, we have } \operatorname{wt}\left(\left.\xi\right|_{w}, \rho_{v}^{\xi \mid w}\right)=\{v\} \tag{3.16}
\end{equation*}
$$

I.B.: Let $v=\varepsilon$. Then $w=u$ and $\rho_{\varepsilon}^{\left.\xi\right|_{u}}: \operatorname{pos}\left(\left.\xi\right|_{u}\right) \rightarrow Q$ maps $\varepsilon$ to $p$ and each other position to $e$. Let $\xi(u)=\kappa$ for some $\kappa \in \Sigma^{(k)}$ with $k \in\{0,1,2\}$. Then we calculate as follows (abbreviating $\rho_{e}^{\left.\xi\right|_{u}}$ by $\nu$ ):

$$
\begin{aligned}
\mathrm{wt}\left(\left.\xi\right|_{u}, \nu\right) & =\mathrm{wt}\left(\left.\xi\right|_{u 1},\left.\nu\right|_{1}\right) \circ^{R} \ldots \circ^{R} \mathrm{wt}\left(\left.\xi\right|_{u k},\left.\nu\right|_{k}\right) \circ^{R} \delta_{k}(\underbrace{e \cdots e}_{k}, \kappa, p) \\
& =\{\varepsilon\} \circ^{R} \ldots \circ^{R}\{\varepsilon\} \circ^{R}\{\varepsilon\}
\end{aligned}
$$

(by the fact that $\left.\nu\right|_{i}=\left.\left(\rho_{e}^{\left.\xi\right|_{u}}\right)\right|_{i}=\rho_{e}^{\left.\xi\right|_{u i}}$ for each $i \in[k]$, by (3.13), and by definition of $\delta$ )

$$
=\{\varepsilon\}
$$

I.S.: Let $v \neq \varepsilon$. Let $\xi(w)=\kappa$ for some $\kappa \in \Sigma^{(k)}$ with $k \in\{1,2\}$. Then there exists a unique $v^{\prime} \in \operatorname{postfix}(u)$ with $v^{\prime} \prec_{\mathrm{d}} v$. Let $i \in[k]$ be such that $v=i v^{\prime}$, then $\operatorname{wt}\left(\left.\xi\right|_{w i}, \rho_{v^{\prime}}^{\left.\xi\right|_{w i}}\right)=\left\{v^{\prime}\right\}$ by the I.H. Then we can calculate as follows (abbreviating $\rho_{v}^{\left.\xi\right|_{w}}$ by $\nu$ ):

$$
\begin{aligned}
\mathrm{wt}\left(\left.\xi\right|_{w}, \nu\right)= & \mathrm{wt}\left(\xi_{w 1},\left.\nu\right|_{1}\right) \circ^{R} \ldots \circ^{R} \mathrm{wt}\left(\xi_{w(i-1)},\left.\nu\right|_{i-1}\right) \circ^{R} \mathrm{wt}\left(\xi_{w i},\left.\nu\right|_{i}\right) \\
& \circ^{R} \mathrm{wt}\left(\xi_{w(i+1)},\left.\nu\right|_{i+1}\right) \circ^{R} \ldots \circ^{R} \mathrm{wt}\left(\xi_{w k},\left.\nu\right|_{k}\right) \circ^{R} \delta_{k}(\underbrace{e \cdots e}_{i-1} p \underbrace{e \cdots e}_{k-i}, \kappa, p) \\
= & \mathrm{wt}\left(\xi_{w i},\left.\nu\right|_{i}\right) \circ^{R} \delta_{k}(\underbrace{e \cdots e}_{i-1} p \underbrace{e \cdots e}_{k-i}, \kappa, p) \\
& \quad \text { because }\left.\nu\right|_{j}=\rho_{e}^{\xi| |_{w j}} \text { for each } j \neq i ; \text { thus } \mathrm{wt}\left(\xi_{w j},\left.\nu\right|_{j}\right)=\{\varepsilon\} \text { by Statement (3.13)) } \\
= & \left\{v^{\prime}\right\} \circ^{R} \delta_{k}(\underbrace{e \cdots e}_{i-1} p \underbrace{e \cdots e}_{k-i}, \kappa, p) \quad \text { (because }\left.\nu\right|_{i}=\rho_{v^{\prime}}^{\left.\xi\right|_{w i}} ; \text { by I.H. wt }\left(\xi_{w i},\left.\nu\right|_{i}\right)=\left\{v^{\prime}\right\}) \\
= & \left\{v^{\prime}\right\} \circ^{R}\{i\}=\left\{i v^{\prime}\right\}=\{v\} . \quad \quad \text { (by definition of } \delta_{k}(\underbrace{e \cdots e}_{i-1} p \underbrace{e \cdots e}_{k-i}, \kappa, p) \text { ) }
\end{aligned}
$$

This finishes the proof of Statement (3.16). Choosing $v=u$ in this statement proves Statement (3.15).
Now, for each $\xi \in \mathrm{T}_{\Sigma}$, we can prove that $\llbracket \mathcal{A} \rrbracket^{\text {run }}(\xi)=\operatorname{pos}(\xi)$ as follows.

$$
\begin{aligned}
& \llbracket \mathcal{A} \rrbracket^{\operatorname{run}}(\xi)=\bigcup\left(\operatorname{wt}(\xi, \rho) \circ^{R} F_{\rho(\varepsilon)} \mid \rho \in \mathrm{R}_{\mathcal{A}}(\xi)\right) \\
= & \bigcup\left(\operatorname{wt}(\xi, \rho) \circ^{R} F_{\rho(\varepsilon)} \mid \rho \in\left\{\rho_{u}^{\xi} \mid u \in \operatorname{pos}(\xi)\right\} \cup\left\{\rho_{e}^{\xi}\right\}\right) \\
= & \bigcup\left(\operatorname{wt}(\xi, \rho) \circ^{R} F_{\rho(\varepsilon)} \mid \rho \in\left\{\rho_{u}^{\xi} \mid u \in \operatorname{pos}(\xi)\right\}\right) \quad \quad \text { (because } \rho_{e}^{\xi}(\varepsilon)=e \text { (3.14)) } \\
= & \left.\bigcup\left(\operatorname{wt}(\xi, \rho) \mid \rho \in\left\{\rho_{u}^{\xi} \mid u \in \operatorname{pos}(\xi)\right\}\right) \quad \quad \text { (because } \rho_{u}^{\xi}(\varepsilon)=\emptyset \text { for each } u \in \operatorname{pos}(\xi) \text { and } F_{p}=\{\varepsilon\}\right) \\
= & \bigcup\left(\operatorname{wt}\left(\xi, \rho_{u}^{\xi}\right) \mid u \in \operatorname{pos}(\xi)\right) \\
= & \bigcup(\{u\} \mid u \in \operatorname{pos}(\xi)) \quad(\text { by }(\underline{3.15})) \\
= & \operatorname{pos}(\xi) .
\end{aligned}
$$

Hence $\llbracket \mathcal{A} \rrbracket^{\text {run }}=$ pos and thus pos $\in \operatorname{Rec}^{\text {run }}(\Sigma$, Pos $)$.


Figure 3.6: The $\left(\Sigma\right.$ Lang $\left._{\Sigma}\right)$-wta $\mathcal{A}=(Q, \delta, F)$ which i-recognizes yield ${ }^{\mathcal{P}}$.

Example 3.2.6. (Yield of trees.) Let $\Sigma=\left\{\sigma^{(2)}, \alpha^{(0)}, \beta^{(0)}\right\}$. We consider the mapping

$$
\operatorname{yield}^{\mathcal{P}}: \mathrm{T}_{\Sigma} \rightarrow \mathcal{P}\left(\left(\Sigma^{(0)}\right)^{*}\right)
$$

defined, for each $\xi \in \mathrm{T}_{\Sigma}$, by yield ${ }^{\mathcal{P}}(\xi)=\{\operatorname{yield}(\xi)\}$. For the definition of yield $(\xi)$ we refer to Section 2.9, For instance, yield ${ }^{\mathcal{P}}(\sigma(\alpha, \sigma(\alpha, \beta))=\{\alpha \alpha \beta\}$.

As weight algebra we use the formal language language semiring $\operatorname{Lang}_{\Sigma}=\left(\mathcal{P}\left(\Sigma^{*}\right), \cup, \cdot, \emptyset,\{\varepsilon\}\right)$. Thus, the mapping yield is a $\left(\Sigma\right.$, Lang $\left._{\Sigma}\right)$-weighted tree language.

We define the $\left(\Sigma\right.$, Lang $\left._{\Sigma}\right)$-wta $\mathcal{A}=(Q, \delta, F)$ which i-recognizes yield ${ }^{\mathcal{P}}$, as follows.

- $Q=\{y\}$ (intuitively, $y$ calculates the yield),
- $\delta_{0}(\varepsilon, \alpha, y)=\{\alpha\}, \delta_{0}(\varepsilon, \beta, y)=\{\beta\}$, and $\delta_{2}(y y, \sigma, y)=\{\varepsilon\}$,
- $F_{y}=\{\varepsilon\}$.

In Figure 3.6 we represent $\mathcal{A}$ as fta-hypergraph. Clearly, $\mathcal{A}$ is bu deterministic, total, and root weight normalized; $\mathcal{A}$ is not crisp deterministic because, e.g., $\delta_{0}(\varepsilon, \alpha, y) \notin\{\emptyset,\{\varepsilon\}\}$.

By induction on $\mathrm{T}_{\Sigma}$, we prove that the following statement holds:
For each $\xi \in \mathrm{T}_{\Sigma}$, we have $\mathrm{h}_{\mathcal{A}}(\xi)_{y}=\{\operatorname{yield}(\xi)\}$.
I.B.: Let $\xi=\alpha$. Then $\mathrm{h}_{\mathcal{A}}(\xi)_{y}=\delta_{0}(\varepsilon, \alpha, y)=\{\alpha\}=\{\operatorname{yield}(\alpha)\}$. For $\xi=\beta$ the proof is similar.
I.S.: Now let $\xi=\sigma\left(\xi_{1}, \xi_{2}\right)$ and assume that $\mathrm{h}_{\mathcal{A}}\left(\xi_{i}\right)_{y}=\left\{\operatorname{yield}\left(\xi_{i}\right)\right\}$ for each $i \in\{1,2\}$. Then

$$
\mathrm{h}_{\mathcal{A}}(\xi)_{y}=\mathrm{h}_{\mathcal{A}}\left(\xi_{1}\right)_{y} \mathrm{~h}_{\mathcal{A}}\left(\xi_{2}\right)_{y} \delta_{2}(y y, \sigma, y)=\left\{\operatorname{yield}\left(\xi_{1}\right)\right\}\left\{\operatorname{yield}\left(\xi_{2}\right)\right\}\{\varepsilon\}=\{\operatorname{yield}(\xi)\}
$$

Then we have $\llbracket \mathcal{A} \rrbracket^{\text {init }}(\xi)=\mathrm{h}_{\mathcal{A}}(\xi)_{y} F_{y}=\{\operatorname{yield}(\xi)\}=\operatorname{yield}^{\mathcal{P}}(\xi)$. Thus yield $^{\mathcal{P}} \in \operatorname{bud}-\operatorname{Rec}^{\text {init }}\left(\Sigma, \operatorname{Lang}_{\Sigma}\right)$.

Example 3.2.7. (Transformation monoid.) We recall from Section 2.11 that a $\Gamma$-fsa is a tuple $A=(Q, I, \delta, F)$ where $Q$ is the finite set of states, $I \subseteq Q$ and $F \subseteq Q$ are the sets of initial states and final states, respectively, and $\delta \subseteq Q \times \Gamma \times Q$ is the set of transitions. The language $\mathrm{L}(A)$ recognized by $A$ is defined in terms of runs.

Alternatively, we wish to describe $\mathrm{L}(A)$ by using the mapping

$$
\bar{\delta}: \Gamma^{*} \rightarrow(\mathcal{P}(Q) \rightarrow \mathcal{P}(Q))
$$

as follows. Intuitively, for every $w \in \Gamma^{*}$ and $U \subseteq Q$, the set $\bar{\delta}(w)(U)$ is the set of all states which $A$ can enter when starting in some state of $U$ and reading the string $w$. Formally, we define $\bar{\delta}$ by induction on $\left(\Gamma^{*}, \prec\right)$ where, for every $w_{1}, w_{2} \in \Gamma^{*}$, we let $w_{1} \prec w_{2}$ if there exists an $a \in \Gamma$ such that $w_{2}=w_{1} a$. Obviously, $\prec$ is well-founded and $\min _{\prec}\left(\Gamma^{*}\right)=\{\varepsilon\}$. For every $a \in \Gamma, w \in \Gamma^{*}$, and $U \subseteq Q$, we define

$$
\bar{\delta}(\varepsilon)(U)=U \quad \text { and } \quad \bar{\delta}(w a)(U)=\{p \in Q \mid(\exists r \in \bar{\delta}(w)(U)):(r, a, p) \in \delta\}
$$

Then it easy to see that $\mathrm{L}(A)=\left\{w \in \Gamma^{*} \mid \bar{\delta}(w)(I) \cap F \neq \emptyset\right\}$.
Let $\Sigma=\left\{\sigma^{(2)}, \alpha^{(0)}, \beta^{(0)}\right\}$ and let $A=(Q, I, \delta, F)$ be a $\Gamma$-fsa with $\Gamma=\{\alpha, \beta\}$. We define the mapping

$$
\operatorname{BPS}_{A}: \mathrm{T}_{\Sigma} \rightarrow(\mathcal{P}(Q) \rightarrow \mathcal{P}(Q)) \quad \text { with } \quad \operatorname{BPS}_{A}(\xi)=\bar{\delta}(\operatorname{yield}(\xi)) \text { for each } \xi \in \mathrm{T}_{\Sigma}
$$

where yield is defined in Section 2.9 Eventually, we will prove that $\mathrm{BPS}_{A}$ is r-recognizable over a suitable strong bimonoid.

The name "BPS" of the mapping refers to Bar-Hillel, Perles, and Shamir BPS61; in that paper they have used a technique to fold the computation of a finite-state automaton onto a derivation tree of a context-free grammar. They used this technique for the proof of the fact that the set of context-free languages is closed under intersection with the set of regular languages. We note that the set of contextfree languages and the set of yields of recognizable tree languages are equal, cf. Bra69, Thm. 3.20] and Don70, Thm. 2.5] (also cf. Corollary 8.3.4 and GS84, Cor. 3.2.4 and Thm. 3.2.5]). Here we illustrate the BPS-technique for the yield of the particular recognizable tree language $\mathrm{T}_{\Sigma}$ (also cf. Section 8.3).

We consider the near semiring $\operatorname{NearSem}_{\mathcal{P}(Q)}=(B, \cup, \diamond, \widetilde{\emptyset}$, id) over the commutative semigroup $(\mathcal{P}(Q), \cup, \emptyset)$ as defined in Example 2.6.10(5), where id abbreviates $\operatorname{id}_{\mathcal{P}(Q)}$. We recall that $B=\{f \mid$ $f: \mathcal{P}(Q) \rightarrow \mathcal{P}(Q), f(\emptyset)=\emptyset\}$.

We note that ( $B, \diamond, \mathrm{id}$ ) is a monoid. Moreover, it is easy to verify that $\bar{\delta}$ is monoid homomorphism from $\left(\Gamma^{*}, \cdot, \varepsilon\right)$ to $(B, \diamond$, id $)$, i.e., $\bar{\delta}(\varepsilon)=\mathrm{id}$ and,

$$
\begin{equation*}
\text { for every } v, w \in \Gamma^{*} \text {, we have } \bar{\delta}(w v)=\bar{\delta}(w) \diamond \bar{\delta}(v) \text {. } \tag{3.17}
\end{equation*}
$$

By induction on $\left(\Gamma^{*}, \prec\right)$, we can now easily prove (3.17).
Now we construct the $\left(\Sigma, \operatorname{NearSem}_{\mathcal{P}(Q)}\right)$-wta $\mathcal{A}=\left(\{*\}, \delta^{\prime}, F\right)$ which r-recognizes $\mathrm{BPS}_{A}$ as follows:

- $\left(\delta^{\prime}\right)_{0}(\varepsilon, \alpha, *)=\bar{\delta}(\alpha),\left(\delta^{\prime}\right)_{0}(\varepsilon, \beta, *)=\bar{\delta}(\beta)$, and $\left(\delta^{\prime}\right)_{2}(* *, \sigma, *)=\mathrm{id}$ and
- $F_{*}=\mathrm{id}$.

In fact, $\mathcal{A}$ is bu deterministic and root weight normalized, and in general $\mathcal{A}$ is not crisp deterministic.
Let $\xi \in \mathrm{T}_{\Sigma}$ and $\rho_{\xi}: \operatorname{pos}(\xi) \rightarrow Q$ be the run which maps each position of $\xi$ to $*$. By induction on $\mathrm{T}_{\Sigma}$, we prove that the following statement holds:

$$
\begin{equation*}
\text { For each } \xi \in \mathrm{T}_{\Sigma} \text {, we have } \mathrm{wt}_{\mathcal{A}}\left(\xi, \rho_{\xi}\right)=\bar{\delta}(\operatorname{yield}(\xi)) . \tag{3.18}
\end{equation*}
$$

Let $\xi=\alpha$. Then $\operatorname{wt}_{\mathcal{A}}\left(\xi, \rho_{\xi}\right)=\left(\delta^{\prime}\right)_{0}(\varepsilon, \alpha, *)=\bar{\delta}(\alpha)=\bar{\delta}(\operatorname{yield}(\xi))$. In the same way we obtain (3.18) for $\xi=\beta$. Now let $\xi=\sigma\left(\xi_{1}, \xi_{2}\right)$ and assume that (3.18) holds for $\xi_{1}$ and $\xi_{2}$. Then

$$
\begin{array}{rlr}
\operatorname{wt}_{\mathcal{A}}\left(\xi, \rho_{\xi}\right) & =\operatorname{wt}_{\mathcal{A}}\left(\xi_{1}, \rho_{\xi_{1}}\right) \diamond \operatorname{wt}_{\mathcal{A}}\left(\xi_{2}, \rho_{\xi_{2}}\right) \diamond\left(\delta^{\prime}\right)_{2}(* *, \sigma, *) & \\
& =\bar{\delta}\left(\operatorname{yield}\left(\xi_{1}\right)\right) \diamond \bar{\delta}\left(\operatorname{yield}\left(\xi_{2}\right)\right) \diamond \operatorname{id} & \text { (by I.H. and construction) } \\
& =\bar{\delta}\left(\operatorname{yield}\left(\xi_{1}\right)\right) \diamond \bar{\delta}\left(\operatorname{yield}\left(\xi_{2}\right)\right) & \quad \text { (because id is the identity) } \\
& =\bar{\delta}\left(\operatorname{yield}\left(\xi_{1}\right) \operatorname{yield}\left(\xi_{2}\right)\right) & \\
& =\bar{\delta}\left(\operatorname{yield}\left(\sigma\left(\xi_{1}, \xi_{2}\right)\right) .\right. & \text { (because } \bar{\delta} \text { is a monoid homomorphism) }
\end{array}
$$

Finally, for each $\xi \in \mathrm{T}_{\Sigma}$ we have

$$
\llbracket \mathcal{A} \rrbracket^{\mathrm{run}}(\xi)=\bigcup_{\rho \in \mathrm{R}_{\mathcal{A}}(\xi)} \operatorname{wt}_{\mathcal{A}}(\xi, \rho)=\mathrm{wt}_{\mathcal{A}}\left(\xi, \rho_{\xi}\right)=\bar{\delta}(\operatorname{yield}(\xi))=\operatorname{BPS}_{A}(\xi) .
$$

Thus $\operatorname{BPS}_{A} \in \operatorname{bud}-\operatorname{Rec}^{\text {run }}\left(\Sigma, \operatorname{NearSem}_{\mathcal{P}(Q)}\right)$.


Figure 3.7: The $(\Sigma$, TropBM $)$-wta $\mathcal{A}$ which r-recognizes exp.

Example 3.2.8. (Exponentiation.) We consider the string ranked alphabet $\Sigma=\left\{\gamma^{(1)}, \alpha^{(0)}\right\}$ and the mapping

$$
\exp : \mathrm{T}_{\Sigma} \rightarrow \mathbb{N} \quad \text { with } \quad \exp (\xi)=2^{n+1} \quad \text { if } \xi=\gamma^{n}(\alpha) \text { for some } n \in \mathbb{N}
$$

As weight algebra we consider the tropical bimonoid $\operatorname{TropBM}=\left(\mathbb{N}_{\infty},+, \min , 0, \infty\right)$ from Example 2.6.10. Thus exp is a $(\Sigma$, TropBM $)$-weighted tree language.

We construct the $(\Sigma$, TropBM)-wta $\mathcal{A}=(Q, \delta, F)$ which r-recognizes exp, as follows.

- $Q=\left\{q_{0}, q_{1}\right\}$,
- $\delta_{0}(\varepsilon, \alpha, p)=\delta_{1}(p, \gamma, q)=1$ for every $p, q \in Q$, and
- $F_{q_{0}}=F_{q_{1}}=1$.

In Figure 3.7 we show the wta $\mathcal{A}$. Obviously, $\mathcal{A}$ is not bu deterministic.
Now let $n \in \mathbb{N}$. We compute $\llbracket \mathcal{A} \rrbracket^{\text {run }}\left(\gamma^{n}(\alpha)\right)$. It is easy to see that $\operatorname{wt}\left(\gamma^{n}(\alpha), \rho\right)=1$ for each run $\rho \in \mathrm{R}_{\mathcal{A}}\left(\gamma^{n}(\alpha)\right)$. Then for each $n \in \mathbb{N}$ we have
where the last but one equality is due to the fact that there exists a bijection between the two finite sets $\mathrm{R}_{\mathcal{A}}\left(\gamma^{n}(\alpha)\right)$ and $\left\{q_{0}, q_{1}\right\}^{n+1}$. Hence $\llbracket \mathcal{A} \rrbracket^{\text {run }}=\exp$ and thus $\exp \in \operatorname{Rec}^{\text {run }}(\Sigma$, TropBM $)$.

Example 3.2.9. (Exponentiation plus one.) Let $\Sigma=\left\{\gamma^{(1)}, \alpha^{(0)}\right\}$ be a string ranked alphabet. We define the mapping

$$
(\exp +1): \mathrm{T}_{\Sigma} \rightarrow \mathbb{N} \quad \text { with } \quad(\exp +1)\left(\gamma^{n}(\alpha)\right)=2^{n}+1 \text { for each } n \in \mathbb{N}
$$

As weight algebra we consider the semiring $\operatorname{Nat}=(\mathbb{N},+, \cdot, 0,1)$. Thus, the mapping exp is a $(\Sigma, \mathbb{N})$ weighted tree language.

We construct the ( $\Sigma$, Nat)-wta $\mathcal{A}=(Q, \delta, F)$ which i-recognizes (exp +1 ), as follows:

- $Q=\{1, e\}$ (intuitively, 1 and $e$ calculate $1 \in \mathbb{N}$ and $2^{n}$, respectively),
- $\delta_{0}(\varepsilon, \alpha, 1)=\delta_{0}(\varepsilon, \alpha, e)=1$ and for every $q_{1}, q \in Q$,

$$
\delta_{1}\left(q_{1}, \gamma, q\right)= \begin{cases}1 & \text { if } q_{1}=q=1 \\ 2 & \text { if } q_{1}=q=e \\ 0 & \text { otherwise }\end{cases}
$$

- $F_{1}=F_{e}=1$.


Figure 3.8: The $(\Sigma$, Nat $)$-wta $\mathcal{A}=(Q, \delta, F)$ which i-recognizes $(\exp +1)$.

In Figure 3.8 we represent $\mathcal{A}$ as fta-hypergraph. Clearly, $\mathcal{A}$ is total and has identity root weights; $\mathcal{A}$ is not bu deterministic and it is not root weight normalized.

We can prove easily that:

$$
\mathrm{h}_{\mathcal{A}}\left(\gamma^{n}(\alpha)\right)_{1}=1 \text { and } \mathrm{h}_{\mathcal{A}}\left(\gamma^{n}(\alpha)\right)_{e}=2^{n} \text { for each } n \in \mathbb{N}
$$

Then $\llbracket \mathcal{A} \rrbracket^{\text {init }}\left(\gamma^{n}(\alpha)\right)=2^{n}+1=(\exp +1)\left(\gamma^{n}(\alpha)\right)$ for each $n \in \mathbb{N}$. Thus $(\exp +1) \in \operatorname{Rec}^{\text {init }}(\Sigma$, Nat $)$.

Example 3.2.10. (Number of $\gamma$ 's modulo 2) Let $\Sigma=\left\{\gamma^{(1)}, \alpha^{(0)}\right\}$ be a string ranked alphabet. We define the mapping

$$
\text { odd }: \mathrm{T}_{\Sigma} \rightarrow\{0,1\} \quad \text { with } \quad \operatorname{odd}\left(\gamma^{n}(\alpha)\right)=\left\{\begin{array}{ll}
1 & \text { if } n \text { is odd } \\
0 & \text { otherwise }
\end{array} \text { for each } n \in \mathbb{N} .\right.
$$

As weight algebra we consider the field $F_{2}=(\{0,1\}, \oplus, \otimes, 0,1)$ (cf. Example 2.6.10(7)). Thus, the mapping odd is a $\left(\Sigma, \mathrm{F}_{2}\right)$-weighted tree language.

We construct the $\left(\Sigma, \mathrm{F}_{2}\right)$-wta $\mathcal{A}=(Q, \delta, F)$ which r-recognizes odd, as follows.

- $Q=\left\{q_{0}, q_{1}\right\}$ (intuitively, on each input tree $\gamma^{n}(\alpha)$, the wta $\mathcal{A}$ starts at the leaf in state $q_{0}$ or $q_{1}$ and, at any of the occurrences of $\gamma$, nondeterministically either stays in $q_{0}$ or switches to $q_{1}$ and remains there),
- for every $p_{1}, p_{2} \in Q$ we define

$$
\delta_{0}\left(\varepsilon, \alpha, q_{0}\right)=1 \text { and } \delta_{0}\left(\varepsilon, \alpha, q_{1}\right)=0 \text { and } \delta_{1}\left(p_{1}, \gamma, p_{2}\right)= \begin{cases}1 & \text { if } p_{1} p_{2} \in\left\{q_{0} q_{0}, q_{0} q_{1}, q_{1} q_{1}\right\} \\ 0 & \text { otherwise }\end{cases}
$$

- $F_{q_{0}}=0$ and $F_{q_{1}}=1$.

In Figure 3.9 we show the $\left(\Sigma, \mathrm{F}_{2}\right)$-wta which r-recognizes odd. Clearly, $\mathcal{A}$ has identity transition weights and it is root weight normalized; $\mathcal{A}$ is not bu deterministic.

Let $n \in \mathbb{N}$ and let $\xi=\gamma^{n}(\alpha)$. Then

$$
\llbracket \mathcal{A} \rrbracket^{\mathrm{run}}(\xi)=\bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}}(\xi)} \mathrm{wt}(\xi, \rho) \otimes F_{\rho(\varepsilon)}=\bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}}\left(q_{1}, \xi\right)} \mathrm{wt}(\xi, \rho)=\bigoplus_{\rho \in[n]} 1=\operatorname{odd}(\xi)
$$

Here we use the additive monoid $(\{0,1\}, \oplus, 0)$ of $\mathrm{F}_{2}$ to count modulo 2. At the same time, for each strong bimonoid $\mathrm{B}=(B, \oplus, \otimes, \mathbb{O}, \mathbb{1})$, we can easily design a crisp deterministic $(\Sigma, \mathrm{B})$-wta with two states and identity root weights such that it simulates counting modulo 2 in its state behaviour. Hence, for each strong bimonoid $B$, the corresponding mapping odd ${ }_{B}: T_{\Sigma} \rightarrow\{0, \mathbb{1}\}$ is r-recognizable by a crisp deterministic ( $\Sigma, \mathrm{B}$ )-wta with identity root weights. We note that we use the state behaviour of a wta for "counting" also in Example 3.2.16.


Figure 3.9: The $\left(\Sigma, \mathrm{F}_{2}\right)$-wta $\mathcal{A}$ which r-recognizes odd.


Figure 3.10: The $(\Sigma, N a t)$-wta $\mathcal{A}=(Q, \delta, F)$ which r-recognizes $\#_{\sigma(., \alpha)}$.

## Example 3.2.11. (Number of occurrences of a pattern, semiring Nat.)

Let $\Sigma=\left\{\sigma^{(2)}, \gamma^{(1)}, \alpha^{(0)}\right\}$. We define the mapping

$$
\#_{\sigma(., \alpha)}: \mathrm{T}_{\Sigma} \rightarrow \mathbb{N} \quad \text { with } \quad \#_{\sigma(., \alpha)}(\xi)=|U(\xi)| \text { for each } \xi \in \mathrm{T}_{\Sigma}
$$

where $U(\xi)=\{u \in \operatorname{pos}(\xi) \mid \xi(u)=\sigma, \xi(u 2)=\alpha\}$. Intuitively, $\#_{\sigma(., \alpha)}(\xi)$ is the number of occurrences of the pattern $\sigma(., \alpha)$ in $\xi$. For instance, $\#_{\sigma(., \alpha)}(\sigma(\gamma(\sigma(\alpha, \alpha)), \sigma(\alpha, \alpha)))=2$.

As weight algebra we use the natural number semiring $\mathrm{Nat}=(\mathbb{N},+, \cdot, 0,1)$. Thus, the mapping $\#_{\sigma(., \alpha)}$ is a $(\Sigma, N a t)$-weighted tree language.

We construct the $(\Sigma, N a t)$-wta $\mathcal{A}=(Q, \delta, F)$ which r-recognizes $\#_{\sigma(., \alpha)}$, as follows.

- $Q=\{\perp, a, f\}$ (intuitively, $\perp$ ignores occurrences of the pattern, a detects an $\alpha$-labeled leaf, and $f$ reports "pattern found" up to the root),
- for every $q_{1}, q_{2}, q \in Q$ we define

$$
\begin{gathered}
\delta_{0}(\varepsilon, \alpha, q)=\left\{\begin{array}{ll}
1 & \text { if } q \in\{\perp, a\} \\
0 & \text { otherwise }
\end{array} \quad \delta_{1}\left(q_{1}, \gamma, q\right)= \begin{cases}1 & \text { if } q_{1} q \in\{\perp \perp, f f\} \\
0 & \text { otherwise }\end{cases} \right. \\
\delta_{2}\left(q_{1} q_{2}, \sigma, q\right)= \begin{cases}1 & \text { if } q_{1} q_{2} q \in\{\perp \perp \perp, \perp a f, \perp f f, f \perp f\} \\
0 & \text { otherwise }\end{cases}
\end{gathered}
$$

- $F_{\perp}=F_{a}=0$ and $F_{f}=1$.

Clearly, $\mathcal{A}$ is root weight normalized; $\mathcal{A}$ is not total, because there does not exist a state $q$ with $\delta_{1}(a, \gamma, q) \neq$ $0 ; \mathcal{A}$ is not bu deterministic. In Figure 3.10 we represent $\mathcal{A}$ as an fta-hypergraph.

Let $\xi \in \mathrm{T}_{\Sigma}$. We prove that $\llbracket \mathcal{A} \rrbracket^{\mathrm{run}}(\xi)=\#_{\sigma(., \alpha)}(\xi)$. For this we define, for each $u \in U(\xi)$, the special run $\rho_{u}^{\xi}: \operatorname{pos}(\xi) \rightarrow Q$ for each $w \in \operatorname{pos}(\xi)$ by

$$
\rho_{u}^{\xi}(w)= \begin{cases}f & \text { if } w \text { is a prefix of } u \\ a & \text { if } w=u 2 \\ \perp & \text { otherwise }\end{cases}
$$

It is obvious that $\operatorname{wt}\left(\xi, \rho_{u}^{\xi}\right)=1$ and $\rho_{u}^{\xi}(\varepsilon)=f$. Hence $\mathrm{wt}\left(\xi, \rho_{u}^{\xi}\right) \cdot F_{\rho_{u}^{\xi}(\varepsilon)}=1$. Moreover, $\mathrm{wt}(\xi, \rho) \cdot F_{\rho(\varepsilon)}=$ 0 for each $\rho \in \mathrm{R}_{\mathcal{A}}(\xi) \backslash\left\{\rho_{u}^{\xi} \mid u \in U(\xi)\right\}$.

Then we can calculate as follows.

Hence $\llbracket \mathcal{A} \rrbracket^{\text {run }}=\#_{\sigma(., \alpha)}$ and thus $\#_{\sigma(., \alpha)} \in \operatorname{Rec}^{\text {run }}(\Sigma$, Nat $)$.
Next we show that $\mathcal{A}$ also i-recognizes $\#_{\sigma(., \alpha)}$. By induction on $\mathrm{T}_{\Sigma}$, we prove that the following statement holds:

$$
\begin{equation*}
\text { For each } \xi \in \mathrm{T}_{\Sigma} \text {, we have } \mathrm{h}_{\mathcal{A}}(\xi)_{\perp}=1, \mathrm{~h}_{\mathcal{A}}(\xi)_{a}=c(\xi) \text {, and } \mathrm{h}_{\mathcal{A}}(\xi)_{f}=\#_{\sigma(., \alpha)}(\xi) \tag{3.19}
\end{equation*}
$$ where $c(\xi)=1$ if $\xi=\alpha$, and 0 otherwise.

I.B.: Let $\xi=\alpha$. Then

$$
\begin{aligned}
\mathrm{h}_{\mathcal{A}}(\alpha)_{\perp} & =\delta_{0}(\varepsilon, \alpha, \perp)=1 \\
\mathrm{~h}_{\mathcal{A}}(\alpha)_{a} & =\delta_{0}(\varepsilon, \alpha, a)=1=c(\alpha), \text { and } \\
\mathrm{h}_{\mathcal{A}}(\alpha)_{f} & =\delta_{0}(\varepsilon, \alpha, f)=0=\#_{\sigma(., \alpha)}(\alpha)
\end{aligned}
$$

I.S.: Let $\xi=\gamma\left(\xi^{\prime}\right)$. Then

$$
\begin{aligned}
\mathrm{h}_{\mathcal{A}}\left(\gamma\left(\xi^{\prime}\right)\right)_{\perp} & =\mathrm{h}_{\mathcal{A}}\left(\xi^{\prime}\right)_{\perp} \cdot \delta_{1}(\perp, \gamma, \perp)=1 \\
\mathrm{~h}_{\mathcal{A}}\left(\gamma\left(\xi^{\prime}\right)\right)_{a} & =0=c\left(\gamma\left(\xi^{\prime}\right)\right), \text { and } \\
\mathrm{h}_{\mathcal{A}}\left(\gamma\left(\xi^{\prime}\right)\right)_{f} & =\mathrm{h}_{\mathcal{A}}\left(\xi^{\prime}\right)_{f} \cdot \delta_{1}(f, \gamma, f)=\#_{\sigma(., \alpha)}\left(\xi^{\prime}\right)=\#_{\sigma(., \alpha)}\left(\gamma\left(\xi^{\prime}\right)\right)
\end{aligned}
$$

Let $\xi=\sigma\left(\xi_{1}, \xi_{2}\right)$. Then

$$
\begin{aligned}
\mathrm{h}_{\mathcal{A}}\left(\sigma\left(\xi_{1}, \xi_{2}\right)\right)_{\perp}= & \mathrm{h}_{\mathcal{A}}\left(\xi_{1}\right)_{\perp} \cdot \mathrm{h}_{\mathcal{A}}\left(\xi_{2}\right)_{\perp} \cdot \delta_{2}(\perp \perp, \sigma, \perp)=1, \\
\mathrm{~h}_{\mathcal{A}}\left(\sigma\left(\xi_{1}, \xi_{2}\right)\right)_{a}= & 0=c\left(\sigma\left(\xi_{1}, \xi_{2}\right)\right), \text { and } \\
\mathrm{h}_{\mathcal{A}}\left(\sigma\left(\xi_{1}, \xi_{2}\right)\right)_{f}= & \mathrm{h}_{\mathcal{A}}\left(\xi_{1}\right)_{\perp} \cdot \mathrm{h}_{\mathcal{A}}\left(\xi_{2}\right)_{a} \cdot \delta_{2}(\perp a, \sigma, f)+ \\
& \mathrm{h}_{\mathcal{A}}\left(\xi_{1}\right)_{\perp} \cdot \mathrm{h}_{\mathcal{A}}\left(\xi_{2}\right)_{f} \cdot \delta_{2}(\perp f, \sigma, f)+ \\
& \mathrm{h}_{\mathcal{A}}\left(\xi_{1}\right)_{f} \cdot \mathrm{~h}_{\mathcal{A}}\left(\xi_{2}\right)_{\perp} \cdot \delta_{2}(f \perp, \sigma, f) \\
= & c\left(\xi_{2}\right)+\#_{\sigma(., \alpha)}\left(\xi_{2}\right)+\#_{\sigma(., \alpha)}\left(\xi_{1}\right) \\
= & \#_{\sigma(., \alpha)}\left(\sigma\left(\xi_{1}, \xi_{2}\right)\right)
\end{aligned}
$$

This proves (3.19). Then for each $\xi \in \mathrm{T}_{\Sigma}$ we have

$$
\llbracket \mathcal{A} \rrbracket^{\mathrm{init}}(\xi)=\underset{q \in Q}{ } \mathrm{~h}_{\mathcal{A}}(\xi)_{q} \cdot F_{q}=\mathrm{h}_{\mathcal{A}}(\xi)_{f}=\#_{\sigma(., \alpha)}(\xi)
$$

Hence $\#_{\sigma(., \alpha)} \in \operatorname{Rec}^{\mathrm{init}}(\Sigma$, Nat $)$.

Example 3.2.12. (Number of occurrences of a pattern, semiring Nat ${ }_{\text {max, }}+\cdot$ ) Again, let $\Sigma=$ $\left\{\sigma^{(2)}, \gamma^{(1)}, \alpha^{(0)}\right\}$ and let us consider the mapping $\#_{\sigma(., \alpha)}: \mathrm{T}_{\Sigma} \rightarrow \mathbb{N}$ defined in Example 3.2.11.

We construct an $\left(\Sigma, \mathrm{Nat}_{\text {max },+}\right)$-wta $\mathcal{A}=(Q, \delta, F)$ which r-recognizes $\#_{\sigma(., \alpha)}$, where $\mathrm{Nat}_{\mathrm{max},+}$ is the arctic semiring. For this, we let

- $Q=\{\bar{\sigma}, \bar{\gamma}, \bar{\alpha}\}$,
- for every $q_{1}, q_{2}, q \in Q$ we define

$$
\begin{gathered}
\delta_{0}(\varepsilon, \alpha, q)= \begin{cases}0 & \text { if } q=\bar{\alpha} \\
-\infty & \text { otherwise }\end{cases} \\
\delta_{1}\left(q_{1}, \gamma, q\right)= \begin{cases}0 & \text { if } q=\bar{\gamma} \\
-\infty & \text { otherwise }\end{cases} \\
\delta_{2}\left(q_{1} q_{2}, \sigma, q\right)= \begin{cases}1 & \text { if } q_{2}=\bar{\alpha} \text { and } q=\bar{\sigma} \\
0 & \text { if } q_{2} \neq \bar{\alpha} \text { and } q=\bar{\sigma} \\
-\infty & \text { otherwise }\end{cases}
\end{gathered}
$$

- for each $q \in Q$ we define $F_{q}=0$.

It is clear that $\mathcal{A}$ is bu deterministic. Since $\delta_{2}\left(q_{1} \bar{\alpha}, \sigma, \bar{\sigma}\right) \notin\{-\infty, 0\}$ for each $q_{1} \in Q$, the wta $\mathcal{A}$ is not crisp deterministic.

Now let $\xi \in \mathrm{T}_{\Sigma}$. We define the particular run $\rho^{\xi}$ on $\xi$ such that $\rho^{\xi}(w)=\overline{\xi(w)}$ for each $w \in \operatorname{pos}(\xi)$. Then, for each $w \in \operatorname{pos}(\xi)$, the weight of the transition induced by $\rho^{\xi}$ on $\xi$ at $w$ is 1 if $\xi(w)=\sigma$ and $\xi(w 2)=\alpha$ (i.e., the pattern $\sigma(., \alpha)$ shows up at $w$ ) and 0 otherwise. Hence wt $\left(\xi, \rho^{\xi}\right)=\#_{\sigma(., \alpha)}(\xi)$. Moreover, for each $\rho \in \mathrm{R}_{\mathcal{A}}(\xi) \backslash\left\{\rho^{\xi}\right\}$ we have $\mathrm{wt}(\xi, \rho)=-\infty$.

Thus we can calculate as follows.

$$
\llbracket \mathcal{A} \rrbracket^{\mathrm{run}}(\xi)=\max \left(\mathrm{wt}(\xi, \rho)+F_{\rho(\varepsilon)} \mid \rho \in \mathrm{R}_{\mathcal{A}}(\xi)\right)=\mathrm{wt}\left(\xi, \rho^{\xi}\right)+F_{\rho \xi(\varepsilon)}=\#_{\sigma(., \alpha)}(\xi)+0=\#_{\sigma(., \alpha)}(\xi),
$$

hence $\llbracket \mathcal{A} \rrbracket^{\text {run }}(\xi)=\#_{\sigma(., \alpha)}(\xi)$. Thus $\#_{\sigma(., \alpha)} \in \operatorname{bud}-\operatorname{Rec}^{\text {run }}\left(\Sigma, \operatorname{Nat}_{\text {max },+}\right)$.

Example 3.2.13. (Difference of numbers of occurrences of nullary symbols.) Let $\Sigma=$ $\left\{\sigma^{(2)}, \alpha^{(0)}, \beta^{(0)}\right\}$ be a ranked alphabet. We define the weighted tree language

$$
\begin{aligned}
\operatorname{diff}: \mathrm{T}_{\Sigma} & \rightarrow \mathbb{Z}_{\infty} \quad \text { such that, for each } \xi \in \mathrm{T}_{\Sigma}, \text { we let } \\
\operatorname{diff}(\xi) & = \begin{cases}\left|\operatorname{pos}_{\alpha}\left(\xi_{1}\right)\right|-\left|\operatorname{pos}_{\beta}\left(\xi_{2}\right)\right| & \text { if } \xi=\sigma\left(\xi_{1}, \xi_{2}\right) \text { for some } \xi_{1}, \xi_{2} \in \mathrm{~T}_{\Sigma} \\
\infty & \text { otherwise }\end{cases}
\end{aligned}
$$

For instance, $\operatorname{diff}(\sigma(\alpha, \sigma(\beta, \beta)))=-1$. As weight algebra we consider the tropical semiring Int $_{\text {min },+}=$ $\left(\mathbb{Z}_{\infty}, \min ,+, \infty, 0\right)$ over $\mathbb{Z}$. Thus, the mapping diff is a $\left(\Sigma, \operatorname{lnt}_{\min ,+}\right)$-weighted tree language.

We construct the $\left(\Sigma, \operatorname{lnt}_{\min ,+}\right)$-wta $\mathcal{A}=(Q, \delta, F)$ which i-recognizes diff, as follows:

- $Q=\left\{\#_{\alpha}, \#_{\beta}, d\right\}$ (intuitively, $\#_{\alpha}$ calculates the number of occurrences of $\alpha$, $\#_{\beta}$ calculates the negative of the number of occurrences of $\beta$, and $d$ calculates the difference),
- for each $q \in Q$ we define

$$
\delta_{0}(\varepsilon, \alpha, q)=\left\{\begin{array}{ll}
1 & \text { if } q=\#_{\alpha} \\
0 & \text { if } q=\#_{\beta} \\
\infty & \text { otherwise }
\end{array} \quad \delta_{0}(\varepsilon, \beta, q)= \begin{cases}0 & \text { if } q=\#_{\alpha} \\
-1 & \text { if } q=\#_{\beta} \\
\infty & \text { otherwise }\end{cases}\right.
$$

and for each $q_{1}, q_{2}, q \in Q$ we define

$$
\delta_{2}\left(q_{1} q_{2}, \sigma, q\right)= \begin{cases}0 & \text { if } q_{1} q_{2} q \in\left\{\#_{\alpha} \#_{\alpha} \#_{\alpha}, \#_{\beta} \#_{\beta} \#_{\beta}, \#_{\alpha} \#_{\beta} d\right\} \\ \infty & \text { otherwise }\end{cases}
$$

- $F_{\#_{\alpha}}=F_{\#_{\beta}}=\infty$ and $F_{d}=0$.


Figure 3.11: The $\left(\Sigma, \operatorname{lnt}_{\min ,+}\right)$-wta $\mathcal{A}=(Q, \delta, F)$ which i-recognizes diff.

Clearly, $\mathcal{A}$ is root weight normalized; $\mathcal{A}$ is not total, because there does not exist a state $q$ such that $\delta_{2}(d d, \sigma, q) \neq \infty ; \mathcal{A}$ is not bu deterministic, because $\delta_{0}\left(\varepsilon, \alpha, \#_{\alpha}\right) \neq \infty \neq \delta_{0}\left(\varepsilon, \alpha, \#_{\beta}\right)$. In Figure 3.11 we represent $\mathcal{A}$ as fta-hypergraph.

First we observe that, for each $\xi \in \mathrm{T}_{\Sigma}$, we have

$$
\begin{equation*}
\mathrm{h}_{\mathcal{A}}(\xi)_{\#_{\alpha}}=\left|\operatorname{pos}_{\alpha}(\xi)\right| \quad \text { and } \mathrm{h}_{\mathcal{A}}(\xi)_{\#_{\beta}}=-\left|\operatorname{pos}_{\beta}(\xi)\right| \tag{3.20}
\end{equation*}
$$

Next we prove that $\llbracket \mathcal{A} \rrbracket^{\text {init }}=$ diff. Let $\xi \in \mathrm{T}_{\Sigma}$. By definition of $F$, we have:

$$
\llbracket \mathcal{A} \rrbracket^{\text {init }}(\xi)=\min \left(\mathrm{h}_{\mathcal{A}}(\xi)_{q}+F_{q} \mid q \in Q\right)=\mathrm{h}_{\mathcal{A}}(\xi)_{d}+F_{d}=\mathrm{h}_{\mathcal{A}}(\xi)_{d}
$$

Let $\xi \in\{\alpha, \beta\}$. Then $\mathrm{h}_{\mathcal{A}}(\xi)_{d}=\delta_{0}(\varepsilon, \xi, d)=\infty$. Hence $\llbracket \mathcal{A} \rrbracket^{\text {init }}(\xi)=\infty=\operatorname{diff}(\xi)$.
Now let $\xi=\sigma\left(\xi_{1}, \xi_{2}\right)$ for some $\xi_{1}, \xi_{2} \in \mathrm{~T}_{\Sigma}$. Then we can calculate as follows:

$$
\begin{align*}
\mathrm{h}_{\mathcal{A}}\left(\sigma\left(\xi_{1}, \xi_{2}\right)\right)_{d} & =\min \left(\mathrm{h}_{\mathcal{A}}\left(\xi_{1}\right)_{q_{1}}+\mathrm{h}_{\mathcal{A}}\left(\xi_{2}\right)_{q_{2}}+\delta_{2}\left(q_{1} q_{2}, \sigma, d\right) \mid q_{1}, q_{2} \in Q\right) \\
& =\mathrm{h}_{\mathcal{A}}\left(\xi_{1}\right)_{\#_{\alpha}}+\mathrm{h}_{\mathcal{A}}\left(\xi_{2}\right)_{\#_{\beta}}+\delta_{2}\left(\#_{\alpha} \#_{\beta}, \sigma, d\right) \\
& =\mathrm{h}_{\mathcal{A}}\left(\xi_{1}\right)_{\#_{\alpha}}+\mathrm{h}_{\mathcal{A}}\left(\xi_{2}\right)_{\#_{\beta}}+0 \\
& =\left|\operatorname{pos}_{\alpha}\left(\xi_{1}\right)\right|-\left|\operatorname{pos}_{\beta}\left(\xi_{2}\right)\right|  \tag{3.20}\\
& =\operatorname{diff}\left(\sigma\left(\xi_{1}, \xi_{2}\right)\right)
\end{align*}
$$

Hence $\llbracket \mathcal{A} \rrbracket^{\text {init }}=$ diff and thus diff $\in \operatorname{Rec}^{\text {init }}\left(\Sigma\right.$, Int $\left._{\text {min },+}\right)$.

Example 3.2.14. (Difference of numbers of occurrences of symbols in monadic trees.) Let $\Sigma=\left\{\gamma^{(1)}, \sigma^{(1)}, \alpha^{(0)}\right\}$ be a string ranked alphabet. We define the mapping

$$
\operatorname{diffm}: \mathrm{T}_{\Sigma} \rightarrow \mathbb{Z} \quad \text { with } \operatorname{diffm}(\xi)=\left|\operatorname{pos}_{\gamma}(\xi)\right|-\left|\operatorname{pos}_{\sigma}(\xi)\right| \text { for each } \xi \in \mathrm{T}_{\Sigma}
$$

As weight algebra we consider the ring Int $=(\mathbb{Z},+, \cdot, 0,1)$ of integers. Thus, the mapping diffm is a ( $\Sigma, \operatorname{lnt}$ )-weighted tree language.

We construct the ( $\Sigma$, Int)-wta $\mathcal{A}=(Q, \delta, F)$ which i-recognizes diffm, as follows.

- $Q=\{1, d\}$ (intuitively, the state 1 calculates the integer 1 and, at each non-leaf position $u$, the state $d$ increases or decreases the difference by 1 depending on the symbol at $u$ ),


Figure 3.12: The $(\Sigma, \operatorname{lnt})$-wta $\mathcal{A}=(Q, \delta, F)$ which i-recognizes diffm.

- $\delta_{0}(\varepsilon, \alpha, 1)=1$ and $\delta_{0}(\varepsilon, \alpha, d)=0$, and for every $q_{1}, q_{2} \in Q$ and $\nu \in\{\gamma, \sigma\}$ we define

$$
\delta_{1}\left(q_{1}, \nu, q_{2}\right)= \begin{cases}1 & \text { if } q_{1} q_{2} \in\{11, d d\} \\ 1 & \text { if } q_{1} q_{2}=1 d \text { and } \nu=\gamma \\ -1 & \text { if } q_{1} q_{2}=1 d \text { and } \nu=\sigma \\ 0 & \text { if } q_{1} q_{2}=d 1\end{cases}
$$

- $F_{1}=0$ and $F_{d}=1$.

Clearly, $\mathcal{A}$ is root weight normalized; $\mathcal{A}$ is not bu deterministic, because $\delta_{1}(1, \gamma, 1) \neq 0 \neq \delta_{1}(1, \gamma, d)$. In Figure 3.12 we represent $\mathcal{A}$ as fta-hypergraph.

First, by induction on $\mathrm{T}_{\Sigma}$, we prove that the following statement holds:

$$
\begin{equation*}
\text { For every } \xi \in \mathrm{T}_{\Sigma} \text {, we have } \mathrm{h}_{\mathcal{A}}(\xi)_{1}=1 \text { and } \mathrm{h}_{\mathcal{A}}(\xi)_{d}=\left|\operatorname{pos}_{\gamma}(\xi)\right|-\left|\operatorname{pos}_{\sigma}(\xi)\right| \tag{3.21}
\end{equation*}
$$

I.B.: Let $\xi=\alpha$. Then $\mathrm{h}_{\mathcal{A}}(\xi)_{1}=\delta_{0}(\varepsilon, \alpha, 1)=1$ and $\mathrm{h}_{\mathcal{A}}(\xi)_{d}=\delta_{0}(\varepsilon, \alpha, d)=0$. Thus (3.21) holds.
I.S.: Let $\xi=\gamma\left(\xi^{\prime}\right)$ for some $\xi^{\prime} \in \mathrm{T}_{\Sigma}$. Then

$$
\begin{aligned}
\mathrm{h}_{\mathcal{A}}(\xi)_{1} & ={\underset{q \in Q}{ } \mathrm{~h}_{\mathcal{A}}\left(\xi^{\prime}\right)_{q} \cdot \delta_{1}(q, \gamma, 1)}=\mathrm{h}_{\mathcal{A}}\left(\xi^{\prime}\right)_{1} \cdot \delta_{1}(1, \gamma, 1) \\
& =1 \cdot 1=1
\end{aligned}
$$

$$
=\mathrm{h}_{\mathcal{A}}\left(\xi^{\prime}\right)_{1} \cdot \delta_{1}(1, \gamma, 1) \quad \text { (by definition of } \delta_{1} \text { and by I.H.) }
$$

and

$$
\begin{array}{rll}
\mathrm{h}_{\mathcal{A}}(\xi)_{d} & =\underset{q \in Q}{ } \mathrm{~h}_{\mathcal{A}}\left(\xi^{\prime}\right)_{q} \cdot \delta_{1}(q, \gamma, d)=\mathrm{h}_{\mathcal{A}}\left(\xi^{\prime}\right)_{1} \cdot \delta_{1}(1, \gamma, d)+\mathrm{h}_{\mathcal{A}}\left(\xi^{\prime}\right)_{d} \cdot \delta_{1}(d, \gamma, d) \\
& =1 \cdot 1+\left(\left|\operatorname{pos}_{\gamma}\left(\xi^{\prime}\right)\right|-\left|\operatorname{pos}_{\sigma}\left(\xi^{\prime}\right)\right|\right) \cdot 1 \quad \quad \text { (by definition of } \delta_{1} \text { and by I.H.) } \\
& =\left(\left|\operatorname{pos}_{\gamma}\left(\xi^{\prime}\right)\right|+1\right)-\left|\operatorname{pos}_{\sigma}\left(\xi^{\prime}\right)\right| \\
& =\left|\operatorname{pos}_{\gamma}(\xi)\right|-\left|\operatorname{pos}_{\sigma}(\xi)\right| .
\end{array}
$$

In a similar way we can prove that (3.21) holds for $\xi=\sigma\left(\xi^{\prime}\right)$.
Using Statement (3.21), we obtain for each $\xi \in \mathrm{T}_{\Sigma}$ :

$$
\llbracket \mathcal{A} \rrbracket^{\mathrm{init}}(\xi)=\underset{q \in Q}{ } \mathrm{~L}_{\mathcal{A}}(\xi)_{q} \cdot F_{q}=\mathrm{h}_{\mathcal{A}}(\xi)_{d}=\left|\operatorname{pos}_{\gamma}(\xi)\right|-\left|\operatorname{pos}_{\sigma}(\xi)\right|=\operatorname{diffm}(\xi)
$$

Thus diffm $\in \operatorname{Rec}^{\text {init }}(\Sigma, \operatorname{lnt})$.


Figure 3.13: The $\left(\Sigma, \operatorname{Nat}_{\text {max },+}\right)$-wta $\mathcal{A}=(Q, \delta, F)$ which i-recognizes zigzag.

Example 3.2.15. (Length of zigzag-path.) Let $\Sigma=\left\{\sigma^{(2)}, \alpha^{(0)}\right\}$ be a ranked alphabet. We define the mapping zigzag: $\mathrm{T}_{\Sigma} \rightarrow \mathbb{N}$ which, intuitively, computes for each tree the length of its zigzag-path (where "zig" means "go to the left child", and "zag" means "go to the right child"). Formally, we first define the auxiliary mappings $\mathrm{zz}_{l}: \mathrm{T}_{\Sigma} \rightarrow\left(\mathbb{N}_{+}\right)^{*}$ and $\mathrm{zz}_{r}: \mathrm{T}_{\Sigma} \rightarrow\left(\mathbb{N}_{+}\right)^{*}$ by induction on $\mathrm{T}_{\Sigma}$ as follows. We let

$$
\mathrm{zz}_{l}(\alpha)=\mathrm{zz}_{r}(\alpha)=\varepsilon
$$

and, for each $\xi=\sigma\left(\xi_{1}, \xi_{2}\right)$, we define

$$
\operatorname{zz}_{l}\left(\sigma\left(\xi_{1}, \xi_{2}\right)\right)=1 \mathrm{zz}_{r}\left(\xi_{1}\right) \quad \text { and } \quad \mathrm{zz}_{r}\left(\sigma\left(\xi_{1}, \xi_{2}\right)\right)=2 \mathrm{zz}_{l}\left(\xi_{2}\right)
$$

Then we let

$$
\operatorname{zigzag}: \mathrm{T}_{\Sigma} \rightarrow \mathbb{N} \quad \text { with } \operatorname{zigzag}(\xi)=\left|\mathrm{zz}_{l}(\xi)\right| \text { for each } \xi \in \mathrm{T}_{\Sigma}
$$

For instance, for $\xi=\sigma\left(\sigma\left(\zeta_{1}, \alpha\right), \zeta_{2}\right)$ and any $\zeta_{1}, \zeta_{2} \in \mathrm{~T}_{\Sigma}$, we have $\operatorname{zigzag}(\xi)=\left|z_{l}(\xi)\right|=|12|=2$.
As weight algebra we use the arctic semiring $\operatorname{Nat}_{\max ,+}=\left(\mathbb{N}_{-\infty}, \max ,+,-\infty, 0\right)$. Thus, the mapping zigzag is a $\left(\Sigma\right.$, Nat $_{\text {max }},+$ )-weighted tree language.

We construct the $\left(\Sigma, \mathrm{Nat}_{\text {max },+}\right)$-wta $\mathcal{A}=(Q, \delta, F)$ which i-recognizes zigzag as follows.

- $Q=\{l, r, 0\}$ (intuitively, $l$ and $r$ stand for "left" and "right", respectively, and 0 calculates the $0 \in \mathbb{N}$ ),
- $\delta_{0}(\varepsilon, \alpha, l)=\delta_{0}(\varepsilon, \alpha, r)=\delta_{0}(\varepsilon, \alpha, 0)=0$ and for every $q_{1}, q_{2}, q \in Q$

$$
\delta_{2}\left(q_{1} q_{2}, \sigma, q\right)= \begin{cases}1 & \text { if } q_{1} q_{2} q \in\{r 0 l, 0 l r\} \\ 0 & \text { if } q_{1} q_{2} q=000 \\ -\infty & \text { otherwise }\end{cases}
$$

- $F_{l}=0, F_{r}=F_{0}=-\infty$.

Clearly, $\mathcal{A}$ is root weight normalized; $\mathcal{A}$ is not bu deterministic. In Figure 3.13 we represent $\mathcal{A}$ as an fta-hypergraph.

By induction on $\mathrm{T}_{\Sigma}$, we can prove that the following statement holds:

$$
\begin{equation*}
\text { For each } \xi \in \mathrm{T}_{\Sigma} \text {, we have } \mathrm{h}_{\mathcal{A}}(\xi)_{l}=\left|\mathrm{zz}_{l}(\xi)\right|, \mathrm{h}_{\mathcal{A}}(\xi)_{r}=\left|\mathrm{zz}_{r}(\xi)\right| \text {, and } \mathrm{h}_{\mathcal{A}}(\xi)_{0}=0 \tag{3.22}
\end{equation*}
$$

I.B.: Let $\xi=\alpha$. Then, for each $q \in Q$, we have $\mathrm{h}_{\mathcal{A}}(\xi)_{q}=\delta_{0}(\varepsilon, \alpha, q)=0=\left|\mathrm{zz}_{l}(\xi)\right|=\left|\mathrm{zz}_{r}(\xi)\right|$.
I.S.: Now let $\xi=\sigma\left(\xi_{1}, \xi_{2}\right)$ for some $\xi_{1}, \xi_{2} \in \mathrm{~T}_{\Sigma}$. Then

$$
\begin{align*}
\mathrm{h}_{\mathcal{A}}\left(\sigma\left(\xi_{1}, \xi_{2}\right)\right)_{l} & =\max \left(\mathrm{h}_{\mathcal{A}}\left(\xi_{1}\right)_{q_{1}}+\mathrm{h}_{\mathcal{A}}(\xi)_{q_{2}}+\delta_{2}\left(q_{1} q_{2}, \sigma, l\right) \mid q_{1}, q_{2} \in Q\right) \\
& =\mathrm{h}_{\mathcal{A}}\left(\xi_{1}\right)_{r}+\mathrm{h}_{\mathcal{A}}\left(\xi_{2}\right)_{0}+\delta_{2}(r 0, \sigma, l) \\
& =\left|\mathrm{zz}_{r}\left(\xi_{1}\right)\right|+0+1  \tag{byI.H.}\\
& =\left|\mathrm{zz}_{l}\left(\sigma\left(\xi_{1}, \xi_{2}\right)\right)\right|
\end{align*}
$$

In a similar way we can prove that $\mathrm{h}_{\mathcal{A}}\left(\sigma\left(\xi_{1}, \xi_{2}\right)\right)_{r}=\left|\mathrm{zz}_{r}\left(\sigma\left(\xi_{1}, \xi_{2}\right)\right)\right|$, and $\mathrm{h}_{\mathcal{A}}\left(\sigma\left(\xi_{1}, \xi_{2}\right)\right)_{0}=0$. This finishes the proof of Statement (3.22).

Now let $\xi \in \mathrm{T}_{\Sigma}$. Then, using (3.22) at the fourth equality below, we obtain

$$
\llbracket \mathcal{A} \rrbracket^{\text {init }}(\xi)=\max \left(\mathrm{h}_{\mathcal{A}}(\xi)_{q}+F_{q} \mid q \in Q\right)=\mathrm{h}_{\mathcal{A}}(\xi)_{l}+F_{l}=\mathrm{h}_{\mathcal{A}}(\xi)_{l}=\left|\operatorname{zz}_{l}(\xi)\right|=\operatorname{zigzag}(\xi)
$$

Thus $\llbracket \mathcal{A} \rrbracket^{\text {init }}=$ zigzag and hence zigzag $\in \operatorname{Rec}^{\text {init }}\left(\Sigma\right.$, Nat $\left._{\text {max },+}\right)$. In DV06, Ex. 3.2] and HMM07, Ex. 2], a ( $\Sigma$, Nat)-wta $N$ (over the semiring of natural numbers) is shown for which $\llbracket N \rrbracket^{\text {init }}=$ zigzag.

Example 3.2.16. (Recognizable step mapping twothree.) Let $\Sigma=\left\{\sigma^{(2)}, \gamma^{(1)}, \alpha^{(0)}\right\}$ be a ranked alphabet. We define the mapping twothree : $\mathrm{T}_{\Sigma} \rightarrow \mathbb{N}$ for each $\xi \in \mathrm{T}_{\Sigma}$ by

$$
\text { twothree }: \mathrm{T}_{\Sigma} \rightarrow \mathbb{N} \quad \text { with } \quad \text { twothree }(\xi)=\left\{\begin{array}{ll}
2 & \text { if }|\operatorname{pos}(\xi)| \text { is even } \\
3 & \text { otherwise }
\end{array} \quad \text { for each } \xi \in \mathrm{T}_{\Sigma}\right.
$$

As weight algebra we use the natural number semiring $\operatorname{Nat}=(\mathbb{N},+, \cdot, 0,1)$. Thus, the mapping twothree is a $(\Sigma, N a t)$-weighted tree language. Since the two $\Sigma$-tree languages

$$
L_{e}=\left\{\xi \in \mathrm{T}_{\Sigma}| | \operatorname{pos}(\xi) \mid \text { is even }\right\} \quad \text { and } \quad L_{o}=\mathrm{T}_{\Sigma} \backslash L_{e}
$$

are recognizable, twothree is even a recognizable step mapping, because

$$
\text { twothree }=\left(2 \cdot \chi_{\mathrm{Nat}}\left(L_{e}\right)+3 \cdot \chi_{\mathrm{Nat}}\left(L_{o}\right)\right)
$$

We construct the ( $\Sigma$, Nat)-wta $\mathcal{A}=(Q, \delta, F)$ which i-recognizes twothree as follows

- $Q=\{e, o\}$ where $e$ and $o$ stand for "even" and "odd", respectively,
- $\delta_{0}(\varepsilon, \alpha, e)=0$ and $\delta_{0}(\varepsilon, \alpha, o)=1$ and for every $q_{1}, q_{2}, q \in Q$ we let

$$
\begin{aligned}
\delta_{1}\left(q_{1}, \gamma, q\right) & = \begin{cases}1 & \text { if }\left(q_{1}=e \text { and } q=o\right) \text { or }\left(q_{1}=o \text { and } q=e\right) \\
0 & \text { otherwise },\end{cases} \\
\delta_{2}\left(q_{1} q_{2}, \sigma, q\right) & = \begin{cases}1 & \text { if }\left(q_{1}=q_{2} \text { and } q=o\right) \text { or }\left(q_{1} \neq q_{2} \text { and } q=e\right) \\
0 & \text { otherwise },\end{cases}
\end{aligned}
$$

and

- $F_{e}=2$ and $F_{o}=3$.

We note that $\mathcal{A}$ is crisp deterministic and consequently also bu deterministic. Figure 3.14 shows the ( $\Sigma$, Nat)-wta $\mathcal{A}$.

Let $\xi \in \mathrm{T}_{\Sigma}$ and $q \in Q$. It is clear that

$$
\mathrm{h}_{\mathcal{A}}(\xi)_{q}= \begin{cases}1 & \text { if }((q=e \text { and }|\operatorname{pos}(\xi)| \text { is even }) \text { or }(q=o \text { and }|\operatorname{pos}(\xi)| \text { is odd })) \\ 0 & \text { otherwise }\end{cases}
$$



Figure 3.14: The ( $\Sigma$, Nat)-wta $\mathcal{A}$ which i-recognizes twothree.

Thus

$$
\begin{aligned}
& \llbracket \mathcal{A} \rrbracket^{\text {init }}(\xi)={\underset{q \in Q}{ }}\left(\mathrm{~h}_{\mathcal{A}}(\xi)_{q} \cdot F_{q}\right) \\
& =\left(\mathrm{h}_{\mathcal{A}}(\xi)_{e} \cdot F_{e}\right)+\left(\mathrm{h}_{\mathcal{A}}(\xi)_{o} \cdot F_{o}\right)=\left(\mathrm{h}_{\mathcal{A}}(\xi)_{e} \cdot 2\right)+\left(\mathrm{h}_{\mathcal{A}}(\xi)_{o} \cdot 3\right) \\
& =\text { twothree }(\xi) \text {. }
\end{aligned}
$$

Hence twothree $\in \operatorname{bud}-\operatorname{Rec}^{\text {init }}(\Sigma, N a t)$.

Example 3.2.17. (Evaluation algebras.) [FSV12, Lm. 4.3] Let ( $\kappa_{k} \mid k \in \mathbb{N}$ ) be an $\mathbb{N}$-indexed family of mappings $\kappa_{k}: \Sigma^{(k)} \rightarrow B$. We recall that $\mathrm{B}=(B, \oplus, \otimes, \mathbb{O}, \mathbb{1})$ is a strong bimonoid, $\mathrm{M}(\Sigma, \kappa)$ is the $(\Sigma, \kappa)$-evaluation algebra and $\mathrm{h}_{\mathrm{M}(\Sigma, \kappa)}: \mathrm{T}_{\Sigma} \rightarrow B$ is the unique $\Sigma$-algebra homomorphism, as defined in Subsection 2.10.2.

We construct the $(\Sigma, \mathrm{B})$-wta $\mathcal{A}=(Q, \delta, F)$ which i-recognizes $\mathrm{h}_{\mathrm{M}(\Sigma, \kappa)}$ as follows.

- $Q=\{q\}$ and $F_{q}=\mathbb{1}$ and
- for every $k \in \mathbb{N}, \sigma \in \Sigma^{(k)}$, we let $\delta_{k}(q \cdots q, \sigma, q)=\kappa_{k}(\sigma)$.

Obviously, $\mathcal{A}$ is bu deterministic and root weight normalized. Moreover, if there exists a $\sigma$ such that $\kappa_{k}(\sigma) \notin\{\mathbb{0}, \mathbb{1}\}$, then $\mathcal{A}$ does not have identity transition weights and $\mathcal{A}$ is not crisp deterministic.

It is easy to see that the vector algebra $\mathrm{V}(\mathcal{A})=\left(B^{Q}, \delta_{\mathcal{A}}\right)$ of $\mathcal{A}$ and the $(\Sigma, \kappa)$-evaluation algebra $\mathrm{M}(\Sigma, \kappa)=(B, \bar{\kappa})$ are isomorphic $\Sigma$-algebras, by identifying each $Q$-vector over $B$ with its one and only component in $B$, and by verifying that, for every $b_{1}, \ldots, b_{k} \in B$, we have

$$
\delta_{\mathcal{A}}(\sigma)\left(\left(b_{1}\right), \ldots,\left(b_{k}\right)\right)_{q}=\left(\bigotimes_{i \in[k]} b_{i}\right) \otimes \delta_{k}(q \cdots q, \sigma, q)=\left(\bigotimes_{i \in[k]} b_{i}\right) \otimes \kappa_{k}(\sigma)=\bar{\kappa}(\sigma)\left(b_{1}, \ldots, b_{k}\right)
$$

Hence, for each $\xi \in \mathrm{T}_{\Sigma}$, we have $\mathrm{h}_{\mathcal{A}}(\xi)_{q}=\mathrm{h}_{\mathrm{M}(\Sigma, \kappa)}(\xi)$. Then, for each $\xi \in \mathrm{T}_{\Sigma}$, we have

$$
\llbracket \mathcal{A} \rrbracket^{\mathrm{init}}(\xi)=\bigoplus_{q \in Q} \mathrm{~h}_{\mathcal{A}}(\xi)_{q} \otimes F_{q}=\mathrm{h}_{\mathcal{A}}(\xi)_{q}=\mathrm{h}_{\mathrm{M}(\Sigma, \kappa)}(\xi)
$$

Thus, in particular, $\mathrm{h}_{\mathrm{M}(\Sigma, \kappa)} \in{\operatorname{bud}-\operatorname{Rec}^{\mathrm{init}}(\Sigma, \mathrm{B}) \text {. }}$
Above we have proved that, for each $\mathbb{N}$-indexed family $\kappa=\left(\kappa_{k} \mid k \in \mathbb{N}\right)$ of mappings $\kappa_{k}: \Sigma^{(k)} \rightarrow B$, the $\Sigma$-algebra homomorphism $\mathrm{h}_{\mathrm{M}(\Sigma, \kappa)}$ is i-recognizable by a bu deterministic ( $\left.\Sigma, \mathrm{B}\right)$-wta. This can be used to show i-recognizability of the mappings size (cf. Example 3.2.3), yield ${ }^{\mathcal{P}}$ (cf. Example 3.2.6), and $\mathrm{BPS}_{A}$ (cf. Example 3.2.7) by instantiating $B$ and providing $\mathbb{N}$-indexed families of mappings appropriately.

- We consider the tropical semiring $\operatorname{Nat}_{\min ,+}=\left(\mathbb{N}_{\infty}, \min ,+, \infty, 0\right)$ and the $\left(\Sigma, \operatorname{Nat}_{\text {min },+}\right)$-weighted tree language

$$
\text { size }: \mathrm{T}_{\Sigma} \rightarrow \mathbb{N}
$$

of Example 3.2.3 with $\Sigma=\left\{\sigma^{(2)}, \alpha^{(0)}\right\}$. We define the $\mathbb{N}$-indexed family $\kappa^{\text {size }}=\left(\kappa_{k}^{\text {size }} \mid k \in \mathbb{N}\right)$ of mappings by $\kappa_{0}^{\text {size }}(\alpha)=\kappa_{2}^{\text {size }}(\sigma)=1$. Then size $=\mathrm{h}_{\mathrm{M}\left(\Sigma, \kappa^{\text {ize }}\right)}$.

- We consider the ranked alphabet $\Sigma=\left\{\sigma^{(2)}, \alpha^{(0)}, \beta^{(0)}\right\}$ and the formal language semiring Lang ${ }_{\Sigma}=$ $\left(\mathcal{P}\left(\Sigma^{*}\right), \cup, \cdot, \emptyset,\{\varepsilon\}\right)$ (where we forget about the rank of symbols). Moreover, we consider the ( $\Sigma$, Lang $_{\Sigma}$ )-weighted tree language

$$
\operatorname{yield}^{\mathcal{P}}: \mathrm{T}_{\Sigma} \rightarrow \mathcal{P}\left(\left(\Sigma^{(0)}\right)^{*}\right)
$$

of Example 3.2.6. We define the $\mathbb{N}$-indexed family $\kappa^{\text {yield }}=\left(\kappa_{k}^{\text {yield }} \mid k \in \mathbb{N}\right)$ of mappings by $\kappa_{0}^{\text {yield }}(\alpha)=\{\alpha\}, \kappa_{0}^{\text {yield }}(\beta)=\{\beta\}$, and $\kappa_{2}^{\text {yield }}(\sigma)=\{\varepsilon\}$. Then yield ${ }^{\mathcal{P}}=\mathrm{h}_{\mathrm{M}\left(\Sigma, \kappa^{\text {yield }}\right)}$.

- Let $A=(Q, I, \delta, F)$ be an $\{\alpha, \beta\}$-fsa. We consider the near semiring $\operatorname{NearSem}_{\mathcal{P}(Q)}=$ $\left(B, \cup, \diamond, \emptyset, \operatorname{id}_{\mathcal{P}(Q)}\right)$ with $B=\{f \mid f: \mathcal{P}(Q) \rightarrow \mathcal{P}(Q), f(\emptyset)=\emptyset\}$ and the $\left(\Sigma, \operatorname{NearSem}_{\mathcal{P}(Q)}\right)$-weighted tree language

$$
\mathrm{BPS}_{A}: \mathrm{T}_{\Sigma} \rightarrow \mathcal{P}(Q)^{\mathcal{P}(Q)}
$$

of Example 3.2.7 with $\Sigma=\left\{\sigma^{(2)}, \alpha^{(0)}, \beta^{(0)}\right\}$. We define the $\mathbb{N}$-indexed family $\kappa^{\mathrm{BPS}}=\left(\kappa_{k}^{\mathrm{BPS}} \mid k \in \mathbb{N}\right)$ of mappings such that, for each $U \in \mathcal{P}(Q)$, we let

$$
\begin{aligned}
& -\kappa_{0}^{\mathrm{BPS}}(\alpha)(U)=\{p \in Q \mid(\exists r \in U):(r, \alpha, p) \in \delta\}, \\
& -\kappa_{0}^{\mathrm{BPS}}(\beta)(U)=\{p \in Q \mid(\exists r \in U):(r, \beta, p) \in \delta\}, \text { and } \\
& \left.-\kappa_{2}^{\mathrm{BPS}}(\sigma)(U)=U \text { (i.e., } \kappa_{2}^{\mathrm{BPS}}(\sigma)=\operatorname{id}_{\mathcal{P}(Q)}\right) .
\end{aligned}
$$

Then $\operatorname{BPS}_{A}=\mathrm{h}_{\mathrm{M}\left(\Sigma, \kappa^{\mathrm{BPS}}\right)}$.
Example 3.2.18. ( $\Sigma$-algebra homomorphism.) Let $\Sigma$ be a ranked alphabet and $\mathbf{B}=(B, \oplus, \otimes, \mathbb{0}, \mathbb{1})$ be a strong bimonoid. Moreover, let $\mathcal{B}=(B, \theta)$ be a $\Sigma$-algebra such that there exists a family $\left(b_{\sigma, w} \mid k \in\right.$ $\left.\mathbb{N}, \sigma \in \Sigma^{(k)}, w \in\{0,1\}^{k}\right)$ over $B$ and, for every $k \in \mathbb{N}, \sigma \in \Sigma^{(k)}$, and $b_{1}, \ldots, b_{k} \in B$, the following holds:

$$
\begin{equation*}
\theta(\sigma)\left(b_{1}, \ldots, b_{k}\right)=\bigoplus_{w \in\{0,1\}^{k}} b_{1}^{w_{1}} \otimes \cdots \otimes b_{k}^{w_{k}} \otimes b_{\sigma, w} \tag{3.23}
\end{equation*}
$$

where $w_{i}$ denotes the $i$-th symbol of $w$ and $b_{i}^{0}=\mathbb{1}$ and $b_{i}^{1}=b_{i}$. Thus $\theta(\sigma)$ is a polynomial over $k$ noncommuting variables with degree at most 1 . Let us denote by $h_{\mathcal{B}}$ the unique $\Sigma$-algebra homomorphism from $T_{\Sigma}$ to $\mathcal{B}$.

We construct the $(\Sigma, \mathrm{B})$-wta $\mathcal{A}=(Q, \delta, F)$ such that $\llbracket \mathcal{A} \rrbracket^{\text {init }}=\mathrm{h}_{\mathcal{B}}$, as follows. We let $Q=\{0,1\}$, $F_{0}=\mathbb{O}$, and $F_{1}=\mathbb{1}$. Moreover, for every $k \in \mathbb{N}, \sigma \in \Sigma^{(k)}$, and $q_{1}, \ldots, q_{k}, q \in Q$, we let

$$
\delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right)= \begin{cases}b_{\sigma, q_{1} \cdots q_{k}} & \text { if } q=1 \\ \mathbb{1} & \text { if } q_{1} \cdots q_{k} q=0^{k+1} \\ \mathbb{O} & \text { otherwise }\end{cases}
$$

By induction on $\mathrm{T}_{\Sigma}$ we prove the following statement:
For each $\xi \in \mathrm{T}_{\Sigma}$, we have: $\mathrm{h}_{\mathcal{A}}(\xi)_{1}=\mathrm{h}_{\mathcal{B}}(\xi)$ and $\mathrm{h}_{\mathcal{A}}(\xi)_{0}=\mathbb{1}$.

Let $\xi=\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)$. Then we can calculate as follows:

$$
\begin{aligned}
& \mathrm{h}_{\mathcal{A}}\left(\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)\right)_{1}=\bigoplus_{q_{1} \cdots q_{k} \in Q^{k}} \mathrm{~h}_{\mathcal{A}}\left(\xi_{1}\right)_{q_{1}} \otimes \cdots \otimes \mathrm{~h}_{\mathcal{A}}\left(\xi_{k}\right)_{q_{k}} \otimes \delta_{k}\left(q_{1} \cdots q_{k}, \sigma, 1\right) \\
&=\bigoplus_{q_{1} \cdots q_{k} \in Q^{k}} \mathrm{~h}_{\mathcal{A}}\left(\xi_{1}\right)_{q_{1}} \otimes \cdots \otimes \mathrm{~h}_{\mathcal{A}}\left(\xi_{k}\right)_{q_{k}} \otimes b_{\sigma, q_{1} \cdots q_{k}} \quad \text { (by construction) } \\
&=\bigoplus_{q_{1} \cdots q_{k} \in Q^{k}} \mathrm{~h}_{\mathcal{B}}\left(\xi_{1}\right)^{q_{1}} \otimes \cdots \otimes \mathrm{~h}_{\mathcal{B}}\left(\xi_{k}\right)^{q_{k}} \otimes b_{\sigma, q_{1} \cdots q_{k}} \\
&=\theta(\sigma)\left(\mathrm{h}_{\mathcal{B}}\left(\xi_{1}\right), \ldots, \mathrm{h}_{\mathcal{B}}\left(\xi_{k}\right)\right) \\
&=\mathrm{h}_{\mathcal{B}}\left(\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)\right) . \quad \text { (by I.H.) } \\
& \\
& \text { (because } \mathrm{h}_{\mathcal{B}} \text { is a } \Sigma \text {-algebra homomorphism) }
\end{aligned}
$$

Moreover we can calculate:

$$
\left.\begin{array}{rlr}
\mathrm{h}_{\mathcal{A}}\left(\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)\right)_{0} & =\bigoplus_{q_{1} \cdots q_{k} \in Q^{k}} \mathrm{~h}_{\mathcal{A}}\left(\xi_{1}\right)_{q_{1}} \otimes \cdots \otimes \mathrm{~h}_{\mathcal{A}}\left(\xi_{k}\right)_{q_{k}} \otimes \delta_{k}\left(q_{1} \cdots q_{k}, \sigma, 0\right) \\
& =\mathrm{h}_{\mathcal{A}}\left(\xi_{1}\right)_{0} \otimes \cdots \otimes \mathrm{~h}_{\mathcal{A}}\left(\xi_{k}\right)_{0} \otimes \mathbb{1} \\
& =\mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes \mathbb{1}=\mathbb{1} \tag{byI.H.}
\end{array} \quad \text { (by construction) }\right) \quad \text { (by I.H.) }
$$

Then, for each $\xi \in \mathrm{T}_{\Sigma}$, we have $\llbracket \mathcal{A} \rrbracket^{\text {init }}(\xi)=\bigoplus_{q \in Q} \mathrm{~h}_{\mathcal{A}}(\xi)_{q} \otimes F_{q}=\mathrm{h}_{\mathcal{A}}(\xi)_{1}=\mathrm{h}_{\mathcal{B}}(\xi)$.
We finish this example with three instances of this general scenario. Each instance results in a wta which we have already constructed in an ad hoc manner, viz., in Example 3.2.17 (evaluation algebras), in Example 3.2.4 (the mapping height : $\mathrm{T}_{\Sigma} \rightarrow \mathbb{N}$ ), and in the proof of Lemma 3.1.5 (the computation of $\left.\langle A\rangle_{\{\oplus, \otimes, 0, \mathbb{1}\}}\right)$.

1. Let $\left(\kappa_{k} \mid k \in \mathbb{N}\right)$ be an $\mathbb{N}$-indexed family of mappings $\kappa_{k}: \Sigma^{(k)} \rightarrow B$. We consider the $(\Sigma, \kappa)$ evaluation algebra $\mathrm{M}(\Sigma, \kappa)=(B, \bar{\kappa})$ and the unique $\Sigma$-algebra homomorphism $\mathrm{h}_{\mathrm{M}(\Sigma, \kappa)}: \mathrm{T}_{\Sigma} \rightarrow B$ (cf. Subsection 2.10.2). We note that $\mathrm{M}(\Sigma, \kappa)$ is a particular $\Sigma$-algebra.
Now we define the family $\left(b_{\sigma, w} \mid k \in \mathbb{N}, \sigma \in \Sigma^{(k)}, w \in\{0,1\}^{k}\right)$ over $B$ such that $b_{\sigma, w}=\kappa_{k}(\sigma)$ if $w=1^{k}$, and $\mathbb{O}$ otherwise, and we define the $\Sigma$-algebra $\mathcal{B}=(B, \theta)$ as in (3.23). Then, for every $b_{1}, \ldots, b_{k} \in B$, we have

$$
\bar{\kappa}(\sigma)\left(b_{1}, \ldots, b_{k}\right)=b_{1} \otimes \cdots \otimes b_{k} \otimes \kappa_{k}(\sigma)=\bigoplus_{w \in\{0,1\}^{k}} b_{1}^{w_{1}} \otimes \cdots \otimes b_{k}^{w_{k}} \otimes b_{\sigma, w}=\theta(\sigma)\left(b_{1}, \ldots, b_{k}\right)
$$

Hence $\kappa=\theta$ and thus $\mathrm{h}_{\mathrm{M}(\Sigma, \kappa)}=\mathrm{h}_{\mathcal{B}}$. Then, using the general scenario, we can construct a $(\Sigma, \mathrm{B})$-wta $\mathcal{A}$ such that $\llbracket \mathcal{A} \rrbracket^{\text {init }}=\mathrm{h}_{\mathrm{M}(\Sigma, \kappa)}$. In this sense, the construction of the current example subsumes the one given in Example 3.2.17 (where the state 1 corresponds to state $q$, and state 0 is superfluous).
2. As strong bimonoid $B$ we consider the semiring $\operatorname{Nat}_{\max ,+}=\left(\mathbb{N}_{-\infty}, \max ,+,-\infty, 0\right)$. Moreover, we extend the $\Sigma$-algebra $\left(\mathbb{N}, \theta_{1}\right)$ defined on page 43 to the $\Sigma$-algebra $\left(\mathbb{N}_{-\infty}, \theta_{1}^{\prime}\right)$ canonically as follows:

- for each $\alpha \in \Sigma^{(0)}$, we let $\theta_{1}^{\prime}(\alpha)()=0$, and
- for every $k \in \mathbb{N}_{+}, \sigma \in \Sigma^{(k)}$, and $n_{1}, \ldots, n_{k} \in \mathbb{N}_{-\infty}$, we let $\theta_{1}^{\prime}(\sigma)\left(n_{1}, \ldots, n_{k}\right)=1+$ $\max \left(n_{1}, \ldots, n_{k}\right)$.
Then the unique $\Sigma$-algebra homomorphism from $\mathrm{T}_{\Sigma}$ to $\left(\mathbb{N}_{-\infty}, \theta_{1}^{\prime}\right)$ is the mapping height : $\mathrm{T}_{\Sigma} \rightarrow \mathbb{N}$.
Now we define the family $\left(b_{\sigma, w} \mid k \in \mathbb{N}, \sigma \in \Sigma^{(k)}, w \in\{0,1\}^{k}\right)$ over $\mathbb{N}_{-\infty}$ such that $b_{\sigma, w}=1$ if $w$ contains exactly one occurrence of 1 , and $-\infty$ otherwise. Moreover, we define the $\Sigma$-algebra $\mathcal{B}=\left(\mathbb{N}_{-\infty}, \theta\right)$ as in (3.23). Then we have

$$
\begin{aligned}
\theta_{1}^{\prime}(\sigma)\left(n_{1}, \ldots, n_{k}\right) & =1+\max \left(n_{1}, \ldots, n_{k}\right)=\max \left(n_{1}+1, \ldots, n_{k}+1\right) \\
& =\max _{w \in\{0,1\}^{k}}\left(n_{1}^{w_{1}}+\cdots+n_{k}^{w_{k}}+b_{\sigma, w}\right)=\theta(\sigma)\left(n_{1}, \ldots, n_{k}\right)
\end{aligned}
$$

Hence $\theta_{1}^{\prime}=\theta$ and thus height $=h_{\mathcal{B}}$. Then, using the general scenario, we can construct a $\left(\Sigma\right.$, Nat $\left._{\text {max },+}\right)$-wta $\mathcal{A}$ such that $\llbracket \mathcal{A} \rrbracket^{\text {init }}=$ height. In fact, for $\Sigma=\left\{\sigma^{(2)}, \alpha^{(0)}\right\}$, this $(\Sigma$, Nat max , $)$-wta $\mathcal{A}$ is the same as the one that we have constructed in Example 3.2.4.
3. Here we wish to show that also the $(\Sigma, B)$-wta $\mathcal{A}$ constructed in the proof of Lemma 3.1.5 can be obtained as an application of our general scenario. For this, let $A \subseteq B$ be a finite subset and $\Sigma$ a ranked alphabet such that $\left|\Sigma^{(0)}\right| \geq|A \cup\{\mathbb{0}, \mathbb{1}\}|$ and $\left|\Sigma^{(2)}\right| \geq 2$. Without loss of generality we can assume that $\left\{a^{(0)} \mid a \in A \cup\{\mathbb{0}, \mathbb{1}\}\right\} \subseteq \Sigma^{(0)}$ and $\left\{+^{(2)}, \times^{(2)}\right\} \subseteq \Sigma^{(2)}$. We let $\Delta$ be the ranked alphabet $\left\{a^{(0)} \mid a \in A \cup\{\mathbb{0}, \mathbb{1}\}\right\} \cup\left\{+^{(2)}, \times^{(2)}\right\}$.
Moreover, we define the family $\left(b_{\sigma, w} \mid k \in \mathbb{N}, \sigma \in \Sigma^{(k)}, w \in\{0,1\}^{k}\right)$ over $B$ such that

- $b_{a, \varepsilon}=a$ for every $a \in A \cup\{\mathbb{O}, \mathbb{1}\}$,
- $b_{+, 01}=b_{+, 10}=\mathbb{1}$ and $b_{+, 00}=b_{+, 11}=\mathbb{O}$,
- $b_{\times, 11}=\mathbb{1}$ and $b_{\times, 00}=b_{\times, 10}=b_{\times, 01}=\mathbb{O}$, and
- for every $k \in \mathbb{N}, \sigma \in \Sigma^{(k)} \backslash \Delta$, and $w \in\{0,1\}^{k}$, we define $b_{\sigma, w}$ to be an arbitrary element of $A \cup\{0, \mathbb{1}\}$.
Using this family, we define the $\Sigma$-algebra $\mathcal{B}=(B, \theta)$ as in (3.23). Then, in particular, the following properties hold:
(P1) $\theta(a)()=a$ for each $a \in A \cup\{\mathbb{O}, \mathbb{1}\}$,
(P2) $\theta(+)\left(b_{1}, b_{2}\right)=b_{1} \oplus b_{2}$ and $\theta(\times)\left(b_{1}, b_{2}\right)=b_{1} \otimes b_{2}$ for every $b_{1}, b_{2} \in B$, and
(P3) for every $k \in \mathbb{N}, \sigma \in \Sigma^{(k)} \backslash \Delta$, the set $\langle A\rangle_{\{\oplus, \otimes, \mathbb{Q}, \mathbb{1}\}}$ is closed under the operation $\theta(\sigma)$.
Let us abbreviate $\langle A\rangle_{\{\oplus, \otimes, \mathbb{Q}, \mathbb{1}\}}$ by $\langle A\rangle$. Next we show that $\operatorname{im}\left(\mathrm{h}_{\mathcal{B}}\right)=\langle A\rangle$.
The inclusion $\operatorname{im}\left(\mathrm{h}_{\mathcal{B}}\right) \subseteq\langle A\rangle$ follows from the definition of $\Sigma$-algebra homomorphism and properties (P1), (P2), and (P3) of the operations of $\mathcal{B}$.
As preparation for the proof of the other inclusion, we consider the well-founded relation $\prec$ on $\langle A\rangle$ which we defined in the proof of Lemma 3.1.5. By induction on $(\langle A\rangle, \prec)$, we prove that the following statement holds:

$$
\begin{equation*}
\text { For each } a \in\langle A\rangle \text {, there exists } \xi \in \mathrm{T}_{\Sigma} \text { such that } \mathrm{h}_{\mathcal{B}}(\xi)=a \tag{3.25}
\end{equation*}
$$

I.B.: Let $a \in A \cup\{\mathbb{O}, \mathbb{1}\}$. Then with $\xi=a$ we have $\mathrm{h}_{\mathcal{B}}(\xi)=\theta(a)()=a$.
I.S.: Now let $a=a_{1} \oplus a_{2}$. By the I.H., for each $i \in\{1,2\}$, there exists a $\xi_{i} \in \mathrm{~T}_{\Sigma}$ such that $\mathrm{h}_{\mathcal{B}}\left(\xi_{i}\right)=a_{i}$. We let $\xi=+\left(\xi_{1}, \xi_{2}\right)$. Then

$$
\left.\begin{array}{rl}
\mathrm{h}_{\mathcal{B}}(\xi) & =\mathrm{h}_{\mathcal{B}}\left(+\left(\xi_{1}, \xi_{2}\right)\right)=\theta(+)\left(\mathrm{h}_{\mathcal{B}}\left(\xi_{1}\right), \mathrm{h}_{\mathcal{B}}\left(\xi_{2}\right)\right)=\theta(+)\left(a_{1}, a_{2}\right) \\
& =a_{1} \oplus a_{2}
\end{array} \quad \quad \text { (by property }(\mathrm{P} 1)\right) \text { ) }
$$

Next let $a=a_{1} \otimes a_{2}$. As before, for each $i \in\{1,2\}$, there exists a $\xi_{i} \in \mathrm{~T}_{\Sigma}$ such that $\mathrm{h}_{\mathcal{B}}\left(\xi_{i}\right)=a_{i}$. We can show that $\mathrm{h}_{\mathcal{B}}(\xi)=a_{1} \otimes a_{2}$ in a similar way as above. This proves (3.25).
Hence $\langle A\rangle \subseteq \operatorname{im}\left(\mathrm{h}_{\mathcal{B}}\right)$ and thus $\langle A\rangle=\operatorname{im}\left(\mathrm{h}_{\mathcal{B}}\right)$.
By using our general scenario, we can construct a $(\Sigma, B)$-wta $\mathcal{A}$ such that $\llbracket \mathcal{A} \rrbracket^{\text {init }}=\mathrm{h}_{\mathcal{B}}$. Thus, in particular, $\operatorname{im}\left(\llbracket \mathcal{A} \rrbracket^{\text {init }}\right)=\langle A\rangle$. In fact, this $(\Sigma, \mathrm{B})$-wta $\mathcal{A}$ is the same as the one which we have constructed in the proof of Lemma 3.1.5.

### 3.3 Weighted string automata

In this section we recall the concept of weighted string automata Sch61, Eil74 and show that, essentially, they are equivalent to wta over string ranked alphabets (cf. [FV09, p. 324]). Moreover, we show that they generalize fsa.

A weighted string automaton (over $\Gamma$ and B ) (for short: ( $\Gamma, \mathrm{B}$ )-wsa or wsa) is a tuple $\mathcal{A}=(Q, \lambda, \mu, \gamma)$, where $Q$ is a finite nonempty set of states such that $Q \cap \Gamma=\emptyset, \mu: \Gamma \rightarrow B^{Q \times Q}$ is the transition mapping, and $\lambda, \gamma \in B^{Q}$ are the initial weight mapping and the final weight mapping, respectively. We define the run semantics and the initial algebra semantics for $\mathcal{A}$ as follows.

Run semantics. Let $w=a_{1} \cdots a_{n}$ be a string in $\Gamma^{*}$ with $n \in \mathbb{N}$. A run of $\mathcal{A}$ on $w$ is a string $\rho \in Q^{n+1}$. Let $\rho=q_{0} \cdots q_{n}$ with $q_{i} \in Q$ for each $i \in[0, n]$. The weight of $\rho$ for $w$, denoted by $\operatorname{wt}_{\mathcal{A}}(w, \rho)$, is the element of $B$ defined by

$$
\mathrm{wt}_{\mathcal{A}}(w, \rho)=\lambda_{q_{0}} \otimes \mathrm{wt}_{\mathcal{A}}^{-}(w, \rho) \otimes \gamma_{q_{n}}
$$

where

$$
\mathrm{wt}_{\mathcal{A}}^{-}(w, \rho)=\mu\left(a_{1}\right)_{q_{0}, q_{1}} \otimes \ldots \otimes \mu\left(a_{n}\right)_{q_{n-1}, q_{n}} .
$$

Thus, in particular, we have that $\mathrm{wt}_{\mathcal{A}}^{-}(\varepsilon, \rho)=\mathbb{1}$ and $\mathrm{wt}_{\mathcal{A}}(\varepsilon, \rho)=\lambda_{q_{0}} \otimes \gamma_{q_{0}}$.
The run semantics of $\mathcal{A}$ is the weighted language $\llbracket \mathcal{A} \rrbracket^{\text {run }}: \Gamma^{*} \rightarrow B$ defined by

$$
\llbracket \mathcal{A} \rrbracket^{\mathrm{run}}(w)=\bigoplus_{\rho \in Q^{|w|+1}} \mathrm{wt}_{\mathcal{A}}(w, \rho)
$$

for every $w \in \Gamma^{*}$. In particular, $\llbracket \mathcal{A} \rrbracket^{\text {run }}(\varepsilon)=\bigoplus_{q \in Q} \lambda_{q} \otimes \gamma_{q}$.
A weighted language $r: \Gamma^{*} \rightarrow B$ is run recognizable (over B ) (for short: r-recognizable) if there exists a $(\Gamma, \mathrm{B})$-wsa $\mathcal{A}$ such that $r=\llbracket \mathcal{A} \rrbracket^{\text {run }}$.

Initial algebra semantics. Let $e \notin \Gamma$. For the definition of the initial algebra semantics, we use the string ranked alphabet $\Gamma_{e}$ and the $\Gamma_{e}$-algebra ( $\Gamma^{*}, \widehat{\Gamma_{e}}$ ) defined in Paragraph "String-like terms" of Section 2.9 .

Also, we define the $\Gamma_{e}$-algebra $\left(B^{Q}, \theta\right)$ with

$$
\theta(e)()=\lambda \quad \text { and } \quad \theta(a)(v)=v \cdot \mu(a) \quad \text { (vector-matrix product, see Section 2.7) }
$$

for every $v \in B^{Q}$ and $a \in \Gamma$. By (2.24), $\left(\Gamma^{*}, \widehat{\Gamma_{e}}\right)$ is initial, hence there exists a unique $\Gamma_{e}$-algebra homomorphism from $\left(\Gamma^{*}, \widehat{\Gamma_{e}}\right)$ to $\left(B^{Q}, \theta\right)$. Let us denote this by $\mathrm{h}_{\mathcal{A}}$. Then the initial algebra semantics of $\mathcal{A}$ is the weighted language $\llbracket \mathcal{A} \rrbracket^{\text {init }}: \Gamma^{*} \rightarrow B$ defined by

$$
\llbracket \mathcal{A} \rrbracket^{\mathrm{init}}(w)=\bigoplus_{q \in Q} \mathrm{~h}_{\mathcal{A}}(w)_{q} \otimes \gamma_{q}
$$

for every $w \in \Gamma^{*}$. We note that, using another nullary symbol $e^{\prime}$ instead of $e$, leads to the same weighted language $\llbracket \mathcal{A} \rrbracket^{\text {init }}$.

A weighted language $r: \Gamma^{*} \rightarrow B$ is initial algebra recognizable (over B ) (for short: i-recognizable) if there exists a $(\Gamma, \mathrm{B})$-wsa $\mathcal{A}$ such that $r=\llbracket \mathcal{A} \rrbracket^{\text {init }}$.

The next lemma shows the connection of the initial algebra semantics of $\mathcal{A}$ to its behaviour semantics as defined in Eil74, VI.6], if B is a semiring (also cf. [Sch61, Def. 1']). We recall that, under this assumption, the matrix multiplication is associative, and hence ( $B^{Q \times Q}, \cdot, \mathrm{M}_{\mathbb{1}}$ ) is a monoid (cf. Section 2.7). Then $\mu: \Gamma^{*} \rightarrow B^{Q \times Q}$ is a monoid homomorphism from $\left(\Gamma^{*}, \cdot, \varepsilon\right)$ to $\left(B^{Q \times Q}, \cdot, \mathrm{M}_{\mathbb{1}}\right)$.

Lemma 3.3.1. Let B be a semiring and $w \in \Gamma^{*}$. Then $\mathrm{h}_{\mathcal{A}}(w)=\lambda \cdot \mu(w)$. Hence $\llbracket \mathcal{A} \rrbracket^{\text {init }}(w)=\lambda \cdot \mu(w) \cdot \gamma$.
Proof. Let $w \in \Gamma^{*}$. We consider the well-founded set ( $\operatorname{prefix}(w), \prec$ ) where, for every $w_{1}, w_{2} \in \operatorname{prefix}(w)$, we let $w_{1} \prec w_{2}$ if $w_{1} a=w_{2}$ for some $a \in \Gamma$. Obviously, $\prec$ is well-founded and $\min _{\prec}(\operatorname{prefix}(w))=\{\varepsilon\}$.

By induction on (prefix $(w), \prec)$, we prove that the following statement holds:
For every $v \in \operatorname{prefix}(w)$, we have $\mathrm{h}_{\mathcal{A}}(v)=\lambda \cdot \mu(v)$.

Let $v \in \operatorname{prefix}(w)$.
I.B.: Let $v=\varepsilon$. Then we can calculate as follows.

$$
\begin{aligned}
\mathrm{h}_{\mathcal{A}}(\varepsilon) & =\mathrm{h}_{\mathcal{A}}(\widehat{e}()) \\
& =\theta(e)() \\
& =\lambda
\end{aligned}
$$

(interpretation of $e$ in the $\Gamma_{e}$-algebra $\left(\Gamma^{*}, \widehat{\Gamma_{e}}\right)$ )
( $\mathrm{h}_{\mathcal{A}}$ is a $\Gamma_{e}$-algebra homomorphism)
(interpretation of $e$ in the $\Gamma_{e}$-algebra $\left(B^{Q}, \theta\right)$ )
I.S.: Let $v=v^{\prime} a$ for some $a \in \Gamma$. Then we can calculate as follows.

$$
\begin{array}{rlrl}
\mathrm{h}_{\mathcal{A}}\left(v^{\prime} a\right) & =\mathrm{h}_{\mathcal{A}}\left(\widehat{a}\left(v^{\prime}\right)\right) & \text { (interpretation of } \left.a \text { in the } \Gamma_{e} \text {-algebra }\left(\Gamma^{*}, \widehat{\Gamma_{e}}\right)\right) \\
& =\theta(a)\left(\mathrm{h}_{\mathcal{A}}\left(v^{\prime}\right)\right) & \left(\mathrm{h}_{\mathcal{A}} \text { is a } \Gamma_{e}\right. \text {-algebra homomorphism) } \\
& =\theta(a)\left(\lambda \cdot \mu\left(v^{\prime}\right)\right) & \\
& =\lambda \cdot \mu\left(v^{\prime}\right) \cdot \mu(a) & \text { (by I.H.) } \\
& =\lambda \cdot \mu\left(v^{\prime} a\right) . & \text { (interpretation of } \left.a \text { in the } \Gamma_{e} \text {-algebra }\left(B^{Q}, \theta\right)\right) \\
\text { (because } \mu \text { is a monoid homomorphism) }
\end{array}
$$

This proves (3.26). Since $w \in \operatorname{prefix}(w)$, Equation (3.26) implies the first statement of the lemma. Finally

$$
\llbracket \mathcal{A} \rrbracket^{\text {init }}(w)=\bigoplus_{q \in Q} \mathrm{~h}_{\mathcal{A}}(w)_{q} \otimes \gamma_{q}=\bigoplus_{q \in Q}(\lambda \cdot \mu(w))_{q} \otimes \gamma_{q}=\lambda \cdot \mu(w) \cdot \gamma
$$

Embedding of wsa into wta. Now let $\mathcal{A}=(Q, \lambda, \mu, \gamma)$ be a ( $\Gamma, \mathrm{B})$-wsa, $\Sigma$ a string ranked alphabet, and $\mathcal{B}=\left(Q^{\prime}, \delta, F\right)$ a $(\Sigma, \mathrm{B})$-wta. We say that $\mathcal{A}$ and $\mathcal{B}$ are related if

- $\Sigma=\Gamma_{e}$ for some $e \notin \Gamma$,
- $Q=Q^{\prime}$,
- for every $a \in \Sigma^{(1)}$ and $p, q \in Q$, we have $\delta_{0}(\varepsilon, e, q)=\lambda_{q}$, and $\delta_{1}(p, a, q)=\mu(a)_{p, q}$, and
- $F=\gamma$.

For the proof of the next lemma, we recall that tree $e_{e}$ is an isomorphism from the $\Gamma_{e}$-algebra $\left(\Gamma^{*}, \widehat{\Gamma_{e}}\right)$ to the $\Gamma_{e}$-term algebra $\left(\mathrm{T}_{\Gamma_{e}}, \theta_{\Gamma_{e}}\right)$, cf. Paragraph "String-like terms" in Section 2.9,

Lemma 3.3.2. Let $\Sigma$ be a string ranked alphabet, $\mathcal{A}=(Q, \lambda, \mu, \gamma)$ a $(\Gamma, \mathrm{B})$-wsa, and $\mathcal{B}=\left(Q^{\prime}, \delta, F\right)$ a $(\Sigma, \mathrm{B})$-wta. If $\mathcal{A}$ and $\mathcal{B}$ are related, then $\llbracket \mathcal{A} \rrbracket^{\text {run }}=\llbracket \mathcal{B} \rrbracket^{\text {run }} \circ$ tree $_{e}$ and $\llbracket \mathcal{A} \rrbracket^{\text {init }}=\llbracket \mathcal{B} \rrbracket^{\text {init }} \circ$ tree $_{e}$, where $e$ denotes the unique nullary element of $\Sigma$.

Proof. To show that $\llbracket \mathcal{A} \rrbracket^{\text {run }}=\llbracket \mathcal{B} \rrbracket^{\text {run }} \circ$ tree $_{e}$, let $w \in \Gamma^{*}$ with $w=a_{1} \cdots a_{n}$ for some $n \in \mathbb{N}$. It is obvious that there exists a bijection $\mapsto$ between $Q^{|w|+1}$ and $\mathrm{R}_{\mathcal{B}}\left(\operatorname{tree}_{e}(w)\right)$ such that, for $\rho=q_{0} \cdots q_{n}$, we have $\rho \mapsto \rho^{\prime}$, where $\rho^{\prime}\left(1^{i}\right)=q_{n-i}$ for every $i \in[0, n]$ (note that $\operatorname{pos}\left(\operatorname{tree}_{e}(w)\right)=\left\{1^{i} \mid i \in[0, n]\right\}$ ). Moreover, if $\rho \mapsto \rho^{\prime}$, then due to Observation 3.1.1 we also have

$$
\mathrm{wt}_{\mathcal{A}}(w, \rho)=\operatorname{wt}_{\mathcal{B}}\left(\operatorname{tree}_{e}(w), \rho^{\prime}\right) \otimes F_{q_{n}}
$$

where $\operatorname{wt}_{\mathcal{B}}\left(\operatorname{tree}_{e}(w), \rho^{\prime}\right)$ denotes the weight of $\rho^{\prime}$ on $\operatorname{tree}_{e}(w)$ for $\mathcal{B}$. Then

$$
\llbracket \mathcal{A} \rrbracket^{\mathrm{run}}(w)=\bigoplus_{\rho \in Q^{|w|+1}} \operatorname{wt}_{\mathcal{A}}(w, \rho)=\bigoplus_{\rho^{\prime} \in \mathrm{R}_{\mathcal{B}}\left(\operatorname{tree}_{e}(w)\right)} \operatorname{wt}_{\mathcal{B}}\left(\operatorname{tree}_{e}(w), \rho^{\prime}\right) \otimes F_{\rho^{\prime}(\varepsilon)}=\llbracket \mathcal{B} \rrbracket^{\mathrm{run}}\left(\operatorname{tree}_{e}(w)\right)
$$

where the second equality follows from the bijection $\mapsto$ described above and the fact that $\rho^{\prime}(\varepsilon)=q_{n}$.
Now we show that $\llbracket \mathcal{A} \rrbracket^{\text {init }}=\llbracket \mathcal{B} \rrbracket^{\text {init }} \circ$ tree $_{e}$. Since tree $e_{e}: \Gamma^{*} \rightarrow \mathrm{~T}_{\Gamma_{e}}$ and $\mathrm{h}_{\mathcal{B}}: \mathrm{T}_{\Gamma_{e}} \rightarrow B^{Q}$ are $\Gamma_{e^{-}}$ algebra homomorphisms, it follows from Theorem 2.6.3 that $\left(\mathrm{h}_{\mathcal{B}} \circ \operatorname{tree}_{e}\right): \Gamma^{*} \rightarrow B^{Q}$ is a $\Gamma_{e}$-algebra homomorphism. Since $\mathrm{h}_{\mathcal{A}}$ is the unique $\Gamma_{e}$-algebra homomorphism from $\left(\Gamma^{*}, \widehat{\Gamma_{e}}\right)$ to $\left(B^{Q}, \theta\right)$, we have $\mathrm{h}_{\mathcal{A}}=\mathrm{h}_{\mathcal{B}} \circ$ tree $_{e}$.

Then we obtain

$$
\llbracket \mathcal{A} \rrbracket^{\text {init }}(w)=\bigoplus_{q \in Q} \mathrm{~h}_{\mathcal{A}}(w)_{q} \otimes \gamma_{q}=\bigoplus_{q \in Q^{\prime}} \mathrm{h}_{\mathcal{B}}\left(\operatorname{tree}_{e}(w)\right)_{q} \otimes F_{q}=\llbracket \mathcal{B} \rrbracket^{\mathrm{init}^{2}}\left(\operatorname{tree}_{e}(w)\right) .
$$

Lemma 3.3.3. Let $\Gamma$ be an alphabet and $B$ be a strong bimonoid. Then the following two statements hold.
(1) For every ( $\Gamma, B$ )-wsa $\mathcal{A}$ and $e \notin \Gamma$, we can construct a $\left(\Gamma_{e}, B\right)$-wta $\mathcal{B}$ such that $\llbracket \mathcal{A} \rrbracket^{\text {run }}=\llbracket \mathcal{B} \rrbracket^{\text {run }} \circ$ tree $_{e}$ and $\llbracket \mathcal{A} \rrbracket^{\text {init }}=\llbracket \mathcal{B} \rrbracket^{\text {init }} \circ$ tree $_{e}$.
(2) For every string ranked alphabet $\Gamma_{e}$ and $\left(\Gamma_{e}, \mathrm{~B}\right)$-wta $\mathcal{B}$, we can construct a $(\Gamma, \mathrm{B})$-wsa $\mathcal{A}$ such that $\llbracket \mathcal{A} \rrbracket^{\text {run }}=\llbracket \mathcal{B} \rrbracket^{\text {run }} \circ$ tree $_{e}$ and $\llbracket \mathcal{A} \rrbracket^{\text {init }}=\llbracket \mathcal{B} \rrbracket^{\text {init }} \circ$ tree $_{e}$.

Proof. First we prove Statement (1). For a given ( $\Gamma, B$ )-wsa $\mathcal{A}$, it is easy to construct a $\left(\Gamma_{e}, B\right)$-wta $\mathcal{B}$ such that $\mathcal{A}$ and $\mathcal{B}$ are related. Then the statement follows from Lemma 3.3.2. Statement (2) follows analogously.

Weighted string automata over the Boolean semiring. Let $\mathcal{A}=(Q, \lambda, \mu, \gamma)$ be a ( $\Gamma$, Boole)-wsa and $A=(Q, I, \delta, F)$ be a $\Gamma$-fsa. We say that $\mathcal{A}$ and $A$ are related if $I=\operatorname{supp}(\lambda), F=\operatorname{supp}(\gamma)$, and for every $a \in \Gamma$ and $q, q^{\prime} \in Q$ we have: $\mu(a)_{q, q^{\prime}}=1 \mathrm{iff}\left(q, a, q^{\prime}\right) \in \delta$.

It is easy to see that, if $\mathcal{A}$ and $A$ are related, then $\operatorname{supp}\left(\llbracket \mathcal{A} \rrbracket^{\text {run }}\right)=\mathrm{L}(A)$. Moreover, for each ( $\Gamma$, Boole)wsa, we can construct a $\Gamma$-fsa $A$ such that $\mathcal{A}$ and $A$ are related; and vice versa, for each $\Gamma$-fsa $A$, we can construct a ( $\Gamma$, Boole)-wsa $\mathcal{A}$ such that $\mathcal{A}$ and $A$ are related. Thus we obtain the following equivalence between $\Gamma$-fsa and ( $\Gamma$, Boole)-wsa.

Observation 3.3.4. Let $L \subseteq \Gamma^{*}$. Then the following two statements are equivalent.
(A) We can construct a $\Gamma$-fsa $A$ such that $L=\mathrm{L}(A)$.
(B) We can construct a $\left(\Gamma\right.$, Boole)-wsa $\mathcal{A}$ such that $L=\operatorname{supp}\left(\llbracket \mathcal{A} \rrbracket^{\text {run }}\right)$.

### 3.4 Weighted tree automata over the Boolean semiring

Here we prove that every wta over the Boolean semiring Boole $=(\mathbb{B}, \vee, \wedge, 0,1)$ is essentially an fta, and vice versa.

Let $\mathcal{A}=(Q, \delta, F)$ be a $(\Sigma, \mathrm{B})$-wta. The support fta of $\mathcal{A}$, denoted by $\operatorname{supp}_{\mathrm{B}}(\mathcal{A})$, is the $\Sigma$-fta $\left(Q, \delta^{\prime}, F^{\prime}\right)$ where for every $k \in \mathbb{N}$ we have $\delta_{k}^{\prime}=\operatorname{supp}_{\mathrm{B}}\left(\delta_{k}\right)$ and $F^{\prime}=\operatorname{supp}_{\mathrm{B}}(F)$. If B is clear from the context, then we drop $B$ from $\operatorname{supp}_{B}$.

Theorem 3.4.1. Let $\mathcal{A}$ be ( $\Sigma$, Boole)-wta. Then $\operatorname{supp}\left(\llbracket \mathcal{A} \rrbracket^{\text {init }}\right)=\operatorname{supp}\left(\llbracket \mathcal{A} \rrbracket^{\text {run }}\right)=\mathrm{L}(\operatorname{supp}(\mathcal{A}))$. Thus $\llbracket \mathcal{A} \rrbracket^{\text {init }}=\llbracket \mathcal{A} \rrbracket^{\text {run }}$.

Proof. Let $\mathcal{A}=(Q, \delta, F)$ and $\operatorname{supp}(\mathcal{A})=\left(Q, \delta^{\prime}, F^{\prime}\right)$. We recall that $\mathrm{L}(\operatorname{supp}(\mathcal{A}))=\mathrm{L}_{\mathrm{i}}(\operatorname{supp}(\mathcal{A}))=$ $\mathrm{L}_{\mathrm{r}}(\operatorname{supp}(\mathcal{A}))$, cf. Lemma 2.13.1.

First we prove $\operatorname{supp}\left(\llbracket \mathcal{A} \rrbracket^{\text {init }}\right)=\mathrm{L}_{\mathrm{i}}(\operatorname{supp}(\mathcal{A}))$. By induction on $\mathrm{T}_{\Sigma}$, we prove that the following statement holds:

$$
\begin{equation*}
\text { For every } \xi \in \mathrm{T}_{\Sigma} \text { and } q \in Q: \mathrm{h}_{\mathcal{A}}(\xi)_{q} \neq 0 \text { iff } q \in \mathrm{~h}_{\operatorname{supp}(\mathcal{A})}(\xi) \tag{3.27}
\end{equation*}
$$

Let $\xi=\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)$.

$$
\operatorname{iff}\left(\mathrm{h}_{\mathcal{A}}(\xi)_{q} \neq 0\right.
$$

$$
\begin{aligned}
& \text { iff } \\
& \quad(*)\left(\exists q_{1} \cdots q_{k} \in Q^{k}\right):\left(\mathrm{h}_{\mathcal{A}}\left(\xi_{1}\right)_{q_{1}} \neq 0\right) \wedge \ldots \wedge\left(\mathrm{h}_{\mathcal{A}}\left(\xi_{k}\right)_{q_{k}} \neq 0\right) \wedge\left(\delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right) \neq 0\right) \\
& \left.\quad \text { iff }\left(\exists q_{1} \cdots q_{k} \in Q^{k}\right):\left(q_{1} \in \mathrm{~h}_{\operatorname{supp}(\mathcal{A})}\left(\xi_{1}\right)\right) \wedge \ldots \wedge\left(q_{k} \in \mathrm{~h}_{\operatorname{supp}(\mathcal{A})}\left(\xi_{k}\right)\right) \wedge\left(\left(q_{1} \cdots q_{k}, \sigma, q\right) \in \delta_{k}^{\prime}\right)\right)
\end{aligned}
$$

(by I.H.)
iff $q \in \mathrm{~h}_{\text {supp }(\mathcal{A})}(\xi)$.
At the implication from right to left of equivalence (*), we have used the fact that the Boolean semiring is positive, i.e., zero-divisor free and zero-sum free. This proves (3.27).

Then, for each $\xi \in \mathrm{T}_{\Sigma}$, we can argue as follows:

$$
\begin{gathered}
\xi \in \operatorname{supp}\left(\left[\mathcal{A} \rrbracket^{\text {init }}\right) \text { iff }\left(\bigvee_{q \in Q} \mathrm{~h}_{\mathcal{A}}(\xi)_{q} \wedge F_{q}\right) \neq 0 \text { iff }\left(\bigvee_{q \in F^{\prime}} \mathrm{h}_{\mathcal{A}}(\xi)_{q}\right) \neq 0\right. \\
\text { iff }^{(* *)}\left(\exists q \in F^{\prime}\right): \mathrm{h}_{\mathcal{A}}(\xi)_{q} \neq 0 \text { iff }\left(\exists q \in F^{\prime}\right): q \in \mathrm{~h}_{\operatorname{supp}(\mathcal{A})}(\xi) \text { iff } \xi \in \mathrm{L}_{\mathrm{i}}(\operatorname{supp}(\mathcal{A}))
\end{gathered}
$$

where the implication from right to left at equivalence (**) uses the fact that Boole is zero-sum free; the last but one equivalence is due to (3.27). This proves $\operatorname{supp}\left(\llbracket \mathcal{A} \rrbracket^{\text {init }}\right)=\mathrm{L}_{\mathrm{i}}(\operatorname{supp}(\mathcal{A}))$.

Next we prove that $\operatorname{supp}\left(\llbracket \mathcal{A} \rrbracket^{\text {run }}\right)=\mathrm{L}_{\mathrm{r}}(\operatorname{supp}(\mathcal{A}))$. We note that $\mathrm{R}_{\mathcal{A}}(\xi)=\mathrm{R}_{\text {supp }(\mathcal{A})}(\xi)$ for every $\xi \in \mathrm{T}_{\Sigma}$. First, by induction on $\mathrm{T}_{\Sigma}$, we prove that the following statement holds:

$$
\begin{equation*}
\text { For every } \xi \in \mathrm{T}_{\Sigma} \text { and } \rho \in \mathrm{R}_{\mathcal{A}}(\xi): \quad \operatorname{wt}_{\mathcal{A}}(\xi, \rho) \neq 0 \text { iff } \rho \in \mathrm{R}_{\operatorname{supp}(\mathcal{A})}^{\mathrm{v}}(\xi) . \tag{3.28}
\end{equation*}
$$

We let $\xi=\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)$. Then

$$
\begin{aligned}
& \quad \operatorname{wt}_{\mathcal{A}}(\xi, \rho) \neq 0 \text { iff }\left(\operatorname{wt}_{\mathcal{A}}\left(\xi_{1},\left.\rho\right|_{1}\right) \wedge \ldots \wedge \operatorname{wt}_{\mathcal{A}}\left(\xi_{k},\left.\rho\right|_{k}\right) \wedge \delta_{k}(\rho(1) \cdots \rho(k), \sigma, \rho(\varepsilon))\right) \neq 0 \\
& \text { iff }(* * *)\left(\operatorname{wt}_{\mathcal{A}}\left(\xi_{1},\left.\rho\right|_{1}\right) \neq 0\right) \wedge \ldots \wedge\left(\operatorname{wt}_{\mathcal{A}}\left(\xi_{k},\left.\rho\right|_{k}\right) \neq 0\right) \wedge\left(\delta_{k}(\rho(1) \cdots \rho(k), \sigma, \rho(\varepsilon)) \neq 0\right) \\
& \quad \text { iff }\left(\left.\rho\right|_{1} \in \mathrm{R}_{\operatorname{supp}(\mathcal{A})}^{\mathrm{v}}\left(\xi_{1}\right)\right) \wedge \ldots \wedge\left(\left.\rho\right|_{k} \in \mathrm{R}_{\operatorname{supp}(\mathcal{A})}^{\mathrm{v}}\left(\xi_{k}\right)\right) \wedge\left((\rho(1) \cdots \rho(k), \sigma, \rho(\varepsilon)) \in \delta_{k}^{\prime}\right) \quad \text { (by I.H.) } \\
& \text { iff } \rho \in \mathrm{R}_{\operatorname{supp}(\mathcal{A})}^{\mathrm{v}}(\xi)
\end{aligned}
$$

where at $(* * *)$ from right to left we have used the fact that Boole is zero-divisor free. This proves (3.28).
Now let $\xi \in \mathrm{T}_{\Sigma}$. Then

$$
\begin{aligned}
& \xi \in \operatorname{supp}\left(\llbracket \mathcal{A} \rrbracket^{\mathrm{run}}\right) \text { iff }\left(\underset{q \in Q}{ } \bigvee_{\rho \in \mathrm{R}_{\mathcal{A}}(q, \xi)} \operatorname{wt}_{\mathcal{A}}(\xi, \rho) \wedge F_{q}\right) \neq 0 \\
& \text { iff }\left(\bigvee_{q \in F^{\prime}} \bigvee_{\rho \in \mathbb{R}_{\mathcal{A}}(q, \xi)} \operatorname{wt}_{\mathcal{A}}(\xi, \rho)\right) \neq 0 \\
& \text { iff }{ }^{(v *)}\left(\exists q \in F^{\prime}\right)\left(\exists \rho \in \mathrm{R}_{\mathcal{A}}(q, \xi)\right): \operatorname{wt}_{\mathcal{A}}(\xi, \rho) \neq 0 \\
& \text { iff }\left(\exists q \in F^{\prime}\right)\left(\exists \rho \in \mathrm{R}_{\operatorname{supp}(\mathcal{A})}(q, \xi)\right): \rho \in \mathrm{R}_{\operatorname{supp}(\mathcal{A})}^{\mathrm{v}}(\xi) \text { iff } \xi \in \mathrm{L}_{\mathrm{r}}(\operatorname{supp}(\mathcal{A}))
\end{aligned}
$$

where at $(v *)$ from right to left we have used the fact that Boole is zero-sum free; the last but one equivalence is due to (3.28). This proves $\operatorname{supp}\left([\mathcal{A}]^{\text {run }}\right)=\mathrm{L}_{\mathrm{r}}(\operatorname{supp}(\mathcal{A}))$.

Finally we have $\llbracket \mathcal{A} \rrbracket^{\text {init }}=\chi\left(\operatorname{supp}\left(\llbracket \mathcal{A} \rrbracket^{\text {init }}\right)\right)=\chi\left(\operatorname{supp}\left(\llbracket \mathcal{A} \rrbracket^{\text {run }}\right)\right)=\llbracket \mathcal{A} \rrbracket^{\text {run }}$.
In Lemma 18.2.3(1) we will generalize the first statement of Theorem 3.4.1 from the Boolean semiring Boole to arbitrary positive strong bimonoids. In Theorem 5.3.2 we will generalize the second statement of Theorem 3.4.1 from Boole to arbitrary semirings.

Corollary 3.4.2. Let $\Sigma$ be a ranked alphabet and $L \subseteq \mathrm{~T}_{\Sigma}$. Then the following three statements are equivalent.
(A) We can construct a $\Sigma$ - $\mathrm{fta} A$ such $L=\mathrm{L}(A)$.
(B) We can construct a $\left(\Sigma\right.$, Boole)-wta $\mathcal{A}$ such that $L=\operatorname{supp}\left(\llbracket \mathcal{A} \rrbracket^{\text {run }}\right)=\operatorname{supp}\left(\llbracket \mathcal{A} \rrbracket^{\text {init }}\right)$.
(C) We can construct a ( $\Sigma$, Boole)-wta $\mathcal{A}$ such that $\chi(L)=\llbracket \mathcal{A} \rrbracket^{\text {run }}=\llbracket \mathcal{A} \rrbracket^{\text {init }}$.
$\operatorname{Moreover}$, we have $\operatorname{supp}\left(\operatorname{Rec}^{\text {init }}(\Sigma, \operatorname{Boole})\right)=\operatorname{supp}\left(\operatorname{Rec}^{\text {run }}(\Sigma, \operatorname{Boole})\right)=\operatorname{Rec}(\Sigma)$.

Proof. Proof of $(\mathrm{A}) \Rightarrow(\mathrm{B})$ : Let $A=(Q, \delta, F)$ be a $\Sigma$-fta such that $L=\mathrm{L}(A)$. We construct the ( $\Sigma$, Boole)wta $\mathcal{A}=\left(Q, \delta^{\prime}, F^{\prime}\right)$ such that $\operatorname{supp}\left(\delta_{k}^{\prime}\right)=\delta_{k}$ for each $k \in \mathbb{N}$, and $\operatorname{supp}\left(F^{\prime}\right)=F$ (note that this determines $\delta^{\prime}$ and $F^{\prime}$ because $\left.\mathbb{B}=\{0,1\}\right)$. Then $A=\operatorname{supp}(\mathcal{A})$ and by Theorem3.4.1 we obtain $\mathrm{L}(A)=\mathrm{L}(\operatorname{supp}(\mathcal{A}))=$ $\operatorname{supp}\left(\llbracket \mathcal{A} \rrbracket^{\text {run }}\right)=\operatorname{supp}\left(\llbracket \mathcal{A} \rrbracket^{\text {init }}\right)$.

Proof of $(\mathrm{B}) \Rightarrow(\mathrm{A})$ : Let $(\Sigma$, Boole $)$-wta $\mathcal{A}$ such that $L=\operatorname{supp}\left(\llbracket \mathcal{A} \rrbracket^{\text {run }}\right)=\operatorname{supp}\left(\llbracket \mathcal{A} \rrbracket^{\text {init }}\right)$. Then, we construct the $\Sigma$-fta $\operatorname{supp}(\mathcal{A})$, and by Theorem 3.4.1 we have that $\mathrm{L}(\operatorname{supp}(\mathcal{A}))=\operatorname{supp}\left(\llbracket \mathcal{A} \rrbracket^{\mathrm{run}}\right)=$ $\operatorname{supp}\left(\llbracket \mathcal{A} \rrbracket^{\text {init }}\right)$.

Proof of $(\mathrm{B}) \Leftrightarrow(\mathrm{C})$ : This follows from the equivalence that for every $r: \mathrm{T}_{\Sigma} \rightarrow \mathbb{B}$ and $L \subseteq \mathrm{~T}_{\Sigma}: r=\chi(L)$ iff $L=\operatorname{supp}(r)$.

Finally, the equality $\operatorname{supp}\left(\operatorname{Rec}^{\text {init }}(\Sigma\right.$, Boole $\left.)\right)=\operatorname{supp}\left(\operatorname{Rec}^{\text {run }}(\Sigma\right.$, Boole $\left.)\right)$ follows from Theorem 3.4.1, and the equality $\operatorname{Rec}(\Sigma)=\operatorname{supp}\left(\operatorname{Rec}^{\text {run }}(\Sigma\right.$, Boole $\left.)\right)$ follows from $(\mathrm{A}) \Leftrightarrow(\mathrm{B})$.

### 3.5 Weighted tree automata over the semiring of natural numbers

In this section we will consider the semiring $\operatorname{Nat}=(\mathbb{N},+, \cdot, 0,1)$ of natural numbers as weight algebra. In Eil74, Thm. 9.1] it was proved that, for every alphabet $\Gamma$ and ( $\Gamma, N a t$ )-wsa $\mathcal{A}$, there exists an unambiguous ( $\Gamma$, Nat)-wsa $\mathcal{B}$ such that $\llbracket \mathcal{A} \rrbracket^{\text {run }}=\llbracket \mathcal{B} \rrbracket^{\text {run }}$. There, unambiguous means that each transition weight and each initial and final weight are elements of $\{0,1\}$. For wta, we call the corresponding property " $\mathcal{B}$ has identity transition weights and identity root weights". Here we generalize Eilenberg's result to wta. However, since we will discuss normal forms of wta only later, we have to start with a ( $\Sigma, N$ Nat)-wta $\mathcal{A}$ which has identity root weights.

Lemma 3.5.1. (cf. Eil74, Thm. 9.1]) For each ( $\Sigma$, Nat)-wta $\mathcal{A}$ which has identity root weights, we can construct a $(\Sigma, N a t)$-wta $\mathcal{B}$ such that $\mathcal{B}$ has identity transition weights and identity root weights and $\llbracket \mathcal{A} \rrbracket^{\text {init }}=\llbracket \mathcal{B} \rrbracket^{\text {init }}$.

Proof. Let $\mathcal{A}=(Q, \delta, F)$. Let $m=\max (\operatorname{wts}(\mathcal{A}))$. If $m=0$, then let $\mathcal{B}=\mathcal{A}$. Otherwise, we construct $\mathcal{B}=$ $\left(Q^{\prime}, \delta^{\prime}, F^{\prime}\right)$ according to the following idea. We supply $\mathcal{B}$ with $m$ copies of each state of $\mathcal{A}$; a copy is a pair $(q, \ell)$ with $q \in Q$ and $\ell \in[m]$. If $\tau=\left(q_{1} \cdots q_{k}, \sigma, q\right)$ is a transition of $\mathcal{A}$ and $\left(q_{1}, \ell_{1}\right), \ldots,\left(q_{k}, \ell_{k}\right)$ are copies of $q_{1}, \ldots, q_{k}$, then $\mathcal{B}$ simulates the weight $\delta_{k}(\tau)$ by nondeterministically branching from $\left(q_{1}, \ell_{1}\right), \ldots,\left(q_{k}, \ell_{k}\right)$ while reading $\sigma$ into $(q, 1), \ldots,\left(q, \delta_{k}(\tau)\right)$, and each of these transitions has weight 1 . Formally, we let:

- $Q^{\prime}=Q \times[m]$,
- for each $(q, \ell) \in Q^{\prime}$, we define $F_{(q, \ell)}^{\prime}=F_{q}$,
- for every $k \in \mathbb{N}, \sigma \in \Sigma^{(k)},\left(q_{1}, \ell_{1}\right), \ldots,\left(q_{k}, \ell_{k}\right),(q, \ell) \in Q^{\prime}$, we define

$$
\left(\delta^{\prime}\right)_{k}\left(\left(q_{1}, \ell_{1}\right) \cdots\left(q_{k}, \ell_{k}\right), \sigma,(q, \ell)\right)= \begin{cases}1 & \text { if } \ell \in\left[\delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right)\right] \\ 0 & \text { otherwise }\end{cases}
$$

By induction on $\mathrm{T}_{\Sigma}$, we prove that the following statement holds:

$$
\begin{equation*}
\text { For every } \xi \in \mathrm{T}_{\Sigma} \text { and } q \in Q \text {, we have: } \mathrm{h}_{\mathcal{A}}(\xi)_{q}={\underset{\ell \in[m]}{ } \mathrm{h}_{\mathcal{B}}(\xi)_{(q, \ell)} . . . . . . . .} \tag{3.29}
\end{equation*}
$$

Let $\xi=\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)$. Then we can calculate as follows.

$$
\begin{aligned}
& \mathrm{h}_{\mathcal{A}}\left(\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)\right)_{q} \\
& =\mathrm{F}_{q_{1}, \ldots, q_{k} \in Q} \mathrm{~h}_{\mathcal{A}}\left(\xi_{1}\right)_{q_{1}} \cdot \ldots \cdot \mathrm{~h}_{\mathcal{A}}\left(\xi_{k}\right)_{q_{k}} \cdot \delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right)
\end{aligned}
$$

$$
\begin{align*}
& ={\underset{q_{1}, \ldots, q_{k} \in Q}{ }}\left(\underset{\ell_{1} \in[m]}{ } \mathrm{h}_{\mathcal{B}}\left(\xi_{1}\right)_{\left(q_{1}, \ell_{1}\right)}\right) \cdot \ldots \cdot\left({\underset{\ell_{k} \in[m]}{ }} \mathrm{h}_{\mathcal{B}}\left(\xi_{k}\right)_{\left(q_{k}, \ell_{k}\right)}\right) \cdot \delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right)  \tag{byI.H.}\\
& =\underset{\left(q_{1}, \ell_{1}\right), \ldots,\left(q_{k}, \ell_{k}\right) \in Q^{\prime}}{ } \mathrm{h}_{\mathcal{B}}\left(\xi_{1}\right)_{\left(q_{1}, \ell_{1}\right)} \cdots \cdot \mathrm{h}_{\mathcal{B}}\left(\xi_{k}\right)_{\left(q_{k}, \ell_{k}\right)} \cdot\left(\underset{\ell \in[m]}{+}\left(\delta^{\prime}\right)_{k}\left(\left(q_{1}, \ell_{1}\right) \cdots\left(q_{k}, \ell_{k}\right), \sigma,(q, \ell)\right)\right) \\
& \text { (by right-distributivity and construction) } \\
& =+_{\ell \in[m]\left(q_{1}, \ell_{1}\right), \ldots,\left(q_{k}, \ell_{k}\right) \in Q^{\prime}} \mathrm{h}_{\mathcal{B}}\left(\xi_{1}\right)_{\left(q_{1}, \ell_{1}\right)} \cdot \cdots \cdot \mathrm{h}_{\mathcal{B}}\left(\xi_{k}\right)_{\left(q_{k}, \ell_{k}\right)} \cdot\left(\delta^{\prime}\right)_{k}\left(\left(q_{1}, \ell_{1}\right) \cdots\left(q_{k}, \ell_{k}\right), \sigma,(q, \ell)\right)
\end{align*}
$$

(by left-distributivity)

$$
=\underset{\ell \in[m]}{+} \mathrm{h}_{\mathcal{B}}\left(\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)\right)_{(q, \ell)}
$$

Now let $\xi \in \mathrm{T}_{\Sigma}$. Then we can calculate as follows.

$$
\begin{array}{rlr}
\llbracket \mathcal{A} \rrbracket^{\text {init }}(\xi) & =\underset{q \in Q}{+} \mathrm{h}_{\mathcal{A}}(\xi)_{q} \cdot F_{q} & \\
& =\underset{q \in Q}{+}\left(\underset{\ell \in[m]}{+} \mathrm{h}_{\mathcal{B}}(\xi)_{(q, \ell)}\right) \cdot F_{q} &  \tag{3.29}\\
& =\underset{q \in Q}{+} \underset{\ell \in[m]}{+} \mathrm{h}_{\mathcal{B}}(\xi)_{(q, \ell)} \cdot F_{q} & \text { (by (because } \left.F_{q} \in\{0,1\}\right) \\
& =\underset{(q, \ell) \in Q^{\prime}}{ } \mathrm{h}_{\mathcal{B}}(\xi)_{(q, \ell)} \cdot\left(F^{\prime}\right)_{(q, \ell)} & \text { (by construction) } \\
& =\llbracket \mathcal{B} \rrbracket^{\text {init }}(\xi) . & \square
\end{array}
$$

Assuming that Theorem 5.3.2 and the normal form theorem Theorem 7.3.1 are already available, Lemma 3.5.1 implies the following corollary. We note that the proofs of the Theorems 5.3.2 and 7.3.1 do not depend on Corollary 3.5.2

Corollary 3.5.2. For each ( $\Sigma, \operatorname{Nat})$-wta $\mathcal{A}$, we can construct a $(\Sigma, N a t)$-wta $\mathcal{B}$ such that $\mathcal{B}$ has identity transition weights and identity root weights and $\llbracket \mathcal{A} \rrbracket^{\text {init }}=\llbracket \mathcal{B} \rrbracket^{\text {init }}$.

Proof. Let $\mathcal{A}$ be a $\left(\Sigma\right.$, Nat)-wta. By Theorem 5.3 .2 we have $\llbracket \mathcal{A} \rrbracket^{\text {init }}=\llbracket \mathcal{A} \rrbracket^{\text {run }}$. By Theorem 7.3 .1 we can construct a root weight normalized $\left(\Sigma\right.$, Nat)-wta $\mathcal{C}$ such that $\llbracket \mathcal{A} \rrbracket^{\text {run }}=\llbracket \mathcal{C} \rrbracket^{\text {run }}$. By Theorem 5 .3.2 we have $\llbracket \mathcal{C} \rrbracket^{\text {run }}=\llbracket \mathcal{C} \rrbracket^{\text {init. }}$. Then, by Lemma 3.5 .1 we can construct a $(\Sigma, N a t)$-wta $\mathcal{B}$ such that $\mathcal{B}$ has identity transition weights and identity root weights and $\llbracket \mathcal{C} \rrbracket^{\text {init }}=\llbracket \mathcal{B} \rrbracket^{\text {init }}$.

Next we define the characteristic wta of a $\Sigma$-fta $A$. For this, let $A=(Q, \delta, F)$ be a $\Sigma$-fta. The characteristic wta of $A$, denoted by $\chi_{\mathrm{Nat}}(A)$, is the ( $\Sigma$, Nat)-wta $\left(Q, \delta^{\prime}, F^{\prime}\right)$ where for every $k \in$ Nat we have $\delta_{k}^{\prime}=\chi_{\mathrm{Nat}}\left(\delta_{k}\right)$ and $F^{\prime}=\chi_{\mathrm{Nat}}(F)$. Note that $\chi_{\mathrm{Nat}}(A)$ has identity transition weights and identity root weights.

The characteristic wta is the "inverse" of the support fta in the following sense. For each ( $\Sigma$, Nat)-wta $\mathcal{A}$ with identity transition weights and identity root weights, we have $\chi_{\mathrm{Nat}}(\operatorname{supp}(\mathcal{A}))=\mathcal{A}$. Moreover, for each $\Sigma$-fta $A$, we have $\operatorname{supp}\left(\chi_{\text {Nat }}(A)\right)=A$.

We recall that, for a $\Sigma$-fta $A$, we denote by $\mathrm{R}_{A}^{\mathrm{a}}(\xi)$ the set of accepting runs of $A$ on a tree $\xi$. Moreover, we recall that the multiplicity mapping of $A$ is defined as the weighted tree language $\#_{R_{A}^{a}}: \mathrm{T}_{\Sigma} \rightarrow \mathbb{N}$ such that, for each $\xi \in \mathrm{T}_{\Sigma}$, we have $\#_{\mathrm{R}_{A}^{\mathrm{a}}}(\xi)=\left|\mathrm{R}_{A}^{\mathrm{a}}(\xi)\right|$ (cf Example 3.2.1).

In Example 3.2.1 we have proved the following result.
Lemma 3.5.3. For each $\Sigma$-fta $A$, we have $\llbracket \chi_{\text {Nat }}(A) \rrbracket^{\text {init }}=\#_{\mathrm{R}_{A}^{a}}$.

We also have the following kind of inverse result.
Lemma 3.5.4. Let $\mathcal{A}$ be a ( $\Sigma, \mathrm{Nat})$-wta which has identity transition weights and identity root weights. Then $\llbracket \mathcal{A} \rrbracket^{\text {init }}=\#_{\mathrm{R}_{\operatorname{supp}(\mathcal{A})}^{\mathrm{a}}}$.

Proof. This follows from the fact that $\mathcal{A}=\chi_{\mathrm{Nat}}(\operatorname{supp}(\mathcal{A}))$ and from Lemma 3.5.3.

For example, the ( $\Sigma, N \mathrm{Nat})$-wta $\mathcal{A}$ in Example 3.2 .11 has identity transition weights and identity root weights. Since $\llbracket \mathcal{A} \rrbracket^{\text {init }}=\#_{\sigma(., \alpha)}$, it follows from Lemma 3.5.4 that $\#_{\sigma(., \alpha)}=\#_{R_{\operatorname{supp}(\mathcal{A})}^{a}}$.

We can conclude the following characterization of $\operatorname{Rec}^{\mathrm{init}}(\Sigma, N a t)$, which says that the i-recognizable ( $\Sigma, \mathrm{Nat})$-weighted tree languages are exactly the multiplicity mappings of $\Sigma$-fta.

Theorem 3.5.5. Let $\Sigma$ be a ranked alphabet and $r: \mathrm{T}_{\Sigma} \rightarrow \mathbb{N}$. Then the following three statements are equivalent.
(A) We can construct $a(\Sigma, \mathrm{Nat})$-wta $\mathcal{A}$ such that $r=\llbracket \mathcal{A} \rrbracket^{\text {init }}$.
(B) We can construct a ( $\Sigma, \mathrm{Nat})-w t a \mathcal{A}$ which has identity transition weights and identity root weights such that $r=\llbracket \mathcal{A} \rrbracket^{\text {init }}$.
(C) We can construct a $\Sigma$-fta $A$ such that $r=\#_{\mathrm{R}_{A}^{\mathrm{a}}}$.

Proof. $(\mathrm{A}) \Rightarrow(\mathrm{B})$ follows from Corollary 3.5 .2 and $(\mathrm{B}) \Rightarrow(\mathrm{A})$ is obvious. $(\mathrm{B}) \Rightarrow(\mathrm{C})$ follows from Lemma 3.5 .4 and $(\mathrm{C}) \Rightarrow(\mathrm{B})$ follows from Lemma 3.5.3,

It will turn out later (cf. Theorem 5.3.2) that $\llbracket \mathcal{A} \rrbracket^{\text {init }}=\llbracket \mathcal{A} \rrbracket^{\text {run }}$ for each $(\Sigma$, Nat)-wta. Hence, Theorem 3.5.5 also shows a characterization of $\operatorname{Rec}^{\text {run }}(\Sigma, \mathrm{Nat})$.

Finally, we show that, for each ( $\Sigma$, Nat)-wta $\mathcal{A}$ and $\xi \in \mathrm{T}_{\Sigma}$, the value $\llbracket \mathcal{A} \rrbracket^{\text {run }}(\xi)$ is bounded exponentially by size $(\xi)$.

Lemma 3.5.6. Let $\mathcal{A}=(Q, \delta, F)$ be a ( $\Sigma, \mathrm{Nat})$-wta. There exists $K \in \mathbb{N}$ such that, for each $\xi \in \mathrm{T}_{\Sigma}$, we have $\llbracket \mathcal{A} \rrbracket^{\text {run }}(\xi) \leq K^{\text {size }(\xi)}$.

Proof. Let $m=\max (\operatorname{wts}(\mathcal{A}))$ and $K=|Q| \cdot m^{2}$. Obviously, for each $\xi \in \mathrm{T}_{\Sigma}$, the number of runs on $\xi$ is $|Q|^{\text {size }(\xi)}$. Moreover, for each $\xi \in \mathrm{T}_{\Sigma}$ and $\rho \in \mathrm{R}_{\mathcal{A}}(\xi)$, we have $\mathrm{wt}(\xi, \rho) \leq m^{\text {size }(\xi)}$. Then

### 3.6 Weighted tree automata over commutative semirings

In the particular case that B is a commutative semiring, we can view each $(\Sigma, \mathrm{B})$-wta as a B -semimodule with multilinear operations and a linear form. The latter combination will be called multilinear representation (cf. BR82, BA89, Boz91, FS11]). Also, vice versa, each multilinear representation can be viewed as a wta. In the next subsections we will formalize these informal statements. We will use notions from Sections 2.7 and 2.8 .

In the rest of this section, we let B denote an arbitrary commutative semiring.

### 3.6.1 Multilinear representations

Let $Q$ be a finite set. We recall that $\mathbb{D}_{Q}$ is the $Q$-vector in $B^{Q}$ which contains $\mathbb{0}$ in each component, and that $\mathbb{1}_{q}$ is the $q$-unit vector in $B^{Q}$ for each $q \in Q$ (cf. Section [2.7). We start this subsection with an important observation.

Observation 3.6.1. For each finite set $Q$, the triple $\left(B^{Q},+, D_{Q}\right)$ is a B-semimodule via scalar multiplication.

Proof. Obviously, $\left(B^{Q},+, 0_{Q}\right)$ is a commutative monoid. Also the properties (2.18)-(2.22) are satisfied, where (2.19) and (2.20) require that B is distributive. Hence $\left(B^{Q},+, 0_{Q}\right)$ is a B -semimodule.

A $(\Sigma, \mathrm{B})$-multilinear representation ${ }^{4}$ is a tuple $(\mathrm{V}, \mu, \gamma)$ where

- $\mathrm{V}=\left(B^{Q},+, \mathbb{O}_{Q}\right)$ for some finite set $Q$,
- $\left(B^{Q}, \mu\right)$ is a $\Sigma$-algebra such that $\mu(\sigma) \in \mathcal{L}\left(\mathrm{V}^{k}, \mathrm{~V}\right)$ for each $k$-ary $\sigma$, i.e., $\mu(\sigma)$ is a $k$-ary multilinear operation over V , and
- $\gamma: B^{Q} \rightarrow B$ is a linear form over V (cf. Section (2.8).

We call $\left(B^{Q}, \mu\right)$ the $\Sigma$-algebra associated with ( $\mathrm{V}, \mu, \gamma$ ), and denote the unique $\Sigma$-algebra homomorphism from $\mathrm{T}_{\Sigma}$ to $\left(B^{Q}, \mu\right)$ by $\mathrm{h}_{\mathrm{V}}$. The weighted tree language recognized by $(V, \mu, \gamma)$ is the weighted tree language $r: \mathrm{T}_{\Sigma} \rightarrow B$ where $r(\xi)=\gamma(\mathrm{hv}(\xi))$ for every $\xi \in \mathrm{T}_{\Sigma}$.

Before showing an example of a multilinear representation, we prove a useful lemma. For this, we note that the algebra ( $B^{\mathrm{T}_{\Sigma}}, \oplus, \widetilde{\mathbb{D}}$ ) is a B-semimodule via $\otimes$, i.e., scalar multiplication from left.

Lemma 3.6.2. (cf. BR82, Prop. 3.1]) The set of weighted tree languages which are recognizable by ( $\Sigma, \mathrm{B}$ )-multilinear representations is a sub-semimodule of the $B$-semimodule ( $B^{\mathrm{T}_{\Sigma}}, \oplus, \widetilde{\mathbb{D}}$ ).

Proof. We show that the subset of $B^{T_{\Sigma}}$ which consists of all weighted tree languages which are recognizable by ( $\Sigma, \mathrm{B}$ )-multilinear representations is closed under (a) scalar multiplication from left and (b) under sum.
(a) Let $r \in B^{T_{\Sigma}}$ and $b \in B$ and assume that $r$ is recognized by the ( $\left.\Sigma, \mathrm{B}\right)$-multilinear representation $(\mathrm{V}, \mu, \gamma)$ with $\mathrm{V}=\left(B^{Q},+, \mathbb{D}_{Q}\right)$. Then the $(\Sigma, \mathrm{B})$-weighted tree language $b \otimes r$ is recognizable by the ( $\Sigma, \mathrm{B}$ )-multilinear representation ( $\mathrm{V}, \mu, \gamma^{\prime}$ ), where $\gamma^{\prime}(v)=b \otimes \gamma(v)$ for each $v \in V$.
(b) Let $r_{1}, r_{2} \in B^{\mathrm{T}_{\Sigma}}$ and assume that, for each $i \in\{1,2\}$, the $(\Sigma, \mathrm{B})$-multilinear representation $\left(\mathrm{V}_{i}, \mu_{i}, \gamma_{i}\right)$ with $\mathrm{V}_{i}=\left(B^{Q_{i}},+, 0_{Q_{i}}\right)$ recognizes $r_{i}$. We assume that $Q_{1} \cap Q_{2}=\emptyset$. We define a ( $\left.\Sigma, \mathrm{B}\right)$ multilinear representation $(\mathrm{V}, \mu, \gamma)$ which recognizes $r_{1} \oplus r_{2}$ as follows. Let

- $\mathrm{V}=\left(B^{Q_{1} \cup Q_{2}},+, 0_{Q_{1} \cup Q_{2}}\right)$ where + is the usual sum of $\left(Q_{1} \cup Q_{2}\right)$-vectors over $B$,
- let $\mu$ be the $\Sigma$-indexed family over the set of all operations over $B$ such that, for every $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $v_{1}, \ldots, v_{k} \in B^{Q_{1} \cup Q_{2}}$ and $q \in Q_{1} \cup Q_{2}$, we have (recalling that vectors are mappings)

$$
\mu(\sigma)\left(v_{1}, \ldots, v_{k}\right)_{q}= \begin{cases}\mu_{1}(\sigma)\left(\left.v_{1}\right|_{Q_{1}}, \ldots,\left.v_{k}\right|_{Q_{1}}\right)_{q} & \text { if } q \in Q_{1} \\ \mu_{2}(\sigma)\left(\left.v_{1}\right|_{Q_{2}}, \ldots,\left.v_{k}\right|_{Q_{2}}\right)_{q} & \text { otherwise }\end{cases}
$$

- and let $\gamma: B^{Q_{1} \cup Q_{2}} \rightarrow B$ be a mapping such that, for each $u \in B^{Q_{1} \cup Q_{2}}$, we have $\gamma(u)=\gamma_{1}\left(\left.u\right|_{Q_{1}}\right) \oplus$ $\gamma_{2}\left(\left.u\right|_{Q_{2}}\right)$.

It is easy to see that $\mu(\sigma) \in \mathcal{L}\left(\mathrm{V}^{k}, \mathrm{~V}\right)$ for every $k \in \mathbb{N}$ and $\sigma \in \Sigma^{(k)}$, and that $\gamma$ is a linear form. Hence $(\mathrm{V}, \mu, \gamma)$ is a $(\Sigma, \mathrm{B})$-multilinear representation.

[^11]It is also easy to show by induction on $\mathrm{T}_{\Sigma}$ that $\mathrm{h}_{\mathrm{V}}(\xi)=\mathrm{h}_{\mathrm{V}_{1}}(\xi) \cup \mathrm{h}_{\mathrm{V}_{2}}(\xi)$ for each $\xi \in \mathrm{T}_{\Sigma}$. Then, for each $\xi \in \mathrm{T}_{\Sigma}$, we have

$$
\gamma(\mathrm{h} \mathrm{~V}(\xi))=\gamma\left(\mathrm{h}_{\mathrm{V}_{1}}(\xi) \cup \mathrm{h}_{\mathrm{V}_{2}}(\xi)\right)=\gamma_{1}\left(\mathrm{~h}_{\mathrm{V}_{1}}(\xi)\right) \oplus \gamma_{2}\left(\mathrm{~h}_{\mathrm{V}_{2}}(\xi)\right)=r_{1}(\xi) \oplus r_{2}(\xi)
$$

i.e., $(\mathrm{V}, \mu, \gamma)$ recognizes $r_{1} \oplus r_{2}$.

Example 3.6.3. BR82, Ex. 4.1] Here we show an example of a multilinear representation for the field Rat of rational numbers. For this, we consider the weighted tree language size : $\mathrm{T}_{\Sigma} \rightarrow \mathbb{Q}$ (cf. Example 3.2.3) over the Rat and we show that size is recognizable by a ( $\Sigma$, Rat)-multilinear representation.

For each $\delta \in \Sigma$, we define the mapping $\operatorname{size}_{\delta}: \mathrm{T}_{\Sigma} \rightarrow \mathbb{Q}$ by $\operatorname{size}_{\delta}(\xi)=\left|\operatorname{pos}_{\delta}(\xi)\right|$. Then we have size $=+_{\delta \in \Sigma} \operatorname{size}_{\delta}$. By Lemma 3.6.2, a finite sum of weighted tree languages which are recognizable by ( $\Sigma$, Rat)-multilinear representation is also recognizable by a ( $\Sigma$, Rat)-multilinear representation. Hence, it suffices to prove, for an arbitrary but fixed $\delta$ that $\operatorname{size}_{\delta}$ is recognizable by some $(\Sigma, \mathbb{Q})$-multilinear representation. From now on let $\delta \in \Sigma$ be arbitrary, but fixed. We assume that $\delta$ has rank $\ell$ for some $\ell \in \mathbb{N}$.

We define the ( $\Sigma$, Rat)-multilinear representation ( $\operatorname{Rat}^{2}, \mu, \gamma$ ) with the Rat-vector space Rat $^{2}=$ $\left(\mathbb{Q}^{2},+,(0,0)\right)$ in the following way. Certainly, it is sufficient to define each multilinear operation $\mu_{k}(\sigma)$ (for $k \in \mathbb{N}$ and $\left.\sigma \in \Sigma^{(k)}\right)$ and $\gamma$ on the basis vectors $e_{1}=(1,0)$ and $e_{2}=(0,1)$. For every $e_{i_{1}}, \ldots, e_{i_{\ell}} \in\left\{e_{1}, e_{2}\right\}$, we define

$$
\mu(\delta)\left(e_{i_{1}}, \ldots, e_{i_{\ell}}\right)= \begin{cases}e_{1}+e_{2} & \text { if } i_{1}=\ldots=i_{\ell}=1 \\ e_{2} & \text { if there exists exactly one } j \in[\ell] \text { with } i_{j}=2 \\ (0,0) & \text { otherwise }\end{cases}
$$

and, for every $\sigma \in \Sigma^{(k)}$ with $k \in \mathbb{N}$ and $\sigma \neq \delta$ and $e_{i_{1}}, \ldots, e_{i_{k}} \in\left\{e_{1}, e_{2}\right\}$, we define

$$
\mu(\sigma)\left(e_{i_{1}}, \ldots, e_{i_{k}}\right)= \begin{cases}e_{1} & \text { if } i_{1}=\ldots=i_{k}=1 \\ e_{2} & \text { if there exists exactly one } j \in[k] \text { with } i_{j}=2 \\ (0,0) & \text { otherwise. }\end{cases}
$$

Thus, in particular, for $\alpha \in \Sigma^{(0)}$, this gives

$$
\mu(\alpha)()= \begin{cases}e_{1}+e_{2} & \text { if } \alpha=\delta  \tag{3.30}\\ e_{1} & \text { otherwise }\end{cases}
$$

Finally, we define $\gamma\left(e_{1}\right)=0$ and $\gamma\left(e_{2}\right)=1$.
Next, by induction on $\mathrm{T}_{\Sigma}$, we prove that the following statement holds:

$$
\begin{equation*}
\text { For every } \xi \in \mathrm{T}_{\Sigma}: \quad \mathrm{h}_{\operatorname{Rat}^{2}}(\xi)=e_{1}+\operatorname{size}_{\delta}(\xi) e_{2} \tag{3.31}
\end{equation*}
$$

I.B.: Let $\xi \in \Sigma^{(0)}$. Then (3.31) follows from (3.30).
I.S.: Now let $\xi=\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)$. Then, using the multilinearity at equations marked by $*$, we can calculate as follows:

$$
\begin{align*}
\mathrm{h}_{\operatorname{Rat}^{2}}(\xi)= & \mu(\sigma)\left(\mathrm{h}_{\operatorname{Rat}^{2}}\left(\xi_{1}\right), \ldots, \mathrm{h}_{\operatorname{Rat}^{2}}\left(\xi_{k}\right)\right) \\
= & \mu(\sigma)\left(e_{1}+\operatorname{size}_{\delta}\left(\xi_{1}\right) e_{2}, \ldots, e_{1}+\operatorname{size}_{\delta}\left(\xi_{k}\right) e_{2}\right)  \tag{byI.H.}\\
= & { }^{*} \mu(\sigma)\left(e_{1}, \ldots, e_{1}\right)+\operatorname{F}_{i \in[k]} \mu(\sigma)\left(e_{1}, \ldots, e_{1}, \operatorname{size}_{\delta}\left(\xi_{i}\right) e_{2}, e_{1}, \ldots, e_{1}\right) \\
& \quad+\operatorname{l}_{1 \leq i<j \leq m} \mu(\sigma)\left(\ldots, \operatorname{size}_{\delta}\left(\xi_{i}\right) e_{2}, \ldots, \operatorname{size}_{\delta}\left(\xi_{j}\right) e_{2}, \ldots\right)
\end{align*}
$$

$$
\begin{aligned}
& ={ }^{*} \mu(\sigma)\left(e_{1}, \ldots, e_{1}\right)+\underset{i \in[k]}{+} \operatorname{size}_{\delta}\left(\xi_{i}\right) \mu(\sigma)\left(e_{1}, \ldots, e_{1}, e_{2}, e_{1}, \ldots, e_{1}\right) \\
& \quad \quad++\ldots \operatorname{size}_{\delta}\left(\xi_{i}\right) \ldots \operatorname{size}_{\delta}\left(\xi_{j}\right) \ldots \mu(\sigma)\left(\ldots, e_{2}, \ldots, e_{2}, \ldots\right) \\
& =\mu(\sigma)\left(e_{1}, \ldots, e_{1}\right)+\underset{i \in[k]}{+} \operatorname{size}_{\delta}\left(\xi_{i}\right) e_{2}+(0,0) \\
& = \begin{cases}e_{1}+e_{2}++_{i \in[k]} \operatorname{size}_{\delta}\left(\xi_{i}\right) e_{2} & \text { if } \sigma=\delta \\
e_{1}++_{i \in[k]} \operatorname{size}_{\delta}\left(\xi_{i}\right) e_{2} & \text { otherwise. }\end{cases}
\end{aligned}
$$

This proves (3.31). Finally, it is easy to see that $\gamma\left(\mathrm{h}_{\operatorname{Rat}^{2}}(\xi)\right)=\operatorname{size}_{\delta}(\xi)$ for each $\xi \in \mathrm{T}_{\Sigma}$. This means that size $\delta$ is recognizable by the ( $\Sigma$, Rat)-multilinear representation ( $\operatorname{Rat}^{2}, \mu, \gamma$ ). Thus size is recognizable by some ( $\Sigma$, Rat)-multilinear representation.

### 3.6.2 Relationship between wta and multilinear representations

Let $(\mathrm{V}, \mu, \gamma)$ be a $(\Sigma, \mathrm{B})$-multilinear representation with $\mathrm{V}=\left(B^{Q},+, \mathrm{D}_{Q}\right)$. Moreover, let $\mathcal{A}=(Q, \delta, F)$ be a $(\Sigma, \mathrm{B})$-wta. We recall that $\mathrm{V}(\mathcal{A})=\left(B^{Q}, \delta_{\mathcal{A}}\right)$ is the vector algebra of $\mathcal{A}$.

We say that $(\mathrm{V}, \mu, \gamma)$ and $\mathcal{A}$ are related if

- for every $k \in \mathbb{N}, \sigma \in \Sigma^{(k)}$, and $q, q_{1}, \ldots, q_{k} \in Q$, the equation

$$
\mu(\sigma)\left(\mathbb{1}_{q_{1}}, \ldots, \mathbb{1}_{q_{k}}\right)_{q}=\delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right)
$$

holds, and

- for every $q \in Q$, the equation $\gamma\left(\mathbb{1}_{q}\right)=F_{q}$ holds.

The intention of the first condition in the definition of relatedness is that $\mu$ and $\delta_{\mathcal{A}}$ should play the same role. In order to formalize this, we first prove that each $\delta_{\mathcal{A}}(\sigma)$ is a multilinear mapping.

Lemma 3.6.4. (cf. Boz99) Let $\mathcal{A}=(Q, \delta, F)$ be a $(\Sigma, \mathrm{B})$-wta. For every $k \in \mathbb{N}$ and $\sigma \in \Sigma^{(k)}$, the operation $\delta_{\mathcal{A}}(\sigma)$ is multilinear.

Proof. Let $k \in \mathbb{N}, \sigma \in \Sigma^{(k)}, i \in[k], b, b^{\prime} \in B, v_{1}, \ldots, v_{k}, v, v^{\prime} \in B^{Q}$, and $q \in Q$. Then we can calculate as follows.

$$
\begin{aligned}
& \delta_{\mathcal{A}}(\sigma)\left(v_{1}, \ldots, v_{i-1}, b v+b^{\prime} v^{\prime}, v_{i+1}, \ldots, v_{k}\right)_{q} \\
= & \bigoplus_{q_{1} \cdots q_{k} \in Q^{k}}\left(\bigotimes_{j \in[1, i-1]}\left(v_{j}\right)_{q_{j}}\right) \otimes\left(b v+b^{\prime} v^{\prime}\right)_{q_{i}} \otimes\left(\bigotimes_{j \in[i+1, k]}\left(v_{j}\right)_{q_{j}}\right) \otimes \delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right) \\
= & \bigoplus_{q_{1} \cdots q_{k} \in Q^{k}}\left(\bigotimes_{j \in[1, i-1]}\left(v_{j}\right)_{q_{j}}\right) \otimes\left(b \otimes v_{q_{i}} \oplus b^{\prime} \otimes\left(v^{\prime}\right)_{q_{i}}\right) \otimes\left(\bigotimes_{j \in[i+1, k]}\left(v_{j}\right)_{q_{j}}\right) \otimes \delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right) \\
= & \bigoplus_{q_{1} \cdots q_{k} \in Q^{k}}\left[\left(\bigotimes_{j \in[1, i-1]}\left(v_{j}\right)_{q_{j}}\right) \otimes b \otimes v_{q_{i}} \otimes\left(\bigotimes_{j \in[i+1, k]}\left(v_{j}\right)_{q_{j}}\right) \otimes \delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right)\right. \\
& \left.\oplus\left(\bigotimes_{j \in[1, i-1]}\left(v_{j}\right)_{q_{j}}\right) \otimes b^{\prime} \otimes\left(v^{\prime}\right)_{q_{i}} \otimes\left(\bigotimes_{j \in[i+1, k]}\left(v_{j}\right)_{q_{j}}\right) \otimes \delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right)\right]
\end{aligned}
$$

(by distributivity of $B$ )

$$
\begin{aligned}
= & \bigoplus_{q_{1} \cdots q_{k} \in Q^{k}}\left[\left(\bigotimes_{j \in[1, i-1]}\left(v_{j}\right)_{q_{j}}\right) \otimes b \otimes v_{q_{i}} \otimes\left(\bigotimes_{j \in[i+1, k]}\left(v_{j}\right)_{q_{j}}\right) \otimes \delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right)\right] \\
& \oplus \bigoplus_{q_{1} \cdots q_{k} \in Q^{k}}\left[\left(\bigotimes_{j \in[1, i-1]}\left(v_{j}\right)_{q_{j}}\right) \otimes b^{\prime} \otimes\left(v^{\prime}\right)_{q_{i}} \otimes\left(\bigotimes_{j \in[i+1, k]}\left(v_{j}\right)_{q_{j}}\right) \otimes \delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
& =b \otimes\left[\bigoplus_{q_{1} \cdots q_{k} \in Q^{k}}\left(\bigotimes_{j \in[1, i-1]}\left(v_{j}\right)_{q_{j}}\right) \otimes v_{q_{i}} \otimes\left(\bigotimes_{j \in[i+1, k]}\left(v_{j}\right)_{q_{j}}\right) \otimes \delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right)\right] \\
& \\
& \oplus b^{\prime} \otimes\left[\bigoplus_{q_{1} \cdots q_{k} \in Q^{k}}\left(\bigotimes_{j \in[1, i-1]}\left(v_{j}\right)_{q_{j}}\right) \otimes\left(v^{\prime}\right)_{q_{i}} \otimes\left(\bigotimes_{j \in[i+1, k]}\left(v_{j}\right)_{q_{j}}\right) \otimes \delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right)\right]
\end{aligned}
$$

(by commutativity and distributivity)

$$
=b \delta_{\mathcal{A}}(\sigma)\left(v_{1}, \ldots, v_{i-1}, v, v_{i+1}, \ldots, v_{k}\right)_{q}+b^{\prime} \delta_{\mathcal{A}}(\sigma)\left(v_{1}, \ldots, v_{i-1}, v^{\prime}, v_{i+1}, \ldots, v_{k}\right)_{q}
$$

Hence $\delta_{\mathcal{A}}(\sigma)$ is multilinear.
Now we can show the semantic implications of the relatedness between a multilinear representation and a wta.

Lemma 3.6.5. Let $(\mathrm{V}, \mu, \gamma)$ with $\mathrm{V}=\left(B^{Q},+, \mathrm{O}_{Q}\right)$ and $\mathcal{A}=(Q, \delta, F)$ be related. Then

- $\mu(\sigma)=\delta_{\mathcal{A}}(\sigma)$ for each $\sigma \in \Sigma$,
- for every $\xi \in \mathrm{T}_{\Sigma}$ and $q \in Q$ we have that $\mathrm{h}_{\mathcal{V}}(\xi)_{q}=\mathrm{h}_{\mathcal{A}}(\xi)_{q}$, and
- for every $\xi \in \mathrm{T}_{\Sigma}$ we obtain: $\gamma\left(\mathrm{h}_{\mathrm{v}}(\xi)\right)=\llbracket \mathcal{A} \rrbracket^{\text {init }}(\xi)$.

Proof. Let $k \in \mathbb{N}, \sigma \in \Sigma^{(k)}$, and $q, q_{1}, \ldots, q_{k} \in Q$. Then we have

$$
\mu(\sigma)\left(\mathbb{1}_{q_{1}}, \ldots, \mathbb{1}_{q_{k}}\right)_{q}=\delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right)=\delta_{\mathcal{A}}(\sigma)\left(\mathbb{1}_{q_{1}}, \ldots, \mathbb{1}_{q_{k}}\right)_{q}
$$

where the first equality holds by relatedness and the second one holds by the definition of $\delta_{\mathcal{A}}(\sigma)$. Since $\mu(\sigma)$ is multilinear (by definition) and $\delta_{\mathcal{A}}(\sigma)$ is multilinear (due to Lemma 3.6.4) and they coincide on unit vectors, we obtain that $\mu(\sigma)=\delta_{\mathcal{A}}(\sigma)$. This proves the first statement.

The second statement follows from the fact that the two $\Sigma$-algebras $\left(B^{Q}, \mu\right)$ and $\left(B^{Q}, \delta_{\mathcal{A}}\right)$ and hence also the two $\Sigma$-algebra homomorphisms $h_{v}$ and $h_{\mathcal{A}}$ are equal. Now let $\xi \in \mathrm{T}_{\Sigma}$. Then

$$
\llbracket \mathcal{A} \rrbracket^{\mathrm{init}}(\xi)=\bigoplus_{q \in Q} \mathrm{~h}_{\mathcal{A}}(\xi)_{q} \otimes F_{q}=\bigoplus_{j \in[\kappa]} \mathrm{h}_{\mathcal{A}}(\xi)_{q_{j}} \otimes F_{q_{j}}=\bigoplus_{j \in[\kappa]} \mathrm{h}_{\mathrm{V}}(\xi)_{q_{j}} \otimes \gamma\left(\mathbb{1}_{q_{j}}\right)=\gamma(\mathrm{h} v(\xi)) .
$$

The next theorem contains a characterization of recognizability in terms of multilinear representations. For the case that the weight algebra B is a field, the proof was sketched in [FV09, Thm. 3.52]. Already in [Bor04, p. 518], direct constructions have been indicated. There, the author deals with the concepts of B-vector space and basis, and thereby gives the impression that B is a field. On the background of this understanding, it is a bit puzzling that the author mentions twice the condition that "the underlying semiring is commutative", as if the indicated constructions would also hold for commutative semirings. Anyway, the characterization for the case that $B$ is a commutative semiring has been (re)discovered in Dro22.

Theorem 3.6.6. Dro22 Let $\Sigma$ be a ranked alphabet, B be a commutative semiring, and $r: \mathrm{T}_{\Sigma} \rightarrow B$. Then the following two statements are equivalent.
(A) We can construct a $(\Sigma, \mathrm{B})$-multilinear representation which recognizes $r$.
(B) We can construct a ( $\Sigma, \mathrm{B})$-wta which i-recognizes $r$.

Proof. Proof of $(\mathrm{A}) \Rightarrow(\mathrm{B})$ : Let $r: \mathrm{T}_{\Sigma} \rightarrow B$ be recognizable by the $(\Sigma, \mathrm{B})$-multilinear representation $(\mathrm{V}, \mu, \gamma)$ with $\mathrm{V}=\left(B^{Q},+, \mathrm{O}_{Q}\right)$ for some finite set $Q$. Hence $r(\xi)=\gamma(\mathrm{h} \mathrm{V}(\xi))$ for every $\xi \in \mathrm{T}_{\Sigma}$.

It is easy to construct a $(\Sigma, \mathrm{B})$-wta $\mathcal{A}=(Q, \delta, F)$ such that $(\mathrm{V}, \mu, \gamma)$ and $\mathcal{A}$ are related. Hence, by Lemma 3.6.5, for every $\xi \in \mathrm{T}_{\Sigma}$ we obtain: $\gamma\left(\mathrm{h}_{\mathrm{V}}(\xi)\right)=\llbracket \mathcal{A} \rrbracket^{\text {init }}(\xi)$.

Proof of $(\mathrm{B}) \Rightarrow(\mathrm{A})$ : Let $r: \mathrm{T}_{\Sigma} \rightarrow B$ be i-recognizable by some $(\Sigma, \mathrm{B})$-wta $\mathcal{A}=(Q, \delta, F)$ with $Q=$ $\left\{q_{1}, \ldots, q_{\kappa}\right\}$. By Observation 3.6.1, $\left(B^{Q},+, \mathbb{O}_{Q}\right)$ is a B-semimodule and for every $k \in \mathbb{N}$ and $\sigma \in \Sigma^{(k)}$,
the operation $\delta_{\mathcal{A}}(\sigma)$ is multilinear. We define the linear form $\gamma: B^{Q} \rightarrow B$ such that, for each $v=$ $b_{1} \mathbb{1}_{q_{1}}+\ldots+b_{\kappa} \mathbb{1}_{q_{\kappa}}$ in $B^{Q}$, we let $\gamma(v)=\bigoplus_{j \in[\kappa]} b_{j} \otimes F_{q_{j}}$. In particular, $\gamma\left(\mathbb{1}_{q_{j}}\right)=F_{q_{j}}$ for each $j \in[\kappa]$. Then $\left(B^{Q}, \delta_{\mathcal{A}}, \gamma\right)$ is a $(\Sigma, \mathrm{B})$-multilinear representation and, moreover, $\left(B^{Q}, \delta_{\mathcal{A}}, \gamma\right)$ and $\mathcal{A}$ are related. Thus it follows from Lemma 3.6 .5 that $\gamma\left(\mathrm{h}_{\mathrm{V}}(\xi)\right)=\llbracket \mathcal{A} \rrbracket^{\text {init }}(\xi)$ for every $\xi \in \mathrm{T}_{\Sigma}$.

Finally, we note that Theorem 3.6.6 might be understood as the tree version of the well-known result for wsa that a weighted string language is run-recognizable iff it has a linear representation $(\lambda, \mu, \gamma)$ (cf. [Eil74, Cor. 6.2], BR88, Ch. I, Sect. 5], and DK21, Thm. 3.2]).

### 3.7 Extension of the weight algebra

We finish this chapter with an easy observation on the extension of the weight algebra of a ( $\Sigma, \mathrm{B}$ )-wta $\mathcal{A}$. If $C$ is a strong bimonoid such that $B$ is a subalgebra of $C$, then we say that $C$ is an extension of $B$. If this is the case, then we can view each $(\Sigma, \mathrm{B})$-wta $\mathcal{A}$ as a $(\Sigma, \mathrm{C})$-wta. Moreover, the run semantics and the initial algebra semantics of the $(\Sigma, C)$-wta $\mathcal{A}$ are the same as the corresponding semantics of the $(\Sigma, \mathrm{B})$-wta $\mathcal{A}$. Thus we obtain the following observation.

Observation 3.7.1. Let the strong bimonoid C be an extension of B . Then $\operatorname{Rec}^{\mathrm{run}}(\Sigma, \mathrm{B}) \subseteq \operatorname{Rec}^{\mathrm{run}}(\Sigma, \mathrm{C})$ and $\operatorname{Rec}^{\text {init }}(\Sigma, \mathrm{B}) \subseteq \operatorname{Rec}^{\text {init }}(\Sigma, \mathrm{C})$.

One might also think of the following inverse problem. Is it true that, for each extension $C$ of $B$, the inclusions

$$
\begin{equation*}
\operatorname{Rec}^{\text {run }}(\Sigma, \mathrm{C}) \cap B^{\mathrm{T}_{\Sigma}} \subseteq \operatorname{Rec}^{\text {run }}(\Sigma, \mathrm{B}) \quad \text { and } \operatorname{Rec}^{\text {init }}(\Sigma, \mathrm{C}) \cap B^{\mathrm{T}_{\Sigma}} \subseteq \operatorname{Rec}^{\text {init }}(\Sigma, \mathrm{B}) \tag{3.32}
\end{equation*}
$$

hold?
An extension C of B for which (3.32) holds, is called Fatou extension of B (cf. [BR88, Ch. V]). For instance, for string ranked alphabets, the ring of integers is not a Fatou extension of the semiring of natural numbers [KS86, Ex. 8.1] (also cf. Lemma 18.2.5). In contrast, for the case that $\Sigma$ is a string ranked alphabet, the semiring Rat ${ }_{\geq 0}$ is a Fatou extension of the semiring Nat of natural numbers Fli75] (also cf. BR88, Thm. 3.3]); moreover, for every two fields B and C, if C is an extension of B, then C is a Fatou extension of B [Fli74] (also cf. [BR88, Thm. 3.1]). Recently, the latter result has been extended to wta in Dro22.

## Chapter 4

## Basic properties of wta

In this chapter we analyse basic properties of an arbitrary $(\Sigma, \mathrm{B})$-wta $\mathcal{A}$. In particular, we show that the annihilation property (i.e., $\mathbb{O} \otimes b=b \otimes \mathbb{O}=\mathbb{O}$ ) propagates over runs of $\mathcal{A}$ on an input tree $\xi \in \Sigma$ and over $Q$-vectors $\mathrm{h}_{\mathcal{A}}(\xi)$ (where $Q$ is the set of states of $\mathcal{A}$; cf. Lemma 4.1.1). If $\mathcal{A}$ is bu deterministic, then for each input tree $\xi \in \mathrm{T}_{\Sigma}$, there exists at most one $\operatorname{run} \rho \in \mathrm{R}_{\mathcal{A}}(\xi)$ with weight different from $\mathbb{O}$, and at most one component $q$ of $\mathrm{h}_{\mathcal{A}}(\xi)$ is different from $\mathbb{D}$; moreover, such a $\rho$ exists if and only if such a component $q$ exists, and if they exist, then $\mathrm{wt}_{\mathcal{A}}(\xi, \rho)=\mathrm{h}_{\mathcal{A}}(\xi)_{q}$ (cf. Lemma 4.2.1). If $\mathcal{A}$ is crisp deterministic, then we can replace in the previous sentence "at most one" by "exactly once" and $\mathrm{wt}_{\mathcal{A}}(\xi, \rho)=\mathrm{h}_{\mathcal{A}}(\xi)_{q}=\mathbb{1}$ (cf. Lemma 4.3.1). Moreover, we prove a characterization result which involves the congruence $\operatorname{ker}\left(\mathrm{h}_{\mathcal{A}}\right)$ (cf. Theorem 4.3.5).

### 4.1 Properties of arbitrary wta

Let $\mathcal{A}=(Q, \delta, F)$ be a $(\Sigma, \mathrm{B})$-wta. For each $\xi \in \mathrm{T}_{\Sigma}$ we define the sets

$$
\begin{aligned}
\mathrm{Q}_{\neq \mathbb{O}}^{\mathrm{h}_{\mathcal{A}}}(\xi) & =\left\{q \in Q \mid \mathrm{h}_{\mathcal{A}}(\xi)_{q} \neq \mathbb{O}\right\}, \text { and } \\
\mathrm{Q}_{\neq \mathbb{D}}^{\mathrm{R}_{\mathcal{A}}}(\xi) & =\left\{q \in Q \mid \exists\left(\rho \in \mathrm{R}_{\mathcal{A}}(q, \xi)\right) \text { such that } \operatorname{wt}(\xi, \rho) \neq \mathbb{O}\right\}
\end{aligned}
$$

Since $\mathbb{O}$ is annihilating with respect to the multiplication of $B$, we obtain the following zero-propagation statements (where the second statement can be compared to [Bor04, Cor. 3.6]).

Lemma 4.1.1. Let $\xi \in \mathrm{T}_{\Sigma}$ and $w \in \operatorname{pos}(\xi)$. Then the following two statements hold.
(1) If $\mathrm{Q}_{\neq 0}^{\mathrm{h}} \mathcal{A}\left(\left.\xi\right|_{w}\right)=\emptyset$, then $\mathrm{Q}_{\neq 0}^{\mathrm{h}}(\xi)=\emptyset$.
(2) If $\mathrm{Q}_{\neq \mathbb{D}}^{\mathrm{R}}\left(\left.\xi\right|_{w}\right)=\emptyset$, then $\mathrm{Q}_{\neq \mathbb{D}}^{\mathrm{R}}(\xi)=\emptyset$.

Proof. Let $\xi \in \mathrm{T}_{\Sigma}$ and $w \in \operatorname{pos}(\xi)$. We define the well-founded set ( $\left.\operatorname{prefix}(w), \prec\right)$ where, for every $w_{1}, w_{2} \in \operatorname{prefix}(w)$, we let $w_{1} \prec w_{2}$ if there exists a $j \in \mathbb{N}$ such that $w_{1}=w_{2} j$. Obviously, $\prec$ is well-founded and $\min _{\prec}(\operatorname{prefix}(w))=\{w\}$.

Proof of (1): We assume that $\mathrm{Q}_{\neq 0}^{\mathrm{h} \mathcal{A}}\left(\left.\xi\right|_{w}\right)=\emptyset$. By induction on ( $\operatorname{prefix}(w), \prec$ ), we prove that the following statement holds:

$$
\begin{equation*}
\text { For each } w^{\prime} \in \operatorname{prefix}(w) \text {, we have } \mathrm{Q}_{\neq 0}^{\mathrm{h}_{\mathcal{A}}}\left(\left.\xi\right|_{w^{\prime}}\right)=\emptyset \tag{4.1}
\end{equation*}
$$

I.B.: Let $w^{\prime}=w$. Then the statement holds by assumption.
I.S.: Now let $w^{\prime} \in \operatorname{prefix}(w) \backslash\{w\}$ and $\xi\left(w^{\prime}\right)=\sigma$ for some $\sigma \in \Sigma^{(k)}$ and $k \in \mathbb{N}$. Since $w^{\prime} \in$ $\operatorname{prefix}(w) \backslash\{w\}$, we have that $k \geq 1$ and there exists a $j \in[k]$ such that $w^{\prime} j \in \operatorname{prefix}(w)$. Moreover, for each $q \in Q$, we have

$$
\mathrm{h}_{\mathcal{A}}\left(\left.\xi\right|_{w^{\prime}}\right)_{q}=\mathrm{h}_{\mathcal{A}}\left(\sigma\left(\left.\xi\right|_{w^{\prime} 1}, \ldots,\left.\xi\right|_{w^{\prime} k}\right)\right)_{q}=\bigoplus_{q_{1} \cdots q_{k} \in Q^{k}}\left(\bigotimes_{i \in[k]} \mathrm{h}_{\mathcal{A}}\left(\left.\xi\right|_{w^{\prime} i}\right)_{q_{i}}\right) \otimes \delta\left(q_{1} \cdots q_{k}, \sigma, q\right)
$$

Then, by the I.H., $\mathrm{Q}_{\neq \mathbb{D}}^{\mathrm{h}}\left(\left.\xi\right|_{w^{\prime} j}\right)=\emptyset$ and thus for each $q_{j} \in Q$ we have $\mathrm{h}_{\mathcal{A}}\left(\left.\xi\right|_{w^{\prime} j}\right)_{q_{j}}=\mathbb{O}$. Hence each of the summands is a product in which the $j$-th factor is $\mathbb{O}$. Consequently, each summand is $\mathbb{O}$ and thus $\mathrm{h}_{\mathcal{A}}\left(\left.\xi\right|_{w^{\prime}}\right)_{q}=\mathbb{0}$. This proves (4.1).

Finally, $\mathrm{Q}_{\neq 0}^{\mathrm{h}_{\mathcal{A}}}(\xi)=\emptyset$ follows from (4.1) with $w^{\prime}=\varepsilon$. Thus Statement (1) holds.
Proof of (2): We assume that $\mathrm{Q}_{\neq 0}^{\mathrm{R}_{\mathcal{A}}}\left(\left.\xi\right|_{w}\right)=\emptyset$. Then, by induction on ( $\operatorname{prefix}(w), \prec$ ), we prove that the following statement holds:

$$
\begin{equation*}
\text { For each } w^{\prime} \in \operatorname{prefix}(w), \text { we have } \mathrm{Q}_{\neq \mathbb{D}}^{\mathrm{R}_{\mathcal{A}}}\left(\left.\xi\right|_{w^{\prime}}\right)=\emptyset \tag{4.2}
\end{equation*}
$$

I.B.: Let $w^{\prime}=w$. Then the statement holds by assumption.
I.S.: Now let $w^{\prime} \in \operatorname{prefix}(w) \backslash\{w\}$. Let $\xi\left(w^{\prime}\right)=\sigma$ for some $\sigma \in \Sigma^{(k)}$ and $k \in \mathbb{N}$. Since $w^{\prime} \in$ $\operatorname{prefix}(w) \backslash\{w\}$, we have that $k \geq 1$ and there exists a $j \in[k]$ such that $w^{\prime} j \in \operatorname{prefix}(w)$.

Let $q \in Q$ and $\rho$ be an arbitrary run in $\mathrm{R}_{\mathcal{A}}\left(q,\left.\xi\right|_{w^{\prime}}\right)$. The weight of $\rho$, by definition, is

$$
\mathrm{wt}\left(\left.\xi\right|_{w^{\prime}}, \rho\right)=\left(\bigotimes_{i \in[k]} \mathrm{wt}\left(\left.\xi\right|_{w^{\prime} i},\left.\rho\right|_{i}\right)\right) \otimes \delta(\rho(1) \cdots \rho(k), \sigma, q)
$$

By the I.H. $\mathrm{Q}_{\neq 0}^{\mathrm{R}_{\mathcal{A}}}\left(\left.\xi\right|_{w^{\prime} j}\right)=\emptyset$, hence $\mathrm{wt}\left(\left.\xi\right|_{w^{\prime} j},\left.\rho\right|_{w^{\prime} j}\right)=\mathbb{0}$. This implies $\mathrm{wt}\left(\left.\xi\right|_{w^{\prime}}, \rho\right)=\mathbb{0}$ and thus (4.2) holds.
Then $\mathrm{Q}_{\neq \mathbb{D}}^{\mathrm{R}_{\mathcal{A}}}(\xi)=\emptyset$ follows from (4.2) for $w^{\prime}=\varepsilon$. Thus Statement (2) holds.

### 4.2 Properties of bu deterministic wta

Next we consider an arbitrary bu deterministic ( $\Sigma, \mathrm{B}$ )-wta $\mathcal{A}=(Q, \delta, F)$. Thus, for every $k \in \mathbb{N}, \sigma \in \Sigma^{(k)}$, and $w \in Q^{k}$ there exists at most one $q \in Q$ such that $\delta_{k}(w, \sigma, q) \neq \mathbb{0}$. This fact has the following consequences.

Lemma 4.2.1. [FKV21, Lm. 3.5] Let $\Sigma$ be a ranked alphabet, $\mathrm{B}=(B, \oplus, \otimes, \mathbb{0}, \mathbb{1})$ a strong bimonoid, $\mathcal{A}=(Q, \delta, F)$ a bu deterministic $(\Sigma, \mathrm{B})$-wta, and $\xi \in \mathrm{T}_{\Sigma}$. Then the following three statements hold.
(1) $\left|\mathrm{Q}_{\neq \mathbb{D}}^{\mathrm{h}_{\mathcal{A}}}(\xi)\right| \leq 1$ (cf. BV03, Thm. 3.6]).
(2) $\left|\mathrm{Q}_{\neq 0}^{\mathrm{R}} \mathrm{A}_{\mathcal{D}}(\xi)\right| \leq 1$.
(3) Either (a) $\mathrm{Q}_{\neq 0}^{\mathrm{h}_{\mathcal{A}}}(\xi)=\emptyset=\mathrm{Q}_{\neq \mathbb{D}}^{\mathrm{R}_{\mathcal{A}}}(\xi)$ or $(\mathrm{b})$ there exists a $q \in Q$ such that $\mathrm{Q}_{\neq \mathbb{D}}^{\mathrm{h}_{\mathcal{A}}}(\xi)=\{q\}=\mathrm{Q}_{\neq 0}^{\mathrm{R}_{\mathcal{A}}}(\xi)$, there exists a $\rho \in \mathrm{R}_{\mathcal{A}}(q, \xi)$ with $\operatorname{wt}(\xi, \rho) \neq \mathbb{O}$ and $\mathrm{h}_{\mathcal{A}}(\xi)_{q}=\operatorname{wt}(\xi, \rho)$, and for each $\rho^{\prime} \in \mathrm{R}_{\mathcal{A}}(q, \xi) \backslash\{\rho\}$ we have $\mathrm{wt}\left(\xi, \rho^{\prime}\right)=\mathbb{0}$.

Proof. Proof of (1): We prove the statement by induction on $\mathrm{T}_{\Sigma}$. Let $\xi=\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)$. We assume that $\left|\mathrm{Q}_{\neq \mathbb{\mathcal { O }}}^{\mathrm{h}}\left(\xi_{i}\right)\right| \leq 1$ for each $i \in[k]$ (I.H.), and we continue by case analysis.



$$
\begin{aligned}
\mathrm{h}_{\mathcal{A}}\left(\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)\right)_{q} & =\bigoplus_{q_{1} \cdots q_{k} \in Q^{k}}\left(\bigotimes_{i \in[k]} \mathrm{h}_{\mathcal{A}}\left(\xi_{i}\right)_{q_{i}}\right) \otimes \delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right) \\
& =\left(\bigotimes_{i \in[k]} \mathrm{h}_{\mathcal{A}}\left(\xi_{i}\right)_{q_{\xi_{i}}}\right) \otimes \delta_{k}\left(q_{\xi_{1}} \cdots q_{\xi_{k}}, \sigma, q\right)
\end{aligned}
$$

Since $\mathcal{A}$ is bu deterministic, there exists at most one $q \in Q$ such that $\delta_{k}\left(q_{\xi_{1}} \cdots q_{\xi_{k}}, \sigma, q\right) \neq \mathbb{O}$. Thus $\left|\mathrm{Q}_{\neq \mathfrak{D}}^{\mathrm{h}_{\mathcal{A}}}\left(\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)\right)\right| \leq 1$. (We note that, since B may contain zero-divisors, the cardinality of this set can be 0 .)

Proof of (2): Its proof is very similar to that of Statement (1).
Proof of (3): We prove the statement by induction on $\mathrm{T}_{\Sigma}$. Let $\xi=\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)$. If $\mathrm{Q}_{\neq \mathfrak{\mathcal { O }}}^{\mathrm{h}_{\mathcal{A}}}(\xi)=$ $\emptyset=\mathrm{Q}_{\neq \emptyset}^{\mathrm{R}_{\mathcal{A}}}(\xi)$, then we are done. Otherwise, first assume that $\mathrm{Q}_{\neq \emptyset}^{\mathrm{h}_{\mathcal{A}}}(\xi) \neq \emptyset$. Then, by Statement (1), $\mathrm{Q}_{\neq 0}^{\mathrm{h} \mathcal{A}}(\xi)=\{q\}$ for some $q \in Q$ and, by Lemma 4.1.1 $(1), \mathrm{Q}_{\neq 0}^{\mathrm{h} \mathcal{A}}\left(\xi_{i}\right) \neq \emptyset$ for every $i \in[k]$. By the I.H, for every $i \in[k]$ there exists a $q_{i} \in Q$ such that $\mathrm{Q}_{\neq 0}^{\mathrm{h} \mathcal{A}}\left(\xi_{i}\right)=\left\{q_{i}\right\}=\mathrm{Q}_{\neq \mathbb{D}}^{\mathrm{R}_{\mathcal{A}}}\left(\xi_{i}\right)$ and there exists exactly one $\rho_{i} \in \mathrm{R}_{\mathcal{A}}\left(q_{i}, \xi_{i}\right)$ with $\operatorname{wt}\left(\xi_{i}, \rho_{i}\right) \neq \mathbb{O}$ and $\mathrm{h}_{\mathcal{A}}\left(\xi_{i}\right)_{q_{i}}=\operatorname{wt}\left(\xi_{i}, \rho_{i}\right)$. Now we define the run $\rho \in \mathrm{R}_{\mathcal{A}}(q, \xi)$ by $\rho(\varepsilon)=q$, and $\rho(i w)=\rho_{i}(w)$ for every $i \in[k]$ and $w \in \operatorname{pos}\left(\xi_{i}\right)$. Then we have

$$
\begin{aligned}
\mathrm{wt}(\xi, \rho) & =\left(\bigotimes_{i \in[k]} \mathrm{wt}\left(\xi_{i},\left.\rho\right|_{i}\right)\right) \otimes \delta_{k}(\rho(1) \cdots \rho(k), \sigma, \rho(\varepsilon)) \\
& =\left(\bigotimes_{i \in[k]} \mathrm{wt}\left(\xi_{i}, \rho_{i}\right)\right) \otimes \delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right) \\
& =\left(\bigotimes_{i \in[k]} \mathrm{h}_{\mathcal{A}}\left(\xi_{i}\right)_{q_{i}}\right) \otimes \delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right)=\mathrm{h}_{\mathcal{A}}(\xi)_{q}
\end{aligned}
$$

Since $\mathrm{h}_{\mathcal{A}}(\xi)_{q} \neq \mathbb{0}$, we have $\mathrm{wt}(\xi, \rho) \neq \mathbb{0}$, i.e., $q \in \mathrm{Q}_{\neq \mathbb{D}}^{\mathrm{R}_{\mathcal{A}}}(\xi)$, and thus, by Statement $(2), \mathrm{Q}_{\neq \mathbb{C}}^{\mathrm{R}_{\mathcal{A}}}(\xi)=\{q\}$. Lastly, let $\rho^{\prime} \in \mathrm{R}_{\mathcal{A}}(q, \xi)$ with $\rho^{\prime} \neq \rho$. Using the definition of $\rho$, the assumption that $\rho_{i}$ is the only run in $\mathrm{R}_{\mathcal{A}}\left(q_{i}, \xi_{i}\right)$ with $\mathrm{wt}\left(\xi_{i}, \rho_{i}\right) \neq \mathbb{O}$ for each $i \in[k]$, and the fact that $\mathcal{A}$ is bu deterministic, it follows easily that $\operatorname{wt}\left(\xi, \rho^{\prime}\right)=\mathbb{0}$.

The case $\mathrm{Q}_{\neq 0}^{\mathrm{R} \mathcal{A}}(\xi) \neq \emptyset$ can be proved similarly.
A consequence of Lemma 4.2.1(3) is that both, the run semantics and the initial algebra semantics of a bu deterministic $(\Sigma, B)$-wta $\mathcal{A}$ are determined by the multiplicative monoid of B ; the additive part does not play any role (apart from its unit element). Hence, if we replace the addition by another operation such that the algebra is still a strong bimonoid, then both the run semantics and the initial algebra semantics of $\mathcal{A}$ remain the same mapping. Below, we prove this statement and show some pairs of strong bimonoids which differ only in the definition of the addition.

Corollary 4.2.2. Let $\mathrm{B}_{1}=(B, \oplus, \otimes, \mathbb{O}, \mathbb{1})$ and $\mathrm{B}_{2}=(B,+, \otimes, \mathbb{O}, \mathbb{1})$ be strong bimonoids and $\mathcal{A}=(Q, \delta, F)$ be a bu deterministic $\left(\Sigma, \mathrm{B}_{1}\right)$-wta. Then, for the bu deterministic $\left(\Sigma, \mathrm{B}_{2}\right)$-wta $\mathcal{B}=(Q, \delta, F)$, we have (1) $\llbracket \mathcal{B} \rrbracket^{\text {run }}=\llbracket \mathcal{A} \rrbracket^{\text {run }}$ and (2) $\llbracket \mathcal{B} \rrbracket^{\text {init }}=\llbracket \mathcal{A} \rrbracket^{\text {init }}$.

Proof. It is obvious that, for every $\xi \in \mathrm{T}_{\Sigma}$ and $q \in Q$, we have $\mathrm{R}_{\mathcal{A}}(q, \xi)=\mathrm{R}_{\mathcal{B}}(q, \xi)$. Moreover, for each $\rho \in \mathrm{R}_{\mathcal{A}}(q, \xi)$, we have

$$
\begin{equation*}
\mathrm{wt}_{\mathcal{A}}(\xi, \rho)=\mathrm{wt}_{\mathcal{B}}(\xi, \rho) \tag{4.3}
\end{equation*}
$$

Hence it follows that

$$
\begin{equation*}
\mathrm{Q}_{\neq 0}^{\mathrm{R}_{\mathcal{A}}}(\xi)=\mathrm{Q}_{\neq 0}^{\mathrm{R}_{\mathcal{B}}}(\xi) \tag{4.4}
\end{equation*}
$$

Now let $\xi \in \mathrm{T}_{\Sigma}$. By Lemma 4.2.1(3), we distinguish two cases.

Case (a): $\mathrm{Q}_{\neq 0}^{\mathrm{h}_{\mathcal{A}}}(\xi)=\emptyset=\mathrm{Q}_{\neq 0}^{\mathrm{R}_{\mathcal{A}}}(\xi)$. Then, by (4.4) and Lemma 4.2.1(3) (applied to $\mathcal{B}$ ), we have $\mathrm{Q}_{\neq 0}^{\mathrm{h}} \overline{\mathcal{B}}(\xi)=\emptyset=\mathrm{Q}_{\neq 0}^{\mathrm{R}_{\mathcal{B}}}(\xi)$. Hence, for each $\rho \in \mathrm{R}_{\mathcal{A}}(\xi)$, we have $\mathrm{wt}_{\mathcal{A}}(\xi, \rho)=\mathbb{O}=\mathrm{wt}_{\mathcal{B}}(\xi, \rho)$. Then


Also, for each $q \in Q$, we have $\mathrm{h}_{\mathcal{A}}(\xi)_{q}=\mathbb{O}=\mathrm{h}_{\mathcal{B}}(\xi)_{q}$. Then, obviously, $\llbracket \mathcal{A} \rrbracket^{\text {init }}(\xi)=\mathbb{O}=\llbracket \mathcal{B} \rrbracket^{\text {init }}(\xi)$.
Case (b): There exists a $q \in Q$ such that $\mathrm{Q}_{\neq \mathbb{\mathcal { D }}}^{\mathrm{h}_{\mathcal{A}}}(\xi)=\{q\}=\mathrm{Q}_{\neq \mathbb{\mathcal { O }}}^{\mathrm{R}_{\mathcal{A}}}(\xi)$, and there exists a $\rho \in \mathrm{R}_{\mathcal{A}}(q, \xi)$ with $\mathrm{wt}_{\mathcal{A}}(\xi, \rho) \neq \mathbb{O}$ and $\mathrm{h}_{\mathcal{A}}(\xi)_{q}=\mathrm{wt}_{\mathcal{A}}(\xi, \rho)$, and for each $\rho^{\prime} \in \mathrm{R}_{\mathcal{A}}(q, \xi) \backslash\{\rho\}$ we have $\mathrm{wt}_{\mathcal{A}}\left(\xi, \rho^{\prime}\right)=\mathbb{0}$.

Again, by (4.4) and Lemma4.2.1(3) (applied to $\mathcal{B}$ ), as well as (4.3), we obtain $\mathrm{Q}_{\neq 0}^{\mathrm{h}_{\mathcal{B}}}(\xi)=\{q\}=\mathrm{Q}_{\neq 0}^{\mathrm{R}_{\mathcal{B}}}(\xi)$, $\operatorname{wt}_{\mathcal{B}}(\xi, \rho)=\operatorname{wt}_{\mathcal{A}}(\xi, \rho) \neq \mathbb{O}$ and $\mathrm{h}_{\mathcal{B}}(\xi)_{q}=\operatorname{wt}_{\mathcal{B}}(\xi, \rho)$, and for each $\rho^{\prime} \in \mathrm{R}_{\mathcal{B}}(q, \xi) \backslash\{\rho\}$ we have $\operatorname{wt}_{\mathcal{B}}\left(\xi, \rho^{\prime}\right)=\mathbb{0}$. Then we have

$$
\begin{aligned}
\llbracket \mathcal{A} \rrbracket^{\mathrm{run}}(\xi) & =\bigoplus_{\rho^{\prime} \in \mathrm{R}_{\mathcal{A}}(\xi)} \mathrm{wt}_{\mathcal{A}}\left(\xi, \rho^{\prime}\right) \otimes F_{\rho^{\prime}(\varepsilon)}=\mathrm{wt}_{\mathcal{A}}(\xi, \rho) \otimes F_{q}=\mathrm{wt}_{\mathcal{B}}(\xi, \rho) \otimes F_{q} \\
& =\prod_{\rho^{\prime} \in \mathrm{R}_{\mathcal{B}}(\xi)} \mathrm{wt}_{\mathcal{B}}\left(\xi, \rho^{\prime}\right) \otimes F_{\rho^{\prime}(\varepsilon)}=\llbracket \mathcal{B} \rrbracket^{\mathrm{run}}(\xi) .
\end{aligned}
$$

Since $\operatorname{wt}_{\mathcal{A}}(\xi, \rho)=\operatorname{wt}_{\mathcal{B}}(\xi, \rho)$, we also have $\mathrm{h}_{\mathcal{A}}(\xi)_{q}=\mathrm{h}_{\mathcal{B}}(\xi)_{q}$. Moreover, for each $p \in Q \backslash\{q\}$, we have $\mathrm{h}_{\mathcal{A}}(\xi)_{p}=\mathbb{O}=\mathrm{h}_{\mathcal{B}}(\xi)_{p}$. Hence, we obtain

Next we show three examples of pairs of strong bimonoids $\mathrm{B}_{1}=(B, \oplus, \otimes, \mathbb{O}, \mathbb{1})$ and $\mathrm{B}_{2}=(B,+, \otimes, \mathbb{0}, \mathbb{1})$ :

- the strong bimonoids $([0,1], \oplus, \cdot, 0,1)$ and $([0,1],+, \cdot, 0,1)$ where $\oplus$ and + are t-conorms (cf. Example 2.6 .10 (3)) and $\cdot$ is the usual multiplication; examples of a t-conorm $u:[0,1] \times[0,1] \rightarrow[0,1]$ are
- the standard union $u(a, b)=\max (a, b)$,
- the algebraic sum $u(a, b)=a+b-a \cdot b$, or
- the bounded sum $u(a, b)=\min (a+b, 1)$,
- the Boolean semiring Boole and the field $F_{2}$ with two elements (cf. Example 2.6.10 (77)), and
- the plus-plus strong bimonoid of natural numbers $\mathrm{PP}_{\mathbb{N}}=\left(\mathbb{N}_{\mathbb{O}}, \oplus,+, \mathbb{0}, 0\right)$ (cf. Example 2.6.10 8) and the semiring $\left(\mathbb{N}_{\mathbb{D}}, \max ^{\prime},+, \mathbb{0}, 0\right)$ where the binary operation $\max ^{\prime}$, if restricted to $\mathbb{N}$, is the usual operation max on natural numbers (e.g. $\max (3,2)=3)$. Moreover, $\max ^{\prime}(\mathbb{O}, x)=\max ^{\prime}(x, \mathbb{O})=x$ for each $x \in \mathbb{N}_{0}$.
Another consequence of Lemma 4.2.1 is that, for each bu deterministic wta, its run semantics coincides with its initial algebra semantics. We will deal with this topic in Section 5.3


### 4.3 Properties of crisp deterministic wta

Here we consider an arbitrary crisp deterministic $(\Sigma, \mathrm{B})$-wta $\mathcal{A}=(Q, \delta, F)$. Thus, for every $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $w \in Q^{k}$ there exists a $q \in Q$ such that $\delta_{k}(w, \sigma, q)=\mathbb{1}$ and for every $q^{\prime} \in Q$ with $q^{\prime} \neq q$ we have $\delta_{k}\left(w, \sigma, q^{\prime}\right)=\mathbb{0}$.

Lemma 4.3.1. Let $\Sigma$ be a ranked alphabet, $\mathrm{B}=(B, \oplus, \otimes, \mathbb{O}, \mathbb{1})$ a strong bimonoid, and $\mathcal{A}=(Q, \delta, F)$ a crisp deterministic $(\Sigma, \mathrm{B})$-wta. For each $\xi \in \mathrm{T}_{\Sigma}$ there exists a $q \in Q$ such that
(1) $\mathrm{Q}_{\neq \mathbb{D}}^{\mathrm{h}_{\mathcal{A}}}(\xi)=\{q\}=\mathrm{Q}_{\neq \mathbb{D}}^{\mathrm{R}_{\mathcal{A}}}(\xi)$ and there exists $\rho \in \mathrm{R}_{\mathcal{A}}(q, \xi)$ with $\mathrm{h}_{\mathcal{A}}(\xi)_{q}=\mathrm{wt}(\xi, \rho)=\mathbb{1}$ and for each $\rho^{\prime} \in \mathrm{R}_{\mathcal{A}}(q, \xi) \backslash\{\rho\}$ we have $\mathrm{wt}\left(\xi, \rho^{\prime}\right)=\mathbb{0}$, and
(2) $\llbracket \mathcal{A} \rrbracket^{\text {init }}(\xi)=F_{q}$.

Thus, in particular, $\operatorname{im}\left(\llbracket \mathcal{A} \rrbracket^{\text {init }}\right) \subseteq \operatorname{im}(F)$.

Proof. Proof of (1): First, by induction on $\mathrm{T}_{\Sigma}$, we prove that the following statement holds:

$$
\begin{equation*}
\text { For each } \xi \in \mathrm{T}_{\Sigma} \text { there exists a } q \in Q \text { such that } \mathrm{Q}_{\neq \mathfrak{D}}^{\mathrm{h}_{\mathcal{A}}}(\xi)=\{q\} \text { and } \mathrm{h}_{\mathcal{A}}(\xi)_{q}=\mathbb{1} \tag{4.5}
\end{equation*}
$$

Let $\xi=\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)$. We assume that $\mathrm{Q}_{\neq \mathfrak{\mathcal { O }}}^{\mathrm{h}_{\mathcal{A}}}\left(\xi_{i}\right)=\left\{q_{\xi_{i}}\right\}$ and $\mathrm{h}_{\mathcal{A}}\left(\xi_{i}\right)_{q_{\xi_{i}}}=\mathbb{1}$ for each $i \in[k]$ (I.H.). Then for every $p \in Q$ :

$$
\mathrm{h}_{\mathcal{A}}\left(\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)\right)_{p}=\bigoplus_{q_{1} \cdots q_{k} \in Q^{k}}\left(\bigotimes_{i \in[k]} \mathrm{h}_{\mathcal{A}}\left(\xi_{i}\right)_{q_{i}}\right) \otimes \delta_{k}\left(q_{1} \cdots q_{k}, \sigma, p\right)=\delta_{k}\left(q_{\xi_{1}} \cdots q_{\xi_{k}}, \sigma, p\right)
$$

where the last equality holds by I.H. Since $\mathcal{A}$ is bu deterministic and total, there exists a $q \in Q$ such that $\delta_{k}\left(q_{\xi_{1}} \cdots q_{\xi_{k}}, \sigma, q\right) \neq \mathbb{O}$ and for each $q^{\prime} \in Q$ with $q \neq q^{\prime}$ we have $\delta_{k}\left(q_{\xi_{1}} \cdots q_{\xi_{k}}, \sigma, q^{\prime}\right)=\mathbb{O}$. Thus $\mathrm{Q}_{\neq 0}^{\mathrm{h}} \mathcal{A}(\xi)=\{q\}$. Since $\mathcal{A}$ has identity transition weights, we have that $\delta_{k}\left(q_{\xi_{1}} \cdots q_{\xi_{k}}, \sigma, q\right)=\mathbb{1}$, and hence $\mathrm{h}_{\mathcal{A}}(\xi)_{q}=\mathbb{1}$. This proves (4.5).

Now let $\xi \in \mathrm{T}_{\Sigma}$. Then Statement (1) follows from Lemma 4.2.1(3).
Proof of (2): Let $\xi \in \mathrm{T}_{\Sigma}$. Then there exists $q \in Q$ such that Statement (1) holds. Thus we have $\llbracket \mathcal{A} \rrbracket^{\text {init }}(\xi)=\mathrm{h}_{\mathcal{A}}(\xi)_{q} \otimes F_{q}=F_{q}$.

Finally, it follows from Statements $(2)$ that $\operatorname{im}\left(\llbracket \mathcal{A} \rrbracket^{\text {init }}\right) \subseteq \operatorname{im}(F)$.
Essentially, each crisp deterministic wta $\mathcal{A}$ is a finite $\Sigma$-algebra combined with a mapping which reflects the root weights. For the formalization of this statement, let us consider the crisp deterministic $(\Sigma, \mathrm{B})$-wta $\mathcal{A}=(Q, \delta, F)$. The state algebra of $\mathcal{A}$ is the $\Sigma$-algebra $\mathrm{S}(\mathcal{A})=\left(Q, \theta_{\mathcal{A}}\right)$ such that, for every $k \in \mathbb{N}, \sigma \in \Sigma^{(k)}, q \in Q$, and $q_{1} \cdots q_{k} \in Q^{k}$, we define

$$
\theta_{\mathcal{A}}(\sigma)\left(q_{1}, \ldots, q_{k}\right)=q, \text { where } q \text { is the unique state for which } \delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right)=\mathbb{1}
$$

We recall that the unique $\Sigma$-algebra homomorphism from the $\Sigma$-term algebra to $\mathrm{S}(\mathcal{A})$ is denoted by $\mathrm{h}_{\mathrm{S}(\mathcal{A})}$. Obviously, $\mathrm{S}(\mathcal{A})$ is isomorphic to the subalgebra $\left(\mathbb{1}_{Q}, \nu_{\mathcal{A}}\right)$ of the vector algebra $\mathrm{V}(\mathcal{A})$, where

- $\mathbb{1}_{Q}=\left\{\mathbb{1}_{q} \mid q \in Q\right\}$ is the set of $q$-unit vectors over $Q$ (cf. Section 2.8), and
- $\nu_{\mathcal{A}}(\sigma)\left(\mathbb{1}_{q_{1}}, \ldots, \mathbb{1}_{q_{k}}\right)_{q}=\delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right)$ for every $k \in \mathbb{N}, \sigma \in \Sigma^{(k)}, q \in Q$, and $q_{1} \cdots q_{k} \in Q^{k}$.

Lemma 4.3.2. Let $\mathcal{A}=(Q, \delta, F)$ be a crisp deterministic $(\Sigma, \mathrm{B})$-wta and $\mathrm{S}(\mathcal{A})=\left(Q, \theta_{\mathcal{A}}\right)$ the state algebra of $\mathcal{A}$. The following two statements hold.
(1) For each $\xi \in \mathrm{T}_{\Sigma}$ and each $q \in Q$, we have

$$
\mathrm{h}_{\mathcal{A}}(\xi)_{q}= \begin{cases}\mathbb{1} & \text { if } q=\mathrm{h}_{(\mathcal{A})}(\xi) \\ \mathbb{0} & \text { otherwise }\end{cases}
$$

(2) $\llbracket \mathcal{A} \rrbracket^{\text {init }}=F \circ \mathrm{~h}_{\mathrm{S}(\mathcal{A})}$.

Proof. Proof of (1): We prove the statement by induction on $\mathrm{T}_{\Sigma}$. Let $\xi=\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)$. Then by I.H. and the definition of state algebra we obtain
$\mathrm{h}_{\mathcal{A}}\left(\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)\right)_{q}=\bigoplus_{q_{1} \cdots q_{k} \in Q^{k}}\left(\bigotimes_{i \in[k]} \mathrm{h}_{\mathcal{A}}\left(\xi_{i}\right)_{q_{i}}\right) \otimes \delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right)=\left(\bigotimes_{i \in[k]} \mathrm{h}_{\mathcal{A}}\left(\xi_{i}\right)_{q_{\xi_{i}}}\right) \otimes \delta_{k}\left(q_{\xi_{1}} \cdots q_{\xi_{k}}, \sigma, q\right)$
(where $q_{\xi_{i}}$ is the unique element of $\mathrm{Q}_{\neq \mathbb{\mathcal { D }}}^{\mathrm{h}}\left(\xi_{i}\right)$ with $\mathrm{h}_{\mathcal{A}}\left(\xi_{i}\right)_{q_{\xi_{i}}}=\mathbb{1}$ for each $i \in[k]$, cf. Lemma 4.3.1(1))

$$
\begin{aligned}
& =\delta_{k}\left(\mathrm{~h}_{\mathrm{S}(\mathcal{A})}\left(\xi_{1}\right) \cdots \mathrm{h}_{\mathrm{S}(\mathcal{A})}\left(\xi_{k}\right), \sigma, q\right) \\
& =\left\{\begin{array}{ll}
\mathbb{1} & \text { if } \theta_{\mathcal{A}}(\sigma)\left(\mathrm{h}_{\mathrm{S}(\mathcal{A})}\left(\xi_{1}\right), \ldots, \mathrm{h}_{\mathrm{S}(\mathcal{A})}\left(\xi_{k}\right)\right)=q \\
\boldsymbol{0} & \text { otherwise }
\end{array} \quad \text { (by the definition of }\left(Q, \theta_{\mathcal{A}}\right)\right)
\end{aligned}
$$

Since $\theta_{\mathcal{A}}(\sigma)\left(\mathrm{h}_{\mathrm{S}(\mathcal{A})}\left(\xi_{1}\right), \ldots, \mathrm{h}_{\mathrm{S}_{(\mathcal{A})}}\left(\xi_{k}\right)\right)=\mathrm{h}_{\mathrm{S}_{(\mathcal{A})}}\left(\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)\right)$, we have proved Statement (1).
Proof of (2): For each $\xi \in \mathrm{T}_{\Sigma}$, we have

$$
\llbracket \mathcal{A} \rrbracket^{\mathrm{init}}(\xi)=\bigoplus_{q \in Q} \mathrm{~h}_{\mathcal{A}}(\xi)_{q} \otimes F_{q}=F_{\mathrm{h}_{\mathrm{S}(\mathcal{A})}(\xi)}=\left(F \circ \mathrm{~h}_{\mathrm{S}(\mathcal{A})}\right)(\xi)
$$

where the second equality follows from Statement (1).
Now we consider two strong bimonoids $B_{1}$ and $B_{2}$ which have the same carrier set; note that the unit elements of $B_{1}$ and $B_{2}$ may not be the same (examples are given below). Moreover, we let $\mathcal{A}$ be a crisp deterministic $\left(\Sigma, \mathrm{B}_{1}\right)$-wta. According to the note on page $62, \mathcal{A}$ is different from the crisp deterministic $\left(\Sigma, \mathrm{B}_{2}\right)$-wta $\mathcal{B}$ which we obtain from $\mathcal{A}$ by replacing the unit elements of $\mathrm{B}_{1}$ in the transitions of $\mathcal{A}$ by the corresponding unit elements of $\mathrm{B}_{2}$. However, due to Lemma4.3.2 (2), the initial algebra semantics of $\mathcal{A}$ and $\mathcal{B}$ viewed as mappings coincide 1 In [FV22a the next corollary was proved for the special case that $B_{1}$ and $B_{2}$ are bounded lattices.

Corollary 4.3.3. FV22a] Let $B_{1}$ and $B_{2}$ be strong bimonoids with the same carrier set. For each crisp deterministic $\left(\Sigma, \mathrm{B}_{1}\right)$-wta $\mathcal{A}$, we can construct a crisp deterministic $\left(\Sigma, \mathrm{B}_{2}\right)$-wta $\mathcal{B}$ such that $\llbracket \mathcal{B} \rrbracket^{\text {init }}=$ $\llbracket \mathcal{A} \rrbracket^{\text {init }}$. In particular, $\operatorname{cd}-\operatorname{Rec}^{\mathrm{init}}\left(\Sigma, \mathrm{B}_{1}\right)=\operatorname{cd}-\operatorname{Rec}^{\mathrm{init}}\left(\Sigma, \mathrm{B}_{2}\right)$.

Proof. Let $\mathrm{B}_{1}=(B, \oplus, \otimes, \mathbb{0}, \mathbb{1})$ and $\mathrm{B}_{2}=(B,+, \times, 0,1)$ be strong bimonoids. Let $\mathcal{A}=(Q, \delta, F)$ be a crisp deterministic ( $\Sigma, \mathrm{B}_{1}$ )-wta. We construct the crisp deterministic $\left(\Sigma, \mathrm{B}_{2}\right)$-wta $\mathcal{B}=\left(Q, \delta^{\prime}, F\right)$ where for every $k \in \mathbb{N}, \sigma \in \Sigma^{(k)}$, and $q_{1}, \ldots, q_{k}, q \in Q$ we let

$$
\left(\delta^{\prime}\right)_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right)= \begin{cases}1 & \text { if } \delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right)=\mathbb{1} \\ 0 & \text { otherwise }\end{cases}
$$

Clearly, $\mathcal{B}$ is crisp deterministic. It is obvious that the state algebra $S(\mathcal{A})$ is equal to the state algebra $\mathrm{S}(\mathcal{B})$. As a consequence, the $\Sigma$-algebra homomorphism from $\mathrm{T}_{\Sigma}$ to $\mathrm{S}(\mathcal{A})$, denoted by $\mathrm{h}_{(\mathcal{A})}$, is the same as the $\Sigma$-algebra homomorphism from $\mathrm{T}_{\Sigma}$ to $\mathrm{S}(\mathcal{B})$, denoted by $\mathrm{h}_{\mathrm{S}(\mathcal{B})}$. Then, by applying Lemma 4.3.2(2) twice, we have $\llbracket \mathcal{A} \rrbracket^{\text {init }}=F \circ \mathrm{~h}_{(\mathcal{A})}=F \circ \mathrm{~h}_{\mathrm{S}_{(\mathcal{B})}}=\llbracket \mathcal{B} \rrbracket^{\text {init }}$.

Examples for pairs $\left(B_{1}, B_{2}\right)$ of different strong bimonoids which have the same carrier set ( - and the last three also have the same unit elements -) are

- the semiring $\left(\mathbb{N}_{\infty},+, \cdot, 0,1\right)$ and the distributive bounded lattice $\left(\mathbb{N}_{\infty}, \max , \min , 0, \infty\right)$,
- the bounded lattices $N_{5}$ and $M_{3}$ from Examples 2.6.15(3) and 2.6.15(4), respectively,
- the tropical semiring $\left(\mathbb{N}_{\infty}, \min ,+, \infty, 0\right)$ and the tropical bimonoid $\left(\mathbb{N}_{\infty},+, \min , 0, \infty\right)$,
- the semiring of formal languages $\left(\mathcal{P}\left(\Gamma^{*}\right), \cup, \cdot, \emptyset,\{\varepsilon\}\right)$ and the semiring $\left(\mathcal{P}\left(\Gamma^{*}\right), \cup, \cap, \emptyset, \Gamma^{*}\right)$,
- the two different strong bimonoids with the carrier set [0, 1] in Example 2.6.10(3) using algebraic sum and bounded sum, respectively,
- the strong bimonoid $([0,1], \oplus, \cdot, 0,1)$ with bounded sum $\oplus(c f$. Example 2.6.10(3) $)$ and the strong bimonoid ( $[0,1], u, i, 0,1$ ) with t-conorm $u$ and t-norm $i$ (cf. Example 2.6.10(4)), and
- the Boolean semiring Boole and the field $F_{2}$ with two elements (cf. Example 2.6.10(7).

In the next theorem we characterize the set of weighted tree languages which are i-recognizable by crisp deterministic wta. As preparation we prove the following lemma which is a kind of inverse of Lemma 4.3.2(2).

[^12]Lemma 4.3.4. For every finite $\Sigma$-algebra $\mathrm{A}=(Q, \theta)$ and mapping $F: Q \rightarrow B$, we can construct a crisp deterministic ( $\Sigma, \mathrm{B}$ )-wta $\mathcal{A}$ such that A is the state algebra of $\mathcal{A}$ (as defined on page 107) and $F \circ \mathrm{~h}_{\mathrm{A}}=\llbracket \mathcal{A} \rrbracket^{\text {init }}$.

Proof. We construct the crisp deterministic ( $\Sigma, \mathrm{B})$-wta $\mathcal{A}=(Q, \delta, F)$ where for every $k \in \mathbb{N}, \sigma \in \Sigma^{(k)}$, $q \in Q$, and $q_{1} \cdots q_{k} \in Q^{k}$ we define

$$
\delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right)= \begin{cases}\mathbb{1} & \text { if } \theta(\sigma)\left(q_{1}, \ldots,, q_{k}\right)=q \\ \mathbb{0} & \text { otherwise } .\end{cases}
$$

Clearly, A is the state algebra of $\mathcal{A}$. Then $\mathrm{h}_{\mathrm{A}}=\mathrm{h}_{\mathrm{S}(\mathcal{A})}$ and thus, by Lemma 4.3.2(2), we obtain $F \circ \mathrm{~h}_{\mathrm{A}}=$ $\llbracket \mathcal{A} \rrbracket^{\text {init }}$.

In the next theorem we characterize the weighted tree languages which are recognized by crisp deterministic wta. Roughly speaking, it shows that each crisp deterministic ( $\Sigma, B$ )-wta consists of a finite $\Sigma$-algebra with carrier set $Q$ and a mapping $F: Q \rightarrow B$. In BLB10. Thm. 2], the combination of (a) a finite $\Sigma$-algebra with carrier set $Q$ and (b) a mapping $F: Q \rightarrow B$ was called "finite representation".

Theorem 4.3.5. Let $\Sigma$ be a ranked alphabet, B be a strong bimonoid, and $r: \mathrm{T}_{\Sigma} \rightarrow B$. Then the following three statements are equivalent.
(A) We can construct a finite $\Sigma$-algebra $\mathrm{A}=(Q, \theta)$ and a mapping $F: Q \rightarrow B$ such that $r=F \circ \mathrm{~h}_{\mathrm{A}}$.
(B) We can construct a crisp deterministic ( $\Sigma, \mathrm{B})$-wta $\mathcal{A}$ such that $r=\llbracket \mathcal{A} \rrbracket^{\text {init }}$.
(C) We can construct $a(\Sigma, \mathrm{~B})$-wta $\mathcal{A}$ such that the congruence $\operatorname{ker}\left(\mathrm{h}_{\mathcal{A}}\right)$ has finite index and $r=\llbracket \mathcal{A} \rrbracket^{\mathrm{init}}$.

Proof. Proof of $(\mathrm{A}) \Rightarrow(\mathrm{B})$ : This follows directly from Lemma 4.3.4.
Proof of $(\mathrm{B}) \Rightarrow(\mathrm{C})$ : Let $\mathcal{A}=(Q, \delta, F)$ be a crisp deterministic $(\Sigma, \mathrm{B})$-wta with $r=\llbracket \mathcal{A}]_{\text {init }}$. The image of $\mathrm{h}_{\mathcal{A}}$ is a finite set because, by Lemma 4.3.1 $\operatorname{im}\left(\mathrm{h}_{\mathcal{A}}\right) \subseteq\{\mathbb{0}, \mathbb{1}\}^{Q}$. Thus $\operatorname{ker}\left(\mathrm{h}_{\mathcal{A}}\right)$ has finite index.

Proof of $(\mathrm{C}) \Rightarrow(\mathrm{A})$ : Let $\mathcal{A}=\left(Q, \delta, F_{\mathcal{A}}\right)$ be a $(\Sigma, \mathrm{B})$-wta such that $\operatorname{ker}\left(\mathrm{h}_{\mathcal{A}}\right)$ has finite index and $r=$ $\llbracket \mathcal{A}]^{\text {init }}$. We consider the accessible subalgebra $\operatorname{aV}(\mathcal{A})=\left(\operatorname{im}\left(\mathrm{h}_{\mathcal{A}}\right), \delta_{\mathrm{aV}(\mathcal{A})}\right)$ of $\mathrm{V}(\mathcal{A})$ (cf. Section 3.1). Since $\operatorname{ker}\left(\mathrm{h}_{\mathcal{A}}\right)$ has finite index, $\operatorname{im}\left(\mathrm{h}_{\mathcal{A}}\right)$ is finite and hence $\operatorname{aV}(\mathcal{A})$ is a finite $\Sigma$-algebra. By Observation 2.9.4, we have $\operatorname{im}\left(\mathrm{h}_{\mathcal{A}}\right)=\langle\emptyset\rangle_{\delta_{\mathcal{A}}(\Sigma)}$, and by Lemma [2.6.1] we have that $\langle\emptyset\rangle_{\delta_{\mathcal{A}}(\Sigma)}$ can be constructed. Thus $\operatorname{im}\left(\mathrm{h}_{\mathcal{A}}\right)$ can be constructed. Finally we define the mapping $F: \operatorname{im}\left(\mathrm{h}_{\mathcal{A}}\right) \rightarrow B$ for every $v \in \operatorname{im}\left(\mathrm{~h}_{\mathcal{A}}\right)$ by $F(v)=\bigoplus_{q \in Q} v_{q} \otimes\left(F_{\mathcal{A}}\right)_{q}$. Then by (3.4), we have $\llbracket \mathcal{A} \rrbracket^{\text {init }}=F \circ \mathrm{~h}_{\mathrm{a}}(\mathcal{A})$.

Finally, we prove that the set of weighted tree languages which are recognized by crisp deterministic $(\Sigma, B)$-wta with identity root weights, is exactly the set of characteristic mappings of recognizable $\Sigma$-tree languages. We note that in the following theorem $(\mathrm{B}) \Rightarrow(\mathrm{A})$ is in fact Theorem [2.13.2.

Theorem 4.3.6. Let $L \subseteq \mathrm{~T}_{\Sigma}$. Then the following three statements are equivalent.
(A) We can construct a total and bu deterministic $\Sigma$-fta $A$ such that $\mathrm{L}(A)=L$.
(B) We can construct a $\Sigma$-fta $A$ such that $\mathrm{L}(A)=L$.
(C) We can construct a crisp deterministic ( $\Sigma, \mathrm{B})-w t a \mathcal{A}$ with identity root weights such that $\llbracket \mathcal{A} \rrbracket^{\text {init }}=\chi(L)$.

Proof. Proof of $(\mathrm{A}) \Rightarrow(\mathrm{B})$ : This is by definition.
Proof of $(\mathrm{B}) \Rightarrow(\mathrm{C})$ : Let $A=(Q, \delta, F)$ be a $\Sigma$-fta such that $L=\mathrm{L}(A)$. We consider the finite $\Sigma$-algebra $\left(\mathcal{P}(Q), \delta_{A}\right)$ associated with $A$ (for the definition of this algebra see page [56] we recall that the unique $\Sigma$-homomorphism from $\mathrm{T}_{\Sigma}$ to $\left(\mathcal{P}(Q), \delta_{A}\right)$ is denoted by $\mathrm{h}_{A}$ ) and the mapping $F^{\prime}: \mathcal{P}(Q) \rightarrow B$ defined for each $P \in \mathcal{P}(Q)$ by

$$
F^{\prime}(P)= \begin{cases}\mathbb{1} & \text { if } P \cap F \neq \emptyset \\ 0 & \text { otherwise } .\end{cases}
$$

Obviously, $\chi(L)=F^{\prime} \circ \mathrm{h}_{A}$. Then, by Lemma 4.3.4 we can construct a crisp deterministic ( $\Sigma, \mathrm{B}$ )-wta $\mathcal{A}^{\prime}=\left(\mathcal{P}(Q), \delta^{\prime}, F^{\prime}\right)$ such that $\llbracket \mathcal{A}^{\prime} \rrbracket^{\text {init }}=F^{\prime} \circ \mathrm{h}_{A}$. Hence $\llbracket \mathcal{A}^{\prime} \rrbracket^{\text {init }}=\chi(L)$. Moreover, $\mathcal{A}^{\prime}$ has identity root weights and thus (C) holds.

Proof of $(\mathrm{C}) \Rightarrow(\mathrm{A})$ : Let $\mathcal{A}=(Q, \delta, F)$ be a crisp deterministic $(\Sigma, \mathrm{B})$-wta with identity root weights and $\llbracket \mathcal{A} \rrbracket^{\text {init }}=\chi(L)$. Let $\mathrm{S}(\mathcal{A})=\left(Q, \theta_{\mathcal{A}}\right)$ be the state algebra of $\mathcal{A}$. By Lemma 4.3.2(2), we have $\llbracket \mathcal{A} \rrbracket^{\text {init }}=F \circ \mathrm{~h}_{\mathrm{S}(\mathcal{A})}$, and by Lemma4.3.2(1), we have for each $\xi \in \mathrm{T}_{\Sigma}$

$$
\mathrm{h}_{\mathcal{A}}(\xi)_{q}= \begin{cases}\mathbb{1} & \text { if } q=\mathrm{h}_{\mathbf{S}(\mathcal{A})}(\xi) \\ \mathbb{0} & \text { otherwise }\end{cases}
$$

We construct the total and bu deterministic $\Sigma$-fta $A=\left(Q, \delta^{\prime}, F^{\prime}\right)$ such that for every $k \in \mathbb{N}, \sigma \in \Sigma^{(k)}$, $q \in Q$, and $q_{1} \cdots q_{k} \in Q^{k}$ we have $\left(\delta^{\prime}\right)_{k}\left(q_{1} \cdots q_{k}, \sigma\right)=q$ iff $\theta_{\mathcal{A}}(\sigma)\left(q_{1}, \ldots, q_{k}\right)=q$, and that $F^{\prime}=\operatorname{supp}(F)$. Since $\left(\delta^{\prime}\right)_{A}=\theta_{\mathcal{A}}$, we have $\mathrm{h}_{A}=\mathrm{h}_{\mathrm{S}_{(\mathcal{A})}}$. Then, for each $\xi \in \mathrm{T}_{\Sigma}$, we have:

$$
\begin{aligned}
\xi \in \mathrm{L}(A) \text { iff } \mathrm{h}_{A}(\xi) \in F^{\prime} \text { iff } F_{\mathrm{h}_{A}(\xi)}=\mathbb{1} \text { iff } \quad F_{\mathrm{h}_{(\mathcal{A})}(\xi)}=\mathbb{1} \\
\text { iff } \bigoplus_{q \in Q} \mathrm{~h}_{\mathcal{A}}(\xi)_{q} \otimes F_{q}=\mathbb{1} \text { iff } \llbracket \mathcal{A} \rrbracket^{\text {init }}(\xi)=\mathbb{1} \text { iff } \xi \in \operatorname{supp}\left(\llbracket \mathcal{A} \rrbracket^{\text {init }}\right)
\end{aligned}
$$

Thus (A) holds.

## Chapter 5

## Comparison of initial algebra semantics and run semantics

In Section 3.2 we gave a number of examples of a $(\Sigma, \mathrm{B})$-wta $\mathcal{A}$ and we computed its run semantics or initial algebra semantics.

In Section 5.1, we start with an analysis of the complexity of two algorithms; the one calculates $\llbracket \mathcal{A} \rrbracket^{\text {init }}(\xi)$ on input $\mathcal{A}$ and $\xi$, and the other calculates $\llbracket \mathcal{A} \rrbracket^{\text {run }}(\xi)$ on input $\mathcal{A}$ and $\xi$ (cf. Theorem 5.1.1); both algorithms follow the definition of the respective semantics in a natural way. We refer the reader to AG20 for investigations on the complexity of the evaluation problem in the general framework of weighted tiling systems.

Second, in Section 5.2, we compare the run semantics and initial algebra semantics of some arbitrary, but fixed wta. In some situations, the semantics are equal. For instance, for the mapping height, in Example 3.2 .4 we constructed a $\left(\Sigma\right.$, Nat $\left._{\text {max },+}\right)$-wta $\mathcal{A}$ and proved that $\llbracket \mathcal{A} \rrbracket^{\text {init }}=\llbracket \mathcal{A} \rrbracket^{\text {run }}=$ height. The question arises whether $\llbracket \mathcal{A} \rrbracket^{\text {init }}=\llbracket \mathcal{A} \rrbracket^{\text {run }}$ for each $(\Sigma, \mathrm{B})$-wta $\mathcal{A}$. In Section 5.2 we will show that the answer is negative, i.e., we show a $(\Sigma, \mathrm{B})$-wta $\mathcal{A}$ such that $\llbracket \mathcal{A} \rrbracket^{\text {init }} \neq \llbracket \mathcal{A} \rrbracket^{\text {run }}$; in fact, we show four such examples. This stimulates the next question: is it possible to construct, for each $(\Sigma, \mathrm{B})$-wta $\mathcal{A}$, a $(\Sigma, \mathrm{B})$-wta $\mathcal{B}$ such that $\llbracket \mathcal{A} \rrbracket^{\text {init }}=\llbracket \mathcal{B} \rrbracket^{\text {run }}$ (and similarly with run semantics and initial algebra semantics exchanged)? In general, the answer to this question is also negative, i.e., we prove that there exist a strong bimonoid B and a $(\Sigma, \mathrm{B})$-wta $\mathcal{A}$ such that $\llbracket \mathcal{A} \rrbracket^{\text {init }} \notin \operatorname{Rec}^{\text {run }}(\Sigma, \mathrm{B})$ (cf. Theorem 5.2.5).

On the positive side, in Section 5.3, we will prove that, under certain conditions (on the wta or the strong bimonoid), the two semantics are equal.

### 5.1 Complexity of calculating run semantics and initial algebra semantics

In this section we analyse the complexity of two natural algorithms: one calculates $\llbracket \mathcal{A} \rrbracket^{\text {run }}(\xi)$ on input $\mathcal{A}$ and $\xi$ (cf. Algorithm (1) and the other calculates $\llbracket \mathcal{A} \rrbracket^{\text {init }}(\xi)$ of input $\mathcal{A}$ and $\xi$ (cf. Algorithm (2). It is obvious that these algorithms are correct.

More specifically, we are interested in the number $\#_{\text {run }}(\mathcal{A}, \xi)$ of occurrences of the strong bimonoid operations $\oplus$ and $\otimes$ that occur in the execution of Algorithm 1 if the inputs are $\mathcal{A}$ and $\xi$. Similarly, we are interested in the number $\#_{\text {init }}(\mathcal{A}, \xi)$ of occurrences of the strong bimonoid operations $\oplus$ and $\otimes$ that occur in the execution of Algorithm 2 if the inputs are $\mathcal{A}$ and $\xi$.

```
Algorithm 1 Computation of the run semantics
Input: \((\Sigma, \mathrm{B})\)-wta \(\mathcal{A}=(Q, \delta, F)\) and \(\xi \in \mathrm{T}_{\Sigma}\)
Variables: \(b, r: B ; w: \operatorname{pos}(\xi)\)
Output: \(\llbracket \mathcal{A} \rrbracket^{\text {run }}(\xi)\)
    \(b \leftarrow \mathbb{O}\)
    for each \(\rho: \operatorname{pos}(\xi) \rightarrow Q\) do
        \(r \leftarrow \mathbb{1}\)
        for each \(w \in \operatorname{pos}(\xi)\) in depth-first post-order do
            let \(k=\operatorname{rk}(\xi(w))\)
            \(r \leftarrow r \otimes \delta_{k}(\rho(w 1) \cdots \rho(w k), \xi(w), \rho(w))\)
        end for
        \(b \leftarrow b \oplus\left(r \otimes F_{\rho(\varepsilon)}\right)\)
    end for
    return \(b\)
```

```
Algorithm 2 Computation of the initial algebra semantics
Input: \((\Sigma, \mathrm{B})\)-wta \(\mathcal{A}=(Q, \delta, F)\) and \(\xi \in \mathrm{T}_{\Sigma}\)
Variables: \(v: \operatorname{pos}(\xi) \rightarrow B^{Q} ; w: \operatorname{pos}(\xi) ; b: B\)
Output: \(\llbracket \mathcal{A} \rrbracket^{\text {init }}(\xi)\)
    for each \(w \in \operatorname{pos}(\xi)\) in depth-first post-order do
        let \(k=\operatorname{rk}(\xi(w))\)
        for each \(q \in Q\) do
            \(v(w)_{q} \leftarrow \mathbb{O}\)
            for each \(q_{1} \cdots q_{k} \in Q^{k}\) do
                \(v(w)_{q} \leftarrow v(w)_{q} \oplus v(w 1)_{q_{1}} \otimes \ldots \otimes v(w k)_{q_{k}} \otimes \delta_{k}\left(q_{1} \cdots q_{k}, \xi(w), q\right)\)
            end for
        end for
    end for
    \(b \leftarrow 0\)
    for each \(q \in Q\) do
        \(b \leftarrow b \oplus\left(v(\varepsilon)_{q} \otimes F_{q}\right)\)
    end for
    return \(b\)
```

Theorem 5.1.1. Let $\Sigma$ be a ranked alphabet, B be a strong bimonoid, and $\mathcal{A}$ be $a(\Sigma, \mathrm{~B})$-wta. For every $\xi \in \mathrm{T}_{\Sigma}$, we have
(1) $\#_{\text {run }}(\mathcal{A}, \xi)=(\operatorname{size}(\xi)+2) \cdot|Q|^{\text {size }(\xi)}$, and
(2) $\#_{\text {init }}(\mathcal{A}, \xi) \leq \operatorname{size}(\xi) \cdot \max (m+1,2) \cdot\left(|Q|^{m+1}+|Q|\right)$ where $m=\operatorname{maxrk}(\Sigma)$.

Proof. Proof of (1): The loop in line 2 of Algorithm 1 is executed $\mid Q{ }^{\text {size }(\xi)}$ many times. In each execution of the body of this loop, $\operatorname{size}(\xi)+1$ times $\otimes$ is executed and 1 times $\oplus$ is executed.

Proof of (2): We abbreviate maxrk $(\Sigma)$ by $m$. By assuming that each position of $w$ of $\xi$ has $m$ children (apart from the leaves of $\xi$ ) we can approximate $\#_{\text {init }}(\mathcal{A}, \xi)$ as follows. The loops in lines 1,3 , and 5 are executed $\operatorname{size}(\xi)$ many times, $|Q|$ many times, and maximally $|Q|^{m}$ many times, respectively. In line 6 there are maximally $m+1$ occurrences of $\oplus$ and $\otimes$. Hence, during execution of the loop in line 1 , there are maximally $\operatorname{size}(\xi) \cdot|Q|^{m+1} \cdot(m+1)$ many occurrences of $\oplus$ and $\otimes$. During execution of the loop in line 11 there are $2 \cdot|Q|$ many occurrences of $\oplus$ and $\otimes$. Thus in total we obtain

$$
\#_{\text {init }}(\mathcal{A}, \xi) \leq \operatorname{size}(\xi) \cdot|Q|^{m+1} \cdot(m+1)+2 \cdot|Q|
$$

$$
\begin{aligned}
& \leq \operatorname{size}(\xi) \cdot|Q|^{m+1} \cdot \max (m+1,2)+\max (m+1,2) \cdot|Q| \cdot \operatorname{size}(\xi) \\
& \leq \operatorname{size}(\xi) \cdot \max (m+1,2) \cdot\left(|Q|^{m+1}+|Q|\right)
\end{aligned}
$$

As we can see from Theorem 5.1.1, the effort of calculating $\llbracket \mathcal{A} \rrbracket^{\mathrm{run}}(\xi)$ by Algorithm 1 is exponential in the size of $\xi$, and the effort of calculating $\llbracket \mathcal{A} \rrbracket^{\text {init }}(\xi)$ by Algorithm 2 is at most linear in the size of $\xi$.

With respect to the discussion on page 63, both algorithms compute the values of the algebraic ftahyperpath problem of an fta-hypergraph where the weights come from a strong bimonoid. We note that the value computation algorithm in [MV21, Alg. 6.1] (also cf. [MV19a, Alg. 1]) computes the values of the fta-algebraic hyperpath problem where the weights are calculated in a multioperator monoid.

### 5.2 Negative results for equality of the two semantics

In general, the run semantics and the initial algebra semantics of a wta over $B$ are different, which is witnessed by the following four examples for string ranked alphabets.

Example 5.2.1. We consider the string ranked alphabet $\Sigma=\left\{\gamma^{(1)}, \alpha^{(0)}\right\}$ and the tropical bimonoid TropBM $=\left(\mathbb{N}_{\infty},+, \min , 0, \infty\right)$ from Example 2.6.10(1). Moreover, let $\mathcal{A}=(Q, \delta, F)$ be the ( $\Sigma$, TropBM)wta from Example 3.2.8, i.e.,

- $Q=\left\{q_{0}, q_{1}\right\}$,
- $\delta_{0}(\varepsilon, \alpha, p)=\delta_{1}(p, \gamma, q)=1$ for every $p, q \in Q$, and
- $F_{q_{0}}=F_{q_{1}}=1$.

From Example 3.2 .8 we know that $\llbracket \mathcal{A} \rrbracket^{\text {run }}\left(\gamma^{n}(\alpha)\right)=2^{n+1}$ for every $n \in \mathbb{N}$. Next we compute $\llbracket \mathcal{A} \rrbracket^{\text {init }}\left(\gamma^{n}(\alpha)\right)$.
where we have to verify $(*)$. For this, by induction on $\mathbb{N}$, we prove that the following statement holds:

$$
\text { For every } n \in \mathbb{N} \text { and } q \in Q \text { we have: } \mathrm{h}_{\mathcal{A}}\left(\gamma^{n}(\alpha)\right)_{q}= \begin{cases}1 & \text { if } n=0 \\ 2 & \text { otherwise }\end{cases}
$$

I.B.: Let $n=0$. Then $\mathrm{h}_{\mathcal{A}}\left(\gamma^{n}(\alpha)\right)_{q}=\mathrm{h}_{\mathcal{A}}(\alpha)_{q}=\delta_{0}(\varepsilon, \alpha, q)=1$ for each $q \in Q$.
I.S.: Let $n \geq 1$. Then for each $q \in Q$ :
where the second equality follows from (a) the I.H. saying that $1 \leq \mathrm{h}_{\mathcal{A}}\left(\gamma^{n-1}(\alpha)\right)_{p} \leq 2$ and (b) the fact that $\delta_{1}(p, \gamma, q)=1$. Hence $\llbracket \mathcal{A} \rrbracket^{\text {run }} \neq \llbracket \mathcal{A} \rrbracket^{\text {init }}$.

Example 5.2.2. (adapted from CDIV10, Ex. 3.1]) Let Unitlnt ${ }_{\mathrm{bs}}=([0,1], \oplus, \cdot, 0,1)$ be the strong bimonoid given in Example 2.6.10(3), where $\oplus$ is the bounded sum, and let $\Sigma=\left\{\gamma^{(1)}, \nu^{(1)}, \alpha^{(0)}\right\}$. We consider the $\left(\Sigma\right.$, Unitlnt $\left._{\text {bs }}\right)$-wta $\mathcal{A}=(Q, \delta, F)$, where

- $Q=\{p, q\}$,
- the transition mappings are given as follows:

$$
\begin{aligned}
& \delta_{0}(\varepsilon, \alpha, p)=0.6, \quad \delta_{0}(\varepsilon, \alpha, q)=0 \\
& \delta_{1}(p, \gamma, p)=1, \quad \delta_{1}(p, \gamma, q)=1, \quad \delta_{1}(q, \gamma, p)=0, \quad \delta_{1}(q, \gamma, q)=0 \\
& \delta_{1}(p, \nu, p)=1, \quad \delta_{1}(p, \nu, q)=0, \quad \delta_{1}(q, \nu, p)=1, \quad \delta_{1}(q, \nu, q)=0
\end{aligned}
$$



Figure 5.1: The ( $\Sigma$, Unitlnt $_{\mathrm{bs}}$ )-wta $\mathcal{A}$ of Example 5.2.2,

- $F_{p}=0.6$ and $F_{q}=0$.

Figure 5.1 shows the fta-hypergraph of $\mathcal{A}$.
We consider the tree $\xi=\nu(\gamma(\alpha))$ and the runs $\rho_{1}$ and $\rho_{2}$ of $\mathcal{A}$ on $\xi$ with $\rho_{1}(\varepsilon)=\rho_{1}(1)=\rho_{1}(11)=p$ and $\rho_{2}(\varepsilon)=p, \rho_{1}(1)=q$, and $\rho_{2}(11)=p$. Then $\operatorname{wt}\left(\xi, \rho_{1}\right)=\operatorname{wt}\left(\xi, \rho_{2}\right)=0.6$. Then we can calculate as follows:

$$
\begin{aligned}
\llbracket \mathcal{A} \rrbracket^{\mathrm{run}}(\xi) & =\bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}}(\xi)} \mathrm{wt}(\xi, \rho) \cdot F_{\rho(\varepsilon)}=\bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}}(p, \xi)} \mathrm{wt}(\xi, \rho) \cdot 0.6 \\
& =\mathrm{wt}\left(\xi, \rho_{1}\right) \cdot 0.6 \oplus \mathrm{wt}\left(\xi, \rho_{2}\right) \cdot 0.6=0.36 \oplus 0.36=0.72 .
\end{aligned}
$$

For the calculation of $\llbracket \mathcal{A} \rrbracket^{\text {init }}(\xi)$ we calculate the following $Q$-vectors where the upper component corresponds to the state $p$ :

$$
\begin{gathered}
\mathrm{h}_{\mathcal{A}}(\alpha)=\binom{0.6}{0} \quad \mathrm{~h}_{\mathcal{A}}(\gamma(\alpha))=\binom{\mathrm{h}_{\mathcal{A}}(\alpha)_{p} \cdot 1}{\mathrm{~h}_{\mathcal{A}}(\alpha)_{p} \cdot 1}=\binom{0.6}{0.6} \\
\mathrm{~h}_{\mathcal{A}}(\nu(\gamma(\alpha)))=\binom{\mathrm{h}_{\mathcal{A}}(\gamma(\alpha))_{p} \cdot 1 \oplus \mathrm{~h}_{\mathcal{A}}(\gamma(\alpha))_{q} \cdot 1}{\mathrm{~h}_{\mathcal{A}}(\gamma(\alpha))_{p} \cdot 0 \oplus \mathrm{~h}_{\mathcal{A}}(\gamma(\alpha))_{q} \cdot 0}=\binom{0.6 \oplus 0.6}{0}=\binom{1}{0}
\end{gathered}
$$

Then $\llbracket \mathcal{A} \rrbracket^{\text {init }}(\xi)=\bigoplus_{q^{\prime} \in Q} \mathrm{~h}_{\mathcal{A}}(\xi)_{q^{\prime}} \cdot F_{q^{\prime}}=\mathrm{h}_{\mathcal{A}}(\xi)_{p} \cdot 0.6=0.6$. Hence $\llbracket \mathcal{A} \rrbracket^{\text {run }} \neq \llbracket \mathcal{A} \rrbracket^{\text {init }}$. This inequality is due to the lack of right-distributivity in UnitInt ${ }_{\text {bs }}$ :

$$
0.72=0.36 \oplus 0.36=(0.6 \cdot 0.6) \oplus(0.6 \cdot 0.6) \neq(0.6 \oplus 0.6) \cdot 0.6=1 \cdot 0.6=0.6
$$

As the next two examples show, the run semantics and the initial algebra semantics can be different even when the strong bimonoid is finite.

Example 5.2.3. We consider the finite strong bimonoid Three $=(\{0,1,2\}$, max, $\cdot, 0,1)$ with

$$
a \hat{\cdot} b=(a \cdot b) \bmod 3
$$

for every $a, b \in\{0,1,2\}$ (cf. Example 2.6.10(7)), and the string ranked alphabet $\Sigma=\left\{\gamma^{(1)}, \alpha^{(0)}\right\}$. Now we let $\mathcal{A}=(Q, \delta, F)$ be the $\left(\Sigma\right.$, Three)-wta such that $Q=\{1,2, q\}, F_{1}=F_{2}=0$ and $F_{q}=2$, and

$$
\delta_{0}(\varepsilon, \alpha, 1)=1, \quad \delta_{0}(\varepsilon, \alpha, 2)=2, \quad \delta_{0}(\varepsilon, \alpha, q)=0, \quad \delta_{1}(1, \gamma, q)=\delta_{1}(2, \gamma, q)=1
$$

and for every other combination $p_{1}, p_{2} \in Q$ we let $\delta_{1}\left(p_{1}, \gamma, p_{2}\right)=0$. Figure 5.2 shows the fta-hypergraph of $\mathcal{A}$.


Figure 5.2: The fta-hypergraph of the ( $\Sigma$, Three)-wta $\mathcal{A}$ of Example 5.2.3.

Now let $\xi=\gamma(\alpha)$. Clearly $\left\{\rho \in \mathrm{R}_{\mathcal{A}}(q, \xi) \mid \operatorname{wt}(\xi, \rho) \neq 0\right\}=\left\{\rho_{1}, \rho_{2}\right\}$ with

$$
\rho_{1}(\varepsilon)=\rho_{2}(\varepsilon)=q, \quad \rho_{1}(1)=1, \quad \rho_{2}(1)=2, \quad \text { and } \quad \operatorname{wt}\left(\xi, \rho_{1}\right)=1, \quad \operatorname{wt}\left(\xi, \rho_{2}\right)=2
$$

Hence $\llbracket \mathcal{A} \rrbracket^{\text {run }}(\xi)=\max \left(\mathrm{wt}\left(\xi, \rho_{1}\right) \hat{\wedge} 2, \operatorname{wt}\left(\xi, \rho_{2}\right) \hat{\wedge} 2\right)=\max (1 \hat{\wedge} 2,2 \hat{\wedge})=\max (2,1)=2$.
For the initial algebra semantics, we have

$$
\mathrm{h}_{\mathcal{A}}(\alpha)_{1}=1, \quad \mathrm{~h}_{\mathcal{A}}(\alpha)_{2}=2, \quad \mathrm{~h}_{\mathcal{A}}(\alpha)_{q}=0
$$

and

$$
\mathrm{h}_{\mathcal{A}}(\xi)_{q}=\max \left(\mathrm{h}_{\mathcal{A}}(\alpha)_{1} \hat{\therefore}, \mathrm{~h}_{\mathcal{A}}(\alpha)_{2} \hat{\therefore} 1\right)=\max (1,2)=2
$$

Hence $\llbracket \mathcal{A} \rrbracket^{\text {init }}(\xi)=\mathrm{h}_{\mathcal{A}}(\xi)_{q} \hat{\wedge} 2=2 \hat{\wedge} 2=1$.
We obtain $\llbracket \mathcal{A} \rrbracket^{\text {run }}(\xi)=2 \neq 1=\llbracket \mathcal{A} \rrbracket^{\text {init }}(\xi)$. This inequality is due to the lack of right-distributivity in Three: $\max (1 \hat{\wedge} 2,2 \hat{\wedge})=\max (2,1)=2 \neq 1=2 \hat{\wedge}=\max (1,2) \cdot 2$.

Example 5.2.4. FV22a We consider the bounded lattice $N_{5}=\left(N_{5}, \vee, \wedge, o, i\right)$ shown in Figure 2.3. Moreover, we consider the string ranked alphabet $\Sigma=\left\{\gamma^{(1)}, \alpha^{(0)}\right\}$. Now we let $\mathcal{A}=(Q, \delta, F)$ be the $\left(\Sigma, \mathrm{N}_{5}\right)$-wta with $Q=\left\{q_{1}, q_{2}, q\right\}, F_{q_{1}}=F_{q_{2}}=o$ and $F_{q}=a$. Moreover, let

- $\delta_{0}\left(\varepsilon, \alpha, q_{1}\right)=b, \delta_{0}\left(\varepsilon, \alpha, q_{2}\right)=c, \delta_{1}\left(q_{1}, \gamma, q\right)=\delta_{1}\left(q_{2}, \gamma, q\right)=i$,
- $\delta_{0}(\varepsilon, \alpha, q)=o$ and, for each $p_{1} p_{2} \in(Q \times Q) \backslash\left\{q_{1} q, q_{2} q\right\}$, we let $\delta_{1}\left(p_{1}, \gamma, p_{2}\right)=o$.

The fta-hypergraph of $\mathcal{A}$ is obtained from the one in Figure 5.2 by replacing the states 1 and 2 by $q_{1}$ and $q_{2}$, respectively, and by adapting the weights appropriately.

Then

$$
\begin{aligned}
\llbracket \mathcal{A} \rrbracket^{\text {init }}(\gamma(\alpha)) & =\bigvee_{p \in Q} \mathrm{~h}_{\mathcal{A}}(\gamma(\alpha))_{p} \wedge F_{p}=\mathrm{h}_{\mathcal{A}}(\gamma(\alpha))_{q} \wedge a=\left(\bigvee_{p \in Q} \mathrm{~h}_{\mathcal{A}}(\alpha)_{p} \wedge \delta_{1}(p, \gamma, q)\right) \wedge a \\
& =\left(\left(\mathrm{h}_{\mathcal{A}}(\alpha)_{q_{1}} \wedge \delta_{1}\left(q_{1}, \gamma, q\right)\right) \vee\left(\mathrm{h}_{\mathcal{A}}(\alpha)_{q_{2}} \wedge \delta_{1}\left(q_{2}, \gamma, q\right)\right)\right) \wedge a \\
& =\left(\left(\delta_{0}\left(\varepsilon, \alpha, q_{1}\right) \wedge i\right) \vee\left(\delta_{0}\left(\varepsilon, \alpha, q_{2}\right) \wedge i\right)\right) \wedge a \\
& =(b \vee c) \wedge a=a
\end{aligned}
$$

Also we have

$$
\begin{aligned}
\llbracket \mathcal{A} \rrbracket^{\mathrm{run}}(\gamma(\alpha)) & =\bigvee_{p \in Q} \bigvee_{\rho \in \mathrm{R}_{\mathcal{A}}(p, \xi)} \mathrm{wt}(\gamma(\alpha), \rho) \wedge F_{p}=\bigvee_{\rho \in \mathrm{R}_{\mathcal{A}}(q, \xi)}(\mathrm{wt}(\gamma(\alpha), \rho) \wedge a) \\
& =\left(\delta_{0}\left(\varepsilon, \alpha, q_{1}\right) \wedge \delta_{1}\left(q_{1}, \gamma, q\right) \wedge a\right) \vee\left(\delta_{0}\left(\varepsilon, \alpha, q_{2}\right) \wedge \delta_{1}\left(q_{2}, \gamma, q\right) \wedge a\right) \\
& =(b \wedge i \wedge a) \vee(c \wedge i \wedge a) \\
& =(b \wedge a) \vee(c \wedge a)=b
\end{aligned}
$$

Hence $\llbracket \mathcal{A} \rrbracket^{\text {init }} \neq \llbracket \mathcal{A} \rrbracket^{\text {run }}$.
For the particular string ranked alphabet $\Sigma$ and the ( $\Sigma, \operatorname{TropBM}$ )-wta $\mathcal{A}$ of Example 5.2.1 it is easy to construct a $\left(\Sigma\right.$, TropBM)-wta $\mathcal{B}$ such that $\llbracket \mathcal{A} \rrbracket^{\text {init }}=\llbracket \mathcal{B} \rrbracket^{\text {run }}$ (just let $\mathcal{B}$ have one state $q$, the weight of each transition is 2 , and the root weight of $q$ is also 2 ). The next theorem shows that, in general, this is not possible. Its proof is a slight adaptation of the proof of DSV10, Ex. 25].

Theorem 5.2.5. [DSV10, Ex. 25] The following two statements hold.
(1) For each $(\Sigma, \mathrm{Stb})-w t a \mathcal{A}$, the set $\operatorname{im}\left(\llbracket \mathcal{A} \rrbracket^{\text {run }}\right)$ is finite.
(2) If $\Sigma$ is nontrivial, then $\operatorname{Rec}^{\text {init }}(\Sigma, \mathrm{Stb}) \backslash \operatorname{Rec}^{\text {run }}(\Sigma, \mathrm{Stb}) \neq \emptyset$.

Proof. Before proving Statements (1) and (2), we recall the strong bimonoid Stb $=(\mathbb{N}, \oplus, \odot, 0,1)$ from [DSV10, Ex. 25] (also cf. Example 2.6.10(9)). The two commutative operations $\oplus$ and $\odot$ on $\mathbb{N}$ are defined as follows. First, let $0 \oplus a=a, 0 \odot a=0$, and $1 \odot a=a$ for every $a \in \mathbb{N}$. If $a, b \in \mathbb{N} \backslash\{0\}$ with $a \leq b$, we put (with + being the usual addition on $\mathbb{N}$ )

$$
a \oplus b= \begin{cases}b & \text { if } b \text { is even } \\ b+1 & \text { if } b \text { is odd. }\end{cases}
$$

If $a, b \in \mathbb{N} \backslash\{0,1\}$ with $a \leq b$, let

$$
a \odot b= \begin{cases}b+1 & \text { if } b \text { is even } \\ b & \text { if } b \text { is odd. }\end{cases}
$$

Proof of $(1)$ : Let $\mathcal{A}=(Q, \delta, F)$ be an arbitrary $(\Sigma, \operatorname{Stb})$-wta. Let $m=\max (\operatorname{wts}(\mathcal{A}))$ be the maximum of all weights which occur in $\mathcal{A}$. Then, by the definition of $\llbracket \mathcal{A} \rrbracket^{\text {run }}$, we have for each $\xi \in \mathrm{T}_{\Sigma}$ that

$$
\llbracket \mathcal{A} \rrbracket^{\mathrm{run}}(\xi)=\bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}}(\xi)} \operatorname{wt}(\xi, \rho) \odot F_{\rho(\varepsilon)} \leq \bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}}(\xi)}\left(\bigodot_{w \in \operatorname{pos}(\xi)} m\right) \odot m,
$$

where the inequality holds because $\odot$ is monotonic in each argument.
Now let $\xi=\alpha$ for some $\alpha \in \Sigma^{(0)}$. Then we can calculate as follows.

$$
\llbracket \mathcal{A} \rrbracket^{\mathrm{run}}(\xi) \leq \bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}}(\alpha)}\left(\bigodot_{w \in\{\varepsilon\}} m\right) \odot m=\bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}}(\alpha)} m \odot m \leq \bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}}(\alpha)}(m+1)=\bigoplus_{q \in Q}(m+1) \leq m+2 .
$$

Now let $\xi \in \mathrm{T}_{\Sigma} \backslash \Sigma^{(0)}$. Then by definition of $\odot$ :

$$
\bigodot_{w \in \operatorname{pos}(\xi)} m=\left\{\begin{array}{ll}
m+1 & \text { if } m \text { is even } \\
m & \text { otherwise }
\end{array} \quad \text { and } \quad\left(\bigodot_{w \in \operatorname{pos}(\xi)} m\right) \odot m= \begin{cases}m+1 & \text { if } m \text { is even } \\
m & \text { otherwise }\end{cases}\right.
$$

We note that in these calculations we have used the fact that $|\operatorname{pos}(\xi)| \geq 2$.
By definition of $\oplus$ we finally have:

$$
\bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}}(\xi)}\left(\bigodot_{w \in \operatorname{pos}(\xi)} m\right) \odot m= \begin{cases}m+2 & \text { if } m \text { is even } \\ m+1 & \text { otherwise } .\end{cases}
$$

Hence $\operatorname{im}\left(\left[\mathcal{A} \rrbracket^{\text {run }}\right)\right.$ is finite.

Proof of (2): Let $\Sigma$ be nontrivial. Now we consider the particular ( $\Sigma, \operatorname{Stb}$ )-wta $\mathcal{A}=(Q, \delta, F)$ with $Q=\left\{q_{1}, q_{2}\right\}$ and $\delta_{k}\left(p_{1} \cdots p_{k}, \sigma, p\right)=F_{p}=2$ for every $k \in \mathbb{N}, \sigma \in \Sigma^{(k)}$, and $p, p_{1}, \ldots, p_{k} \in Q$. By induction on $\mathrm{T}_{\Sigma}$, we prove that the following statement holds:

$$
\begin{equation*}
\text { For every } \xi \in \mathrm{T}_{\Sigma} \text { and } p \in Q \text {, we have }: \mathrm{h}_{\mathcal{A}}(\xi)_{p}=2 \cdot \operatorname{height}(\xi)+2 \tag{5.1}
\end{equation*}
$$

I.B.: Let $\xi=\alpha$ be in $\Sigma^{(0)}$. Then $\mathrm{h}_{\mathcal{A}}(\xi)_{p}=\delta_{0}(\varepsilon, \alpha, p)=2=2 \cdot \operatorname{height}(\xi)+2$.
I.S.: Let $\xi=\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)$ with $k \geq 1$ and $p \in Q$. Moreover, let $j \in[k]$ be such that $\operatorname{height}\left(\xi_{j}\right)=$ $\max \left(\right.$ height $\left.\left(\xi_{i}\right) \mid i \in[k]\right)$. Then,

$$
\begin{aligned}
\mathrm{h}_{\mathcal{A}}\left(\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)\right)_{p} & =\bigoplus_{p_{1} \cdots p_{k} \in Q} \mathrm{~h}_{\mathcal{A}}\left(\xi_{1}\right)_{p_{1}} \odot \ldots \odot \mathrm{~h}_{\mathcal{A}}\left(\xi_{k}\right)_{p_{k}} \odot \delta_{k}\left(p_{1} \cdots p_{k}, \sigma, p\right) \\
& =\bigoplus_{p_{1} \cdots p_{k} \in Q}\left(2 \cdot \operatorname{height}\left(\xi_{1}\right)+2\right) \odot \ldots \odot\left(2 \cdot \operatorname{height}\left(\xi_{k}\right)+2\right) \odot 2
\end{aligned}
$$

(by I.H. and definition of $\delta_{k}$ )

$$
=\bigoplus_{p_{1} \cdots p_{k} \in Q}\left(2 \cdot \operatorname{height}\left(\xi_{j}\right)+3\right)
$$

(by definition of $\odot$ and our choice of $j$; note that $2 \cdot \operatorname{height}\left(\xi_{j}\right)+2$ is even)

$$
=2 \cdot \operatorname{height}\left(\xi_{j}\right)+4
$$

(by definition of $\oplus$ and the fact that there are at least two summands due to $k \geq 1$ and $|Q|=2$ )

$$
=2 \cdot \operatorname{height}\left(\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)\right)+2
$$

This finished the proof of (5.1).
Hence, for each $\xi \in \mathrm{T}_{\Sigma}$, we have

$$
\llbracket \mathcal{A} \rrbracket^{\text {init }}(\xi)=\bigoplus_{p \in Q} \mathrm{~h}_{\mathcal{A}}(\xi)_{q} \odot F_{p}=\bigoplus_{p \in Q}(2 \cdot \operatorname{height}(\xi)+2) \odot 2=\bigoplus_{p \in Q}(2 \cdot \operatorname{height}(\xi)+3)=2 \cdot \operatorname{height}(\xi)+4
$$

Since $\Sigma$ is not trivial, this implies that the set $\operatorname{im}\left(\llbracket \mathcal{A} \rrbracket^{\text {init }}\right)$ is infinite.
By using Statement (1), we obtain that

$$
\llbracket \mathcal{A} \rrbracket^{\text {init }} \in \operatorname{Rec}^{\text {init }}(\Sigma, \operatorname{Stb}) \backslash \operatorname{Rec}^{\text {run }}(\Sigma, \mathrm{Stb})
$$

The result which is dual to Theorem 5.2 .5 would be that there exists a ranked alphabet $\Sigma$ and a strong bimonoid B such that $\operatorname{Rec}^{\text {run }}(\Sigma, \mathrm{B}) \backslash \operatorname{Rec}^{\text {init }}(\Sigma, \mathrm{B}) \neq \emptyset$. The corresponding statement for wsa was shown in DSV10, Ex. 26]. However, we were not able to reproduce that proof, and thus, we cannot include its version for wta into the book.

### 5.3 Positive results for equality of the two semantics

In this section we show that the run semantics and the initial algebra semantics of a $(\Sigma, \mathrm{B})$-wta $\mathcal{A}$ are equal if $\mathcal{A}$ is bu deterministic or B is distributive.

Theorem 5.3.1. (cf. FKV21, Thm. 3.6]) Let $\Sigma$ be a ranked alphabet and B be a strong bimonoid. For each bu deterministic $(\Sigma, \mathrm{B})-w t a \mathcal{A}$ we have $\llbracket \mathcal{A} \rrbracket^{\text {run }}=\llbracket \mathcal{A} \rrbracket^{\text {init }}$. Thus, in particular, $\operatorname{bud}^{\text {-Rec }}{ }^{\text {run }}(\Sigma, \mathrm{B})=$ bud-Rec ${ }^{\text {init }}(\Sigma, \mathrm{B})$ and $\operatorname{cd}-\operatorname{Rec}^{\text {run }}(\Sigma, \mathrm{B})=\operatorname{cd}-\operatorname{Rec}^{\text {init }}(\Sigma, \mathrm{B})$.

Proof. Let $\mathcal{A}=(Q, \delta, F)$ and $\xi \in \mathrm{T}_{\Sigma}$. Using $\mathrm{Q}_{\neq \mathbb{\perp}}^{\mathrm{R}_{\mathcal{A}}}(\xi)$ and $\mathrm{Q}_{\neq \mathbb{D}}^{\mathrm{h}_{\mathcal{D}}}(\xi)$ (cf. Section 4.1), we have

$$
\llbracket \mathcal{A} \rrbracket^{\mathrm{run}}(\xi)=\bigoplus_{q \in \mathrm{Q}_{\neq 0}^{\mathrm{R}, \mathcal{A}}(\xi)} \bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}}(q, \xi)} \mathrm{wt}(\xi, \rho) \otimes F_{q} \quad \text { and } \quad \llbracket \mathcal{A} \rrbracket^{\mathrm{init}}(\xi)=\bigoplus_{q \in \mathrm{Q}_{\neq 0}^{\mathrm{h} \mathcal{A}}(\xi)} \mathrm{h}_{\mathcal{A}}(\xi)_{q} \otimes F_{q}
$$

Then we proceed by case analysis. Using Lemma 4.2.1(3) we distinguish the following two cases. $\underline{\text { Case (a): Let } Q_{\neq 0}^{R} \mathcal{A}}(\xi)=\emptyset=Q_{\neq 0}^{\mathrm{h}_{\mathcal{D}}}(\xi)$. Then

$$
\llbracket \mathcal{A} \rrbracket^{\mathrm{run}}(\xi)=\bigoplus_{q \in \emptyset} \bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}}(q, \xi)} \mathrm{wt}(\xi, \rho) \otimes F_{q}=\mathbb{O}=\bigoplus_{q \in \emptyset} \mathrm{~h}_{\mathcal{A}}(\xi)_{q} \otimes F_{q}=\llbracket \mathcal{A} \rrbracket^{\mathrm{init}}(\xi)
$$

Case (b): Let $\mathrm{Q}_{\neq \mathcal{D}}^{\mathrm{R}_{\mathcal{A}}}(\xi)=\{q\}=\mathrm{Q}_{\neq \mathcal{D}}^{\mathrm{h}_{\mathcal{A}}}(\xi)$ for some $q \in Q$ and there exists exactly one $\rho \in \mathrm{R}_{\mathcal{A}}(q, \xi)$ with $\mathrm{h}_{\mathcal{A}} \overline{(\xi)_{q}=\mathrm{wt}}(\xi, \rho)$. Then

$$
\llbracket \mathcal{A} \rrbracket^{\mathrm{run}}(\xi)=\operatorname{wt}(\xi, \rho) \otimes F_{q}=\mathrm{h}_{\mathcal{A}}(\xi)_{q} \otimes F_{q}=\llbracket \mathcal{A} \rrbracket^{\mathrm{init}}(\xi) .
$$

Due to Theorem 5.3.1, if $\mathcal{A}$ is bu deterministic, then we write $\llbracket \mathcal{A} \rrbracket$ instead of $\llbracket \mathcal{A} \rrbracket^{\text {run }}$ and $\llbracket \mathcal{A} \rrbracket^{\text {init }}$. Moreover, for a weighted tree language $r$, we say that $r$ is bu deterministically recognizable (instead of bu deterministically i-recognizable and bu deterministically r-recognizable). Also, we write bud- $\operatorname{Rec}(\Sigma, B)$ for bud- $^{-\operatorname{Rec}^{\text {run }}}(\Sigma, B)$ (and hence, for bud-Rec ${ }^{\text {init }}(\Sigma, B)$ ). Similarly, we say that $r$ is crisp deterministically recognizable (instead of crisp deterministically i-recognizable and crisp deterministically r-recognizable). Also, we write $\operatorname{cd}-\operatorname{Rec}(\Sigma, \mathrm{B})$ for $\operatorname{cd}-\operatorname{Rec}^{\mathrm{run}}(\Sigma, \mathrm{B})$ and $\left.\operatorname{cd}-\operatorname{Rec}^{\text {init }}(\Sigma, \mathrm{B})\right)$.

Also the two semantics are equal for each ( $\Sigma, \mathrm{B}$ )-wta if B is a semiring (as, e.g., for the Boolean semiring, cf. Lemma 2.13.1). In the next theorem we will prove that even the "converse" result holds. The theorem was proved in [DSV10, Lm. 4] for the particular case that $\Sigma$ is a string ranked alphabet; there only right-distributivity is needed. It turns out that in the tree case also left-distributivity is needed if $\Sigma$ is not monadic. Moreover, in [LP05, Thm. 3.1(i),(iii)], the implication (A) $\Rightarrow$ (B) of the theorem was proved for string ranked alphabets and lattice-ordered monoids in which the unit element of the monoid equals the upper bound of the lattice.

Theorem 5.3.2. (cf. Rad10, Thm. 4.1] and Bor05 Lm. 4.1.13]) Let $\Sigma$ be a ranked alphabet and B be a strong bimonoid. The following two statements are equivalent:
(A) If $\Sigma$ is not trivial, then B is right-distributive, and if $\Sigma$ is not monadic, then B is left-distributive.
(B) For each $(\Sigma, \mathrm{B})-$ wta $\mathcal{A}$ we have $\llbracket \mathcal{A} \rrbracket^{\text {run }}=\llbracket \mathcal{A} \rrbracket^{\text {init }}$.

Proof. Proof of $(\mathrm{A}) \Rightarrow(\mathrm{B})$ : Let $\xi \in \mathrm{T}_{\Sigma}$. Since

$$
\llbracket \mathcal{A} \rrbracket^{\mathrm{run}}(\xi)=\bigoplus_{q \in Q} \bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}}(q, \xi)} \mathrm{wt}(\xi, \rho) \otimes F_{q} \quad \text { and } \quad \llbracket \mathcal{A} \rrbracket^{\mathrm{init}}(\xi)=\bigoplus_{q \in Q} \mathrm{~h}_{\mathcal{A}}(\xi)_{q} \otimes F_{q}
$$

it suffices to show that

$$
\text { for each } q \in Q \text {, we have } \mathrm{h}_{\mathcal{A}}(\xi)_{q} \otimes F_{q}=\bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}}(q, \xi)} \mathrm{wt}(\xi, \rho) \otimes F_{q} .
$$

If $\Sigma$ is trivial, then $\left|\mathrm{R}_{\mathcal{A}}(q, \xi)\right|=1$. Otherwise B is right-distributive. Thus in both cases we have

$$
\bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}}(q, \xi)} \mathrm{wt}(\xi, \rho) \otimes F_{q}=\left(\bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}}(q, \xi)} \mathrm{wt}(\xi, \rho)\right) \otimes F_{q}
$$

Hence, by induction on $\mathrm{T}_{\Sigma}$, we prove that the following statement holds:

$$
\begin{equation*}
\text { For every } \xi \in \mathrm{T}_{\Sigma} \text { and } q \in Q \text {, we have } \mathrm{h}_{\mathcal{A}}(\xi)_{q}=\bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}}(q, \xi)} \mathrm{wt}(\xi, \rho) \text {. } \tag{5.2}
\end{equation*}
$$

For this, let $\xi=\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)$ and $q \in Q$. Then

$$
\text { (by right-distributivity, in case } k \geq 1 \text { ) }
$$

$$
\text { (by left-distributivity, in case } k \geq 2 \text { ) }
$$

Proof of $(\mathrm{B}) \Rightarrow(\mathrm{A})$ : We have to show:
(i) If $\Sigma$ is not trivial, then $(a \oplus b) \otimes c=a \otimes c \oplus b \otimes c$ for every $a, b, c \in B$, and
(ii) if $\Sigma$ is not monadic, then $a \otimes(b \oplus c)=a \otimes b \oplus a \otimes c$ for every $a, b, c \in B$.

To prove (i), we assume that $\Sigma$ is not trivial. Hence, $\Sigma \neq \Sigma^{(0)}$ and, by our convention on page 40, we also have $\Sigma^{(0)} \neq \emptyset$. Thus there exist $\alpha \in \Sigma^{(0)}$ and $\gamma \in \Sigma^{(n)}$ for some $n \in \mathbb{N}_{+}$. Let $a, b, c \in B$. We construct the $(\Sigma, \mathrm{B})$-wta $\mathcal{A}=(Q, \delta, F)$ as follows: $Q=\{(a, 1),(b, 2), c, \mathbb{1}\}, F_{(a, 1)}=F_{(b, 2)}=F_{\mathbb{1}}=\mathbb{O}$, and $F_{c}=c$. (We use $(a, 1)$ and $(b, 2)$ in order to keep the states separate even if $a=b$.) Moreover, we define the transition mappings as follows:

- $\delta_{0}(\varepsilon, \alpha,(a, 1))=a, \delta_{0}(\varepsilon, \alpha,(b, 2))=b$, and $\delta_{0}(\varepsilon, \alpha, \mathbb{1})=\mathbb{1}$
- $\delta_{0}(\varepsilon, \alpha, c)=\mathbb{0}$,
- $\delta_{n}((a, 1) \underbrace{\mathbb{1} \cdots \mathbb{1}}_{n-1}, \gamma, c)=\delta_{n}((b, 2) \underbrace{\mathbb{1} \cdots \mathbb{1}}_{n-1}, \gamma, c)=\mathbb{1}$,
- $\delta_{n}(w, \gamma, q)=\mathbb{0}$ for each $(w, q) \in\left(Q^{n} \times Q\right) \backslash\{((a, 1) \mathbb{1} \cdots \mathbb{1}, c),((b, 2) \mathbb{1} \cdots \mathbb{1}, c)\}$, and
- for every other input symbol $\sigma \in \Sigma^{(k)}$ with $k \in \mathbb{N}$ and $(w, r) \in Q^{k} \times Q$ we can define $\delta_{k}(w, \sigma, r)$ arbitrarily.
We consider the particular input tree $\xi=\gamma(\alpha, \ldots, \alpha)$ in $\mathrm{T}_{\Sigma}$ and calculate its initial algebra semantics.

$$
\begin{aligned}
\llbracket \mathcal{A} \rrbracket^{\text {init }}(\xi)= & \bigoplus_{q \in Q} \mathrm{~h}_{\mathcal{A}}(\xi)_{q} \otimes F_{q}=\mathrm{h}_{\mathcal{A}}(\xi)_{c} \otimes F_{c}=\left(\bigoplus_{q_{1} \cdots q_{n} \in Q^{n}}\left(\bigotimes_{i \in[n]} \mathrm{h}_{\mathcal{A}}(\alpha)_{q_{i}}\right) \otimes \delta_{n}\left(q_{1} \cdots q_{n}, \gamma, c\right)\right) \otimes c \\
= & (\mathrm{h}_{\mathcal{A}}(\alpha)_{(a, 1)} \otimes \underbrace{\mathrm{h}_{\mathcal{A}}(\alpha)_{\mathbb{1}} \otimes \ldots \otimes \mathrm{h}_{\mathcal{A}}(\alpha)_{\mathbb{1}}}_{n-1} \otimes \delta_{n}((a, 1) \mathbb{1} \cdots \mathbb{1}, \gamma, c) \\
& \oplus \mathrm{h}_{\mathcal{A}}(\alpha)_{(b, 2)} \otimes \underbrace{\mathrm{h}_{\mathcal{A}}(\alpha)_{\mathbb{1}} \otimes \ldots \otimes \mathrm{h}_{\mathcal{A}}(\alpha)_{\mathbb{1}}}_{n-1} \otimes \delta_{n}((b, 2) \mathbb{1} \cdots \mathbb{1}, \gamma, c)) \otimes c \\
= & (a \otimes \mathbb{1} \otimes \ldots \otimes \mathbb{1} \otimes \mathbb{1} \oplus b \otimes \mathbb{1} \otimes \ldots \otimes \mathbb{1} \otimes \mathbb{1}) \otimes c=(a \oplus b) \otimes c .
\end{aligned}
$$

For the calculation of the run semantics of $\xi$ we consider the particular runs $\rho_{(a, 1)}$ and $\rho_{(b, 2)}$ of $\mathcal{A}$ on

$$
\begin{aligned}
& \mathrm{h}_{\mathcal{A}}(\xi)_{q} \\
& =\bigoplus_{q_{1} \cdots q_{k} \in Q^{k}}\left(\bigotimes_{i \in[k]} \mathrm{h}_{\mathcal{A}}\left(\xi_{i}\right)_{q_{i}}\right) \otimes \delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right) \\
& =\bigoplus_{q_{1} \cdots q_{k} \in Q^{k}}\left(\bigoplus_{\rho_{1} \in \mathrm{R}_{\mathcal{A}}\left(q_{1}, \xi_{1}\right)} \cdots \bigoplus_{\rho_{k} \in \mathrm{R}_{\mathcal{A}}\left(q_{k}, \xi_{k}\right)}\left(\bigotimes_{i \in[k]} \mathrm{wt}\left(\xi_{i}, \rho_{i}\right)\right) \otimes \delta_{k}\left(\rho_{1}(\varepsilon) \cdots \rho_{k}(\varepsilon), \sigma, q\right)\right) \\
& =\bigoplus_{q_{1} \cdots q_{k} \in Q^{k}}\left(\bigoplus_{\substack{\rho \in \mathrm{R}_{\mathcal{A}}(q, \xi), \rho(1)=q_{1}, \ldots, \rho(k)=q_{k}}}\left(\bigotimes_{i \in[k]} \mathrm{wt}\left(\xi_{i},\left.\rho\right|_{i}\right)\right) \otimes \delta_{k}(\rho(1) \cdots \rho(k), \sigma, \rho(\varepsilon))\right) \\
& =\bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}}(q, \xi)}\left(\bigotimes_{i \in[k]} \mathrm{wt}\left(\xi_{i},\left.\rho\right|_{i}\right)\right) \otimes \delta_{k}(\rho(1) \cdots \rho(k), \sigma, \rho(\varepsilon))=\bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}}(q, \xi)} \mathrm{wt}(\xi, \rho) .
\end{aligned}
$$

(a)

(b)


Figure 5.3: (a) Particular runs $\rho_{(a, 1)}$ and $\rho_{(b, 2)}$, (b) particular runs $\rho_{a \otimes(b, 1)}$ and $\rho_{a \otimes(c, 2)}$.
$\xi$ (cf. Figure 5.3(a)) defined by

- $\rho_{(a, 1)}(\varepsilon)=c, \rho_{(a, 1)}(1)=(a, 1)$, and $\rho_{(a, 1)}(i)=\mathbb{1}$ for every $i \in[2, n]$ and
- $\rho_{(b, 2)}(\varepsilon)=c, \rho_{(b, 2)}(1)=(b, 2)$, and $\rho_{(b, 2)}(i)=\mathbb{1}$ for every $i \in[2, n]$.

We note that, for every $\rho \in \mathrm{R}_{\mathcal{A}}(\xi) \backslash\left\{\rho_{(a, 1)}, \rho_{(b, 2)}\right\}$, we have $\mathrm{wt}(\xi, \rho)=\mathbb{O}$. Then

$$
\begin{aligned}
\llbracket \mathcal{A} \rrbracket^{\mathrm{run}}(\xi)= & \bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}}(\xi)} \mathrm{wt}(\xi, \rho) \otimes F_{\rho(\varepsilon)}=\bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}}(c, \xi)} \mathrm{wt}(\xi, \rho) \otimes c=\mathrm{wt}\left(\xi, \rho_{(a, 1)}\right) \otimes c \oplus \mathrm{wt}\left(\xi, \rho_{(b, 2)}\right) \otimes c \\
= & \operatorname{wt}\left(\alpha,\left.\rho_{(a, 1)}\right|_{1}\right) \otimes \operatorname{wt}\left(\alpha,\left.\rho_{(a, 1)}\right|_{2}\right) \otimes \ldots \otimes \operatorname{wt}\left(\alpha,\left.\rho_{(a, 1)}\right|_{n}\right) \otimes \delta_{n}((a, 1) \mathbb{1} \cdots \mathbb{1}, \gamma, c) \otimes c \\
& \oplus \operatorname{wt}\left(\alpha,\left.\rho_{(b, 2)}\right|_{1}\right) \otimes \operatorname{wt}\left(\alpha,\left.\rho_{(b, 2)}\right|_{2}\right) \otimes \ldots \otimes \operatorname{wt}\left(\alpha,\left.\rho_{(b, 2)}\right|_{n}\right) \otimes \delta_{n}((b, 2) \mathbb{1} \cdots \mathbb{1}, \gamma, c) \otimes c \\
= & a \otimes \mathbb{1} \otimes \ldots \otimes \mathbb{1} \otimes \mathbb{1} \otimes c \oplus b \otimes \mathbb{1} \otimes \ldots \otimes \mathbb{1} \otimes \mathbb{1} \otimes c=a \otimes c \oplus b \otimes c .
\end{aligned}
$$

Since $\llbracket \mathcal{A} \rrbracket^{\text {init }}=\llbracket \mathcal{A} \rrbracket^{\text {run }}$, we obtain that Statement (i) holds.
For the proof of (ii), we assume $\Sigma$ is not monadic. Hence, $\Sigma \neq \Sigma^{(0)} \cup \Sigma^{(1)}$ and, by our convention on page 40, we also have $\Sigma^{(0)} \neq \emptyset$. Thus there exist $\alpha \in \Sigma^{(0)}$ and $\sigma \in \Sigma^{(m)}$ for some $m \geq 2$. Let $a, b, c \in B$. We construct the $(\Sigma, \mathrm{B})$-wta $\mathcal{A}=(Q, \delta, F)$ with $Q=\{a,(b, 1),(c, 2), p, q, \mathbb{1}\}, F_{q}=\mathbb{1}, F_{u}=\mathbb{0}$ for every $u \in Q \backslash\{q\}$, and $\delta$ is defined by:

- $\delta_{0}(\varepsilon, \alpha, x)=x$ for every $x \in\{a, \mathbb{1}\}, \delta_{0}(\varepsilon, \alpha,(b, 1))=b, \delta_{0}(\varepsilon, \alpha,(c, 2))=c$, and $\delta_{0}(\varepsilon, \alpha, p)=\delta_{0}(\varepsilon, \alpha, q)=\mathbb{0}$,
- $\delta_{m}((b, 1) \underbrace{\mathbb{1} \cdots \mathbb{1}}_{m-1}, \sigma, p)=\delta_{m}((c, 2) \underbrace{\mathbb{1} \cdots \mathbb{1}}_{m-1}, \sigma, p)=\delta_{m}(a \underbrace{\mathbb{1} \cdots \mathbb{1}}_{m-2} p, \sigma, q)=\mathbb{1}$,
- for every other $(w, r) \in Q^{m} \times Q$, we let $\delta_{m}(w, \sigma, r)=\mathbb{0}$, and
- for every input symbol $\tau \in \Sigma \backslash\{\alpha, \sigma\}$ (with some rank $k \geq 0$ ) and every $(w, r) \in Q^{k} \times Q$ we can define $\delta_{k}(w, \tau, r)$ arbitrarily.
Now we consider the particular input tree $\xi=\sigma(\underbrace{\alpha, \ldots, \alpha}_{m-1}, \sigma(\underbrace{\alpha, \ldots, \alpha}_{m})) \in \mathrm{T}_{\Sigma}$.

$$
\begin{aligned}
\mathrm{h}_{\mathcal{A}}(\sigma(\underbrace{\alpha, \ldots, \alpha}_{m}))_{p}= & \bigoplus_{q_{1} \cdots q_{m} \in Q^{m}} \mathrm{~h}_{\mathcal{A}}(\alpha)_{q_{1}} \otimes \mathrm{~h}_{\mathcal{A}}(\alpha)_{q_{2}} \otimes \ldots \otimes \mathrm{~h}_{\mathcal{A}}(\alpha)_{q_{m}} \otimes \delta_{m}\left(q_{1} q_{2} \cdots q_{m}, \sigma, p\right) \\
= & \mathrm{h}_{\mathcal{A}}(\alpha)_{(b, 1)} \otimes \mathrm{h}_{\mathcal{A}}(\alpha)_{\mathbb{1}} \otimes \ldots \otimes \mathrm{h}_{\mathcal{A}}(\alpha)_{\mathbb{1}} \otimes \delta_{m}((b, 1) \mathbb{1} \cdots \mathbb{1}, \sigma, p) \\
& \oplus \mathrm{h}_{\mathcal{A}}(\alpha)_{(c, 2)} \otimes \mathrm{h}_{\mathcal{A}}(\alpha)_{\mathbb{1}} \otimes \ldots \otimes \mathrm{h}_{\mathcal{A}}(\alpha)_{\mathbb{1}} \otimes \delta_{m}((c, 2) \mathbb{1} \cdots \mathbb{1}, \sigma, p)=b \oplus c
\end{aligned}
$$

Then

$$
\begin{aligned}
\llbracket \mathcal{A} \rrbracket^{\mathrm{init}}(\xi) & =\mathrm{h}_{\mathcal{A}}(\xi)_{q} \\
& =\mathrm{h}_{\mathcal{A}}(\alpha)_{a} \otimes \mathrm{~h}_{\mathcal{A}}(\alpha)_{\mathbb{1}} \otimes \ldots \otimes \mathrm{h}_{\mathcal{A}}(\alpha)_{\mathbb{1}} \otimes \mathrm{h}_{\mathcal{A}}(\sigma(\alpha, \ldots, \alpha))_{p} \otimes \delta_{m}(a \mathbb{1} \cdots \mathbb{1} p, \sigma, q)=a \otimes(b \oplus c)
\end{aligned}
$$

Now we consider the particular runs $\rho_{a \otimes(b, 1)}, \rho_{a \otimes(c, 2)} \in \mathrm{R}_{\mathcal{A}}(q, \xi)$ (cf. Figure5.3(b)) such that

- $\rho_{a \otimes(b, 1)}(\varepsilon)=q, \rho_{a \otimes(b, 1)}(1)=a, \rho_{a \otimes(b, 1)}(i)=\mathbb{1}$ for every $i \in[2, m-1], \rho_{a \otimes(b, 1)}(m)=p$, $\rho_{a \otimes(b, 1)}(m 1)=(b, 1)$, and $\rho_{a \otimes(b, 1)}(m j)=\mathbb{1}$ for every $j \in[2, m]$, and
- $\rho_{a \otimes(c, 1)}$ is the same as $\rho_{a \otimes(b, 1)}$ except that $\rho_{a \otimes(c, 2)}(m 1)=(c, 2)$.

It is clear that $\operatorname{wt}\left(\xi, \rho_{a \otimes(b, 1)}\right)=a \otimes b$ and $\operatorname{wt}\left(\xi, \rho_{a \otimes(c, 2)}\right)=a \otimes c$. Then
$\llbracket \mathcal{A} \rrbracket^{\mathrm{run}}(\xi)=\bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}}(\xi)} \mathrm{wt}(\xi, \rho) \otimes F_{\rho(\varepsilon)}=\bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}}(q, \xi)} \mathrm{wt}(\xi, \rho)=\mathrm{wt}\left(\xi, \rho_{a \otimes(b, 1)}\right) \oplus \mathrm{wt}\left(\xi, \rho_{a \otimes(c, 2)}\right)=a \otimes b \oplus a \otimes c$.
Since $\llbracket \mathcal{A} \rrbracket^{\text {run }}=\llbracket \mathcal{A} \rrbracket^{\text {init }}$, we obtain $a \otimes(b \oplus c)=a \otimes b \oplus a \otimes c$. This proves Statement (ii).

In Examples 3.2.4 and 3.2.11 we gave a wta over the semiring $\mathrm{Nat}_{\mathrm{max},+}$ and a wta over the semiring Nat, respectively, and we have proved that for both wta the run semantics coincides with the initial algebra semantics. The following corollary shows that this coincidence is valid for each wta over an arbitrary semiring.

Corollary 5.3.3. Let $\Sigma$ be a ranked alphabet and $\mathrm{B}=(B, \oplus, \otimes, \mathbb{O}, \mathbb{1})$ be a semiring. For each $(\Sigma, \mathrm{B})$-wta $\mathcal{A}$, we have
(1) $\mathrm{h}_{\mathcal{A}}(\xi)_{q}=\bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}}(q, \xi)} \mathrm{wt}(\xi, \rho)$ for every $\xi \in \mathrm{T}_{\Sigma}$ and state $q$ of $\mathcal{A}$ and
(2) $\llbracket \mathcal{A} \rrbracket^{\text {run }}=\llbracket \mathcal{A} \rrbracket^{\text {init }}$.

Thus, in particular, $\operatorname{Rec}^{\mathrm{run}}(\Sigma, \mathrm{B})=\operatorname{Rec}^{\text {init }}(\Sigma, \mathrm{B})$.

Proof. Since B is a semiring, Equation (5.2) holds, and hence Statement (1) holds. Moreover, Theo$\operatorname{rem} 5.3 .2((\mathrm{~A}) \Rightarrow(\mathrm{B}))$ implies that $\llbracket \mathcal{A} \rrbracket^{\text {run }}=\llbracket \mathcal{A} \rrbracket^{\text {init }}$ and $\operatorname{Rec}^{\text {run }}(\Sigma, \mathrm{B})=\operatorname{Rec}^{\text {init }}(\Sigma, \mathrm{B})$.

Due to Corollary 5.3.3, if B is a semiring and $\mathcal{A}$ is a $(\Sigma, \mathrm{B})$-wta, then we write $\llbracket \mathcal{A} \rrbracket$ instead of $\llbracket \mathcal{A} \rrbracket^{\text {run }}$ and $\llbracket \mathcal{A} \rrbracket^{\text {init }}$. Moreover, for an $i$-recognizable or $r$-recognizable weighted tree language $r$, we say that $r$ is recognizable. Also, we denote the set $\operatorname{Rec}^{\mathrm{run}}(\Sigma, \mathrm{B})\left(\right.$ and hence, $\operatorname{Rec}^{\text {init }}(\Sigma, \mathrm{B})$ ) by $\operatorname{Rec}(\Sigma, \mathrm{B})$.

With respect to the complexity of calculating the semantics $\llbracket \mathcal{A} \rrbracket$ of some $(\Sigma, \mathrm{B})$-wta $\mathcal{A}$ where B is a semiring, Theorem 5.1.1 implies the following: in general, it is more efficient to calculate $\llbracket \mathcal{A} \rrbracket$ according to Algorithm 2 (initial algebra semantics) than to calculate it according to Algorithm 1 (run semantics).

In Section 3.3 we have recalled the definition of wsa and the definitions of their run semantics and initial algebra semantics. Moreover, we have shown that wsa can be considered as particular wta (cf. Lemma 3.3.2). As an easy consequence of previous results we obtain that the two semantics of wsa over semirings are equal.

Corollary 5.3.4. (cf. DSV10, Lm. 4]) Let $\Gamma$ be an alphabet and $B$ be a right-distributive strong bimonoid. For each ( $\Gamma, B$ )-wsa $\mathcal{A}$ we have $\llbracket \mathcal{A} \rrbracket^{\text {run }}=\llbracket \mathcal{A} \rrbracket^{\text {init }}$.

Proof. Let $\mathcal{A}$ be a $(\Gamma, \mathrm{B})$-wsa. Moreover, let $e \notin \Gamma$. By Lemma 3.3.2 we can construct a $\left(\Gamma_{e}, \mathrm{~B}\right)$-wta $\mathcal{B}$ such that $\llbracket \mathcal{A} \rrbracket^{\text {run }}=\llbracket \mathcal{B} \rrbracket^{\text {run }} \circ$ tree $_{e}$ and $\llbracket \mathcal{A} \rrbracket^{\text {init }}=\llbracket \mathcal{B} \rrbracket^{\text {init }} \circ$ tree $_{e}$. We note that $\Gamma \neq \emptyset$ and hence $\Gamma_{e}$ is not trivial. Then, by Theorem 5.3 .2 and again by Lemma 3.3.2, we obtain that $\llbracket \mathcal{A} \rrbracket^{\text {run }}=\llbracket \mathcal{A} \rrbracket^{\text {init }}$.

Due to Corollary 5.3.4, if B is a semiring and $\mathcal{A}$ is a $(\Gamma, \mathrm{B})$-wsa, then we write $\llbracket \mathcal{A} \rrbracket$ instead of $\llbracket \mathcal{A} \rrbracket^{\text {run }}$ and $\llbracket \mathcal{A} \rrbracket^{\text {init }}$.

## Chapter 6

## Pumping lemmas

In this chapter we show three pumping lemmas, which have different flavors: the pumping lemma for runs [Bor04, Sec. 5] (cf. Theorem 6.1.4) and, as a corollary, a pumping lemma for wta over positive strong bimonoids (cf. Corollary 6.1.6), and the pumping lemma for wta over fields BR82] (cf. Theorem 6.2.9).

The pumping lemma in Theorem 6.1.4 concerns runs of weighted tree automata over B. It says that, given a big tree and a run on that tree, both the tree and the run can be decomposed such that both, the tree and the run can be pumped up along its components, and the weight of a pumped run on the corresponding pumped tree can be constructed as the product of the weights of the components of the run. As a corollary of Theorem 6.1.4 we prove a pumping lemma for the support of weighted tree languages recognizable by wta over a positive strong bimonoid B (cf. Corollary 6.1.6). Essentially this is the classical pumping lemma for recognizable $\Sigma$-tree languages [GS84, Lm. 2.10.1].

The pumping lemma in Theorem 6.2 .9 concerns the support of weighted tree languages recognizable by wta over a field B. It says that, given a recognizable weighted tree language $r$ and a big tree in the support of $r$, the tree can be decomposed such that it can be pumped up along its components and infinitely many of the pumped trees belong to the support of $r$.

We mention that in MR18, CMMR21 five pumping lemmas for wsa over particular important semirings are shown. They concern weighted languages (a) which are recognizable by wsa over Nat ${ }_{\infty}$, (b) which have the form $\min \left(r_{1}, \ldots, r_{m}\right)$, where $r_{1}, \ldots, r_{m}$ are recognizable by wsa over $\mathrm{Nat}_{\infty}$, (c) which are recognizable by polynomially-ambiguous wsa over $\mathrm{Nat}_{\text {min },+}$, (d) which are recognizable by finitely-ambiguous wsa over $\mathrm{Nat}_{\text {max },+}$, and (e) which are recognizable by polynomially-ambiguous wsa over $\mathrm{Nat}_{\text {max },+}$, respectively. Each of these pumping lemmas is used to show that certain weighted languages do not belong to the corresponding set of recognizable weighted languages.

We recall that $\mathrm{C}_{\Sigma} \subseteq \mathrm{T}_{\Sigma}(\{z\})$ denotes the set of all $\Sigma$-contexts (cf. page 48). For each $c \in \mathrm{C}_{\Sigma}$, we denote by depth $(c)$ the length of the unique $z$-labeled position of $c$, i.e. depth $(c)=\left|\operatorname{pos}_{z}(c)\right|$. We say that $c$ is elementary if depth $(c)=1$, i.e., there exist $k \in \mathbb{N}_{+}, \sigma \in \Sigma^{(k)}, i \in[k]$, and $\xi_{1}, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots, \xi_{k} \in \mathrm{~T}_{\Sigma}$ such that

$$
c=\sigma\left(\xi_{1}, \ldots, \xi_{i-1}, z, \xi_{i+1}, \ldots, \xi_{k}\right)
$$

We denote the set of all elementary $\Sigma$-contexts by $\mathrm{eC}_{\Sigma}$. Lastly we define the binary relation $\prec_{\mathrm{C}_{\Sigma}}$ on $\mathrm{C}_{\Sigma}$ such that, for every $c_{1}, c_{2} \in \mathrm{C}_{\Sigma}$, we have $c_{1} \prec_{\mathrm{C}_{\Sigma}} c_{2}$ if there exists an elementary context $c \in \mathrm{eC}_{\Sigma}$ such that $c_{2}=c\left[c_{1}\right]$. Obviously, $\prec_{\mathrm{C}_{\Sigma}}$ is well-founded and $\min _{\prec_{\mathrm{C}_{\Sigma}}}\left(\mathrm{C}_{\Sigma}\right)=\{z\}$.

### 6.1 Pumping lemma for runs and wta over strong bimonoids

The pumping lemma of this section is based on the idea in Bor04, Sec. 5]. The form in which we present it is adapted from DFKV20, DFKV22.

$$
\text { In this section, we let } \mathcal{A}=(Q, \delta, F) \text { be an arbitrary }(\Sigma, \mathrm{B}) \text {-wta. }
$$

We extend the concept of run (cf. Section (2.9) to runs on trees in $\mathrm{T}_{\Sigma}(\{z\})$, where $z$ is a variable. For this, let $\xi \in \mathrm{T}_{\Sigma}(\{z\})$ and $q \in Q$. A run (of $\left.\mathcal{A}\right)$ on $\xi$ is a mapping $\rho: \operatorname{pos}(\xi) \rightarrow Q$, and it is a $q$-run if $\rho(\varepsilon)=q$. Also, we denote the set of all runs on $\xi$ by $\mathrm{R}_{\mathcal{A}}(\xi)$ and the set of all $q$-runs on $\xi$ by $\mathrm{R}_{\mathcal{A}}(q, \xi)$. For every $\xi \in \mathrm{T}_{\Sigma}(\{z\}), \rho \in \mathrm{R}_{\mathcal{A}}(\xi)$, and $w \in \operatorname{pos}(\xi)$, the run induced by $\rho$ at position $w$, denoted by $\left.\rho\right|_{w}$, is defined in a similar way as shown on page 64.

Next we define the weight of a run on trees in $\mathrm{T}_{\Sigma}(\{z\})$ in a similar way as shown on page 64. We consider the well-founded set $\left(\mathrm{TR}_{z}, \prec\right)$ where $\mathrm{TR}_{z}=\left\{(\xi, \rho) \mid \xi \in \mathrm{T}_{\Sigma}(\{z\}), \rho \in \mathrm{R}_{\mathcal{A}}(\xi)\right\}$ and $\prec$ is the binary relation on $\mathrm{TR}_{z}$ defined by

$$
\prec=\left\{\left(\left(\left.\xi\right|_{i},\left.\rho\right|_{i}\right),(\xi, \rho)\right) \mid(\xi, \rho) \in \mathrm{TR}_{z}, i \in[\operatorname{rk}(\xi(\varepsilon))]\right\}
$$

Obviously, $\prec$ is well-founded and $\min _{\prec}\left(\operatorname{TR}_{z}\right)=\left\{(\alpha, \rho) \mid \alpha \in \Sigma^{(0)} \cup\{z\}, \rho:\{\varepsilon\} \rightarrow Q\right\}$. We define the mapping

$$
\mathrm{wt}_{\mathcal{A}}: \mathrm{TR}_{z} \rightarrow B
$$

by induction on $\left(\mathrm{TR}_{z}, \prec\right)$ for every $\xi \in \mathrm{T}_{\Sigma}(\{z\})$ and $\rho \in \mathrm{R}_{\mathcal{A}}(\xi)$ as follows:
I.B.: If $\xi=z$, then $\operatorname{wt}_{\mathcal{A}}(\xi, \rho)=\mathbb{1}$. If $\xi=\alpha$ is in $\Sigma^{(0)}$, then $\operatorname{wt}_{\mathcal{A}}(\xi, \rho)=\delta_{0}(\varepsilon, \alpha, \rho(\varepsilon))$.
I.S.: Let $\xi=\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)$ with $k \geq 1$. Then we define

$$
\begin{equation*}
\operatorname{wt}_{\mathcal{A}}(\xi, \rho)=\left(\bigotimes_{i \in[k]} \mathrm{wt}_{\mathcal{A}}\left(\left.\xi\right|_{i},\left.\rho\right|_{i}\right)\right) \otimes \delta_{k}(\rho(1) \cdots \rho(k), \sigma, \rho(\varepsilon)) \tag{6.1}
\end{equation*}
$$

We call $\mathrm{wt}_{\mathcal{A}}(\xi, \rho)$ the weight of $\rho(b y \mathcal{A}$ on $\xi)$.
Now let $c \in \mathrm{C}_{\Sigma}$ be a context with $\operatorname{pos}_{z}(c)=v$ and let $\rho \in \mathrm{R}_{\mathcal{A}}(q, c)$ for some $q \in Q$. If $\rho(v)=p$, then we call $\rho$ a $(q, p)$-run on $c$ and we denote the set of all such $(q, p)$-runs by $\mathrm{R}_{\mathcal{A}}(q, c, p)$. A $(q, q)$-run is called a loop. If $\rho \in \mathrm{R}_{\mathcal{A}}(z)$ with $\rho(\varepsilon)=q$ for some $q \in Q$, then sometimes we write $\widetilde{q}$ for $\rho$.

Let $c \in \mathrm{C}_{\Sigma}, \zeta \in \mathrm{T}_{\Sigma}, v=\operatorname{pos}_{z}(c), q^{\prime}, q \in Q, \rho \in \mathrm{R}_{\mathcal{A}}\left(q^{\prime}, c, q\right)$, and $\theta \in \mathrm{R}_{\mathcal{A}}(q, \zeta)$. The combination of $\rho$ and $\theta$, denoted by $\rho[\theta]$, is the mapping $\rho[\theta]: \operatorname{pos}(c[\zeta]) \rightarrow Q$ defined for every $u \in \operatorname{pos}(c[\zeta])$ as follows: if $u=v w$ for some $w$, then we define $\rho[\theta](u)=\theta(w)$, otherwise we define $\rho[\theta](u)=\rho(u)$. Clearly, $\rho[\theta] \in \mathrm{R}_{\mathcal{A}}\left(q^{\prime}, c[\zeta]\right)$.

Let $\xi \in \mathrm{T}_{\Sigma}$ and $v \in \operatorname{pos}(\xi)$. We denote by $\left.\xi\right|^{v}$ the context $\xi[z]_{v}$ obtained by replacing the subtree of $\xi$ at $v$ by $z$ (cf. page 44). Moreover, for each $\rho \in \mathrm{R}_{\mathcal{A}}(\xi)$, we define the run $\left.\rho\right|^{v}$ on the context $\left.\xi\right|^{v}$ such that for every $w \in \operatorname{pos}\left(\left.\xi\right|^{v}\right)$ we set $\left.\rho\right|^{v}(w)=\rho(w)$.

Let $c \in \mathrm{C}_{\Sigma}, v=\operatorname{pos}_{z}(c)$, and $\rho \in \mathrm{R}_{\mathcal{A}}(c)$. We define two mappings

$$
l_{c, \rho}: \operatorname{prefix}(v) \rightarrow B \quad \text { and } \quad r_{c, \rho}: \operatorname{prefix}(v) \rightarrow B
$$

(cf. Bor04, p. 526] for deterministic wta). Intuitively, the product (6.1) which yields the element $\mathrm{wt}(c, \rho) \in B$, can be split into a left subproduct $l_{c, \rho}(\varepsilon)$ and a right subproduct $r_{c, \rho}(\varepsilon)$, where the border is given by the factor $\mathbb{1}$ coming from the weight of $z$. Figure 6.1 illustrates the mappings $l_{c, \rho}$ and $r_{c, \rho}$.

Formally, we define the well-founded set $(\operatorname{prefix}(v), \prec)$ where, for every $w_{1}, w_{2} \in \operatorname{prefix}(v)$, we let $w_{1} \prec$ $w_{2}$ if there exists an $i \in \mathbb{N}$ such that $w_{1}=w_{2} i$. Obviously, $\prec$ is well-founded and $\min _{\prec}(\operatorname{prefix}(v))=\{v\}$ (cf. the proof of Lemma4.1.1). Then we define $l_{c, \rho}$ and $r_{c, \rho}$ by induction on ( $\operatorname{prefix}(v), \prec$ ) as follows. Let $w \in \operatorname{prefix}(v)$ and assume that $c(w)=\sigma$ and $\operatorname{rk}_{\Sigma}(\sigma)=k$. Then we define

$$
\begin{gathered}
l_{c, \rho}(w)= \begin{cases}\mathbb{1} & \text { if } w=v \\
\bigotimes_{j \in[1, i-1]} \mathrm{wt}\left(\left.c\right|_{w j},\left.\rho\right|_{w j}\right) \otimes l_{c, \rho}(w i) & \text { if } w i \in \operatorname{prefix}(v) \text { for some } i \in \mathbb{N}_{+}\end{cases} \\
r_{c, \rho}(w)= \begin{cases}\mathbb{1} & \text { if } w=v \\
r_{c, \rho}(w i) \otimes \\
\bigotimes_{j \in[i+1, k]} \operatorname{wt}\left(\left.c\right|_{w j},\left.\rho\right|_{w j}\right) \otimes \delta_{k}(\rho(w 1) \cdots \rho(w k), \sigma, \rho(w))\end{cases} \\
\text { if } w i \in \operatorname{prefix}(v) \text { for some } i \in \mathbb{N}_{+}
\end{gathered}
$$

In the sequel, we abbreviate $l_{c, \rho}(\varepsilon)$ and $r_{c, \rho}(\varepsilon)$ by $l_{c, \rho}$ and $r_{c, \rho}$, respectively.
Observation 6.1.1. Let $c \in \mathrm{C}_{\Sigma}$ and $\rho \in \mathrm{R}_{\mathcal{A}}(c)$. Then $\mathrm{wt}(c, \rho)=l_{c, \rho} \otimes r_{c, \rho}$.
Lemma 6.1.2. (cf. Bor04, Lm. 5.1]) Let $c \in \mathrm{C}_{\Sigma}, \zeta \in \mathrm{T}_{\Sigma}, q^{\prime}, q \in Q, \rho \in \mathrm{R}_{\mathcal{A}}\left(q^{\prime}, c, q\right)$, and $\theta \in \mathrm{R}_{\mathcal{A}}(q, \zeta)$. Then $\operatorname{wt}(c[\zeta], \rho[\theta])=l_{c, \rho} \otimes \mathrm{wt}(\zeta, \theta) \otimes r_{c, \rho}$.

Proof. We prove the statement by induction on $\left(\mathrm{C}_{\Sigma}, \prec_{\mathrm{C}_{\Sigma}}\right)$.
I.B.: Let $c=z$. Then

$$
\begin{aligned}
\mathrm{wt}(c[\zeta], \rho[\theta]) & =\mathrm{wt}(\zeta, \theta) & & (\text { since } c[\zeta]=\zeta \text { and } \rho[\theta]=\theta) \\
& =\mathbb{1} \otimes \mathrm{wt}(\zeta, \theta) \otimes \mathbb{1} & & \\
& =l_{z, \rho} \otimes \mathrm{wt}(\zeta, \theta) \otimes r_{z, \rho} & & \left(\text { since } l_{z, \rho}=\mathbb{1} \text { and } r_{z, \rho}=\mathbb{1}\right)
\end{aligned}
$$

I.S.: Let $c=\sigma\left(\xi_{1}, \ldots, \xi_{i-1}, c^{\prime}, \xi_{i+1}, \ldots, \xi_{k}\right)$ with $k \in \mathbb{N}_{+}, i \in[k], \xi_{1}, \ldots, \xi_{k} \in \mathrm{~T}_{\Sigma}$, and $c^{\prime} \in \mathrm{C}_{\Sigma}$. Then we have

$$
\begin{align*}
& \mathrm{wt}(c[\zeta], \rho[\theta])=\left(\bigotimes_{j \in[1, i-1]} \mathrm{wt}\left(\xi_{j},\left.\rho\right|_{j}\right)\right) \otimes \mathrm{wt}\left(c^{\prime}[\zeta],\left(\left.\rho\right|_{i}\right)[\theta]\right) \\
&\left.\otimes\left(\bigotimes_{j \in[i+1, k]} \mathrm{wt}\left(\xi_{j},\left.\rho\right|_{j}\right)\right) \otimes \delta_{k}\left(\rho(1) \cdots \rho(k), \sigma, q^{\prime}\right) \quad \text { (by the definition of } c[\zeta] \text { and } \rho[\theta]\right) \\
&=\left(\bigotimes_{j \in[1, i-1]} \mathrm{wt}\left(\xi_{j},\left.\rho\right|_{j}\right)\right) \otimes l_{c^{\prime},\left.\rho\right|_{i}} \otimes \mathrm{wt}(\zeta, \theta) \otimes r_{c^{\prime},\left.\rho\right|_{i}} \\
& \otimes\left(\bigotimes_{j \in[i+1, k]} \mathrm{wt}\left(\xi_{j},\left.\rho\right|_{j}\right)\right) \otimes \delta_{k}\left(\rho(1) \cdots \rho(k), \sigma, q^{\prime}\right)  \tag{byI.H.}\\
&= \quad \text { (by I.H.) } \\
& \quad \begin{array}{l}
\left.\quad \text { (by the definition of } l_{c, \rho} \text { and } r_{c, \rho}\right) \\
\\
\quad \\
\left.\quad \text { (note that } l_{\left.c\right|_{i},\left.\rho\right|_{i}}=l_{c^{\prime},\left.\rho\right|_{i}} \text { and } r_{\left.c\right|_{i},\left.\rho\right|_{i}}=r_{c^{\prime},\left.\rho\right|_{i}}\right)
\end{array}
\end{align*}
$$

Let $c \in \mathrm{C}_{\Sigma}$. For each $n \in \mathbb{N}$, we define the $n$-th power of $c$, denoted by $c^{n}$, by induction on $\mathbb{N}$ as follows: $c^{0}=z$ and $c^{n+1}=c\left[c^{n}\right]$. Moreover, let $q \in Q$ and $\rho \in \mathrm{R}_{\mathcal{A}}(q, c, q)$ be a loop. For each $n \in \mathbb{N}$, the $n$-th power of $\rho$, denoted by $\rho^{n}$, is the run on $c^{n}$ defined by induction on $\mathbb{N}$ as follows: $\rho^{0}=\widetilde{q}$ (note that $c^{0}=z$ ) and $\rho^{n+1}=\rho\left[\rho^{n}\right]$. Next we apply the previous results to the weights of powers of loops.

Theorem 6.1.3. (cf. Bor04 Lm. 5.3]) Let $c^{\prime}, c \in \mathrm{C}_{\Sigma}$ and $\zeta \in \mathrm{T}_{\Sigma}, q^{\prime}, q \in Q, \rho^{\prime} \in \mathrm{R}_{\mathcal{A}}\left(q^{\prime}, c^{\prime}, q\right)$, $\rho \in \mathrm{R}_{\mathcal{A}}(q, c, q)$, and $\theta \in \mathrm{R}_{\mathcal{A}}(q, \zeta)$. Then, for each $n \in \mathbb{N}$,

$$
\begin{equation*}
\mathrm{wt}\left(c^{\prime}\left[c^{n}[\zeta]\right], \rho^{\prime}\left[\rho^{n}[\theta]\right]\right)=l_{c^{\prime}, \rho^{\prime}} \otimes\left(l_{c, \rho}\right)^{n} \otimes \mathrm{wt}(\zeta, \theta) \otimes\left(r_{c, \rho}\right)^{n} \otimes r_{c^{\prime}, \rho^{\prime}} \tag{6.2}
\end{equation*}
$$



Figure 6.1: Illustration of mappings $l_{c, \rho}$ and $r_{c, \rho}$ (cf. DFKV20, DFKV22])

Proof. First, by induction on $\mathbb{N}$, we prove that the following statement holds:

$$
\begin{equation*}
\text { For each } n \in \mathbb{N} \text { we have } \mathrm{wt}\left(c^{n}[\zeta], \rho^{n}[\theta]\right)=\left(l_{c, \rho}\right)^{n} \otimes \mathrm{wt}(\zeta, \theta) \otimes\left(r_{c, \rho}\right)^{n} . \tag{6.3}
\end{equation*}
$$

I.B.: Let $n=0$. Then $c^{0}=z$ and we have

$$
\begin{array}{rlrl}
\mathrm{wt}\left(c^{0}[\zeta], \rho^{0}[\theta]\right) & =\mathrm{wt}(\zeta, \theta) & & \left(\text { since } c^{0}[\zeta]=\zeta \text { and } \rho^{0}[\theta]=\theta\right) \\
& =\mathbb{1} \otimes \mathrm{wt}(\zeta, \theta) \otimes \mathbb{1} & \\
& =\left(l_{c, \rho}\right)^{0} \otimes \mathrm{wt}(\zeta, \theta) \otimes\left(r_{c, \rho}\right)^{0} & \quad\left(\text { since }\left(l_{c, \rho}\right)^{0}=\mathbb{1} \text { and }\left(r_{c, \rho}\right)^{0}=\mathbb{1} .\right)
\end{array}
$$

I.S.: We assume that the equality holds for $n$. Since $c^{n+1}=c\left[c^{n}\right]$ and $\rho^{n+1}=\rho\left[\rho^{n}\right]$, we have

$$
\begin{aligned}
\mathrm{wt}\left(c^{n+1}[\zeta], \rho^{n+1}[\theta]\right) & =l_{c, \rho} \otimes \operatorname{wt}\left(c^{n}[\zeta], \rho^{n}[\theta]\right) \otimes r_{c, \rho} \\
& =l_{c, \rho} \otimes\left(l_{c, \rho}\right)^{n} \otimes \operatorname{wt}(\zeta, \theta) \otimes\left(r_{c, \rho}\right)^{n} \otimes r_{c, \rho} \\
& =\left(l_{c, \rho}\right)^{n+1} \otimes \operatorname{wt}(\zeta, \theta) \otimes\left(r_{c, \rho}\right)^{n+1}
\end{aligned}
$$

This proves (6.3). Now let $n \in \mathbb{N}$. Then we have

$$
\begin{aligned}
\mathrm{wt}\left(c^{\prime}\left[c^{n}[\zeta]\right], \rho^{\prime}\left[\rho^{n}[\theta]\right]\right) & =l_{c^{\prime}, \rho^{\prime}} \otimes \operatorname{wt}\left(c^{n}[\zeta], \rho^{n}[\theta]\right) \otimes r_{c^{\prime}, \rho^{\prime}} \\
& =l_{c^{\prime}, \rho^{\prime}} \otimes\left(l_{c, \rho}\right)^{n} \otimes \mathrm{wt}(\zeta, \theta) \otimes\left(r_{c, \rho}\right)^{n} \otimes r_{c^{\prime}, \rho^{\prime}}
\end{aligned}
$$

Next, we prove our pumping lemma for runs of $\mathcal{A}$ on trees in $\mathrm{T}_{\Sigma}$ which are large enough. We note that B need not be commutative.

Theorem 6.1.4. (cf. Bor04, Lm. 5.5]) Let $\Sigma$ be a ranked alphabet, $\mathrm{B}=(B, \oplus, \otimes, \mathbb{O}, \mathbb{1})$ be a strong bimonoid, and $\mathcal{A}$ be a $(\Sigma, \mathrm{B})$-wta. Let $\xi \in \mathrm{T}_{\Sigma}, q^{\prime} \in Q$, and $\kappa \in \mathrm{R}_{\mathcal{A}}\left(q^{\prime}, \xi\right)$. If height $(\xi) \geq|Q|$, then there exist $c^{\prime}, c \in \mathrm{C}_{\Sigma}, \zeta \in \mathrm{T}_{\Sigma}, q \in Q, \rho^{\prime} \in \mathrm{R}_{\mathcal{A}}\left(q^{\prime}, c^{\prime}, q\right), \rho \in \mathrm{R}_{\mathcal{A}}(q, c, q)$, and $\theta \in \mathrm{R}_{\mathcal{A}}(q, \zeta)$ such that $\xi=c^{\prime}[c[\zeta]]$, $\kappa=\rho^{\prime}[\rho[\theta]]$, height $(c)>0$, height $(c[\zeta])<|Q|$, and, for each $n \in \mathbb{N}$,

$$
\mathrm{wt}\left(c^{\prime}\left[c^{n}[\zeta]\right], \rho^{\prime}\left[\rho^{n}[\theta]\right]\right)=l_{c^{\prime}, \rho^{\prime}} \otimes\left(l_{c, \rho}\right)^{n} \otimes \operatorname{wt}(\zeta, \theta) \otimes\left(r_{c, \rho}\right)^{n} \otimes r_{c^{\prime}, \rho^{\prime}}
$$

Proof. Since $\operatorname{height}(\xi) \geq|Q|$ there exist $u, w \in \mathbb{N}_{+}^{*}$ such that $u w \in \operatorname{pos}(\xi),|w|>0, \operatorname{height}\left(\left.\xi\right|_{u}\right)<|Q|$, and $\kappa(u)=\kappa(u w)$. Then we let $c^{\prime}=\left.\xi\right|^{u}, c=\left.\left(\left.\xi\right|_{u}\right)\right|^{w}, \zeta=\left.\xi\right|_{u w}$. Clearly, $\xi=c^{\prime}[c[\zeta]]$, height $(c)>0$ because $|w|>0$, and height $(c[\zeta])<|Q|$ because height $\left(\left.\xi\right|_{u}\right)<|Q|$. Moreover, we set $q=\kappa(u), \rho^{\prime}=\left.\kappa\right|^{u}$, $\rho=\left.\left(\left.\kappa\right|_{u}\right)\right|^{w}$ and $\theta=\left.\kappa\right|_{u w}$. Then the statement follows from Theorem 6.1.3.

By Example 3.2.4 for $\Sigma=\left\{\sigma^{(2)}, \alpha^{(0)}\right\}$, the weighted tree language height: $\mathrm{T}_{\Sigma} \rightarrow \mathbb{N}$ is initial algebra recognizable by a ( $\left.\Sigma, \mathrm{Nat}_{\mathrm{max},+}\right)$-wta. As an application of Theorem 6.1.3, we show that height cannot be recognized by a bu deterministic ( $\Sigma, N \operatorname{Nat}_{\mathrm{max},+}$ )-wta. In contrast, if we consider an arbitrary string ranked alphabet $\Sigma$, then height : $\mathrm{T}_{\Sigma} \rightarrow \mathbb{N}$ is in $\operatorname{bud}-\operatorname{Rec}\left(\Sigma, \operatorname{Nat}_{\text {max },+}\right)$.
Corollary 6.1.5. Bor04, Ex. 5.9] For the ranked alphabet $\Sigma=\left\{\sigma^{(2)}, \alpha^{(0)}\right\}$, we have that height $\notin$ $\operatorname{bud}-\operatorname{Rec}\left(\Sigma\right.$, Nat $\left._{\text {max },+}\right)$.

Proof. We prove by contradiction. For this, let us assume that there exists a bu deterministic $\left(\Sigma, \mathrm{Nat}_{\mathrm{max},+}\right)$-wta $\mathcal{A}=(Q, \delta, F)$ such that $\llbracket \mathcal{A} \rrbracket=$ height.

Let $\xi=\sigma\left(\zeta_{1}, \zeta_{2}\right)$ be a tree such that height $\left(\zeta_{1}\right)=|Q|$ and height $\left(\zeta_{2}\right)=2|Q|$. By Lemma 4.2.1(3)(b), there exist a $q \in Q$ and a run $\kappa \in \mathrm{R}_{\mathcal{A}}(q, \xi)$ such that

$$
\llbracket \mathcal{A} \rrbracket(\xi)=\mathrm{wt}(\xi, \kappa)+F_{q}
$$

Moreover, there exist $u \in\left(\mathbb{N}_{+}\right)^{+}$and $v \in\left(\mathbb{N}_{+}\right)^{+}$such that $u v \in \operatorname{pos}(\xi)$ and $u=1 u^{\prime}$ for some $u^{\prime} \in \operatorname{pos}\left(\zeta_{1}\right)$ (i.e., $u$ is located in $\zeta_{1}$ ), and $\kappa(u)=\kappa(u v)$. Let us introduce


Figure 6.2: Decomposition of the tree $\xi$ and the run $\kappa$.

- the contexts $c^{\prime}=\left.\xi\right|^{u}, c=\left.\left(\left.\xi\right|_{u}\right)\right|^{v}$, the tree $\zeta=\left.\xi\right|_{u v}$, as well as,
- the runs $\rho^{\prime}=\left.\kappa\right|^{u}, \rho=\left.\left(\left.\kappa\right|_{u}\right)\right|^{v}$, and $\theta=\left.\kappa\right|_{u v}$, see Figure 6.2 and
- let $\xi_{n}=c^{\prime}\left[c^{n}[\zeta]\right]$ and $\kappa_{n}=\rho^{\prime}\left[\rho^{n}[\theta]\right]$ for each $n \in \mathbb{N}$.

Note that $\xi_{1}=\xi$ and $\kappa_{1}=\kappa$. By Theorem 6.1.3, we have

$$
\mathrm{wt}\left(\xi_{n}, \kappa_{n}\right)=l_{c^{\prime}, \rho^{\prime}}+n \cdot l_{c, \rho}+\mathrm{wt}(\zeta, \theta)+n \cdot r_{c, \rho}+r_{c^{\prime}, \rho^{\prime}}
$$

for each $n \in \mathbb{N}$. Since height $(\xi)=2|Q|+1$, we have

$$
\llbracket \mathcal{A} \rrbracket(\xi)=l_{c^{\prime}, \rho^{\prime}}+l_{c, \rho}+\mathrm{wt}(\zeta, \theta)+r_{c, \rho}+r_{c^{\prime}, \rho^{\prime}}+F_{q}=2|Q|+1
$$

Thus, each of $l_{c^{\prime}, \rho^{\prime}}, l_{c, \rho}, \mathrm{wt}(\zeta, \theta), r_{c, \rho}, r_{c^{\prime}, \rho^{\prime}}$, and $F_{q}$ is in $\mathbb{N}$. Therefore $\operatorname{wt}\left(\xi_{n}, \kappa_{n}\right) \neq-\infty$ for each $n \in \mathbb{N}$ and thus, by Lemma 4.2.1 (3)(b), we have

$$
\llbracket \mathcal{A} \rrbracket\left(\xi_{n}\right)=l_{c^{\prime}, \rho^{\prime}}+n \cdot l_{c, \rho}+\mathrm{wt}(\zeta, \theta)+n \cdot r_{c, \rho}+r_{c^{\prime}, \rho^{\prime}}+F_{q}
$$

for each $n \in \mathbb{N}$. It follows that $l_{c, \rho} \neq 0$ or $r_{c, \rho} \neq 0$ because otherwise height $\left(\xi_{n}\right)$ would be the same number for each $n \in \mathbb{N}$. But then $\llbracket \mathcal{A} \rrbracket\left(\xi_{0}\right)<\llbracket \mathcal{A} \rrbracket\left(\xi_{1}\right)$, which is a contradiction because height $\left(\xi_{0}\right)=\operatorname{height}\left(\xi_{1}\right)$.

Contrary to Corollary 6.1.5, for each string ranked alphabet $\Sigma$, the weighted tree language height : $\mathrm{T}_{\Sigma} \rightarrow \mathbb{N}$ is in $\operatorname{bud}-\operatorname{Rec}\left(\Sigma, \operatorname{Nat}_{\mathrm{max},+}\right)$. To see this, let $\Sigma$ be a string ranked alphabet and assume that $\Sigma^{(0)}=\{\alpha\}$. Then we consider the bu deterministic $\left(\Sigma, \operatorname{Nat}_{\mathrm{max},+}\right)$-wta $\mathcal{A}=(\{q\}, \delta, F)$ with $\delta_{0}(\varepsilon, \alpha, q)=0$, and $\delta_{1}(q, \gamma, q)=1$ for each $\gamma \in \Sigma^{(1)} ;$ moreover, we let $F_{q}=0$. It is easy to see that $\llbracket \mathcal{A} \rrbracket^{\text {run }}=$ height.

As a corollary of Theorem 6.1.4 we prove a pumping lemma for the supports of weighted tree languages recognizable by wta over positive strong bimonoids.

Corollary 6.1.6. Let $\Sigma$ be a ranked alphabet, B be a positive strong bimonoid, and $L \subseteq \mathrm{~T}_{\Sigma}$. If $L \in$ $\operatorname{supp}\left(\operatorname{Rec}^{\text {run }}(\Sigma, \mathrm{B})\right)$, then there exists a $p \in \mathbb{N}_{+}$such that for each $\xi \in L$ with height $(\xi) \geq p$, there exist $c^{\prime}, c \in \mathrm{C}_{\Sigma}, \zeta \in \mathrm{T}_{\Sigma}$ such that

- $\xi=c^{\prime}[c[\zeta]]$,
- height $(c)>0$ and $\operatorname{height}(c[\zeta])<p$, and
- for each $n \in \mathbb{N}$, we have $c^{\prime}\left[c^{n}[\zeta]\right] \in L$.

Proof. Since $L \in \operatorname{supp}\left(\operatorname{Rec}^{\text {run }}(\Sigma, \mathrm{B})\right)$, there exists a $(\Sigma, \mathrm{B})$-wta $\mathcal{A}=(Q, \delta, F)$ such that $L=\operatorname{supp}\left(\llbracket \mathcal{A} \rrbracket^{\text {run }}\right)$. Let $p=|Q|$.

Let $\xi \in L$ with height $(\xi) \geq p$. There there exist $q^{\prime} \in Q$ and $\kappa \in \mathrm{R}_{\mathcal{A}}\left(q^{\prime}, \xi\right)$ such that wt $\mathcal{A}_{\mathcal{A}}(\xi, \kappa) \otimes F_{q^{\prime}} \neq \mathbb{0}$. Thus wt $\mathcal{A}(\xi, \kappa) \neq \mathbb{0}$ and $F_{q^{\prime}} \neq \mathbb{0}$.

By Theorem 6.1.4 there exist $c^{\prime}, c \in \mathrm{C}_{\Sigma}, \zeta \in \mathrm{T}_{\Sigma}, q \in Q, \rho^{\prime} \in \mathrm{R}_{\mathcal{A}}\left(q^{\prime}, c^{\prime}, q\right), \rho \in \mathrm{R}_{\mathcal{A}}(q, c, q)$, and $\theta \in \mathrm{R}_{\mathcal{A}}(q, \zeta)$ such that $\xi=c^{\prime}[c[\zeta]], \kappa=\rho^{\prime}[\rho[\theta]], \operatorname{height}(c)>0$, $\operatorname{height}(c[\zeta])<p$, and, for each $n \in \mathbb{N}$,
$\mathrm{wt}\left(c^{\prime}\left[c^{n}[\zeta]\right], \rho^{\prime}\left[\rho^{n}[\theta]\right]\right)=l_{c^{\prime}, \rho^{\prime}} \otimes\left(l_{c, \rho}\right)^{n} \otimes \operatorname{wt}(\zeta, \theta) \otimes\left(r_{c, \rho}\right)^{n} \otimes r_{c^{\prime}, \rho^{\prime}}$. Together with $\mathrm{wt}_{\mathcal{A}}(\xi, \kappa) \neq \mathbb{O}$, this implies (for $n=1$ ) that each of the three values $l_{c^{\prime}, \rho^{\prime}}, \mathrm{wt}(\zeta, \theta)$, and $r_{c, \rho}$ is different from $\mathbb{0}$. Since B is zero-divisor free, we obtain that

$$
l_{c^{\prime}, \rho^{\prime}} \otimes\left(l_{c, \rho}\right)^{n} \otimes \mathrm{wt}(\zeta, \theta) \otimes\left(r_{c, \rho}\right)^{n} \otimes r_{c^{\prime}, \rho^{\prime}} \neq \mathbb{O}
$$

for each $n \in \mathbb{N}$, and hence $\operatorname{wt}\left(c^{\prime}\left[c^{n}[\zeta]\right], \rho^{\prime}\left[\rho^{n}[\theta]\right]\right) \neq \mathbb{O}$ (using Theorem6.1.4 again). Since $F_{q^{\prime}} \neq \mathbb{O}$ and B is zero-divisor free, we obtain that

$$
\operatorname{wt}\left(c^{\prime}\left[c^{n}[\zeta]\right], \rho^{\prime}\left[\rho^{n}[\theta]\right]\right) \otimes F_{q^{\prime}} \neq \mathbb{O}
$$

Since B is zero-sum free, we obtain:

$$
\left.\llbracket \mathcal{A} \rrbracket^{\mathrm{run}}\left(c^{\prime}\left[c^{n}[\zeta]\right]\right]\right)=\bigoplus_{\nu \in \mathrm{R}_{\mathcal{A}}(\xi)} \mathrm{wt}\left(c^{\prime}\left[c^{n}[\zeta]\right], \nu\right) \otimes F_{\nu(\varepsilon)} \neq \mathbb{O}
$$

Thus, for each $n \in \mathbb{N}$, we have that $c^{\prime}\left[c^{n}[\zeta]\right] \in L$.
By Corollary 3.4.2 $\operatorname{supp}\left(\operatorname{Rec}^{\text {run }}(\Sigma\right.$, Boole $\left.)\right)=\operatorname{Rec}(\Sigma)$. Thus, for the case that B is the Boolean semiring, Corollary 6.1.6 shows a slight improvement of the pumping lemma for recognizable tree languages [GS84, Lm. 2.10.1] (because GS84, Lm. 2.10.1] does not show the condition height $(c[\zeta])<p$ ). Hence in case $\mathrm{B}=$ Boole, by using the contraposition of Corollary 6.1.6, we can give a shorter proof of that a certain tree language is not recognizable, than by using [GS84, Lm. 2.10.1]. The contraposition is the following statement:

Let $L \subseteq \mathrm{~T}_{\Sigma}$. If for each $p \in \mathbb{N}_{+}$, there exists $\xi \in L$ with height $(\xi) \geq p$ such that for every $c^{\prime}, c \in \mathrm{C}_{\Sigma}, \zeta \in \mathrm{T}_{\Sigma}$ with $\xi=c^{\prime}[c[\zeta]]$, height $(c)>0$, and height $(c[\zeta])<p$, there exists an $n \in \mathbb{N}$ such that $c^{\prime}\left[c^{n}[\zeta]\right] \notin L$, then $L$ is not recognizable.

Let us give an example of this application (cf. [HMU14, Ex. 4.2]).
Example 6.1.7. We consider the string ranked alphabet $\Sigma=\left\{\sigma^{(1)}, \gamma^{(1)}, \alpha^{(0)}\right\}$ and the $\Sigma$-tree language

$$
L=\left\{\xi \in \mathrm{T}_{\Sigma}| | \operatorname{pos}_{\gamma}(\xi)\left|=\left|\operatorname{pos}_{\sigma}(\xi)\right|\right\}\right.
$$

Now we apply the contraposition of Corollary 6.1.6 (for $\mathrm{B}=$ Boole). For this let $p \in \mathbb{N}_{+}$. We consider the tree $\xi=\sigma^{p+1} \gamma^{p+1} \alpha$. Clearly, $\xi \in L$ and $\operatorname{height}(\xi) \geq p$.

Now we consider an arbitrary decomposition of $\xi$ of the form $\xi=c^{\prime}[c[\zeta]]$ for some $c^{\prime}, c \in \mathrm{C}_{\Sigma}, \zeta \in \mathrm{T}_{\Sigma}$ such that height $(c)>0$ and height $(c[\zeta])<p$. By our choice of $\xi$, we have that $c=\gamma^{k}\left(z_{1}\right)$ for some $k \geq 1$ (we recall that height $\left(z_{1}\right)=0$ ).

Now we choose $n=2$. Obviously, $c^{\prime}\left[c^{2}[\zeta]\right]$ contains more $\gamma$ s than $\sigma$ s, hence $c^{\prime}\left[c^{2}[\zeta]\right] \notin L$. Hence, by the contraposition of Corollary 6.1.6 (for $\mathrm{B}=$ Boole), the tree language $L$ is not recognizable.

It will turn out later (when Theorem 18.2 .4 is available), that $\operatorname{supp}\left(\operatorname{Rec}^{\text {run }}(\Sigma, B)\right)=\operatorname{Rec}(\Sigma)$ for each positive semiring $B$, hence Corollary 6.1.6 is the same slight improvement of the classical pumping lemma for recognizable $\Sigma$-tree languages [GS84, Lm. 2.10.1] which we described for the case $\mathrm{B}=$ Boole.

### 6.2 Pumping lemma for wta over fields

Here we deal with weighted tree automata over fields and a pumping lemma which guarantees the existence of infinitely many trees in the support of the semantics of such wta. The pumping lemma is based on Reu80, BR82].

Let B be a field and $(\mathrm{V}, \mu, \gamma)$ be a $(\Sigma, \mathrm{B})$-multilinear representation with $\mathrm{V}=\left(B^{Q},+, \widetilde{0}\right)$. Then V is a $|Q|$-dimensional B -vector space. Vice versa, let V be a $\kappa$-dimensional B -vector space with $\kappa \in \mathbb{N}$. According to our convention on representing vector spaces (cf. page 39), we have $\mathrm{V}=\left(B^{\kappa},+, \mathbb{O}_{\kappa}\right)$. This justifies to use, as first component of a ( $\Sigma, \mathrm{B}$ )-multilinear representation, also an arbitrary finite dimensional $B$-vector space V . We will use this point of view in this section.

In the rest of this section, we let B be a field and $(\mathrm{V}, \mu, \gamma)$ be a $(\Sigma, \mathrm{B})$-multilinear representation where $\mathrm{V}=(V,+, 0)$ is a $\kappa$-dimensional B -vector space for some $\kappa \in \mathbb{N}_{+}$, unless specified otherwise.

We recall from Section 3.6.1 that $(V, \mu)$ is a $\Sigma$-algebra and that $\mathrm{h}_{\mathrm{V}}$ is the unique $\Sigma$-algebra homomorphism from $\mathrm{T}_{\Sigma}$ to $(V, \mu)$. Moreover, $(\mathrm{V}, \mu, \gamma)$ recognizes the weighted tree language $r: \mathrm{T}_{\Sigma} \rightarrow B$ where $r(\xi)=\gamma\left(\mathrm{h}_{\mathrm{V}}(\xi)\right)$ for every $\xi \in \mathrm{T}_{\Sigma}$. We also recall that $\mathcal{L}(\mathrm{V}, \mathrm{V})$ is the set of all linear mappings $f: V \rightarrow V$ from V to V .

### 6.2.1 The set of contexts viewed as monoid

Each tree $\xi$ can be considered as a sequence $c$ of contexts which are substituted consecutively into each other, and a nullary symbol $\alpha$ for turning the resulting context into $\xi$. The monoid $\left(\mathrm{C}_{\Sigma}, \circ_{z}, z\right)$ formalizes the substitution of contexts, where we also denote by $\circ_{z}$ the restriction of the binary operation $\circ_{z}$ on $\mathrm{T}_{\Sigma}(\{z\})$ (cf. page 48) to $\mathrm{C}_{\Sigma}$. In the next lemma we prove that $\left(\mathrm{C}_{\Sigma}, \circ_{z}, z\right)$ is freely generated by comparing it with the monoid $\left(\left(\mathrm{eC}_{\Sigma}\right)^{*}, \cdot, \varepsilon\right)$. In the sequel we abbreviate $\left(\mathrm{eC}_{\Sigma}\right)^{*}$ by $\mathrm{eC}_{\Sigma}^{*}$.

Lemma 6.2.1. BR82, Prop. 9.1] The monoid $\left(\mathrm{C}_{\Sigma}, \circ_{z}, z\right)$ is freely generated by $\mathrm{e}_{\Sigma}$ over the set of all monoids.

Proof. We prove that the monoid $\left(\mathrm{C}_{\Sigma}, \mathrm{o}_{z}, z\right)$ is isomorphic to the monoid $\left(\mathrm{eC}_{\Sigma}^{*}, \cdot, \varepsilon\right)$. Since $\left(\mathrm{eC}_{\Sigma}^{*}, \cdot, \varepsilon\right)$ is freely generated by $\mathrm{eC}_{\Sigma}$ over the set of all monoids, this implies the statement of the lemma.

We define the mapping $\psi: \mathrm{eC}_{\Sigma} \rightarrow \mathrm{C}_{\Sigma}$ with $\psi(e)=e$ for each $e \in \mathrm{eC}_{\Sigma}$. Since $\left(\mathrm{eC}_{\Sigma}^{*}, \cdot, \varepsilon\right)$ is freely generated by $\mathrm{eC}_{\Sigma}$ over the set of all monoids, there exists a unique monoid homomorphism $\llbracket . \rrbracket: \mathrm{eC}_{\Sigma}^{*} \rightarrow \mathrm{C}_{\Sigma}$ which extends $\psi$. For each $c \in \mathrm{eC}_{\Sigma}^{*}$, we write $\llbracket c \rrbracket$ instead of $\llbracket . \rrbracket(c)$. Then, for every $n \in \mathbb{N}_{+}$and $e_{1}, \ldots, e_{n} \in \mathrm{eC}_{\Sigma}$ we have

$$
\llbracket e_{1} \cdots e_{n} \rrbracket=e_{1} \circ_{z} \cdots \circ_{z} e_{n} \text { and } \llbracket \varepsilon \rrbracket=z
$$

We prove that the mapping 【.』 is surjective. For this, by induction on $\left(\mathrm{C}_{\Sigma}, \prec_{\mathrm{C}_{\Sigma}}\right)$ (cf. page 123), we prove that the following statement holds:

$$
\begin{equation*}
\text { For each } c \in \mathrm{C}_{\Sigma} \text {, we have } c \in \operatorname{im}(\mathbb{I} \cdot \rrbracket) \text {. } \tag{6.4}
\end{equation*}
$$

I.B.: Let $c=z$. Then the statement holds because $\llbracket \varepsilon \rrbracket=z$.
I.S.: Let $c=e\left[c^{\prime}\right]$ for some $e \in \mathrm{eC}_{\Sigma}$ and $c^{\prime} \in \mathrm{C}_{\Sigma}$. By the I.H. we have $c^{\prime} \in \operatorname{im}(\llbracket . \rrbracket)$ and thus there exists a $d \in \mathrm{eC}_{\Sigma}^{*}$ such that $\llbracket d \rrbracket=c^{\prime}$. Since $\llbracket e d \rrbracket=\llbracket e \rrbracket \circ_{z} \llbracket d \rrbracket=e \circ_{z} c^{\prime}=c$, we have that $c \in \operatorname{im}(\llbracket \cdot \rrbracket)$.

Hence (6.4) holds and $\llbracket . \rrbracket$ is surjective.
Finally, we prove that $\llbracket . \rrbracket$ is injective. Let $e_{1}, \ldots, e_{n} \in \mathrm{eC}_{\Sigma}$ and $e_{1}^{\prime}, \ldots, e_{m}^{\prime} \in \mathrm{eC}_{\Sigma}$ such that $\llbracket e_{1} \cdots e_{n} \rrbracket=$ $\llbracket e_{1}^{\prime} \cdots e_{m}^{\prime} \rrbracket$. Since $\operatorname{depth}\left(\llbracket e_{1} \cdots e_{n} \rrbracket\right)=n$ and $\operatorname{depth}\left(\llbracket e_{1}^{\prime} \cdots e_{m}^{\prime} \rrbracket\right)=m$, it follows that $n=m$. Now assume that there exists $i \in[n]$ such that $e_{i} \neq e_{i}^{\prime}$. Then there exists a $w \in \operatorname{pos}\left(e_{i}\right) \cap \operatorname{pos}\left(e_{i}^{\prime}\right)$ such that $e_{i}(w) \neq e_{i}^{\prime}(w)$. Using this fact, it is easy to see that $\llbracket e_{1} \cdots e_{n} \rrbracket \neq \llbracket e_{1}^{\prime} \cdots e_{m}^{\prime} \rrbracket$. Thus $e_{i}=e_{i}^{\prime}$ for each $i \in[n]$. Hence, $\llbracket . \rrbracket$ is injective.

So we have proved that $\llbracket . \rrbracket$ is an isomorphism from $\left(\mathrm{eC}_{\Sigma}^{*}, \cdot, \varepsilon\right)$ to $\left(\mathrm{C}_{\Sigma}, \circ_{z}, z\right)$.

Each context $c \in \mathrm{C}_{\Sigma}$ determines a linear mapping in the following way. We define the mapping $\theta: \mathrm{eC}_{\Sigma} \rightarrow \mathcal{L}(\mathrm{V}, \mathrm{V})$ for every $e \in \mathrm{eC}_{\Sigma}$ and $v \in V$ by

$$
(\theta(e))(v)=\mu(\sigma)\left(\mathrm{h} v\left(\xi_{1}\right), \ldots, \operatorname{hv}\left(\xi_{i-1}\right), v, \operatorname{hv}\left(\xi_{i+1}\right), \ldots, \mathrm{h} v\left(\xi_{k}\right)\right)
$$

if $e$ has the form $\sigma\left(\xi_{1}, \ldots, \xi_{i-1}, z, \xi_{i+1}, \ldots, \xi_{k}\right)$. Then $\theta(e)$ is an endomorphism on V because $\mu(\sigma)$ is multilinear.

By Lemma 6.2.1, the monoid $\left(\mathrm{C}_{\Sigma}, \mathrm{o}_{z}, z\right)$ is freely generated by $\mathrm{eC}_{\Sigma}$. Thus there exists a unique monoid homomorphism (.) from the monoid $\left(\mathrm{C}_{\Sigma}, \circ_{z}, z\right)$ to the monoid $\left.\left(\mathcal{L}(\mathrm{V}, \mathrm{V}), \circ, \mathrm{id}_{V}\right)\right)$ which extends $\theta$. Instead of $().(c)$ we write $(c)$ for each $c \in \mathrm{C}_{\Sigma}$.

In the next example we illustrate the evaluation of $(c)(v)$ for a multilinear representation which is related to a particular ( $\Sigma$, Rat)-wta. Before that, we would like to discuss a phenomenon: the reversing of the roles of rows and columns when starting from weighted string automata and moving towards multilinear representations.

Let $\mathcal{A}=(Q, \lambda, \mu, \tau)$ be a ( $\Gamma, \mathrm{B}$ )-wsa (where we have used untypically $\tau$ as final weight mapping because we use $\gamma$ as part of the multilinear representation). Then by Lemma 3.3.1 we have $\mathrm{h}_{\mathcal{A}}(w)=\lambda \cdot \mu(w)$ for each $w \in \Gamma^{*}$. For instance, let $\Gamma=\{a, b\}$. Since $\mu$ is a monoid homomorphism, we have

$$
\lambda \cdot \mu(a b b)=\lambda \cdot \mu(a) \cdot \mu(b) \cdot \mu(b)
$$

The intuition behind the matrix, say, $\mu(a)$, is that, for each row $q \in Q$ and each column $p \in Q$, the value $\mu(a)_{q, p}$ is the weight of the transition from state $q$ to state $p$ when reading $a$.

Now we might turn $\mathcal{A}$ into the related $\left(\Gamma_{e}, B\right.$-wta $\mathcal{B}$ (using tree $e_{e}$ and Lemma 3.3.3) and then we can look at the $\left(\Gamma_{e}, \mathrm{~B}\right)$-multilinear representation which is related to $\mathcal{B}$ (cf. Figure 6.3). When turning $\mathcal{A}$ into $\mathcal{B}$, the string $a b b$ is turned into the tree $b(b(a(e)))$. Obviously, when reading the string $b(b(a(e)))$ from left to right, the order of $a$ and the $b$ 's is reversed with respect to the order of these symbols in the string $a b b$. Hence, the order of the matrices $\mu(a)$ and $\mu(b)$ is also reversed. In order to keep the correct value, the matrices $\mu(a)$ and $\mu(b)$ have to be transposed. (Since we do not distinguish between row- and column-vectors, we do not have to transpose $\lambda$.) Thus,

$$
\lambda \cdot \mu(a) \cdot \mu(b) \cdot \mu(b)=\mu(b)^{\mathrm{T}} \cdot \mu(b)^{\mathrm{T}} \cdot \mu(a)^{\mathrm{T}} \cdot \lambda=\ b(z) D(0 b(z) D(0 a(z) D(\lambda)))
$$

where the linear mappings $(a(z))$ and $\left(b(z) D\right.$ are represented by the matrices $\mu(a)^{\mathrm{T}}$ and $\mu(b)^{\mathrm{T}}$, respectively. In particular, an entry $\left(\mu(b)^{\mathrm{T}}\right)_{q, p}$ shows the transition weight on the elementary context $b(z)$ when starting at the $z$-labeled leaf in the $q$-column of $\mu(b)$ and ending at the $b$-labeled root in the $p$-row of $\mu(b)$. In this sense the order of rows and columns is reversed.
Example 6.2.2. Here we consider the mapping $\#_{\sigma(., \alpha)}: \mathrm{T}_{\Sigma} \rightarrow \mathbb{N}$ and the ( $\left.\Sigma, \mathrm{Nat}\right)$-wta $\mathcal{A}$ as defined in Example 3.2.11 with two slight modifications. One modification is that we replace in $\Sigma$ the unary symbol $\gamma \in \Sigma^{(1)}$ by the unary symbol $\omega$ because we use $\gamma$ as part of the multilinear representation. The other modification is that we view $\mathcal{A}$ as a ( $\Sigma$, Rat)-wta, which is possible because Rat is an extension of Nat (cf. Section 3.7). Thus, $\Sigma=\left\{\sigma^{(2)}, \omega^{(1)}, \alpha^{(0)}\right\}$ and

$$
\begin{aligned}
\#_{\sigma(., \alpha)}: \mathrm{T}_{\Sigma} & \rightarrow \mathbb{Q} \\
\xi & \mapsto|U(\xi)| \text { for every } \xi \in T_{\Sigma}
\end{aligned}
$$

where $U(\xi)=\{u \in \operatorname{pos}(\xi) \mid \xi(u)=\sigma, \xi(u 2)=\alpha\}$. We recall that $\mathcal{A}=(Q, \delta, F)$, where

- $Q=\{\perp, a, f\}$ (intuitively, $\perp$ ignores occurrences of the pattern, $a$ detects an $\alpha$-labeled leaf, and $f$ reports "pattern found" up to the root),
- for every $q_{1}, q_{2}, q \in Q$ we define

$$
\begin{gathered}
\delta_{0}(\varepsilon, \alpha, q)=\left\{\begin{array}{ll}
1 & \text { if } q \in\{\perp, a\} \\
0 & \text { otherwise }
\end{array} \quad \delta_{1}\left(q_{1}, \omega, q\right)= \begin{cases}1 & \text { if } q_{1} q \in\{\perp \perp, f f\} \\
0 & \text { otherwise }\end{cases} \right. \\
\delta_{2}\left(q_{1} q_{2}, \sigma, q\right)= \begin{cases}1 & \text { if } q_{1} q_{2} q \in\{\perp \perp \perp, \perp a f, \perp f f, f \perp f\} \\
0 & \text { otherwise }\end{cases}
\end{gathered}
$$



Figure 6.3: The change of roles of rows and columns when starting from wsa and moving towards multilinear representations; (1) from strings to trees and (2) reversing order and transposing the matrices.

- $F_{\perp}=F_{a}=0$ and $F_{f}=1$.

Now we define the ( $\Sigma$, Rat)-multilinear representation $\left(\mathrm{V}, \delta_{\mathcal{A}}, \gamma\right)$ where

- $\mathrm{V}=\left(\mathbb{Q}^{Q},+, 0_{3}\right)$ is a Rat-vector space, where the components of each $v \in \mathbb{Q}^{Q}$ are ordered according to the sequence $(\perp, a, f)$, and $0_{3}=(0,0,0)$ and
- $\gamma(v)=v_{f}$.

Then $\left(\mathrm{V}, \delta_{\mathcal{A}}, \gamma\right)$ and $\mathcal{A}$ are related. Hence, by Lemma 3.6.5 for each $\xi \in \mathrm{T}_{\Sigma}$, we have $\mathrm{h}_{\mathcal{A}}(\xi)=\mathrm{h}_{\mathrm{V}}(\xi)$ and $\llbracket \mathcal{A} \rrbracket^{\text {init }}(\xi)=\gamma\left(\mathrm{h}_{\mathcal{A}}(\xi)\right)=\#_{\sigma(., \alpha)}(\xi)$.

For each $\zeta \in \mathrm{C}_{\Sigma}$, we define the $(Q \times Q)$-matrix :

$$
M_{\zeta}=\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & a(\zeta) & 0 \\
\#_{\sigma(., \alpha)}(\zeta) & b(\zeta) & 1
\end{array}\right)
$$

where
(a) the rows and columns are ordered according to the sequence $(\perp, a, f)$,
(b) we have extended $\#_{\sigma(., \alpha)}$ in the obvious way such that it is also applicable to contexts, and
(c) $a(\zeta)=1$ if $\zeta=z$, and 0 otherwise; and $b(\zeta)=1$ if there exists a position $w \in \operatorname{pos}(\zeta)$ such that $z=\zeta(w 2)$, and 0 otherwise.
Intuitively, for each $q, p \in Q$, the entry $\left(M_{\zeta}\right)_{q, p}$ is the sum of the weights of all $(q, p)$-runs of $\mathcal{A}$ on $\zeta$. In particular, $M_{z}=\mathrm{M}_{1}$.

In Figure 6.4 we illustrate the evaluation of $\ \zeta)(v)$ for $\zeta=\sigma(\sigma(\sigma(\alpha, z), \alpha), \alpha)$ and $v=(1,0,2)$ (where, e.g., $(1,0,2)=\mathrm{h}_{\mathcal{A}}(\xi)$ with $\left.\xi=\sigma(\sigma(\alpha, \alpha), \alpha)\right)$.

By induction on ( $\mathrm{C}_{\Sigma}, \prec_{\mathrm{C}_{\Sigma}}$ ), we prove that the following statement holds:

$$
\begin{equation*}
\text { For each } \zeta \in \mathrm{C}_{\Sigma} \text { and } v \in \mathbb{Q}^{Q}, \text { we have } \backslash \zeta \emptyset(v)=M_{\zeta} \cdot v . \tag{6.5}
\end{equation*}
$$

I.B.: Let $\zeta=z$ and $v \in \mathbb{Q}^{Q}$. Then $(\zeta)(v)=(z z)(v)=\operatorname{id}_{\mathbb{Q}^{Q}}(v)=v=M_{\zeta} \cdot v$.
I.S.: Let $\zeta=e\left[\zeta^{\prime}\right]$ for some $e \in \mathrm{e}_{\Sigma}$ and $\zeta^{\prime} \in \mathrm{C}_{\Sigma}$. Let $e=\sigma(z, \xi)$ for some $\xi \in \mathrm{T}_{\Sigma}$. Using the I.H., we obtain:

$$
\left(\zeta \emptyset(v)=\left(e\left[\zeta^{\prime}\right] D(v)=(e\rangle\left(\Omega \zeta^{\prime} D(v)\right)=(e\rangle\left(M_{\zeta^{\prime}} \cdot v\right)=\delta_{\mathcal{A}}(\sigma)\left(M_{\zeta^{\prime}} \cdot v, \mathrm{~h}_{\mathcal{A}}(\xi)\right)\right.\right.
$$

$$
M_{\zeta} \cdot h_{\delta_{\mathcal{A}}}(\xi): \quad\left(\begin{array}{l}
1 \\
0 \\
4
\end{array}\right)
$$

$$
M_{\zeta}:\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & 0 & 0 \\
2 & 1 & 1
\end{array}\right) \quad h_{\delta_{\mathcal{A}}}(\xi):\left(\begin{array}{l}
1 \\
0 \\
2
\end{array}\right)
$$



Figure 6.4: An illustration of the evaluation of $(\zeta \zeta)(v)$ for $\zeta=\sigma(\sigma(\sigma(\alpha, z), \alpha), \alpha)$ and $v=(1,0,2)$.

We recall that, by (3.19), for each $\xi \in \mathrm{T}_{\Sigma}$, we have $\mathrm{h}_{\mathcal{A}}(\xi)_{\perp}=1, \mathrm{~h}_{\mathcal{A}}(\xi)_{a}=c(\xi)$ and $\mathrm{h}_{\mathcal{A}}(\xi)_{f}=\#{ }_{\sigma(\cdot, \alpha)}(\xi)$, where $c(\xi)=1$ if $\xi=\alpha$ and $c(\xi)=0$ otherwise. Then we obtain:

$$
\left.\begin{array}{rl}
(\zeta)(v)_{\perp} & =\delta_{\mathcal{A}}(\sigma)\left(M_{\zeta^{\prime}} \cdot v, \mathrm{~h}_{\mathcal{A}}(\xi)\right)_{\perp}=\left(M_{\zeta^{\prime}} \cdot v\right)_{\perp} \cdot \mathrm{h}_{\mathcal{A}}(\xi)_{\perp} \cdot 1=v_{\perp}=\left(M_{\zeta} \cdot v\right)_{\perp} \\
(\zeta)(v)_{a} & \left.=\delta_{\mathcal{A}}(\sigma)\left(M_{\zeta^{\prime}} \cdot v, \mathrm{~h}_{\mathcal{A}}(\xi)\right)_{a}=0=\left(M_{\zeta} \cdot v\right)_{a} \quad \text { (the last equality holds because } a(\zeta)=0\right) \\
\left(\zeta \emptyset(v)_{f}\right. & =\delta_{\mathcal{A}}(\sigma)\left(M_{\zeta^{\prime}} \cdot v, \mathrm{~h}_{\mathcal{A}}(\xi)\right)_{f} \\
& =\left(M_{\zeta^{\prime}} \cdot v\right)_{\perp} \cdot \mathrm{h}_{\mathcal{A}}(\xi)_{a}+\left(M_{\zeta^{\prime}} \cdot v\right)_{\perp} \cdot \mathrm{h}_{\mathcal{A}}(\xi)_{f}+\left(M_{\zeta^{\prime}} \cdot v\right)_{f} \cdot \mathrm{~h}_{\mathcal{A}}(\xi)_{\perp} \\
& =\left(M_{\zeta^{\prime}} \cdot v\right)_{\perp} \cdot\left(c(\xi)+\#{ }_{\sigma(\cdot, \alpha)}(\xi)\right)+\left(M_{\zeta^{\prime}} \cdot v\right)_{f} \\
& =\left(\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & 0 & 0 \\
c(\xi)+\#\left(M_{\zeta^{\prime}} \cdot v\right)
\end{array}\right)_{f(., \alpha)}(\xi)\right. \\
0 & 1
\end{array}\right) \quad \begin{gathered}
\\
\\
\\
\\
=\left(M_{e} \cdot\left(M_{\zeta^{\prime}} \cdot v\right)\right)_{f}={ }^{(*)}\left(\left(M_{e} \cdot M_{\zeta^{\prime}}\right) \cdot v\right)_{f}=\left(M_{e\left[\zeta^{\prime}\right]} \cdot v\right)_{f}=\left(M_{\zeta} \cdot v\right)_{f},
\end{gathered}
$$

where we have used (2.17) at the equality marked by $(*)$ (note that $\mathbb{Q}$ is distributive).
Now let $e=\sigma(\xi, z)$ for some $\xi \in \mathrm{T}_{\Sigma}$. Using the I.H., we obtain:

$$
\langle\zeta)(v)=\left(e\left[\zeta^{\prime}\right] D(v)=(e \ell)\left(\left\langle\zeta^{\prime} D(v)\right)=(e)\left(M_{\zeta^{\prime}} \cdot v\right)=\delta_{\mathcal{A}}(\sigma)\left(\mathrm{h}_{\mathcal{A}}(\xi), M_{\zeta^{\prime}} \cdot v\right)\right.\right.
$$

Then, using (3.19), we obtain:

$$
\begin{aligned}
(\zeta \zeta)(v)_{\perp} & =\delta_{\mathcal{A}}(\sigma)\left(\mathrm{h}_{\mathcal{A}}(\xi), M_{\zeta^{\prime}} \cdot v\right)_{\perp}=\mathrm{h}_{\mathcal{A}}(\xi)_{\perp} \cdot\left(M_{\zeta^{\prime}} \cdot v\right)_{\perp} \cdot 1=v_{\perp}=\left(M_{\zeta} \cdot v\right)_{\perp} \\
(\zeta)(v)_{a} & \left.=\delta_{\mathcal{A}}(\sigma)\left(\mathrm{h}_{\mathcal{A}}(\xi), M_{\zeta^{\prime}} \cdot v\right)_{a}=0=\left(M_{\zeta} \cdot v\right)_{a} \quad \quad \quad \text { (because } a(\zeta)=0\right) \\
(\zeta)(v)_{f} & =\delta_{\mathcal{A}}(\sigma)\left(\mathrm{h}_{\mathcal{A}}(\xi), M_{\zeta^{\prime}} \cdot v\right)_{f} \\
& =\mathrm{h}_{\mathcal{A}}(\xi)_{\perp} \cdot\left(M_{\zeta^{\prime}} \cdot v\right)_{a}+\mathrm{h}_{\mathcal{A}}(\xi)_{\perp} \cdot\left(M_{\zeta^{\prime}} \cdot v\right)_{f}+\mathrm{h}_{\mathcal{A}}(\xi)_{f} \cdot\left(M_{\zeta^{\prime}} \cdot v\right)_{\perp}
\end{aligned}
$$

$$
\begin{aligned}
& =\left(\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & 0 & 0 \\
\# \sigma(., \alpha) & (\xi) & 1
\end{array}\right) \cdot\left(M_{\zeta^{\prime}} \cdot v\right)\right)_{f} \\
& =\left(M_{e} \cdot\left(M_{\zeta^{\prime}} \cdot v\right)\right)_{f}={ }^{(*)}\left(\left(M_{e} \cdot M_{\zeta^{\prime}}\right) \cdot v\right)_{f}=\left(M_{e\left[\zeta^{\prime}\right]} \cdot v\right)_{f}=\left(M_{\zeta} \cdot v\right)_{f},
\end{aligned}
$$

where, again, we have used (2.17) at the equality marked by (*).
Now let $e=\omega(z)$. Using the I.H., we obtain:

$$
\left.\langle\zeta\rangle(v)=\left(e\left[\zeta^{\prime}\right] D(v)=\ e\right\rangle\left(\left\langle\zeta^{\prime}\right)(v)\right)=\ e\right\rangle\left(M_{\zeta^{\prime}} \cdot v\right)=\delta_{\mathcal{A}}(\omega)\left(M_{\zeta^{\prime}} \cdot v\right) .
$$

Then we obtain:

$$
\begin{aligned}
&(\zeta \zeta)(v)_{\perp}=\delta_{\mathcal{A}}(\omega)\left(M_{\zeta^{\prime}} \cdot v\right)_{\perp}=\left(M_{\zeta^{\prime}} \cdot v\right)_{\perp} \cdot 1=\left(M_{\zeta^{\prime}} \cdot v\right)_{\perp} \\
&=\left(\left(\begin{array}{lll}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{array}\right) \cdot\left(M_{\zeta^{\prime}} \cdot v\right)\right)_{\perp} \\
&=\left(M_{e} \cdot\left(M_{\zeta^{\prime}} \cdot v\right)_{\perp}={ }^{(*)}\left(\left(M_{e} \cdot M_{\zeta^{\prime}}\right) \cdot v\right)_{\perp}=\left(M_{e\left[\zeta^{\prime}\right]} \cdot v\right)_{\perp}=\left(M_{\zeta} \cdot v\right)_{\perp},\right. \\
&\quad \text { (because } a(\zeta)=0) \\
& \text { OS) }(v)_{a}=\delta_{\mathcal{A}}(\omega)\left(M_{\zeta^{\prime}} \cdot v\right)_{a}=0=\left(M_{\zeta} \cdot v\right)_{a} \quad \\
&\text { OSD(v) })=\delta_{\mathcal{A}}(\omega)\left(M_{\zeta^{\prime}} \cdot v\right)_{f}=\left(M_{\zeta^{\prime}} \cdot v\right)_{f}=\left(M_{e} \cdot\left(M_{\zeta^{\prime}} \cdot v\right)\right)_{f}=\left(M_{\zeta} \cdot v\right)_{f},
\end{aligned}
$$

where, again, we have used (2.17) at the equality marked by (*).
This proves (6.5).
Lemma 6.2.3. For every $c \in \mathrm{C}_{\Sigma}$ and $\xi \in \mathrm{T}_{\Sigma}$, we have $(c)\left(\mathrm{h}_{\mathcal{V}}(\xi)\right)=\mathrm{h}_{\mathrm{V}}(c[\xi])$.
Proof. We prove this statement by induction on ( $\mathrm{C}_{\Sigma}, \prec_{\mathrm{C}_{\Sigma}}$ ).
I.B.: Let $c=z$. Then the statement is obvious because $(z z)$ is the identity mapping on $v$.
I.S.: Now let $c=e\left[c^{\prime}\right]$, where $e=\sigma\left(\xi_{1}, \ldots, \xi_{i-1}, z, \xi_{i+1}, \ldots, \xi_{k}\right)$ is an elementary $\Sigma$-context and $c^{\prime} \in \mathrm{C}_{\Sigma}$. Then, for each $\xi \in \mathrm{T}_{\Sigma}$, we have

$$
\begin{align*}
& (c)(\operatorname{hv}(\xi))=\left(e \ell\left(\left(\left\langle c^{\prime}\right\rangle(\operatorname{hv}(\xi))\right) \quad \text { (because } \ .\right)\right. \text { is a monoid homomorphism) } \\
& \left.\left.=\theta(e)\left(\left(c^{\prime}\right)\left(h_{\mathrm{V}}(\xi)\right)\right) \quad \text { (because } 0 .\right) \text { extends } \theta\right) \\
& =\mu(\sigma)\left(h_{\mathrm{V}}\left(\xi_{1}\right), \ldots, \mathrm{h}_{\mathrm{V}}\left(\xi_{i-1}\right),\left(c^{\prime}\right)\left(\mathrm{h}_{\mathrm{V}}(\xi)\right), \mathrm{h}_{\mathrm{V}}\left(\xi_{i+1}\right), \ldots, \mathrm{h}_{\mathrm{v}}\left(\xi_{k}\right)\right) \\
& =\mu(\sigma)\left(\operatorname{hv}\left(\xi_{1}\right), \ldots, \operatorname{hv}\left(\xi_{i-1}\right), \operatorname{hyv}_{\mathrm{v}}\left(c^{\prime}[\xi]\right), \mathrm{hv}\left(\xi_{i+1}\right), \ldots, \mathrm{h}_{\mathrm{V}}\left(\xi_{k}\right)\right)  \tag{byI.H}\\
& =\mathrm{h}_{\mathrm{V}}\left(\sigma\left(\xi_{1}, \ldots, \xi_{i-1}, c^{\prime}[\xi], \xi_{i+1}, \ldots, \xi_{k}\right)\right) \\
& =\mathrm{h} v(c[\xi]) \text {. }
\end{align*}
$$

### 6.2.2 Linear recurrence equation

Weighted tree languages that are recognizable over multilinear operations by a vector space satisfy a particular linear recurrence equation. In the following lemma, for a context $\zeta \in \mathrm{C}_{\Sigma}$, we view $(\zeta) \in \mathcal{L}(\mathrm{V}, \mathrm{V})$ as an element of $B^{\kappa \times \kappa}$, i.e., as a $[\kappa]$-square matrix over $B$ (cf. page 39). We recall that the characteristic polynomial of such a $(\zeta)$ is $\operatorname{char}_{(\zeta\rangle}(x)=\operatorname{det}\left((\zeta)-x \mathrm{M}_{\mathbb{I}}\right)$ (cf. page 37).

Lemma 6.2.4. Let $\Sigma$ be a ranked alphabet, $\mathrm{B}=(B, \oplus, \otimes, \mathbb{Q}, \mathbb{1})$ be a field, V be a $\kappa$-dimensional B-vector space, and $r: \mathrm{T}_{\Sigma} \rightarrow B$. Let $(\mathrm{V}, \mu, \gamma)$ be a $(\Sigma, \mathrm{B})$-multilinear representation which recognizes $r$. Moreover, let $\zeta \in \mathrm{C}_{\Sigma}$ and let $\operatorname{char}_{(\zeta)}(x)=(-\mathbb{1})^{\kappa} \otimes x^{\kappa} \oplus b_{1} \otimes x^{\kappa-1} \oplus \cdots \oplus b_{\kappa}$. Then, for every $\theta \in \mathrm{C}_{\Sigma}, \alpha \in \Sigma^{(0)}$, and $n \in \mathbb{N}$, we have

$$
(-\mathbb{1})^{\kappa} \otimes r\left(\theta\left[\zeta^{n+\kappa}[\alpha]\right]\right) \oplus b_{1} \otimes r\left(\theta\left[\zeta^{n+\kappa-1}[\alpha]\right]\right) \oplus \cdots \oplus b_{\kappa} \otimes r\left(\theta\left[\zeta^{n}[\alpha]\right]\right)=0 .
$$

Proof. For the sake of brevity we write $\operatorname{char}_{(\zeta D}(x)=\bigoplus_{i \in[0, \kappa]} b_{\kappa-i} \otimes x^{i}$ where we let $b_{0}=(-\mathbb{1})^{\kappa}$.
By Theorem 2.7.1 (i.e., the theorem of Cayley and Hamilton), we have char $\left.{ }_{\Omega \zeta \emptyset}(\Omega \zeta)\right)=\mathrm{M}_{\bullet}$. Now let $\theta \in \mathrm{C}_{\Sigma}, \alpha \in \Sigma^{(0)}$, and $n \in \mathbb{N}$. Then we can calculate as follows (where we abbreviate by $v_{0}$ the $[\kappa]$-vector in $B^{\kappa}$ with 0 in each of its entries):

$$
\begin{aligned}
& \left.\operatorname{char}_{\ \zeta D}(0 \zeta)\right)=\mathrm{M}_{0}
\end{aligned}
$$

$$
\begin{aligned}
& \left.\Rightarrow\left(\underset{i \in[0, \kappa]}{+} b_{\kappa-i} \cdot(\zeta)^{i}\right) \cdot \mathrm{h}_{\mathrm{V}}(\alpha)=v_{0} \quad \quad \text { (matrix-vector multiplication with } \mathrm{h}_{\mathrm{V}}(\alpha)\right) \\
& \left.\Rightarrow \Omega \theta\left[\zeta^{n}\right] D \cdot\left(\underset{i \in[0, \kappa]}{\nmid} b_{\kappa-i} \cdot \Omega \zeta\right)^{i}\right) \cdot \mathrm{h}_{\mathrm{V}}(\alpha)=v_{0} \quad \quad \text { (matrix-vector multiplication with }\left(\theta\left[\zeta^{n}\right] \mathrm{D}\right) \\
& \Rightarrow\left(\underset{i \in[0, c}{ } b_{\kappa-i} \cdot\left(\theta\left[\zeta^{n}\right] D \cdot(\zeta)^{i} \cdot \mathrm{~h}_{\mathrm{V}}(\alpha)\right)=v_{0} \quad \quad \text { (because }\left(\theta\left[\zeta^{n}\right]\right)\right. \text { is a linear mapping) } \\
& \Rightarrow\left(\underset{i \in[0, \kappa]}{+} b_{\kappa-i} \cdot\left(\theta\left[\zeta^{n+i}\right]\right) \cdot \operatorname{h\vee }(\alpha)\right)=v_{0} \quad \quad \text { (because (0.) is a monoid homomorphism) } \\
& \Rightarrow\left(\underset{i \in[0, \kappa]}{ } b_{\kappa-i} \cdot h_{\mathrm{V}}\left(\theta\left[\zeta^{n+i}[\alpha]\right]\right)\right)=v_{0} \quad \quad \text { (by Lemma 6.2.3) } \\
& \Rightarrow \gamma\left(\underset{i \in[0, \kappa]}{+} b_{\kappa-i} \cdot h_{\mathrm{V}}\left(\theta\left[\zeta^{n+i}[\alpha]\right]\right)\right)=0 \quad \text { (by applying the linear form } \gamma \text { to both sides) } \\
& \Rightarrow \bigoplus_{i \in[0, \kappa]} b_{\kappa-i} \otimes \gamma\left(\mathrm{~h} v\left(\theta\left[\zeta^{n+i}[\alpha]\right]\right)\right)=\mathbb{0} \quad \quad \text { (because } \gamma \text { is a linear form) } \\
& \Rightarrow \bigoplus_{i \in[0, \kappa]} b_{\kappa-i} \otimes r\left(\theta\left[\zeta^{n+i}[\alpha]\right]\right)=0 \quad \quad \quad \text { (because }(V, \mu, \gamma) \text { recognizes } r \text { ) }
\end{aligned}
$$

Example 6.2.5. We continue with Example 6.2 .2 and compute the characteristic polynomial of $(\zeta)$ for each context $\zeta$. This characteristic polynomial has the following form:

- if $\zeta \neq z$, then $a(\zeta)=0$ and $\operatorname{char}_{(\zeta)}(x)=-x^{3}+2 x^{2}-x$ and
- if $\zeta=z$, then $a(\zeta)=1$ and $\operatorname{char}_{\langle\zeta \emptyset}(x)=-x^{3}+3 x^{2}-3 x+1$.

Then, due to Lemma 6.2.4 we obtain the following linear recurrence equation for $\#_{\sigma(., \alpha)}$. For every $\theta, \zeta \in \mathrm{C}_{\Sigma}$ with $\zeta \neq z$ and every $n \in \mathbb{N}$ we have

$$
\#_{\sigma(., \alpha)}\left(\theta\left[\zeta^{n+3}[\alpha]\right]\right)=2 \cdot \#_{\sigma(., \alpha)}\left(\theta\left[\zeta^{n+2}[\alpha]\right]\right)-\#_{\sigma(., \alpha)}\left(\theta\left[\zeta^{n+1}[\alpha]\right]\right)
$$

In the following corollary, height $: \mathrm{T}_{\Sigma} \rightarrow \mathbb{N}$ is the weighted tree language defined in Example 3.2.4

Corollary 6.2.6. BR82, Ex. 9.2] For the ranked alphabet $\Sigma=\left\{\sigma^{(2)}, \alpha^{(0)}\right\}$, we have that height $\notin$ $\operatorname{Rec}(\Sigma$, Real $)$.

Proof. We prove by contradiction. For this we assume that there exists a ( $\Sigma$, Real)-multilinear representation $(\mathrm{V}, \mu, \gamma)$ such that height $(\xi)=\gamma(\mathrm{h} V(\xi))$ for each $\xi \in \mathrm{T}_{\Sigma}$. Let V be $\kappa$-dimensional for some $\kappa \in \mathbb{N}$.

Let $\xi \in \mathrm{T}_{\Sigma}$ be a tree with height $(\xi)=\kappa$. Due to the definition of $\Sigma$ such a $\xi$ exists. We consider the two elementary $\Sigma$-contexts $\theta=\sigma(z, \xi)$ and $\zeta=\sigma(z, \alpha)$. For each $n \in \mathbb{N}$ we define the tree $\xi_{n}=\theta\left[\zeta^{n}[\alpha]\right]$ in $\mathrm{T}_{\Sigma}$. Then

$$
\operatorname{height}\left(\xi_{n}\right)=1+\max (n, \kappa) \quad \text { for every } n \in \mathbb{N}
$$

By Lemma 6.2.4 there exist $b_{1}, \ldots, b_{\kappa} \in \mathbb{R}$ such that

$$
\begin{equation*}
(-1)^{\kappa} \cdot \operatorname{height}\left(\xi_{n+\kappa}\right)+b_{1} \cdot \operatorname{height}\left(\xi_{n+\kappa-1}\right)+\cdots+b_{\kappa} \cdot \operatorname{height}\left(\xi_{n}\right)=0 \quad \text { for every } n \in \mathbb{N} \tag{6.6}
\end{equation*}
$$

By using (6.6) for $n=0$ and $n=1$, we obtain

$$
\begin{align*}
& (-1)^{\kappa} \cdot(1+\kappa)+b_{1} \cdot(1+\kappa)+\cdots+b_{\kappa} \cdot(1+\kappa)=0, \text { and }  \tag{6.7}\\
& (-1)^{\kappa} \cdot(1+(\kappa+1))+b_{1} \cdot(1+\kappa)+\cdots+b_{\kappa} \cdot(1+\kappa)=0 \tag{6.8}
\end{align*}
$$

respectively. Then Equations (6.7) and (6.8) imply that $\kappa=\kappa+1$ which is a contradiction.
It follows that, for $\Sigma=\left\{\sigma^{(2)}, \alpha^{(0)}\right\}$ and any strong bimonoid B such that Real is an extension of B, the weighted tree language height is not recognizable by any ( $\Sigma, \mathrm{B}$ )-wta (cf. Observation 3.7.1). In particular, height is not recognizable by any $(\Sigma, \operatorname{Rat})$-wta and $(\Sigma, N a t)$-wta. Recall that, at the same time, there is a ( $\Sigma, \mathrm{Nat}_{\mathrm{max},+}$ )-wta which recognizes height (cf. Example 3.2.4).

### 6.2.3 The pumping lemma

Lemma 6.2.1 justifies to use the following theorem and lemma from Reu80 (using $A=\mathrm{eC}_{\Sigma}$ ). We will use the fact that a matrix $M \in B^{\kappa \times \kappa}$ is pseudo-regular if and only if $x^{2}$ does not divide its characteristic polynomial char ${ }_{M}$ (cf. Reu80, Prop. 1]). We do not show the proof of this statement.
Theorem 6.2.7. Reu80, Thm. 3] Let $\kappa \in \mathbb{N}_{+}$. There exists $N \in \mathbb{N}$ such that for every set $A$, monoid homomorphism $\mu: A^{*} \rightarrow B^{\kappa \times \kappa}$, string $w \in A^{*}$ of length at least $N$, the string $w$ has a factor $v \neq \varepsilon$ such that $\mu(v)$ is a pseudo-regular matrix.
Lemma 6.2.8. Reu80, Lm. 1] Let $\kappa \in \mathbb{N}_{+}, M \in B^{\kappa \times \kappa}$, and $\lambda, \gamma \in B^{\kappa}$. If $M$ is pseudo-regular and $\lambda \cdot M \cdot \gamma \neq \mathbb{O}$, then there exist infinitely many $n \in \mathbb{N}$ such that $\lambda \cdot M^{n} \cdot \gamma \neq \mathbb{0}$.

Proof. Let

$$
\operatorname{char}_{M}(x)=(-\mathbb{1})^{\kappa} \otimes x^{\kappa} \oplus b_{1} \otimes x^{\kappa-1} \oplus \cdots \oplus b_{\kappa-1} \otimes x \oplus b_{\kappa}
$$

By our assumption $b_{\kappa-1} \neq \mathbb{O}$ or $b_{\kappa} \neq \mathbb{O}$. Let us abbreviate $\lambda \cdot M^{n} \cdot \gamma$ by $a_{n}$ for each $n \in \mathbb{N}$. We show that, for each $n \in \mathbb{N}$, if $a_{n} \neq \mathbb{O}$, then there exists $m>n$ such that $a_{m} \neq \mathbb{O}$. Since $a_{1} \neq \mathbb{O}$, this proves the lemma.

Let $n \in \mathbb{N}$ and assume that $a_{n} \neq \mathbb{O}$. In addition, assume that $b_{\kappa} \neq \mathbb{O}$. By Theorem 2.7.1 we have

$$
(-\mathbb{1})^{\kappa} \otimes a_{n+\kappa} \oplus b_{1} \otimes a_{n+\kappa-1} \oplus \cdots \oplus b_{\kappa-1} \otimes a_{n+1} \oplus b_{\kappa} \otimes a_{n}=0
$$

Then $a_{m} \neq \mathbb{O}$ for some $m \in[n+1, n+\kappa]$. Now assume that $b_{\kappa}=\mathbb{O}$ and $b_{\kappa-1} \neq \mathbb{O}$. By our assumption $\kappa>1$ and by Theorem 2.7.1, we have

$$
(-\mathbb{1})^{\kappa} \otimes a_{n+\kappa-1} \oplus b_{1} \otimes a_{n+\kappa-2} \oplus \cdots \oplus b_{\kappa-1} \otimes a_{n}=\mathbb{0}
$$

Then $a_{m} \neq \mathbb{O}$ for some $m \in[n+1, n+\kappa-1]$.
Let $\xi \in \mathrm{T}_{\Sigma}, c \in \mathrm{C}_{\Sigma}$, and $\alpha \in \Sigma^{(0)}$. The pair $(c, \alpha)$ is a walk in $\xi$ if $\xi=c[\alpha]$. The depth of $(c, \alpha)$ is $\operatorname{depth}(c)$. Let $c \in \mathrm{C}_{\Sigma}$. We recall that $c^{0}=z$, and for every $n \in \mathbb{N}$, we have $c^{n+1}=c\left[c^{n}\right]$.

Theorem 6.2.9. BR82, Thm. 9.2] Let $\Sigma$ be a ranked alphabet, B be a field, and $r \in \operatorname{Rec}(\Sigma, \mathrm{~B})$. There exists a constant $N \in \mathbb{N}$ such that for every $\xi \in \operatorname{supp}(r)$ and every walk $(c, \alpha)$ in $\xi$ of depth at least $N$, there exist $c_{1}, c_{2}, c_{3} \in \mathrm{C}_{\Sigma}$ such that $c_{2} \neq z, c=c_{1} \circ_{z} c_{2} \circ_{z} c_{3}$, and the set $\left\{\left(c_{1} \circ_{z} c_{2}^{n} \circ_{z} c_{3}\right)[\alpha] \mid n \in \mathbb{N}\right\} \cap \operatorname{supp}(r)$ is infinite.

Proof. By Theorem 3.6 .6 there exists a $(\Sigma, \mathrm{B})$-multilinear representation $(\mathrm{V}, \mu, \gamma)$ where $\mathrm{V}=(V,+, 0)$ is a $\kappa$-dimensional $B$-vector space for some $\kappa \in \mathrm{N}$ and

$$
\begin{equation*}
r(\xi)=\gamma\left(\mathrm{h}_{\mathrm{V}}(\xi)\right) \text { for each } \xi \in \mathrm{T}_{\Sigma} \tag{6.9}
\end{equation*}
$$

Since the two monoids $\left(B^{\kappa \times \kappa}, \cdot, \mathrm{M}_{\mathbb{1}}\right)$ and $(\mathcal{L}(\mathrm{V}, \mathrm{V}), \circ, \mathrm{id})$ are isomorphic (cf Section 2.8), we identify them.

Then we can apply Theorem 6.2.7. Let $N \in \mathbb{N}$ be the integer from that theorem. Moreover, let $\xi \in \operatorname{supp}(r)$ and $(c, \alpha)$ be a walk of $\xi$ of depth at least $N$. By Theorem6.2.7 there exists $c_{1}, c_{2}, c_{3} \in \mathrm{C}_{\Sigma}$ such that $c=c_{1} \circ_{z} c_{2} \circ_{z} c_{3}$ with $c_{2} \neq z$ and $\left(c_{2}\right)$ is a pseudo-regular endomorphism. In particular, we have $\xi=c_{1}\left[c_{2}\left[c_{3}[\alpha]\right]\right]$.

For each $n \in \mathbb{N}$ we define the element $u_{n} \in B$ by

$$
u_{n}=\gamma\left(\left(c_{1}\right)\left(\left(c_{2}\right)^{n}\left(\left(c_{3}\right)\left(\mathrm{h}_{\vee}(\alpha)\right)\right)\right)\right.
$$

In particular,

$$
\left.\left.u_{1}=\gamma\left(\| c_{1}\right)\left(\| c_{2}\right\rangle\left(\left(c_{3}\right)\left(\mathrm{h}_{\vee}(\alpha)\right)\right)\right)=\gamma\left(\mathrm{h}_{\vee}\left(c_{1}\left[c_{2}\left[c_{3}[(\alpha)]\right]\right]\right)\right)\right)=r\left(c_{1}\left[c_{2}\left[c_{3}[(\alpha)]\right]\right]\right)=r(\xi) \neq \mathbb{O}
$$

where the second and third equality are due to Lemma 6.2.3 and (6.9), respectively.
Thus, we can apply Lemma 6.2.8 (using $M=\left(c_{2}\right)$ and $\left.\lambda=\left(c_{3}\right) \cdot h_{\mathrm{V}}(\alpha)\right)$ and hence, there exist infinitely many $n$ such that $u_{n} \neq \mathbb{0}$.

Since $u_{n}=r\left(c_{1}\left[c_{2}^{n}\left[c_{3}[(\alpha)]\right]\right]\right)$ (by a similar calculation as for $u_{1}$ ), we obtain the statement of the theorem.

Theorem 6.2.9 can be used to show that certain tree languages cannot be the support of recognizable B-weighted tree languages.

Corollary 6.2.10. Let $\Sigma=\left\{\gamma^{(1)}, \sigma^{(1)}, \alpha^{(0)}\right\}$. For each field $B$, the tree language $L=\left\{\sigma^{n} \gamma^{n}(\alpha) \mid n \geq 0\right\}$ is not in $\operatorname{supp}(\operatorname{Rec}(\Sigma, B))$.

Proof. Let B be a field. We continue to prove by contradiction. For this, we assume that there exists an $r \in \operatorname{Rec}(\Sigma, \mathrm{~B})$ such that $L=\operatorname{supp}(r)$. Let $N$ be the number for $r$ appearing in Theorem 6.2.9 and consider the tree $\xi=\sigma^{N} \gamma^{N}(\alpha)$ and the walk $(c, \alpha)$ in $\xi$, where $c=\sigma^{N} \gamma^{N}(z)$. By Theorem 6.2.9, there exist $c_{1}, c_{2}, c_{3} \in \mathrm{C}_{\Sigma}$ such that $c_{2} \neq z, c=c_{1} \circ_{z} c_{2} \circ_{z} c_{3}$ and the set $\left\{\left(c_{1} \circ_{z} c_{2}^{n} \circ_{z} c_{3}\right)[\alpha] \mid n \in \mathbb{N}\right\} \cap L$ is infinite. On the other hand, for any decomposition $c=c_{1} \circ_{z} c_{2} \circ_{z} c_{3}$ of $c$ with $c_{2} \neq z$ we have $\left\{\left(c_{1} \circ_{z} c_{2}^{n} \circ_{z} c_{3}\right)[\alpha] \mid n \in \mathbb{N}\right\} \cap L \subseteq\left\{\left(c_{1} \circ_{z} c_{3}\right)[\alpha], \xi\right\}$. This is a contradiction, hence $L \notin \operatorname{supp}(\operatorname{Rec}(\Sigma, \mathrm{~B}))$.

By similar arguments as in Corollary 6.2.10, we can show that the tree language FB is not in $\operatorname{supp}(\operatorname{Rec}(\Sigma, \mathrm{B}))$. The tree language FB of fully balanced trees over $\left\{\sigma^{(2)}, \alpha^{(0)}\right\}$ is the smallest $\Sigma$-tree language $L$ satisfying (i) $\alpha \in L$, and (ii) $\sigma(\xi, \xi) \in L$ for each $\xi \in L$.

## Chapter 7

## Normal forms of wta

In this chapter we prove some useful normal forms of wta. More precisely, we define several useful properties which a given wta either has or does not have. Then, for each of these properties, we show that, under some conditions on the weight algebra, any given wta can be transformed into an equivalent wta which has this property. In this sense, the given wta is turned into a normal form. Sometimes normal forms of wta are useful for reducing the complexity of proofs which start from a given wta.

We deal with the following properties: trim, total, root weight normalized, and identity transition weights.

### 7.1 Trimming a wta

In this section we elaborate two trimming methods for wta. Intuitively, a trimming method takes a wta as input and constructs a run equivalent trim wta as output. A wta is trim if it contains only useful states, i.e., states which occur in a successful run on some input tree.

In the literature trimming methods are known for finite-state automata [Sak09, Prop. 1.9], context-free grammars Har78, Thm. 3.2.3], regular tree grammars $\mathrm{CDG}^{+} 07$, Prop. 2.1.3] and [Dre06, Lm. A.2.6], fta [Sei89, Prop. 1.1], weighted string automata over semirings [Sak09, p. 408], and wta over strong bimonoids DFKV22, Thm. 4.2]. Except for the last two shown references, each of them deals with the unweighted case.

Our first trimming method is applicable to wta over zero-cancellation free strong bimonoids and it leads to a "strong form" of trim wta. Our second method is applicable to wta over arbitrary strong bimonoids and it leads to a "weak form" of trim wta. We start by defining some useful concepts.

### 7.1.1 Basic definitions

Let $\mathcal{A}=(Q, \delta, F)$ be a $(\Sigma, \mathrm{B})$-wta. Let $q \in Q, \xi \in \mathrm{~T}_{\Sigma}$, and $\rho \in \mathrm{R}_{\mathcal{A}}(q, \xi)$. We say that

- $\rho$ is successful if $\mathrm{wt}(\xi, \rho) \otimes F_{q} \neq \mathbb{0}$,
- $\rho$ is local-successful if (a) for each $v \in \operatorname{pos}(\xi)$ we have $\delta_{k}(\rho(v 1) \cdots \rho(v k), \xi(v), \rho(v)) \neq \mathbb{O}$ where $k=\operatorname{rk}_{\Sigma}(\xi(v))$ and (b) $F_{q} \neq 0$.
Let $p \in Q$. We say that
- $p$ is useful if there exist $q \in Q, \xi \in \mathrm{~T}_{\Sigma}$, and a successful run $\rho \in \mathrm{R}_{\mathcal{A}}(q, \xi)$ such that $p \in \operatorname{im}(\rho)$.
- $p$ is local-useful if there exist $q \in Q, \xi \in \mathrm{~T}_{\Sigma}$, and a local-successful run $\rho \in \mathrm{R}_{\mathcal{A}}(q, \xi)$ such that $p \in \operatorname{im}(\rho)$.
- $p$ is accessible if there exist $\xi \in \mathrm{T}_{\Sigma}$ and $\rho \in \mathrm{R}_{\mathcal{A}}(p, \xi)$ such that $\mathrm{wt}(\xi, \rho) \neq \mathbb{0}$.

$$
\text { trim wta } \underset{H}{\longrightarrow} \quad \text { weak-trim wta } \quad \underset{H}{\longrightarrow} \quad \text { local-trim wta } \quad \underset{H}{\longrightarrow} \quad \text { wta }
$$

Figure 7.1: An illustration of the relationships between trim wta, weak-trim wta, local-trim wta, and wta where $\longrightarrow$ means "implies" (cf. Observation 7.1.1 (1)-(4)).

- $p$ is local-accessible if there exist $\xi \in \mathrm{T}_{\Sigma}$ and $\rho \in \mathrm{R}_{\mathcal{A}}(p, \xi)$ such that, for each $v \in \operatorname{pos}(\xi)$, we have $\delta_{k}(\rho(v 1) \cdots \rho(v k), \xi(v), \rho(v)) \neq \mathbb{O}$ where $k=\operatorname{rk}_{\Sigma}(\xi(v))$.
- $p$ is co-accessible if there exist $q \in Q$, a context $c \in \mathrm{C}_{\Sigma}$, and a run $\rho \in \mathrm{R}_{\mathcal{A}}(q, c, p)$ such that $\mathrm{wt}(c, \rho) \otimes F_{q} \neq \mathbb{0}$.
- $p$ is weakly useful if it is accessible and co-accessible.

Clearly, useful implies weakly useful, and weakly useful implies local-useful. Moreover, if B is zerodivisor free, then the three notions are equivalent; thus, in particular, this holds if $B$ is the Boolean semiring Boole.

A $(\Sigma, \mathrm{B})$-wta $\mathcal{A}$ is called trim if each of its states is useful; in a similar way, we define weak-trim and local-trim by employing weakly useful and local-useful, respectively. We note that the notions of trim in Sak09 and in DFKV22 correspond to the notions weak-trim and local-trim, respectively. In the next observation we formally compare the different trim properties (cf. Figure 7.1).

Observation 7.1.1. The following five statements hold.
(1) Each trim $(\Sigma, \mathrm{B})$-wta is weak-trim, and each weak-trim $(\Sigma, \mathrm{B})$-wta is local-trim.
(2) There exist a trivial ranked alphabet $\Sigma$ and a $(\Sigma, \mathrm{B})$-wta $\mathcal{A}$ such that $\llbracket \mathcal{A} \rrbracket^{\text {run }} \neq \widetilde{\mathbb{O}}$ and $\mathcal{A}$ is not local-trim.
(3) There exist a string ranked alphabet $\Sigma$, a strong bimonoid B , and a local-trim $(\Sigma, \mathrm{B})$-wta which is not weak-trim.
(4) There exist a string ranked alphabet $\Sigma$, a strong bimonoid $B$, and a weak-trim ( $\Sigma, \mathrm{B})$-wta which is not trim.
(5) If B is zero-divisor free, then for each $(\Sigma, \mathrm{B})$-wta $\mathcal{A}$ we have: $\mathcal{A}$ is trim iff $\mathcal{A}$ is weak-trim iff $\mathcal{A}$ is local-trim.

Proof. Statements (1) and (5) are obvious.
Proof of Statement (2): We consider the ranked alphabet $\Sigma=\left\{\alpha^{(0)}\right\}$ and the ( $\left.\Sigma, \mathrm{B}\right)$-wta $\mathcal{A}=(Q, \delta, F)$, where $Q=\{q, p\}, \delta(\varepsilon, \alpha, q)=\mathbb{1}, \delta(\varepsilon, \alpha, p)=\mathbb{O}$, and $F_{q}=\mathbb{1}$ and $F_{p}=\mathbb{O}$. Then $\llbracket \mathcal{A} \rrbracket^{\text {run }}(\alpha)=\mathbb{1}$, i.e., $\llbracket \mathcal{A} \rrbracket^{\text {run }} \neq \widetilde{\mathbb{0}}$. We note that $p$ is not local-useful and hence $\mathcal{A}$ is not local-trim.

Proof of Statement (3): We consider the string ranked alphabet $\Sigma=\left\{\gamma^{(1)}, \alpha^{(0)}\right\}$ and the ring Intmod4 $=\left(\{0,1,2,3\},{ }_{4}, \cdot 4,0,1\right)$ defined in Example 2.6.9(5). This Intmod4 is not zero-divisor free because $2 \cdot 42=0$.

Then we let $\mathcal{A}=(Q, \delta, F)$ be the ( $\Sigma$, Intmod4)-wta with $Q=\left\{q, q_{1}, p, q_{2}\right\}$ and for each $f, g \in Q$ we define

$$
\delta_{0}(\varepsilon, \alpha, f)=\left\{\begin{array}{ll}
1 & \text { if } f=q \\
0 & \text { otherwise }
\end{array} \quad \text { and } \quad \delta_{1}(f, \gamma, g)= \begin{cases}2 & \text { if } f g \in\left\{q q_{1}, q_{1} p, p q_{2}, q_{2} q\right\} \\
0 & \text { otherwise }\end{cases}\right.
$$

and $F_{q}=1$ and $F_{q_{1}}=F_{p}=F_{q_{2}}=0$. Then each $f \in Q$ is local-useful and hence $\mathcal{A}$ is local-trim. The state $q$ is weakly useful. The state $q_{1}$ is accessible; $p$ and $q_{2}$ are not accessible; $q_{2}$ is co-accessible; $p$ and $q_{1}$ are not co-accessible. Hence $\mathcal{A}$ is not weak-trim.

Proof of Statement (4): We consider the ranked alphabet $\Sigma$ and the semiring Intmod4 as in the proof of Statement (3).


Figure 7.2: Cutting out contexts above $v$ repeatedly.


Figure 7.3: Cutting out contexts below $\widehat{v}$ repeatedly.

Then we let $\mathcal{A}=(Q, \delta, F)$ be the ( $\Sigma$, Intmod4)-wta with $Q=\{q, p\}$ and for each $f, g \in Q$ we define

$$
\delta_{0}(\varepsilon, \alpha, f)=\left\{\begin{array}{ll}
1 & \text { if } f=q \\
0 & \text { otherwise }
\end{array} \quad \text { and } \quad \delta_{1}(f, \gamma, g)= \begin{cases}2 & \text { if } f g \in\{q p, p q\} \\
0 & \text { otherwise }\end{cases}\right.
$$

and $F_{q}=1$ and $F_{p}=0$. Then $p$ and $q$ are accessible and co-accessible, and hence, weakly useful. Thus $\mathcal{A}$ is weak-trim. The state $p$ is not useful and hence $\mathcal{A}$ is not trim.

### 7.1.2 Trimming for zero-cancellation free strong bimonoids

Here we show our first trimming method. It takes a $(\Sigma, \mathrm{B})$-wta over some zero-cancellation free strong bimonoid B , and constructs a run equivalent trim $(\Sigma, \mathrm{B})$-wta. The method employs the pumping lemma to cut out in a careful way contexts from successful runs; due to zero-cancellation freeness of $B$, this results in smaller successful runs. As preparation we characterize useful states.

Lemma 7.1.2. Let B be zero-cancellation free, $\mathcal{A}=(Q, \delta, F)$ be a $(\Sigma, \mathrm{B})$-wta, and $p \in Q$. Then $p$ is useful if and only if there exist $q \in Q, \xi \in \mathrm{~T}_{\Sigma}$, and a successful run $\rho \in \mathrm{R}_{\mathcal{A}}(q, \xi)$ such that $p \in \operatorname{im}(\rho)$ and height $(\xi)<2|Q|$.

Proof. The "if"-direction holds trivially. For the "only-if"-direction we prove the following.
For every $q \in Q, \xi \in \mathrm{~T}_{\Sigma}, v \in \operatorname{pos}(\xi)$, and successful run $\kappa \in \mathrm{R}_{\mathcal{A}}(q, \xi)$,
there exist $\widehat{\xi} \in \mathrm{T}_{\Sigma}, \widehat{v} \in \operatorname{pos}(\widehat{\xi})$, and successful run $\widehat{\kappa} \in \mathrm{R}_{\mathcal{A}}(q, \widehat{\xi})$
such that height $(\widehat{\xi})<2|Q|$, and $\widehat{\kappa}(\widehat{v})=\kappa(v)$.
Let $q \in Q, \xi \in \mathrm{~T}_{\Sigma}, v \in \operatorname{pos}(\xi)$, and $\kappa \in \mathrm{R}_{\mathcal{A}}(q, \xi)$ be a successful run. We prove (7.1) in three steps. Intuitively, in Step 1 we cut out repeatedly contexts above $v$ such that eventually the state $\kappa(v)$ occurs at $\widehat{v}$ and $|\widehat{v}|<|Q|$. In Step 2 we cut out repeatedly contexts below $\widehat{v}$ such that eventually the subtree below $\widehat{v}$ has height smaller than $|Q|$. In Step 3 we cut out repeatedly contexts aside of $\widehat{v}$ such that eventually we obtain the desired result. Since B is zero-cancellation free, cutting out a context transforms a successful run into a successful run. We illustrate the three steps in Figures [7.2, 7.3, and 7.4 respectively.


Figure 7.4: Cutting out contexts aside of $\widehat{v}$ repeatedly.
$\underline{\text { Step 1: We construct a } \xi^{\prime} \in \mathrm{T}_{\Sigma} \text {, a } \widehat{v} \in \operatorname{pos}\left(\xi^{\prime}\right) \text {, and a successful run } \kappa^{\prime} \in \mathrm{R}_{\mathcal{A}}\left(q, \xi^{\prime}\right) \text { such that }|\widehat{v}|<|Q|}$ and $\overline{\kappa^{\prime}(\widehat{v})}=\kappa(v)$ as follows.

If $|v|<|Q|$, then we let $\xi^{\prime}=\xi, \widehat{v}=v$, and $\kappa^{\prime}=\kappa$ and we are ready.
Otherwise, there exist $u, x \in \mathbb{N}_{+}^{*}$ and $w \in \mathbb{N}_{+}^{+}$such that $v=u w x$ and $\kappa(u)=\kappa(u w)$. Let $c^{\prime}=\left.\xi\right|^{u}$, $c=\left.\left(\left.\xi\right|_{u}\right)\right|^{w}$, and $\zeta=\left.\xi\right|_{u w}$, and let $\rho^{\prime}=\left.\kappa\right|^{u}, \rho=\left.\left(\left.\kappa\right|_{u}\right)\right|^{w}$, and $\theta=\left.\kappa\right|_{u w}$. Then $\xi=c^{\prime}[c[\zeta]]$ and $\kappa=\rho^{\prime}[\rho[\theta]]$, where $\rho$ is a loop due to the condition $\rho(u)=\rho(u w)$. Thus, by Theorem 6.1.3 (equality (6.2) for $n=1$ ), we have

$$
\operatorname{wt}(\xi, \kappa) \otimes F_{q}=l_{c^{\prime}, \rho^{\prime}} \otimes l_{c, \rho} \otimes \operatorname{wt}(\zeta, \theta) \otimes r_{c, \rho} \otimes r_{c^{\prime}, \rho^{\prime}} \otimes F_{q} .
$$

Now we cut out the context $c$ and the corresponding loop $\rho$. Formally, let $\xi^{\prime}=c^{\prime}[\zeta]$ and $\kappa^{\prime}=\rho^{\prime}[\theta]$. By equality (6.2) for $n=0$, we obtain

$$
\mathrm{wt}\left(\xi^{\prime}, \kappa^{\prime}\right) \otimes F_{q}=l_{c^{\prime}, \rho^{\prime}} \otimes \mathrm{wt}(\zeta, \theta) \otimes r_{c^{\prime}, \rho^{\prime}} \otimes F_{q} .
$$

Since $\kappa$ is a successful run, i.e., $\operatorname{wt}(\xi, \kappa) \otimes F_{q} \neq \mathbb{0}$, and B is zero-cancellation free, we obtain that $\kappa^{\prime}$ is also successful. Moreover, for $\widehat{v}=u x$, we have $\kappa^{\prime}(\widehat{v})=\kappa(v)$ by the definition of $\kappa^{\prime}$. If $|\widehat{v}| \geq|Q|$, then we repeat the above procedure with $\xi^{\prime}, \widehat{v}$, and $\kappa^{\prime}$. After finitely many steps, we obtain $\xi^{\prime}, \widehat{v}$, and $\kappa^{\prime}$ such that $|\widehat{v}|<|Q|$.

Step 2: Given $\xi^{\prime} \in \mathrm{T}_{\Sigma}, \widehat{v} \in \operatorname{pos}\left(\xi^{\prime}\right)$, and the successful run $\kappa^{\prime} \in \mathrm{R}_{\mathcal{A}}\left(q, \xi^{\prime}\right)$ constructed in Step 1, we construct $\xi^{\prime \prime} \in \mathrm{T}_{\Sigma}$ and successful run $\kappa^{\prime \prime} \in \mathrm{R}_{\mathcal{A}}\left(q, \xi^{\prime \prime}\right)$ such that $\widehat{v} \in \operatorname{pos}\left(\xi^{\prime \prime}\right), \kappa^{\prime \prime}(\widehat{v})=\kappa^{\prime}(\widehat{v})$, and $\operatorname{height}\left(\xi^{\prime \prime} \mid \widehat{v}\right)<|Q|$ as follows.

If height $\left(\xi^{\prime} \mid \widehat{v}\right)<|Q|$, then we let $\xi^{\prime \prime}=\xi^{\prime}$ and $\kappa^{\prime \prime}=\kappa^{\prime}$ and we are ready.
Otherwise, there exist $u \in \mathbb{N}_{+}^{*}$ and $w \in \mathbb{N}_{+}^{+}$such that $\widehat{v} u w \in \operatorname{pos}\left(\xi^{\prime}\right)$ and $\kappa^{\prime}(\widehat{v} u)=\kappa^{\prime}(\widehat{v} u w)$. Similarly to the construction in Step 1, by cutting out the context $\left.\left(\xi^{\prime} \mid \widehat{v} u\right)\right|^{w}$ and the loop $\left(\kappa^{\prime}|\widehat{v} u|^{w}\right.$ from $\xi^{\prime}$ and $\kappa^{\prime}$, respectively, we obtain $\xi^{\prime \prime} \in \mathrm{T}_{\Sigma}$ and successful run $\kappa^{\prime \prime} \in \mathrm{R}_{\mathcal{A}}\left(q, \xi^{\prime \prime}\right)$ such that $\widehat{v} \in \operatorname{pos}\left(\xi^{\prime \prime}\right), \kappa^{\prime \prime}(\widehat{v})=\kappa^{\prime}(\widehat{v})$ and $\operatorname{size}\left(\left.\xi^{\prime \prime}\right|_{\hat{v}}\right)<\operatorname{size}\left(\xi^{\prime} \mid \widehat{v}\right)$. If height $\left(\xi^{\prime \prime} \mid \widehat{v}\right) \geq|Q|$, then we repeat the above procedure with $\xi^{\prime \prime}$ and $\kappa^{\prime \prime}$. After finitely many steps, we obtain $\xi^{\prime \prime}$ and $\kappa^{\prime \prime}$ such that height $\left(\xi^{\prime \prime} \mid \widehat{v}\right)<|Q|$.

Step 3: Given $\xi^{\prime \prime} \in \mathrm{T}_{\Sigma}, \widehat{v} \in \operatorname{pos}\left(\xi^{\prime \prime}\right)$, and the successful run $\kappa^{\prime \prime} \in \mathrm{R}_{\mathcal{A}}\left(q, \xi^{\prime \prime}\right)$ constructed in Step 2, we construct $\widehat{\xi} \in \mathrm{T}_{\Sigma}$ and the successful run $\widehat{\kappa} \in \mathrm{R}_{\mathcal{A}}(q, \widehat{\xi})$ such that $\widehat{v} \in \operatorname{pos}(\widehat{\xi}), \widehat{\kappa}(\widehat{v})=\kappa^{\prime \prime}(\widehat{v})$, and height $(\widehat{\xi})<2|Q|$.

Let $x \in \operatorname{pos}\left(\xi^{\prime \prime}\right)$ be such that $x \notin \operatorname{prefix}(\widehat{v})$. Let us denote by $\operatorname{lcp}(\hat{v}, x)$ the longest common prefix of $\widehat{v}$ and $x$. Moreover, let rest $(\widehat{v}, x)$ be the unique string in $\mathbb{N}_{+}^{+}$such that $x=\operatorname{lcp}(\widehat{v}, x) \operatorname{rest}(\widehat{v}, x)$. We note that if $\widehat{v} \in \operatorname{prefix}(x)$, i.e., $\operatorname{lcp}(\widehat{v}, x)=\widehat{v}$, then $|\operatorname{rest}(\widehat{v}, x)|<|Q|$ due to the condition height $\left(\xi^{\prime \prime} \mid \widehat{v}\right)<|Q|$.

If, for each $x \in \operatorname{pos}\left(\xi^{\prime \prime}\right)$ with $x \notin \operatorname{prefix}(\widehat{v})$ and $\widehat{v} \notin \operatorname{prefix}(x)$, we also have $|\operatorname{rest}(\widehat{v}, x)|<|Q|+1$, then we are ready because $|\hat{v}|<|Q|$.

Otherwise, there exist $u \in \mathbb{N}_{+}^{+}$and $w \in \mathbb{N}_{+}^{+}$such that $\operatorname{lcp}(\widehat{v}, x) u w \in \operatorname{prefix}(x)$ and $\kappa^{\prime \prime}(\operatorname{lcp}(\widehat{v}, x) u)=$ $\kappa^{\prime \prime}(\operatorname{lcp}(\widehat{v}, x) u w)$. Similarly to the construction in Step 1, by cutting out the context $\left.\left(\xi^{\prime \prime} \mid \operatorname{lcp}(\widehat{v}, x) u\right)\right|^{w}$ and the loop $\left.\left(\left.\kappa^{\prime \prime}\right|_{\operatorname{lcp}(\widehat{v}, x) u}\right)\right|^{w}$ from $\xi^{\prime \prime}$ and $\kappa^{\prime \prime}$, respectively, we obtain $\widehat{\xi} \in \mathrm{T}_{\Sigma}$ and successful run $\widehat{\kappa} \in \mathrm{R}_{\mathcal{A}}(q, \widehat{\xi})$ such that $\widehat{v} \in \operatorname{pos}(\widehat{\xi}), \widehat{\kappa}(\widehat{v})=\kappa^{\prime \prime}(\widehat{v})$ and $\operatorname{size}(\widehat{\xi})<\operatorname{size}\left(\xi^{\prime \prime}\right)$. The condition $\widehat{v} \in \operatorname{pos}(\widehat{\xi})$ is ensured by $u \in \mathbb{N}_{+}^{+}$.

If there still exists an $x \in \operatorname{pos}(\widehat{\xi})$ such that $x \notin \operatorname{prefix}(\widehat{v}), \widehat{v} \notin \operatorname{prefix}(x)$, and $|\operatorname{rest}(\widehat{v}, x)| \geq|Q|+1$, then we repeat the above procedure with $\widehat{\xi}$ and $\widehat{\kappa}$. After finitely many steps, we obtain $\widehat{\xi}$ and $\widehat{\kappa}$ as desired.

Now we can prove the trimming result for zero-cancellation free strong bimonoids. By Observation 2.6.11(1) this also covers commutative strong bimonoids.

Theorem 7.1.3. Let $\Sigma$ be a ranked alphabet, B be a zero-cancellation free strong bimonoid, and $\mathcal{A}$ be a ( $\Sigma, \mathrm{B})-$ wta. If $\mathcal{A}$ contains a useful state, then we can construct $a(\Sigma, \mathrm{~B})-w t a \mathcal{A}^{\prime}$ such that $\mathcal{A}^{\prime}$ is trim and $\llbracket \mathcal{A}^{\prime} \rrbracket^{\text {run }}=\llbracket \mathcal{A} \rrbracket^{\text {run }}$.

Proof. Let $\mathcal{A}=(Q, \delta, F)$. We construct the set

$$
Q^{\prime}=\bigcup\left(\operatorname{im}(\rho)\left|\xi \in \mathrm{T}_{\Sigma}, \operatorname{height}(\xi)<2\right| Q \mid, \rho \in \mathrm{R}_{\mathcal{A}}(q, \xi), \mathrm{wt}(\xi, \rho) \otimes F_{\rho(\varepsilon)} \neq \mathbb{0}\right)
$$

By Lemma 7.1.2, $Q^{\prime}$ is the set of all useful states of $\mathcal{A}$. Due to our assumption on $\mathcal{A}$, we have $Q^{\prime} \neq \emptyset$. Then we construct the $(\Sigma, \mathrm{B})$-wta $\mathcal{A}^{\prime}=\left(Q^{\prime}, \delta^{\prime}, F^{\prime}\right)$ such that

- $\delta_{k}^{\prime}=\delta_{k}$ restricted to $\left(Q^{\prime}\right)^{k} \times \Sigma^{(k)} \times Q^{\prime}$ for each $k \in \mathbb{N}$, and
- $F^{\prime}=\left.F\right|_{Q^{\prime}}$.

Clearly, $\mathcal{A}^{\prime}$ is trim. Let $\xi \in \mathrm{T}_{\Sigma}$. Then

$$
\begin{aligned}
\llbracket \mathcal{A} \rrbracket^{\mathrm{run}}(\xi) & =\bigoplus_{q \in Q} \bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}}(q, \xi)} \mathrm{wt}_{\mathcal{A}}(\xi, \rho) \otimes F_{q}=\bigoplus_{q \in Q} \bigoplus_{\substack{\rho \in \mathrm{R}_{\mathcal{A}}(q, \xi): \\
\rho \text { is successful }}} \mathrm{wt}_{\mathcal{A}}(\xi, \rho) \otimes F_{q} \\
& =\bigoplus_{q \in Q^{\prime}} \bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}^{\prime}}(q, \xi)} \mathrm{wt}_{\mathcal{A}^{\prime}}(\xi, \rho) \otimes F_{q}^{\prime}=\llbracket \mathcal{A}^{\prime} \rrbracket^{\mathrm{run}}(\xi),
\end{aligned}
$$

where the third equality follows from the fact that, if $\rho$ is successful, then all states in $\operatorname{im}(\rho)$ are useful and that for every $q \in Q^{\prime}$ and $\rho \in \mathrm{R}_{\mathcal{A}^{\prime}}(q, \xi)$ we have $\mathrm{wt}_{\mathcal{A}^{\prime}}(\xi, \rho)=\mathrm{wt}_{\mathcal{A}}(\xi, \rho)$.

### 7.1.3 Local-trimming for arbitrary strong bimonoids

Here we show our second trimming method. It takes a $(\Sigma, B)$-wta over an arbitrary strong bimonoid $B$ as input and constructs a run equivalent $\operatorname{local}-\operatorname{trim}(\Sigma, B)$-wta. This trimming method is adapted from Har78, Thm. 3.2.3] where it is shown how to construct an equivalent reduced context-free grammar from a given context-free grammar.

Theorem 7.1.4. Let $\mathcal{A}$ be $a(\Sigma, \mathrm{~B})-w t a$. If $\mathcal{A}$ contains a local-useful state, then we can construct $a$ $(\Sigma, \mathrm{B})$-wta $\mathcal{A}^{\prime}$ such that $\mathcal{A}^{\prime}$ is local-trim and $\llbracket \mathcal{A}^{\prime} \rrbracket^{\text {run }}=\llbracket \mathcal{A} \rrbracket^{\text {run }}$.

Proof. The following proof is based on Dro22. Let $\mathcal{A}=(Q, \delta, F)$. First, in a bottom-up process, we construct the set $Q_{1}$ of all local-accessible states of $\mathcal{A}$ (by using a mapping $f$ ). Second, in a top-down process, we construct the set of all local-useful states (by using a mapping $g$ ).

We define the mapping $f: \mathcal{P}(Q) \rightarrow \mathcal{P}(Q)$ for each $U \in \mathcal{P}(Q)$ by

$$
f(U)=U \cup\left\{q \in Q \mid \text { there exist } k \in \mathbb{N}, q_{1}, \ldots, q_{k} \in U, \text { and } \sigma \in \Sigma^{(k)} \text { such that } \delta_{k}\left(q_{1} \ldots q_{k}, \sigma, q\right) \neq \mathbb{O}\right\}
$$

Trivially, $f$ is order-preserving and hence continuous, as $Q$ is finite.
Let $Q_{1}$ be the smallest subset of $Q$ which is closed under $f$. Thus, by Theorem 2.6.17, we have that $Q_{1}=\bigcup\left(f^{n}(\emptyset) \mid n \in \mathbb{N}\right)$. Since $\bigcup\left(f^{n}(\emptyset) \mid n \in \mathbb{N}\right)=\bigcup\left(f^{n}(\emptyset)|0 \leq n \leq|Q|)\right.$, we can construct $Q_{1}$. Moreover, each state in $Q_{1}$ is local-accessible in $\mathcal{A}$. Conversely, working up a run, it follows that each local-accessible state of $\mathcal{A}$ belongs to $Q_{1}$. Hence $Q_{1}$ is the set of all local-accessible states of $\mathcal{A}$.

Next we define $g: \mathcal{P}\left(Q_{1}\right) \rightarrow \mathcal{P}\left(Q_{1}\right)$ for each $U \in \mathcal{P}\left(Q_{1}\right)$ by

$$
g(U)=\left\{\begin{array}{cc}
\left\{q \in Q_{1} \mid F_{q} \neq \mathbb{O}\right\} & \text { if } U=\emptyset \\
U \cup\left\{q_{i} \in Q_{1} \mid \text { there exist } k \in \mathbb{N}_{+}, q \in U, q_{1}, \ldots, q_{k} \in Q_{1}, i \in[k]\right. \text { and } & \\
\left.\sigma \in \Sigma^{(k)} \text { such that } \delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right) \neq \mathbb{O}\right\} & \text { otherwise }
\end{array}\right.
$$

The mapping $g$ is order-preserving and hence continuous, as $Q_{1}$ is finite. Let $Q^{\prime}$ be the smallest subset of $Q_{1}$ which is closed under $g$. Thus, by Theorem 2.6.17, we have that $Q^{\prime}=\bigcup\left(g^{n}(\emptyset) \mid n \in \mathbb{N}\right)$. Since $\bigcup\left(g^{n}(\emptyset) \mid n \in \mathbb{N}\right)=\bigcup\left(g^{n}(\emptyset)\left|0 \leq n \leq\left|Q_{1}\right|\right)\right.$, we can construct $Q^{\prime}$.

We claim that

$$
\begin{equation*}
Q^{\prime} \text { is the set of all local-useful states of } \mathcal{A} \text {. } \tag{7.2}
\end{equation*}
$$

(a) First we show that each $q \in Q^{\prime}$ is local-useful. For this, by induction on $\mathbb{N}$, we prove the following statement.

For each $n \in \mathbb{N}_{+}$, we have that each state in $g^{n}(\emptyset)$ is local-useful.
I.B.: Clearly, each state in $g(\emptyset)=\left\{q \in Q_{1} \mid F_{q} \neq \mathbb{O}\right\}$ is local-useful.
I.S.: Let $n \geq 1$ and $p \in g^{n+1}(\emptyset)$. We may assume that $p \in g^{n+1}(\emptyset) \backslash g^{n}(\emptyset)$ because otherwise the statement follows by the I.H. immediately. Then there exist $k \in \mathbb{N}_{+}, j \in[k], q \in g^{n}(\emptyset), q_{1}, \ldots, q_{k} \in Q_{1}$, and $\sigma \in \Sigma^{(k)}$ such that $\delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right) \neq \mathbb{O}$ and $p=q_{j}$.

Since $q_{1}, \ldots, q_{k}$ are in $Q_{1}$, i.e., local-accessible, for each $i \in[k]$ there is a tree $\xi_{i} \in \mathrm{~T}_{\Sigma}$ and run $\rho_{i} \in \mathrm{R}_{\mathcal{A}}\left(q_{i}, \xi_{i}\right)$ such that for every $v \in \operatorname{pos}\left(\xi_{i}\right)$, the weight $\delta_{\ell}\left(\rho_{i}(v 1) \cdots \rho_{i}(v \ell), \xi_{i}(v), \rho_{i}(v)\right) \neq \mathbb{O}$, where $\ell=\operatorname{rk}(\xi(v))$.

By the I.H. $q$ is local-useful, hence there is a $\xi \in \mathrm{T}_{\Sigma}$, and a local-successful run $\rho \in \mathrm{R}_{\mathcal{A}}(\xi)$ such that $q \in \operatorname{im}(\rho)$. Let $w \in \operatorname{pos}(\xi)$ with $\rho(w)=q$.

Now consider the tree $\zeta=\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)$ and define the run $\kappa \in \mathrm{R}_{\mathcal{A}}(\zeta)$ such that $\kappa(\varepsilon)=q$ and for each $i \in[k]$ and $v \in \operatorname{pos}\left(\xi_{i}\right)$ we have $\kappa(i v)=\rho_{i}(v)$.

Note that $\kappa(j)=p$ and

$$
\begin{equation*}
\text { for every } v \in \operatorname{pos}(\zeta), \text { the weight } \delta_{\ell}(\kappa(v 1) \cdots \kappa(v \ell), \zeta(v), \kappa(v)) \neq \mathbb{O} \text {, where } \ell=\operatorname{rk}\left(\xi^{\prime}(v)\right) \tag{7.4}
\end{equation*}
$$

Lastly, we consider the tree $\xi^{\prime}=\xi[\zeta]_{w}$ and the run $\rho^{\prime} \in \mathrm{R}_{\mathcal{A}}\left(\xi^{\prime}\right)$ defined, for each $v \in \operatorname{pos}\left(\xi^{\prime}\right)$, by

$$
\rho^{\prime}(v)= \begin{cases}\kappa(u) & \text { if }\left(\exists u \in \mathbb{N}^{*}\right): v=w u \\ \rho(v) & \text { otherwise }\end{cases}
$$

Since $\rho$ is local-successful and $\kappa$ has property (7.4), the run $\rho^{\prime}$ is also local-successful. Moreover, $\rho^{\prime}(w j)=$ $\kappa(j)=p$, hence $p$ is local-useful. This finishes the proof of (7.3).

Since $Q^{\prime}=\bigcup\left(g^{n}(\emptyset) \mid n \in \mathbb{N}\right)$, we obtain that each state of $Q^{\prime}$ is local-useful in $\mathcal{A}$.
(b) Next we show that each local-useful state of $\mathcal{A}$ is in $Q^{\prime}$.

For this, let $\xi \in \mathrm{T}_{\Sigma}$ and $\rho \in \mathrm{R}_{\mathcal{A}}(\xi)$ be a local-successful run on $\xi$. By induction on ( $\left.\operatorname{pos}(\xi),<_{\text {pref }}\right)$ we show that

$$
\begin{equation*}
\text { for each } w \in \operatorname{pos}(\xi), \text { there exists } n \in \mathbb{N} \text { such that state } \rho(w) \in g^{n}(\emptyset) \tag{7.5}
\end{equation*}
$$

We note that, since $\rho$ is local-successful, $\rho(v) \in Q_{1}$ for each $v \in \operatorname{pos}(\xi)$.
I.B.: Let $w=\varepsilon$. Again, since $\rho$ is local-successful, we have $F_{\rho(\varepsilon)} \neq \mathbb{O}$. Hence $\rho(\varepsilon) \in g(\emptyset)$.
I.S.: Let $w=v i$ for some $v \in \operatorname{pos}(\xi)$ and $i \in \operatorname{maxrk}(\Sigma)$. By I.H., there exists $n \in \mathbb{N}$ such that the state $\rho(v) \in g^{n}(\emptyset)$. Let $\sigma=\xi(v)$ and $k=\operatorname{rk}(\sigma)$. Since $\rho$ is local-successful, we have $\delta_{k}(\rho(v 1) \ldots \rho(v k), \sigma, \rho(v)) \neq$ 0. Then $\rho(v 1), \ldots, \rho(v k) \in g^{n+1}(\emptyset)$ and thus, in particular, $\rho(w) \in g^{n+1}(\emptyset)$.

With this we proved that (7.2) holds. By our assumption on $\mathcal{A}$, the set $Q^{\prime}$ is not empty.

Now we construct the $(\Sigma, \mathrm{B})$-wta $\mathcal{A}^{\prime}=\left(Q^{\prime}, \delta^{\prime}, F^{\prime}\right)$ such that, for each $k \in \mathbb{N}, \delta_{k}^{\prime}=\left.\delta_{k}\right|_{\left(Q^{\prime}\right)^{k} \times \Sigma^{(k)} \times Q^{\prime}}$, and $F^{\prime}=\left.F\right|_{Q^{\prime}}$. Then $\mathcal{A}^{\prime}$ is local-trim.

Finally, we prove that $\llbracket \mathcal{A} \rrbracket^{\text {run }}=\llbracket \mathcal{A}^{\prime} \rrbracket^{\text {run }}$. Let $\xi \in \mathrm{T}_{\Sigma}$. Obviously, $\mathrm{R}_{\mathcal{A}^{\prime}}(\xi) \subseteq \mathrm{R}_{\mathcal{A}}(\xi)$ and for each $\rho \in \mathrm{R}_{\mathcal{A}^{\prime}}(\xi)$ we have $\mathrm{wt}_{\mathcal{A}^{\prime}}(\xi, \rho)=\mathrm{wt}_{\mathcal{A}}(\xi, \rho)$. If $\rho \in \mathrm{R}_{\mathcal{A}}(\xi) \backslash \mathrm{R}_{\mathcal{A}^{\prime}}(\xi)$, then there exists $p \in \operatorname{im}(\rho)$ such that $p$ is not local-useful. Then $\mathrm{wt}_{\mathcal{A}}(\xi, \rho)=\mathbb{O}$ or $F_{\rho(\varepsilon)}=\mathbb{O}$ and hence $\mathrm{wt}_{\mathcal{A}}(\xi, \rho) \otimes F_{\rho(\varepsilon)}=\mathbb{O}$. Thus we can compute

$$
\llbracket \mathcal{A} \rrbracket^{\mathrm{run}}(\xi)=\bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}}(\xi)} \operatorname{wt}_{\mathcal{A}}(\xi, \rho) \otimes F_{\rho(\varepsilon)}=\bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}^{\prime}}(\xi)} \mathrm{wt}_{\mathcal{A}^{\prime}}(\xi, \rho) \otimes F_{\rho(\varepsilon)}^{\prime}=\llbracket \mathcal{A}^{\prime} \rrbracket^{\mathrm{run}}(\xi)
$$

### 7.2 Transforming wta into total wta

We recall that a $(\Sigma, \mathrm{B})$-wta $\mathcal{A}=(Q, \delta, F)$ is total if for every $k \in \mathbb{N}, \sigma \in \Sigma^{(k)}$, and $w \in Q^{k}$ there exists at least one state $q$ such that $\delta_{k}(w, \sigma, q) \neq \mathbb{0}$.
Theorem 7.2.1. Let $\mathcal{A}$ be a $(\Sigma, \mathrm{B})$-wta. We can construct $a(\Sigma, \mathrm{~B})$-wta $\mathcal{A}^{\prime}$ such that $\mathcal{A}^{\prime}$ is total, $\llbracket \mathcal{A} \rrbracket^{\mathrm{init}}=$ $\llbracket \mathcal{A}^{\prime} \rrbracket^{\text {init }}$, and $\llbracket \mathcal{A} \rrbracket^{\text {run }}=\llbracket \mathcal{A}^{\prime} \rrbracket^{\text {run }}$. Moreover, if $\mathcal{A}$ is bu deterministic (or crisp deterministic), then so is $\mathcal{A}^{\prime}$.

Proof. Let $\mathcal{A}=(Q, \delta, F)$. We construct the ( $\Sigma, \mathrm{B})$-wta $\mathcal{A}^{\prime}=\left(Q^{\prime}, \delta^{\prime}, F^{\prime}\right)$ as follows.

- $Q^{\prime}=Q \cup\left\{q_{\perp}\right\}$ where $q_{\perp} \notin Q$,
- $\delta^{\prime}=\left(\delta_{k}^{\prime} \mid k \in \mathbb{N}\right)$ such that for every $k \in \mathbb{N}, \sigma \in \Sigma^{(k)}$, and $q \in Q^{\prime}$, and $q_{1} \cdots q_{k} \in\left(Q^{\prime}\right)^{k}$ :

$$
\left(\delta^{\prime}\right)_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right)= \begin{cases}\delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right) & \text { if } q_{1} \cdots q_{k} \in Q^{k}, q \in Q \\ \mathbb{1} & \text { if } q_{1} \cdots q_{k} \in Q^{k}, q=q_{\perp}, \text { and } \\ & \left(\forall q^{\prime} \in Q\right): \delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q^{\prime}\right)=\mathbb{0} \\ \mathbb{1} & \text { if } q_{\perp} \in\left\{q_{1}, \ldots, q_{k}\right\} \text { and } q=q_{\perp} \\ \mathbb{O} & \text { otherwise }\end{cases}
$$

- $F_{q}^{\prime}=F_{q}$ for each $q \in Q$ and $F_{q_{\perp}}^{\prime}=\mathbb{0}$.

Obviously, if $\mathcal{A}$ is bu deterministic, then so is $\mathcal{A}^{\prime}$. The construction also preserves crisp determinism.
First we prove that $\llbracket \mathcal{A} \rrbracket^{\text {init }}=\llbracket \mathcal{A}^{\prime} \rrbracket^{\text {init }}$. By induction on $\mathrm{T}_{\Sigma}$, we prove that the following statement holds:

$$
\begin{equation*}
\text { For every } \xi \in \mathrm{T}_{\Sigma} \text { and } q \in Q: \mathrm{h}_{\mathcal{A}}(\xi)_{q}=\mathrm{h}_{\mathcal{A}^{\prime}}(\xi)_{q} \tag{7.6}
\end{equation*}
$$

Let $\xi=\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)$ and assume that (7.6) holds for $\xi_{1}, \ldots, \xi_{k}$. Let $q \in Q$. Then we can calculate:

$$
\begin{aligned}
\mathrm{h}_{\mathcal{A}^{\prime}}\left(\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)\right)_{q} & =\bigoplus_{q_{1} \cdots q_{k} \in\left(Q \cup\left\{q_{\perp}\right\}\right)^{k}}\left(\bigotimes_{i \in[k]} \mathrm{h}_{\mathcal{A}^{\prime}}\left(\xi_{i}\right)_{q_{i}}\right) \otimes \delta_{k}^{\prime}\left(q_{1} \cdots q_{k}, \sigma, q\right) \\
= & \bigoplus_{q_{1} \cdots q_{k} \in Q^{k}}\left(\bigotimes_{i \in[k]} \mathrm{h}_{\mathcal{A}^{\prime}}\left(\xi_{i}\right)_{q_{i}}\right) \otimes \delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right) \\
& \text { (note that } \left.\delta_{k}^{\prime}\left(q_{1} \cdots q_{k}, \sigma, q\right)=0 \text { if one of the } q_{i} \text { is } q_{\perp}\right) \\
& \bigoplus_{q_{1} \cdots q_{k} \in Q^{k}}\left(\bigotimes_{i \in[k]} \mathrm{h}_{\mathcal{A}}\left(\xi_{i}\right)_{q_{i}}\right) \otimes \delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right) \\
& =\mathrm{h}_{\mathcal{A}}\left(\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)\right)_{q} .
\end{aligned}
$$

Then for each $\xi \in \mathrm{T}_{\Sigma}$.

$$
\llbracket \mathcal{A}^{\prime} \rrbracket^{\text {init }}(\xi)=\bigoplus_{q \in Q \cup\left\{q_{\perp}\right\}} \mathrm{h}_{\mathcal{A}^{\prime}}(\xi)_{q} \otimes F_{q}^{\prime}=\bigoplus_{q \in Q} \mathrm{~h}_{\mathcal{A}^{\prime}}(\xi)_{q} \otimes F_{q}=\bigoplus_{q \in Q} \mathrm{~h}_{\mathcal{A}}(\xi)_{q} \otimes F_{q}=\llbracket \mathcal{A} \rrbracket^{\text {init }}(\xi)
$$

where the second equality holds because $F_{q_{\perp}}^{\prime}=\mathbb{0}$ and $\left.F^{\prime}\right|_{Q}=F$.

Next we prove that $\llbracket \mathcal{A} \rrbracket^{\text {run }}=\llbracket \mathcal{A}^{\prime} \rrbracket^{\text {run }}$. First, it is easy to see that

$$
\begin{align*}
& \text { for every } \xi \in \mathrm{T}_{\Sigma} \text { and } q \in Q: \mathrm{R}_{\mathcal{A}}(q, \xi) \subseteq \mathrm{R}_{\mathcal{A}^{\prime}}(q, \xi) \text {, and } \\
& \text { for each } \rho \in \mathrm{R}_{\mathcal{A}^{\prime}}(q, \xi) \backslash \mathrm{R}_{\mathcal{A}}(q, \xi): \mathrm{wt}_{\mathcal{A}^{\prime}}(\xi, \rho)=\mathbb{0} \tag{7.7}
\end{align*}
$$

Also it is easy to see that

$$
\begin{equation*}
\text { for each } \xi \in \mathrm{T}_{\Sigma}, q \in Q, \text { and } \rho \in \mathrm{R}_{\mathcal{A}}(q, \xi): \mathrm{wt}_{\mathcal{A}^{\prime}}(\xi, \rho)=\mathrm{wt}_{\mathcal{A}}(\xi, \rho) \text {, } \tag{7.8}
\end{equation*}
$$

where $\mathrm{wt}_{\mathcal{A}^{\prime}}$ and $\mathrm{wt}_{\mathcal{A}}$ use $\delta^{\prime}$ and $\delta$, respectively. Now let $\xi \in \mathrm{T}_{\Sigma}$. Then we can calculate as follows.

$$
\begin{array}{rlrl}
\llbracket \mathcal{A}^{\prime} \rrbracket^{\mathrm{run}}(\xi) & =\bigoplus_{q \in Q^{\prime}} \bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}^{\prime}}(q, \xi)} \mathrm{wt}_{\mathcal{A}^{\prime}}(\xi, \rho) \otimes F_{q}^{\prime} & & \\
& =\bigoplus_{q \in Q} \bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}^{\prime}}(q, \xi)} \mathrm{wt}_{\mathcal{A}^{\prime}}(\xi, \rho) \otimes F_{q} & & \text { (because } \left.F_{q \perp}^{\prime}=0 \text { and }\left.F^{\prime}\right|_{Q}=F\right) \\
& =\bigoplus_{q \in Q} \bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}}(q, \xi)} \mathrm{wt}_{\mathcal{A}^{\prime}}(\xi, \rho) \otimes F_{q} &  \tag{7.7}\\
& =\bigoplus_{q \in Q} \bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}}(q, \xi)} \mathrm{wt}_{\mathcal{A}}(\xi, \rho) \otimes F_{q} & \\
& =\llbracket \mathcal{A} \rrbracket^{\text {run }}(\xi) . & &
\end{array}
$$

### 7.3 Normalizing root weights of wta

Next we want to show that each wta can be transformed into a run equivalent root weight normalized wta. In DPV05, Lm. 4.8] the same statement is proved if $B$ is a semiring. (In fact, in the proof of DPV05, Lm. 4.8] only right-distributivity is used.) Here we present a slightly more complicated construction which allows to prove the statement for an arbitrary strong bimonoid.

Theorem 7.3.1. Let $\Sigma$ be a ranked alphabet, B be a strong bimonoid, and $\mathcal{A}$ be a $(\Sigma, \mathrm{B})$-wta. We can construct $a(\Sigma, B)$-wta $\mathcal{A}^{\prime}$ such that $\mathcal{A}^{\prime}$ is root weight normalized and $\llbracket \mathcal{A} \rrbracket^{\text {run }}=\llbracket \mathcal{A}^{\prime} \rrbracket$ run .

Proof. Let $\mathcal{A}=(Q, \delta, F)$. The idea for the construction of $\mathcal{A}^{\prime}$ is the following. The wta $\mathcal{A}^{\prime}$ simulates $\mathcal{A}$. Additionally, at each leaf of the given input tree $\xi$, the wta $\mathcal{A}^{\prime}$ guesses a state $p$ of $\mathcal{A}$ and stores it in its states; at each non-nullary symbol of $\Sigma$, the wta $\mathcal{A}^{\prime}$ checks whether the guesses in the subtrees are consistent and propagates the guessed state; and at the root, the wta $\mathcal{A}^{\prime}$ multiplies the last transition weight with $F_{p}$ (if $p$ is the state guessed at each leaf) and moves to the final state $q_{f}$. (We note that the idea for maintaining the guessed final states was used in [HVD19, Lm. 9] where initial- and final-state normalization was proved for weighted string automata with storage over unital valuation monoids.)

We construct the $(\Sigma, \mathrm{B})$-wta $\mathcal{A}^{\prime}=\left(Q^{\prime}, \delta^{\prime}, F^{\prime}\right)$ as follows:

- $Q^{\prime}=(Q \times Q) \cup\left\{q_{f}\right\}$ where $q_{f} \notin Q$; for each $q^{\prime} \in Q \times Q$ we let $\left(q^{\prime}\right)_{i}$ denote the $i$-th component of $q^{\prime}$ for $i \in\{1,2\}$;
- $\delta^{\prime}=\left(\delta_{k}^{\prime} \mid k \in \mathbb{N}\right)$ where for each $k \in \mathbb{N}, \sigma \in \Sigma^{(k)}, q_{0}^{\prime} \in Q^{\prime}$ and $q_{1}^{\prime} \cdots q_{k}^{\prime} \in\left(Q^{\prime}\right)^{k}$ we define $\delta_{k}^{\prime}\left(q_{1}^{\prime} \cdots q_{k}^{\prime}, \sigma, q_{0}^{\prime}\right)$ by case analysis as follows. $\underline{\text { Case (a): Let } k=0 \text {. Then }}$

$$
\delta_{0}^{\prime}\left(\varepsilon, \sigma, q_{0}^{\prime}\right)= \begin{cases}\delta_{0}\left(\varepsilon, \sigma,\left(q_{0}^{\prime}\right)_{1}\right) & \text { if } q_{0}^{\prime} \in Q \times Q \\ \bigoplus_{p \in Q} \delta_{0}(\varepsilon, \sigma, p) \otimes F_{p} & \text { if } q_{0}^{\prime}=q_{f}\end{cases}
$$

$\underline{\text { Case (b): Let } k \geq 1 \text {. Then }}$

$$
\begin{aligned}
& \delta_{k}^{\prime}\left(q_{1}^{\prime} \cdots q_{k}^{\prime}, \sigma, q_{0}^{\prime}\right) \\
& = \begin{cases}\delta_{k}\left(\left(q_{1}^{\prime}\right)_{1} \cdots\left(q_{k}^{\prime}\right)_{1}, \sigma,\left(q_{0}^{\prime}\right)_{1}\right) & \text { if }(\exists p \in Q)(\forall i \in[0, k])\left(\exists q_{i} \in Q\right): q_{i}^{\prime}=\left(q_{i}, p\right) \\
\delta_{k}\left(\left(q_{1}^{\prime}\right)_{1} \cdots\left(q_{k}^{\prime}\right)_{1}, \sigma,\left(q_{1}^{\prime}\right)_{2}\right) \otimes F_{\left(q_{1}^{\prime}\right)_{2}} & \text { if }(\exists p \in Q)(\forall i \in[k])\left(\exists q_{i} \in Q\right): q_{i}^{\prime}=\left(q_{i}, p\right) \text { and } q_{0}^{\prime}=q_{f} \\
0 & \text { otherwise, }\end{cases}
\end{aligned}
$$

where in the second case we have used the assumption that $k \geq 1$,

- $F_{q_{f}}^{\prime}=\mathbb{1}$ and $F_{q^{\prime}}^{\prime}=\mathbb{O}$ for each $q^{\prime} \in Q^{\prime} \backslash\left\{q_{f}\right\}$.

It is clear that, for each position of $\xi \in \mathrm{T}_{\Sigma}$ except at its root, $\mathcal{A}^{\prime}$ behaves exactly as $\mathcal{A}$. To make this precise, for each $p \in Q$, we define the family ( $\psi_{p, \xi} \mid \xi \in \mathrm{T}_{\Sigma}$ ) of mappings

$$
\psi_{p, \xi}: \mathrm{R}_{\mathcal{A}}(\xi) \rightarrow \mathrm{R}_{\mathcal{A}^{\prime}}(\xi)
$$

such that, for every $\rho \in \mathrm{R}_{\mathcal{A}}(\xi)$ and $w \in \operatorname{pos}(\xi)$, we let $\psi_{p, \xi}(\rho)(w)=(\rho(w), p)$. Obviously, $\psi_{p, \xi}$ is injective. Also it is obvious that, for every $\xi=\sigma\left(\xi_{1}, \ldots, \xi_{k}\right), i \in[k]$, and $\rho \in \mathrm{R}_{\mathcal{A}}(\xi)$, the following holds:

$$
\begin{equation*}
\left.\psi_{p, \xi}(\rho)\right|_{i}=\psi_{p, \xi_{i}}\left(\left.\rho\right|_{i}\right) \tag{7.9}
\end{equation*}
$$

By induction on $\mathrm{T}_{\Sigma}$, we prove that the following statement holds:
For every $\xi \in \mathrm{T}_{\Sigma}, \rho \in \mathrm{R}_{\mathcal{A}}(\xi)$, and $p \in Q$, we have $\mathrm{wt}_{\mathcal{A}}(\xi, \rho)=\mathrm{wt}_{\mathcal{A}^{\prime}}\left(\xi, \psi_{p, \xi}(\rho)\right)$.
Let $\xi=\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)$. Then we can calculate as follows:

$$
\begin{aligned}
\mathrm{wt}_{\mathcal{A}}(\xi, \rho) & =\left(\bigotimes_{i \in[k]} \mathrm{wt}_{\mathcal{A}}\left(\xi_{i},\left.\rho\right|_{i}\right)\right) \otimes \delta_{k}(\rho(1) \cdots \rho(k), \sigma, \rho(\varepsilon)) \\
& =\left(\bigotimes_{i \in[k]} \operatorname{wt}_{\mathcal{A}^{\prime}}\left(\xi_{i}, \psi_{p, \xi_{i}}\left(\left.\rho\right|_{i}\right)\right)\right) \otimes \delta_{k}^{\prime}((\rho(1), p) \cdots(\rho(k), p), \sigma,(\rho(\varepsilon), p))
\end{aligned}
$$

(by I.H. and construction)

$$
=\left(\bigotimes_{i \in[k]} \mathrm{wt}_{\mathcal{A}^{\prime}}\left(\xi_{i},\left.\psi_{p, \xi}(\rho)\right|_{i}\right)\right) \otimes \delta_{k}^{\prime}((\rho(1), p) \cdots(\rho(k), p), \sigma,(\rho(\varepsilon), p))
$$

$$
=\mathrm{wt}_{\mathcal{A}^{\prime}}\left(\xi, \psi_{p, \xi}(\rho)\right)
$$

Next we prove that $\llbracket \mathcal{A} \rrbracket^{\text {run }}(\xi)=\llbracket \mathcal{A}^{\prime} \rrbracket^{\text {run }}(\xi)$ for each $\xi \in \mathrm{T}_{\Sigma}$ by case analysis.
Case (a): Let $\xi=\alpha$ for some $\alpha \in \Sigma^{(0)}$. Since, for each $p \in Q$, we have $\mathrm{R}_{\mathcal{A}}(p, \alpha)=\{\rho\}$ with $\rho(\varepsilon)=p$ and $\left.\overline{\mathrm{wt}_{\mathcal{A}}(\alpha, \rho}\right)=\delta_{0}(\varepsilon, \alpha, p)$, and similarly, $\mathrm{R}_{\mathcal{A}^{\prime}}\left(q_{f}, \alpha\right)=\left\{\rho^{\prime}\right\}$ with $\rho^{\prime}(\varepsilon)=q_{f}$ and $\mathrm{wt}_{\mathcal{A}^{\prime}}\left(\alpha, \rho^{\prime}\right)=\delta_{0}^{\prime}\left(\varepsilon, \alpha, q_{f}\right)$, we obtain

$$
\begin{aligned}
\llbracket \mathcal{A} \rrbracket^{\mathrm{run}}(\alpha) & =\bigoplus_{p \in Q} \bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}}(p, \xi)} \mathrm{wt}_{\mathcal{A}}(\alpha, \rho) \otimes F_{p}=\bigoplus_{p \in Q} \delta_{0}(\varepsilon, \alpha, p) \otimes F_{p}=\delta_{0}^{\prime}\left(\varepsilon, \alpha, q_{f}\right) \\
& =\delta_{0}^{\prime}\left(\varepsilon, \alpha, q_{f}\right) \otimes F_{q_{f}}^{\prime}=\bigoplus_{q^{\prime} \in Q^{\prime}} \bigoplus_{\rho^{\prime} \in \mathrm{R}_{\mathcal{A}^{\prime}}\left(q^{\prime}, \xi\right)} \mathrm{wt}_{\mathcal{A}^{\prime}}\left(\alpha, \rho^{\prime}\right) \otimes F_{q^{\prime}}^{\prime}=\llbracket \mathcal{A}^{\prime} \rrbracket^{\mathrm{run}}(\alpha) .
\end{aligned}
$$

 assign states of the form $(q, p)$ to inner nodes of $\xi$ for some $q \in Q$. Formally, we define the mapping

$$
\varphi_{p}: \mathrm{R}_{\mathcal{A}}(p, \xi) \rightarrow \mathrm{R}_{\mathcal{A}^{\prime}}^{p}\left(q_{f}, \xi\right)
$$

where $\mathrm{R}_{\mathcal{A}^{\prime}}^{p}\left(q_{f}, \xi\right)=\left\{\rho^{\prime} \in \mathrm{R}_{\mathcal{A}^{\prime}}\left(q_{f}, \xi\right) \mid(\forall w \in \operatorname{pos}(\xi) \backslash\{\varepsilon\}): \rho^{\prime}(w) \in Q \times\{p\}\right\}$, and for every $\rho \in \mathrm{R}_{\mathcal{A}}(p, \xi)$ and $w \in \operatorname{pos}(\xi)$ we let $\left(\varphi_{p}(\rho)\right)(w)=q_{f}$ if $w=\varepsilon$, and $(\rho(w), p)$ otherwise. It is obvious that, for every $p \in Q$, the mapping $\varphi_{p}$ is bijective (note that $\xi \notin \Sigma^{(0)}$ ) and

$$
\begin{equation*}
\text { for every } \rho^{\prime} \in \mathrm{R}_{\mathcal{A}^{\prime}}\left(q_{f}, \xi\right) \backslash \bigcup_{p \in Q} \mathrm{R}_{\mathcal{A}^{\prime}}^{p}\left(q_{f}, \xi\right) \text { we have } \mathrm{wt}_{\mathcal{A}^{\prime}}\left(\xi, \rho^{\prime}\right)=0 \tag{7.11}
\end{equation*}
$$

Moreover, the family $\left(\mathrm{R}_{\mathcal{A}^{\prime}}^{p}\left(q_{f}, \xi\right) \mid p \in Q\right)$ partitions $\bigcup_{p \in Q} \mathrm{R}_{\mathcal{A}^{\prime}}^{p}\left(q_{f}, \xi\right)$, i.e.,

$$
\begin{equation*}
\mathrm{R}_{\mathcal{A}^{\prime}}^{p}\left(q_{f}, \xi\right) \cap \mathrm{R}_{\mathcal{A}^{\prime}}^{p^{\prime}}\left(q_{f}, \xi\right)=\emptyset \text { for every } p, p^{\prime} \in Q \text { with } p \neq p^{\prime} \tag{7.12}
\end{equation*}
$$

Next we prove a correspondence between the weights of runs of $\mathcal{A}$ and $\mathcal{A}^{\prime}$ on $\xi$. We claim:

$$
\begin{equation*}
\text { for every } p \in Q, \text { and } \rho \in \mathrm{R}_{\mathcal{A}}(p, \xi), \text { we have } \mathrm{wt}_{\mathcal{A}}(\xi, \rho) \otimes F_{p}=\mathrm{wt}_{\mathcal{A}^{\prime}}\left(\xi, \varphi_{p}(\rho)\right) \tag{7.13}
\end{equation*}
$$

For the proof let $\xi=\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)$ (note that $k \geq 1$ ). Let $p \in Q$, and $\rho \in \mathrm{R}_{\mathcal{A}}(p, \xi)$ (hence $\left.\rho(\varepsilon)=p\right)$. Then we can calculate as follows:

$$
\begin{aligned}
\mathrm{wt}_{\mathcal{A}}(\xi, \rho) \otimes F_{p} & =\left(\left(\bigotimes_{i \in[k]} \mathrm{wt}_{\mathcal{A}}\left(\xi_{i},\left.\rho\right|_{i}\right)\right) \otimes \delta_{k}(\rho(1) \cdots \rho(k), \sigma, p)\right) \otimes F_{p} \\
& =\left(\bigotimes_{i \in[k]} \mathrm{wt}_{\mathcal{A}}\left(\xi_{i},\left.\rho\right|_{i}\right)\right) \otimes\left(\delta_{k}(\rho(1) \cdots \rho(k), \sigma, p) \otimes F_{p}\right) \\
& =\left(\bigotimes_{i \in[k]} \mathrm{wt}_{\mathcal{A}^{\prime}}\left(\xi_{i}, \psi_{p, \xi_{i}}\left(\left.\rho\right|_{i}\right)\right)\right) \otimes\left(\delta_{k}(\rho(1) \cdots \rho(k), \sigma, p) \otimes F_{p}\right) \quad \quad \text { (by (7.10)) } \\
& =\left(\bigotimes_{i \in[k]} \mathrm{wt}_{\mathcal{A}^{\prime}}\left(\xi_{i}, \psi_{p, \xi_{i}}\left(\left.\rho\right|_{i}\right)\right)\right) \otimes \delta_{k}^{\prime}\left((\rho(1), p) \cdots(\rho(k), p), \sigma, q_{f}\right) \quad \text { (by construction) } \\
& =\left(\bigotimes_{i \in[k]} \mathrm{wt}_{\mathcal{A}^{\prime}}\left(\xi_{i},\left.\varphi_{p}(\rho)\right|_{i}\right)\right) \otimes \delta_{k}^{\prime}\left(\varphi_{p}(\rho)(1) \cdots \varphi_{p}(\rho)(k), \sigma, \varphi_{p}(\rho)(\varepsilon)\right) \\
& =\mathrm{wt}_{\mathcal{A}^{\prime}}\left(\xi, \varphi_{p}(\rho)\right) .
\end{aligned}
$$

Finally, we can prove that $\llbracket \mathcal{A} \rrbracket^{\text {run }}(\xi)=\llbracket \mathcal{A}^{\prime} \rrbracket^{\text {run }}(\xi)$.

$$
\left.\left.\begin{array}{rl}
\llbracket \mathcal{A}^{\prime} \rrbracket^{\mathrm{run}}(\xi) & =\bigoplus_{q^{\prime} \in Q^{\prime}} \bigoplus_{\rho^{\prime} \in \mathrm{R}_{\mathcal{A}^{\prime}}\left(q^{\prime}, \xi\right)} \mathrm{wt}_{\mathcal{A}^{\prime}}\left(\xi, \rho^{\prime}\right) \otimes F_{q^{\prime}}^{\prime} \\
& \left.=\bigoplus_{\rho^{\prime} \in \mathrm{R}_{\mathcal{A}^{\prime}\left(q_{f}, \xi\right)}^{\prime}} \mathrm{wt}_{\mathcal{A}^{\prime}}\left(\xi, \rho^{\prime}\right) \quad \text { (because } F_{q_{f}}^{\prime}=\mathbb{1} \text { and } F_{q^{\prime}}^{\prime}=0 \text { for each } q^{\prime} \in Q^{\prime} \backslash\left\{q_{f}\right\}\right) \\
& =\bigoplus_{\rho^{\prime} \in \cup_{p \in Q} \mathrm{R}_{\mathcal{A}^{\prime}}^{p}\left(q_{f}, \xi\right)} \mathrm{wt}_{\mathcal{A}^{\prime}}\left(\xi, \rho^{\prime}\right) \\
& =\bigoplus_{p \in Q} \bigoplus_{\rho^{\prime} \in \mathrm{R}_{\mathcal{A}^{\prime}}^{p}\left(q_{f}, \xi\right)} \mathrm{wt}_{\mathcal{A}^{\prime}}\left(\xi, \rho^{\prime}\right) \\
& =\bigoplus_{p \in Q} \bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}}(p, \xi)} \mathrm{wt}_{\mathcal{A}^{\prime}}\left(\xi, \varphi_{p}(\rho)\right) \\
& =\bigoplus_{p \in Q} \bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}}(p, \xi)} \mathrm{wt}_{\mathcal{A}}(\xi, \rho) \otimes F_{p}  \tag{7.13}\\
& =\llbracket \mathcal{A} \rrbracket^{\mathrm{run}}(\xi) .
\end{array} \quad \text { (by (because (11)) } \varphi_{p} \text { is a bijection) }\right) \quad \text { (by (7.12)) }\right)
$$

An analysis of the construction in the proof of Theorem 7.3.1 shows that it does not preserve bu determinism. If we restrict the set of weight algebras to the set of commutative semifields, then we can transform each wta into an equivalent one with identity root weights and this transformation preserves bu determinism. The transformation uses the technique of weight pushing HMQ18, Def. 4.1] (also cf. Moh97, p. 296] for the case that B is the tropical semiring on the set of non-negative real numbers), which is of interest of its own.

Formally, let $\mathcal{A}=(Q, \delta, F)$ be a $(\Sigma, \mathrm{B})$-wta over some commutative semifield B. Moreover, let $\lambda$ : $Q \rightarrow B \backslash\{\mathbb{O}\}$ be a mapping such that $\lambda(q)=F_{q}$ for each $q \in \operatorname{supp}(F)$. We define the wta $\operatorname{push}_{\lambda}(\mathcal{A})$ to be the $(\Sigma, \mathrm{B})$-wta $\left(Q, \delta^{\prime}, F^{\prime}\right)$ as follows. For each $k \in \mathbb{N}, \sigma \in \Sigma^{(k)}, q \in Q$, and $q_{1} \cdots q_{k} \in Q$ :

$$
\delta_{k}^{\prime}\left(q_{1} \cdots q_{k}, \sigma, q\right)=\left(\bigotimes_{i \in[k]} \lambda\left(q_{i}\right)^{-1}\right) \otimes \delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right) \otimes \lambda(q)
$$

and for each $q \in Q$

$$
F_{q}^{\prime}= \begin{cases}\mathbb{1} & \text { if } q \in \operatorname{supp}(F) \\ \mathbb{0} & \text { otherwise }\end{cases}
$$

Lemma 7.3.2. (cf. HMQ18, Prop. 4.2]) Let $\mathcal{A}$ and $\lambda$ be defined as above. Then $\llbracket \mathcal{A} \rrbracket=\llbracket \operatorname{push}_{\lambda}(\mathcal{A}) \rrbracket$.
Proof. We abbreviate $\operatorname{push}_{\lambda}(\mathcal{A})$ by $\mathcal{A}^{\prime}$. By induction on $\mathrm{T}_{\Sigma}$, we prove that the following statement holds:

$$
\begin{equation*}
\text { For each } \xi \in \mathrm{T}_{\Sigma} \text { and } q \in Q, \text { we have: } \mathrm{h}_{\mathcal{A}^{\prime}}(\xi)_{q}=\lambda(q) \otimes \mathrm{h}_{\mathcal{A}}(\xi)_{q} \tag{7.14}
\end{equation*}
$$

Let $\xi=\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)$. Then

$$
\begin{aligned}
& \mathrm{h}_{\mathcal{A}^{\prime}}\left(\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)\right)_{q} \\
& =\bigoplus_{q_{1} \cdots q_{k} \in Q^{k}}\left(\bigotimes_{i \in[k]}^{\left.\mathrm{h}_{\mathcal{A}^{\prime}}\left(\xi_{i}\right)_{q_{i}}\right) \otimes \delta^{\prime}\left(q_{1} \cdots q_{k}, \sigma, q\right)}\right. \\
& =\bigoplus_{q_{1} \cdots q_{k} \in Q^{k}}\left(\bigotimes_{i \in[k]} \lambda\left(q_{i}\right) \otimes \mathrm{h}_{\mathcal{A}}\left(\xi_{i}\right)_{q_{i}}\right) \otimes\left(\left(\bigotimes_{i \in[k]} \lambda\left(q_{i}\right)^{-1}\right) \otimes \delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right) \otimes \lambda(q)\right)
\end{aligned}
$$

(by I.H. and construction of $\operatorname{push}_{\lambda}(\mathcal{A})$ )

$$
=\lambda(q) \otimes \bigoplus_{q_{1} \ldots q_{k} \in Q^{k}}\left(\bigotimes_{i \in[k]} \mathrm{h}_{\mathcal{A}}\left(\xi_{i}\right)_{q_{i}}\right) \otimes \delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right)
$$

(by commutativity and associativity of $\otimes$ and by distributivity)

$$
=\lambda(q) \otimes \mathrm{h}_{\mathcal{A}}\left(\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)\right)_{q}
$$

Then, for each $\xi \in \mathrm{T}_{\Sigma}$, we have

$$
\begin{array}{lr}
\llbracket \operatorname{push}_{\lambda}(\mathcal{A}) \rrbracket(\xi) \\
=\bigoplus_{q \in Q} \mathrm{~h}_{\mathcal{A}^{\prime}}(\xi)_{q} \otimes F_{q}^{\prime}=\bigoplus_{q \in Q} \mathrm{~h}_{\mathcal{A}}(\xi)_{q} \otimes \lambda(q) \otimes F_{q}^{\prime} & \text { (by (7.14) and commutativity of } \otimes) \\
=\bigoplus_{q \in \operatorname{supp}(F)} \mathrm{h}_{\mathcal{A}}(\xi)_{q} \otimes \lambda(q) & \\
=\bigoplus_{q \in \operatorname{supp}(F)} \mathrm{h}_{\mathcal{A}}(\xi)_{q} \otimes F_{q} & \text { (by the definition of } \left.F^{\prime}\right) \\
=\llbracket \mathcal{A} \rrbracket(\xi) . & \square
\end{array}
$$

Theorem 7.3.3. Let B be a commutative semifield and $\mathcal{A}$ be a $(\Sigma, \mathrm{B})$-wta. We can construct $a(\Sigma, \mathrm{~B})$-wta $\mathcal{A}^{\prime}$ such that $\mathcal{A}^{\prime}$ has identity root weights and $\llbracket \mathcal{A} \rrbracket=\llbracket \mathcal{A}^{\prime} \rrbracket$. If $\mathcal{A}$ is bu deterministic, then so is $\mathcal{A}^{\prime}$.

Proof. Let $\mathcal{A}=(Q, \delta, F)$ and let $\lambda: Q \rightarrow B \backslash\{\mathbb{O}\}$ be any mapping such that $\lambda(q)=F_{q}$ for each $q \in \operatorname{supp}(F)$. We note that $\operatorname{push}_{\lambda}(\mathcal{A})$ has identity root weights. It is obvious that $\operatorname{push}_{\lambda}(\mathcal{A})$ is bu deterministic, if $\mathcal{A}$ is so. By Lemma 7.3 .2 we have $\llbracket \mathcal{A} \rrbracket=\llbracket \operatorname{push}_{\lambda}(\mathcal{A}) \rrbracket$, and hence we can choose $\mathcal{A}^{\prime}=$ $\operatorname{push}_{\lambda}(\mathcal{A})$.

### 7.4 Normalizing transition weights of wta

Finally, we want to show that each $(\Sigma, B)$-wta $\mathcal{A}$ can be transformed into a run equivalent ( $\Sigma, \mathrm{B}$ )-wta which only has identity transition weights provided that the set

$$
\begin{equation*}
\mathrm{H}(\mathcal{A})=\left\{\operatorname{wt}(\xi, \rho) \mid \xi \in \mathrm{T}_{\Sigma}, \rho \in \mathrm{R}_{\mathcal{A}}(\xi)\right\} \tag{7.15}
\end{equation*}
$$

is finite. Due to this requirement on $\mathcal{A}$, we can code the weights of runs into the states. First we show constructivity of $\mathrm{H}(\mathcal{A})$ if this set is finite.

Lemma 7.4.1. DFKV22, Lm. 6.5] Let $\mathcal{A}=(Q, \delta, F)$ be a $(\Sigma, \mathrm{B})$-wta. If $\mathrm{H}(\mathcal{A})$ is finite, then we can construct the set $\mathrm{H}(\mathcal{A})$.

Proof. Let $\mathrm{H}(\mathcal{A})$ be finite. For every $n \in \mathbb{N}$ and $q \in Q$ let

$$
H_{n, q}=\left\{\operatorname{wt}(\xi, \rho) \mid \xi \in \mathrm{T}_{\Sigma}, \operatorname{height}(\xi) \leq n, \rho \in \mathrm{R}_{\mathcal{A}}(q, \xi)\right\}
$$

Clearly, we have $H_{0, q} \subseteq H_{1, q} \subseteq \ldots \subseteq \mathrm{H}(\mathcal{A})$ for each $q \in Q$. We claim that

$$
\begin{equation*}
\text { for each } n \in \mathbb{N} \text {, if for each } q \in Q: H_{n, q}=H_{n+1, q} \text {, then for each } q \in Q: H_{n+1, q}=H_{n+2, q} \text {. } \tag{7.16}
\end{equation*}
$$

To show this, let $n \in \mathbb{N}, q \in Q$, and $b \in H_{n+2, q}$. There exist $\xi \in \mathrm{T}_{\Sigma}$ and $\rho \in \mathrm{R}_{\mathcal{A}}(q, \xi)$ such that $\operatorname{height}(\xi) \leq n+2$ and $\operatorname{wt}(\xi, \rho)=b$. We may assume that $\operatorname{height}(\xi)=n+2$. Hence $\xi=\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)$ such that height $\left(\xi_{i}\right) \leq n+1$ for each $i \in[k]$. Clearly, for each $i \in[k]$, we have $\mathrm{wt}\left(\xi_{i},\left.\rho\right|_{i}\right) \in H_{n+1, \rho(i)}$, so by our assumption there exist $\zeta_{i} \in \mathrm{~T}_{\Sigma}$ with $\operatorname{height}\left(\zeta_{i}\right) \leq n$ and run $\theta_{i} \in \mathrm{R}_{\mathcal{A}}\left(\rho(i), \zeta_{i}\right)$ such that $\mathrm{wt}\left(\xi_{i},\left.\rho\right|_{i}\right)=\mathrm{wt}\left(\zeta_{i}, \theta_{i}\right)$.

Now let $\zeta=\sigma\left(\zeta_{1}, \ldots, \zeta_{k}\right)$. Obviously, height $(\zeta) \leq n+1$. Moreover, let $\theta \in \mathrm{R}_{\mathcal{A}}(q, \zeta)$ such that $\left.\theta\right|_{i}=\theta_{i}$ for each $i \in[k]$. Clearly, $\operatorname{wt}(\zeta, \theta) \in H_{n+1, q}$, and we calculate

$$
\begin{aligned}
\mathrm{wt}(\zeta, \theta) & =\left(\bigotimes_{i \in[k]} \mathrm{wt}\left(\zeta_{i},\left.\theta\right|_{i}\right)\right) \otimes \delta_{k}(\theta(1) \cdots \theta(i), \sigma, q) \\
& =\left(\bigotimes_{i \in[k]} \mathrm{wt}\left(\xi_{i},\left.\rho\right|_{i}\right)\right) \otimes \delta_{k}(\rho(1) \cdots \rho(i), \sigma, q)=\mathrm{wt}(\xi, \rho)=b
\end{aligned}
$$

This shows that $b \in H_{n+1, q}$, proving (7.16).
We recall that $H_{0, q} \subseteq H_{1, q} \subseteq \ldots \subseteq \mathrm{H}(\mathcal{A})$ for each $q \in Q$. Obviously, we can construct $H_{n, q}$ for every $n \in \mathbb{N}$ and $q \in Q$.

Then, since $\mathrm{H}(\mathcal{A})$ is finite, by constructing $H_{0, q}$ for each $q \in Q, H_{1, q}$ for each $q \in Q$, and so on, we can find the least number $n_{m} \in \mathbb{N}$ such that $H_{n_{m}, q}=H_{n_{m}+1, q}$ for each $q \in Q$ and thus by (7.16) we have $H_{n_{m}, q}=H_{j, q}$ for every $q \in Q$ and $j \in \mathrm{~N}$ with $j \geq n_{m}$.

We show that $\mathrm{H}(\mathcal{A})=\bigcup_{q \in Q} H_{n_{m}, q}$. For this, let $b \in \mathrm{H}(\mathcal{A})$, i.e., $b=\operatorname{wt}(\xi, \rho)$ for some $\xi \in \mathrm{T}_{\Sigma}$ with $\operatorname{height}(\xi)=j, q \in Q$ and $\rho \in \mathrm{R}_{\mathcal{A}}(q, \xi)$. Then $b \in H_{j, q}=H_{n_{m}, q}$. The other inclusion is obvious. Since we can construct the set $\bigcup_{q \in Q} H_{n_{m}, q}$, the set $\mathrm{H}(\mathcal{A})$ can be constructed.

Now we can prove the following normal form theorem.

Theorem 7.4.2. Let $\Sigma$ be a ranked alphabet and B be a strong bimonoid. Moreover, let $\mathcal{A}$ be a $(\Sigma, \mathrm{B})$-wta $\mathcal{A}$ such that $\mathrm{H}(\mathcal{A})$ is finite. Then we can construct $a(\Sigma, \mathrm{~B})$-wta $\mathcal{B}$ such that $\llbracket \mathcal{B} \rrbracket^{\mathrm{run}}=\llbracket \mathcal{A} \rrbracket^{\mathrm{run}}$ and $\mathcal{B}$ has identity transition weights.

Proof. Let $\mathrm{B}=(B, \oplus, \otimes, \mathbb{O}, \mathbb{1})$ and $\mathcal{A}=(Q, \delta, F)$. Since $\mathrm{H}(\mathcal{A})$ is finite, by Lemma 7.4.1, we can construct $\mathrm{H}(\mathcal{A})$.

Now we construct the $(\Sigma, \mathrm{B})$-wta $\mathcal{B}=\left(Q^{\prime}, \delta^{\prime}, F^{\prime}\right)$ as follows.

- $Q^{\prime}=Q \times \mathrm{H}(\mathcal{A})$,
- for each $(q, b) \in Q^{\prime}$, we let $\left(F^{\prime}\right)_{(q, b)}=b \otimes F_{q}$, and
- for every $k \in \mathbb{N}, \sigma \in \Sigma^{(k)}$, and $\left(q_{1}, b_{1}\right), \ldots,\left(q_{k}, b_{k}\right),(q, b) \in Q^{\prime}$ we define

$$
\left(\delta^{\prime}\right)_{k}\left(\left(q_{1}, b_{1}\right) \cdots\left(q_{k}, b_{k}\right), \sigma,(q, b)\right)= \begin{cases}\mathbb{1} & \text { if } b=\left(\otimes_{i \in[k]} b_{i}\right) \otimes \delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right) \\ 0 & \text { otherwise } .\end{cases}
$$

Obviously, $\mathcal{B}$ has identity transition weights.
As preparation for the proof of $\llbracket \mathcal{A} \rrbracket^{\text {run }}=\llbracket \mathcal{B} \rrbracket^{\text {run }}$, we define a bijection between the set of runs of $\mathcal{A}$ on an input tree and the set of runs of $\mathcal{B}$ on that tree. Indeed, it is easy to see that for every $q \in Q, \xi \in \mathrm{~T}_{\Sigma}$, and $b \in \mathrm{H}(\mathcal{A})$, the mapping

$$
\varphi:\left\{\rho \in \mathrm{R}_{\mathcal{A}}(q, \xi) \mid \mathrm{wt}_{\mathcal{A}}(\xi, \rho)=b\right\} \rightarrow\left\{\rho^{\prime} \in \mathrm{R}_{\mathcal{B}}((q, b), \xi) \mid \mathrm{wt}_{\mathcal{B}}\left(\xi, \rho^{\prime}\right)=\mathbb{1}\right\}
$$

defined, for each $w \in \operatorname{pos}(\xi)$, by $\varphi(\rho)(w)=\left(\rho(w), \mathrm{wt}_{\mathcal{A}}\left(\left.\xi\right|_{w},\left.\rho\right|_{w}\right)\right)$ is a bijection.
Then we can calculate as follows.

$$
\begin{aligned}
& \llbracket \mathcal{A} \rrbracket^{\mathrm{run}}(\xi)=\bigoplus_{q \in Q} \bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}}(q, \xi)} \operatorname{wt}_{\mathcal{A}}(\xi, \rho) \otimes F_{q} \\
& =\bigoplus_{q \in Q \in \mathcal{Q}} \bigoplus_{b \in \mathrm{H}(\mathcal{A})} \bigoplus_{\begin{array}{c}
\rho \in \mathrm{R}_{\mathcal{A}}(q, \xi): \\
\text { wt } \\
\text { ( } \xi, \rho)=b
\end{array}} \mathrm{wt}_{\mathcal{A}}(\xi, \rho) \otimes F_{q} \\
& \left.=\bigoplus b \otimes F_{q} \quad \quad \text { (by the definition of } Q^{\prime}\right) \\
& (q, b) \in Q^{\prime} \begin{array}{c}
\rho \in \mathrm{R}_{\mathcal{A}}(q, \xi) ; \\
\mathrm{wt} \mathcal{A}(\xi, \rho)=b
\end{array} \\
& =\bigoplus_{\substack{(q, b) \in Q^{\prime} \\
\rho^{\prime} \in \in \in_{\mathcal{B}}((q, b), \xi): \\
\mathrm{wt}_{\mathcal{B}}\left(\xi, \rho^{\prime}\right)=\mathbb{1}}} b \otimes F_{q} \quad \quad \text { (because } \varphi \text { is a bijection) } \\
& =\bigoplus_{\substack{(q, b) \in Q^{\prime} \\
\rho^{\prime} \in \mathrm{R}_{\mathcal{B}}((q, b), \xi): \\
\mathrm{wt}_{\mathcal{B}}\left(\xi, \rho^{\prime}\right)=\mathbb{1}}} \operatorname{wt}_{\mathcal{B}}\left(\xi, \rho^{\prime}\right) \otimes b \otimes F_{q} \\
& =\bigoplus_{(q, b) \in Q^{\prime}} \bigoplus_{\rho^{\prime} \in \mathrm{R}_{\mathcal{B}}((q, b), \xi)} \operatorname{wt}_{\mathcal{B}}\left(\xi, \rho^{\prime}\right) \otimes b \otimes F_{q} \quad \quad\left(\text { because } \operatorname{wt}_{\mathcal{B}}\left(\xi, \rho^{\prime}\right) \in\{0, \mathbb{1}\}\right) \\
& =\bigoplus_{(q, b) \in Q^{\prime}} \bigoplus_{\rho^{\prime} \in \mathrm{R}_{\mathcal{B}}((q, b), \xi)} \operatorname{wt}_{\mathcal{B}}\left(\xi, \rho^{\prime}\right) \otimes\left(F^{\prime}\right)_{(q, b)} \quad \quad \text { (by definition of } F^{\prime} \text { ) } \\
& =\llbracket \mathcal{B} \rrbracket^{\mathrm{run}}(\xi)
\end{aligned}
$$

If $\mathrm{H}(\mathcal{A})$ is finite, then we can characterize the run semantics of a $(\Sigma, \mathrm{B})$-wta $\mathcal{A}$ in an easy way. We recall that $n b$ denotes the $n$-fold summation of $b \in B$ (cf. (2.15)).

Theorem 7.4.3. Let $\Sigma$ be a ranked alphabet and $\mathrm{B}=(B, \oplus, \otimes, 0, \mathbb{1})$ be a strong bimonoid. Moreover, let $\mathcal{A}=(Q, \delta, F)$ be $a(\Sigma, \mathrm{~B})$-wta such that $\mathrm{H}(\mathcal{A})$ is finite. Then we can construct a $\Sigma$-fta $D$ such that, for each $\xi \in \mathrm{T}_{\Sigma}$, we have $\llbracket \mathcal{A} \rrbracket^{\mathrm{run}}(\xi)=\bigoplus_{q \in Q}\left|\mathrm{R}_{D}^{\mathrm{v}}(q, \xi)\right| F_{q}$.

Proof. By Theorem 7.4.2 we can construct a $(\Sigma, \mathrm{B})$-wta $\mathcal{B}$ such that $\mathcal{B}$ has identity transition weights and $\llbracket \mathcal{B} \rrbracket^{\text {run }}=\llbracket \mathcal{A} \rrbracket^{\text {run }}$. Now we let $D=\operatorname{supp}_{\mathrm{B}}(\mathcal{B})$ (cf. Section 3.4 for the definition of $\operatorname{supp}_{\mathrm{B}}(\mathcal{B})$ ).

Let $\xi \in \mathrm{T}_{\Sigma}$. The following statements are obvious.

$$
\begin{equation*}
\text { For each } \rho \in \mathrm{R}_{\mathcal{B}}(\xi) \text {, we have } \operatorname{wt}_{\mathcal{B}}(\xi, \rho) \in\{0, \mathbb{1}\} \tag{7.17}
\end{equation*}
$$

$$
\begin{equation*}
\text { For every } q \in Q \text { and } \rho \in \mathrm{R}_{\mathcal{B}}(q, \xi) \text {, we have: } \operatorname{wt}_{\mathcal{B}}(\xi, \rho)=\mathbb{1} \text { iff } \rho \in \mathrm{R}_{D}^{\mathrm{v}}(q, \xi) \tag{7.18}
\end{equation*}
$$

Then we can calculate as follows:

$$
\begin{array}{rlr}
\llbracket \mathcal{A} \rrbracket^{\mathrm{run}}(\xi) & =\llbracket \mathcal{B} \rrbracket^{\mathrm{run}}(\xi)=\bigoplus_{q \in Q} \bigoplus_{\rho \in \mathrm{R}_{\mathcal{B}}(q, \xi)} \mathrm{wt}_{\mathcal{B}}(\xi, \rho) \otimes F_{q} \\
& =\bigoplus_{q \in Q} \bigoplus_{\rho \in \mathrm{R}_{D}^{\mathrm{v}}(q, \xi)} F_{q} & \quad(\text { by (7.17) and (7.18)) } \\
& =\bigoplus_{q \in Q}\left|\mathrm{R}_{D}^{\mathrm{v}}(q, \xi)\right| F_{q} .
\end{array}
$$

When additionally restricting the summation to be idempotent, we obtain the following consequence.
Corollary 7.4.4. Let $\mathrm{B}=(B, \oplus, \otimes, \mathbb{0}, \mathbb{1})$ be a strong bimonoid such that $\oplus$ is idempotent. Moreover, let $\mathcal{A}$ be a $(\Sigma, B)$-wta such that $H(\mathcal{A})$ is finite. Then $\llbracket \mathcal{A} \rrbracket^{\text {run }}$ is a recognizable step mapping.

Proof. By Theorem 7.4 .3 we can construct a $\Sigma$-fta $D$ such that, for each $\xi \in \mathrm{T}_{\Sigma}$, we have $\llbracket \mathcal{A} \rrbracket^{\text {run }}(\xi)=$ $\bigoplus_{q \in Q}\left|\mathrm{R}_{D}^{\mathrm{v}}(q, \xi)\right| F_{q}$. Let $D=(Q, \delta, F)$. For each $q \in Q$, we define the $\Sigma$-fta $D^{q}=(Q, \delta,\{q\})$. Then

$$
\begin{equation*}
\text { for each } \xi \in \mathrm{T}_{\Sigma} \text {, we have } \mathrm{R}_{D}^{\mathrm{v}}(q, \xi) \neq \emptyset \text { iff } \xi \in \mathrm{L}\left(D^{q}\right) \text { iff } \chi_{\mathrm{B}}\left(\mathrm{~L}\left(D^{q}\right)\right)(\xi)=\mathbb{1} \tag{7.19}
\end{equation*}
$$

Then, for each $\xi \in \mathrm{T}_{\Sigma}$, we can calculate as follows.

$$
\begin{align*}
\llbracket \mathcal{A} \rrbracket^{\mathrm{run}}(\xi) & =\bigoplus_{q \in Q}\left|\mathrm{R}_{D}^{\mathrm{v}}(q, \xi)\right| F_{q} \\
& =\bigoplus_{\substack{q \in Q: \\
\mathrm{R}_{D}^{\mathrm{v}}(q, \xi) \neq \emptyset}} F_{q} \quad \quad \text { (because } \oplus \text { is idempotent) } \\
& =\bigoplus_{q \in Q} F_{q} \otimes \chi_{\mathrm{B}}\left(\mathrm{~L}\left(D^{q}\right)\right)(\xi) \\
& =\left(\bigoplus_{q \in Q} F_{q} \otimes \chi_{\mathrm{B}}\left(\mathrm{~L}\left(D^{q}\right)\right)\right)(\xi) . \tag{7.19}
\end{align*}
$$

Thus $\llbracket \mathcal{A} \rrbracket^{\text {run }}$ is a recognizable step mapping.
Since for each $(\Sigma, B)$-wta $\mathcal{A}$ where B has a locally finite multiplication, the set $\mathrm{H}(\mathcal{A})$ is finite, Corollary 7.4 .4 implies the following: for each $(\Sigma, B)$-wta $\mathcal{A}$ where B has an idempotent summation and a locally finite multiplication, the run semantics $\llbracket \mathcal{A} \rrbracket^{\text {run }}$ is a recognizable step mapping. Next we show some examples of such strong bimonoids:

- each bounded lattice.
- the semiring $\operatorname{Nat}_{\max ,+, n}=\left([0, n]_{-\infty}, \max , \hat{+}_{n},-\infty, 0\right)$ from Example 2.6.9 (9).
- the semiring $\operatorname{Nat}_{\max , \min }=\left(\mathbb{N}_{\infty}, \max , \min , 0, \infty\right)$ from Example 2.6.9 (11).
- the strong bimonoid $(B, \max , \odot, 0,1)$ where
$-B=\{0\} \cup\{b \in \mathbb{R} \mid \lambda \leq b \leq 1\}$ for some $\lambda \in \mathbb{R}$ with $0<\lambda<\frac{1}{2}$ and
$-a \odot b=a \cdot b$ if $a \cdot b \geq \lambda$, and 0 otherwise,
and $\cdot$ is the usual multiplication of real numbers (thus, $\odot$ is the same as the multiplication of Trunc ${ }_{\lambda}$ of Example 2.6.10(2)).
We note that in Theorem 16.2 .6 we will strengthen Corollary 7.4 .4 by relaxing the requirement that the summation of the strong bimonoid is idempotent.


## Chapter 8

## Weighted context-free grammars

In the theory of tree languages it is a fundamental theorem that the yield of a recognizable tree language is a context-free language and, vice versa, each context-free language can be obtained in this way [Tha67] (also cf. Eng75b, Thm. 3.28] and [GS84, Sect. 3.2]). This fundamental theorem has been generalized to the weighted setting in ÉK03 for continuous and commutative semirings and in FG18 for arbitrary semirings. Here we generalize it for arbitrary strong bimonoids.

First, we recall the concept of weighted context-free grammars (cf. Section 8.1) and prove a number of normal form lemmas: nonterminal form, start-separated, local-reduced, chain-free, and $\varepsilon$-free (cf. Section 8.2). Then we prove that a weighted language is context-free if and only if it is the yield of an r-recognizable weighted tree language (cf. Theorem 8.3.1).

We note that in [RT19] weighted context-free grammars over bimonoids were investigated; roughly speaking, a bimonoid is a strong bimonoid in which the addition need not be commutative and the annihilation law $(b \otimes \mathbb{O}=\mathbb{O} \otimes b=\mathbb{O})$ need not hold. Moreover, we note that we will use the concept of weighted context-free grammar to define the concept of weighted regular tree grammar (cf. Chapter 9) and the concept of weighted projective bimorphism (cf. Section 10.13).

### 8.1 The grammar model

Weighted context-free grammars were introduced in [S63] where the weights reflected the degree of ambiguity (also cf [Sha67]). Similar to such grammars, finite systems of algebraic equations were investigated in SS78, KS86, ÉK03. We build our definitions along DV14a.

A weighted context-free grammar over $\Gamma$ and B (for short: ( $\Gamma, \mathrm{B}$ )-wcfg, or: wcfg) is a tuple $\mathcal{G}=$ $(N, S, R, w t)$, where

- $(N, S, R)$ is a $\Gamma$-cfg which has a terminal ruld 1 and
- wt: $R \rightarrow B$ is the weight mapping.

The $c f g$ underlying $\mathcal{G}$, denoted by $\mathcal{G}^{\mathrm{u}}$, is the $\Gamma$-cfg $(N, S, R)$.
Since each ( $\Gamma, B$ )-wcfg $\mathcal{G}$ is an extension of some $\Gamma$-cfg (viz. by some weight mapping $w t$ ), the concepts and abbreviations which are defined for context-free grammars (cf. Section 2.12) are also available for weighted-context free grammars. This concerns, in particular,

- the abbreviations $\operatorname{lhs}(r)$ and $\operatorname{rhs}_{N, i}(r)$ for some rule $r$ of $\mathcal{G}$,
- the explicit form $A \rightarrow u_{0} A_{1} u_{1} \cdots A_{k} u_{k}$ of a rule $r$ of $\mathcal{G}$,
- the mapping $\pi_{\mathcal{G}^{\mathrm{u}}}: \mathrm{T}_{R} \rightarrow \Gamma^{*}$, which we will call projection of $\mathcal{G}$ and denote by $\pi_{\mathcal{G}}$,
- the concept of rule tree; we will denote the sets $\mathrm{RT}_{\mathcal{G}^{\mathrm{u}}}(A, u), \mathrm{RT}_{\mathcal{G}^{\mathrm{u}}}\left(N^{\prime}, L\right), \mathrm{RT}_{\mathcal{G}^{\mathrm{u}}}(L)$, and $\mathrm{RT}_{\mathcal{G}^{\mathrm{u}}}$ by

[^13]$\mathrm{RT}_{\mathcal{G}}(A, u), \mathrm{RT}_{\mathcal{G}}\left(N^{\prime}, L\right), \mathrm{RT}_{\mathcal{G}}(L)$, and $\mathrm{RT}_{\mathcal{G}}$, respectively.
Next we define the weight of rule trees by using the concept of evaluation algebra (cf. Section 2.9). For this we consider the mapping $w t$ as $\mathbb{N}$-indexed family ( $w t_{k} \mid k \in \mathbb{N}$ ) of mappings $w t_{k}: R^{(k)} \rightarrow B$ by defining $w t_{k}=\left.w t\right|_{R^{(k)}}$. Then, for each $d \in \mathrm{~T}_{R}$, the weight of $d$ is the value $\mathrm{h}_{\mathrm{M}(R, w t)}(d)$ in $B$, where $\mathrm{M}(R, w t)$ is the $(R, w t)$-evaluation algebra. For convenience, we will abbreviate $\mathrm{h}_{\mathrm{M}(R, w t)}$ by $\mathrm{wt}_{\mathcal{G}}$. We note that, for each $d=r\left(d_{1}, \ldots, d_{k}\right)$ in $\mathrm{T}_{R}$, we have
\[

$$
\begin{equation*}
\mathrm{wt}_{\mathcal{G}}(d)=\mathrm{wt}_{\mathcal{G}}\left(d_{1}\right) \otimes \cdots \otimes \mathrm{wt}_{\mathcal{G}}\left(d_{k}\right) \otimes w t(r) \tag{8.1}
\end{equation*}
$$

\]

by (2.26). If there is no confusion, then we drop the index $\mathcal{G}$ from $\mathrm{wt}_{\mathcal{G}}$ and just write wt $(d)$ for the weight of $d$.

We define the weighted rule tree language of $\mathcal{G}$, denoted by $\llbracket \mathcal{G} \rrbracket^{\text {wrt }}$, as the weighted tree language $\llbracket \mathcal{G} \rrbracket^{\mathrm{wrt}}: \mathrm{T}_{R} \rightarrow B$ defined by the Hadamard product

$$
\llbracket \mathcal{G} \rrbracket^{\mathrm{wrt}}=\mathrm{wt}_{\mathcal{G}} \otimes \chi\left(\mathrm{RT}_{\mathcal{G}}\right)
$$

We say that $\mathcal{G}$ is finite-derivational if the set $\operatorname{RT}_{\mathcal{G}}(u)$ is finite for every $u \in \Gamma^{*}$. We note that, if $\mathcal{G}$ is finite-derivational, then $\llbracket \mathcal{G} \rrbracket^{\text {wrt }}$ is $\chi\left(\pi_{\mathcal{G}}\right)$-summable (cf. Section 2.10.3) because $\pi_{\mathcal{G}}^{-1}(u) \cap \operatorname{supp}\left(\llbracket \mathcal{G} \rrbracket^{\text {wrt }}\right) \subseteq$ $\mathrm{RT}_{\mathcal{G}}(u)$ for each $u \in \Gamma^{*}$.

If $\mathcal{G}$ is finite-derivational or B is $\sigma$-complete, then the weighted language generated by $\mathcal{G}$ is the weighted language $\llbracket \mathcal{G} \rrbracket^{\mathrm{s}}: \Gamma^{*} \rightarrow B$ defined by

$$
\llbracket \mathcal{G} \rrbracket^{\mathrm{s}}=\chi\left(\pi_{\mathcal{G}}\right)\left(\llbracket \mathcal{G} \rrbracket^{\mathrm{wrt}}\right)
$$

Thus, for each $u \in \Gamma^{*}$, we have

$$
\begin{align*}
& \llbracket \mathcal{G} \rrbracket^{\mathrm{s}}(u)=\chi\left(\pi_{\mathcal{G}}\right)\left(\llbracket \mathcal{G} \rrbracket^{\mathrm{wrt}}\right)(u)=\sum_{d \in \pi_{\mathcal{G}}^{-1}(u)}^{\oplus} \llbracket \mathcal{G} \rrbracket^{\mathrm{wrt}}(d) \\
& =\sum_{d \in \pi_{\mathcal{G}}^{-1}(u)}^{\oplus}\left(\mathrm{wt}_{\mathcal{G}} \otimes \chi\left(\mathrm{RT}_{\mathcal{G}}\right)\right)(d)=\sum_{d \in \mathrm{RT}_{\mathcal{G}}(u)}^{\oplus} \mathrm{wt}_{\mathcal{G}}(d) \tag{8.2}
\end{align*}
$$

where in the second equality we have used (2.30). In the last expression of (8.2), the sum is well defined: if B is $\sigma$-complete, then the sum is defined on page 21 and, if in addition $\mathcal{G}$ is finite-derivational, then it is equal to $\bigoplus_{d \in \mathrm{RT}_{\mathcal{G}}(u)} \mathrm{wt}(d)$ by (2.6.8) ; if B is not $\sigma$-complete, then $\mathcal{G}$ is finite-derivational and the sum denotes $\bigoplus_{d \in \mathrm{RT}_{\mathcal{G}}(u)} \mathrm{wt}(d)$ by our convention (cf. page 21). Thus, the sum is well defined for arbitrary B. Figure 8.1 illustrates the definition of $\llbracket \mathcal{G} \rrbracket^{\mathrm{s}}(u)$.

Let $\mathcal{G}_{1}$ and $\mathcal{G}_{2}$ be two ( $\Gamma, \mathrm{B}$ )-wcfg such that both $\mathcal{G}_{1}$ and $\mathcal{G}_{2}$ are finite-derivational or B is $\sigma$-complete. Then $\mathcal{G}_{1}$ and $\mathcal{G}_{2}$ are equivalent if $\llbracket \mathcal{G}_{1} \rrbracket^{\mathrm{s}}=\llbracket \mathcal{G}_{2} \rrbracket^{\mathrm{s}}$.

A weighted language $s: \Gamma^{*} \rightarrow B$ is called $(\Gamma, \mathrm{B})$-weighted context-free language if there exists a $(\Gamma, \mathrm{B})-\mathrm{wcfg} \mathcal{G}$ which is finite-derivational if B is not $\sigma$-complete such that $s=\llbracket \mathcal{G} \rrbracket^{\mathrm{s}}$.

Since the Boolean semiring Boole $=(\mathbb{B}, \vee, \wedge, 0,1)$ is $\sigma$-complete, for each ( $\Gamma$, Boole)-wcfg $\mathcal{G}$, the weighted language $\llbracket \mathcal{G} \rrbracket^{\mathrm{s}}: \Gamma^{*} \rightarrow \mathbb{B}$ is defined, and the support of $\llbracket \mathcal{G} \rrbracket^{\mathrm{s}}$ is a context-free language. Vice versa, each context-free language over $\Gamma$ is the support of some ( $\Gamma$, Boole)-weighted context-free language. Indeed, the concept of weighted context-free grammars generalizes the concept of context-free grammars in the following sense.

Observation 8.1.1. Let $L \subseteq \Gamma^{*}$. Then the following two statements are equivalent.
(A) We can construct a $\Gamma$-cfg $G$ which has a terminal rule such that $\mathrm{L}(G)=L$.
(B) We can construct a $(\Gamma$, Boole $)-w c f g \mathcal{G}$ such that $\operatorname{supp}\left(\llbracket \mathcal{G} \rrbracket^{\mathrm{s}}\right)=L$.


Figure 8.1: An illustration of the definition of $\llbracket \mathcal{G} \rrbracket^{\mathrm{s}}(u)$.

Proof. Proof of $(\mathrm{A}) \Rightarrow(\mathrm{B})$ : Let $G=(N, S, R)$ be a $\Gamma$-cfg with a terminal rule. Then we construct the ( $\Gamma$, Boole) $-\operatorname{wcfg} \mathcal{G}=(N, S, R, w t)$ such that for each $r \in R$ we let $w t(r)=1$. Obviously, $\mathrm{RT}_{\mathcal{G}}=\mathrm{RT}_{G}$ and, for each $d \in \mathrm{RT}_{\mathcal{G}}$, we have $\mathrm{wt}_{\mathcal{G}}(d)=1$. Then we can prove $\mathrm{L}(G)=\operatorname{supp}\left(\llbracket \mathcal{G} \rrbracket^{\mathrm{s}}\right)$ as follows.

$$
\left.\left.\begin{array}{rl}
\operatorname{supp}\left(\llbracket \mathcal{G} \rrbracket^{\mathrm{s}}\right) & =\operatorname{supp}\left(\chi\left(\pi_{\mathcal{G}}\right)\left(\mathrm{wt}_{\mathcal{G}} \otimes \chi\left(\mathrm{RT}_{\mathcal{G}}\right)\right)\right)=\operatorname{supp}\left(\chi\left(\pi_{\mathcal{G}}\right)\left(\chi\left(\mathrm{RT}_{\mathcal{G}}\right)\right)\right) \\
& =\pi_{\mathcal{G}}\left(\operatorname{supp}\left(\chi\left(\mathrm{RT}_{\mathcal{G}}\right)\right)\right) \\
& =\pi_{G}\left(\mathrm{RT}_{G}\right)=\mathrm{L}(G) .
\end{array} \quad \text { (because } \pi_{\mathcal{G}}=\pi_{G} \text { and } \mathrm{RT}_{\mathcal{G}}=\mathrm{RT}_{G}\right) \text { (2.31) }\right) ~ l
$$

Proof of $(\mathrm{B}) \Rightarrow(\mathrm{A})$ : Let $\mathcal{G}=(N, S, R, w t)$ be a ( $\Gamma$, Boole)-wcfg. We distinguish two cases.
Case (a): For each terminal rule $r \in R$, we have $w t(r)=0$. Then $\operatorname{supp}\left(\llbracket \mathcal{G} \rrbracket^{\mathrm{s}}\right)=\emptyset$. We construct the $\Gamma$-cfg $G=\left(\left\{A, S_{0}\right\},\left\{S_{0}\right\}, R^{\prime}\right)$ where $R^{\prime}=\{A \rightarrow \varepsilon\}$. Obviously, $\mathrm{L}(G)=\emptyset$.
Case (b): There exists a terminal rule $r \in R$ for which $w t(r)=1$. Then we construct the $\Gamma$-cfg $G=\left(N, S, R^{\prime}\right)$ where $R^{\prime}=\operatorname{supp}(w t)$. We can view $R^{\prime}$ as a ranked alphabet. Hence $\mathrm{RT}_{G}=\mathrm{RT}_{\mathcal{G}} \cap \mathrm{T}_{R^{\prime}}$. We note that, for each $d \in \mathrm{RT}_{\mathcal{G}}$, we have $\mathrm{wt}_{\mathcal{G}}(d)=1$ if $d \in \mathrm{~T}_{R^{\prime}}$, and $\mathrm{wt}_{\mathcal{G}}(d)=0$ otherwise. Then we can prove

$$
\begin{equation*}
\operatorname{supp}\left(\mathrm{wt}_{\mathcal{G}} \otimes \chi\left(\mathrm{RT}_{\mathcal{G}}\right)\right)=\mathrm{RT}_{G} \tag{8.3}
\end{equation*}
$$

For this let $d \in \mathrm{~T}_{R}$. Then

$$
d \in \operatorname{supp}\left(\mathrm{wt}_{\mathcal{G}} \otimes \chi\left(\mathrm{RT}_{\mathcal{G}}\right)\right) \quad \text { iff } \mathrm{wt}_{\mathcal{G}}(d)=1 \wedge d \in \mathrm{RT}_{\mathcal{G}} \quad \text { iff } d \in \mathrm{~T}_{R^{\prime}} \wedge d \in \mathrm{RT}_{\mathcal{G}} \quad \text { iff } \quad d \in \mathrm{RT}_{G}
$$

Then we can prove $\mathrm{L}(G)=\operatorname{supp}\left(\llbracket \mathcal{G} \rrbracket^{\mathrm{s}}\right)$ as follows.

$$
\begin{array}{rlr}
\operatorname{supp}\left(\llbracket \mathcal{G} \rrbracket^{\mathrm{s}}\right) & =\operatorname{supp}\left(\chi\left(\pi_{\mathcal{G}}\right)\left(\mathrm{wt}_{\mathcal{G}} \otimes \chi\left(\mathrm{RT}_{\mathcal{G}}\right)\right)\right) \\
& =\pi_{\mathcal{G}}\left(\operatorname{supp}\left(\operatorname{wt}_{\mathcal{G}} \otimes \chi\left(\mathrm{RT}_{\mathcal{G}}\right)\right)\right) \\
& =\pi_{\mathcal{G}}\left(\mathrm{RT}_{G}\right) & \quad(\text { by (2.31) }) \\
& =\pi_{G}\left(\mathrm{RT}_{G}\right)=\mathrm{L}(G) . & \quad \text { (by (8.3)) } \\
\text { (because } \left.\pi_{\mathcal{G}}=\pi_{G}\right)
\end{array}
$$

Example 8.1.2. Let $G=(N, S, R)$ be an arbitrary $\Gamma$-cfg which has a terminal rule.

1. (cf. CS63, Sec. 2.3]) We consider the $\sigma$-complete semiring $\mathrm{Nat}_{\infty}=\left(\mathbb{N}_{\infty},+, \cdot, 0,1\right)$ of natural numbers and we define the $\left(\Gamma, \operatorname{Nat}_{\infty}\right)-\operatorname{wcfg} \mathcal{G}=(N, S, R, w t)$ such that $w t(r)=1$ for each $r \in R$. Then $\llbracket \mathcal{G} \rrbracket^{\text {wrt }}=\chi\left(\mathrm{RT}_{\mathcal{G}}\right)$, i.e., the characteristic mapping of $\mathrm{RT}_{\mathcal{G}}$. Obviously, the sets $\mathrm{D}_{G, 1}(S, u)$ and
$\mathrm{RT}_{\mathcal{G}}(u)$ are in a one-to-one correspondence for each $u \in \Gamma^{*}$, and $\mathrm{L}(G)=\pi_{\mathcal{G}}\left(\mathrm{RT}_{\mathcal{G}}\right)$. Moreover, for each $u \in \Gamma^{*}$, we have

$$
\llbracket \mathcal{G} \rrbracket^{\mathrm{s}}(u)=\sum_{d \in \mathrm{RT}_{\mathcal{G}}(u)}^{+}=\left|\mathrm{RT}_{\mathcal{G}}(u)\right|=\left|\mathrm{D}_{G, 1}(S, u)\right|
$$

i.e., $\llbracket \mathcal{G} \rrbracket^{\mathrm{s}}(u)$ is the number of leftmost derivations of $u$ in $G$.
2. We consider the tropical semiring $\operatorname{Nat}_{\min ,+}=\left(\mathbb{N}_{\infty}, \min ,+, \infty, 0\right)$, which is $\sigma$-complete. Moreover, we define the $\left(\Gamma, \operatorname{Nat}_{\min ,+}\right)$-wcfg $\mathcal{G}=(N, S, R, w t)$ such that $w t(r)=1$ for each $r \in R$ (as above). Then, for every $u \in \Gamma^{*}$ and $d \in \operatorname{RT}_{\mathcal{G}}(u)$, we have $\operatorname{wt}_{\mathcal{G}}(d)=\operatorname{size}(d)$ and

$$
\llbracket \mathcal{G} \rrbracket^{\mathrm{s}}(u)=\sum_{d \in \mathrm{RT}_{\mathcal{G}}(u)}^{\min } \operatorname{size}(d)
$$

(We recall that $\sum^{\text {min }}$ is the extension of min to an arbitrary countable index set.) Thus $\llbracket \mathcal{G} \rrbracket^{\mathrm{s}}(u)$ is the minimal size of a rule tree for $u$. Since $\operatorname{size}(d)$ is equal to the length of the leftmost derivation which corresponds to $d$, also the number $\llbracket \mathcal{G} \rrbracket^{\mathrm{s}}(u)$ is the minimal length of a leftmost derivation of $u$ in $\mathcal{G}$.

Example 8.1.3. (cf. [FG18, Ex. 1]) Let $\Gamma=\{a, b\}$. It is known that the language $L=\left\{\left.w \in \Gamma^{*}| | w\right|_{a}=\right.$ $\left.|w|_{b}\right\}$ is context-free. It can be generated, for instance, by the $\Gamma$ - $\operatorname{cfg} G=(\{S\}, S, R)$, where $R$ is the set of the following rules:

$$
r_{1}: S \rightarrow S S, r_{2}: S \rightarrow a S b, r_{3}: S \rightarrow b S a, \text { and } r_{4}: S \rightarrow \varepsilon
$$

Now we consider the tropical semiring $\mathrm{Nat}_{\mathrm{min},+}$ and define the ( $\Gamma$, $\mathrm{Nat}_{\mathrm{min},+}$ )-wcfg $\mathcal{G}=(\{S\}, S, R, w t)$, where $w t\left(r_{1}\right)=w t\left(r_{2}\right)=w t\left(r_{3}\right)=0$ and $w t\left(r_{4}\right)=1$.

In Figure 8.2, we show two rule trees $d_{1}$ and $d_{2}$ of $\mathcal{G}$, for which the following hold: $\pi_{\mathcal{G}}\left(d_{1}\right)=a b$, $\pi_{\mathcal{G}}\left(d_{2}\right)=a b a b$, hence $d_{1} \in \operatorname{RT}_{\mathcal{G}}(a b)$ and $d_{2} \in \mathrm{RT}_{\mathcal{G}}(a b a b)$. Clearly, $\mathrm{wt}\left(d_{1}\right)=\mathrm{wt}\left(d_{2}\right)=2$.

Note that $\mathcal{G}$ is not finite-derivational. In fact, for each $u \in L$, the set $\operatorname{RT}_{\mathcal{G}}(u)$ is not finite. However, the semiring $\mathrm{Nat}_{\mathrm{min},+}$ is $\sigma$-complete. Moreover, due to the fact that there exists only one nonterminal, we have $\mathrm{RT}_{\mathcal{G}}=\mathrm{T}_{R}$ (recall that we view $R$ as a ranked alphabet).

It is clear that, for each $d \in \mathrm{RT}_{\mathcal{G}}$, the weight $\mathrm{wt}(d)$ is the number of the occurrences of $r_{4}$ (i.e., the erasing rule) in $d$. Let us denote this number by $\#_{\text {ers }}(d)$. Hence, for each $d \in \mathrm{~T}_{R}$, we have

$$
\llbracket \mathcal{G} \rrbracket^{\mathrm{wrt}}(d)=\#_{\mathrm{ers}}(d)
$$

Moreover, for each $u \in \Gamma^{*}$ we have $\llbracket \mathcal{G} \rrbracket^{\mathrm{s}}(u)=\sum_{d \in \mathrm{RT}_{\mathcal{G}}(u)}^{\min } \#_{\mathrm{ers}}(d)$.

### 8.2 Normal forms of wcfg

Now we define wcfg which satisfy particular properties. Let $\mathcal{G}=(N, S, R, w t)$ be a ( $\Sigma, \mathrm{B})$-wcfg. We say that $\mathcal{G}$ is

- in nonterminal form if, for each rule $A \rightarrow \alpha$ in $R$, either $\alpha \in \Gamma$ or $\alpha \in N^{*}$,
- start-separated if it has exactly one initial nonterminal and this nonterminal does not occur in the right-hand side of a rule,
- chain-free if $\mathcal{G}$ does not have chain rules, and
- $\varepsilon$-free if $\mathcal{G}$ does not have $\varepsilon$-rules.


Figure 8.2: Rule trees for $a b$ and $a b a b$ of Example 8.1.3,

Moreover,

- a rule tree $d \in \mathrm{RT}_{\mathcal{G}}$ is local-successful if $w t(d(w)) \neq \mathbb{1}$ for each $w \in \operatorname{pos}(d)$,
- a nonterminal $A \in N$ is local-useful if there exists a local-successful $d \in \mathrm{RT}_{\mathcal{G}}$ such that $A$ occurs in $d$, and
- $\mathcal{G}$ is local-reduced if $\operatorname{supp}(w t)=R$ and each nonterminal in $N$ is local-useful.

Finally, we call a nonterminal $A \in N$ nullable if $\operatorname{RT}_{\mathcal{G}}(A, \varepsilon) \neq \emptyset$.
We note that $\mathcal{G}$ has a local-useful nonterminal if and only if it has a local-successful rule tree.
Observation 8.2.1. Let $\mathcal{G}=(N, S, R, w t)$ be a finite-derivational and local-reduced ( $\Gamma, \mathrm{B}$ )-wcfg. Then, for every $A \in N$ and $u \in \Gamma^{*}$, the set $\operatorname{RT}_{\mathcal{G}}(A, u)$ is finite.

Proof. We prove by contradiction. We assume that there exist $A \in N$ and $u \in \Gamma^{*}$ such that $\mathrm{RT}_{\mathcal{G}}(A, u)$ is not finite. Since $\mathcal{G}$ is local-reduced, there exists an $x \in \Gamma^{*}$ and a rule tree $d \in \mathrm{RT}_{\mathcal{G}}(x)$ such that $A$ occurs in $d$, i.e., there exists a $w \in \operatorname{pos}(d)$ with $\operatorname{lhs}(d(w))=A$. Then there exists a $y \in \Gamma^{*}$ such that for each $d^{\prime} \in \operatorname{RT}_{\mathcal{G}}(A, u)$, we have $d\left[d^{\prime}\right]_{w} \in \mathrm{RT}_{\mathcal{G}}(y)$. Hence $\mathrm{RT}_{\mathcal{G}}(y)$ is not finite which contradicts that $\mathcal{G}$ is finite-derivational.

Observation 8.2.2. If the $(\Gamma, B)$-wcfg $\mathcal{G}$ is chain-free and $\varepsilon$-free, then it is finite-derivational.
Proof. Let $\mathcal{G}=(N, S, R, w t)$ be chain-free and $\varepsilon$-free. By induction on $\mathrm{T}_{R}$, we prove that the following statement holds:

$$
\begin{equation*}
\text { For each } d \in \mathrm{~T}_{R} \text {, we have } \operatorname{size}(d) \leq 2^{\left|\pi_{\mathcal{G}}(d)\right|} \tag{8.4}
\end{equation*}
$$

For this, let $d=r\left(d_{1}, \ldots, d_{k}\right)$ be in $\mathrm{T}_{R}$ with $r=\left(A \rightarrow u_{0} A_{1} u_{1} \cdots A_{k} u_{k}\right)$. Since $\mathcal{G}$ is chain-free, in case $k=1$ we have $\left|u_{0} u_{1}\right|>0$. Moreover, since $\mathcal{G}$ is $\varepsilon$-free, we have that $\left|\pi_{\mathcal{G}}\left(d_{i}\right)\right|>0$ for each $i \in[k]$. Then we can calculate as follows:

$$
\begin{aligned}
\operatorname{size}\left(r\left(d_{1}, \ldots, d_{k}\right)\right) & =1+\underset{i \in[k]}{+} \operatorname{size}\left(d_{i}\right) \\
& \leq 1+\prod_{i \in[k]} 2^{\left|\pi_{\mathcal{G}}\left(d_{i}\right)\right|} \\
& \leq 2^{\left|u_{0} u_{1} \cdots u_{k}\right|} \cdot 2^{\left|\pi_{\mathcal{G}}\left(d_{1}\right)\right|} \ldots \ldots \cdot 2^{\left|\pi_{\mathcal{G}}\left(d_{k}\right)\right|} \quad \text { (by I.H.) } \\
& =2^{\left|u_{0} u_{1} \cdots u_{k}\right|++_{i \in[k] \mid}\left|\pi_{\mathcal{G}}\left(d_{i}\right)\right|}=2^{\left|\pi_{\mathcal{G}}\left(r\left(d_{1}, \ldots, d_{k}\right)\right)\right|}
\end{aligned} \quad \text { (since } \mathcal{G} \text { is chain-free and } \varepsilon \text {-free) } \quad .
$$

By (8.4), for every $u \in \Gamma^{*}$ and $d \in \operatorname{RT}_{\mathcal{G}}(u)$, we have that $\operatorname{size}(d) \leq 2^{|u|}$, and hence $\mathcal{G}$ is finite-derivational.

Next we show the following normal form constructions for wcfg:

- nonterminal form,
- start-separated,
- local-reduced,
- chain-free, and
- $\varepsilon$-free.

Lemma 8.2.3. Let $\mathcal{G}$ be a $(\Gamma, B)$-wcfg such that $\mathcal{G}$ is finite-derivational or B is $\sigma$-complete. We can construct a $(\Gamma, \mathrm{B})$-wcfg $\mathcal{G}^{\prime}$ such that $\mathcal{G}^{\prime}$ is in nonterminal form and $\llbracket \mathcal{G}^{\prime} \rrbracket^{\mathrm{s}}=\llbracket \mathcal{G} \rrbracket^{\mathrm{s}}$. Moreover, the construction preserves the properties finite-derivational, start-separated, chain-free, $\varepsilon$-free, and local-reduced.

Proof. Let $\mathcal{G}=(N, S, R, w t)$. We construct the $(\Gamma, B)-\operatorname{wcfg} \mathcal{G}^{\prime}=\left(N^{\prime}, S, R^{\prime}, w t^{\prime}\right)$ such that $N^{\prime}=N \cup\left\{A_{a} \mid\right.$ $a \in \Gamma\}$ and, for each rule $r=(A \rightarrow \alpha)$ in $R$,

- if $\alpha \in \Gamma$, then $r$ is in $R^{\prime}$; we let $w t^{\prime}(r)=w t(r)$,
- otherwise the rule $r^{\prime}=\left(A \rightarrow \alpha^{\prime}\right)$ is in $R^{\prime}$ where $\alpha^{\prime}$ is obtained from $\alpha$ by replacing each symbol $a \in \Gamma$ by $A_{a}$; we let $w t^{\prime}\left(r^{\prime}\right)=w t(r)$.
Moreover, for each $a \in \Gamma$, the rule $A_{a} \rightarrow a$ is in $R^{\prime}$ with $w t^{\prime}\left(A_{a} \rightarrow a\right)=\mathbb{1}$.
Clearly, $\mathcal{G}^{\prime}$ is in nonterminal form and the construction preserves the mentioned properties. In particular, if $\mathcal{G}$ is finite-derivational, then so is $\mathcal{G}^{\prime}$. Hence $\llbracket \mathcal{G}^{\prime} \rrbracket^{\mathrm{s}}$ is defined. It is clear that, for each $u \in \Gamma^{*}$, the sets $\mathrm{RT}_{\mathcal{G}}(u)$ and $\mathrm{RT}_{\mathcal{G}^{\prime}}(u)$ are in a one-to-one correspondence, and $\mathrm{wt}_{\mathcal{G}}(d)=\mathrm{wt}_{\mathcal{G}^{\prime}}\left(d^{\prime}\right)$ if $d$ and $d^{\prime}$ correspond to each other. Thus $\llbracket \mathcal{G} \rrbracket^{\mathrm{s}}=\llbracket \mathcal{G}^{\prime} \rrbracket^{\mathrm{s}}$.

Lemma 8.2.4. Let $\mathcal{G}$ be a $(\Gamma, B)$-wcfg such that $\mathcal{G}$ is finite-derivational or B is $\sigma$-complete. We can construct a $(\Gamma, \mathrm{B})-\operatorname{wcfg} \mathcal{G}^{\prime}$ such that $\mathcal{G}^{\prime}$ is start-separated and $\llbracket \mathcal{G}^{\prime} \rrbracket^{\mathrm{s}}=\llbracket \mathcal{G} \rrbracket^{\mathrm{s}}$. Moreover, the construction preserves the properties finite-derivational, nonterminal form, $\varepsilon$-free, and local-reduced. The construction does not preserve the property chain-free.

Proof. Let $\mathcal{G}=(N, S, R, w t)$. We construct the $(\Gamma, \mathrm{B})-\mathrm{wcfg} \mathcal{G}^{\prime}=\left(N^{\prime}, S_{0}, R^{\prime}, w t^{\prime}\right)$ as follows. We let $N^{\prime}=N \cup\left\{S_{0}\right\}$ where $S_{0} \notin N \cup \Gamma$. The set $R^{\prime}$ contains the following rules.

- For each $A \in S$, the rule $r=\left(S_{0} \rightarrow A\right)$ is in $R^{\prime}$ with $w t^{\prime}(r)=\mathbb{1}$.
- Each rule $r \in R$ is in $R^{\prime}$ with $w t^{\prime}(r)=w t(r)$.

It is clear that $\mathcal{G}^{\prime}$ is start-separated and that the construction preserves the mentioned properties. In particular, if $\mathcal{G}$ is finite-derivational, then so is $\mathcal{G}^{\prime}$. Hence $\llbracket \mathcal{G}^{\prime} \rrbracket^{\mathrm{s}}$ is defined. Moreover, it is clear that,
for every $A \in N$ and $u \in \Gamma^{*}$, we have $\operatorname{RT}_{\mathcal{G}}(A, u)=\operatorname{RT}_{\mathcal{G}^{\prime}}(A, u)$
and $\mathrm{wt}_{\mathcal{G}}(d)=\mathrm{wt}_{\mathcal{G}^{\prime}}(d)$ for each $d \in \mathrm{RT}_{\mathcal{G}}(A, u)$.
We can prove that $\llbracket \mathcal{G} \rrbracket^{\mathrm{s}}=\llbracket \mathcal{G}^{\prime} \rrbracket^{\mathrm{s}}$ as follows. Let $u \in \Gamma^{*}$. Then

$$
\begin{aligned}
\llbracket \mathcal{G} \rrbracket^{\mathrm{s}}(u) & =\sum_{d \in \mathrm{RT}_{\mathcal{G}}(S, u)}^{\oplus} \mathrm{wt}_{\mathcal{G}^{\prime}}(d) \\
& \left.=\bigoplus_{A \in S} \sum_{d \in \mathrm{RT}_{\mathcal{G}}(A, u)}^{\oplus} \mathrm{wt}_{\mathcal{G}}(d) \quad \text { (because the family }\left(\mathrm{RT}_{\mathcal{G}}(A, u) \mid A \in S\right) \text { partitions } \mathrm{RT}_{\mathcal{G}}(S, u) .\right) \\
& =\bigoplus_{A \in S} \sum_{d^{\prime} \in \mathrm{RT}_{\mathcal{G}^{\prime}}(A, u)}^{\oplus} \mathrm{wt}_{\mathcal{G}^{\prime}}\left(d^{\prime}\right) \\
& =\bigoplus_{A \in S} \sum_{d^{\prime} \in \mathrm{RT}_{\mathcal{G}^{\prime}}(A, u)}^{\oplus} \mathrm{wt}_{\mathcal{G}^{\prime}}\left(d^{\prime}\right) \otimes w t^{\prime}\left(S_{0} \rightarrow A\right) \\
& =\sum_{d^{\prime \prime} \in \mathrm{RT}_{\mathcal{G}^{\prime}}\left(S_{0}, u\right)} \mathrm{wt}_{\mathcal{G}^{\prime}}\left(d^{\prime \prime}\right)
\end{aligned}
$$

(because the family $\left(\left\{\left(S_{0} \rightarrow A\right)\left(d^{\prime}\right) \mid d^{\prime} \in \mathrm{RT}_{\mathcal{G}^{\prime}}(A, u)\right\} \mid A \in S\right)$ partitions $\left.\mathrm{RT}_{\mathcal{G}^{\prime}}\left(S_{0}, u\right)\right)$

$$
=\llbracket \mathcal{G}^{\prime} \rrbracket^{\mathrm{s}}(u) .
$$

Next we will prove that for each wcfg we can construct an equivalent local-reduced wcfg. We follow the outline of the proof of constructing an equivalent local-trim wta from a given wta (cf. Theorem 7.1.4). As an auxiliary tool, we associate to each ( $\Gamma, \mathrm{B}$ )-wcfg $\mathcal{G}=(N, S, R, w t)$, the context-free grammar $\mathrm{G}(\mathcal{G})=\left(N \cup\left\{S_{0}\right\}, S_{0}, P\right)$ over $\Gamma$, where $S_{0}$ is a new symbol and $P$ is the smallest set of rules such that

- for each $A \in S$, the rule $S_{0} \rightarrow A$ is in $P$ and
- $\operatorname{supp}(w t) \subseteq P$.

Then it is obvious that a nonterminal $A \in N$ is local-useful in $\mathcal{G}$ if and only if it is useful in $\mathrm{G}(\mathcal{G})$. This implies that $\mathcal{G}$ is local-reduced if and only if $\mathrm{G}(\mathcal{G})$ is reduced.

Lemma 8.2.5. Let $\mathcal{G}$ be a $(\Gamma, B)$-wcfg such that $\mathcal{G}$ is finite-derivational or B is $\sigma$-complete. If $\mathcal{G}$ has a local-successful rule tree, then we can construct a $(\Gamma, B)$-wcfg $\mathcal{G}^{\prime}$ such that $\mathcal{G}^{\prime}$ is local-reduced and $\llbracket \mathcal{G}^{\prime} \rrbracket^{\mathrm{s}}=\llbracket \mathcal{G} \rrbracket^{\mathrm{s}}$. Moreover, the construction preserves the properties finite-derivational, nonterminal form, start-separated, chain-free, and $\varepsilon$-free.

Proof. Let $\mathcal{G}=(N, S, R, w t)$. We can construct the context-free grammar $G(\mathcal{G})=\left(N \cup\left\{S_{0}\right\}, S_{0}, P\right)$ associated to $\mathcal{G}$. Due to our assumption on $\mathcal{G}$ we have $\mathrm{L}(\mathrm{G}(\mathcal{G})) \neq \emptyset$. Thus, by Theorem 2.12.1, a contextfree grammar $G^{\prime}=\left(N^{\prime}, S_{0}, P^{\prime}\right)$ can be constructed such that $G^{\prime}$ is reduced and $\mathrm{L}\left(G^{\prime}\right)=\mathrm{L}(\mathrm{G}(\mathcal{G}))$. By the proof of that theorem, we know that $N^{\prime}=N_{u} \cup\left\{S_{0}\right\}$, where $N_{u}$ is the set of all useful nonterminals in $N$. Moreover, $N_{u} \neq \emptyset$ by our assumption on $\mathcal{G}$.

Now let $\mathcal{G}^{\prime}=\left(N_{u}, S \cap N_{u}, R^{\prime}, w t^{\prime}\right)$ be the ( $\Gamma, \mathrm{B}$ )-wcfg such that

- $R^{\prime}=\left\{(A \rightarrow \alpha) \in R \mid A \in N_{u}, \alpha \in\left(N_{u} \cup \Gamma\right)^{*}\right\}$ and
- $w t^{\prime}(r)=w t(r)$ for each $r \in R^{\prime}$.

We note that, since $\mathcal{G}$ has a local-successful rule tree, there exists a terminal rule $B \rightarrow u$ in $R$ and $B$ is useful in $\mathrm{G}(\mathcal{G})$. This rule is also in $R^{\prime}$, hence $R^{\prime}$ contains a terminal rule. Moreover, it is obvious that $\mathcal{G}^{\prime}$ is local-reduced and the mentioned properties are preserved. Thus, in particular, $\llbracket \mathcal{G}^{\prime} \rrbracket$ is defined.

Lastly, we prove that $\llbracket \mathcal{G} \rrbracket^{\mathrm{s}}=\llbracket \mathcal{G}^{\prime} \rrbracket^{\mathrm{s}}$. Let $u \in \Gamma^{*}$. Obviously, $\mathrm{RT}_{\mathcal{G}^{\prime}}(u) \subseteq \mathrm{RT}_{\mathcal{G}}(u)$ and for each $d \in$ $\operatorname{RT}_{\mathcal{G}^{\prime}}(u)$ we have $\operatorname{wt}_{\mathcal{G}^{\prime}}(d)=\operatorname{wt}_{\mathcal{G}}(d)$. For each $d \in \mathrm{RT}_{\mathcal{G}}(u) \backslash \mathrm{RT}_{\mathcal{G}^{\prime}}(u)$, there exists $A^{\prime} \in N$ such that $A^{\prime}$ occurs in $d$ and $A^{\prime}$ is not local-useful. Hence $\mathrm{wt}_{\mathcal{G}}(d)=\mathbb{0}$. Thus we can compute

$$
\llbracket \mathcal{G} \rrbracket^{\mathrm{s}}(u)=\bigoplus_{d \in \mathrm{RT}_{\mathcal{G}}(u)} \mathrm{wt}_{\mathcal{G}}(d)=\bigoplus_{d \in \mathrm{RT}_{\mathcal{G}^{\prime}}(u)} \mathrm{wt}_{\mathcal{G}^{\prime}}(d)=\llbracket \mathcal{G}^{\prime} \rrbracket^{\mathrm{s}}(u) .
$$

Next we show how a wcfg can be transformed into an equivalent chain-free wcfg.

Theorem 8.2.6. (cf. KS86, Thm. 14.2]) Let $\Gamma$ be an alphabet, B be a semiring, and $\mathcal{G}$ be a ( $\Gamma, B$ )wcfg such that (a) $\mathcal{G}$ is finite-derivational or (b) B is $\sigma$-complete. Then there exists a ( $\Gamma, \mathrm{B}$ )-wcfg $\mathcal{G}^{\prime}$ such that $\mathcal{G}^{\prime}$ is chain-free and $\llbracket \mathcal{G}^{\prime} \rrbracket^{\mathrm{s}}=\llbracket \mathcal{G} \rrbracket^{\mathrm{s}}$. If $\mathcal{G}$ has one of the following properties, then also $\mathcal{G}^{\prime}$ has it: finite-derivational, nonterminal form, start-separated, $\varepsilon$-free, and local-reduced. Moreover, if $\mathcal{G}$ is finite-derivational, then we can even construct $\mathcal{G}^{\prime}$.

Proof. Let $\mathcal{G}=(N, S, R, w t)$. By standard pumping methods, we can decide whether $\mathcal{G}$ has a localsuccessful rule tree. If the answer is no, then $\llbracket \mathcal{G} \rrbracket^{s}=\widetilde{\mathbb{D}}$ and thus the statement of the lemma holds obviously. Otherwise, by Lemma 8.2.5 we can construct a ( $\Gamma, B$ )-wcfg which is local-reduced and equivalent to $\mathcal{G}$. So we may assume that $\mathcal{G}$ is local-reduced.

We will prove the theorem by case analysis according to the Cases (a) and (b). Before doing so, we make some common preparations. Since $B$ is a semiring, we can consider the semiring $\left(B^{N \times N},+, \cdot, \mathrm{M}_{\mathbb{D}}, \mathrm{M}_{\mathbb{I}}\right)$ of all $N$-square matrices over $B$ (where we assume an arbitrary but fixed linear
order on $N) 2$ We define the $N$-square matrix $M$ over $B$ such that, for every $A, A^{\prime} \in N$ :

$$
M_{A^{\prime}, A}= \begin{cases}w t(r) & \text { if } r=\left(A \rightarrow A^{\prime}\right) \text { is in } R \\ \mathbb{0} & \text { otherwise }\end{cases}
$$

Then, for each $n \in \mathbb{N}$, we define the $N$-square matrix $M^{n}$ over $B$ such that $M^{0}=\mathrm{M}_{\mathbb{1}}$ and $M^{n+1}=$ $M^{n} \cdot M$. Since $\otimes$ is distributive with respect to $\oplus$, for each $n \in \mathbb{N}_{+}$and $A, A^{\prime} \in N$, we have

$$
\begin{equation*}
\left(M^{n}\right)_{A^{\prime}, A}=\bigoplus_{\substack{C_{1}, \ldots, C_{n+1} \in N: \\ C_{1}=A, C_{n+1}=A^{\prime}}} \bigotimes_{j \in[0, n-1]} M_{C_{n+1-j}, C_{n-j}} \tag{8.6}
\end{equation*}
$$

Hence, $\left(M^{n}\right)_{A^{\prime}, A}$ is the sum of all products

$$
\begin{equation*}
\mathrm{wt}\left(r_{n}\right) \otimes \cdots \otimes \mathrm{wt}\left(r_{1}\right) \tag{8.7}
\end{equation*}
$$

where, for each $i \in[n], r_{i}$ is a chain-rule of the form $C_{i} \rightarrow C_{i+1}$ such that $C_{1}=A$ and $C_{n+1}=A^{\prime}$.
We start the proof with Case (b), i.e., we assume that B is $\sigma$-complete. Then also the semiring $\left(B^{N \times N},+, \cdot, \mathrm{M}_{\mathbb{D}}, \mathrm{M}_{\mathbb{I}}\right)$ is $\sigma$-complete and hence the matrix $M^{*}$ is well defined where $M^{*}=\sum_{n \in \mathbb{N}}^{+} M^{n}$. Hence, $\left(M^{*}\right)_{A^{\prime}, A}$ is the sum of all products (8.7) for some $n \in \mathbb{N}$.

Then we eliminate the chain rules from $\mathcal{G}$ (cf., e.g., ÉK03, Thm. 3.2], FMV11, Lm. 3.2], and [FHV18, Thm. 6.3]). We define the ( $\Gamma, \mathrm{B})-\mathrm{wcfg} \mathcal{G}^{\prime}=\left(N, S, R^{\prime}, w t^{\prime}\right)$ as follows. Let

$$
R^{\prime}=\left\{A \rightarrow \alpha \mid A \in N, \alpha \notin N,\left(\exists A^{\prime} \in N\right): A^{\prime} \rightarrow \alpha \text { is in } R\right\}
$$

Moreover, for each rule $A \rightarrow \alpha$ in $R^{\prime}$, we define

$$
w t^{\prime}(A \rightarrow \alpha)=\bigoplus_{\substack{A^{\prime} \in N: \\\left(A^{\prime} \rightarrow \alpha\right) \in R}} w t\left(A^{\prime} \rightarrow \alpha\right) \otimes\left(M^{*}\right)_{A^{\prime}, A}
$$

Obviously, $\mathcal{G}^{\prime}$ is chain-free and each mentioned property is preserved. In particular, $\llbracket \mathcal{G}^{\prime} \rrbracket^{\mathrm{s}}$ is defined.
In fact, the above definition of $\mathcal{G}^{\prime}$ is the same as the corresponding one in the proof of [FHV18, Thm. 6.3] for the case that (i) the storage type is the trivial one and (ii) the M-monoid $K$ is the M-monoid associated with the $\sigma$-complete semiring B (for the concept of "M-monoid associated with a semiring" cf. [FMV09, Def. 8.5] or [FSV12, p. 261]). By instantiating the correctness proof of [FHV18, Thm. 6.3] to the case specified by (i) and (ii), we obtain a proof of $\llbracket \mathcal{G}^{\prime} \rrbracket^{\mathrm{s}}=\llbracket \mathcal{G} \rrbracket^{\mathrm{s}}$. For the sake of $\sigma$-completeness, we repeat this correctness proof here.

We note that, in virtue of the definition of $\mathcal{G}^{\prime}$, we cannot construct $\mathcal{G}^{\prime}$. This is due to the fact that the definition of the weights of rules involves the matrix $M^{*}$, and, although $M^{*}$ is well defined, in general, we cannot compute it algorithmically.

For the inductive definition of a mapping that relates rule trees of $\mathcal{G}$ with rule trees of $\mathcal{G}^{\prime}$, we employ the well-founded set $\left(\mathrm{RT}_{\mathcal{G}}\left(N, \Gamma^{*}\right), \prec\right)$ where we let

$$
\prec=\prec_{R} \cap\left(\operatorname{RT}_{\mathcal{G}}\left(N, \Gamma^{*}\right) \times \operatorname{RT}_{\mathcal{G}}\left(N, \Gamma^{*}\right)\right)
$$

(for the definition of $\prec_{R}$ we refer to page 433). Obviously, $\prec$ is well-founded and $\min _{\prec}\left(\mathrm{RT}_{\mathcal{G}}\left(N, \Gamma^{*}\right)\right.$ ) is the set of terminal rules of $R$, which is not empty. We define the mapping eff : $\mathrm{RT}_{\mathcal{G}}\left(N, \Gamma^{*}\right) \rightarrow \mathrm{RT}_{\mathcal{G}^{\prime}}\left(N, \Gamma^{*}\right)$ by induction on $\left(\operatorname{RT}_{\mathcal{G}}\left(N, \Gamma^{*}\right), \prec\right)$ as follows. Let $d \in \operatorname{RT}_{\mathcal{G}}\left(N, \Gamma^{*}\right)$. Then

- there exist $n \in \mathbb{N}$ and rules $r_{1}=\left(C_{1} \rightarrow C_{2}\right), \ldots, r_{n}=\left(C_{n} \rightarrow C_{n+1}\right)$, and $C_{n+1} \rightarrow \alpha$ with $\alpha=u_{0} A_{1} u_{1} \ldots A_{k} u_{k}$ in $R$ such that $\alpha \notin N$ and

[^14]- for each $i \in[k]$ there exists a rule tree $d_{i} \in \operatorname{RT}_{\mathcal{G}}\left(A_{i}, \Gamma^{*}\right)$
such that $d=r_{1} \ldots r_{n}\left(C_{n+1} \rightarrow \alpha\right)\left(d_{1}, \ldots, d_{k}\right)$. We define

$$
\operatorname{eff}(d)=\left(C_{1} \rightarrow \alpha\right)\left(\operatorname{eff}\left(d_{1}\right), \ldots, \operatorname{eff}\left(d_{k}\right)\right)
$$

Then $\operatorname{lhs}\left(\operatorname{eff}\left(d_{i}\right)(\varepsilon)\right)=A_{i}$ for each $i \in[k]$. Moreover,

$$
\begin{align*}
& \text { for every } A \in N, u \in \Gamma^{*} \text {, and } d \in \operatorname{RT}_{\mathcal{G}}\left(N, \Gamma^{*}\right) \text { we have: } \\
& d \in \operatorname{RT}_{\mathcal{G}}(A, u) \text { if and only if eff }(d) \in \operatorname{RT}_{\mathcal{G}^{\prime}}(A, u) \text {. } \tag{8.8}
\end{align*}
$$

Next we will prove a relationship between the weights of rule trees that are related by eff. For this we use the well-founded set $\left(\mathrm{RT}_{\mathcal{G}^{\prime}}\left(N, \Gamma^{*}\right), \prec^{\prime}\right)$ where we let

$$
\prec^{\prime}=\prec_{R^{\prime}} \cap\left(\mathrm{RT}_{\mathcal{G}^{\prime}}\left(N, \Gamma^{*}\right) \times \mathrm{RT}_{\mathcal{G}^{\prime}}\left(N, \Gamma^{*}\right)\right)
$$

Again, $\prec^{\prime}$ is well-founded and $\min _{\prec^{\prime}}\left(\operatorname{RT}_{\mathcal{G}^{\prime}}\left(N, \Gamma^{*}\right)\right.$ ) is the set of terminal rules of $R^{\prime}$, which is not empty. Then, by induction on $\left(\operatorname{RT}_{\mathcal{G}^{\prime}}\left(N, \Gamma^{*}\right), \prec^{\prime}\right)$, we prove that the following statement holds.

$$
\begin{equation*}
\text { For every } d^{\prime} \in \operatorname{RT}_{\mathcal{G}^{\prime}}\left(N, \Gamma^{*}\right) \text { we have } \sum_{\substack{d \in \operatorname{RT}_{\mathcal{G}}\left(N, \Gamma^{*}\right): \\ \operatorname{eff}(d)=d^{\prime}}}^{\oplus} \mathrm{wt}(d)=\mathrm{wt}^{\prime}\left(d^{\prime}\right) \tag{8.9}
\end{equation*}
$$

Let $d^{\prime} \in \operatorname{RT}_{\mathcal{G}^{\prime}}\left(N, \Gamma^{*}\right)$. Hence there exist $A \in N$ and $u \in \Gamma^{*}$ such that $d^{\prime} \in \operatorname{RT}_{\mathcal{G}^{\prime}}(A, u)$. Then

- there exists a rule $r^{\prime}=(A \rightarrow \alpha)$ in $R^{\prime}$ with $\alpha=u_{0} A_{1} u_{1} \ldots A_{k} u_{k}, k \in \mathbb{N}_{+}$, and $\alpha \notin N$ and
- for each $i \in[k]$ there exist $v_{i} \in \Gamma^{*}$ and $d_{i}^{\prime} \in \operatorname{RT}_{\mathcal{G}^{\prime}}\left(A_{i}, v_{i}\right)$
such that $u=u_{0} v_{1} u_{1} \cdots v_{k} u_{k}$ and $d^{\prime}=r^{\prime}\left(d_{1}^{\prime}, \ldots, d_{k}^{\prime}\right)$. Then we can calculate as follows:
(by renaming of $C_{n+1}$ by $A^{\prime}$ )

$$
=\left(\bigotimes_{i \in[k]} \mathrm{wt}^{\prime}\left(d_{i}^{\prime}\right)\right) \otimes\left(\bigoplus_{A^{\prime} \in N} w t\left(A^{\prime} \rightarrow u_{0} A_{1} u_{1} \ldots A_{k} u_{k}\right)\right) \otimes \sum_{n \in \mathbb{N}}^{\oplus} \bigoplus_{\substack{C_{1}, \ldots, C_{n+1} \in N^{\prime}: j \in[0, n-1] \\ C_{1}=A, C_{n+1}=A^{\prime}}} \bigotimes_{C_{n+1-j}, C_{n-j}}
$$

$$
\begin{aligned}
& \sum_{\substack{d \in \operatorname{RT}_{\mathcal{G}}\left(N, \Gamma^{*}\right): \\
\text { eff }(d)=d^{\prime}}}^{\oplus} \mathrm{wt}(d)=\sum_{\substack{d \in \mathrm{RT}_{\mathcal{G}}(A, u): \\
\operatorname{eff}(d)=r^{\prime}\left(d_{1}^{\prime}, \ldots, d_{k}^{\prime}\right)}}^{\oplus} \mathrm{wt}(d) \\
& =\sum_{n \in \mathbb{N}}^{\oplus} \bigoplus_{\substack{C_{1}, \ldots, C_{n+1} \in N: \\
C_{1}=A}} \sum_{\substack{d_{1} \in \mathrm{RT}_{\mathcal{G}}\left(A_{1}, v_{1}\right), \ldots, d_{k} \in \mathrm{RT}_{\mathcal{G}}\left(A_{k}, v_{k}\right): \\
(\forall i \in[k]): \mathrm{eff}\left(d_{i}\right)=d_{i}^{\prime}}}^{\oplus} \\
& \left(\bigotimes_{i \in[k]} \mathrm{wt}\left(d_{i}\right)\right) \otimes w t\left(C_{n+1} \rightarrow u_{0} A_{1} u_{1} \ldots A_{k} u_{k}\right) \otimes \bigotimes_{j \in[0, n-1]} M_{C_{n+1-j}, C_{n-j}} \\
& =\sum_{n \in \mathbb{N}}^{\oplus} \bigoplus_{\substack{C_{1}, \ldots, C_{n+1} \in N: \\
C_{1}=A}}\left(\bigotimes_{\substack{i \in[k] \\
d_{i} \in \operatorname{RT}_{\mathcal{G}}\left(A_{i}, v_{i}\right): \\
\operatorname{eff}\left(d_{i}\right)=d_{i}^{\prime}}}^{\oplus} \mathrm{wt}\left(d_{i}\right)\right) \\
& \otimes w t\left(C_{n+1} \rightarrow u_{0} A_{1} u_{1} \ldots A_{k} u_{k}\right) \otimes \bigotimes_{j \in[0, n-1]} M_{C_{n+1-j}, C_{n-j}} \quad \text { (because B is distributive) } \\
& =\sum_{n \in \mathbb{N}}^{\oplus} \bigoplus_{\substack{C_{1}, \ldots, C_{n+1} \in N: \\
C_{1}=A}}\left(\bigotimes_{i \in[k]} \mathrm{wt}^{\prime}\left(d_{i}^{\prime}\right)\right) \otimes w t\left(C_{n+1} \rightarrow u_{0} A_{1} u_{1} \ldots A_{k} u_{k}\right) \otimes \bigotimes_{j \in[0, n-1]} M_{C_{n+1-j}, C_{n-j}} \text { (by I.H.) } \\
& =\bigoplus_{A^{\prime} \in N} \sum_{n \in \mathbb{N}}^{\oplus} \bigoplus_{\substack{C_{1}, \ldots, C_{n+1} \in N: \\
C_{1}=A C_{n}}}\left(\bigotimes_{i \in[k]} \mathrm{wt}^{\prime}\left(d_{i}^{\prime}\right)\right) \otimes w t\left(A^{\prime} \rightarrow u_{0} A_{1} u_{1} \ldots A_{k} u_{k}\right) \otimes \bigotimes_{j \in[0, n-1]} M_{C_{n+1-j}, C_{n-j}}
\end{aligned}
$$

$$
\begin{align*}
& =\left(\bigotimes_{i \in[k]} \mathrm{wt}^{\prime}\left(d_{i}^{\prime}\right)\right) \otimes\left(\bigoplus_{A^{\prime} \in N} w t\left(A^{\prime} \rightarrow u_{0} A_{1} u_{1} \ldots A_{k} u_{k}\right)\right) \otimes \sum_{n \in \mathbb{N}}^{\oplus}\left(M^{n}\right)_{A^{\prime}, A}  \tag{8.6}\\
& =\left(\bigotimes_{i \in[k]} \mathrm{wt}^{\prime}\left(d_{i}^{\prime}\right)\right) \otimes\left(\bigoplus_{A^{\prime} \in N} w t\left(A^{\prime} \rightarrow u_{0} A_{1} u_{1} \ldots A_{k} u_{k}\right)\right) \otimes\left(\sum_{n \in \mathbb{N}}^{+}\left(M^{n}\right)\right)_{A^{\prime}, A} \\
& =\left(\bigotimes_{i \in[k]} \mathrm{wt}^{\prime}\left(d_{i}^{\prime}\right)\right) \otimes\left(\bigoplus_{A^{\prime} \in N} w t\left(A^{\prime} \rightarrow u_{0} A_{1} u_{1} \ldots A_{k} u_{k}\right) \otimes\left(M^{*}\right)_{A^{\prime}, A}\right) \\
& =\left(\bigotimes_{i \in[k]} \mathrm{wt}^{\prime}\left(d_{i}^{\prime}\right)\right) \otimes w t^{\prime}\left(r^{\prime}\right) \\
& =\mathrm{wt}^{\prime}\left(r^{\prime}\left(d_{1}^{\prime}, \ldots, d_{k}^{\prime}\right)\right) .
\end{align*}
$$

This proves (8.9).
Then we can prove for each $u \in \Gamma^{*}$ :

$$
\begin{align*}
& \llbracket \mathcal{G} \rrbracket^{\mathrm{s}}(u)=\sum_{d \in \mathrm{RT}_{\mathcal{G}}(\xi)}^{\oplus} \mathrm{wt}(d)=\bigoplus_{A \in S} \sum_{d \in \mathrm{RT}_{\mathcal{G}}(A, u)}^{\oplus} \mathrm{wt}(d)=\bigoplus_{A \in S} \sum_{d^{\prime} \in \mathrm{RT}_{\mathcal{G}^{\prime}}(A, u)}^{\oplus} \sum_{\substack{ \\
\begin{array}{c}
\text { RTf } \\
\text { eff }(d)=d^{\prime}
\end{array}}{ }^{\oplus} \mathrm{wt}(d)} \\
& =\bigoplus_{A \in S} \sum_{d^{\prime} \in \mathrm{RT}_{\mathcal{G}^{\prime}}(A, u)}^{\oplus} \mathrm{wt}^{\prime}\left(d^{\prime}\right)  \tag{8.9}\\
& =\sum_{d^{\prime} \in \mathrm{RT}_{\mathcal{G}^{\prime}}(u)}^{\oplus} \mathrm{wt}^{\prime}\left(d^{\prime}\right)=\llbracket \mathcal{G}^{\prime} \rrbracket^{\mathrm{s}}(u) .
\end{align*}
$$

Now we continue the proof with Case (a), i.e., we assume that $\mathcal{G}$ is finite-derivational. Since $\mathcal{G}$ is localreduced and finite-derivational, the length of sequences of chain rules in $A$-rule trees of $\mathcal{G}$ is bounded by $|N|-1$ for each $A \in N$. We show this statement by contradiction. For this, let us assume that there exist a string $u \in \Gamma^{*}$ and a $d \in \operatorname{RT}_{\mathcal{G}}(A, u)$ such that $d$ contains a sequence of chain rules of length at least $|N|$. Then a nonterminal is repeated in that sequence and, by applying the standard pumping argument, we obtain that $\operatorname{RT}_{\mathcal{G}}(A, u)$ is not finite. However, this contradicts Observation 8.2.1,

This boundedness property gives us the possibility to construct $\mathcal{G}^{\prime}$, because, in order to eliminate the chain rules from $\mathcal{G}$, now we can use the matrix $\underset{n \in[0,|N|-1]}{+} M^{n}$.

Thus, we construct the $(\Gamma, \mathrm{B})-\operatorname{wcfg} \mathcal{G}^{\prime}=\left(N, S, R^{\prime}, w t^{\prime}\right)$ as follows. As in Case (b), we let

$$
R^{\prime}=\left\{A \rightarrow \alpha \mid A \in N, \alpha \notin N,\left(\exists A^{\prime} \in N\right): A^{\prime} \rightarrow \alpha \text { is in } R\right\}
$$

Moreover, for each rule $A \rightarrow \alpha$ in $R^{\prime}$, we define

$$
w t^{\prime}(A \rightarrow \alpha)=\bigoplus_{\substack{A^{\prime} \in N: \\\left(A^{\prime} \rightarrow \alpha\right) \in R}} w t\left(A^{\prime} \rightarrow \alpha\right) \otimes\left({\underset{c}{n \in[0,|N|-1]}} M^{n}\right)_{A^{\prime}, A}
$$

Obviously, $\mathcal{G}^{\prime}$ is chain-free and, again, all the mentioned properties are preserved. Thus, since $\mathcal{G}$ is finite-derivational, also $\mathcal{G}^{\prime}$ is so, hence $\llbracket \mathcal{G}^{\prime} \rrbracket^{\mathrm{s}}$ is defined. By induction on $\left(\mathrm{RT}_{\mathcal{G}^{\prime}}\left(N, \Gamma^{*}\right), \prec^{\prime}\right)$ (where $\prec^{\prime}$ is the well-founded relation defined in the proof of Case (b) above), we prove that the following statement holds:

$$
\begin{equation*}
\text { For every } d^{\prime} \in \operatorname{RT}_{\mathcal{G}^{\prime}}\left(N, \Gamma^{*}\right) \text {, we have } \bigoplus_{\substack{d \in \operatorname{RT}_{\mathcal{G}}\left(N, \Gamma^{*}\right): \\ \text { eff }(d)=d^{\prime}}} \mathrm{wt}(d)=\mathrm{wt}^{\prime}\left(d^{\prime}\right) \tag{8.10}
\end{equation*}
$$

The proof of (8.10) is the same as the proof of (8.9) except that the four infinite summations

$$
\sum_{\substack{d \in \mathrm{RT}_{\mathcal{G}}(A, u): \\ \operatorname{eff}(d)=r^{\prime}\left(d_{1}^{\prime}, \ldots, d_{k}^{\prime}\right)}}^{\oplus} \text { and } \sum_{\substack{d_{i} \in \operatorname{RT}_{\mathcal{G}}\left(A_{i}, v_{i}\right): \\ \operatorname{eff}\left(d_{i}\right)=d_{i}^{\prime}}}^{\oplus} \text { and } \sum_{n \in \mathbb{N}}^{\oplus} \text { and } \sum_{n \in \mathbb{N}}^{+}
$$

are replaced by the finite summations

as well as,

$$
\left(M^{*}\right)_{A^{\prime}, A} \text { is replaced by }\left(\prod_{n \in[0,|N|-1]} M^{n}\right)_{A^{\prime}, A}
$$

By a similar modification of the final calculation of (b) we obtain the proof for $\llbracket \mathcal{G} \rrbracket^{\mathrm{s}}=\llbracket \mathcal{G}^{\prime} \rrbracket^{\mathrm{s}}$.
In the next theorem we deal with the transformation of a ( $\Gamma, B$ )-wcfg into an $\varepsilon$-free one. More precisely, for a given $(\Gamma, \mathrm{B})-\operatorname{wcfg} \mathcal{G}=(N, S, R, w t)$ such that (a) B is a commutative semiring and (b) $\mathcal{G}$ is finitederivational or B is $\sigma$-complete, we define an $\varepsilon$-free $(\Gamma, B)$-wcfg $\mathcal{G}^{\prime}$ such that $\llbracket \mathcal{G}^{\prime} \rrbracket^{\mathrm{s}}=\llbracket \mathcal{G} \rrbracket^{\mathrm{s}} \otimes \chi_{\mathrm{B}}\left(\Gamma^{+}\right)$. We follow the idea of [BPS61, Lm. 4.1] (also cf. HMU07, Subsec. 7.1.3]) where context-free grammars are transformed into (almost) equivalent $\varepsilon$-free context-free grammars. However, here we have to be careful about the weights and have to cope with two phenomena.

We explain the first phenomenon by means of an example. Let $\mathcal{G}$ contain the initial nonterminal $A$ and the three rules

$$
r=(A \rightarrow a C C), r^{\prime}=(C \rightarrow b), \text { and } r^{\prime \prime}=(C \rightarrow \varepsilon)
$$

Then $\operatorname{RT}_{\mathcal{G}}(A, a b)=\left\{d_{1}, d_{2}\right\}$ with $d_{1}=r\left(r^{\prime}, r^{\prime \prime}\right)$ and $d_{2}=r\left(r^{\prime \prime}, r^{\prime}\right)$. Hence

$$
\llbracket \mathcal{G} \rrbracket^{\mathrm{s}}(a b)=\mathrm{wt}_{\mathcal{G}}\left(d_{1}\right) \oplus \mathrm{wt}_{\mathcal{G}}\left(d_{2}\right)
$$

According to the construction of [BPS61, Lm. 4.1], the wcfg $\mathcal{G}^{\prime}$ contains the rules $r, r^{\prime}$, and $\bar{r}=(A \rightarrow a C)$, where $\bar{r}$ results from $r$ by erasing either the first occurrence of $C$ or the second one. Thus $\mathrm{RT}_{\mathcal{G}^{\prime}}(A, a b)=$ $\left\{\bar{r}\left(r^{\prime}\right)\right\}$ and

$$
\llbracket \mathcal{G}^{\prime} \rrbracket^{\mathbf{s}}(a b)=\mathrm{wt}_{\mathcal{G}^{\prime}}\left(\bar{r}\left(r^{\prime}\right)\right)
$$

We observe that this construction does not take care of the different ways in which a right-hand side (e.g. $a C$ ) results from erasing nonterminals from the right-hand side of the $\mathcal{G}$-rule (e.g. $a C C$ ). In this particular example, we could define the weights of the rules of $\mathcal{G}^{\prime}$ such that the weight balance $\llbracket \mathcal{G} \rrbracket^{\mathrm{s}}(a b)=\llbracket \mathcal{G}^{\prime} \rrbracket^{\mathrm{s}}(a b)$ is satisfied. But in general, rules of $\mathcal{G}$ call each other recursively, and then this method of defining weights is not successful. Instead, we will code the occurrences of nonterminals which are selected for erasing into the rules of $\mathcal{G}^{\prime}$, and thereby we keep the different ways in which a rule for $\mathcal{G}^{\prime}$ was obtained separately.

The second phenomenon is the fact that the empty string $\varepsilon$ can be derived in many different ways, and $\mathcal{G}^{\prime}$ has to sum up the weights of all the corresponding rule trees. For instance, if there exists a rule $r=(C \rightarrow a A b)$ in $R$ and we have detected that $A$ is nullable, i.e., $\mathrm{RT}_{\mathcal{G}}(A, \varepsilon) \neq \emptyset$, then according to the construction in [BPS61, Lm. 4.1] the rule $r^{\prime}=(C \rightarrow a b)$ will be in $\mathcal{G}^{\prime}$ and the weight of $r^{\prime}$ in $\mathcal{G}^{\prime}$ will be $w t(r) \otimes W(A)$, where we define

$$
W(A)=\sum_{d \in \mathrm{RT}_{\mathcal{G}}(A, \varepsilon)}^{\oplus} \mathrm{wt}_{\mathcal{G}}(d)
$$

By our assumptions on $\mathcal{G}$ and B , the value $W(A)$ is well defined. We note that, if $\mathrm{RT}_{\mathcal{G}}(A, \varepsilon)=\emptyset$, then $W(A)=\mathbb{O}$. If $\mathcal{G}$ is not finite-derivational (and hence B is $\sigma$-complete), it is in general not clear how to construct $W(A)$. However, if $\mathcal{G}$ is finite-derivational, then $W(A)$ can be computed.

Observation 8.2.7. Let $\mathcal{G}=(N, S, R, w t)$ be a finite-derivational and local-reduced ( $\Gamma, \mathrm{B}$ )-wcfg. Then, for each $A \in N$, we can compute $W(A)$.

Proof. First we show that

$$
\begin{equation*}
\operatorname{RT}_{\mathcal{G}}(A, \varepsilon)=\left\{d \in \operatorname{RT}_{\mathcal{G}}(A, \varepsilon)|\operatorname{height}(d) \leq|N|-1\}\right. \tag{8.11}
\end{equation*}
$$

We prove (8.11) by contradiction and assume that there exists a $d \in \mathrm{RT}_{\mathcal{G}}(A, \varepsilon)$ such that height $(d) \geq|N|$. Then there exist $w \in \operatorname{pos}(d)$ and $v \in \mathbb{N}^{+}$such that $\operatorname{lhs}(d(w))=\operatorname{lhs}(d(w v))$. Then, using $c$ as abbreviation for the context $\left(\left.d\right|_{w}\right)[z]_{v}$, we obtain that $d\left[c^{n}\left[\left.d\right|_{w v}\right]\right]_{w} \in \operatorname{RT}_{\mathcal{G}}(A, \varepsilon)$ for each $n \in \mathbb{N}$. Hence $\operatorname{RT}_{\mathcal{G}}(A, \varepsilon)$ is not finite, which contradicts Observation 8.2.1. This proves (8.11).

Then we have

$$
\begin{aligned}
W(A) & =\sum_{d \in \operatorname{RT}_{\mathcal{G}}(A, \varepsilon)}^{\oplus} \mathrm{wt}_{\mathcal{G}}(d) \\
& =\sum_{\substack{d \in \mathrm{RT}_{\mathcal{G}}(A, \varepsilon): \\
\operatorname{height}(d) \leq|N|-1}}^{\oplus} \mathrm{wt}_{\mathcal{G}}(d)
\end{aligned}
$$

$$
=\sum_{d \in \operatorname{RT}_{\mathcal{G}}(A, \varepsilon):}^{\oplus} \mathrm{wt}_{\mathcal{G}}(d)
$$

$$
=\bigoplus_{d \in \operatorname{RT}_{\mathcal{G}}(A, \varepsilon):} \mathrm{wt}_{\mathcal{G}}(d) \quad \text { (by Observation 2.6.8(2)) }
$$

Obviously, $\bigoplus_{d \in \mathrm{RT}_{\mathcal{G}}(A, \varepsilon):} \mathrm{wt}_{\mathcal{G}}(d)$ can be computed.

The next theorem has been achieved in KS86, Thm. 14.6] for wcfg over a partially ordered, continuous, and commutative semiring B. We follow the construction given in [KS86, Thm. 14.6].

Theorem 8.2.8. (cf. KS86, Thm. 14.6]) Let $\Gamma$ be an alphabet, $B$ be a commutative semiring, and $\mathcal{G}$ be a ( $\Gamma, \mathrm{B}$ )-wcfg such that $\mathcal{G}$ is finite-derivational or B is $\sigma$-complete. Then there exists an $\varepsilon$-free $(\Gamma, \mathrm{B})$-wcfg $\mathcal{G}^{\prime}$ such that $\llbracket \mathcal{G}^{\prime} \rrbracket^{\mathrm{s}}=\llbracket \mathcal{G} \rrbracket^{\mathrm{s}} \otimes \chi_{\mathrm{B}}\left(\Gamma^{+}\right)$. If $\mathcal{G}$ is finite-derivational, then we can construct $\mathcal{G}^{\prime}$.

Proof. Let $\mathcal{G}=(N, S, R, w t)$. By standard pumping methods, we can decide whether $\mathcal{G}$ has a localsuccessful rule tree. If the answer is no, then $\llbracket \mathcal{G} \rrbracket^{s}=\widetilde{\mathbb{D}}$. Hence, the statement of the theorem holds obviously because we can construct an $\varepsilon$-free $(\Gamma, B)$-wcfg $\mathcal{G}^{\prime}$ such that $\llbracket \mathcal{G}^{\prime} \rrbracket^{\mathrm{s}}=\widetilde{0}$.

Otherwise, by Lemma 8.2.3 and Lemma 8.2.5, we can construct a ( $\Gamma, B$ )-wcfg which is in nonterminal form, local-reduced, and equivalent to $\mathcal{G}$. Hence we can assume that $\mathcal{G}$ is in nonterminal form and local-reduced. Then we distinguish two cases.
$\underline{\text { Case (a): Each terminal rule of } \mathcal{G} \text { is an } \varepsilon \text {-rule. Then } \llbracket \mathcal{G} \rrbracket^{\mathrm{s}} \otimes \chi_{\mathrm{B}}\left(\Gamma^{+}\right)=\widetilde{0} \text {. Again, the statement of the }}$ theorem holds obviously.

Case (b): There exists a terminal rule of $\mathcal{G}$, say $A \rightarrow u$, such that $u \neq \varepsilon$. Since $\mathcal{G}$ is in nonterminal form, we have $u \in \Gamma$. We will use this property when we construct $\mathcal{G}^{\prime}$. But before this we need some preparations.

As in the case of context-free grammars, we construct the set of nullable nonterminal symbols (cf. BPS61, Lm. 4.1]), i.e., the set

$$
E=\left\{A \in N \mid \operatorname{RT}_{\mathcal{G}}(A, \varepsilon) \neq \emptyset\right\}
$$

In order to prepare for the above mentioned first phenomenon, we introduce the concept of $E$-selection. Intuitively, an $E$-selection is a string over $\{0,1\}$ and it shows, for a given right-hand side $\alpha \in(N \cup \Gamma)^{*}$ of a rule and each of its nonterminal occurrences, whether that occurrence is selected for erasing (represented by 1) or not (represented by 0 ). Formally, let $\alpha=u_{0} A_{1} u_{1} \ldots A_{k} u_{k}$ be in $(N \cup \Gamma)^{*}$. An E-selection for
$\alpha$ is a string $f=f_{1} \cdots f_{k}$ with $f_{i} \in\{0,1\}$ such that, for every $i \in[k]$, if $f_{i}=1$, then $A_{i} \in E$, i.e., only occurrences of nullable nonterminals can be selected for erasing. In particular, for each $u \in \Gamma^{*}$, the only $E$-selection is $f=\varepsilon$.

For instance, if $\alpha=a A b D C C, E=\{C, D\}$, then $f=0110$ is an $E$-selection. It indicates that the occurrence of $D$ and the leftmost occurrence of $C$ are selected for erasing, and the other two occurrences of nonterminals are not.

Let $\alpha=u_{0} A_{1} u_{1} \ldots A_{k} u_{k}$ in $(N \cup \Gamma)^{*}$ and $f$ be an $E$-selection for $\alpha$. Then we define the application of $f$ to $\alpha$, denoted by $f(\alpha)$, to be the string $u_{0} X_{1} u_{1} \ldots X_{k} u_{k}$, where for each $i \in[k]$ :

$$
X_{i}= \begin{cases}\varepsilon & \text { if } f_{i}=1 \\ A_{i} & \text { otherwise }\end{cases}
$$

In particular, for each $u \in \Gamma^{*}$, we have $\varepsilon(u)=u$. For $\alpha$ and $f$ in the above example we have $f(\alpha)=a A b C$.
For each nonterminal which is not selected by $f$ for erasing, we will have to know its target position in $f(\alpha)$. Formally, let $f=f_{1} \cdots f_{k}$ be an $E$-selection for $\alpha$. We let $\operatorname{pos}_{0}(f)=\left\{i \in[k] \mid f_{i}=0\right\}$. Then we define the mapping $g_{f}: \operatorname{pos}_{0}(f) \rightarrow\left[\left|\operatorname{pos}_{0}(f)\right|\right]$ for each $i \in \operatorname{pos}_{0}(f)$ by $g_{f}(i)=j$ if the $j$ th occurrence of 0 in $f$ (counted from left to right) has position $i$. We note that $g_{f}$ is bijective. For $f$ in the above example, we have $\operatorname{pos}_{0}(f)=\{1,4\}$ and $g_{f}(1)=1$ and $g_{f}(4)=2$.

Now we define the $(\Gamma, \mathrm{B})$-wcfg $\mathcal{G}^{\prime}=\left(N^{\prime}, S, R^{\prime}, w t^{\prime}\right)$ as follows.

- $N^{\prime}=N \cup\{[\alpha, f] \mid(\exists(A \rightarrow \alpha)$ in $R): f$ is an $E$-selection for $\alpha\}$ and
- $R^{\prime}$ and $w t^{\prime}$ are defined as follows. For every $r=(A \rightarrow \alpha)$ with $\alpha=u_{0} A_{1} u_{1} \ldots A_{k} u_{k}$ in $R$ and $\alpha \neq \varepsilon$, and $E$-selection $f$ for $\alpha$ such that $f(\alpha) \neq \varepsilon$,
- the rule $r^{\prime}=(A \rightarrow[\alpha, f])$ is in $R^{\prime}$ with $w t^{\prime}\left(r^{\prime}\right)=w t(r)$ and
- the rule $r^{\prime \prime}=([\alpha, f] \rightarrow f(\alpha))$ is in $R^{\prime}$ with

$$
w t^{\prime}\left(r^{\prime \prime}\right)=\bigotimes_{\substack{i \in[k]: \\ f_{i}=1}} W\left(A_{i}\right)
$$

In particular, if $\left\{i \in[k] \mid f_{i}=1\right\}=\emptyset$, then $w t^{\prime}\left(r^{\prime \prime}\right)=\mathbb{1}$.
Since $R$ contains a terminal rule $A \rightarrow u$ with $u \in \Gamma$, the set $R^{\prime}$ contains the rule $[u, \varepsilon] \rightarrow u$, and hence the $\Gamma$-cfg $\left(N^{\prime}, S, R^{\prime}\right)$ contains a terminal rule. Thus, $\mathcal{G}^{\prime}$ is a $(\Gamma, \mathrm{B})$-wcfg. Moreover, if $\mathcal{G}$ is finite-derivational, then $w t^{\prime}\left(r^{\prime \prime}\right)$ can be constructed by Observation 8.2.7, and hence $\mathcal{G}^{\prime}$ can be constructed.

We note that the construction of $\mathcal{G}^{\prime}$ is essentially the same as the construction of the algebraic system ( $y_{i}=q_{i} \mid 1 \leq i \leq n$ ) in the proof of Theorem KS86, Thm. 14.6]. For an explanation, we assume that $N=\left\{A_{1}, \ldots, A_{n}\right\}$ and $A_{j} \rightarrow \alpha_{1}, \ldots, A_{j} \rightarrow \alpha_{\ell}$ are all the rules of $\mathcal{G}$ with left-hand side $A_{j}$. Then the corresponding algebraic system has the set $Y=N$ as set of variables and it contains the equation

$$
A_{j}=p_{j} \quad \text { where } \quad p_{j}=\alpha_{1}+\ldots+\alpha_{\ell}
$$

The set of all right-hand sides $f\left(\alpha_{i}\right)$ where $1 \leq i \leq \ell$ and $f$ is an $E$-selection for $\alpha_{i}$, is in one-toone correspondence to the set of all monomials in the polynomial $q_{i}$ (in the proof of Theorem KS86, Thm. 14.6]); the latter is obtained by substituting $(\sigma, \varepsilon) \varepsilon+Y$ into $p_{j}$. The coefficient $(\sigma, \varepsilon)$ is the $N$ vector $\left(W\left(A_{1}\right), \ldots, W\left(A_{n}\right)\right)$, which is relevant for all those nonterminals which are selected for erasing, and the part $Y$ reflects the case that a nonterminal is not erased.

Next we prove $\llbracket \mathcal{G}^{\prime} \rrbracket^{\mathrm{s}}=\llbracket \mathcal{G} \rrbracket^{\mathrm{s}} \otimes \chi_{\mathrm{B}}\left(\Gamma^{+}\right)$. As preparation, we define the binary relation $\prec$ on $\mathrm{RT}_{\mathcal{G}}\left(N, \Gamma^{+}\right)$ by

$$
\prec=\prec_{R} \cap \mathrm{RT}_{\mathcal{G}}\left(N, \Gamma^{+}\right) \times \mathrm{RT}_{\mathcal{G}}\left(N, \Gamma^{+}\right) .
$$

Since $\prec_{R}$ is well-founded, also $\prec$ has this property. Moreover, we have $\min _{\prec}\left(\mathrm{RT}_{\mathcal{G}}\left(N, \Gamma^{+}\right)\right)=\{(A \rightarrow u) \in$ $\left.R \mid u \in \Gamma^{+}\right\}$, which is not empty. We define the mapping

$$
\varphi: \mathrm{RT}_{\mathcal{G}}\left(N, \Gamma^{+}\right) \rightarrow \mathrm{RT}_{\mathcal{G}^{\prime}}\left(N, \Gamma^{+}\right)
$$



Figure 8.3: An illustration of $\varphi(d)=d^{\prime}$ in the proof of Theorem 8.2.8.
by induction on $\left(\operatorname{RT}_{\mathcal{G}}\left(N, \Gamma^{+}\right), \prec\right)$ as follows (cf. Figure 8.3). Let $d \in \mathrm{RT}_{\mathcal{G}}\left(N, \Gamma^{+}\right)$. Then there exists a rule $(A \rightarrow \alpha)$ in $R$ with $\alpha=u_{0} A_{1} u_{1} \ldots A_{k} u_{k}$, and for every $i \in[k]$ there exists $d_{i} \in \operatorname{RT}_{\mathcal{G}}\left(A_{i}, \Gamma^{*}\right)$ such that

$$
d=(A \rightarrow \alpha)\left(d_{1}, \ldots, d_{k}\right)
$$

We can assume that $\varphi\left(d_{i}\right)$ is defined for each $d_{i} \in \operatorname{RT}_{\mathcal{G}}\left(N, \Gamma^{+}\right)$.
Since $\pi_{\mathcal{G}}(d) \neq \varepsilon$, we have $u_{0} u_{1} \cdots u_{k} \neq \varepsilon$ or there exists $i \in[k]$ such that $d_{i} \in \operatorname{RT}_{\mathcal{G}}\left(A_{i}, \Gamma^{+}\right)$. Let $f=f_{1} \cdots f_{k}$ be the $E$-selection for $\alpha$ defined, for each $i \in[k]$, by

$$
f_{i}= \begin{cases}1 & \text { if } d_{i} \in \mathrm{RT}_{\mathcal{G}}\left(A_{i}, \varepsilon\right) \\ 0 & \text { otherwise }\end{cases}
$$

Then the rule $A \rightarrow[\alpha, f]$ is in $R^{\prime}$ and we define

$$
\varphi(d)=(A \rightarrow[\alpha, f])(([\alpha, f] \rightarrow f(\alpha)) \bar{\zeta})
$$

where the sequence $\bar{\zeta}$ of rule trees of $\mathcal{G}^{\prime}$ is obtained from the sequence $\left(d_{1}, \ldots, d_{k}\right)$ by
(a) dropping each rule tree $d_{i}$ with $d_{i} \in \mathrm{RT}_{\mathcal{G}}\left(A_{i}, \varepsilon\right)$ and
(b) replacing each remaining $d_{i}$ by $\varphi\left(d_{i}\right)$ (which is defined because $d_{i} \in \operatorname{RT}_{\mathcal{G}}\left(N, \Gamma^{+}\right)$).

For instance, if $r=(S \rightarrow a A b C D D), d_{1} \in \mathrm{RT}_{\mathcal{G}}\left(A, \Gamma^{+}\right), d_{2} \in \mathrm{RT}_{\mathcal{G}}(C, \varepsilon), d_{3} \in \mathrm{RT}_{\mathcal{G}}(D, \varepsilon)$, and $d_{4} \in$ $\mathrm{RT}_{\mathcal{G}}\left(D, \Gamma^{+}\right)$, then $f=0110$ and $\bar{\zeta}=\left(\varphi\left(d_{1}\right), \varphi\left(d_{4}\right)\right)$, cf. Figure 8.3.

In the particular case that $d_{i} \in \operatorname{RT}_{\mathcal{G}}\left(A_{i}, \Gamma^{+}\right)$for each $i \in[k]$, we have $f=0^{k}$ and

$$
\varphi(d)=(A \rightarrow[\alpha, f])\left(([\alpha, f] \rightarrow \alpha)\left(\varphi\left(d_{1}\right), \ldots, \varphi\left(d_{k}\right)\right)\right)
$$

and in the particular case that $d_{i} \in \operatorname{RT}_{\mathcal{G}}\left(A_{i}, \varepsilon\right)$ for each $i \in[k]$, we have $f=1^{k}$ and

$$
\varphi(d)=(A \rightarrow[\alpha, f])\left([\alpha, f] \rightarrow u_{0} u_{1} \ldots u_{k}\right)
$$

Next we prove that $\varphi$ is surjective. More precisely, by induction on $\left(\mathrm{RT}_{\mathcal{G}^{\prime}}\left(N, \Gamma^{+}\right)\right.$, $\left.\prec\right)$, we prove that the following statement holds:

For every $d^{\prime} \in \operatorname{RT}_{\mathcal{G}^{\prime}}\left(N, \Gamma^{+}\right)$, there exists a $d \in \operatorname{RT}_{\mathcal{G}}\left(A, \Gamma^{+}\right)$such that $\varphi(d)=d^{\prime}$.
Let $d^{\prime} \in \operatorname{RT}_{\mathcal{G}^{\prime}}\left(N, \Gamma^{+}\right)$. Then $d^{\prime}$ has the form

$$
(A \rightarrow[\alpha, f])\left(\left([\alpha, f] \rightarrow v_{0} C_{1} v_{1} \ldots C_{n} v_{n}\right)\left(d_{1}^{\prime}, \ldots, d_{n}^{\prime}\right)\right)
$$

where $n \in \mathbb{N}, f$ is an $E$-selection for $\alpha$; moreover $v_{0} C_{1} v_{1} \ldots C_{n} v_{n}=f(\alpha)$ and $d_{1}^{\prime} \in \operatorname{RT}_{\mathcal{G}^{\prime}}\left(C_{j}, \Gamma^{+}\right), \ldots$, $d_{n}^{\prime} \in \operatorname{RT}_{\mathcal{G}^{\prime}}\left(C_{n}, \Gamma^{+}\right)$. Let $\alpha=u_{0} A_{1} u_{1} \ldots A_{k} u_{k}$ and $f=f_{1} \cdots f_{k}$. Hence, $C_{j}=A_{g_{f}^{-1}(j)}$ for each $j \in[n]$.

By I.H. we can assume that, for every $j \in[n]$, there exists $e_{j} \in \operatorname{RT}_{\mathcal{G}}\left(C_{j}, \Gamma^{+}\right)$such that $\varphi\left(e_{j}\right)=d_{j}^{\prime}$. Moreover, for each $i \in[k]$ such that $f_{i}=1$, we choose an arbitrary rule tree $d_{i}^{\prime \prime}$ from $\operatorname{RT}_{\mathcal{G}}\left(A_{i}, \varepsilon\right)$. We note that $\operatorname{RT}_{\mathcal{G}}\left(A_{i}, \varepsilon\right) \neq \emptyset$ because $f$ is an $E$-selection for $\alpha$. Then we construct the rule tree $d=(A \rightarrow$ $\alpha)\left(d_{1}, \ldots, d_{k}\right)$ in $\operatorname{RT}_{\mathcal{G}}\left(A, \Gamma^{+}\right)$where for each $i \in[k]$ we let

$$
d_{i}= \begin{cases}d_{i}^{\prime \prime} & \text { if } f_{i}=1 \\ e_{g_{f}(i)} & \text { otherwise }\end{cases}
$$

Then $\varphi(d)=d^{\prime}$. This proves that $\varphi$ is surjective. We mention that, in general, $\varphi$ is not injective because there may be several rule trees for $\varepsilon$ with the same nonterminal on the left-hand side of the rule at their root.

From the construction of $\mathcal{G}^{\prime}$, it is obvious that

$$
\begin{equation*}
\text { for every } A \in N \text { and } d \in \operatorname{RT}_{\mathcal{G}}\left(A, \Gamma^{+}\right) \text {, we have } \pi_{\mathcal{G}}(d)=\pi_{\mathcal{G}^{\prime}}(\varphi(d)) \tag{8.13}
\end{equation*}
$$

and, by using (8.13), we have that

$$
\begin{align*}
& \text { for every } A \in N \text { and } u \in \Gamma^{+}, \text {the sets } \operatorname{RT}_{\mathcal{G}}(A, u) \text { and }  \tag{8.14}\\
& \left\{\left(d^{\prime}, d\right) \mid d^{\prime} \in \operatorname{RT}_{\mathcal{G}^{\prime}}(A, u), d \in \varphi^{-1}\left(d^{\prime}\right)\right\} \text { are in a one-to-one correspondence. }
\end{align*}
$$

Finally, by induction on $\left(\operatorname{RT}_{\mathcal{G}^{\prime}}\left(N, \Gamma^{+}\right), \prec\right)$, we prove that the following statement holds:

$$
\begin{equation*}
\text { For every } d^{\prime} \in \operatorname{RT}_{\mathcal{G}^{\prime}}\left(N, \Gamma^{+}\right) \text {we have } \mathrm{wt}_{\mathcal{G}^{\prime}}\left(d^{\prime}\right)=\sum_{d \in \varphi^{-1}\left(d^{\prime}\right)}^{\oplus} \mathrm{wt}_{\mathcal{G}}(d) \tag{8.15}
\end{equation*}
$$

Let $d^{\prime} \in \operatorname{RT}_{\mathcal{G}^{\prime}}\left(N, \Gamma^{+}\right)$. Then $d^{\prime}$ has the form

$$
d^{\prime}=(A \rightarrow[\alpha, f])\left(([\alpha, f] \rightarrow f(\alpha))\left(d_{1}^{\prime}, \ldots, d_{n}^{\prime}\right)\right)
$$

with $\alpha=u_{0} A_{1} u_{1} \ldots A_{k} u_{k}$. Hence $f(\alpha)$ contains $n$ occurrences of 0 . Moreover, for each $j \in[n]$, we have $d_{j}^{\prime} \in \operatorname{RT}_{\mathcal{G}^{\prime}}\left(A_{g_{f}^{-1}(j)}, \Gamma^{*}\right)$. Then we can calculate as follows.

$$
\begin{aligned}
& \mathrm{wt}_{\mathcal{G}^{\prime}}\left(d^{\prime}\right) \\
= & \left(\bigotimes_{j \in[n]} \mathrm{wt}_{\mathcal{G}^{\prime}}\left(d_{j}^{\prime}\right)\right) \otimes w t^{\prime}([\alpha, f] \rightarrow f(\alpha)) \otimes w t^{\prime}(A \rightarrow[\alpha, f]) \\
= & \left(\bigotimes_{\substack{i \in[k]: \\
f_{i}=0}} \mathrm{wt}_{\mathcal{G}^{\prime}}\left(d_{g_{f}(i)}^{\prime}\right)\right) \otimes\left(\bigotimes_{\substack{i \in[k]: \\
f_{i}=1}} W\left(A_{i}\right)\right) \otimes w t(A \rightarrow \alpha) \quad \quad \text { (by definition of } f \text { and construction) } \\
= & \left(\bigotimes_{i \in[k]} c_{i}\right) \otimes w t(A \rightarrow \alpha)
\end{aligned}
$$

where

$$
c_{i}=\left\{\begin{array}{lc}
\mathrm{wt}_{\mathcal{G}^{\prime}}\left(d_{g_{f}(i)}^{\prime}\right) & \text { if } f_{i}=0 \\
W\left(A_{i}\right) & \text { otherwise }
\end{array}\right.
$$

By I.H. and by definition of $W\left(A_{i}\right)$, we have

$$
c_{i}=\left\{\begin{array}{cl}
\sum_{d_{i} \in \varphi^{-1}\left(d_{g_{f}(i)}^{\prime}\right)}^{\oplus} \mathrm{wt}_{\mathcal{G}}\left(d_{i}\right) & \text { if } f_{i}=0 \\
\sum_{d_{i} \in \mathrm{RT}_{\mathcal{G}}\left(A_{i}, \varepsilon\right)}^{\oplus} \mathrm{wt}_{\mathcal{G}}\left(d_{i}\right) & \text { otherwise }
\end{array}\right.
$$

Then, by distributivity and by surjectivity of $\varphi$ (see (8.12)), we obtain

$$
\left(\bigotimes_{i \in[k]} c_{i}\right) \otimes w t(A \rightarrow \alpha)=\sum_{d \in \varphi^{-1}\left(d^{\prime}\right)}^{\oplus} \mathrm{wt}_{\mathcal{G}}(d)
$$

which proves (8.15). Then for each $u \in \Gamma^{+}$

$$
\begin{align*}
\llbracket \mathcal{G}^{\prime} \rrbracket^{\mathrm{s}}(u) & =\bigoplus_{A \in S} \sum_{d^{\prime} \in \mathrm{RT}_{\mathcal{G}^{\prime}}(A, u)}^{\oplus} \mathrm{wt}_{\mathcal{G}^{\prime}}\left(d^{\prime}\right) \\
& =\bigoplus_{A \in S} \sum_{d^{\prime} \in \mathrm{RT}_{\mathcal{G}^{\prime}}(A, u)}^{\oplus} \sum_{d \in \varphi^{-1}\left(d^{\prime}\right)}^{\oplus}  \tag{8.14}\\
& =\bigoplus_{A \in S} \sum_{d \in \mathrm{RT}_{\mathcal{G}}(A, u)}^{\oplus} \mathrm{wt}_{\mathcal{G}}(d) \\
& =\llbracket \mathcal{G} \rrbracket^{\mathrm{s}}(u) .
\end{align*}
$$

$$
=\bigoplus_{A \in S} \sum_{d^{\prime} \in \mathrm{RT}_{\mathcal{G}^{\prime}}(A, u)}^{\oplus} \sum_{d \in \varphi^{-1}\left(d^{\prime}\right)}^{\oplus} \mathrm{wt}_{\mathcal{G}}(d)
$$

Since $\llbracket \mathcal{G}^{\prime} \rrbracket^{\mathrm{s}}(\varepsilon)=\mathbb{0}$, we obtain $\llbracket \mathcal{G}^{\prime} \rrbracket^{\mathrm{s}}=\llbracket \mathcal{G} \rrbracket^{\mathrm{s}} \otimes \chi_{\mathrm{B}}\left(\Gamma^{+}\right)$.

### 8.3 Yields of recognizable weighted tree languages

In this section we state the main result of this chapter: the yield of an r-recognizable weighted tree language is a weighted context-free language, and vice versa, each weighted context-free language can be obtained in this way. This result generalizes [Bra69, Thm. 3.20] and [Don70, Thm. 2.5] from the unweighted case to the weighted case (also cf. [GS84, Thm. 3.2.7]), it generalizes ÉK, Thm. 6.8.6] (also cf. ÉK03, Thm. 8.6]) from commutative, continuous semirings to strong bimonoids, and it generalizes Theorem [FG18, Thm. $1(1) \Leftrightarrow(5)$ ] from semirings to strong bimonoids. (We also refer to Boz99, Thm. 30].)

We recall the mapping yield ${ }_{\Gamma}$ from Section 2.9. Let $\Gamma \subseteq \Sigma^{(0)}$. Then the mapping yield ${ }_{\Gamma}: \mathrm{T}_{\Sigma} \rightarrow \Gamma^{*}$ is defined for each $\xi=\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)$ in $\mathrm{T}_{\Sigma}$ by

$$
\operatorname{yield}_{\Gamma}(\xi)= \begin{cases}\operatorname{yield}_{\Gamma}\left(\xi_{1}\right) \cdots \operatorname{yield}_{\Gamma}\left(\xi_{k}\right) & \text { if } k \geq 1 \\ \sigma & \text { if } k=0 \text { and } \sigma \in \Gamma \\ \varepsilon & \text { otherwise }\end{cases}
$$

Let $r: \mathrm{T}_{\Sigma} \rightarrow B$ be a weighted tree language. Also we recall from (2.30) that, if $r$ is $\chi\left(\right.$ yield $\left._{\Gamma}\right)-$ summable or B is $\sigma$-complete, then $\chi\left(\operatorname{yield}_{\Gamma}\right)(r): \Gamma^{*} \rightarrow B$ is a weighted language such that, for each $u \in \Gamma^{*}$, we have

$$
\chi\left(\operatorname{yield}_{\Gamma}\right)(r)(u)=\sum_{\xi \in \operatorname{yield}_{\Gamma}^{-1}(u)}^{\oplus} r(\xi)
$$

Theorem 8.3.1. FG18, Thm. 1] Let $\Gamma$ be an alphabet and B be a strong bimonoid. For each weighted language $s: \Gamma^{*} \rightarrow B$ the following two statements are equivalent.
(A) We can construct a $(\Gamma, \mathrm{B})-w c f g \mathcal{G}$ such that $\mathcal{G}$ is finite-derivational or B is $\sigma$-complete, and $s=\llbracket \mathcal{G} \rrbracket^{\mathrm{s}}$.
(B) We can construct a ranked alphabet $\Sigma$ with $\Gamma \subseteq \Sigma^{(0)}$ and a $(\Sigma, \mathrm{B})$-wta $\mathcal{A}$ such that $\llbracket \mathcal{A} \rrbracket^{\text {run }}$ is $\chi\left(\right.$ yield $\left._{\Gamma}\right)$-summable or B is $\sigma$-complete, and $s=\chi\left(\operatorname{yield}_{\Gamma}\right)\left(\llbracket \mathcal{A} \rrbracket^{\text {run }}\right)$.

Proof. This follows from Lemmas 8.3 .2 and 8.3 .3

Lemma 8.3.2. Let $\mathcal{G}$ be a $(\Gamma, B)$-wcfg such that $\mathcal{G}$ is finite-derivational or B is $\sigma$-complete. We can construct a ranked alphabet $\Sigma$ with $\Gamma \subseteq \Sigma^{(0)}$ and a root weight normalized $(\Sigma, \mathrm{B})$-wta $\mathcal{A}=(Q, \delta, F)$ such that (a) $\llbracket \mathcal{A} \rrbracket^{\text {run }}$ is $\chi\left(\right.$ yield $\left._{\Gamma}\right)$-summable if $\mathcal{G}$ is finite-derivational and (b) $\llbracket \mathcal{G} \rrbracket^{\mathrm{s}}=\chi\left(\right.$ yield $\left._{\Gamma}\right)\left(\llbracket \mathcal{A} \rrbracket^{\text {run }}\right)$.

Proof. Let $\mathcal{G}=(N, S, R, w t)$. By Lemmas 8.2 .3 and 8.2.4, we can construct a $(\Gamma, B)$-wcfg which is in nonterminal form, start-separated, and equivalent to $\mathcal{G}$. Hence we can assume that $\mathcal{G}$ is in nonterminal form and start-separated.

We define the $\mathbb{N}$-indexed family $\left(R_{k} \mid k \in \mathbb{N}\right)$ such that

$$
R_{k}=\{(A \rightarrow \alpha) \in R| | \alpha \mid=k\}
$$

We note that, if $\mathcal{G}$ is $\varepsilon$-free, then $\left(R_{k} \mid k \in \mathbb{N}\right)$ is not a ranked alphabet, because $R_{0}=\emptyset$. Moreover, a terminal rule $A \rightarrow a$ with $a \in \Gamma$ is in $R_{1}$ (and not in $R^{(0)}$ as defined in Section 8.1).

We construct the ranked alphabet $\Sigma$ by $\Sigma^{(k)}=R_{k}$ for each $k \in \mathbb{N}_{+}$, and $\Sigma^{(0)}=R_{0} \cup \Gamma$. We construct the $(\Sigma, \mathrm{B})$-wta $\mathcal{A}=(Q, \delta, F)$ as follows.

- $Q=N \cup \bar{\Gamma}$ where $\bar{\Gamma}=\{\bar{a} \mid a \in \Gamma\}$ (note that $Q \cap \Sigma=\emptyset$ ),
- for every $k \in \mathbb{N}, \sigma \in \Sigma^{(k)}$, and $q, q_{1}, \ldots, q_{k} \in Q$, we let

$$
\delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right)= \begin{cases}w t(\sigma) & \text { if } q, q_{1}, \ldots, q_{k} \in N \text { and } \sigma=\left(q \rightarrow q_{1} \cdots q_{k}\right) \text { is in } R \\ w t(\sigma) & \text { if } k=1, q \in N, q_{1}=\bar{a} \text { for some } a \in \Gamma \text { such that } \sigma=(q \rightarrow a) \text { is in } R \\ \mathbb{1} & \text { if } k=0, q=\bar{\sigma}, \text { and } \sigma \in \Gamma \\ \mathbb{0} & \text { otherwise }\end{cases}
$$

- for each $q \in Q$, we define $F_{q}=\mathbb{1}$ if $q=S$, and $\mathbb{O}$ otherwise.

Obviously, $\mathcal{A}$ is root weight normalized.
For the proof of (a) and (b) we need some preparations. For each $\xi \in \mathrm{T}_{\Sigma}$, we define the run $\rho_{\xi}: \operatorname{pos}(\xi) \rightarrow Q$ such that, for each $w \in \operatorname{pos}(\xi)$, we let

$$
\rho_{\xi}(w)= \begin{cases}\operatorname{lns}(\xi(w)) & \text { if } \xi(w) \in R \\ \xi(w) & \text { otherwise }\end{cases}
$$

We define the mapping $\varphi: \operatorname{RT}_{\mathcal{G}} \rightarrow\left\{(\xi, \rho) \mid \xi \in \mathrm{T}_{\Sigma}, \rho \in \mathrm{R}_{\mathcal{A}}(\xi)\right\}$ for every $d \in \mathrm{RT}_{\mathcal{G}}$ by $\varphi(d)=\left(\xi, \rho_{\xi}\right)$ where $\xi$ is obtained from $d$ by replacing each leaf which is labeled by a rule of the form $A \rightarrow a$ with $a \in \Gamma$ by the tree $(A \rightarrow a)(a)$ (cf. Figure 8.4). In particular, for each $\left(\xi, \rho_{\xi}\right) \in \operatorname{im}(\varphi)$, we have that $\rho_{\xi} \in \mathrm{R}_{\mathcal{A}}(S, \xi)$. Obviously, $\varphi$ is injective.

Next we define $\varphi^{\prime}: \operatorname{RT}_{\mathcal{G}} \rightarrow \operatorname{im}(\varphi)$ by $\varphi^{\prime}(d)=\varphi(d)$ for each $d \in \mathrm{RT}_{\mathcal{G}}$. Obviously, $\varphi^{\prime}$ is bijective. Moreover,

$$
\begin{equation*}
\text { for each } d \in \operatorname{RT}_{\mathcal{G}} \text {, if } \varphi^{\prime}(d)=\left(\xi, \rho_{\xi}\right) \text {, then } \pi(d)=\operatorname{yield}_{\Gamma}(\xi) \tag{8.16}
\end{equation*}
$$

For the proof of (a), assume that $\mathcal{G}$ is finite-derivational. Then, (8.16) and the fact that $\varphi^{\prime}$ is bijective imply that $\operatorname{yield}_{\Gamma}^{-1}(u) \cap \operatorname{supp}\left(\llbracket \mathcal{A} \rrbracket^{\text {run }}\right)$ is finite for each $u \in \Gamma^{*}$, i.e., that $\llbracket \mathcal{A} \rrbracket^{\text {run }}$ is $\chi\left(\right.$ yield $\left._{\Gamma}\right)$-summable.

Finally, we prove (b). It is easy to see that

$$
\begin{equation*}
\text { for each } d \in \mathrm{RT}_{\mathcal{G}} \text {, we have } \mathrm{wt}_{\mathcal{G}}(d)=\mathrm{wt}_{\mathcal{A}}\left(\varphi^{\prime}(d)\right) . \tag{8.17}
\end{equation*}
$$

Moreover,

$$
\begin{equation*}
\text { for every } \xi \in \mathrm{T}_{\Sigma} \text { and } \rho \in \mathrm{R}_{\mathcal{A}}(S, \xi) \text { : if }(\xi, \rho) \notin \operatorname{im}(\varphi) \text {, then } \mathrm{wt}_{\mathcal{A}}(\xi, \rho)=\mathbb{0} . \tag{8.18}
\end{equation*}
$$

Then, for each $u \in \Gamma^{*}$, we can calculate as follows.

$$
\begin{equation*}
\llbracket \mathcal{G} \rrbracket^{\mathrm{s}}(u)=\sum_{d \in \mathrm{RT}_{\mathcal{G}}(u)}^{\oplus} \mathrm{wt}_{\mathcal{G}}(d) \tag{8.2}
\end{equation*}
$$



Figure 8.4: A visualization of $\varphi(d)=\left(\xi, \rho_{\xi}\right)$ in the proof of Lemma 8.3 .2 with $d \in \mathrm{R}_{\mathcal{A}}(a b)$ for $a, b \in \Gamma$, $\xi \in \mathrm{T}_{\Sigma}$, and $\rho_{\xi} \in \mathrm{R}_{\mathcal{A}}(\xi)$. The states of $\rho_{\xi}$ are circled.

$$
\begin{align*}
& =\sum_{\substack{d \in \mathrm{RT}_{\mathcal{G}}: \\
\pi(d)=u}}^{\oplus} \mathrm{wt}_{\mathcal{G}}(d) \\
& =\sum_{\left(\xi, \rho_{\xi}\right) \in \operatorname{im}(\varphi):}^{\oplus} \mathrm{wt}_{\mathcal{G}}\left(\left(\varphi^{\prime}\right)^{-1}\left(\left(\xi, \rho_{\xi}\right)\right)\right) \quad \quad \text { (because } \varphi^{\prime} \text { is bijective and by (8.16) ) } \\
& \text { yield }_{\Gamma}(\xi)=u \\
& \begin{array}{ll}
=\sum_{\substack{\left(\xi, \rho_{\xi}\right) \in \operatorname{im}(\varphi): \\
\text { yield }(\xi)=u}}^{\oplus} \mathrm{wt}_{\mathcal{A}}\left(\xi, \rho_{\xi}\right) & \text { (by (8.17) and because } \left.\varphi\left(\left(\varphi^{\prime}\right)^{-1}\left(\left(\xi, \rho_{\xi}\right)\right)\right)=\left(\xi, \rho_{\xi}\right)\right) \\
=\sum_{\xi \in \operatorname{yield}_{\Gamma}^{-1}(u)} \bigoplus_{\rho \in \operatorname{Re}_{\mathcal{A}}(S, \xi)}^{\oplus} \operatorname{wt}_{\mathcal{A}}(\xi, \rho) & \quad \text { (by (8.18)) }
\end{array} \\
& \left.=\sum_{\xi \in \operatorname{yield}_{\Gamma}^{-1}(u)}^{\oplus} \bigoplus_{q \in Q} \bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}}(q, \xi)} \mathrm{wt}_{\mathcal{A}}(\xi, \rho) \otimes F_{q} \quad \text { (because } F_{S}=\mathbb{1} \text { and } F_{q}=0 \text { for each } q \in Q \backslash\{S\}\right) \\
& =\sum_{\xi \in \text { yield }_{\Gamma}^{-1}(u)}^{\oplus} \llbracket \mathcal{A} \rrbracket^{\text {run }}(\xi) \quad \quad \text { (by definition of } \llbracket \mathcal{A} \rrbracket^{\text {run }} \text { ) } \\
& =\left(\chi\left(\operatorname{yield}_{\Gamma}\right)\left(\llbracket \mathcal{A} \rrbracket^{\text {run }}\right)\right)(u) . \tag{2.30}
\end{align*}
$$

Lemma 8.3.3. Let $\Gamma \subseteq \Sigma^{(0)}$. Moreover, let $\mathcal{A}=(Q, \delta, F)$ be a $(\Sigma, \mathrm{B})$-wta such that $\llbracket \mathcal{A} \rrbracket^{\text {run }}$ is $\chi\left(\right.$ yield $\left._{\Gamma}\right)$ summable or B is $\sigma$-complete. We can construct a $(\Gamma, \mathrm{B})$-wcfg in nonterminal form such that (a) $\mathcal{G}$ is finite-derivational if $\llbracket \mathcal{A} \rrbracket^{\text {run }}$ is $\chi\left(\right.$ yield $\left._{\Gamma}\right)$-summable and $(\mathrm{b}) \llbracket \mathcal{G} \rrbracket^{\mathrm{s}}=\chi\left(\operatorname{yield}_{\Gamma}\right)\left(\llbracket \mathcal{A} \rrbracket^{\text {run }}\right)$.

Proof. Let $\mathcal{A}=(Q, \delta, F)$. By Theorem 7.3.1] we can construct a $(\Sigma, \mathrm{B})$-wta which is root weight normalized and r-equivalent to $\mathcal{A}$. Hence, we can assume that $\mathcal{A}$ is root weight normalized. Thus, $\operatorname{supp}(F)=\left\{q_{f}\right\}$ for some $q_{f} \in Q$ and $F_{q_{f}}=\mathbb{1}$.

We construct the ( $\Gamma, \mathrm{B}$ )-wcfg $\mathcal{G}=(N, S, R, w t)$ as follows.

- $N=Q \times \Sigma$ where each element has the form $[q, \sigma]$ for some $q \in Q$ and $\sigma \in \Sigma$,
- $S=\left\{q_{f}\right\} \times \Sigma$,
- $R$ is the smallest set such that the following conditions hold; simultaneously, we define $w t$.
- For every $k \in \mathbb{N}_{+}, \sigma \in \Sigma^{(k)}, q, q_{1}, \ldots, q_{k} \in Q$, and $\sigma_{1}, \ldots, \sigma_{k} \in \Sigma$, the rule $r=\left([q, \sigma] \rightarrow\left[q_{1}, \sigma_{1}\right] \cdots\left[q_{k}, \sigma_{k}\right]\right)$ is in $R$ and $w t(r)=\delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right)$.
- For every $\sigma \in \Gamma$ and $q \in Q$, the rule $r=([q, \sigma] \rightarrow \sigma)$ is in $R$ and $w t(r)=\delta_{0}(\varepsilon, \sigma, q)$.
- For every $\sigma \in \Sigma^{(0)} \backslash \Gamma$ and $q \in Q$, the rule $r=([q, \sigma] \rightarrow \varepsilon)$ is in $R$ and $w t(r)=\mathbb{1}$.

We define the mapping $\varphi:\left\{(\xi, \rho) \mid \xi \in \mathrm{T}_{\Sigma}, \rho \in \mathrm{R}_{\mathcal{A}}\left(q_{f}, \xi\right)\right\} \rightarrow \mathrm{RT}_{\mathcal{G}}$ for every $(\xi, \rho)$ by $\varphi(\xi, \rho)=d$ where $d$ is determined by the tree domain $W$ and the $R$-tree mapping $d^{\prime}: W \rightarrow R$ (recall that there is a bijective representation of trees as tree domains and tree mappings, cf. Section 2.9). We define $W=\operatorname{pos}(\xi)$ and for each $w \in W$ (using $k$ as abbreviation for $\operatorname{rk}_{\Sigma}(\xi(w))$ ) we let

$$
d^{\prime}(w)= \begin{cases}{[\rho(w), \xi(w)] \rightarrow[\rho(w 1), \xi(w 1)] \cdots[\rho(w k), \xi(w k)]} & \text { if } k \geq 1 \\ {[\rho(w), \xi(w)] \rightarrow \xi(w)} & \text { if } k=0 \text { and } \xi(w) \in \Gamma \\ {[\rho(w), \xi(w)] \rightarrow \varepsilon} & \text { if } k=0 \text { and } \xi(w) \in \Sigma^{(0)} \backslash \Gamma\end{cases}
$$

Obviously, $\varphi$ is injective and surjective, hence bijective (cf. Figure 8.5). Moreover,

$$
\begin{align*}
& \text { for every } \xi \in \mathrm{T}_{\Sigma} \text { and } \rho \in \mathrm{R}_{\mathcal{A}}(\xi) \\
& \text { if } \varphi(\xi, \rho)=d \text {, then } \pi(d)=\operatorname{yield}_{\Gamma}(\xi) \text { and } \mathrm{wt}_{\mathcal{G}}(d)=\mathrm{wt}_{\mathcal{A}}(\xi, \rho) \text {. } \tag{8.19}
\end{align*}
$$

Thus, since $\varphi$ is bijective, the assumption that $\llbracket \mathcal{A} \rrbracket^{\text {run }}$ is $\chi\left(\right.$ yield $\left._{\Gamma}\right)$-summable implies that $\mathcal{G}$ is finitederivational.

Then, for each $u \in \Gamma^{*}$, we can calculate as follows.

$$
\begin{aligned}
& \llbracket \mathcal{G} \rrbracket^{\mathrm{s}}(u)=\sum_{d \in \mathrm{RT}_{\mathcal{G}}(u)}^{\oplus} \mathrm{wt}_{\mathcal{G}}(d)=\sum_{\substack{d \in \mathrm{RT}_{\mathcal{G}}: \\
\pi(d)=u}}^{\oplus} \mathrm{wt}_{\mathcal{G}}(d) \\
& =\sum_{\xi \in T_{\Sigma}:}^{\oplus} \bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}}\left(q_{f}, \xi\right)} \mathrm{wt}_{\mathcal{A}}(\xi, \rho) \quad \quad \text { (because } \varphi \text { is bijective and by (8.19)) } \\
& =\sum_{\xi \in \operatorname{yield}_{\Gamma}^{-1}(u)}^{\oplus} \bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}}\left(q_{f}, \xi\right)} \mathrm{wt}_{\mathcal{A}}(\xi, \rho) \\
& \left.=\sum_{\xi \in \operatorname{yield}_{\Gamma}^{-1}(u)}^{\oplus} \bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}}(\xi)} \operatorname{wt}_{\mathcal{A}}(\xi, \rho) \otimes F_{\rho(\varepsilon)} \quad \text { (because } \operatorname{supp}(F)=\left\{q_{f}\right\} \text { and } F_{q_{f}}=\mathbb{1}\right) \\
& =\sum_{\xi \in \text { yield }_{\Gamma}^{-1}(u)}^{\oplus} \llbracket \mathcal{A} \rrbracket^{\text {run }}(\xi) \quad \text { (by definition of } \llbracket \mathcal{A} \rrbracket^{\text {run }} \text { ) } \\
& =\chi\left(\operatorname{yield}_{\Gamma}\right)\left(\left[\mathcal{A} \rrbracket^{\text {run }}\right)(u)\right. \text {. (by (2.30)) }
\end{aligned}
$$

Finally, we verify that Theorem 8.3.1 generalizes Bra69, Thm. 3.20] and [Don70, Thm. 2.5]. We achieve this by proving that the latter results are equivalent to Theorem 8.3.1 for the case that B is the Boolean semiring Boole.

Corollary 8.3.4. For each language $L \subseteq \Gamma^{*}$ the following two statements are equivalent.
(A) We can construct a context-free grammar $G$ such that $\mathrm{L}(G)=L$.
(B) We can construct a ranked alphabet $\Sigma$ with $\Gamma \subseteq \Sigma^{(0)}$ and a $\Sigma$-fta $A$ such that $L=\operatorname{yield}_{\Gamma}(\mathrm{L}(A))$.

Proof. Proof of $(\mathrm{A}) \Rightarrow(\mathrm{B})$ : Let $G$ be a $\Gamma$-cfg such that $L=\mathrm{L}(G)$. By Observation 8.1.1(A) $\Rightarrow(\mathrm{B})$, we can construct a $(\Gamma$, Boole $)$-wcfg $\mathcal{G}$ such that $L=\operatorname{supp}\left(\llbracket \mathcal{G} \rrbracket^{\mathrm{s}}\right)$. Since Boole is $\sigma$-complete, Theorem 8.3.1 $(\mathrm{A}) \Rightarrow(\mathrm{B})$ implies that we can construct a ranked alphabet $\Sigma$ with $\Gamma \subseteq \Sigma^{(0)}$ and a ( $\Sigma$, Boole)wta $\mathcal{A}$ such that $L=\operatorname{supp}\left(\chi\left(\operatorname{yield}_{\Gamma}\right)(\llbracket \mathcal{A} \rrbracket)\right.$ ). By (2.31) (using $r=\llbracket \mathcal{A} \rrbracket$ and $g=$ yield $_{\Gamma}$ ), we obtain $L=$ yield $_{\Gamma}(\operatorname{supp}(\llbracket \mathcal{A} \rrbracket))$. By Corollary $3.4 .2(\mathrm{~B}) \Rightarrow(\mathrm{A})$, we can construct a $\Sigma$-fta $A$ such that $\operatorname{supp}(\llbracket \mathcal{A} \rrbracket)=\mathrm{L}(A)$. Thus $L=\operatorname{yield}_{\Gamma}(\mathrm{L}(A))$.
$\xi \in \mathrm{T}_{\Sigma}: \quad \rho \in \mathrm{R}_{\mathcal{A}}(\xi):$


$$
\Gamma=\{\alpha\}
$$

Figure 8.5: A visualization of $\varphi(\xi, \rho)=d$ in the proof of Lemma 8.3.3 with $\xi \in \mathrm{T}_{\Sigma}, \rho \in \mathrm{R}_{\mathcal{A}}(\xi), \Gamma=\{\alpha\}$, and $d \in \operatorname{RT}_{\mathcal{G}}(\alpha)$. The states of $\rho$ are circled.

Proof of $(\mathrm{B}) \Rightarrow(\mathrm{A})$ : Let $A$ be a $\Sigma$-fta such that $L=\operatorname{yield}_{\Gamma}(\mathrm{L}(A))$. By Corollary 3.4.2(A) $\Rightarrow(\mathrm{B})$, we can construct a $\left(\Sigma\right.$, Boole)-wta $\mathcal{A}$ such that $L=\operatorname{yield}_{\Gamma}(\operatorname{supp}(\llbracket \mathcal{A} \rrbracket))$. Then we can apply (2.31) and obtain that $L=\operatorname{supp}\left(\chi\left(\operatorname{yield}_{\Gamma}\right)(\llbracket \mathcal{A} \rrbracket)\right)$. Since Boole is $\sigma$-complete, Theorem 8.3.1 $(\mathrm{B}) \Rightarrow(\mathrm{A})$ implies that we can construct a $\left(\Gamma\right.$, Boole)-wcfg $\mathcal{G}$ such that $L=\operatorname{supp}\left(\llbracket \mathcal{G} \rrbracket^{\mathrm{s}}\right)$. Finally, Observation 8.1.1(B) $\Rightarrow(\mathrm{A})$, we can construct a $\Gamma$-cfg $G$ such that $L=\mathrm{L}(G)$.

## Chapter 9

## Weighted regular tree grammars

Here we introduce weighted regular tree grammars as particular wcfg. This possibility is based on the simple fact that each tree is a particular string. We indicate that the normal forms for wcfg are also normal forms for weighted regular tree grammars. We define two more restricted forms of weighted regular tree grammars: alphabetic and tree automata form, and prove that they are normal forms (cf. Lemmas 9.2 .2 and 9.2 .3 respectively). In particular, we prove that weighted regular tree grammars in tree automata form are essentially wta (cf. Theorem 9.2.9).

### 9.1 The grammar model

By definition, each tree $\xi$ over $\Sigma$ is a particular string over the alphabet $\Sigma \cup \Xi$ where $\Xi$ contains the opening and closing parentheses and the comma. For convenience, we abbreviate $\Sigma \cup \Xi$ by $\Sigma^{\Xi}$. Then, by definition, we have $\mathrm{T}_{\Sigma} \subseteq\left(\Sigma^{\Xi}\right)^{*}$. Of course, $\left(\Sigma^{\Xi}\right)^{*} \backslash \mathrm{~T}_{\Sigma} \neq \emptyset$.

This aspect of the definition of trees (being particular strings) allows one to define particular wcfg, viz. those which generate weighted tree languages. We call them weighted regular tree grammars.

Formally, a weighted regular tree grammar over $\Sigma$ and B (for short: $(\Sigma, \mathrm{B})-\mathrm{wrtg}$, or: wrtg) is a ( $\left.\Sigma^{\Xi}, \mathrm{B}\right)$ $\operatorname{wcfg} \mathcal{G}=(N, S, R, w t)$ where each rule in $R$ has the form $A \rightarrow \xi$ with $\xi \in \mathrm{T}_{\Sigma}(N)$. Obviously, $\mathcal{G}$ is $\varepsilon$-free because $\varepsilon \notin \mathrm{T}_{\Sigma}(N)$. Also here we sometimes want to show the occurrences of elements of $N$ in the right-hand side of a rule more explicitly. Then we will write a rule in the form

$$
A \rightarrow \xi\left[A_{1}, \ldots, A_{k}\right]
$$

where $k \in \mathbb{N}, \xi \in \mathrm{C}_{\Sigma}\left(Z_{k}\right)$, and $A_{1}, \ldots, A_{k} \in N$.
Let $\mathcal{G}=(N, S, R, w t)$ be a ( $\Sigma, \mathrm{B}$ )-wrtg. We recall that we considered $R$ as ranked alphabet (where the rank of a rule is the number of nonterminal occurrences in its right-hand side) and that the projection of $\mathcal{G}$ has the type $\pi_{\mathcal{G}}: \mathrm{T}_{R} \rightarrow\left(\Sigma^{\Xi}\right)^{*}$. Due to the special form of the rules, we have $\operatorname{im}\left(\pi_{\mathcal{G}}\right) \subseteq \mathrm{T}_{\Sigma}$, and thus we can view $\pi_{\mathcal{G}}$ as mapping of type

$$
\pi_{\mathcal{G}}: \mathrm{T}_{R} \rightarrow \mathrm{~T}_{\Sigma}
$$

and hence $\mathrm{RT}_{\mathcal{G}}=\mathrm{RT}_{\mathcal{G}}\left(\mathrm{T}_{\Sigma}\right)$. Moreover, it is easy to see that $\pi_{\mathcal{G}}$ is determined by the $(R, \Sigma)$-tree homomorphism $\pi_{\mathcal{G}}=\left(\left(\pi_{\mathcal{G}}\right)_{k} \mid k \in \mathbb{N}\right)$ defined, for every $k \in \mathbb{N}$ and $r \in R^{(k)}$ of the form $r=(A \rightarrow$ $\left.\xi\left[A_{1}, \ldots, A_{k}\right]\right)$, by $\left(\pi_{\mathcal{G}}\right)_{k}(r)=\xi$.

By Observation 8.2.2 and since $\mathcal{G}$ is $\varepsilon$-free, if $\mathcal{G}$ is chain-free, then it is finite-derivational.
If $\mathcal{G}$ is finite-derivational or B is $\sigma$-complete, then the weighted tree language generated by $\mathcal{G}$, denoted by $\llbracket \mathcal{G} \rrbracket^{\mathrm{t}}$, is the mapping $\llbracket \mathcal{G} \rrbracket^{\mathrm{t}}: \mathrm{T}_{\Sigma} \rightarrow B$ defined for each $\xi \in \mathrm{T}_{\Sigma}$ by

$$
\llbracket \mathcal{G} \rrbracket^{\mathrm{t}}(\xi)=\llbracket \mathcal{G} \rrbracket^{\mathrm{s}}(\xi) .
$$

Hence $\llbracket \mathcal{G} \rrbracket^{\mathrm{t}}=\left.\llbracket \mathcal{G} \rrbracket^{\mathrm{s}}\right|_{\mathrm{T}_{\Sigma}}$ and for each $\xi \in \mathrm{T}_{\Sigma}$, using (8.2), we have

$$
\llbracket \mathcal{G} \rrbracket^{\mathrm{t}}(\xi)=\sum_{d \in \mathrm{RT}_{\mathcal{G}}(\xi)}^{\oplus} \mathrm{wt}_{\mathcal{G}}(d)
$$

Let $r$ be a $(\Sigma, \mathrm{B})$-weighted tree language. It is called regular if there exists a $(\Sigma, \mathrm{B})$-wrtg $\mathcal{G}$ which is finite-derivational if B is not $\sigma$-complete and for which $r=\llbracket \mathcal{G} \rrbracket^{\mathrm{t}}$.

Since in this book we will use $\llbracket \mathcal{G} \rrbracket^{\mathrm{t}}$ many times, we make the following convention.
In the rest of this book, for each wrtg $\mathcal{G}$, we will abbreviate $\llbracket \mathcal{G} \rrbracket^{\mathrm{t}}$ by $\llbracket \mathcal{G} \rrbracket$.
We denote by (i) $\operatorname{Reg}_{\mathrm{nc}}\left(\Sigma\right.$, B), (ii) $\operatorname{Reg}_{\mathrm{fd}}(\Sigma, \mathrm{B})$, and (iii) $\operatorname{Reg}(\Sigma, \mathrm{B})$ the sets of $(\Sigma, \mathrm{B})$-weighted tree languages which can be generated by
(i) chain-free $(\Sigma, \mathrm{B})$-wrtg
(ii) finite-derivational $(\Sigma, \mathrm{B})$-wrtg, and
(iii) $(\Sigma, \mathrm{B})-$ wrtg which are finite-derivational if B is not $\sigma$-complete, respectively.

Then $\operatorname{Reg}_{\mathrm{nc}}(\Sigma, \mathrm{B}) \subseteq \operatorname{Reg}_{\mathrm{fd}}(\Sigma, \mathrm{B}) \subseteq \operatorname{Reg}(\Sigma, \mathrm{B}) ;$ moreover, $\operatorname{Reg}_{\mathrm{fd}}(\Sigma, \mathrm{B})=\operatorname{Reg}(\Sigma, \mathrm{B})$ if B is not $\sigma$-complete.
Example 9.1.1. We consider the ranked alphabet $\Sigma=\left\{\sigma^{(2)}, \gamma^{(1)}, \alpha^{(0)}\right\}$ and the set of trees $U \subset \mathrm{~T}_{\Sigma}$ such that each tree $\xi \in U$ has an upper part, which only contains $\sigma$-labeled positions, and a number of lower parts, which only contain $\gamma$ - and $\alpha$-labeled positions. Formally,

$$
U=\left\{\zeta\left[\zeta_{1}, \ldots, \zeta_{n}\right] \mid n \in \mathbb{N}, \zeta \in \mathrm{C}_{\Sigma}\left(Z_{n}\right), \operatorname{pos}_{\{\gamma, \alpha\}}(\zeta)=\emptyset, \text { and } \zeta_{1}, \ldots, \zeta_{n} \in \mathrm{~T}_{\{\gamma, \alpha\}}\right\}
$$

Let $\xi \in U$. There are unique $n, \zeta$, and $\zeta_{1}, \ldots, \zeta_{n}$ such that $\xi=\zeta\left[\zeta_{1}, \ldots, \zeta_{n}\right]$. Then $\zeta$ is the upper part of $\xi$ and $\zeta_{1}, \ldots, \zeta_{n}$ are the lower parts of $\xi$. We denote by $\operatorname{bor}(\xi)$ the set of positions of $\xi$ which are at the border between the upper part and the lower parts, i.e., $\operatorname{bor}(\xi)=\operatorname{pos}_{Z}(\zeta)$.

Now we wish to determine, for each $\xi \in U$, the minimal height of a lower part of $\xi$. For this we consider the tropical semiring $\operatorname{Nat}_{\min ,+}=\left(\mathbb{N}_{\infty}, \min ,+, \infty, 0\right)$ and the mapping $f: \mathrm{T}_{\Sigma} \rightarrow \mathbb{N}_{\infty}$ defined for each $\xi \in \mathrm{T}_{\Sigma}$ by

$$
f(\xi)= \begin{cases}\min \left(\operatorname{height}\left(\left.\xi\right|_{w}\right) \mid w \in \operatorname{bor}(\xi)\right) & \text { if } \xi \in U \\ \infty & \text { otherwise }\end{cases}
$$

We will show that $f$ is a regular $\left(\Sigma, \operatorname{Nat}_{\min ,+}\right)$-weighted tree language. For this we construct the following $\left(\Sigma, \operatorname{Nat}_{\min ,+}\right)-\operatorname{wrtg} \mathcal{G}=(N,\{S\}, R, w t)$ with $N=\{S, A, B, C, D\}$ and the following rules and weights:

$$
\begin{array}{ll}
S \rightarrow \sigma(A, S): 0, & S \rightarrow \sigma(S, A): 0, \quad S \rightarrow B: 0 \\
A \rightarrow \sigma(A, A): 0, & A \rightarrow C: 0 \\
B \rightarrow \gamma(B): 1, & B \rightarrow \alpha: 0 \\
C \rightarrow \gamma(C): 0, & C \rightarrow \alpha: 0
\end{array}
$$

We note that $\mathcal{G}$ contains two chain-rules. However $\mathcal{G}$ is finite-derivational because of the following. For each $\xi \in U$, there exists a bijection between the sets $\operatorname{RT}_{\mathcal{G}}(\xi)$ and $\operatorname{bor}(\xi)$ (in particular, based on $\Sigma$, we can reconstruct from the set $\operatorname{bor}(\xi)$ of positions the tree $\xi)$. Thus $\left|\operatorname{RT}_{\mathcal{G}}(\xi)\right|=|\operatorname{bor}(\xi)|$ for each $\xi \in U$, and $\left|\operatorname{RT}_{\mathcal{G}}(\xi)\right|=\emptyset$ for each $\xi \in \mathrm{T}_{\Sigma} \backslash U$. Hence $\mathcal{G}$ is finite-derivational. Moreover, it is easy to see that $\operatorname{supp}(\llbracket \mathcal{G} \rrbracket)=U$.

Now let $\xi \in U$. Then, for every $d \in \operatorname{RT}_{\mathcal{G}}(\xi)$, there exists a unique position $w \in \operatorname{pos}(d)$ such that $d(w)=(S \rightarrow B)$ and $w \in \operatorname{bor}(\xi)$ (i.e., $\left.\xi\right|_{w}$ is a lower part of $\xi$ ); this is due to the form of the $S$-rules. Let us denote this unique position by $w_{d}$. Vice versa, for each $w \in \operatorname{bor}(\xi)$, there exists a $d \in \mathrm{RT}_{\mathcal{G}}(\xi)$ such that $d(w)=(S \rightarrow B)$. Let us denote this $d$ by $d_{w}$. Clearly, $w_{d_{w}}=w$. We refer to Figure 9.1 for an illustration.


Figure 9.1: An illustration of a rule tree $d \in \operatorname{RT}_{\mathcal{G}}(\xi)$ for some $\xi \in U$.

It is clear that, for each $d \in \operatorname{RT}_{\mathcal{G}}\left(B, \mathrm{~T}_{\{\gamma, \alpha\}}\right)$ we have $\operatorname{wt}_{\mathcal{G}}(d)=\operatorname{height}\left(\pi_{\mathcal{G}}(d)\right)$. Moreover, for each $d \in \mathrm{RT}_{\mathcal{G}}\left(C, \mathrm{~T}_{\{\gamma, \alpha\}}\right)$ we have $\mathrm{wt}_{\mathcal{G}}(d)=0$. Since each of the rules

$$
\begin{array}{lll}
S \rightarrow \sigma(A, S) & S \rightarrow \sigma(S, A) & S \rightarrow B \\
A \rightarrow \sigma(A, A) & A \rightarrow C &
\end{array}
$$

has weight 0 , we obtain the following:

$$
\begin{equation*}
\text { for each } d \in \operatorname{RT}_{\mathcal{G}}(\xi) \text { we have } \mathrm{wt}_{\mathcal{G}}(d)=\operatorname{height}\left(\left.\xi\right|_{w_{d}}\right) . \tag{9.1}
\end{equation*}
$$

Then we have

$$
\begin{aligned}
\llbracket \mathcal{G} \rrbracket(\xi) & =\sum_{d \in \operatorname{RT}_{\mathcal{G}}(S, \xi)}^{\min } \mathrm{wt}_{\mathcal{G}}(d)=\min \left(\mathrm{wt}_{\mathcal{G}}(d) \mid d \in \mathrm{RT}_{\mathcal{G}}(\xi)\right) \\
& =\min \left(\operatorname{wt}_{\mathcal{G}}\left(d_{w}\right) \mid w \in \operatorname{bor}(\xi)\right) \\
& =\min \left(\operatorname{height}\left(\left.\xi\right|_{w}\right) \mid w \in \operatorname{bor}(\xi)\right) \\
& =f(\xi) .
\end{aligned}
$$

Lemma 9.1.2. For every $(\Sigma, \mathrm{B})-\operatorname{wrtg} \mathcal{G}$ it is decidable whether $\mathcal{G}$ is finite-derivational.
Proof. Let $\mathcal{G}=(N, S, R, w t)$ be a $(\Sigma, \mathrm{B})$-wrtg. By standard methods, we can decide whether there exists a $\xi \in \mathrm{T}_{\Sigma}$ such that $\mathrm{RT}_{\mathcal{G}}(\xi) \neq \emptyset$. If for every $\xi \in \mathrm{T}_{\Sigma}$ we have $\mathrm{RT}_{\mathcal{G}}(\xi)=\emptyset$, then $\mathcal{G}$ is finite-derivational.

Now let us assume that there exists a $\xi \in \mathrm{T}_{\Sigma}$ with $\mathrm{RT}_{\mathcal{G}}(\xi) \neq \emptyset$. The idea is to transform $\mathcal{G}$ into a context-free grammar and to apply the construction in the proof of Theorem 2.12.1 for reducing that context-free grammar. In fact, we view each tree $\xi \in \mathrm{T}_{\Sigma}(N)$ as a string over the alphabet $\Sigma^{\Xi} \cup N$. Formally, we construct the $\Sigma^{\Xi}-\operatorname{cfg} G=\left(N \cup\left\{S_{0}\right\}, S_{0}, P\right)$, where $S_{0}$ is a new nonterminal and $P=$ $R \cup\left\{S_{0} \rightarrow A \mid A \in S\right\}$. (We note that $G$ is not the context-free grammar associated to $\mathcal{G}$ in the sense of Chapter 8, ) Then $\mathrm{L}(G) \subseteq \mathrm{T}_{\Sigma}$ and, due to our assumption on $\mathcal{G}$, we have $\mathrm{L}(G) \neq \emptyset$.

Then we apply the construction (cf. Theorem 2.12.1) to obtain a reduced $\Sigma^{\Xi}$-cfg $G^{\prime}=\left(N^{\prime}, S_{0}, P^{\prime}\right)$ such that $G^{\prime}$ and $G$ are equivalent. Since $G^{\prime}$ is reduced, each nonterminal in $N^{\prime}$ occurs in some rule tree $d \in \mathrm{RT}_{G^{\prime}}\left(S_{0}, \Delta^{*}\right)$.

Due to the construction, we have that $N^{\prime} \subseteq N$ and $P^{\prime} \subseteq P$. Then we construct the ( $\Sigma, \mathrm{B}$ )-wrtg $\mathcal{G}^{\prime}=\left(N^{\prime}, S^{\prime}, R^{\prime}, w t^{\prime}\right)$, where $S^{\prime}=S \cap N^{\prime}, R^{\prime}=P^{\prime} \backslash\left\{S_{0} \rightarrow A \mid A \in S^{\prime}\right\}$ (hence $R^{\prime} \subseteq R$ ), and $w t^{\prime}=\left.w t\right|_{R^{\prime}}$. Since $G^{\prime}$ is reduced, the following property holds for $\mathcal{G}^{\prime}$ :

$$
\begin{equation*}
\text { for each } A \in N^{\prime} \text { there exist } \xi \in \mathrm{T}_{\Sigma} \text { and } d \in \mathrm{RT}_{\mathcal{G}^{\prime}}(\xi) \text { such that } A \text { occurs in } d \text {. } \tag{9.2}
\end{equation*}
$$

Moreover, $\operatorname{RT}_{\mathcal{G}^{\prime}}(\xi)=\operatorname{RT}_{\mathcal{G}}(\xi)$ for each $\xi \in \mathrm{T}_{\Sigma}$. Thus $\mathcal{G}^{\prime}$ is equivalent to $\mathcal{G}$, and $\mathcal{G}^{\prime}$ is finite-derivational iff $\mathcal{G}$ is finite-derivational.

Since $\mathcal{G}^{\prime}$ satisfies (9.2) and it is $\varepsilon$-free, it is finite-derivational iff there do not exist $n \in \mathbb{N}_{+}$and a sequence $A_{1} \rightarrow A_{2}, \ldots, A_{n-1} \rightarrow A_{n}$ of chain rules of $\mathcal{G}^{\prime}$ such that $A_{1}=A_{n}$. This latter property of $\mathcal{G}^{\prime}$ is decidable by a standard algorithm on finite graphs War62].

Finally, we mention that in [Kós22, Thm. V.5., V.6] the following was proved for the unweighted case: it is decidable whether the language generated by a given $\Sigma^{\Xi}$-cfg is a $\Sigma$-tree language. Furthermore, if the answer to this question is positive, then a regular tree grammar can be constructed which generates that tree language.

### 9.2 Normal forms of wrtg

Since each $(\Sigma, \mathrm{B})-\operatorname{wrtg} \mathcal{G}$ is a particular $\left(\Sigma^{\Xi}, \mathrm{B}\right)$-wcfg, we can apply the normal form lemmas proved in Chapter 8 also to $\mathcal{G}$. Then it is a question whether the wcfg $\mathcal{G}^{\prime}$ constructed in the proof of such a lemma is a wrtg or not. Next we tailor those lemmas for wrtg for which the answer to the question is positive.

Lemma 9.2.1. Let $\mathcal{G}$ be a $(\Sigma, \mathrm{B})$-wrtg such that $\mathcal{G}$ is finite-derivational or B is $\sigma$-complete. Then the following three statements hold.
(1) We can construct a start-separated $(\Sigma, \mathrm{B})$-wrtg $\mathcal{G}^{\prime}$ such that $\llbracket \mathcal{G} \rrbracket=\llbracket \mathcal{G}^{\prime} \rrbracket$. The construction preserves the properties finite-derivational and local-reduced. The construction does not preserve the property chain-free.
(2) We can construct a local-reduced $(\Sigma, \mathrm{B})$-wrtg $\mathcal{G}^{\prime}$ such that $\llbracket \mathcal{G} \rrbracket=\llbracket \mathcal{G}^{\prime} \rrbracket$. The construction preserves the properties finite-derivational, start-separated, and chain-free.
(3) If B is a semiring, then there exists a chain-free $(\Sigma, \mathrm{B})$-wrtg $\mathcal{G}^{\prime}$ such that $\llbracket \mathcal{G} \rrbracket=\llbracket \mathcal{G}^{\prime} \rrbracket$. If $\mathcal{G}$ has one of the properties: start-separated and local-reduced, then also $\mathcal{G}^{\prime}$ has it. Moreover, if $\mathcal{G}$ is finite-derivational, then we can construct $\mathcal{G}^{\prime}$.

Proof. Statement (1) is a corollary of Lemma 8.2 .4 because if $\mathcal{G}$ is a wrtg, then the construction in the proof of the lemma yields a wrtg $\mathcal{G}^{\prime}$ with the properties stated in (1). Similarly, Statements (2) and (3) are corollaries of Lemma 8.2.5 and Theorem 8.2.6, respectively.

We continue with the definition of two more restricted forms of wrtg. Let $\mathcal{G}=(N, S, R, w t)$ be a $(\Sigma, \mathrm{B})$-wrtg. We say that $\mathcal{G}$ is

- alphabetic if each rule contains at most one occurrence of a symbol in $\Sigma$.
- in tree automata form if $R=\left\{A \rightarrow \sigma\left(A_{1}, \ldots, A_{k}\right) \mid k \in \mathbb{N}, \sigma \in \Sigma^{(k)}, A, A_{1}, \ldots, A_{k} \in N\right\}$. Thus, there exists a bijection from $R$ to the set $\bigcup_{k \in \mathbb{N}} N \times \Sigma^{(k)} \times N^{k}$.
Obviously, if $\mathcal{G}$ is in tree automata form, then it is alphabetic and chain-free.
Lemma 9.2.2. (cf. MW67, Lm. 3.1], IF75, Thm. 1], and AB87, Prop. 1.2]) Let $\mathcal{G}$ be a ( $\Sigma$, B)-wrtg such that $\mathcal{G}$ is finite-derivational or B is $\sigma$-complete. We can construct a $(\Sigma, \mathrm{B})$-wrtg $\mathcal{G}^{\prime}$ such that $\mathcal{G}^{\prime}$ is alphabetic and $\llbracket \mathcal{G}^{\prime} \rrbracket=\llbracket \mathcal{G} \rrbracket$. The construction preserves the properties finite-derivational, start-separated, chain-free, and local-reduced.

Proof. Let $\mathcal{G}=(N, S, R, w t)$ be a $(\Sigma, \mathrm{B})$-wrtg. As a degree for "how much" a $(\Sigma, \mathrm{B})$-wrtg $\mathcal{G}$ does not match the condition of being alphabetic, we define, for each rule $r=(A \rightarrow \xi)$, the degree of $r$, denoted by $\operatorname{deg}(r)$, by

$$
\operatorname{deg}(r)= \begin{cases}0 & \text { if } r \text { is a chain-rule } \\ \left|\operatorname{pos}_{\Sigma}(\xi)\right|-1 & \text { otherwise }\end{cases}
$$

and we define $\operatorname{deg}(\mathcal{G})=\sum_{r \in R} \operatorname{deg}(r)$. Then we have the following equivalence: $\operatorname{deg}(\mathcal{G})=0$ iff each rule of $\mathcal{G}$ contains at most one occurrence of a symbol in $\Sigma$ (i.e., $\mathcal{G}$ is alphabetic). Thus, we will construct a $(\Sigma, \mathrm{B})$-wrtg $\mathcal{G}^{\prime}$ such that $\mathcal{G}^{\prime}$ is equivalent to $\mathcal{G}$ and $\operatorname{deg}\left(\mathcal{G}^{\prime}\right)=0$.

For the construction of $\mathcal{G}^{\prime}$, we pick a rule from $\mathcal{G}$ with more than one occurrence of a symbol in $\Sigma$ and apply the usual decomposition to its right-hand side (cf. [GS84, Lm. 2.3.4] and Eng75b, Thm. 3.22]). More precisely, if $\operatorname{deg}(\mathcal{G})>0$, i.e., there exists a rule

$$
r=\left(A \rightarrow \sigma\left(\xi_{1}, \ldots, \xi_{i-1}, \xi_{i}, \xi_{i+1}, \ldots, \xi_{k}\right)\right)
$$

in $R$ such that $\xi_{i} \notin N$ for some $i \in[k]$, then we replace this rule by the rules

$$
r_{1}=\left(A \rightarrow \sigma\left(\xi_{1}, \ldots, \xi_{i-1}, A^{\prime}, \xi_{i+1}, \ldots, \xi_{k}\right)\right) \quad \text { and } \quad r_{2}=\left(A^{\prime} \rightarrow \xi_{i}\right),
$$

where $A^{\prime}$ is a new nonterminal. Moreover, we let $w t\left(r_{1}\right)=w t(r)$ and $w t\left(r_{2}\right)=\mathbb{1}$. It is clear that this construction preserves each of the mentioned properties. Moreover, it does not change the semantics of $\mathcal{G}$. Since $\operatorname{deg}\left(r_{1}\right)+\operatorname{deg}\left(r_{2}\right)=\operatorname{deg}(r)-1$, the $\operatorname{degree} \operatorname{deg}(\mathcal{G})$ has decreased by one. Now we repeat this decomposition until we reach eventually a ( $\Sigma, \mathrm{B})$-wrtg with degree 0 ; this $(\Sigma, \mathrm{B})$-wrtg we call $\mathcal{G}^{\prime}$.

Next we relate wrtg with wta.

Lemma 9.2.3. Let $\Sigma$ be a ranked alphabet, B be a strong bimonoid, and $\mathcal{G}$ be a $(\Sigma, \mathrm{B})$-wrtg. If (a) $\mathcal{G}$ is chain-free or (b) B is a semiring and $\mathcal{G}$ is finite-derivational or (c) B is a $\sigma$-complete semiring, then there exists a $(\Sigma, \mathrm{B})$-wrtg $\mathcal{G}^{\prime}$ such that $\mathcal{G}^{\prime}$ is in tree automata form and $\llbracket \mathcal{G}^{\prime} \rrbracket=\llbracket \mathcal{G} \rrbracket$. In Cases (a) and (b) we can even construct $\mathcal{G}^{\prime}$.

Proof. Let $\mathcal{G}=(N, S, R, w t)$. In Case (a), by Lemma 9.2 .2 we can construct an alphabetic and chain-free ( $\Sigma, \mathrm{B})$-wrtg which is equivalent to $\mathcal{G}$; thus we can assume that $\mathcal{G}$ is alphabetic and chain-free.

In Cases (b) and (c), by Lemma 9.2.1(3), there exists a chain-free ( $\Sigma, \mathrm{B}$ )-wrtg which is equivalent to $\mathcal{G}$. Then, by Lemma 9.2.2, there exists an alphabetic and chain-free $(\Sigma, \mathrm{B})$-wrtg which is equivalent to $\mathcal{G}$. Moreover, in Case (b) we can even assume that $\mathcal{G}$ is alphabetic and chain-free, because then Lemma 9.2.1(3) is constructive.

Next we complete the set $R$ of rules by adding rules with weight $\mathbb{0}$. Formally, we construct the $(\Sigma, \mathrm{B})-\operatorname{wrtg} \mathcal{G}^{\prime}=\left(N, S, R^{\prime}, w t^{\prime}\right)$ such that $R^{\prime}$ is the set of all rules $r=\left(A \rightarrow \sigma\left(A_{1}, \ldots, A_{k}\right)\right)$ with $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $A, A_{1}, \ldots, A_{k} \in N$. Moreover, for each $r \in R^{\prime}$ we let $w t^{\prime}(r)=w t(r)$ if $r \in R$, and $w t^{\prime}(r)=\mathbb{0}$ otherwise. Clearly, $\mathcal{G}^{\prime}$ is in tree automata form and $\llbracket \mathcal{G}^{\prime} \rrbracket=\llbracket \mathcal{G} \rrbracket$.

Let $\mathcal{A}=(Q, \delta, F)$ be a $(\Sigma, \mathrm{B})$-wta with identity root weights and $\mathcal{G}=(N, S, R, w t)$ be a $(\Sigma, \mathrm{B})$-wrtg in tree automata form. We call $\mathcal{A}$ and $\mathcal{G}$ related if $Q=N$, for each $q \in Q$ we have $F_{q}=\mathbb{1}$ if and only if $q \in S$, and

$$
\delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right)=w t\left(q \rightarrow \sigma\left(q_{1}, \ldots, q_{k}\right)\right)
$$

for every $k \in \mathbb{N}, \sigma \in \Sigma^{(k)}, q \in Q$, and $q_{1} \cdots q_{k} \in Q^{k}$. We note that, if $q \notin S$, then $F_{q}=\mathbb{O}$ because $\mathcal{A}$ has identity root weights. Moreover, if $\mathcal{A}$ and $\mathcal{G}$ are related, then $\mathcal{A}$ is local-trim iff $\mathcal{G}$ is reduced.

Example 9.2.4. We give an example of a wta and a wrtg which are related. As wta, we choose the root weight normalized ( $\Sigma, \mathrm{Nat})$-wta $\mathcal{A}=(Q, \delta, F)$ of Example [3.2.11, which we recall here:

- $\Sigma=\left\{\sigma^{(2)}, \gamma^{(1)}, \alpha^{(0)}\right\}$,
- $Q=\{\perp, a, f\}$ and $F_{\perp}=F_{a}=0$ and $F_{f}=1$,
- for every $q_{1}, q_{2}, q \in Q$ we define

$$
\begin{gathered}
\delta_{0}(\varepsilon, \alpha, q)=\left\{\begin{array}{ll}
1 & \text { if } q \in\{\perp, a\} \\
0 & \text { otherwise }
\end{array} \quad \delta_{1}\left(q_{1}, \gamma, q\right)= \begin{cases}1 & \text { if } q_{1} q \in\{\perp \perp, f f\} \\
0 & \text { otherwise }\end{cases} \right. \\
\delta_{2}\left(q_{1} q_{2}, \sigma, q\right)= \begin{cases}1 & \text { if } q_{1} q_{2} q \in\{\perp \perp \perp, \perp a f, \perp f f, f \perp f\} \\
0 & \text { otherwise } .\end{cases}
\end{gathered}
$$

The $(\Sigma, N a t)-w r t g \mathcal{G}=(Q, f, R, w t)$ in tree automata form which is related to $\mathcal{A}$ is defined by:

- $R=\left\{q \rightarrow \theta\left(q_{1}, \ldots, q_{k}\right) \mid k \in \mathbb{N}, \theta \in \Sigma^{(k)}, q, q_{1}, \ldots, q_{k} \in Q\right\}$ and
- for every $q_{1}, q_{2}, q \in Q$

$$
\begin{gathered}
w t(q \rightarrow \alpha)=\left\{\begin{array}{lll}
1 & \text { if } q \in\{\perp, a\} \\
0 & \text { otherwise }
\end{array} \quad \text { wt }\left(q \rightarrow \gamma\left(q_{1}\right)\right)= \begin{cases}1 & \text { if } q_{1} q \in\{\perp \perp, f f\} \\
0 & \text { otherwise }\end{cases} \right. \\
w t\left(q \rightarrow \sigma\left(q_{1}, q_{2}\right)\right)= \begin{cases}1 & \text { if } q_{1} q_{2} q \in\{\perp \perp \perp, \perp a f, \perp f f, f \perp f\} \\
0 & \text { otherwise } .\end{cases} \\
\hline
\end{gathered}
$$

Lemma 9.2.5. Let $\Sigma$ be a ranked alphabet and B be a strong bimonoid. Moreover, let $\mathcal{A}$ be a $(\Sigma, \mathrm{B})$-wta with identity root weights and $\mathcal{G}$ be a $(\Sigma, \mathrm{B})$-wrtg in tree automata form. If $\mathcal{A}$ and $\mathcal{G}$ are related, then $\llbracket \mathcal{A} \rrbracket^{\mathrm{run}}=\llbracket \mathcal{G} \rrbracket$.

Proof. Let $\mathcal{A}=(Q, \delta, F)$ and $\mathcal{G}=(N, S, R, w t)$. Since $\mathcal{G}$ is in tree automata form, we have $\operatorname{pos}(d)=\operatorname{pos}(\xi)$ for every $\xi \in \mathrm{T}_{\Sigma}$ and $d \in \operatorname{RT}_{\mathcal{G}}(\xi)$. Moreover, since $\mathcal{A}$ and $\mathcal{G}$ are related, there exists a one-to-one correspondence between (a) the rules tree of $\mathcal{G}$ and (b) pairs of $\Sigma$-trees and runs on them.

For the formalization we recall the set TR defined in Section 3.1]on p.63. We have TR $=\{(\xi, \rho) \mid \xi \in$ $\left.\mathrm{T}_{\Sigma}, \rho \in \mathrm{R}_{\mathcal{A}}(\xi)\right\}$. Then we define the mapping $\varphi: \mathrm{RT}_{\mathcal{G}}\left(N, \mathrm{~T}_{\Sigma}\right) \rightarrow \mathrm{TR}$ such that, for every $A \in N, \xi \in \mathrm{~T}_{\Sigma}$, and $d \in \operatorname{RT}_{\mathcal{G}}(A, \xi)$, we let

$$
\varphi(d)=(\xi, \rho)
$$

where, for each $w \in \operatorname{pos}(\xi)$, we let $\rho(w)=\operatorname{lhs}(d(w))$ (cf. Figure 9.2). Hence, $\rho \in \mathrm{R}_{\mathcal{A}}(A, \xi)$. Since $\mathcal{A}$ and $\mathcal{G}$ are related, $\varphi$ is bijective.

Next we will prove a relationship between the weights of rule trees of $\mathcal{G}$ and weights of runs of $\mathcal{A}$ that are related by $\varphi$. For this we use the well-founded set $\left(\mathrm{RT}_{\mathcal{G}}\left(N, \mathrm{~T}_{\Sigma}\right), \prec\right)$, where we let

$$
\prec=\prec_{R} \cap\left(\mathrm{RT}_{\mathcal{G}}\left(N, \mathrm{~T}_{\Sigma}\right) \times \mathrm{RT}_{\mathcal{G}}\left(N, \mathrm{~T}_{\Sigma}\right)\right) .
$$

Obviously, $\prec$ is well-founded and $\min _{\prec}\left(\mathrm{RT}_{\mathcal{G}}\left(N, \mathrm{~T}_{\Sigma}\right)\right)$ is the set of terminal rules of $R$, which is not empty. Then, by induction on $\left(\mathrm{RT}_{\mathcal{G}}\left(N, \mathrm{~T}_{\Sigma}\right), \prec\right)$, we prove that the following statement holds:

$$
\begin{equation*}
\text { For each } d \in \operatorname{RT}_{\mathcal{G}}\left(N, \mathrm{~T}_{\Sigma}\right) \text {, we have: } \operatorname{wt}_{\mathcal{G}}(d)=\operatorname{wt}_{\mathcal{A}}(\varphi(d)) . \tag{9.3}
\end{equation*}
$$

Let $d \in \operatorname{RT}_{\mathcal{G}}\left(N, \mathrm{~T}_{\Sigma}\right)$, i.e., $d \in \operatorname{RT}_{\mathcal{G}}(A, \xi)$ for some $A \in N$ and $\xi=\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)$. Since $\operatorname{pos}(\xi)=\operatorname{pos}(d)$, there exists a rule $r=\left(A \rightarrow \sigma\left(B_{1} \ldots, B_{k}\right)\right)$ in $R$ and for each $i \in[k]$ there exists a rule tree $d_{i} \in$ $\mathrm{RT}_{\mathcal{G}}\left(B_{i}, \xi_{i}\right)$ such that $d=r\left(d_{1}, \ldots, d_{k}\right)$. Then

$$
\begin{equation*}
\mathrm{wt}_{\mathcal{G}}(d)=\bigotimes_{i \in[k]} \mathrm{wt}_{\mathcal{G}}\left(d_{i}\right) \otimes w t\left(A \rightarrow \sigma\left(B_{1} \ldots, B_{k}\right)\right) \tag{8.11}
\end{equation*}
$$

$d \in \operatorname{RT}_{\mathcal{G}}(\xi):$

$$
\xi \in \mathrm{T}_{\Sigma}: \quad \rho \in \mathrm{R}_{\mathcal{A}}(\xi):
$$



Figure 9.2: A visualization of $\varphi(d)=(\xi, \rho)$ in the proof of Lemma 9.2.5 with $\xi=\sigma(\gamma(\alpha), \beta), d \in \mathrm{RT}_{\mathcal{G}}(\xi)$, and $\rho \in \mathrm{R}_{\mathcal{A}}(A, \xi)$. The states of $\rho$ are circled.

$$
=\bigotimes_{i \in[k]} \mathrm{wt}_{\mathcal{A}}\left(\varphi\left(d_{i}\right)\right) \otimes \delta_{k}\left(B_{1} \cdots B_{k}, \sigma, A\right)
$$

$$
=\operatorname{wt}_{\mathcal{A}}(\varphi(d)) . \quad \text { (by definition of } \varphi \text { and (3.1) }
$$

This proves (9.3).
Let $\xi \in \mathrm{T}_{\Sigma}$. Then

$$
\begin{aligned}
& \llbracket \mathcal{A} \rrbracket^{\mathrm{run}}(\xi)=\bigoplus_{\rho \in R_{\mathcal{A}}(\xi)} \operatorname{wt}_{\mathcal{A}}(\xi, \rho) \otimes F_{\rho(\xi)} \\
&=\bigoplus_{A \in S} \bigoplus_{\rho \in R_{\mathcal{A}}(A, \xi)} \mathrm{wt}_{\mathcal{A}}(\xi, \rho) \quad \text { (because } \mathcal{A} \text { has identity root weights and } F_{A}=\mathbb{1} \text { iff } A \in S \text { ) } \\
&=\bigoplus_{A \in S} \bigoplus_{d \in \operatorname{RT}_{\mathcal{G}}(A, \xi)} \mathrm{wt}_{\mathcal{A}}(\xi, \varphi(d)) \\
&=\bigoplus_{A \in S} \bigoplus_{d \in \operatorname{RT}_{\mathcal{G}}(A, \xi)}{ }^{\left(\text {because } \varphi \text { is a bijection and } \varphi\left(\operatorname{RT}_{\mathcal{G}}(A, \xi)\right)=\mathrm{R}_{\mathcal{A}}(A, \xi) \text { for each } A \in S\right)} \\
&=\llbracket \mathcal{G} \rrbracket(\xi) . \\
& \text { (by Equation (9.3)) }
\end{aligned}
$$

Lemma 9.2.6. For each $(\Sigma, \mathrm{B})-$ wta $\mathcal{A}$, we can construct a $(\Sigma, \mathrm{B})-\operatorname{wrtg} \mathcal{G}$ such that $\mathcal{G}$ is in tree automata form and $\llbracket \mathcal{A} \rrbracket^{\text {run }}=\llbracket \mathcal{G} \rrbracket$.

Proof. By Theorem 7.3.1, we can construct a root weight normalized $(\Sigma, \mathrm{B})$-wta $\mathcal{A}^{\prime}$ such that $\llbracket \mathcal{A} \rrbracket^{\text {run }}=$ $\llbracket \mathcal{A}^{\prime} \rrbracket^{\text {run }}$. Thus, in particular, $\mathcal{A}^{\prime}$ has identity root weights. It is obvious how to construct the $(\Sigma, \mathrm{B})$-wrtg $\mathcal{G}$ in tree automata form such that $\mathcal{A}^{\prime}$ and $\mathcal{G}$ are related. By Lemma 9.2.5, we have $\llbracket \mathcal{A}^{\prime} \rrbracket^{\text {run }}=\llbracket \mathcal{G} \rrbracket$.

Corollary 9.2.7. Each r-recognizable ( $\Sigma, \mathrm{B}$ )-weighted tree language is a $\left(\Sigma^{\Xi}, \mathrm{B}\right)$-weighted context-free language.

Proof. By Lemma 9.2.6. for each $(\Sigma, \mathrm{B})$-wta $\mathcal{A}$ we can construct a $(\Sigma, \mathrm{B})$-wrtg $\mathcal{G}$ such that $\mathcal{G}$ is in tree automata form and $\llbracket \mathcal{A} \rrbracket^{\text {run }}=\llbracket \mathcal{G} \rrbracket$. Since $\mathcal{G}$ is a particular $\left(\Sigma^{\Xi}, B\right)$-wcfg, we obtain the result.

Lemma 9.2.8. Let $\mathcal{G}$ be a $(\Sigma, \mathrm{B})$-wrtg. If (a) $\mathcal{G}$ is chain-free or (b) B is a semiring and $\mathcal{G}$ is finitederivational or (c) B is a $\sigma$-complete semiring, then there exists a ( $\Sigma, \mathrm{B}$ )-wta $\mathcal{A}$ such that $\llbracket \mathcal{G} \rrbracket=\llbracket \mathcal{A} \rrbracket^{\text {run }}$. In Cases (a) and (b) we can even construct $\mathcal{A}$.

Proof. By Lemma 9.2.3, in each of the Cases (a), (b), and (c) there exists a ( $\Sigma, \mathrm{B}$ )-wrtg $\mathcal{G}^{\prime}$ in tree automata form such that $\llbracket \mathcal{G}^{\prime} \rrbracket=\llbracket \mathcal{G} \rrbracket$. In Cases (a) and (b) we can even construct $\mathcal{G}^{\prime}$. Then, it is obvious
how to construct a $(\Sigma, \mathrm{B})$-wta $\mathcal{A}$ with identity root weights such that $\mathcal{A}$ and $\mathcal{G}^{\prime}$ are related. Finally, by Lemma 9.2.5, we have $\llbracket \mathcal{G}^{\prime} \rrbracket=\llbracket \mathcal{A} \rrbracket^{\text {run }}$.

Now we can prove the equivalence of $(\Sigma, \mathrm{B})$-wta and chain-free $(\Sigma, \mathrm{B})$-wrtg. For the case that B is the Boolean semiring, this is GS84, Thm. 2.3.6] and Eng75b, Thm. 3.25].

Theorem 9.2.9. Let $\Sigma$ be a ranked alphabet and B be a strong bimonoid. Then the following two statements hold.
(1) $\operatorname{Reg}_{\mathrm{nc}}(\Sigma, \mathrm{B})=\operatorname{Rec}^{\mathrm{run}}(\Sigma, \mathrm{B})$.
(2) If B is a semiring, then $\operatorname{Reg}(\Sigma, \mathrm{B})=\operatorname{Rec}(\Sigma, \mathrm{B})$.

Proof. First we prove the inclusions from left to right. In Statement (1), the inclusion follows from Lemma 9.2.8(a). In Statement (2), if B is not $\sigma$-complete, then $\operatorname{Reg}(\Sigma, \mathrm{B})=\operatorname{Reg}_{\mathrm{fd}}(\Sigma, \mathrm{B})$, hence the inclusion follows from Lemma 9.2 .8 (b). In Statement (2), if B is $\sigma$-complete, then it follows from Lemma 9.2 .8 (c). The inclusions from right to left follow from Lemma 9.2 .6 and the fact that, if a wrtg is in tree automata form, then it is both chain-free and finite-derivational.

Finally, we mention that, for every $(\Sigma, \mathrm{B})$-wta $\mathcal{A}$ with identity root weights and ( $\Sigma, \mathrm{B}$ )-wrtg $\mathcal{G}$ in tree automata form, if $\mathcal{A}$ and $\mathcal{G}$ are related, then the following equivalence holds:

$$
\mathcal{A} \text { is local-trim iff } \mathcal{G} \text { is local-reduced. }
$$

The following result generalizes [IF75, Thm. 2] (also cf. Eng75b, Thm. 3.57] and [GS84, Thm. 3.2.2] for the unweighted case). It says that the weighted rule tree language of each wcfg is a regular weighted tree language. In Lemma 11.2 .2 we will prove that each such weighted rule tree language can be determined by some weighted local system.

Theorem 9.2.10. Let $\Gamma$ be an alphabet and B a strong bimonoid. Let $\Gamma$ be an alphabet and B a strong bimonoid. For each $(\Gamma, \mathrm{B})-w c f g \mathcal{G}$ with rule set $R$, we can construct an $(R, \mathrm{~B})$-wrtg $\mathcal{G}^{\prime}$ such that $\llbracket \mathcal{G}^{\prime} \rrbracket=$ $\llbracket \mathcal{G} \rrbracket^{\mathrm{wrt}}$.

Proof. Let $\mathcal{G}=(N, S, R, w t)$. We consider $R$ as ranked alphabet such that, for each $r \in R$, the rank of $r$ is the number of nonterminals in the right-hand side of $r$.

We construct the $(R, \mathrm{~B})$-wrtg $\mathcal{G}^{\prime}=\left(N, S, R^{\prime}, \mathrm{wt}^{\prime}\right)$ as follows. If $r=\left(A \rightarrow u_{0} A_{1} u_{1} \cdots A_{k} u_{k}\right)$ is in $R$, then $r^{\prime}=\left(A \rightarrow r\left(A_{1}, \ldots, A_{k}\right)\right)$ is in $R^{\prime}$, and we let $\mathrm{wt}^{\prime}\left(r^{\prime}\right)=\mathrm{wt}(r)$. In the usual way, we also consider $R^{\prime}$ as ranked alphabet.

Let $\xi \in \mathrm{RT}_{\mathcal{G}}$. We define $d_{\xi} \in \mathrm{RT}_{\mathcal{G}^{\prime}}$ such that $\operatorname{pos}\left(d_{\xi}\right)=\operatorname{pos}(\xi)$ and, for each $w \in \operatorname{pos}\left(d_{\xi}\right)$, if $\xi(w)=r$ and $r=\left(A \rightarrow u_{0} A_{1} u_{1} \cdots A_{k} u_{k}\right)$, then we let $d_{\xi}(w)=\left(A \rightarrow r\left(A_{1}, \ldots, A_{k}\right)\right)$. Obviously, $\mathrm{RT}_{\mathcal{G}^{\prime}}(\xi)=\left\{d_{\xi}\right\}$ and $\mathrm{wt}_{\mathcal{G}^{\prime}}\left(d_{\xi}\right)=\mathrm{wt}_{\mathcal{G}}(\xi)$.

Now let $\xi \in \mathrm{T}_{R}$. If $\xi \notin \mathrm{RT}_{\mathcal{G}}$, then $\mathrm{RT}_{\mathcal{G}^{\prime}}(\xi)=\emptyset$ and

$$
\llbracket \mathcal{G}^{\prime} \rrbracket(\xi)=\sum_{d \in \mathrm{RT}_{\mathcal{G}^{\prime}}(\xi)}^{\oplus} \mathrm{wt}_{\mathcal{G}^{\prime}}(d)=\mathbb{0}=\left(\mathrm{wt}_{\mathcal{G}} \otimes \chi\left(\mathrm{RT}_{\mathcal{G}}\right)\right)(\xi)=\llbracket \mathcal{G} \rrbracket^{\mathrm{wrt}}(\xi)
$$

If $\xi \in \mathrm{RT}_{\mathcal{G}}$, then

$$
\llbracket \mathcal{G}^{\prime} \rrbracket(\xi)=\sum_{d \in \mathrm{RT}_{\mathcal{G}^{\prime}}(\xi)}^{\oplus} \mathrm{wt}_{\mathcal{G}^{\prime}}(d)=\mathrm{wt}_{\mathcal{G}^{\prime}}\left(d_{\xi}\right)=\mathrm{wt}_{\mathcal{G}}(\xi)=\left(\mathrm{wt}_{\mathcal{G}} \otimes \chi\left(\mathrm{RT}_{\mathcal{G}}\right)\right)(\xi)=\llbracket \mathcal{G} \rrbracket^{\mathrm{wrt}}(\xi)
$$

## Chapter 10

## Closure properties

In this chapter we prove that the set of recognizable weighted tree languages is closed under several operations. For some of the closure results it is technically easier first to prove them in the setting of wrtg and second to instantiate them to wta by using results on the connection between wrtg and wta (cf. Lemmas 9.2.6 and 9.2.8).

Since some of the operations change the ranked alphabet or the strong bimonoid, we define corresponding sets of recognizable weighted tree languages in which these parameters are left open (using the underscore). Formally, we abbreviate by $\operatorname{Rec}^{\text {run }}(-, B)$ the set of all r-recognizable ( $\Sigma, B$ )-weighted tree languages for some ranked alphabet $\Sigma$. Similarly we define the abbreviations $\operatorname{Rec}^{\text {init }}(-, B)$ and $\operatorname{Rec}\left(\_, B\right)$. Moreover, we abbreviate by $\operatorname{Rec}^{\text {run }}\left(\Sigma,,_{-}\right)$the set of all r-recognizable $(\Sigma, \mathrm{B})$-weighted tree language for some strong bimonoid B. Similarly we define the abbreviations $\operatorname{Rec}^{\text {init }}\left(\Sigma,{ }_{-}\right)$and $\operatorname{Rec}\left(\Sigma,{ }_{\boldsymbol{H}}\right)$.

### 10.1 Closure under sum

In this section we will prove that the sets $\operatorname{Rec}^{\text {run }}(\Sigma, \mathrm{B})$ and $\operatorname{Rec}^{\text {init }}(\Sigma, \mathrm{B})$ are closed under sum. A set $\mathcal{L}$ of B -weighted tree languages is closed under sum if the following holds: for every $(\Sigma, \mathrm{B})$-weighted tree languages $r_{1}$ and $r_{2}$, if $r_{1}, r_{2} \in \mathcal{L}$, then $\left(r_{1} \oplus r_{2}\right) \in \mathcal{L}$.

Theorem 10.1.1. Let $\Sigma$ be a ranked alphabet, $\mathrm{B}=(B, \oplus, \otimes, \mathbb{O}, \mathbb{1})$ be a strong bimonoid, and $\mathcal{A}_{1}$ and $\mathcal{A}_{2}$ be two ( $\Sigma, \mathrm{B})$-wta. Then the following two statements hold.
(1) (cf. $\llbracket$ Rad10, Lm. $5.1(1)])$ We can construct $a(\Sigma, \mathrm{~B})-$ wta $\mathcal{B}$ such that $\llbracket \mathcal{B} \rrbracket^{\mathrm{run}}=\llbracket \mathcal{A}_{1} \rrbracket^{\mathrm{run}} \oplus \llbracket \mathcal{A}_{2} \rrbracket^{\mathrm{run}}$ and $\llbracket \mathcal{B} \rrbracket^{\text {init }}=\llbracket \mathcal{A}_{1} \rrbracket^{\text {init }} \oplus \llbracket \mathcal{A}_{2} \rrbracket^{\text {init }}$.
(2) If $\mathcal{A}_{1}$ and $\mathcal{A}_{2}$ are crisp deterministic, then we can construct a crisp deterministic $(\Sigma, \mathrm{B})$-wta $\mathcal{B}$ such that $\llbracket \mathcal{B} \rrbracket=\llbracket \mathcal{A}_{1} \rrbracket \oplus \llbracket \mathcal{A}_{2} \rrbracket$.
Thus, in particular, the sets $\operatorname{Rec}^{\text {run }}(\Sigma, \mathrm{B})$ and $\operatorname{Rec}^{\text {init }}(\Sigma, \mathrm{B})$ are closed under sum.

Proof. Let $\mathcal{A}_{1}=\left(Q_{1}, \delta_{1}, F_{1}\right)$ and $\mathcal{A}_{2}=\left(Q_{2}, \delta_{2}, F_{2}\right)$ such that $Q_{1} \cap Q_{2}=\emptyset$.
Proof of (1): We construct the ( $\Sigma, \mathrm{B}$ )-wta $\mathcal{B}=(Q, \delta, F)$ as follows:

- $Q=Q_{1} \cup Q_{2}$,
- for every $k \in \mathbb{N}, \sigma \in \Sigma^{(k)}, q \in Q$, and $q_{1} \cdots q_{k} \in Q^{k}$ we define

$$
\delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right)= \begin{cases}\left(\delta_{1}\right)_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right) & \text { if } q_{1}, \ldots, q_{k}, q \in Q_{1} \\ \left(\delta_{2}\right)_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right) & \text { if } q_{1}, \ldots, q_{k}, q \in Q_{2} \\ 0 & \text { otherwise }\end{cases}
$$

- for each $q \in Q$ we define $F_{q}=\left(F_{1}\right)_{q}$ if $q \in Q_{1}$, and $F_{q}=\left(F_{2}\right)_{q}$ if $q \in Q_{2}$.

We prove that $\llbracket \mathcal{B} \rrbracket^{\text {run }}=\llbracket \mathcal{A}_{1} \rrbracket^{\text {run }} \oplus \llbracket \mathcal{A}_{2} \rrbracket^{\text {run }}$. Let $\xi \in \mathrm{T}_{\Sigma}$ and $\rho \in \mathrm{R}_{\mathcal{B}}(\xi)$. Obviously, $\mathrm{R}_{\mathcal{A}_{1}}(\xi) \cap \mathrm{R}_{\mathcal{A}_{2}}(\xi)=\emptyset$ and $\mathrm{R}_{\mathcal{A}_{1}}(\xi) \cup \mathrm{R}_{\mathcal{A}_{2}}(\xi) \subseteq \mathrm{R}_{\mathcal{B}}(\xi)$. Moreover, $\operatorname{wt}_{\mathcal{B}}(\xi, \rho)=\mathrm{wt}_{\mathcal{A}_{i}}(\xi, \rho)$ if $\rho \in \mathrm{R}_{\mathcal{A}_{i}}(\xi)$ for each $i \in\{1,2\}$, and $\operatorname{wt}_{\mathcal{B}}(\xi, \rho)=\mathbb{O}$ if $\rho \in \mathrm{R}_{\mathcal{B}}(\xi) \backslash\left(\mathrm{R}_{\mathcal{A}_{1}}(\xi) \cup \mathrm{R}_{\mathcal{A}_{2}}(\xi)\right)$. Thus

$$
\begin{aligned}
\llbracket \mathcal{B} \rrbracket^{\mathrm{run}}(\xi) & =\bigoplus_{\rho \in \mathrm{R}_{\mathcal{B}}(\xi)} \operatorname{wt}_{\mathcal{B}}(\xi, \rho) \otimes F_{\rho(\varepsilon)} \\
& =\bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}_{1}}(\xi)} \mathrm{wt}_{\mathcal{A}_{1}}(\xi, \rho) \otimes\left(F_{1}\right)_{\rho(\varepsilon)} \oplus \bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}_{2}}(\xi)} \mathrm{wt}_{\mathcal{A}_{2}}(\xi, \rho) \otimes\left(F_{2}\right)_{\rho(\varepsilon)} \\
& =\llbracket \mathcal{A}_{1} \rrbracket^{\mathrm{run}}(\xi) \oplus \llbracket \mathcal{A}_{2} \rrbracket^{\mathrm{run}}\left(\xi_{2}\right) .
\end{aligned}
$$

Next we prove $\llbracket \mathcal{B} \rrbracket^{\text {init }}=\llbracket \mathcal{A}_{1} \rrbracket^{\text {init }} \oplus \llbracket \mathcal{A}_{2} \rrbracket^{\text {init }}$. Let $\xi \in \mathrm{T}_{\Sigma}$. Obviously, for every $q \in Q$,

$$
\mathrm{h}_{\mathcal{B}}(\xi)_{q}= \begin{cases}\mathrm{h}_{\mathcal{A}_{1}}(\xi)_{q} & \text { if } q \in Q_{1} \\ \mathrm{~h}_{\mathcal{A}_{2}}(\xi)_{q} & \text { if } q \in Q_{2}\end{cases}
$$

Thus,

$$
\begin{aligned}
\llbracket \mathcal{B} \rrbracket^{\text {init }}(\xi) & =\bigoplus_{q \in Q} \mathrm{~h}_{\mathcal{B}}(\xi)_{q} \otimes F_{q}=\bigoplus_{q \in Q_{1}} \mathrm{~h}_{\mathcal{A}_{1}}(\xi)_{q} \otimes\left(F_{1}\right)_{q} \oplus \bigoplus_{q \in Q_{2}} \mathrm{~h}_{\mathcal{A}_{2}}(\xi)_{q} \otimes\left(F_{2}\right)_{q} \\
& =\llbracket \mathcal{A}_{1} \rrbracket^{\mathrm{init}}(\xi) \oplus \llbracket \mathcal{A}_{2} \rrbracket^{\text {init }}(\xi) .
\end{aligned}
$$

Proof of (2): Let $\mathcal{A}_{1}$ and $\mathcal{A}_{2}$ be crisp deterministic. We construct the $(\Sigma, \mathrm{B})$ wta $\mathcal{B}=(Q, \delta, F)$ as follows:

- $Q=Q_{1} \times Q_{2}$; for each $q \in Q$ we denote its first and second component by $q^{1}$ and $q^{2}$, respectively,
- for every $k \in \mathbb{N}, \sigma \in \Sigma^{(k)}, q \in Q$ and $q_{1} \cdots q_{k} \in Q^{k}$ we define

$$
\delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right)=\left(\delta_{1}\right)_{k}\left(q_{1}^{1} \cdots q_{k}^{1}, \sigma, q^{1}\right) \otimes\left(\delta_{2}\right)_{k}\left(q_{1}^{2} \cdots q_{k}^{2}, \sigma, q^{2}\right)
$$

- for each $q \in Q$ we define $F_{q}=\left(F_{1}\right)_{q^{1}} \oplus\left(F_{2}\right)_{q^{2}}$.

Obviously, $\mathcal{B}$ is crisp deterministic.
Let $\xi \in \mathrm{T}_{\Sigma}$ and $i \in\{1,2\}$. Since $\mathcal{A}_{i}$ is crisp deterministic, by Lemma4.3.1 there exists a state $q_{\xi}^{i}$ such that $\mathrm{Q}_{\neq 0}^{\mathrm{h}_{\mathcal{A}_{i}}}(\xi)=\left\{q_{\xi}^{i}\right\}$ and $\mathrm{h}_{\mathcal{A}_{i}}(\xi)_{q_{\xi}^{i}}=\mathbb{1}$. Then, obviously, $\mathrm{Q}_{\neq 0}^{\mathrm{h}_{\mathcal{B}}}(\xi)=\left\{\left(q_{\xi}^{1}, q_{\xi}^{2}\right)\right\}$ and $\mathrm{h}_{\mathcal{B}}(\xi)_{\left(q_{\xi}^{1}, q_{\xi}^{2}\right)}(\xi)=\mathbb{1}$. Then we can calculate as follows:

$$
\llbracket \mathcal{B} \rrbracket(\xi)=\mathrm{h}_{\mathcal{B}}(\xi)_{\left(q_{\xi}^{1}, q_{\xi}^{2}\right)} \otimes F_{\left(q_{\xi}^{1}, q_{\xi}^{2}\right)}=F_{\left(q_{\xi}^{1}, q_{\xi}^{2}\right)}=\left(F_{1}\right)_{q_{\xi}^{1}} \oplus\left(F_{2}\right)_{q_{\xi}^{2}}=\llbracket \mathcal{A}_{1} \rrbracket(\xi) \oplus \llbracket \mathcal{A}_{2} \rrbracket(\xi)
$$

We note that, in the proof of Theorem 10.1.1(1), we cannot use the product construction of the proof of Theorem 10.1.1 (2), because that would require the strong bimonoid B to be commutative. Also, vice versa, in the proof of Theorem 10.1.1(2), we cannot use the union construction of the proof of Theorem 10.1.1(1), because in general that does not preserve bu determinism: on a nullary symbol the constructed $(\Sigma, \mathrm{B})$-wta $\mathcal{B}$ can start with some $q \in Q_{1}$ or with some $q^{\prime} \in Q_{2}$.

Further, we note that in Gho22, Thm. 4.1] closure of $\operatorname{Rec}^{\text {init }}(\Sigma, B)$ under sum was proved for the particular case that B is a $\sigma$-complete orthomodular lattice. Since each $\sigma$-complete orthomodular lattice is a particular strong bimonoid, Gho22, Thm. 4.1] follows from Theorem 10.1.1 and from Rad10, Lm. 5.1(1)].

For the case of bu deterministic wta we get the following negative result. (Hence [Rad10, Lm. 5.1(2)] is wrong, because the construction of that lemma does not preserve bu determinism.)
Theorem 10.1.2. There exists a string ranked alphabet $\Sigma$ such that $\operatorname{bud} \operatorname{Rec}(\Sigma, \operatorname{Rat})$ is not closed under sum.

Proof. We let $\Sigma=\left\{\gamma^{(1)}, \alpha^{(0)}\right\}$. It is easy to define the bu deterministic ( $\Sigma$, Rat)-wta $\mathcal{A}_{1}$ and $\mathcal{A}_{2}$ over $\Sigma$ and $\mathbb{Q}$ such that $\llbracket \mathcal{A}_{1} \rrbracket\left(\gamma^{n} \alpha\right)=2^{n}$ and $\llbracket \mathcal{A}_{2} \rrbracket\left(\gamma^{n} \alpha\right)=1$ for each $n \in \mathbb{N}$. Clearly, $\llbracket \mathcal{A}_{1} \rrbracket+\llbracket \mathcal{A}_{2} \rrbracket=(\exp +1)$ where the weighted tree language $(\exp +1)$ is defined in Example 3.2.9. However, as Theorem 17.2 .2 shows, $(\exp +1)$ is not in bud-Rec $(\Sigma$, Rat $)$.

### 10.2 Closure under scalar multiplications

In this section we will prove that the sets $\operatorname{Rec}^{\text {init }}(\Sigma, \mathrm{B})$ and $\operatorname{Rec}^{\text {run }}(\Sigma, \mathrm{B})$ are closed under scalar multiplications from the left and from the right. A set $\mathcal{L}$ of B -weighted tree languages is closed under scalar multiplications from the left if for every $b \in B$ and $r \in \mathcal{L}$, the B-weighted tree language $b \otimes r$ is in $\mathcal{L}$. Similarly we define the concept closed under scalar multiplications from the right. A set $\mathcal{L}$ is closed under scalar multiplications if it is closed under scalar multiplications from the left and closed under scalar multiplications from the right.

Theorem 10.2.1. (cf. Rad10, Thm. 5.4]) Let $\Sigma$ be a ranked alphabet, $\mathrm{B}=(B, \oplus, \otimes, \mathbb{O}, \mathbb{1})$ be a strong bimonoid, and $\mathcal{A}$ be $a(\Sigma, \mathrm{~B})-w t a$. Moreover, let $b \in B$. Then the following four statements hold.
(1) If (a) B is left-distributive or (b) $\mathcal{A}$ is bu deterministic, then we can construct $a(\Sigma, \mathrm{~B})$-wta $\mathcal{B}$ such that $\llbracket \mathcal{B} \rrbracket^{\text {init }}=b \otimes \llbracket \mathcal{A} \rrbracket^{\text {init }}$ and $\llbracket \mathcal{B} \rrbracket^{\mathrm{run}}=b \otimes \llbracket \mathcal{A} \rrbracket^{\text {run }}$.
(2) If B is right-distributive, then we can construct $a(\Sigma, \mathrm{~B})$-wta $\mathcal{B}$ such that $\llbracket \mathcal{B} \rrbracket^{\mathrm{init}}=\llbracket \mathcal{A} \rrbracket^{\mathrm{init}} \otimes b$ and $\llbracket \mathcal{B} \rrbracket^{\text {run }}=\llbracket \mathcal{A} \rrbracket^{\text {run }} \otimes b$.
(3) If $\mathcal{A}$ is bu deterministic, then we can construct a bu deterministic $(\Sigma, \mathrm{B})$-wta $\mathcal{B}$ such that $\llbracket \mathcal{B} \rrbracket=$ $\llbracket \mathcal{A} \rrbracket \otimes b$. Moreover, if $\mathcal{A}$ is crisp deterministic, then so is $\mathcal{B}$.
(4) If $\mathcal{A}$ is crisp deterministic, then we can construct a crisp deterministic $(\Sigma, B)$-wta $\mathcal{B}$ such that $\llbracket \mathcal{B} \rrbracket=b \otimes \llbracket \mathcal{A} \rrbracket$.
Thus, in particular, if B is a semiring, then the set $\operatorname{Rec}(\Sigma, \mathrm{B})$ is closed under scalar multiplications.

Proof. Proof of (1): Let $\mathcal{A}=(Q, \delta, F)$. We construct the ( $\Sigma, \mathrm{B}$ )-wta $\mathcal{B}=\left(Q^{\prime}, \delta^{\prime}, F^{\prime}\right)$ such that, if (a) or (b) holds, then $\llbracket \mathcal{B} \rrbracket^{\text {init }}=b \otimes \llbracket \mathcal{A} \rrbracket^{\text {init }}$ and $\llbracket \mathcal{B} \rrbracket^{\text {run }}=b \otimes \llbracket \mathcal{A} \rrbracket^{\text {run }}$, using the following idea. The wta $\mathcal{B}$ simulates $\mathcal{A}$ but, at the leftmost leaf of each input tree, it multiplies the weight of the applied transition with $b$ from the left. The information about this multiplication is propagated up by using states of the form $q^{\ell}$ for each $q \in Q$. Formally, we let

- $Q^{\prime}=Q^{0} \cup Q^{\ell}$ where $Q^{0}=\left\{q^{0} \mid q \in Q\right\}$ and $Q^{\ell}=\left\{q^{\ell} \mid q \in Q\right\}$,
- for every $q \in Q$ and $\alpha \in \Sigma^{(0)}$, we let $\delta_{0}^{\prime}\left(\varepsilon, \alpha, q^{0}\right)=\delta_{0}(\varepsilon, \alpha, q)$ and $\delta_{0}^{\prime}\left(\varepsilon, \alpha, q^{\ell}\right)=b \otimes \delta_{0}(\varepsilon, \alpha, q)$, for every $k \in \mathbb{N}_{+}, \sigma \in \Sigma^{(k)}$, and $q_{1}^{s_{1}}, \ldots, q_{k}^{s_{k}}, q^{s} \in Q^{\prime}$, we let

$$
\delta_{k}^{\prime}\left(q_{1}^{s_{1}} \cdots q_{k}^{s_{k}}, \sigma, q^{s}\right)= \begin{cases}\delta_{k}\left(q_{1} \ldots q_{k}, \sigma, q\right) & \text { if }\left(s_{1}=s=\ell \wedge(\forall i \in[2, k]): s_{i}=0\right) \\ & \vee\left(s=0 \wedge(\forall i \in[k]): s_{i}=0\right) \\ 0 & \text { otherwise }\end{cases}
$$

- for every $q \in Q$ we let $\left(F^{\prime}\right)_{q^{\ell}}=F_{q}$ and $\left(F^{\prime}\right)_{q^{0}}=\mathbb{0}$.

We note that this construction does not preserve bu determinism.
Now we assume that B is left-distributive or $\mathcal{A}$ is bu deterministic. First, we prove that $\llbracket \mathcal{B} \rrbracket^{\text {init }}=$ $b \otimes \llbracket \mathcal{A} \rrbracket^{\text {init }}$. For this, by induction on $\mathrm{T}_{\Sigma}$, we can prove that the following statement holds:

$$
\begin{equation*}
\text { For every } \xi \in \mathrm{T}_{\Sigma} \text { and } q \in Q \text {, we have } \mathrm{h}_{\mathcal{B}}(\xi)_{q^{0}}=\mathrm{h}_{\mathcal{A}}(\xi)_{q} \tag{10.1}
\end{equation*}
$$

Since this proof is straightforward, we do not show it. Next, by induction on $T_{\Sigma}$, we prove that the following statement holds:

$$
\begin{equation*}
\text { For every } \xi \in \mathrm{T}_{\Sigma} \text { and } q \in Q \text {, we have } \mathrm{h}_{\mathcal{B}}(\xi)_{q^{\ell}}=b \otimes \mathrm{~h}_{\mathcal{A}}(\xi)_{q} \tag{10.2}
\end{equation*}
$$

Let $\xi=\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)$. Then

$$
\begin{aligned}
\mathrm{h}_{\mathcal{B}}(\xi)_{q^{\ell}} & =\bigoplus_{\substack{q_{1} \cdots q_{k} \in Q^{k} \\
s_{1}, \ldots, s_{k} \in\{0, \ell\}}} \mathrm{h}_{\mathcal{B}}\left(\xi_{1}\right)_{q_{1}^{s_{1}}} \otimes \ldots \otimes \mathrm{~h}_{\mathcal{B}}\left(\xi_{k}\right)_{q_{k}^{s_{k}}} \otimes \delta_{k}^{\prime}\left(q_{1}^{s_{1}} \cdots q_{k}^{s_{k}}, \sigma, q^{\ell}\right) \\
& =\bigoplus_{q_{1} \cdots q_{k} \in Q^{k}} \mathrm{~h}_{\mathcal{B}}\left(\xi_{1}\right)_{q_{1}^{e}} \otimes \mathrm{~h}_{\mathcal{B}}\left(\xi_{2}\right)_{q_{2}^{0}} \otimes \ldots \otimes \mathrm{~h}_{\mathcal{B}}\left(\xi_{k}\right)_{q_{k}^{0}} \otimes \delta_{k}^{\prime}\left(q_{1}^{\ell} q_{2}^{0} \cdots q_{k}^{0}, \sigma, q^{\ell}\right) \\
& =\bigoplus_{q_{1} \cdots q_{k} \in Q^{k}} b \otimes \mathrm{~h}_{\mathcal{A}}\left(\xi_{1}\right)_{q_{1}} \otimes \mathrm{~h}_{\mathcal{A}}\left(\xi_{2}\right)_{q_{2}} \otimes \ldots \otimes \mathrm{~h}_{\mathcal{A}}\left(\xi_{k}\right)_{q_{k}} \otimes \delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right)
\end{aligned}
$$

(by I.H., (10.1), and construction)
We continue with case analysis.
Case (a): Let B be left-distributive. Then we can continue as follows:

$$
\begin{aligned}
& \bigoplus_{q_{1} \cdots q_{k} \in Q^{k}} b \otimes \mathrm{~h}_{\mathcal{A}}\left(\xi_{1}\right)_{q_{1}} \otimes \ldots \otimes \mathrm{~h}_{\mathcal{A}}\left(\xi_{k}\right)_{q_{k}} \otimes \delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right) \\
= & b \otimes \bigoplus_{q_{1} \cdots q_{k} \in Q^{k}} \mathrm{~h}_{\mathcal{A}}\left(\xi_{1}\right)_{q_{1}} \otimes \ldots \otimes \mathrm{~h}_{\mathcal{A}}\left(\xi_{k}\right)_{q_{k}} \otimes \delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right) \\
= & b \otimes \mathrm{~h}_{\mathcal{A}}(\xi)_{q}
\end{aligned}
$$

Case (b): Let $\mathcal{A}$ be bu deterministic. By Lemma 4.2.1(3), there are two subcases.
$\overline{\text { Case (b1): There exists } i \in[k] \text { such that for each } q \in Q \text { we have } \mathrm{h}_{\mathcal{A}}\left(\xi_{i}\right)_{q}=0 \text {. Then we can continue }}$ as follows:

$$
\begin{aligned}
& \bigoplus_{q_{1} \cdots q_{k} \in Q^{k}} b \otimes \mathrm{~h}_{\mathcal{A}}\left(\xi_{1}\right)_{q_{1}} \otimes \ldots \otimes \mathrm{~h}_{\mathcal{A}}\left(\xi_{i}\right)_{q_{i}} \otimes \ldots \otimes \mathrm{~h}_{\mathcal{A}}\left(\xi_{k}\right)_{q_{k}} \otimes \delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right) \\
&= \bigoplus_{q_{1} \cdots q_{k} \in Q^{k}} b \otimes \mathrm{~h}_{\mathcal{A}}\left(\xi_{1}\right)_{q_{1}} \otimes \ldots \otimes \mathcal{O} \otimes \ldots \otimes \mathrm{~h}_{\mathcal{A}}\left(\xi_{k}\right)_{q_{k}} \otimes \delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right) \\
&=0 \\
&= b \otimes \bigoplus_{q_{1} \cdots q_{k} \in Q^{k}} \mathrm{~h}_{\mathcal{A}}\left(\xi_{1}\right)_{q_{1}} \otimes \ldots \otimes \mathcal{O} \otimes \ldots \otimes \mathrm{~h}_{\mathcal{A}}\left(\xi_{k}\right)_{q_{k}} \otimes \delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right) \\
&= b \otimes \mathrm{~h}_{\mathcal{A}}(\xi)_{q} .
\end{aligned}
$$

Case (b2): For each $i \in[k]$ there exists exactly one $\bar{q}_{i} \in Q$ such that $\mathrm{h}_{\mathcal{A}}\left(\xi_{i}\right)_{\overline{q_{i}}} \neq \mathbb{0}$. Then we can continue as follows:

$$
\begin{aligned}
& \quad \bigoplus_{q_{1} \cdots q_{k} \in Q^{k}} b \otimes \mathrm{~h}_{\mathcal{A}}\left(\xi_{1}\right)_{q_{1}} \otimes \ldots \otimes \mathrm{~h}_{\mathcal{A}}\left(\xi_{k}\right)_{q_{k}} \otimes \delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right) \\
& =b \otimes \mathrm{~h}_{\mathcal{A}}\left(\xi_{1}\right)_{\bar{q}_{1}} \otimes \ldots \otimes \mathrm{~h}_{\mathcal{A}}\left(\xi_{k}\right)_{\bar{q}_{k}} \otimes \delta_{k}\left(\overline{q_{1}} \cdots \overline{q_{k}}, \sigma, q\right) \\
& b \otimes \bigoplus_{q_{1} \cdots q_{k} \in Q^{k}} \mathrm{~h}_{\mathcal{A}}\left(\xi_{1}\right)_{q_{1}} \otimes \ldots \otimes \mathrm{~h}_{\mathcal{A}}\left(\xi_{k}\right)_{q_{k}} \otimes \delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right) \\
& =b \otimes \mathrm{~h}_{\mathcal{A}}(\xi)_{q} .
\end{aligned}
$$

This finishes the proof of (10.2). Now we prove that $\llbracket \mathcal{B} \rrbracket^{\text {init }}=b \otimes \llbracket \mathcal{A} \rrbracket^{\text {init. }}$. For each $\xi \in \mathrm{T}_{\Sigma}$ we have:

$$
\llbracket \mathcal{B} \rrbracket^{\mathrm{init}}(\xi)=\bigoplus_{\substack{s \in\{0, \ell\} \\ q^{s} \in Q^{\prime}}} \mathrm{h}_{\mathcal{B}}(\xi)_{q^{s}} \otimes\left(F^{\prime}\right)_{q^{s}}=\bigoplus_{q^{e} \in Q^{e}} \mathrm{~h}_{\mathcal{B}}(\xi)_{q^{e}} \otimes F_{q^{e}}=\bigoplus_{q \in Q} b \otimes \mathrm{~h}_{\mathcal{A}}(\xi)_{q} \otimes F_{q},
$$

where the last equality follows from (10.2) and the definition of $\mathcal{B}$.
We proceed by case analysis as above.
Case (a): Let B be left-distributive. Then we can continue as follows:

$$
\bigoplus_{q \in Q} b \otimes \mathrm{~h}_{\mathcal{A}}(\xi)_{q} \otimes F_{q}=b \otimes \bigoplus_{q \in Q} \mathrm{~h}_{\mathcal{A}}(\xi)_{q} \otimes F_{q}=b \otimes \llbracket \mathcal{A} \rrbracket^{\text {init }}(\xi)
$$

Case (b): Let $\mathcal{A}$ be bu deterministic. Again, by Lemma4.2.1(3), there are two cases.


$$
\bigoplus_{q \in Q} b \otimes \mathrm{~h}_{\mathcal{A}}(\xi)_{q} \otimes F_{q}=\mathbb{O}=b \otimes \bigoplus_{q \in Q} \mathrm{~h}_{\mathcal{A}}(\xi)_{q} \otimes F_{q}=b \otimes \llbracket \mathcal{A} \rrbracket^{\mathrm{init}}(\xi)
$$

Case (b2): There exists exactly one $\bar{q} \in Q$ such that $\mathrm{h}_{\mathcal{A}}(\xi)_{\bar{q}} \neq \mathbb{O}$. Then we can continue as follows:

$$
\bigoplus_{q \in Q} b \otimes \mathrm{~h}_{\mathcal{A}}(\xi)_{q} \otimes F_{q}=b \otimes \mathrm{~h}_{\mathcal{A}}(\xi)_{\bar{q}} \otimes F_{\bar{q}}=b \otimes \bigoplus_{q \in Q} \mathrm{~h}_{\mathcal{A}}(\xi)_{q} \otimes F_{q}=b \otimes \llbracket \mathcal{A} \rrbracket^{\mathrm{init}}(\xi)
$$

This finishes the proof of $\llbracket \mathcal{B} \rrbracket^{\text {init }}=b \otimes \llbracket \mathcal{A} \rrbracket^{\text {init }}$.
Second, we prove that $\llbracket \mathcal{B} \rrbracket^{\text {run }}=b \otimes \llbracket \mathcal{A} \rrbracket^{\text {run }}$. For each $\xi \in \mathrm{T}_{\Sigma}$ we define the sets

$$
\begin{aligned}
\operatorname{pos}_{\text {left }}(\xi) & =\left\{w \in \operatorname{pos}(\xi) \mid w=1^{n} \text { for some } n \in \mathbb{N}\right\} \\
\mathrm{R}_{\mathcal{B}}^{0}(\xi) & =\left\{\rho \in \mathrm{R}_{\mathcal{B}}(\xi) \mid \operatorname{im}(\rho) \subseteq Q^{0}\right\}, \text { and } \\
\mathrm{R}_{\mathcal{B}}^{\ell}(\xi) & =\left\{\rho \in \mathrm{R}_{\mathcal{B}}(\xi) \mid\left(\forall w \in \operatorname{pos}_{\text {left }}(\xi)\right): \rho(w) \in Q^{\ell} \text { and }\left(\forall w \in \operatorname{pos}(\xi) \backslash \operatorname{pos}_{\text {left }}(\xi)\right): \rho(w) \in Q^{0}\right\}
\end{aligned}
$$

Moreover, for each $\xi \in \mathrm{T}_{\Sigma}$, we define the mapping $\overline{(.)}: \mathrm{R}_{\mathcal{B}}(\xi) \rightarrow \mathrm{R}_{\mathcal{A}}(\xi)$ such that, for every $\rho \in \mathrm{R}_{\mathcal{B}}(\xi)$, we abbreviate $\overline{(.)}(\rho)$ by $\bar{\rho}$ and the run $\bar{\rho}$ of $\mathcal{A}$ is obtained from $\rho$ by dropping the upper index (in $\{0, \ell\}$ ) from each state.

By the construction of $\mathcal{B}$, the following observations are easy to see.
For every $\xi \in \mathrm{T}_{\Sigma}$ and $\rho \in \mathrm{R}_{\mathcal{B}}(\xi) \backslash\left(\mathrm{R}_{\mathcal{B}}^{0}(\xi) \cup \mathrm{R}_{\mathcal{B}}^{\ell}(\xi)\right)$ we have $\mathrm{wt}_{\mathcal{B}}(\xi, \rho)=\mathbb{0}$.
For every $\xi \in \mathrm{T}_{\Sigma}$ and $\rho \in \mathrm{R}_{\mathcal{B}}^{0}(\xi)$ we have $\mathrm{wt}_{\mathcal{B}}(\xi, \rho)=\mathrm{wt}_{\mathcal{A}}(\xi, \bar{\rho})$.
For every $\xi \in \mathrm{T}_{\Sigma}$ and $\rho \in \mathrm{R}_{\mathcal{B}}^{\ell}(\xi)$ we have $\mathrm{wt}_{\mathcal{B}}(\xi, \rho)=b \otimes \mathrm{wt}_{\mathcal{A}}(\xi, \bar{\rho})$.
For the proof of (10.5) we need (10.3), and (10.4). Then for each $\xi \in \mathrm{T}_{\Sigma}$ we can compute as follows.

$$
\begin{aligned}
& \llbracket \mathcal{B} \rrbracket^{\mathrm{run}}(\xi)=\bigoplus_{\rho \in \mathrm{R}_{\mathcal{B}}(\xi)} \operatorname{wt}_{\mathcal{B}}(\xi, \rho) \otimes\left(F^{\prime}\right)_{\rho(\varepsilon)} \\
& =\bigoplus_{\rho \in \mathbb{R}_{\mathcal{B}}(\xi) ;} \operatorname{wt}_{\mathcal{B}}(\xi, \rho) \otimes\left(F^{\prime}\right)_{\rho(\varepsilon)} \quad \quad \quad \text { (because }\left(F^{\prime}\right)_{q^{s}}=\mathbb{O} \text { for each } q^{s} \in Q^{0} \text { ) } \\
& \underset{\substack{\rho \in \mathrm{R}_{\mathcal{B}}(\xi): \\
\rho(\varepsilon) \in \mathrm{Q}^{\ell}}}{ } \\
& =\bigoplus_{\substack{\rho \in \mathrm{R}_{\mathcal{B}}(\xi): \\
\rho(\varepsilon) \in \mathrm{Q}^{\ell}}} \mathrm{wt}_{\mathcal{B}}(\xi, \rho) \otimes F_{\bar{\rho}(\varepsilon)} \\
& =\bigoplus_{\rho \in \mathrm{R}_{\mathcal{B}}^{\mathcal{B}}(\xi)} \mathrm{wt}_{\mathcal{B}}(\xi, \rho) \otimes F_{\bar{\rho}(\varepsilon)} \\
& =\bigoplus_{\rho \in \mathrm{R}_{\mathcal{B}}^{e}(\xi)} b \otimes \mathrm{wt}_{\mathcal{A}}(\xi, \bar{\rho}) \otimes F_{\bar{\rho}(\varepsilon)}
\end{aligned}
$$

$$
=\bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}}(\xi)} b \otimes \operatorname{wt}_{\mathcal{A}}(\xi, \rho) \otimes F_{\rho(\varepsilon)} \quad \text { (because the restriction of } \overline{(.)} \text { to } \mathrm{R}_{\mathcal{B}}^{\ell}(\xi) \text { is a bijection) }
$$

We proceed by case analysis.
Case (a): Let B be left-distributive. Then we can continue as follows:

$$
\bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}}(\xi)} b \otimes \mathrm{wt}_{\mathcal{A}}(\xi, \rho) \otimes F_{\rho(\varepsilon)}=b \otimes \bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}}(\xi)} \mathrm{wt}_{\mathcal{A}}(\xi, \rho) \otimes F_{\rho(\varepsilon)}=b \otimes \llbracket \mathcal{A} \rrbracket^{\mathrm{run}}(\xi)
$$

where we use left-distributivity at the first equality.
Case (b): Let $\mathcal{A}$ be bu deterministic. By Lemma 4.2.1(3), there are two cases.
$\underline{\text { Case (b1): For each } \rho \in \mathrm{R}_{\mathcal{A}}(\xi) \text { we have } \mathrm{wt}_{\mathcal{A}}(\xi, \rho)=\mathbb{O} \text {. Then we can continue as follows: }}$

$$
\begin{aligned}
\bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}}(\xi)} b \otimes \mathrm{wt}_{\mathcal{A}}(\xi, \rho) \otimes F_{\rho(\varepsilon)} & =\bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}}(\xi)} b \otimes \mathbb{O} \otimes F_{\rho(\varepsilon)}=b \otimes \bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}}(\xi)} \mathbb{O} \otimes F_{\rho(\varepsilon)} \\
& =b \otimes \bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}}(\xi)} \mathrm{wt}_{\mathcal{A}}(\xi, \rho) \otimes F_{\rho(\varepsilon)}=b \otimes \llbracket \mathcal{A} \rrbracket^{\mathrm{run}}(\xi) .
\end{aligned}
$$

Case (b2): There exist exactly one $\bar{q} \in Q$ and one $\bar{\rho} \in \mathrm{R}_{\mathcal{A}}(\bar{q}, \xi)$ such that $\mathrm{wt}_{\mathcal{A}}(\xi, \bar{\rho}) \neq \mathbb{O}$; then $\mathrm{wt}_{\mathcal{A}} \overline{(\xi, \bar{\rho})}=\mathrm{h}_{\mathcal{A}}(\xi)_{\bar{q}}$. (Note that $\bar{\rho}(\varepsilon)=\bar{q}$.) Then we continue as follows:

$$
\bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}}(\xi)} b \otimes \mathrm{wt}_{\mathcal{A}}(\xi, \rho) \otimes F_{\rho(\varepsilon)}=b \otimes \mathrm{wt}_{\mathcal{A}}(\xi, \bar{\rho}) \otimes F_{\bar{q}}=b \otimes \bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}}(\xi)} \mathrm{wt}_{\mathcal{A}}(\xi, \rho) \otimes F_{\rho(\varepsilon)}=b \otimes \llbracket \mathcal{A} \rrbracket^{\mathrm{run}}(\xi)
$$

This finishes the proof of $\llbracket \mathcal{B} \rrbracket^{\text {run }}=b \otimes \llbracket \mathcal{A} \rrbracket^{\text {run }}$.

Proof of (2): Let $\mathcal{A}=(Q, \delta, F)$. We construct $\mathcal{B}=\left(Q, \delta, F^{\prime}\right)$ by letting $\left(F^{\prime}\right)_{q}=F_{q} \otimes b$ for each $q \in Q$. First, we prove that $\llbracket \mathcal{B} \rrbracket^{\text {init }}=\llbracket \mathcal{A} \rrbracket^{\text {init }} \otimes b$. Obviously,

$$
\begin{equation*}
\text { for each } q \in Q \text { we have } \mathrm{h}_{\mathcal{B}}(\xi)_{q}=\mathrm{h}_{\mathcal{A}}(\xi)_{q} \tag{10.6}
\end{equation*}
$$

(We note that the proof of (10.6) does not need the assumption that $B$ is right-distributive.) Now we assume that B is right-distributive.

Then, for each $\xi \in \mathrm{T}_{\Sigma}$, we can calculate as follows.

$$
\begin{array}{rlr}
\llbracket \mathcal{B} \rrbracket^{\text {init }}(\xi) & =\bigoplus_{q \in Q} \mathrm{~h}_{\mathcal{B}}(\xi)_{q} \otimes\left(F^{\prime}\right)_{q} \\
& =\bigoplus_{q \in Q} \mathrm{~h}_{\mathcal{A}}(\xi)_{q} \otimes F_{q} \otimes b & \quad \text { (by (10.6) and the definition of } \mathcal{B}) \\
& =\left(\bigoplus_{q \in Q} \mathrm{~h}_{\mathcal{A}}(\xi)_{q} \otimes F_{q}\right) \otimes b & \quad \text { (by right-distributivity) } \\
& =\llbracket \mathcal{A} \rrbracket^{\text {init }}(\xi) \otimes b .
\end{array}
$$

Second, we prove that $\llbracket \mathcal{B} \rrbracket^{\text {run }}=\llbracket \mathcal{A} \rrbracket^{\text {run }} \otimes b$. Obviously,

$$
\begin{align*}
& \text { for each } \xi \in \mathrm{T}_{\Sigma} \text { we have } \mathrm{R}_{\mathcal{A}}(\xi)=\mathrm{R}_{\mathcal{B}}(\xi) \text { and }  \tag{10.7}\\
& \text { for each } \rho \in \mathrm{R}_{\mathcal{A}}(\xi) \text { we have } \mathrm{wt}_{\mathcal{A}}(\xi, \rho)=\operatorname{wt}_{\mathcal{B}}(\xi, \rho)
\end{align*}
$$

Then, for each $\xi \in \mathrm{T}_{\Sigma}$, we can calculate as follows.

$$
\llbracket \mathcal{B} \rrbracket^{\mathrm{run}}(\xi)=\bigoplus_{\rho \in \mathrm{R}_{\mathcal{B}}(\xi)} \mathrm{wt}_{\mathcal{B}}(\xi, \rho) \otimes\left(F^{\prime}\right)_{\rho(\varepsilon)}
$$

$=\bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}}(\xi)} \mathrm{wt}_{\mathcal{A}}(\xi, \rho) \otimes F_{\rho(\varepsilon)} \otimes b$
(by (10.7))

$$
=\left(\bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}}(\xi)} \mathrm{wt}_{\mathcal{A}}(\xi, \rho) \otimes F_{\rho(\varepsilon)}\right) \otimes b
$$

$$
=\llbracket \mathcal{A} \rrbracket^{\mathrm{run}}(\xi) \otimes b
$$

Proof of (3): Let $\mathcal{A}=(Q, \delta, F)$ be bu deterministic. We construct $\mathcal{B}=\left(Q, \delta, F^{\prime}\right)$ in the same way as in the proof of Statement (2). It is obvious that this construction preserves bu determinism and crisp determinism. Let $\xi \in \mathrm{T}_{\Sigma}$. By Lemma 4.2.1 there are two cases.
$\underline{\text { Case (a): For each } q \in Q \text { we have } \mathrm{h}_{\mathcal{A}}(\xi)_{q}=\mathbb{0} \text {. Then by (10.6) we obtain: }}$

$$
\llbracket \mathcal{B} \rrbracket(\xi)=\bigoplus_{q \in Q} \mathrm{~h}_{\mathcal{B}}(\xi)_{q} \otimes\left(F^{\prime}\right)_{q}=\mathbb{O}=\left(\bigoplus_{q \in Q} \mathrm{~h}_{\mathcal{A}}(\xi)_{q} \otimes F_{q}\right) \otimes b=\llbracket \mathcal{A} \rrbracket(\xi) \otimes b
$$

Case (b): There exists exactly one $\bar{q} \in Q$ such that $\mathrm{h}_{\mathcal{A}}(\xi)_{\bar{q}} \neq \mathbb{0}$. Then, using (10.6) again, we can continue as follows:

$$
\llbracket \mathcal{B} \rrbracket(\xi)=\bigoplus_{q \in Q} \mathrm{~h}_{\mathcal{B}}(\xi)_{q} \otimes\left(F^{\prime}\right)_{q}=\mathrm{h}_{\mathcal{B}}(\xi)_{\bar{q}} \otimes F_{\bar{q}}^{\prime}=\mathrm{h}_{\mathcal{A}}(\xi)_{\bar{q}} \otimes F_{\bar{q}} \otimes b=\left(\bigoplus_{q \in Q} \mathrm{~h}_{\mathcal{A}}(\xi)_{q} \otimes F_{q}\right) \otimes b=\llbracket \mathcal{A} \rrbracket(\xi) \otimes b
$$

Proof of (4). Let $\mathcal{A}=(Q, \delta, F)$ be crisp deterministic. We construct $\mathcal{B}=\left(Q, \delta, F^{\prime}\right)$ where $F_{q}^{\prime}=b \otimes F_{q}$ for each $q \in Q$. Obviously, $\mathcal{B}$ is crisp deterministic and (10.6) also holds for this $\mathcal{B}$. Due to Lemma 4.3.1, there exists a state $q_{\xi}$ such that $\mathrm{Q}_{\neq 0}^{\mathrm{h}} \mathcal{A}(\xi)=\left\{q_{\xi}\right\}$ and $\mathrm{h}_{\mathcal{A}}(\xi)_{q_{\xi}}=\mathbb{1}$, and hence $\llbracket \mathcal{A} \rrbracket(\xi)=F_{q_{\xi}}$. Then:

$$
\llbracket \mathcal{B} \rrbracket(\xi)=\mathrm{h}_{\mathcal{B}}(\xi)_{q_{\xi}} \otimes\left(F^{\prime}\right)_{q_{\xi}}=\mathrm{h}_{\mathcal{A}}(\xi)_{q_{\xi}} \otimes b \otimes F_{q_{\xi}}=b \otimes F_{q_{\xi}}=b \otimes \llbracket \mathcal{A} \rrbracket(\xi)
$$

### 10.3 Characterization of recognizable step mappings

As a kind of application of closure under sum and under scalar multiplications, we show that recognizable step mappings and crisp deterministic wta are closely related (cf. Theorem 10.3.1). Moreover, we prove a characterization of the set of all recognizable step mappings in terms of characteristic mappings of recognizable tree languages and closure under sum and scalar multiplication (cf. Theorem 10.3.3).

We recall that a weighted tree language $r: \mathrm{T}_{\Sigma} \rightarrow B$ is a recognizable step mapping (cf. Section 2.14) if there exist $n \in \mathbb{N}_{+}, b_{1}, \ldots, b_{n} \in B$, and recognizable $\Sigma$-tree languages $L_{1}, \ldots, L_{n}$ such that

$$
r=\bigoplus_{i \in[n]} b_{i} \otimes \chi\left(L_{i}\right)
$$

In the next theorem, the equivalences $(\mathrm{A}) \Leftrightarrow(\mathrm{B}) \Leftrightarrow(\mathrm{C})$ were proved in DSV10, Lm. 8 and Prop. 9], respectively, for the string case. The implication $(B) \Rightarrow(D)$ of Theorem 10.3 .1 was proved in DV06, Lm. 3.1] for the case that B is a semiring. A similar result was proved in Li08a, Thm. 2.1] for wsa over a pair of t-conorm and t-norm. We recall that a weighted tree language $r: \mathrm{T}_{\Sigma} \rightarrow B$ has the preimage property, if $r^{-1}(b)$ is a recognizable $\Sigma$-tree language for each $b \in B$.

Theorem 10.3.1. Let $\Sigma$ be a ranked alphabet and $\mathrm{B}=(B, \oplus, \otimes, \mathbb{O}, \mathbb{1})$ be a strong bimonoid. Let $r: \mathrm{T}_{\Sigma} \rightarrow$ $B$. Then the following four statements are equivalent.
(A) We can construct a crisp deterministic $(\Sigma, \mathrm{B})-w t a \mathcal{A}$ such that $r=\llbracket \mathcal{A} \rrbracket$.
(B) $r$ is a recognizable step mapping and we can construct $n \in \mathbb{N}_{+}, b_{1}, \ldots, b_{n} \in B$, and $\Sigma$-fta $A_{1}, \ldots, A_{n}$ such that $r=\bigoplus_{i \in[n]} b_{i} \otimes \chi\left(\mathrm{~L}\left(A_{i}\right)\right)$.
(C) $\operatorname{im}(r)$ is finite, $r$ has the preimage property, and we can construct $n \in \mathbb{N}_{+}$and elements $b_{1}, \ldots, b_{n} \in$ $B$ such that $\operatorname{im}(r)=\left\{b_{1}, \ldots, b_{n}\right\}$ and, moreover, for each $b \in B$ we can construct a $\Sigma$-fta which recognizes $r^{-1}(b)$.
(D) $r$ is a recognizable step mapping with pairwise disjoint step languages and we can construct $n \in \mathbb{N}_{+}$, $b_{1}, \ldots, b_{n} \in B$, and $\Sigma$-fta $A_{1}, \ldots, A_{n}$ such that $r=\bigoplus_{i \in[n]} b_{i} \otimes \chi\left(\mathrm{~L}\left(A_{i}\right)\right)$.
Thus, in particular, $\operatorname{cd}-\operatorname{Rec}(\Sigma, \mathrm{B})=\operatorname{RecStep}(\Sigma, \mathrm{B})$.

Proof. Proof of $(\mathrm{A}) \Rightarrow(\mathrm{D})$ : Let $\mathcal{A}$ be a crisp deterministic $(\Sigma, \mathrm{B})$-wta such that $r=\llbracket \mathcal{A} \rrbracket$. By Theorem 4.3.5, we can construct a finite $\Sigma$-algebra $(Q, \theta)$ and a mapping $F: Q \rightarrow B$ such that $\llbracket \mathcal{A} \rrbracket=F \circ \mathrm{~h}_{Q}$.

Let $n=|Q|$ and let $p_{1}, \ldots, p_{n}$ be a list of the elements of $Q$. For each $i \in[n]$ we construct the bu deterministic and total $\Sigma$-fta $A_{i}=\left(Q, \delta,\left\{p_{i}\right\}\right)$, where for every $k \in \mathbb{N}, \sigma \in \Sigma^{(k)}$, and $q \in Q$ and $q_{1} \cdots q_{k} \in Q^{k}$, we have

$$
\delta_{k}\left(q_{1} \cdots q_{k}, \sigma\right)=q \text { iff } \theta(\sigma)\left(q_{1}, \ldots, q_{k}\right)=q
$$

Since each fta $A_{i}$ is bu deterministic, it is obvious that, for every $i, i^{\prime} \in[n]$ with $i \neq i^{\prime}$, the sets $\mathrm{L}\left(A_{i}\right)$ and $\mathrm{L}\left(A_{i^{\prime}}\right)$ are disjoint, i.e., $\mathrm{L}\left(A_{i}\right) \cap \mathrm{L}\left(A_{i^{\prime}}\right)=\emptyset$.

Let us abbreviate $F_{p_{i}}$ by $b_{i}$ for every $i \in[n]$. Next we prove that $r=\bigoplus_{i \in[n]} b_{i} \otimes \chi\left(\mathrm{~L}\left(A_{i}\right)\right)$. Obviously, for every $i \in[n]$, we have $\mathrm{h}_{Q}=\mathrm{h}_{A_{i}}$, where $\mathrm{h}_{A_{i}}$ is the unique homomorphism from $\mathrm{T}_{\Sigma}$ to the $\Sigma$-algebra $\left(\mathcal{P}(Q), \delta_{A_{i}}\right)$ defined in Section 2.13,

Now let $\xi \in \mathrm{T}_{\Sigma}$ and let $\mathrm{h}_{Q}(\xi)=p_{j}$. Then, in particular, $\mathrm{h}_{Q}=\mathrm{h}_{A_{j}}$. By definition of $A_{j}$ it follows that $\xi \in \mathrm{L}\left(A_{j}\right)$, and thus $\chi\left(\mathrm{L}\left(A_{j}\right)\right)(\xi)=\mathbb{1}$. Moreover, for each $p_{i} \in Q$ with $p_{i} \neq \mathrm{h}_{Q}(\xi)$, we have $\xi \notin \mathrm{L}\left(A_{i}\right)$ and thus $\chi\left(\mathrm{L}\left(A_{i}\right)\right)(\xi)=\mathbb{0}$. Thus we have

$$
r(\xi)=\left(F \circ \mathrm{~h}_{Q}\right)(\xi)=F_{p_{j}}=b_{j} \otimes \chi\left(\mathrm{~L}\left(A_{j}\right)\right)(\xi)=\bigoplus_{i \in[n]} b_{i} \otimes \chi\left(\mathrm{~L}\left(A_{i}\right)\right)(\xi)
$$

Hence $r$ is a recognizable step mapping with pairwise disjoint step languages.
Proof of $(\mathrm{D}) \Rightarrow(\mathrm{C})$ : For each $i \in[n]$, let us abbreviate $\mathrm{L}\left(A_{i}\right)$ by $L_{i}$. Hence $r=\bigoplus_{i \in[n]} b_{i} \otimes \chi\left(L_{i}\right)$, where the step languages $L_{1}, \ldots, L_{n}$ are pairwise disjoint. Then $\operatorname{im}(r)=\left\{b_{1}, \ldots, b_{n}\right\}$ if $\mathrm{T}_{\Sigma}=\bigcup_{i \in[n]} L_{i}$ and $\operatorname{im}(r)=\left\{b_{1}, \ldots, b_{n}, \mathbb{O}\right\}$ if $\mathrm{T}_{\Sigma} \backslash \bigcup_{i \in[n]} L_{i} \neq \emptyset$; in both cases $\operatorname{im}(r)$ is finite.

Now let $b \in B$. Then

$$
r^{-1}(b)= \begin{cases}\bigcup_{i \in[n]: b_{i}=b} L_{i} & \text { if } b \neq 0 \\ \mathrm{~T}_{\Sigma} \backslash \bigcup_{i \in[n]: b_{i} \neq 0} L_{i} & \text { otherwise } .\end{cases}
$$

Since the set of recognizable $\Sigma$-tree languages is effectively closed under union and complement GS84, Thm. 2.4.2], in each case we can construct a $\Sigma$-fta which recognizes $r^{-1}(b)$. Thus $r$ has the preimage property.

Proof of $(\mathrm{C}) \Rightarrow(\mathrm{B})$ : Let $\operatorname{im}(r)=\left\{b_{1}, \ldots, b_{n}\right\}$ for some $n \in \mathbb{N}_{+}$and $b_{1}, \ldots, b_{n} \in B$. By our assumption, for each $i \in[n]$, we can construct a $\Sigma$-fta $A_{i}$ with $\mathrm{L}\left(A_{i}\right)=r^{-1}\left(b_{i}\right)$. Then $r=\bigoplus_{i \in[n]} b_{i} \otimes \chi\left(\mathrm{~L}\left(A_{i}\right)\right)$. Hence $r$ is a recognizable step mapping.

Proof of $(\mathrm{B}) \Rightarrow(\mathrm{A})$ : Let $n \in \mathbb{N}_{+}$and $r=\bigoplus_{i \in[n]} b_{i} \otimes \chi\left(\mathrm{~L}\left(A_{i}\right)\right)$ for some $\Sigma$-fta $A_{1}, \ldots, A_{n}$ and coefficients $b_{1}, \ldots, b_{n} \in B$.

Let $i \in[n]$. By Theorem4.3.6, we can construct a crisp deterministic ( $\Sigma, \mathrm{B})$-wta $\mathcal{A}_{i}$ with identity root weights such that $\llbracket \mathcal{A}_{i} \rrbracket=\chi\left(\mathrm{L}\left(A_{i}\right)\right)$. By Theorem 10.2.1(4), we can construct a crisp deterministic ( $\left.\Sigma, \mathrm{B}\right)$ -
wta $\mathcal{A}_{i}^{\prime}$ such that $\llbracket \mathcal{A}_{i}^{\prime} \rrbracket=b_{i} \otimes \chi\left(\mathrm{~L}\left(A_{i}\right)\right)$. By Theorem 10.1.1(2), we can construct a crisp deterministic $(\Sigma, \mathrm{B})$-wta $\mathcal{A}$ such that $\llbracket \mathcal{A} \rrbracket=\bigoplus_{i \in[n]} b_{i} \otimes \chi\left(\mathrm{~L}\left(A_{i}\right)\right)$.

We note that Theorem $10.3 .1(\mathrm{~B}) \Rightarrow(\mathrm{D})$ ) also follows from Observation 2.14.1.
Corollary 10.3.2. $\operatorname{Rec} \operatorname{Step}(\Sigma, B)$ is closed under sum and scalar multiplications.

Proof. By Theorem $10.3 .1(\mathrm{~A}) \Leftrightarrow(\mathrm{B}), \operatorname{RecStep}(\Sigma, \mathrm{B})$ is equal to the set of $(\Sigma, \mathrm{B})$-weighted tree languages recognizable by crisp deterministic wta. By Theorem 10.1.1(2) the latter is closed under sum. Moreover, by Theorem 10.2.1(3) and (4) it is closed under scalar multiplications from the right and the left, respectively.

We finish this section with a characterization of $\operatorname{RecStep}(\Sigma, \mathrm{B})$ which results from several closure properties.

Theorem 10.3.3. Let $\Sigma$ be a ranked alphabet and B be a strong bimonoid. The set $\operatorname{RecStep}(\Sigma, \mathrm{B})$ of ( $\Sigma, \mathrm{B})$-recognizable step mappings is the smallest set of $(\Sigma, \mathrm{B})$-weighted tree languages which contains, for each recognizable $\Sigma$-tree language $L$, the characteristic mapping $\chi(L)$ and is closed under sum and scalar multiplication from the left.

Proof. For the sake of convenience, we denote by $\mathcal{C}$ the smallest set of ( $\Sigma, \mathrm{B}$ )-weighted tree languages which contains, for each recognizable $\Sigma$-tree language $L$, the characteristic mapping $\chi(L)$ and is closed under sum and scalar multiplications from the left.

First we prove that $\operatorname{RecStep}(\Sigma, \mathrm{B}) \subseteq \mathcal{C}$. Let $r \in \operatorname{RecStep}(\Sigma, \mathrm{~B})$ with $r=\bigoplus_{i \in[n]} b_{i} \otimes \chi\left(L_{i}\right)$ for some $n \in \mathbb{N}_{+}, b_{1}, \ldots, b_{n} \in B$, and $L_{1}, \ldots, L_{n}$ recognizable $\Sigma$-tree languages. Then, for each $i \in[n]$, we have $\chi\left(L_{i}\right) \in \mathcal{C}$ and $b_{i} \otimes \chi\left(L_{i}\right) \in \mathcal{C}$. Since $\mathcal{C}$ is closed under sum, we have $r \in \mathcal{C}$.

Second we prove that $\mathcal{C} \subseteq \operatorname{RecStep}(\Sigma, \mathrm{B})$. Obviously, for each recognizable $\Sigma$-tree language $L$, the characteristic mapping $\chi(L)$ is in $\operatorname{RecStep}(\Sigma, \mathrm{B})$. Moreover, by Corollary 10.3.2. $\operatorname{RecStep}(\Sigma, \mathrm{B})$ is closed under sum and scalar multiplications from the left. Since $\mathcal{C}$ is the smallest set with these properties, we obtain that $\mathcal{C} \subseteq \operatorname{RecStep}(\Sigma, \mathrm{B})$.

### 10.4 Closure under Hadamard product

In this section we will prove that the set $\operatorname{Rec}(\Sigma, \mathrm{B})$ is closed under Hadamard product if B is a commutative semiring. A set $\mathcal{L}$ of B -weighted tree languages is closed under Hadamard product if the following holds: for every $(\Sigma, \mathrm{B})$-weighted tree languages $r_{1}$ and $r_{2}$, if $r_{1}, r_{2} \in \mathcal{L}$, then $\left(r_{1} \otimes r_{2}\right) \in \mathcal{L}$.

Theorem 10.4.1. Let $\Sigma$ be a ranked alphabet, $\mathrm{B}=(B, \oplus, \otimes, \mathbb{O}, \mathbb{1})$ be a strong bimonoid, and $\mathcal{A}_{1}$ and $\mathcal{A}_{2}$ be two ( $\Sigma, \mathrm{B})$-wta. Then the following three statements hold.
(1) (cf. Bor04, Cor. 3.9]) If B is a commutative semiring, then we can construct $a(\Sigma, \mathrm{~B})-w t a \mathcal{A}$ such that $\llbracket \mathcal{A} \rrbracket=\llbracket \mathcal{A}_{1} \rrbracket \otimes \llbracket \mathcal{A}_{2} \rrbracket$.
(2) (cf. [Rad10, Lm. 5.3]) If B is commutative and $\mathcal{A}_{1}$ and $\mathcal{A}_{2}$ are bu deterministic, then we can construct a bu deterministic $(\Sigma, \mathrm{B})-$ wta $\mathcal{A}$ such that $\llbracket \mathcal{A} \rrbracket=\llbracket \mathcal{A}_{1} \rrbracket \otimes \llbracket \mathcal{A}_{2} \rrbracket$.
(3) If $\mathcal{A}_{1}$ and $\mathcal{A}_{2}$ are crisp deterministic, then we can construct a crisp deterministic $(\Sigma, \mathrm{B})$-wta $\mathcal{A}$ such that $\llbracket \mathcal{A} \rrbracket=\llbracket \mathcal{A}_{1} \rrbracket \otimes \llbracket \mathcal{A}_{2} \rrbracket$.
Thus, in particular, if B is a commutative semiring, then the set $\operatorname{Rec}(\Sigma, \mathrm{B})$ is closed under Hadamard product.

Proof. Let $\mathcal{A}_{1}=\left(Q_{1}, \delta_{1}, F_{1}\right)$ and $\mathcal{A}_{2}=\left(Q_{2}, \delta_{2}, F_{2}\right)$. We construct the $(\Sigma, \mathrm{B})$-wta $\mathcal{A}=(Q, \delta, F)$ which we will use in the proof of each of the Statements (1), (2), and (3). We define:

- $Q=Q_{1} \times Q_{2}$; for each $q \in Q$ we denote its first and second component by $q^{1}$ and $q^{2}$, respectively, - for every $k \in \mathbb{N}, \sigma \in \Sigma^{(k)}, q \in Q$, and $q_{1} \cdots q_{k} \in Q^{k}$ we define

$$
\delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right)=\left(\delta_{1}\right)_{k}\left(q_{1}^{1} \cdots q_{k}^{1}, \sigma, q^{1}\right) \otimes\left(\delta_{2}\right)_{k}\left(q_{1}^{2} \cdots q_{k}^{2}, \sigma, q^{2}\right)
$$

- for each $q \in Q$ we define $F_{q}=\left(F_{1}\right)_{q^{1}} \otimes\left(F_{2}\right)_{q^{2}}$.

Obviously, if $\mathcal{A}_{1}$ and $\mathcal{A}_{2}$ are bu deterministic (or crisp deterministic), then so is $\mathcal{A}$.

Proof of (1): We assume that $B$ is a commutative semiring. First, by induction on $T_{\Sigma}$, we prove that the following statement holds.

$$
\begin{equation*}
\text { For every } \xi \in \mathrm{T}_{\Sigma} \text { and } q=\left(q^{1}, q^{2}\right) \in Q, \text { we have: } \mathrm{h}_{\mathcal{A}}(\xi)_{q}=\mathrm{h}_{\mathcal{A}_{1}}(\xi)_{q^{1}} \otimes \mathrm{~h}_{\mathcal{A}_{2}}(\xi)_{q^{2}} \tag{10.8}
\end{equation*}
$$

Let $\xi=\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)$ and $q=\left(q^{1}, q^{2}\right) \in Q$. Then:

$$
\begin{aligned}
& \mathrm{h}_{\mathcal{A}}(\xi)_{q} \\
& =\bigoplus_{q_{1} \cdots q_{k} \in Q^{k}}\left(\bigotimes_{i \in[k]} \mathrm{h}_{\mathcal{A}}\left(\xi_{i}\right)_{q_{i}}\right) \otimes \delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right) \\
& =\bigoplus_{q_{1} \cdots q_{k} \in Q^{k}}\left(\bigotimes_{i \in[k]}\left(\mathrm{h}_{\mathcal{A}_{1}}\left(\xi_{i}\right)_{q_{i}^{1}} \otimes \mathrm{~h}_{\mathcal{A}_{2}}\left(\xi_{i}\right)_{q_{i}^{2}}\right)\right) \otimes\left(\bigotimes_{j \in[2]}\left(\delta_{j}\right)_{k}\left(q_{1}^{j} \cdots q_{k}^{j}, \sigma, q^{j}\right)\right) \\
& \text { (by I.H. and construction) } \\
& =\bigoplus_{q_{1}^{1} \cdots q_{k}^{1} \in\left(Q_{1}\right)^{k}} \bigoplus_{q_{1}^{2} \cdots q_{k}^{2} \in\left(Q_{2}\right)^{k}}\left(\bigotimes_{i \in[k]}\left(\mathrm{h}_{\mathcal{A}_{1}}\left(\xi_{i}\right)_{q_{i}^{1}} \otimes \mathrm{~h}_{\mathcal{A}_{2}}\left(\xi_{i}\right)_{q_{i}^{2}}\right)\right) \otimes\left(\bigotimes_{j \in[2]}\left(\delta_{j}\right)_{k}\left(q_{1}^{j} \cdots q_{k}^{j}, \sigma, q^{j}\right)\right) \\
& =\bigoplus_{q_{1}^{1} \cdots q_{k}^{1} \in\left(Q_{1}\right)^{k}} \bigoplus_{q_{1}^{2} \cdots q_{k}^{2} \in\left(Q_{2}\right)^{k}} \\
& \left(\bigotimes_{i \in[k]} \mathrm{h}_{\mathcal{A}_{1}}\left(\xi_{i}\right)_{q_{i}^{1}}\right) \otimes\left(\delta_{1}\right)_{k}\left(q_{1}^{1} \cdots q_{k}^{1}, \sigma, q^{1}\right) \otimes\left(\bigotimes_{i \in[k]} \mathrm{h}_{\mathcal{A}_{2}}\left(\xi_{i}\right)_{q_{i}^{2}}\right) \otimes\left(\delta_{2}\right)_{k}\left(q_{1}^{2} \cdots q_{k}^{2}, \sigma, q^{2}\right) \\
& \text { (by commutativity) } \\
& =\bigoplus_{q_{1}^{1} \cdots q_{k}^{1} \in\left(Q_{1}\right)^{k}}\left[\left(\bigotimes_{i \in[k]} \mathrm{h}_{\mathcal{A}_{1}}\left(\xi_{i}\right)_{q_{i}^{1}}\right) \otimes\left(\delta_{1}\right)_{k}\left(q_{1}^{1} \cdots q_{k}^{1}, \sigma, q^{1}\right)\right] \otimes \\
& \left(\bigoplus_{q_{1}^{2} \cdots q_{k}^{2} \in\left(Q_{2}\right)^{k}}\left(\bigotimes_{i \in[k]} \mathrm{h}_{\mathcal{A}_{2}}\left(\xi_{i}\right)_{q_{i}^{2}}\right) \otimes\left(\delta_{2}\right)_{k}\left(q_{1}^{2} \cdots q_{k}^{2}, \sigma, q^{2}\right)\right) \quad \text { (by left-distributivity) } \\
& =\left(\underset{q_{1}^{1} \cdots q_{k}^{1} \in\left(Q_{1}\right)^{k}}{\bigoplus}\left(\bigotimes_{i \in[k]} \mathrm{h}_{\mathcal{A}_{1}}\left(\xi_{i}\right)_{q_{i}^{1}}\right) \otimes\left(\delta_{1}\right)_{k}\left(q_{1}^{1} \cdots q_{k}^{1}, \sigma, q^{1}\right)\right) \otimes \\
& \left(\bigoplus_{q_{1}^{2} \cdots q_{k}^{2} \in\left(Q_{2}\right)^{k}}\left(\bigotimes_{i \in[k]} \mathrm{h}_{\mathcal{A}_{2}}\left(\xi_{i}\right)_{q_{i}^{2}}\right) \otimes\left(\delta_{2}\right)_{k}\left(q_{1}^{2} \cdots q_{k}^{2}, \sigma, q^{2}\right)\right) \quad \text { (by right-distributivity) } \\
& =\mathrm{h}_{\mathcal{A}_{1}}(\xi)_{q^{1}} \otimes \mathrm{~h}_{\mathcal{A}_{2}}(\xi)_{q^{2}} .
\end{aligned}
$$

This proves (10.8). Then for each $\xi \in \mathrm{T}_{\Sigma}$ :

$$
\llbracket \mathcal{A} \rrbracket(\xi)=\bigoplus_{q \in Q} \mathrm{~h}_{\mathcal{A}}(\xi)_{q} \otimes F_{q}=\bigoplus_{q \in Q}\left(\mathrm{~h}_{\mathcal{A}_{1}}(\xi)_{q^{1}} \otimes \mathrm{~h}_{\mathcal{A}_{2}}(\xi)_{q^{2}}\right) \otimes\left(\left(F_{1}\right)_{q^{1}} \otimes\left(F_{2}\right)_{q^{2}}\right)
$$

(by Equation (10.8) and construction of $F$ )

$$
=\left(\bigoplus_{q \in Q_{1}} \mathrm{~h}_{\mathcal{A}_{1}}(\xi)_{q} \otimes\left(F_{1}\right)_{q}\right) \otimes\left(\bigoplus_{q \in Q_{2}} \mathrm{~h}_{\mathcal{A}_{2}}(\xi)_{q} \otimes\left(F_{2}\right)_{q}\right)
$$

(by commutativity and distributivity of $B$ )
$=\llbracket \mathcal{A}_{1} \rrbracket(\xi) \otimes \llbracket \mathcal{A}_{2} \rrbracket(\xi)$.

Proof of (2): Let B be commutative and $\mathcal{A}_{1}$ and $\mathcal{A}_{2}$ be bu deterministic. Then $\mathcal{A}$ is bu deterministic. Here we prove (10.8) as follows. Let $\xi=\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)$ and $q=\left(q^{1}, q^{2}\right) \in Q$. As above we obtain

$$
\begin{align*}
\mathrm{h}_{\mathcal{A}}(\xi)_{q} & =\bigoplus_{q_{1}^{1} \cdots q_{k}^{1} \in\left(Q_{1}\right)^{k}} \tag{10.9}
\end{align*} \bigoplus_{q_{1}^{2} \cdots q_{k}^{2} \in\left(Q_{2}\right)^{k}} .
$$

We note that this part uses commutativity and it does not use distributivity. By Lemma 4.2.1(1) (applied to $\mathcal{A}_{1}$ and $\mathcal{A}_{2}$ ), there are four cases. We can combine the four cases into the following two cases.

Case (a): (i) There exists an $i \in[k]$ such that, for every $q_{1} \in Q_{1}$, we have $\mathrm{h}_{\mathcal{A}_{1}}\left(\xi_{i}\right)_{q_{1}}=\mathbb{O}$ or (ii) there exists a $j \in[k]$ such that, for every $q_{2} \in Q_{2}$, we have $\mathrm{h}_{\mathcal{A}_{2}}\left(\xi_{j}\right)_{q_{2}}=\mathbb{0}$.

Due to (10.9) we have $\mathrm{h}_{\mathcal{A}}(\xi)_{q}=\mathbb{0}$. In Case (i), $\mathrm{h}_{\mathcal{A}_{1}}(\xi)_{q^{1}}=0$ by Lemma 4.1.1(1). Similarly, in Case (ii), $\mathrm{h}_{\mathcal{A}_{2}}(\xi)_{q^{2}}=\mathbb{O}$. In both cases we have $\mathrm{h}_{\mathcal{A}_{1}}(\xi)_{q^{1}} \otimes \mathrm{~h}_{\mathcal{A}_{2}}(\xi)_{q^{2}}=\mathbb{0}$.

Case (b): For every $i \in[k]$ there exist exactly one $p_{1}^{i} \in Q_{1}$ and exactly one $p_{2}^{i} \in Q_{2}$ such that $\mathrm{h}_{\mathcal{A}_{1}} \overline{\left(\xi_{i}\right)_{p_{1}^{i}} \neq \mathbb{O}}$ and $\mathrm{h}_{\mathcal{A}_{2}}\left(\xi_{i}\right)_{p_{2}^{i}} \neq \mathbb{O}$. (The states $p_{j}^{i}$ are unique by Lemma 4.2.1(1).) Then, together with (10.9), we have

$$
\begin{aligned}
\mathrm{h}_{\mathcal{A}}(\xi)_{q} & =\left(\bigotimes_{i \in[k]} \mathrm{h}_{\mathcal{A}_{1}}\left(\xi_{i}\right)_{p_{i}^{1}}\right) \otimes\left(\delta_{1}\right)_{k}\left(p_{1}^{1} \cdots p_{k}^{1}, \sigma, q^{1}\right) \otimes\left(\bigotimes_{i \in[k]} \mathrm{h}_{\mathcal{A}_{2}}\left(\xi_{i}\right)_{p_{i}^{2}}\right) \otimes\left(\delta_{2}\right)_{k}\left(p_{1}^{2} \cdots p_{k}^{2}, \sigma, q^{2}\right) \\
& =\mathrm{h}_{\mathcal{A}_{1}}(\xi)_{q^{1}} \otimes \mathrm{~h}_{\mathcal{A}_{2}}(\xi)_{q^{2}}
\end{aligned}
$$

This proves (10.8).
Now let $\xi \in \mathrm{T}_{\Sigma}$. We can prove $\llbracket \mathcal{A} \rrbracket(\xi)=\llbracket \mathcal{A}_{1} \rrbracket(\xi) \otimes \llbracket \mathcal{A}_{2} \rrbracket(\xi)$ by case analysis.
Case (a): Let (i) $\mathrm{h}_{\mathcal{A}_{1}}(\xi)_{q_{1}}=\mathbb{O}$ for every $q_{1} \in Q_{1}$ or (ii) $\mathrm{h}_{\mathcal{A}_{2}}(\xi)_{q_{2}}=\mathbb{O}$ for every $q_{2} \in Q_{2}$. Then $\llbracket \mathcal{A}_{1} \rrbracket(\xi)=\llbracket \mathcal{A}_{2} \rrbracket(\xi)=\mathbb{O}$. By (10.8) we have that $\mathrm{h}_{\mathcal{A}}(\xi)_{\left(q_{1}, q_{2}\right)}=\mathrm{h}_{\mathcal{A}_{1}}(\xi)_{q_{1}} \otimes \mathrm{~h}_{\mathcal{A}_{2}}(\xi)_{q_{2}}=\mathbb{0}$ for every $\left(q_{1}, q_{2}\right) \in Q$, and hence $\llbracket \mathcal{A} \rrbracket(\xi)=\mathbb{0}$.

Case (b): There exist exactly one $p_{1} \in Q_{1}$ and exactly one $p_{2} \in Q_{2}$ such that $\mathrm{h}_{\mathcal{A}_{1}}\left(\xi_{i}\right)_{p_{1}} \neq \mathbb{O}$ and $\mathrm{h}_{\mathcal{A}_{2}} \overline{\left(\xi_{i}\right)_{p_{2}} \neq 0}$. Then, by (10.8), we have that $\mathrm{h}_{\mathcal{A}}(\xi)_{\left(q_{1}, q_{2}\right)}=\mathbb{0}$ for each $\left(q_{1}, q_{2}\right) \in Q \backslash\left\{\left(p_{1}, p_{2}\right)\right\}$. Then

$$
\begin{array}{rlr}
\llbracket \mathcal{A} \rrbracket(\xi) & =\mathrm{h}_{\mathcal{A}}(\xi)_{\left(p_{1}, p_{2}\right)} \otimes F_{\left(p_{1}, p_{2}\right)} \\
& \left.=\left(\mathrm{h}_{\mathcal{A}_{1}}(\xi)_{p_{1}} \otimes \mathrm{~h}_{\mathcal{A}_{2}}(\xi)_{p_{2}}\right) \otimes\left(\left(F_{1}\right)_{p_{1}} \otimes\left(F_{2}\right)_{p_{2}}\right) \quad \text { (by Equation (10.8) and construction of } F\right) \\
& =\left(\mathrm{h}_{\mathcal{A}_{1}}(\xi)_{p_{1}} \otimes\left(F_{1}\right)_{p_{1}}\right) \otimes\left(\mathrm{h}_{\mathcal{A}_{1}}(\xi)_{p_{1}} \otimes\left(F_{2}\right)_{p_{2}}\right) \\
& =\llbracket \mathcal{A}_{1} \rrbracket(\xi) \otimes \llbracket \mathcal{A}_{2} \rrbracket(\xi) . & \text { (by commutativity of B) }
\end{array}
$$

Proof of (3): Now let $\mathcal{A}_{1}$ and $\mathcal{A}_{2}$ be crisp deterministic. Then also $\mathcal{A}$ is crisp deterministic. Let $\xi \in \mathrm{T}_{\Sigma}$. By Lemma 4.3.1 there exist states $p_{1} \in Q_{1}$ and $p_{2} \in Q_{2}$ such that $\mathrm{h}_{\mathcal{A}_{1}}(\xi)_{p_{1}}=\mathbb{1}=\mathrm{h}_{\mathcal{A}_{2}}(\xi)_{p_{2}}$ and for every $\left(q_{1}, q_{2}\right) \in Q \backslash\left\{\left(p_{1}, p_{2}\right)\right\}$ we have $\mathrm{h}_{\mathcal{A}_{1}}(\xi)_{q_{1}}=\mathbb{0}=\mathrm{h}_{\mathcal{A}_{2}}(\xi)_{q_{2}}$. Then we can prove (10.8) and $\llbracket \mathcal{A} \rrbracket(\xi)=\llbracket \mathcal{A}_{1} \rrbracket(\xi) \otimes \llbracket \mathcal{A}_{2} \rrbracket\left(\xi_{2}\right)$ as in the proof of Statement (2) but without the use of commutativity.

We note that in Gho22, Thm. 4.2(2)] closure of $\operatorname{Rec}^{\text {init }}(\Sigma, \mathrm{B})$ under Hadamard product was proved for the particular case that B is a $\sigma$-complete distributive orthomodular lattice. Since each $\sigma$-complete
distributive orthomodular lattice is a particular commutative semiring, Gho22, Thm. 4.2(2)] follows from Theorem 10.4.1 and from Bor04, Cor. 3.9].

Corollary 10.4.2. RecStep $(\Sigma, \mathrm{B})$ is closed under Hadamard product.
Proof. By Theorem $10.3 .1(\mathrm{~A}) \Leftrightarrow(\mathrm{B}), \operatorname{RecStep}(\Sigma, \mathrm{B})$ is equal to the set of $(\Sigma, \mathrm{B})$-weighted tree languages recognizable by crisp deterministic wta. By Theorem 10.4.1(3) the latter is closed under Hadamard product.

The next theorem can be compared to [DGMM11, Thm. 5.12.1(b)].

Theorem 10.4.3. Let $\Sigma$ be a ranked alphabet, $\mathrm{B}=(B, \oplus, \otimes, \mathbb{O}, \mathbb{1})$ a strong bimonoid, $\mathcal{A}$ a ( $\Sigma, \mathrm{B})$-wta, and $\mathcal{B}$ a crisp deterministic ( $\Sigma, \mathrm{B})$-wta. Moreover, let $D$ be a $\Sigma$-fta. Then the following three statements hold.
(1) If B is right-distributive, then we can construct $a(\Sigma, \mathrm{~B})$-wta $\mathcal{C}$ such that $\llbracket \mathcal{C} \rrbracket^{\text {run }}=\llbracket \mathcal{A} \rrbracket^{\text {run }} \otimes \llbracket \mathcal{B} \rrbracket$.
(2) We can construct $a(\Sigma, \mathrm{~B})-w t a \mathcal{C}$ such that $\llbracket \mathcal{C} \rrbracket^{\text {run }}=\llbracket \mathcal{A} \rrbracket^{\text {run }} \otimes \chi(\mathrm{L}(D))$.
(3) We can construct $a(\Sigma, \mathrm{~B})-w t a \mathcal{C}$ such that $\llbracket \mathcal{C} \rrbracket^{\text {run }}=\chi(\mathrm{L}(D)) \otimes \llbracket \mathcal{A} \rrbracket^{\text {run }}$.
(4) 【Dro22] If B is left-distributive, then we can construct $a(\Sigma, B)$-wta $\mathcal{C}$ such that $\llbracket \mathcal{C} \rrbracket^{\text {run }}=\llbracket \mathcal{B} \rrbracket \otimes \llbracket \mathcal{A} \rrbracket^{\text {run }}$. In Statements (1), (2), and (3), if $\mathcal{A}$ is bu deterministic (or crisp deterministic), then so is $\mathcal{C}$.

Proof. Proof of (1): Let B be right-distributive. Moreover, let $\mathcal{A}=\left(Q_{1}, \delta_{1}, F_{1}\right)$ and $\mathcal{B}=\left(Q_{2}, \delta_{2}, F_{2}\right)$. We construct the $(\Sigma, \mathrm{B})$-wta $\mathcal{C}=(Q, \delta, F)$ in the same way as in the proof of Theorem 10.4.1.

- $Q=Q_{1} \times Q_{2}$; for each $q \in Q$ we denote its first and second component by $q^{1}$ and $q^{2}$, respectively,
- for every $k \in \mathbb{N}, \sigma \in \Sigma^{(k)}, q \in Q$, and $q_{1} \cdots q_{k} \in Q^{k}$ we define

$$
\delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right)=\left(\delta_{1}\right)_{k}\left(q_{1}^{1} \cdots q_{k}^{1}, \sigma, q^{1}\right) \otimes\left(\delta_{2}\right)_{k}\left(q_{1}^{2} \cdots q_{k}^{2}, \sigma, q^{2}\right)
$$

- for each $q \in Q$ we define $F_{q}=\left(F_{1}\right)_{q^{1}} \otimes\left(F_{2}\right)_{q^{2}}$.

Obviously, if $\mathcal{A}$ is bu deterministic (or crisp deterministic), then so is $\mathcal{C}$.
Let $\xi \in \mathrm{T}_{\Sigma}$. By Lemma 4.3.1 there exits $p \in Q_{2}$ and a $\rho \in \mathrm{R}_{\mathcal{B}}(p, \xi)$ such that (a) $\operatorname{wt}_{\mathcal{B}}(\xi, \rho)=\mathbb{1}$ and (b) for each $\kappa \in \mathrm{R}_{\mathcal{B}}(\xi) \backslash\{\rho\}$ we have $\mathrm{wt}_{\mathcal{B}}(\xi, \kappa)=\mathbb{0}$. Since (a) and (b) determine $p$ and $\rho$ uniquely, we will denote them by $p_{\xi}$ and $\rho_{\xi}$, respectively. Then $p_{\xi}=\rho_{\xi}(\varepsilon)$, hence we obtain

$$
\begin{equation*}
\llbracket \mathcal{B} \rrbracket(\xi)=\left(\bigoplus_{\kappa \in \mathrm{R}_{\mathcal{B}}(\xi)} \operatorname{wt}_{\mathcal{B}}(\xi, \kappa) \otimes\left(F_{2}\right)_{\kappa(\varepsilon)}\right)=\mathrm{wt}_{\mathcal{B}}\left(\xi, \rho_{\xi}\right) \otimes\left(F_{2}\right)_{\rho_{\xi}(\varepsilon)}=\left(F_{2}\right)_{p_{\xi}} \tag{10.10}
\end{equation*}
$$

Then we define the mapping $\psi: \mathrm{R}_{\mathcal{A}}(\xi) \rightarrow \mathrm{R}_{\mathcal{C}}(\xi)$ for every $\rho \in \mathrm{R}_{\mathcal{A}}(\xi)$ and $w \in \operatorname{pos}(\xi)$ by

$$
\psi(\rho)(w)=\left(\rho(w), p_{\left.\xi\right|_{w}}\right)
$$

(cf. Figure 10.1). Obviously, $\psi$ is injective. Moreover, we define the mapping $\psi^{\prime}: \mathrm{R}_{\mathcal{A}}(\xi) \rightarrow \operatorname{im}(\psi)$ such that $\psi^{\prime}(\rho)=\psi(\rho)$ for each $\rho \in \mathrm{R}_{\mathcal{A}}(\xi)$. It follows that $\psi^{\prime}$ is bijective.

The following statement is easy to see.

$$
\text { For every } \xi \in \mathrm{T}_{\Sigma} \text { and } \rho \in \mathrm{R}_{\mathcal{C}}(\xi): \quad \operatorname{wt}_{\mathcal{C}}(\xi, \rho)= \begin{cases}\mathrm{wt}_{\mathcal{A}}\left(\xi,\left(\psi^{\prime}\right)^{-1}(\rho)\right) & \text { if } \rho \in \operatorname{im}\left(\psi^{\prime}\right)  \tag{10.11}\\ \mathbb{0} & \text { otherwise }\end{cases}
$$

Then, for each $\xi \in \mathrm{T}_{\Sigma}$, we can compute as follows.

$$
\llbracket \mathcal{C} \rrbracket^{\mathrm{run}}(\xi)=\bigoplus_{\rho \in \mathrm{R}_{\mathcal{C}}(\xi)} \mathrm{wt}_{\mathcal{C}}(\xi, \rho) \otimes F_{\rho(\varepsilon)}
$$



Figure 10.1: An illustration of the mapping $\psi^{\prime}$.

$$
\begin{aligned}
& =\bigoplus_{\rho \in \operatorname{im}\left(\psi^{\prime}\right)} \mathrm{wt}_{\mathcal{A}}\left(\xi,\left(\psi^{\prime}\right)^{-1}(\rho)\right) \otimes F_{\rho(\varepsilon)} \\
& =\bigoplus_{\rho \in \operatorname{im}\left(\psi^{\prime}\right)} \operatorname{wt}_{\mathcal{A}}\left(\xi,\left(\psi^{\prime}\right)^{-1}(\rho)\right) \otimes\left(F_{1}\right)_{\rho(\varepsilon)^{1}} \otimes\left(F_{2}\right)_{\rho(\varepsilon)^{2}} \quad \quad \text { (by the construction of } F \text { ) } \\
& =\bigoplus_{\rho \in \operatorname{im}\left(\psi^{\prime}\right)} \operatorname{wt}_{\mathcal{A}}\left(\xi,\left(\psi^{\prime}\right)^{-1}(\rho)\right) \otimes\left(F_{1}\right)_{\rho(\varepsilon)^{1}} \otimes\left(F_{2}\right)_{p_{\xi}} \quad \text { (because } \rho(\varepsilon)^{2}=p_{\xi} \text { by the definition of } \psi \text { ) }
\end{aligned}
$$

$$
=\bigoplus_{\rho^{\prime} \in \mathbb{R}_{\mathcal{A}}(\xi)} \operatorname{wt}_{\mathcal{A}}\left(\xi,\left(\psi^{\prime}\right)^{-1}\left(\psi^{\prime}\left(\rho^{\prime}\right)\right)\right) \otimes\left(F_{1}\right)_{\psi^{\prime}\left(\rho^{\prime}\right)(\varepsilon)^{1}} \otimes\left(F_{2}\right)_{p_{\xi}} \quad \quad \text { (because } \psi^{\prime} \text { is bijective) }
$$

$$
=\bigoplus_{\rho^{\prime} \in \mathrm{R}_{\mathcal{A}}(\xi)} \mathrm{wt}_{\mathcal{A}}\left(\xi, \rho^{\prime}\right) \otimes\left(F_{1}\right)_{\rho^{\prime}(\varepsilon)} \otimes\left(F_{2}\right)_{p_{\xi}} \quad \quad \quad \quad \text { (because } \psi^{\prime}\left(\rho^{\prime}\right)(\varepsilon)^{1}=\rho^{\prime}(\varepsilon) \text { ) }
$$

$$
=^{* *}\left(\bigoplus_{\rho^{\prime} \in \mathrm{R}_{\mathcal{A}}(\xi)} \mathrm{wt}_{\mathcal{A}}\left(\xi, \rho^{\prime}\right) \otimes\left(F_{1}\right)_{\rho^{\prime}(\varepsilon)}\right) \otimes\left(F_{2}\right)_{p_{\xi}} \quad \quad \quad \quad \text { (by right-distributivity) }
$$

$$
\begin{equation*}
=\llbracket \mathcal{A} \rrbracket^{\mathrm{run}}(\xi) \otimes \llbracket \mathcal{B} \rrbracket(\xi) \tag{10.10}
\end{equation*}
$$

The (*) at the last but one equality will be used and explained in the proof of (2).

Proof of (2): By Theorem 4.3.6 we can construct a crisp deterministic ( $\Sigma, \mathrm{B}$ )-wta $\mathcal{B}=\left(Q_{2}, \delta_{2}, F_{2}\right)$ with identity root weights (i.e., $\left.\operatorname{im}\left(F_{2}\right) \subseteq\{\mathbb{O}, \mathbb{1}\}\right)$ such that $\llbracket \mathcal{B} \rrbracket=\chi(\mathrm{L}(D))$. We construct the $(\Sigma, \mathrm{B})$-wta $\mathcal{C}$ and define the mappings $\psi$ and $\psi^{\prime}$ as in the proof of Statement (1). Obviously, (10.11) also holds. Then we can prove that $\llbracket \mathcal{C} \rrbracket^{\text {run }}=\llbracket \mathcal{A} \rrbracket^{\text {run }} \otimes \chi(L(D))$ in the same way as we have proved $\llbracket \mathcal{C} \rrbracket^{\text {run }}=\llbracket \mathcal{A} \rrbracket^{\text {run }} \otimes \llbracket \mathcal{B} \rrbracket$ in Statement (1) except that the equality $(*)$ is justified because $\left(F_{2}\right)_{p_{\xi}} \in\{\mathbb{O}, \mathbb{1}\}$. Thus no right-distributivity of $B$ is used.

Proof of (3): As in the proof of (2), by Theorem4.3.6 we can construct a crisp deterministic ( $\Sigma$, B)-wta $\mathcal{B}=\left(Q_{2}, \delta_{2}, F_{2}\right)$ with identity root weights (i.e., $\left.\operatorname{im}\left(F_{2}\right) \subseteq\{0, \mathbb{1}\}\right)$ such that $\llbracket \mathcal{B} \rrbracket=\chi(\mathrm{L}(D))$. Then, in the same way as in the proof of Statement (1), we construct the $(\Sigma, \mathrm{B})$-wta $\mathcal{C}$. We mention that we do not have to reverse the order of $\left(\delta_{1}\right)_{k}$ and $\left(\delta_{2}\right)_{k}$ in the right-hand side of the definition of $\delta_{k}$, because $\operatorname{im}\left(\left(\delta_{2}\right)_{k}\right) \in\{\mathbb{O}, \mathbb{1}\}$ and thus

$$
\left(\delta_{2}\right)_{k}\left(q_{1}^{2} \cdots q_{k}^{2}, \sigma, q^{2}\right) \otimes\left(\delta_{1}\right)_{k}\left(q_{1}^{1} \cdots q_{k}^{1}, \sigma, q^{1}\right)=\left(\delta_{1}\right)_{k}\left(q_{1}^{1} \cdots q_{k}^{1}, \sigma, q^{1}\right) \otimes\left(\delta_{2}\right)_{k}\left(q_{1}^{2} \cdots q_{k}^{2}, \sigma, q^{2}\right)
$$

Next we define the mappings $\varphi$ and $\psi$ in the same way as in the proof of (1), and of course, Equality (10.11) holds also in this case. Due to the first six steps in the final calulation of the proof of (1), we obtain:

$$
\llbracket \mathcal{C} \rrbracket^{\mathrm{run}}(\xi)=\bigoplus_{\rho^{\prime} \in \mathrm{R}_{\mathcal{A}}(\xi)} \mathrm{wt}_{\mathcal{A}}\left(\xi, \rho^{\prime}\right) \otimes\left(F_{1}\right)_{\rho^{\prime}(\varepsilon)} \otimes\left(F_{2}\right)_{p_{\xi}}
$$

Then we finish the proof as follows:

$$
\begin{array}{rlr} 
& \bigoplus_{\rho^{\prime} \in \mathcal{R}_{\mathcal{A}}(\xi)} \mathrm{wt}_{\mathcal{A}}\left(\xi, \rho^{\prime}\right) \otimes\left(F_{1}\right)_{\rho^{\prime}(\varepsilon)} \otimes\left(F_{2}\right)_{p_{\xi}} & \\
= & \bigoplus_{\rho^{\prime} \in \mathrm{R}_{\mathcal{A}}(\xi)}\left(F_{2}\right)_{p_{\xi}} \otimes \mathrm{wt}_{\mathcal{A}}\left(\xi, \rho^{\prime}\right) \otimes\left(F_{1}\right)_{\rho^{\prime}(\varepsilon)} & \\
= & \left(F_{2}\right)_{p_{\xi}} \otimes \bigoplus_{\rho^{\prime} \in \mathcal{R}_{\mathcal{A}}(\xi)} \mathrm{wt}_{\mathcal{A}}\left(\xi, \rho^{\prime}\right) \otimes\left(F_{1}\right)_{\rho^{\prime}(\varepsilon)} & \text { (because } \left.\left(F_{2}\right)_{p_{\xi}} \in\{0, \mathbb{1}\}\right) \\
= & \llbracket \mathcal{B} \rrbracket(\xi) \otimes \llbracket \mathcal{A} \rrbracket^{\mathrm{run}}(\xi) . & \text { (by ( } \overline{\text { (10.10) })}
\end{array}
$$

Thus no commutativity and no left-distributivity of $B$ is used.

Proof of (4): Let B be left-distributive. By Theorem $10.3 .1(\mathrm{~A}) \Rightarrow(\mathrm{D})$, we can construct $n \in \mathbb{N}_{+}$, $b_{1}, \ldots, b_{n} \in B$, and $\Sigma$-fta $A_{1}, \ldots, A_{n}$ such that $\llbracket \mathcal{B} \rrbracket=\bigoplus_{i \in[n]} b_{i} \otimes \chi\left(\mathrm{~L}\left(A_{i}\right)\right)$ and the step languages $\mathrm{L}\left(A_{i}\right)$ are pairwise disjoint. Due to the disjointness of the step languages and by associativity of $\otimes$, we have

$$
\left(\bigoplus_{i \in[n]} b_{i} \otimes \chi\left(\mathrm{~L}\left(A_{i}\right)\right)\right) \otimes \llbracket \mathcal{A} \rrbracket^{\mathrm{run}}=\bigoplus_{i \in[n]} b_{i} \otimes\left(\chi\left(\mathrm{~L}\left(A_{i}\right)\right) \otimes \llbracket \mathcal{A} \rrbracket^{\mathrm{run}}\right)
$$

By Theorem $10.4 .3(3)$, we can construct a $(\Sigma, \mathrm{B})$-wta $\mathcal{B}_{i}$ such that $\llbracket \mathcal{B}_{i} \rrbracket^{\text {run }}=\chi\left(\mathrm{L}\left(A_{i}\right)\right) \otimes \llbracket \mathcal{A} \rrbracket^{\text {run }}$. Hence

$$
\bigoplus_{i \in[n]} b_{i} \otimes\left(\chi\left(\mathrm{~L}\left(A_{i}\right)\right) \otimes \llbracket \mathcal{A} \rrbracket^{\text {run }}\right)=\bigoplus_{i \in[n]} b_{i} \otimes \llbracket \mathcal{B}_{i} \rrbracket^{\mathrm{run}}
$$

Since B is left-distributive, by Theorem $10.2 .1(1)$ we can construct, for each $i \in[n]$, a $(\Sigma, \mathrm{B})$-wta $\mathcal{C}_{i}$ such that $\llbracket \mathcal{C}_{i} \rrbracket^{\text {run }}=b_{i} \otimes \llbracket \mathcal{B}_{i} \rrbracket^{\text {run }}$. Hence

$$
\bigoplus_{i \in[n]} b_{i} \otimes \llbracket \mathcal{B}_{i} \rrbracket^{\mathrm{run}}=\bigoplus_{i \in[n]} \llbracket \mathcal{C}_{i} \rrbracket^{\mathrm{run}}
$$

By Theorem 10.1 .1 we can construct a $(\Sigma, \mathrm{B})$-wta $\mathcal{C}$ such that $\llbracket \mathcal{C} \rrbracket^{\text {run }}=\bigoplus_{i \in[n \rrbracket} \llbracket \mathcal{C}_{i} \rrbracket^{\mathrm{run}}$.
The next corollary shows that recognizable step mappings are closed under Hadamard product if the weight structure is a semiring. This result is a special instance of Theorem 10.4 .3 (the proof of which uses Theorem 10.3.1). Nevertheless, here we show a proof of the weaker result without referring to Theorem 10.3 .1 its proof exploits the special structure of recognizable step mappings. The reader may view this proof as an exercise.
Corollary 10.4.4. (cf. DGMM11, Thm. 5.12] and Her20a, Lm. 1.4.22]) Let $\Sigma$ be a ranked alphabet and $\mathrm{B}=(B, \oplus, \otimes, \mathbb{0}, \mathbb{1})$ be a semiring. Moreover, let $r: \mathrm{T}_{\Sigma} \rightarrow B$ and $r^{\prime}: \mathrm{T}_{\Sigma} \rightarrow B$ be recognizable $(\Sigma, \mathrm{B})$-weighted tree languages. If $r$ or $r^{\prime}$ is a recognizable step mapping, then $r \otimes r^{\prime}$ is $(\Sigma, \mathrm{B})$-recognizable.

Proof. First we assume that $r$ is a recognizable step mapping. Hence, there exist $n \in \mathbb{N}_{+}, b_{1}, \ldots, b_{n} \in B$, and recognizable $\Sigma$-tree languages $L_{1}, \ldots, L_{n}$ such that $r=\bigoplus_{i \in[n]} b_{i} \otimes \chi\left(L_{i}\right)$. Then, for each $\xi \in \mathrm{T}_{\Sigma}$, we can calculate as follows.

$$
\begin{aligned}
\left(r \otimes r^{\prime}\right)(\xi) & =\left(\bigoplus_{i \in[n]} b_{i} \otimes \chi\left(L_{i}\right)(\xi)\right) \otimes r^{\prime}(\xi) \\
& =\bigoplus_{i \in[n]} b_{i} \otimes \chi\left(L_{i}\right)(\xi) \otimes r^{\prime}(\xi) \quad \quad \text { (by right-distributivity) } \\
& =\left(\bigoplus_{i \in[n]} b_{i} \otimes\left(\chi\left(L_{i}\right) \otimes r^{\prime}\right)\right)(\xi)
\end{aligned}
$$

By Theorem 10.4.3(3), the weighted tree language $\chi\left(L_{i}\right) \otimes r^{\prime}$ is recognizable. Since B is left-distributive, we can apply Theorem10.2.1(1a) and obtain that the weighted tree language $b_{i} \otimes\left(\chi\left(L_{i}\right) \otimes r^{\prime}\right)$ is recognizable. Finally, by Theorem 10.1.1, the weighted tree language $\bigoplus_{i \in[n]} b_{i} \otimes\left(\chi\left(L_{i}\right) \otimes r^{\prime}\right)$ is recognizable. Hence $r \otimes r^{\prime}$ is recognizable.

Now we assume that $r^{\prime}$ is a recognizable step mapping. Hence, there are $n \in \mathbb{N}_{+}, b_{1}, \ldots, b_{n} \in B$, and recognizable $\Sigma$-tree languages $L_{1}, \ldots, L_{n}$ such that $r^{\prime}=\bigoplus_{i \in[n]} b_{i} \otimes \chi\left(L_{i}\right)$. Then, for each $\xi \in \mathrm{T}_{\Sigma}$, we can calculate as follows.

$$
\begin{aligned}
\left(r \otimes r^{\prime}\right)(\xi) & =r(\xi) \otimes \bigoplus_{i \in[n]} b_{i} \otimes \chi\left(L_{i}\right)(\xi) \\
& =\bigoplus_{i \in[n]} r(\xi) \otimes b_{i} \otimes \chi\left(L_{i}\right)(\xi) \\
& =\left(\bigoplus_{i \in[n]}\left(r \otimes b_{i}\right) \otimes \chi\left(L_{i}\right)\right)(\xi)
\end{aligned}
$$

(by left-distributivity)

Since B is right-distributive, we can apply Theorem $10.2 .1(2)$ and obtain that the weighted tree language $r \otimes b_{i}$ is recognizable. By Theorem 10.4.3(2), the weighted tree language $\left(r \otimes b_{i}\right) \otimes \chi\left(L_{i}\right)$ is recognizable. Finally, by Theorem 10.1.1, the weighted tree language $\bigoplus_{i \in[n]}\left(r \otimes b_{i}\right) \otimes \chi\left(L_{i}\right)$ is recognizable. Hence $r \otimes r^{\prime}$ is recognizable.

Finally, we show an overview of the closure results for the Hadamard product, cf. Figure 10.2,

| Theorem | restriction on ( $\Sigma, \mathrm{B}$ )-wta $\mathcal{A}_{1}$ | restriction on ( $\Sigma, \mathrm{B}$ )-wta $\mathcal{A}_{2}$ | subset of <br> strong bimonoids | $\llbracket \mathcal{A}_{1} \rrbracket^{\text {run }} \otimes \llbracket \mathcal{A}_{2} \rrbracket^{\text {run }}$ <br> included in |
| :---: | :---: | :---: | :---: | :---: |
| 10.4.1(1) | - | - | commutative semirings | $\operatorname{Rec}(\Sigma, \mathrm{B})$ |
| 10.4.1 (2) | bu deterministic | bu deterministic | commutative | bud-Rec ( $\Sigma$, B) |
| 10.4.1(3) | crisp deterministic | crisp deterministic | all strong bimonoids | cd-Rec ( $\Sigma, \mathrm{B}$ ) |
| 10.4.3(1) | - | crisp deterministic | right-distributive | $\operatorname{Rec}^{\text {run }}(\Sigma, \mathrm{B})$ |
| 10.4.3(2) | - | $\begin{aligned} & \llbracket \mathcal{A}_{2} \rrbracket^{\text {run }}=\chi(L) \\ & \text { for some } L \in \operatorname{Rec}(\Sigma) \end{aligned}$ | all strong bimonoids | $\operatorname{Rec}^{\text {run }}(\Sigma, \mathrm{B})$ |
| 10.4.3(3) | $\begin{aligned} & \llbracket \mathcal{A}_{1} \rrbracket^{\text {run }}=\chi(L) \\ & \text { for some } L \in \operatorname{Rec}(\Sigma) \end{aligned}$ | ( | all strong bimonoids | $\operatorname{Rec}^{\text {run }}(\Sigma, \mathrm{B})$ |
| 10.4.3(4) | crisp deterministic | - | left-distributive | $\operatorname{Rec}^{\text {run }}(\Sigma, \mathrm{B})$ |

Figure 10.2: An overview of the closure results for the Hadamard product of $\llbracket \mathcal{A}_{1} \rrbracket^{\text {run }}$ and $\llbracket \mathcal{A}_{2} \rrbracket^{\text {run }}$ for two $(\Sigma, \mathrm{B})$-wta $\mathcal{A}_{1}$ and $\mathcal{A}_{2}$.

### 10.5 Closure under top-concatenations

Let $k \in \mathbb{N}, \sigma \in \Sigma^{(k)}$, and $r_{1}, \ldots, r_{k}$ be ( $\left.\Sigma, \mathrm{B}\right)$-weighted tree languages. The top-concatenation of $r_{1}, \ldots, r_{k}$ with $\sigma$ is the $(\Sigma, \mathrm{B})$-weighted tree language $\operatorname{top}_{\sigma}\left(r_{1}, \ldots, r_{k}\right): \mathrm{T}_{\Sigma} \rightarrow B$ defined, for each $\xi \in \mathrm{T}_{\Sigma}$, as follows:

$$
\operatorname{top}_{\sigma}\left(r_{1}, \ldots, r_{k}\right)(\xi)= \begin{cases}r_{1}\left(\xi_{1}\right) \otimes \cdots \otimes r_{k}\left(\xi_{k}\right) & \text { if } \xi=\sigma\left(\xi_{1}, \ldots, \xi_{k}\right) \\ 0 & \text { otherwise }\end{cases}
$$

If $\alpha \in \Sigma^{(0)}$, then we call $\operatorname{top}_{\alpha}()$ the top-concatenation with $\alpha$. For each $\xi \in \mathrm{T}_{\Sigma}$, we have

$$
\operatorname{top}_{\alpha}()(\xi)= \begin{cases}\mathbb{1} & \text { if } \xi=\alpha \\ \mathbb{0} & \text { otherwise }\end{cases}
$$

i.e., $\operatorname{top}_{\alpha}()=\mathbb{1} . \alpha$. It follows that $\operatorname{top}_{\sigma}\left(r_{1}, \ldots, r_{k}\right)(\xi)=\mathbb{O}$ for each $\xi \in \mathrm{T}_{\Sigma}$ with $\xi(\varepsilon) \neq \sigma$.

A set $\mathcal{L}$ of B -weighted tree languages is closed under top-concatenations if for every $k \in \mathbb{N}, \sigma \in \Sigma^{(k)}$, and $(\Sigma, \mathrm{B})$-weighted tree languages $r_{1}, \ldots, r_{k}$ in $\mathcal{L}$, the $(\Sigma, \mathrm{B})$-weighted tree language top ${ }_{\sigma}\left(r_{1}, \ldots, r_{k}\right)$ is in $\mathcal{L}$. Thus, in particular, a set $\mathcal{L}$ of $(\Sigma, B)$-weighted tree languages which is closed under top-concatenations contains the $(\Sigma, \mathrm{B})$-weighted tree language $\operatorname{top}_{\alpha}()$ for each $\alpha \in \Sigma^{(0)}$.

Theorem 10.5.1. Let B be a semiring. Moreover, let $k \in \mathbb{N}$ and $\sigma \in \Sigma^{(k)}$. Also, for each $i \in[k]$, let $\mathcal{G}_{i}$ be a ( $\left.\Sigma, \mathrm{B}\right)$-wrtg such that (a) for each $i \in[k]$ the wrtg $\mathcal{G}_{i}$ is finite-derivational or (b) B is $\sigma$-complete. We can construct a $(\Sigma, \mathrm{B})$-wrtg $\mathcal{G}$ such that $\llbracket \mathcal{G} \rrbracket=\operatorname{top}_{\sigma}\left(\llbracket \mathcal{G}_{1} \rrbracket, \ldots, \llbracket \mathcal{G}_{k} \rrbracket\right)$.

Thus, in particular, if B is a semiring, then the set $\operatorname{Reg}(\Sigma, \mathrm{B})$ is closed under top-concatenations.
Proof. For each $i \in[k]$, let $\mathcal{G}_{i}=\left(N_{i}, S_{i}, R_{i}, \mathrm{wt}_{i}\right)$ be a $(\Sigma, \mathrm{B})$-wrtg such that $N_{i} \cap N_{j}=\emptyset$ for each $i, j \in[k]$ with $i \neq j$. By Lemma 9.2.1(1), for each $i \in[k]$, we can construct a wrtg which is start-separated and equivalent to $\mathcal{G}_{i}$. The construction preserves the property finite-derivational. So we can assume that each $\mathcal{G}_{i}$ is start-separated, i.e., $S_{i}$ is a singleton (and then we assume that $S_{i}$ denotes the only initial nonterminal).

We let $S$ be a new nonterminal, i.e., $S \notin \bigcup_{i \in[k]} N_{i}$, and construct the ( $\left.\Sigma, \mathrm{B}\right)$-wrtg $\mathcal{G}=(N, S, R$, wt $)$ as follows.

- $N=\{S\} \cup \bigcup_{i \in[k]} N_{i}$ and
- $R$ is the smallest set $R^{\prime}$ of rules which satisfies the following conditions:
$-r=\left(S \rightarrow \sigma\left(S_{1}, \ldots, S_{k}\right)\right)$ is in $R^{\prime}$ and $\mathrm{wt}(r)=\mathbb{1}$ and
- for every $i \in[k]$ and $r \in R_{i}$, we let $r \in R^{\prime}$ and $\mathrm{wt}(r)=\mathrm{wt}_{i}(r)$.

In case $k=0$ we have $N=\{S\}$ and $R=\{r\}$ with $r=(S \rightarrow \sigma)$ and $\mathrm{wt}(r)=\mathbb{1}$.
It is obvious that, if for each $i \in[k]$ the $\operatorname{wrtg} \mathcal{G}_{i}$ is finite-derivational, then $\mathcal{G}$ is finite-derivational. Hence $\llbracket \mathcal{G} \rrbracket$ is defined.

Next we prove that

$$
\begin{equation*}
\llbracket \mathcal{G} \rrbracket(\xi)=\operatorname{top}_{\sigma}\left(\llbracket \mathcal{G}_{1} \rrbracket, \ldots, \llbracket \mathcal{G}_{k} \rrbracket\right)(\xi) \text { for each } \xi \in \mathrm{T}_{\Sigma} . \tag{10.12}
\end{equation*}
$$

For this, we mention some properties which are easy to see.

$$
\begin{equation*}
\text { For every } \xi \in \mathrm{T}_{\Sigma} \text { with } \xi(\varepsilon) \neq \sigma \text { we have } \operatorname{RT}_{\mathcal{G}}(S, \xi)=\emptyset \tag{10.13}
\end{equation*}
$$

For every $\xi=\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)$ in $\mathrm{T}_{\Sigma}$ and $d \in \operatorname{RT}_{\mathcal{G}}(S, \xi)$, we have $\operatorname{lhs}(d(i))=S_{i}$ for each $i \in[k]$.

For every $\xi \in \mathrm{T}_{\Sigma}$ and $i \in[k]$, we have $\operatorname{RT}_{\mathcal{G}}\left(S_{i}, \xi\right)=\mathrm{RT}_{\mathcal{G}_{i}}\left(S_{i}, \xi\right)$.

For every $\xi \in \mathrm{T}_{\Sigma}, i \in[k]$, and $d \in \mathrm{RT}_{\mathcal{G}_{i}}\left(S_{i}, \xi\right)$, we have $\mathrm{wt}_{\mathcal{G}}(d)=\mathrm{wt}_{\mathcal{G}_{i}}(d)$.
Now we prove (10.12). Let $\xi \in \mathrm{T}_{\Sigma}$. Then we have:

$$
\llbracket \mathcal{G} \rrbracket(\xi)=\bigoplus_{d \in \mathrm{RT}_{\mathcal{G}}(\xi)} \mathrm{wt}_{\mathcal{G}}(d)=\bigoplus_{d \in \mathrm{RT}_{\mathcal{G}}(S, \xi)} \mathrm{wt}_{\mathcal{G}}(\xi)
$$

If $\xi(\varepsilon) \neq \sigma$, then $\llbracket \mathcal{G} \rrbracket(\xi)=\mathbb{O}$ by (10.13) . Since $\operatorname{top}_{\sigma}\left(\llbracket \mathcal{G}_{1} \rrbracket, \ldots, \llbracket \mathcal{G}_{k} \rrbracket\right)(\xi)=\mathbb{0}$, the equality 10.12 is proved for that case.

Now we assume that $\xi=\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)$. Then we continue with

$$
\bigoplus_{d \in \mathrm{RT}_{\mathcal{G}}(S, \xi)} \mathrm{wt}_{\mathcal{G}}(d)
$$

$$
\begin{aligned}
& =\bigoplus_{\substack{d \in \mathrm{RT}_{\mathcal{G}}(S, \xi): \\
(\forall i \in[k]): \operatorname{lhs}(d(i))=S_{i}}} \mathrm{wt}_{\mathcal{G}}(d) \\
& =\bigoplus_{\substack{d \in \operatorname{RT_{\mathcal {G}}(S,\xi ):} \\
(\forall i \in[k]): \operatorname{lhs}(d(i))=S_{i}}} \mathrm{wt}_{\mathcal{G}}\left(\left.d\right|_{1}\right) \otimes \ldots \otimes \mathrm{wt}_{\mathcal{G}}\left(\left.d\right|_{k}\right) \otimes \mathrm{wt}\left(S \rightarrow \sigma\left(S_{1}, \ldots, S_{k}\right)\right) \\
& =\quad \bigoplus \quad \mathrm{wt}_{\mathcal{G}}\left(\left.d\right|_{1}\right) \otimes \ldots \otimes \mathrm{wt}_{\mathcal{G}}\left(\left.d\right|_{k}\right) \quad\left(\text { since wt }\left(S \rightarrow \sigma\left(S_{1}, \ldots, S_{k}\right)\right)=\mathbb{1}\right) \\
& (\forall i \in[k]): \operatorname{lhs}(d(i))=S_{i} \\
& =\bigoplus_{d_{1} \in \mathrm{RT}_{\mathcal{G}}\left(S_{1}, \xi_{1}\right)} \ldots \bigoplus_{d_{k} \in \mathrm{RT}_{\mathcal{G}}\left(S_{k}, \xi_{k}\right)} \mathrm{wt}_{\mathcal{G}}\left(d_{1}\right) \otimes \ldots \otimes \mathrm{wt}_{\mathcal{G}}\left(d_{k}\right) \\
& =\bigoplus_{d_{1} \in \mathrm{RT}_{\mathcal{G}_{1}}\left(S_{1}, \xi_{1}\right)} \ldots \bigoplus_{d_{k} \in \mathrm{RT}_{\mathcal{G}_{k}}\left(S_{k}, \xi_{k}\right)} \mathrm{wt}_{\mathcal{G}}\left(d_{1}\right) \otimes \ldots \otimes \mathrm{wt}_{\mathcal{G}}\left(d_{k}\right) \quad \quad \text { (by (10.15)) } \\
& =\bigoplus_{d_{1} \in \mathrm{RT}_{\mathcal{G}_{1}}\left(S_{1}, \xi_{1}\right)} \ldots \bigoplus_{d_{k} \in \mathrm{RT}_{\mathcal{G}_{k}}\left(S_{k}, \xi_{k}\right)} \mathrm{wt}_{\mathcal{G}_{1}}\left(d_{1}\right) \otimes \ldots \otimes \mathrm{wt}_{\mathcal{G}_{k}}\left(d_{k}\right) \quad \quad \text { (by (10.16) ) } \\
& =\left(\bigoplus_{d_{1} \in \mathrm{RT}_{\mathcal{G}_{1}}\left(S_{1}, \xi_{1}\right)} \mathrm{wt}_{\mathcal{G}_{1}}\left(d_{1}\right)\right) \otimes \ldots \otimes\left(\bigoplus_{d_{k} \in \mathrm{RT}_{\mathcal{G}_{k}}\left(S_{k}, \xi_{k}\right)} \mathrm{wt}_{\mathcal{G}_{k}}\left(d_{k}\right)\right) \quad \quad \text { (by distributivity) } \\
& =\llbracket \mathcal{G}_{1} \rrbracket\left(\xi_{1}\right) \otimes \ldots \otimes \llbracket \mathcal{G}_{k} \rrbracket\left(\xi_{k}\right) \\
& =\operatorname{top}_{\sigma}\left(\llbracket \mathcal{G}_{1} \rrbracket, \ldots, \llbracket \mathcal{G}_{k} \rrbracket\right)(\xi) \quad\left(\text { because } \xi=\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)\right)
\end{aligned}
$$

This proves (10.12) also in the case that $\xi=\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)$.

Corollary 10.5.2. (cf. DPV05, Lm. 6.2]) Let $\Sigma$ be a ranked alphabet and B be a semiring. Moreover, let $k \in \mathbb{N}$ and $\sigma \in \Sigma^{(k)}$. Also, for each $i \in[k]$, let $\mathcal{A}_{i}$ be a $(\Sigma, \mathrm{B})$-wta. Then we can construct a ( $\left.\Sigma, \mathrm{B}\right)$-wta $\mathcal{A}$ such that $\llbracket \mathcal{A} \rrbracket=\operatorname{top}_{\sigma}\left(\llbracket \mathcal{A}_{1} \rrbracket, \ldots, \llbracket \mathcal{A}_{k} \rrbracket\right)$. Thus, in particular, if B is a semiring, then the $\operatorname{set} \operatorname{Rec}(\Sigma, \mathrm{B})$ is closed under top-concatenations.

Proof. By Lemma 9.2.6, for each $i \in[k]$, we can construct $(\Sigma, \mathrm{B})$-wrtg $\mathcal{G}_{i}$ such that $\mathcal{G}_{i}$ is in tree automata form and $\llbracket \mathcal{A}_{i} \rrbracket=\llbracket \mathcal{G}_{i} \rrbracket$. Then, in particular, $\mathcal{G}_{i}$ is finite-derivational. By Theorem 10.5.1 we can construct a finite-derivational $(\Sigma, \mathrm{B})$-wrtg $\mathcal{G}$ such that $\llbracket \mathcal{G} \rrbracket=\operatorname{top}_{\sigma}\left(\llbracket \mathcal{G}_{1} \rrbracket, \ldots, \llbracket \mathcal{G}_{k} \rrbracket\right)$. Then, by Lemma 9.2.8, we can construct a $(\Sigma, \mathrm{B})$-wta $\mathcal{A}$ such that $\llbracket \mathcal{G} \rrbracket=\llbracket \mathcal{A} \rrbracket$.

### 10.6 Closure under tree concatenations

In this section we recall the definition of tree concatenation and show that $\operatorname{Rec}(\Sigma, \mathrm{B})$ is closed under tree concatenation. The definition of tree concatenation involves a technical problem which is due to a difference between the concatenation of strings and tree concatenation [TW68, p. 59]. The concatenation of two strings $\xi$ and $\zeta$ is simply the string $\xi \zeta$. This also applies if $\xi$ and $\zeta$ are particular strings, viz. trees. However, $\xi \zeta$ is not a tree. In order to define tree concatenation as operation on trees, in TW68 a nullary symbol $\alpha$ is used and each occurrence of $\alpha$ in $\xi$ is replaced by $\zeta$. Hence, tree concatenation consumes the occurrences of $\alpha$ in $\xi$; in contrast, string concatenation does not consume symbols.

Let $r_{1}$ and $r_{2}$ be ( $\Sigma, \mathrm{B}$ )-weighted tree languages and $\alpha \in \Sigma^{(0)}$. We will recall the definition of the $\alpha$-concatenation of $r_{1}$ and $r_{2}$, denoted by $r_{1} \circ_{\alpha} r_{2}$. This is a $(\Sigma, \mathrm{B})$-weighted tree language and hence, for each $\xi \in \mathrm{T}_{\Sigma}$, the value $\left(r_{1} \circ_{\alpha} r_{2}\right)(\xi)$ has to be defined. Thus, intuitively, $\xi$ has to be decomposed and the resulting parts of $\xi$ have to be evaluated appropriately in $r_{1}$ or $r_{2}$. As preparation for this, we define the concept of $\alpha$-cut which helps to identify the mentioned parts of $\xi$.


Figure 10.3: An illustration of an $\alpha$-cut through $\xi$. Position 22 covers the $\alpha$ at positions 221 and 2221. Moreover, position 3 covers $\alpha$ at position 3 .

Let $\alpha \in \Sigma^{(0)}$ and $\xi \in \mathrm{T}_{\Sigma}$. Intuitively, an $\alpha$-cut through $\xi$ is a sequence $\left(w_{1}, \ldots, w_{n}\right)$ of pairwise different positions of $\xi$ such that neither $w_{i}$ is a prefix of $w_{j}$ nor vice versa (for each index pair $i, j$ with $i \neq j$ ); moreover, each $\alpha$-labeled position is covered by some $w_{i}$ (see Figure 10.3). Formally, we define the set of $\alpha$-cuts through $\xi$, denoted by $\operatorname{cut}_{\alpha}(\xi)$, to be the set

$$
\begin{aligned}
\operatorname{cut}_{\alpha}(\xi)=\left\{\left(w_{1}, \ldots, w_{n}\right) \mid\right. & n \in \mathbb{N}, w_{1}, \ldots, w_{n} \in \operatorname{pos}(\xi), \text { such that } \\
& w_{1}<\text { lex } \cdots<\text { lex } w_{n} \text { and } \\
& (\forall i, j \in[n]): \text { if } i \neq j, \text { then } w_{i} \notin \operatorname{prefix}\left(w_{j}\right) \text { and } \\
& \left.\left(\forall w \in \operatorname{pos}_{\alpha}(\xi)\right)(\exists i \in[n]): w_{i} \text { is a prefix of } w\right\} .
\end{aligned}
$$

Let $\widetilde{w}=\left(w_{1}, \ldots, w_{n}\right)$ be an element of $\operatorname{cut}_{\alpha}(\xi)$. Then we will abbreviate $\xi[\alpha]_{w_{1}} \ldots[\alpha]_{w_{n}}$ by $\xi[\alpha]_{\widetilde{w}}$. We point out some properties of $\operatorname{cut}_{\alpha}(\xi)$.

- The set $\operatorname{cut}_{\alpha}(\xi)$ is finite and nonempty; in particular, $(\varepsilon) \in \operatorname{cut}_{\alpha}(\xi)$ and $\xi[\alpha]_{(\varepsilon)}=\xi[\alpha]_{\varepsilon}=\alpha$.
- For each $\widetilde{w}=\left(w_{1}, \ldots, w_{n}\right)$ in $\operatorname{cut}_{\alpha}(\xi)$, we have $\operatorname{pos}_{\alpha}\left(\xi[\alpha]_{\widetilde{w}}\right)=\left\{w_{1}, \ldots, w_{n}\right\}$.
- The following equivalence holds: $\operatorname{pos}_{\alpha}(\xi)=\emptyset$ if, and only if ()$\in \operatorname{cut}_{\alpha}(\xi)$; moreover, $\xi[\alpha]_{()}=\xi$.
- $\operatorname{cut}_{\alpha}(\alpha)=\{(\varepsilon)\}$ and, for each $\beta \in \Sigma^{(0)} \backslash\{\alpha\}$, we have $\operatorname{cut}_{\alpha}(\beta)=\{(),(\varepsilon)\}$.

Let $r_{1}: \mathrm{T}_{\Sigma} \rightarrow B$ and $r_{2}: \mathrm{T}_{\Sigma} \rightarrow B$ be weighted tree languages. Moreover, let $\alpha \in \Sigma^{(0)}$. The $\alpha$-concatenation of $r_{1}$ and $r_{2}$ DPV05 is the weighted tree language ( $r_{1} \circ_{\alpha} r_{2}$ ) : $\mathrm{T}_{\Sigma} \rightarrow B$ defined for every $\xi \in \mathrm{T}_{\Sigma}$ by

$$
\left(r_{1} \circ_{\alpha} r_{2}\right)(\xi)=\bigoplus_{\left(w_{1}, \ldots, w_{n}\right) \in \operatorname{cut}_{\alpha}(\xi)} r_{1}\left(\xi[\alpha]_{\left(w_{1}, \ldots, w_{n}\right)}\right) \otimes r_{2}\left(\left.\xi\right|_{w_{1}}\right) \otimes \ldots \otimes r_{2}\left(\left.\xi\right|_{w_{n}}\right)
$$

Since, for each $\xi \in \mathrm{T}_{\Sigma}$, the index set $\operatorname{cut}_{\alpha}(\xi)$ is finite, we do not need the condition that B is $\sigma$-complete for the summation. Moreover, we have $\left(r_{1} \circ_{\alpha} r_{2}\right)(\alpha)=r_{1}(\alpha) \otimes r_{2}(\alpha)$ and, for each $\beta \in \Sigma^{(0)} \backslash\{\alpha\}$, we have $\left(r_{1} \circ_{\alpha} r_{2}\right)(\beta)=r_{1}(\beta) \oplus r_{1}(\alpha) \otimes r_{2}(\beta)$. The tree concatenation of $r_{1}$ and $r_{2}$ is the $\alpha$-concatenation of $r_{1}$ and $r_{2}$ for some $\alpha \in \Sigma^{(0)}$.

A set $\mathcal{L}$ of B -weighted tree languages is closed under tree concatenations if the following holds: for every ( $\Sigma, \mathrm{B}$ )-weighted tree languages $r_{1}$ and $r_{2}$ and for each $\alpha \in \Sigma^{(0)}$, if $r_{1}, r_{2} \in \mathcal{L}$, then $\left(r_{1} \circ_{\alpha} r_{2}\right) \in \mathcal{L}$.

Theorem 10.6.1. (cf. Eng75b, Thm. 3.35], GS84, Thm. 2.4.6], and [FV22b, Lm. 6.6]) Let B be a commutative semiring and let $\alpha \in \Sigma^{(0)}$. Let $\mathcal{G}_{1}$ and $\mathcal{G}_{2}$ be two $(\Sigma, \mathrm{B})$-wrtg such that $\mathcal{G}_{1}$ and $\mathcal{G}_{2}$ are finite-derivational or B is $\sigma$-complete. Then the following two statements hold.


Figure 10.4: Illustration for the proof of $\llbracket \mathcal{G} \rrbracket=\llbracket \mathcal{G}_{1} \rrbracket \circ_{\alpha} \llbracket \mathcal{G}_{2} \rrbracket$.
(1) There exists a $(\Sigma, \mathrm{B})-$ wrtg $\mathcal{G}$ such that $\llbracket \mathcal{G} \rrbracket=\llbracket \mathcal{G}_{1} \rrbracket \circ_{\alpha} \llbracket \mathcal{G}_{2} \rrbracket$.
(2) If $\mathcal{G}_{1}$ and $\mathcal{G}_{2}$ are finite-derivational, then we can construct a finite-derivational $(\Sigma, \mathrm{B})$-wrtg $\mathcal{G}$ such that $\llbracket \mathcal{G} \rrbracket=\llbracket \mathcal{G}_{1} \rrbracket \circ_{\alpha} \llbracket \mathcal{G}_{2} \rrbracket$.

Proof. Proof of (1): Let $\mathcal{G}_{1}=\left(N_{1}, S_{1}, R_{1}, w t_{1}\right)$ and $\mathcal{G}_{2}=\left(N_{2}, S_{2}, R_{2}\right.$, wt $t_{2}$ ) be two ( $\Sigma$, B)-wrtg with $N_{1} \cap N_{2}=\emptyset$. By Lemmas 9.2.1(1) and 9.2.2 we can assume that $\mathcal{G}_{1}$ and $\mathcal{G}_{2}$ are start-separated and alphabetic. By Lemma 9.2 .1 (3) we can furthermore assume that $\mathcal{G}_{1}$ and $\mathcal{G}_{2}$ are chain-free. We call a rule $A \rightarrow \alpha$ of $R_{1}$ an $\alpha$-rule.

We define the $(\Sigma, \mathrm{B})-\operatorname{wrtg} \underline{\mathcal{G}}=\left(N, S_{1}, R, w t\right)$ such that $\llbracket \mathcal{G} \rrbracket=\llbracket \mathcal{G}_{1} \rrbracket \circ_{\alpha} \llbracket \mathcal{G}_{2} \rrbracket$ as follows. We let $N=$ $N_{1} \cup N_{2}$. Moreover, we let $R=\overline{R_{1}} \cup R_{2}$ where $\overline{R_{1}}$ is obtained from $R_{1}$ by replacing each $\alpha$-rule $r=(A \rightarrow \alpha)$ by the rule $r^{\prime}=\left(A \rightarrow S_{2}\right)$, and we define $w t\left(r^{\prime}\right)=w t_{1}(r)$. Each rule $r \in R$ which is not of the form $A \rightarrow S_{2}$ keeps the weight from its original grammar. We call each rule of the form $A \rightarrow S_{2}$ a $S_{2}$-rule. It is obvious that, if $\mathcal{G}_{1}$ and $\mathcal{G}_{2}$ are finite-derivational, then also $\mathcal{G}$ is finite-derivational. Hence $\llbracket \mathcal{G} \rrbracket$ is defined.

Now we prove that $\llbracket \mathcal{G} \rrbracket=\llbracket \mathcal{G}_{1} \rrbracket \circ_{\alpha} \llbracket \mathcal{G}_{2} \rrbracket$ (see Figure 10.4). For this, let $\xi \in \mathrm{T}_{\Sigma}$ and $\widetilde{w}=\left(w_{1}, \ldots, w_{n}\right) \in$ $\operatorname{cut}_{\alpha}(\xi)$. We define

$$
\operatorname{RT}_{\mathcal{G}}^{\widetilde{w}}(\xi)=\left\{d \in \operatorname{RT}_{\mathcal{G}}(\xi) \mid\left\{w_{1}, \ldots, w_{n}\right\}=\left\{u \in \operatorname{pos}(d) \mid d(u) \text { is a } S_{2} \text {-rule }\right\}\right\}
$$

Obviously, if ()$\in \operatorname{cut}_{\alpha}(\xi)$, i.e., $\operatorname{pos}_{\alpha}(\xi)=\emptyset$, then $\operatorname{RT}_{\mathcal{G}}^{()}(\xi)=\mathrm{RT}_{\mathcal{G}_{1}}(\xi)$. The tree in the lower part of Figure 10.4 illustrates an element of $\mathrm{RT}_{\mathcal{G}}^{(1,22)}(\xi)$, where $\xi=\sigma(\gamma(\beta), \delta(\beta, \gamma(\beta)))$. The following is obvious:

$$
\begin{equation*}
\left(\operatorname{RT}_{\mathcal{G}}^{\widetilde{w}}(\xi) \mid \widetilde{w} \in \operatorname{cut}_{\alpha}(\xi)\right) \text { is a partitioning of } \operatorname{RT}_{\mathcal{G}}(\xi) \tag{10.17}
\end{equation*}
$$

We will show that there exists a bijection

$$
\Phi: \mathrm{RT}_{\mathcal{G}_{1}}\left(\xi[\alpha]_{\widetilde{w}}\right) \times \mathrm{RT}_{\mathcal{G}_{2}}\left(\left.\xi\right|_{w_{1}}\right) \times \ldots \times \mathrm{RT}_{\mathcal{G}_{2}}\left(\left.\xi\right|_{w_{n}}\right) \rightarrow \mathrm{RT}_{\mathcal{G}}^{\widetilde{w}}(\xi)
$$

For this, let $d \in \operatorname{RT}_{\mathcal{G}_{1}}\left(\xi[\alpha]_{\tilde{w}}\right), d_{1} \in \operatorname{RT}_{\mathcal{G}_{2}}\left(\left.\xi\right|_{w_{1}}\right), \ldots, d_{n} \in \operatorname{RT}_{\mathcal{G}_{2}}\left(\left.\xi\right|_{w_{n}}\right)$. Since $\mathcal{G}_{1}$ is alphabetic and chainfree, we have $\operatorname{pos}(d)=\operatorname{pos}\left(\xi[\alpha]_{\tilde{w}}\right)$ and, moreover,

$$
\left\{w_{1}, \ldots, w_{n}\right\}=\{u \in \operatorname{pos}(d) \mid d(u) \text { is an } \alpha \text {-rule }\}
$$

First we define a tree $d^{\prime} \in \mathrm{T}_{R}$ by specifying a tree domain and an $R$-tree mapping and then using the bijective representation of trees as tree domains (cf. Section 2.9). We define the tree domain

$$
\begin{equation*}
W=\operatorname{pos}(d) \cup \bigcup_{i \in[n]}\left\{w_{i} 1 v \mid v \in \operatorname{pos}\left(d_{i}\right)\right\} \tag{10.18}
\end{equation*}
$$

and we define the mapping $d^{\prime}: W \rightarrow \Sigma$ for each $v \in W$ by

$$
d^{\prime}(v)= \begin{cases}d(v) & \text { if }(\forall i \in[n]): \neg\left(w_{i} \leq_{\text {pref }} v\right)  \tag{10.19}\\ d_{i}(u) & \text { if }(\exists i \in[n])\left(\exists u \in \mathbb{N}^{+}\right): v=w_{i} 1 u \\ t_{i} & \text { if }(\exists i \in[n]): v=w_{i}\end{cases}
$$

where $t_{i}$ is defined as follows: if $d\left(w_{i}\right)=(A \rightarrow \alpha)$, then $t_{i}=\left(A \rightarrow S_{2}\right)$.
Since $W$ is a tree domain and $d^{\prime}$ is an $R$-tree mapping, we can view $d^{\prime}$ as a tree over $R$, i.e., $d^{\prime} \in \mathrm{T}_{R}$. It is obvious that $d^{\prime} \in \operatorname{RT}_{\mathcal{G}}^{\widetilde{w}}(\xi)$. Then we define $\Phi$ by letting

$$
\begin{equation*}
\Phi\left(d, d_{1}, \ldots, d_{k}\right)=d^{\prime} \tag{10.20}
\end{equation*}
$$

It is easy to see that $\Phi$ is bijective. To show that it is injective, we assume that (10.20) holds and that also $\Phi\left(\hat{d}, \hat{d}_{1}, \ldots, \hat{d}_{k}\right)=d^{\prime}$ for some $\hat{d} \in \operatorname{RT}_{\mathcal{G}_{1}}(\xi[\alpha] \widetilde{w}), \hat{d}_{1} \in \operatorname{RT}_{\mathcal{G}_{2}}\left(\left.\xi\right|_{w_{1}}\right), \ldots, \hat{d}_{n} \in \operatorname{RT}_{\mathcal{G}_{2}}\left(\left.\xi\right|_{w_{n}}\right)$. We prove that $d=\hat{d}$ by contradiction. If $d \neq \hat{d}$, then there exists an $u \in \operatorname{pos}(d) \cap \operatorname{pos}(\hat{d})$ such that $d(u) \neq \hat{d}(u)$. We note that $d(u), \hat{d}(u) \in R_{1}$ and that, by (10.18), also $u \in \operatorname{pos}\left(d^{\prime}\right)$. The following four cases are possible:

- Neither $d(u)$ nor $\hat{d}(u)$ is an $\alpha$-rule. Then, by (10.19), $d^{\prime}(u)=d(u)$ and $d^{\prime}(u)=\hat{d}(u)$, which is a contradiction.
- $d(u)=(A \rightarrow \alpha)$ and $\hat{d}(u)$ is not an $\alpha$-rule. Then, by (10.19), $d^{\prime}(u)=\left(A \rightarrow S_{2}\right)$. On the other hand, $d^{\prime}(u)=\hat{d}(u)$ is a rule in $R_{1}$, a contradiction.
- $d(u)$ is not an $\alpha$-rule and $\hat{d}(u)$ is an $\alpha$-rule. By symmetry, this also leads to a contradiction.
- $d(u)=(A \rightarrow \alpha)$ and $\hat{d}(u)=(B \rightarrow \alpha)$, where $A \neq B$. Then by (10.19), $d^{\prime}(u)=\left(A \rightarrow S_{2}\right)$ and also $d^{\prime}(u)=\left(B \rightarrow S_{2}\right)$, which is a contradiction.
So we have $d=\hat{d}$. Then, by (10.18) and (10.19), $d_{i}=\hat{d}_{i}$ which proves that $\Phi$ is injective. Also, it is surjective, because for each $d^{\prime} \in \operatorname{RT}_{\mathcal{G}}^{\widetilde{w}}(\xi)$, the positions $w_{1}, \ldots, w_{n}$ determine $d \in \mathrm{RT}_{\mathcal{G}_{1}}\left(\xi[\alpha]_{\widetilde{w}}\right)$, $d_{1} \in \operatorname{RT}_{\mathcal{G}_{2}}\left(\left.\xi\right|_{w_{1}}\right), \ldots, d_{n} \in \operatorname{RT}_{\mathcal{G}_{2}}\left(\left.\xi\right|_{w_{n}}\right)$ such that $\Phi\left(d, d_{1}, \ldots, d_{k}\right)=d^{\prime}$. Thus $\Phi$ is bijective.

Moreover, if (10.20) holds, then

$$
\begin{equation*}
\mathrm{wt}_{\mathcal{G}}\left(d^{\prime}\right)=\mathrm{wt}_{\mathcal{G}_{1}}(d) \otimes \bigotimes_{i \in[n]} \mathrm{wt}_{\mathcal{G}_{2}}\left(d_{i}\right) \tag{10.21}
\end{equation*}
$$

because $B$ is commutative.
Then we can compute as follows (keeping in mind that $\mathcal{G}_{1}, \mathcal{G}_{2}$, and $\mathcal{G}$ are finite-derivational if B is not $\sigma$-complete):

$$
\begin{aligned}
&\left(\llbracket \mathcal{G}_{1} \rrbracket \circ_{\alpha} \llbracket \mathcal{G}_{2} \rrbracket\right)(\xi) \\
&=\bigoplus_{\widetilde{w}=\left(w_{1}, \ldots, w_{n}\right) \in \operatorname{cut}_{\alpha}(\xi)}^{\llbracket \mathcal{G}_{1} \rrbracket\left(\xi[\alpha]_{\widetilde{w}}\right) \otimes \llbracket \mathcal{G}_{2} \rrbracket\left(\left.\xi\right|_{w_{1}}\right) \otimes \ldots \otimes \llbracket \mathcal{G}_{2} \rrbracket\left(\left.\xi\right|_{w_{n}}\right)} \\
&=\bigoplus_{\widetilde{w}=\left(w_{1}, \ldots, w_{n}\right) \in \operatorname{cut}_{\alpha}(\xi)}\left(\sum_{d \in \mathrm{RT}_{\mathcal{G}_{1}}(\xi[\alpha] \widetilde{w})}^{\oplus} \mathrm{wt}_{\mathcal{G}_{1}}(d)\right) \otimes \bigotimes_{i \in[n]}\left(\sum_{d_{i} \in \mathrm{RT}_{\mathcal{G}_{2}}\left(\left.\xi\right|_{w_{i}}\right)}^{\oplus} \mathrm{wt}_{\mathcal{G}_{2}}\left(d_{i}\right)\right)
\end{aligned}
$$

$$
\left.\left.\begin{array}{l}
=\bigoplus_{\widetilde{w}=\left(w_{1}, \ldots, w_{n}\right) \in \operatorname{cut}_{\alpha}(\xi)}^{\substack{d \in \mathrm{RT}_{\left(\mathcal{G}_{1}(\xi[\alpha] \widetilde{w}),\right.}^{(\forall i \in[n]): d_{i} \in \mathrm{RT}_{\mathcal{G}_{2}}\left(\left.\xi\right|_{w_{i}}\right)}}} \sum_{\substack{\oplus}}\left(\mathrm{wt}_{\mathcal{G}_{1}}(d) \otimes \bigotimes_{i \in[n]} \mathrm{wt}_{\mathcal{G}_{2}}\left(d_{i}\right)\right) \quad \text { (by distributivity) } \\
=\bigoplus_{\widetilde{w} \in \operatorname{cut}_{\alpha}(\xi)} \sum_{d^{\prime} \in \mathrm{RT}_{\mathcal{G}}^{\tilde{w}}(\xi)}^{\oplus} \mathrm{wt}_{\mathcal{G}}\left(d^{\prime}\right) \\
=\sum_{d^{\prime} \in \mathrm{RT}_{\mathcal{G}}(\xi)}^{\oplus} \mathrm{wt}_{\mathcal{G}}\left(d^{\prime}\right)  \tag{10.17}\\
=\llbracket \mathcal{G} \rrbracket(\xi) .
\end{array} \quad \text { (because } \Phi \text { is a bijection and by (10.21)) }\right) \text { (by (10.17)) }\right)
$$

Thus $\llbracket \mathcal{G} \rrbracket=\llbracket \mathcal{G}_{1} \rrbracket \circ_{\alpha} \llbracket \mathcal{G}_{2} \rrbracket$.

Proof of (2): Now let $\mathcal{G}_{1}$ and $\mathcal{G}_{2}$ finite-derivational. Then, by Lemma 9.2.1(3), we can even construct equivalent chain-free wrtg. Consequently, the definition of $\mathcal{G}$, as it is given in the proof of (1), is constructive. Finally, as pointed out in the proof of (1), the $\operatorname{wrtg} \mathcal{G}$ is finite-derivational.

Corollary 10.6.2. (cf. DPV05, Lm. 6.5]) Let $\Sigma$ be a ranked alphabet, B be a commutative semiring, and $\mathcal{A}_{1}$ and $\mathcal{A}_{2}$ be two $(\Sigma, \mathrm{B})$-wta. Moreover, let $\alpha \in \Sigma^{(0)}$. Then we can construct a $(\Sigma, \mathrm{B})$-wta $\mathcal{A}$ such that $\llbracket \mathcal{A} \rrbracket=\llbracket \mathcal{A}_{1} \rrbracket \circ_{\alpha} \llbracket \mathcal{A}_{2} \rrbracket$. Thus, in particular, if B is a commutative semiring, then the $\operatorname{set} \operatorname{Rec}(\Sigma, \mathrm{B})$ is closed under tree concatenations.

Proof. By Lemma 9.2 .6 we can construct $(\Sigma, \mathrm{B})$-wrtg $\mathcal{G}_{1}$ and $\mathcal{G}_{2}$ such that $\mathcal{G}_{1}$ and $\mathcal{G}_{2}$ are in tree automata form and $\llbracket \mathcal{A}_{1} \rrbracket=\llbracket \mathcal{G}_{1} \rrbracket$ and $\llbracket \mathcal{A}_{2} \rrbracket=\llbracket \mathcal{G}_{2} \rrbracket$. Then, in particular, $\mathcal{G}_{1}$ and $\mathcal{G}_{2}$ are finite-derivational. By Theorem 10.6 .1 we can construct a finite-derivational $(\Sigma, \mathrm{B})$-wrtg $\mathcal{G}$ such that $\llbracket \mathcal{G} \rrbracket=\llbracket \mathcal{G}_{1} \rrbracket \circ_{\alpha} \llbracket \mathcal{G}_{2} \rrbracket$. Then, by Lemma 9.2 .8 , we can construct a $(\Sigma, \mathrm{B})$-wta $\mathcal{A}$ such that $\llbracket \mathcal{G} \rrbracket=\llbracket \mathcal{A} \rrbracket$.

### 10.7 Closure under Kleene-stars

In this section we recall the definition of Kleene-star and show that $\operatorname{Rec}(\Sigma, \mathrm{B})$ is closed under Kleene-stars. We follow the approach developed in BR82, Eng03, DPV05.

Let $r: \mathrm{T}_{\Sigma} \rightarrow B$ and $\alpha \in \Sigma^{(0)}$. We define the family $\left(r_{\alpha}^{\ell} \mid \ell \in \mathbb{N}\right)$ of weighted tree languages $r_{\alpha}^{\ell}: \mathrm{T}_{\Sigma} \rightarrow B$ (called the $\ell$-th iteration of $r$ at $\alpha$ ) by induction on $\mathbb{N}$ as follows:
I.B.: $r_{\alpha}^{0}=\widetilde{\mathbb{0}}$ and
I.S.: $r_{\alpha}^{\ell+1}=\left(r \circ_{\alpha} r_{\alpha}^{\ell}\right) \oplus \chi(\{\alpha\})$ for each $\ell \in \mathbb{N}$.

We mention that iteration can also be defined in different ways. For instance, if one generalizes the iteration of tree languages as presented in Eng75b, p. 38] to the weighted case, then this would read (cf. DPV05, Def. 3.7]):
I.B.: $r_{\alpha}^{\mathrm{E}, 0}=\chi(\{\alpha\})$ and
I.S.: $r_{\alpha}^{\mathrm{E}, \ell+1}=r_{\alpha}^{\mathrm{E}, \ell} \circ_{\alpha}(r \oplus \chi(\{\alpha\})$ for each $\ell \in \mathbb{N}$.

Another possibility is to generalize the iteration of tree languages as presented in [TW68, p. 66] to the weighted case; this reads as follows (cf. DPV05, Def. 3.7]):
I.B.: $r_{\alpha}^{\mathrm{TW}, 0}=\chi(\{\alpha\})$ and
I.S.: $r_{\alpha}^{\mathrm{TW}, \ell+1}=\left(r \oplus \chi(\{\alpha\}) \circ_{\alpha} r_{\alpha}^{\mathrm{TW}, \ell}\right.$ for each $\ell \in \mathbb{N}$.

For the Boolean semiring, all three definitions are equivalent. This is due to the facts that disjunction is idempotent and that tree concatenation is associative. If B is a commutative semiring, then $r_{\alpha}^{\mathrm{E}, \ell}=$ $r_{\alpha}^{\mathrm{TW}, \ell}$ for each $\ell \in \mathbb{N}$ (cf. [DPV05, Lm. 3.8]). If B is a commutative and idempotent semiring, then
$r_{\alpha}^{\ell} \leq r_{\alpha}^{\mathrm{TW}, \ell} \leq r_{\alpha}^{\ell+1}$ for each $\ell \in \mathbb{N}$, where $\leq$ is defined componentwise: for every $a, b \in B$, we let $a \leq b$ if $a \oplus b=b$. For a more detailed comparison we refer the reader to [DPV05, Sec. 3].

We call $r: \mathrm{T}_{\Sigma} \rightarrow B \alpha$-proper if $r(\alpha)=0$. For each $\alpha$-proper weighted tree language $r$, the application of the iteration of $r$ to a tree $\xi$ becomes stable after height $(\xi)+1$ steps.

Lemma 10.7.1. (cf. DPV05, Lm. 3.10]) Let $r: \mathrm{T}_{\Sigma} \rightarrow B$ be $\alpha$-proper. For every $\xi \in \mathrm{T}_{\Sigma}$ and $\ell \in \mathbb{N}$, if $\ell \geq \operatorname{height}(\xi)+1$, then $r_{\alpha}^{\ell+1}(\xi)=r_{\alpha}^{\ell}(\xi)$. In particular, for every $\ell \in \mathbb{N}_{+}$and $\beta \in \Sigma^{(0)} \backslash\{\alpha\}$, we have $r_{\alpha}^{\ell}(\alpha)=\mathbb{1}$ and $r_{\alpha}^{\ell}(\beta)=r(\beta)$.

Proof. We prove the first statement of the lemma by induction on $\left(\mathrm{T}_{\Sigma}, \prec_{\Sigma}^{+}\right)$; the second statement of the lemma is proved in the induction base.
I.B.: Let $\xi \in \Sigma^{(0)}$. We distinguish two cases.


$$
\begin{aligned}
r_{\alpha}^{\ell}(\alpha) & =\left(\left(r \circ_{\alpha} r_{\alpha}^{\ell-1}\right) \oplus \chi(\{\alpha\})\right)(\alpha)=\left(r \circ_{\alpha} r_{\alpha}^{\ell-1}\right)(\alpha) \oplus \chi(\{\alpha\})(\alpha) \\
& =r(\alpha) \otimes r_{\alpha}^{\ell-1}(\alpha) \oplus \mathbb{1}=\mathbb{0} \otimes r_{\alpha}^{\ell-1}(\alpha) \oplus \mathbb{1}=\mathbb{1}
\end{aligned}
$$

where the third equality is due to the facts that $\operatorname{cut}_{\alpha}(\xi)=\{(\varepsilon)\}$ and $\chi(\{\alpha\})(\alpha)=\mathbb{1}$. Since height $(\xi)+1=$ 1 , this implies that, for each $\ell \geq \operatorname{height}(\xi)+1$, we have $r_{\alpha}^{\ell+1}(\alpha)=r_{\alpha}^{\ell}(\alpha)$.


$$
\begin{aligned}
r_{\alpha}^{\ell}(\xi) & =\left(\left(r \circ_{\xi} r_{\alpha}^{\ell-1}\right) \oplus \chi(\{\alpha\})\right)(\xi)=\left(r \circ_{\alpha} r_{\alpha}^{\ell-1}\right)(\xi) \oplus \chi(\{\alpha\})(\xi)=\left(r \circ_{\alpha} r_{\alpha}^{\ell-1}\right)(\xi) \\
& =r(\xi) \oplus r(\alpha) \otimes r_{\alpha}^{\ell-1}(\xi)=r(\xi) \oplus \mathbb{O} \otimes r_{\alpha}^{\ell-1}(\xi)=r(\xi)
\end{aligned}
$$

where the fourth equality is due to the fact that $\operatorname{cut}_{\alpha}(\xi)=\{(),(\varepsilon)\}$. Since height $(\xi)+1=1$, this implies that, for each $\ell \geq \operatorname{height}(\xi)+1$, we have $r_{\alpha}^{\ell+1}(\xi)=r_{\alpha}^{\ell}(\xi)$.
I.S.: Now let $\xi=\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)$ with $k \in \mathbb{N}_{+}$. Let $\ell \in \mathbb{N}$ with $\ell \geq \operatorname{height}(\xi)+1$.

$$
\begin{aligned}
r_{\alpha}^{\ell+1}(\xi)= & \left(r \circ r_{\alpha}^{\ell}\right)(\xi) \oplus \chi(\{\alpha\})(\xi)=\left(r \circ r_{\alpha}^{\ell}\right)(\xi) \\
= & \bigoplus_{\left(w_{1}, \ldots, w_{n}\right) \in \operatorname{cut}_{\alpha}(\xi)} r\left(\xi[\alpha]_{\left(w_{1}, \ldots, w_{n}\right)}\right) \otimes r_{\alpha}^{\ell}\left(\left.\xi\right|_{w_{1}}\right) \otimes \ldots \otimes r_{\alpha}^{\ell}\left(\left.\xi\right|_{w_{n}}\right) \\
= & r\left(\xi[\alpha]_{(\varepsilon)}\right) \otimes r_{\alpha}^{\ell}\left(\left.\xi\right|_{\varepsilon}\right) \oplus \\
& \bigoplus_{\left(w_{1}, \ldots, w_{n}\right) \in \operatorname{cut}_{\alpha}(\xi) \backslash\{(\varepsilon)\}} r\left(\xi[\alpha]_{\left(w_{1}, \ldots, w_{n}\right)}\right) \otimes r_{\alpha}^{\ell}\left(\left.\xi\right|_{w_{1}}\right) \otimes \ldots \otimes r_{\alpha}^{\ell}\left(\left.\xi\right|_{w_{n}}\right) \\
= & r(\alpha) \otimes r_{\alpha}^{\ell}\left(\left.\xi\right|_{\varepsilon}\right) \oplus \bigoplus_{\left(w_{1}, \ldots, w_{n}\right) \in \operatorname{cut}_{\alpha}(\xi) \backslash\{(\varepsilon)\}} r\left(\xi[\alpha]_{\left(w_{1}, \ldots, w_{n}\right)}\right) \otimes r_{\alpha}^{\ell}\left(\left.\xi\right|_{w_{1}}\right) \otimes \ldots \otimes r_{\alpha}^{\ell}\left(\left.\xi\right|_{w_{n}}\right) \\
= & \left.\bigoplus_{\left(w_{1}, \ldots, w_{n}\right) \in \operatorname{cut}_{\alpha}(\xi) \backslash\{(\varepsilon)\}} r\left(\xi[\alpha]_{\left(w_{1}, \ldots, w_{n}\right)}\right) \otimes r_{\alpha}^{\ell}\left(\left.\xi\right|_{w_{1}}\right) \otimes \ldots \otimes r_{\alpha}^{\ell}\left(\left.\xi\right|_{w_{n}}\right) \quad \quad \text { (because } r(\alpha)=\mathbb{O}\right) \\
= & \bigoplus_{\left(w_{1}, \ldots, w_{n}\right) \in \operatorname{cut}_{\alpha}(\xi) \backslash\{(\varepsilon)\}} r\left(\xi[\alpha]_{\left(w_{1}, \ldots, w_{n}\right)}\right) \otimes r_{\alpha}^{\ell-1}\left(\left.\xi\right|_{w_{1}}\right) \otimes \ldots \otimes r_{\alpha}^{\ell-1}\left(\left.\xi\right|_{w_{n}}\right)
\end{aligned}
$$

(because, for each $i \in[\ell]$, we have $\left|w_{i}\right| \geq 1$ and hence $\ell-1 \geq \operatorname{height}\left(\left.\xi\right|_{w_{i}}\right)+1$, and then by I.H.)

$$
=\bigoplus_{\left(w_{1}, \ldots, w_{n}\right) \in \operatorname{cut}_{\alpha}(\xi)} r\left(\xi[\alpha]_{\left(w_{1}, \ldots, w_{n}\right)}\right) \otimes r_{\alpha}^{\ell-1}\left(\left.\xi\right|_{w_{1}}\right) \otimes \ldots \otimes r_{\alpha}^{\ell-1}\left(\left.\xi\right|_{w_{n}}\right)
$$

$$
\text { (because } r\left(\xi[\alpha]_{(\varepsilon)}\right) \otimes r_{\alpha}^{\ell-1}\left(\left.\xi\right|_{\varepsilon}\right)=r(\alpha) \otimes r_{\alpha}^{\ell-1}\left(\left.\xi\right|_{\varepsilon}\right)=\mathbb{O} \text { as above) }
$$

$$
=\left(r \circ_{\alpha} r_{\alpha}^{\ell-1}\right)(\xi)=\left(r \circ r_{\alpha}^{\ell-1}\right)(\xi) \oplus \chi(\{\alpha\})(\xi)=r_{\alpha}^{\ell}(\xi)
$$

Lemma 10.7 .1 justifies to define the operation Kleene-star as follows. Let $r: \mathrm{T}_{\Sigma} \rightarrow B$ be $\alpha$-proper. The $\alpha$-Kleene star of $r$, denoted by $r_{\alpha}^{*}$, is the weighted tree language $r_{\alpha}^{*}: \mathrm{T}_{\Sigma} \rightarrow B$ defined, for every $\xi \in \mathrm{T}_{\Sigma}$, by $r_{\alpha}^{*}(\xi)=r_{\alpha}^{\text {height }(\xi)+1}(\xi)$ Eng03 (cf. DPV05, Def. 3.11]). Thus, in particular, by Lemma 10.7.1, we have

$$
\begin{equation*}
r_{\alpha}^{*}(\alpha)=r_{\alpha}^{1}(\alpha)=\mathbb{1} \tag{10.22}
\end{equation*}
$$

Moreover, for each $\beta \in \Sigma^{(0)}$ with $\beta \neq \alpha$, by Lemma 10.7.1, we have

$$
\begin{equation*}
r_{\alpha}^{*}(\beta)=r_{\alpha}^{1}(\beta)=r(\beta) \tag{10.23}
\end{equation*}
$$

Example 10.7.2. Let $\Sigma=\left\{\sigma^{(2)}, \alpha^{(0)}\right\}$. We consider the ( $\Sigma, \mathrm{Nat}$ )-weighted tree language $r=1 . \sigma(\alpha, \alpha)$, which is a monomial, and compute $r_{\alpha}^{*}(\sigma(\alpha, \alpha))$ :

$$
\begin{aligned}
& r_{\alpha}^{*}(\sigma(\alpha, \alpha))=r_{\alpha}^{2}(\sigma(\alpha, \alpha))=\left(r \circ_{\alpha} r_{\alpha}^{1}\right)(\sigma(\alpha, \alpha))+\chi(\{\alpha\})(\sigma(\alpha, \alpha)) \\
& =\left(r \circ{ }_{\alpha} r_{\alpha}^{1}\right)(\sigma(\alpha, \alpha)) \\
& =r(\alpha) \cdot r_{\alpha}^{1}(\sigma(\alpha, \alpha))+r(\sigma(\alpha, \alpha)) \cdot r_{\alpha}^{1}(\alpha) \cdot r_{\alpha}^{1}(\alpha) \quad\left(\text { because } \operatorname{cut}_{\alpha}(\sigma(\alpha, \alpha))=\{(\varepsilon),(1,2)\}\right) \\
& =r(\sigma(\alpha, \alpha)) \cdot r_{\alpha}^{1}(\alpha) \cdot r_{\alpha}^{1}(\alpha) \quad(\text { because } r(\alpha)=0) \\
& =1 \cdot r_{\alpha}^{1}(\alpha) \cdot r_{\alpha}^{1}(\alpha) \\
& =1 \text {. } \\
& \text { (by Lemma 10.7.1) }
\end{aligned}
$$

We might wish to compare the calculation of $r_{\alpha}^{2}(\sigma(\alpha, \alpha))$ with that of $r_{\alpha}^{3}(\sigma(\alpha, \alpha))$ :

$$
\begin{array}{lr}
r_{\alpha}^{3}(\sigma(\alpha, \alpha))=\left(r \circ_{\alpha} r_{\alpha}^{2}\right)(\sigma(\alpha, \alpha))+\chi(\{\alpha\})(\sigma(\alpha, \alpha)) & \\
=\left(r \circ_{\alpha} r_{\alpha}^{2}\right)(\sigma(\alpha, \alpha)) & \\
=r(\alpha) \cdot r_{\alpha}^{2}(\sigma(\alpha, \alpha))+r(\sigma(\alpha, \alpha)) \cdot r_{\alpha}^{2}(\alpha) \cdot r_{\alpha}^{2}(\alpha) & \\
=r(\sigma(\alpha, \alpha)) \cdot r_{\alpha}^{2}(\alpha) \cdot r_{\alpha}^{2}(\alpha) & \\
=1 \cdot r_{\alpha}^{2}(\alpha) \cdot r_{\alpha}^{2}(\alpha) & \\
=1 . & \text { (because cut } \left.{ }_{\alpha}(\sigma(\alpha, \alpha))=\{(\varepsilon),(1,2)\}\right) \\
\end{array}
$$

Next we compute $r_{\alpha}^{*}(\sigma(\alpha, \sigma(\alpha, \alpha)))$ :

$$
\begin{aligned}
r_{\alpha}^{*} & (\sigma(\alpha, \sigma(\alpha, \alpha)))=r_{\alpha}^{3}(\sigma(\alpha, \sigma(\alpha, \alpha))) \\
= & \left(r \circ_{\alpha} r_{\alpha}^{2}\right)(\sigma(\alpha, \sigma(\alpha, \alpha)))+\chi(\{\alpha\})(\sigma(\alpha, \sigma(\alpha, \alpha))) \\
= & \left(r \circ_{\alpha} r_{\alpha}^{2}\right)(\sigma(\alpha, \sigma(\alpha, \alpha))) \\
= & r(\alpha) \cdot r_{\alpha}^{2}(\sigma(\alpha, \sigma(\alpha, \alpha))) \\
& +r(\sigma(\alpha, \alpha)) \cdot r_{\alpha}^{2}(\alpha) \cdot r_{\alpha}^{2}(\sigma(\alpha, \alpha)) \\
& +r(\sigma(\alpha, \sigma(\alpha, \alpha))) \cdot r_{\alpha}^{2}(\alpha) \cdot r_{\alpha}^{2}(\alpha) \cdot r_{\alpha}^{2}(\alpha) \\
=r(\sigma(\alpha, \alpha)) \cdot r_{\alpha}^{2}(\alpha) \cdot r_{\alpha}^{2}(\sigma(\alpha, \alpha)) & \text { (because } \left.c^{2} t_{\alpha}(\sigma(\alpha, \sigma(\alpha, \alpha)))=\{(\varepsilon),(1,2),(1,21,22)\}\right) \\
=r_{\alpha}^{2}(\alpha) \cdot r_{\alpha}^{2}(\sigma(\alpha, \alpha)) & \quad \text { (because } r(\alpha)=r(\sigma(\alpha, \sigma(\alpha, \alpha)))=0) \\
=r_{\alpha}^{2}(\sigma(\alpha, \alpha)) . & \text { (because } r(\sigma(\alpha, \alpha))=1) \\
=1 & \text { (because } \left.r_{\alpha}^{2}(\alpha)=1\right) \\
& \text { (as above) })
\end{aligned}
$$

A set $\mathcal{L}$ of B -weighted tree languages is closed under Kleene-stars if the following holds: for each ( $\Sigma, \mathrm{B}$ )-weighted tree language $r$ and for each $\alpha \in \Sigma^{(0)}$ such that $r$ is $\alpha$-proper, if $r \in \mathcal{L}$, then $r_{\alpha}^{*} \in \mathcal{L}$.

Lemma 10.7.3. DPV05, Lm. 3.13] Let $r: \mathrm{T}_{\Sigma} \rightarrow B$ be $\alpha$-proper. Then $r_{\alpha}^{*}=\left(r \circ_{\alpha} r_{\alpha}^{*}\right) \oplus \chi(\{\alpha\})$.

Proof. Let $\xi \in \mathrm{T}_{\Sigma}$ and let $\ell=\operatorname{height}(\xi)+1$. Then we can calculate as follows.

$$
r_{\alpha}^{*}(\xi)=r_{\alpha}^{\ell}(\xi)=r_{\alpha}^{\ell+1}(\xi)=\left(r \circ_{\alpha} r_{\alpha}^{\ell}\right)(\xi) \oplus \chi(\{\alpha\})(\xi)
$$

where the second equality holds by Lemma 10.7.1. Now we proceed with the subexpression $\left(r \circ_{\alpha} r_{\alpha}^{\ell}\right)(\xi)$ as follows.

$$
\begin{aligned}
& \left(r \circ_{\alpha} r_{\alpha}^{\ell}\right)(\xi) \\
& =\bigoplus_{\left(w_{1}, \ldots, w_{n}\right) \in \operatorname{cut}_{\alpha}(\xi)} r\left(\xi[\alpha]_{\left(w_{1}, \ldots, w_{n}\right)}\right) \otimes r_{\alpha}^{\ell}\left(\left.\xi\right|_{w_{1}}\right) \otimes \ldots \otimes r_{\alpha}^{\ell}\left(\left.\xi\right|_{w_{n}}\right) \\
& =\bigoplus_{\left(w_{1}, \ldots, w_{n}\right) \in \operatorname{cut}_{\alpha}(\xi) \backslash\{(\xi)\}} r\left(\xi[\alpha]_{\left(w_{1}, \ldots, w_{n}\right)}\right) \otimes r_{\alpha}^{\ell}\left(\left.\xi\right|_{w_{1}}\right) \otimes \ldots \otimes r_{\alpha}^{\ell}\left(\left.\xi\right|_{w_{n}}\right)
\end{aligned}
$$

$$
\text { (because } \left.r\left(\xi[\alpha]_{(\varepsilon)}\right) \otimes r_{\alpha}^{\ell}\left(\left.\xi\right|_{\varepsilon}\right)=r(\alpha) \otimes r_{\alpha}^{\ell}\left(\left.\xi\right|_{\varepsilon}\right)=\mathbb{O}\right)
$$

$$
=\bigoplus_{\left(w_{1}, \ldots, w_{n}\right) \in \operatorname{cut}_{\alpha}(\xi) \backslash\{(\varepsilon)\}} r\left(\xi[\alpha]_{\left(w_{1}, \ldots, w_{n}\right)}\right) \otimes r_{\alpha}^{\operatorname{height}\left(\left.\xi\right|_{w_{1}}\right)+1}\left(\left.\xi\right|_{w_{1}}\right) \otimes \ldots \otimes r_{\alpha}^{\operatorname{height}\left(\left.\xi\right|_{w_{n}}\right)+1}\left(\left.\xi\right|_{w_{n}}\right)
$$

$$
\text { (because, for each } i \in[n] \text {, we have }\left|w_{i}\right| \geq 1 \text { and hence } \ell \geq \operatorname{height}\left(\left.\xi\right|_{w_{i}}\right)+1, \text { ) }
$$

( and then by Lemma 10.7.1.)

$$
=\bigoplus_{\left(w_{1}, \ldots, w_{n}\right) \in \operatorname{cut}_{\alpha}(\xi) \backslash\{(\varepsilon)\}} r\left(\xi[\alpha]_{\left(w_{1}, \ldots, w_{n}\right)}\right) \otimes r_{\alpha}^{*}\left(\left.\xi\right|_{w_{1}}\right) \otimes \ldots \otimes r_{\alpha}^{*}\left(\left.\xi\right|_{w_{n}}\right) \quad \text { (by definition of } \alpha \text {-Kleene-star) }
$$

$$
=\bigoplus_{\left(w_{1}, \ldots, w_{n}\right) \in \operatorname{cut}_{\alpha}(\xi)} r\left(\xi[\alpha]_{\left(w_{1}, \ldots, w_{n}\right)}\right) \otimes r_{\alpha}^{*}\left(\left.\xi\right|_{w_{1}}\right) \otimes \ldots \otimes r_{\alpha}^{*}\left(\left.\xi\right|_{w_{n}}\right)
$$

$$
=\left(r \circ_{\alpha} r_{\alpha}^{*}\right)(\xi)
$$

Hence $r_{\alpha}^{*}(\xi)=\left(r \circ_{\alpha} r_{\alpha}^{\ell}\right)(\xi) \oplus \chi(\{\alpha\})(\xi)=\left(r \circ_{\alpha} r_{\alpha}^{*}\right)(\xi) \oplus \chi(\{\alpha\})(\xi)$, which proves the lemma.

In the next theorem we prove that, under certain conditions, the $\alpha$-Kleene star of a regular ( $\Sigma, \mathrm{B}$ )weighted tree language is regular. For the Boolean semiring this was proved in TW68, Lm. 12] (also cf. Eng75b, Thm. 3.41] and [GS84, Thm. 2.4.8]) and for wta this was proved in [DPV05, Lm. 6.7]. We follow the construction in [FV22b Lm. 6.7] (for the trivial storage type) and close a gap in the given correctness proof.

Theorem 10.7.4. Let B be a commutative semiring and let $\alpha \in \Sigma^{(0)}$. Let $\mathcal{G}$ be a ( $\left.\Sigma, \mathrm{B}\right)$-wrtg such that (a) $\llbracket \mathcal{G} \rrbracket$ is $\alpha$-proper and (b) $\mathcal{G}$ is finite-derivational or B is $\sigma$-complete. Then the following two statements hold.
(1) There exists a $(\Sigma, \mathrm{B})$-wrtg $\mathcal{G}^{\prime}$ such that $\llbracket \mathcal{G}^{\prime} \rrbracket=\llbracket \mathcal{G} \rrbracket_{\alpha}^{*}$.
(2) If $\mathcal{G}$ is finite-derivational, then we can construct a finite-derivational $(\Sigma, \mathrm{B})$-wrtg $\mathcal{G}^{\prime}$ such that $\llbracket \mathcal{G}^{\prime} \rrbracket=$ $\llbracket \mathcal{G} \rrbracket_{\alpha}^{*}$.

Proof. Proof of (1): Let $\mathcal{G}=(N, S, R, w t)$ be a $(\Sigma, \mathrm{B})$-wrtg and $\alpha \in \Sigma^{(0)}$ such that $\llbracket \mathcal{G} \rrbracket$ is $\alpha$-proper. By Lemmas 9.2.1 (1) and 9.2.2, we can assume that $\mathcal{G}$ is start-separated and alphabetic. By Lemma 9.2.1(3) we can furthermore assume that $\mathcal{G}$ is chain-free. Clearly, each mentioned construction preserves $\alpha$-properness. We call a rule of the form $A \rightarrow \alpha$ in $R$ an $\alpha$-rule.

Since $\mathcal{G}$ is chain-free, for each $\beta \in \Sigma^{(0)}$, we have $\operatorname{RT}_{\mathcal{G}}(\beta) \subseteq\{S \rightarrow \beta\}$, i.e., if $\mathrm{RT}_{\mathcal{G}}(\beta) \neq \emptyset$, then $d=(S \rightarrow \beta)$ is the only rule tree such that $\pi(d)=\beta$. Hence, if $(S \rightarrow \alpha)$ is in $R$, then

$$
w t(S \rightarrow \alpha)=\mathrm{wt}_{\mathcal{G}}(S \rightarrow \alpha)=\sum_{d \in \mathrm{RT}_{\mathcal{G}}(\alpha)}^{\oplus} \mathrm{wt}_{\mathcal{G}}(d)=\llbracket \mathcal{G} \rrbracket(\alpha)=\mathbb{0}
$$

where the last equality holds because $\mathcal{G}$ is $\alpha$-proper. Thus, $\llbracket \mathcal{G} \rrbracket=\llbracket \widetilde{\mathcal{G}} \rrbracket$ where $\widetilde{\mathcal{G}}$ is the $(\Sigma, \mathrm{B})$-wrtg $(N, S, \widetilde{R}, \widetilde{w t})$ with $\widetilde{R}=R \backslash\{S \rightarrow \alpha\}$ and $\widetilde{w t}=\left.w t\right|_{\widetilde{R}}$. Hence we can assume that $\mathcal{G}$ does not contain the rule $S \rightarrow \alpha$. Due to our assumptions on $\mathcal{G}$, for every $\xi \in \mathrm{T}_{\Sigma}$ and $d \in \mathrm{RT}_{\mathcal{G}}(\xi)$, we have $\operatorname{pos}(d)=\operatorname{pos}(\xi)$.

We define the $(\Sigma, \mathrm{B})$-wrtg $\mathcal{G}^{\prime}=\left(N^{\prime},\left\{S, S_{0}\right\}, R^{\prime}\right.$, wt'), where $S_{0} \notin N$ is a new nonterminal, $N^{\prime}=$ $N \cup\left\{S_{0}\right\}$, and $R^{\prime}=R \cup R_{\alpha} \cup R_{\text {init }}$, where

- For each rule $r \in R$ we let $w t^{\prime}(r)=w t(r)$.
- $R_{\alpha}$ : If $r=(A \rightarrow \alpha)$ is in $R$, then $r^{\prime}=(A \rightarrow S)$ is in $R_{\alpha}$ with $w t^{\prime}\left(r^{\prime}\right)=w t(r)$.
- $R_{\text {init }}$ : This set contains the rule $S_{0} \rightarrow \alpha$ with $w t^{\prime}\left(S_{0} \rightarrow \alpha\right)=\mathbb{1}$.

We call each rule of the form $A \rightarrow S$ an $S$-rule.
For each $\xi \in \mathrm{T}_{\Sigma}$, we have

$$
\operatorname{RT}_{\mathcal{G}^{\prime}}(\xi)= \begin{cases}\left\{S_{0} \rightarrow \alpha\right\} & \text { if } \xi=\alpha  \tag{10.24}\\ \operatorname{RT}_{\mathcal{G}^{\prime}}(S, \xi) & \text { otherwise }\end{cases}
$$

Moreover, since $\mathcal{G}$ does not contain chain rules and the rule $S \rightarrow \alpha$, it is easy to see that $\mathrm{RT}_{\mathcal{G}^{\prime}}(S, \beta)=$ $\mathrm{RT}_{\mathcal{G}}(\beta)$.

Let $\xi \in \mathrm{T}_{\Sigma}$ and $\widetilde{w}=\left(w_{1}, \ldots, w_{n}\right) \in \operatorname{cut}_{\alpha}(\xi)$. We define

$$
\begin{aligned}
& \operatorname{RT}_{\mathcal{G}^{\prime}}^{\widetilde{w}}(S, \xi)=\left\{d^{\prime} \in \operatorname{RT}_{\mathcal{G}^{\prime}}(S, \xi) \mid(\forall i \in[n]): w_{i} \in \operatorname{pos}\left(d^{\prime}\right) \wedge\left(d^{\prime}\left(w_{i}\right) \text { is an } \alpha \text {-rule or a } S \text {-rule }\right)\right. \\
&\left.\wedge\left(\left(\forall v \in \operatorname{prefix}\left(w_{i}\right) \backslash\left\{w_{i}\right\}\right): d^{\prime}(v) \text { is not a } S \text {-rule }\right)\right\}
\end{aligned}
$$

We have that

$$
\begin{equation*}
\left(\operatorname{RT}_{\mathcal{G}^{\prime}}^{\widetilde{w}}(S, \xi) \mid \widetilde{w} \in \operatorname{cut}_{\alpha}(\xi)\right) \text { is a partitioning of } \mathrm{RT}_{\mathcal{G}^{\prime}}(S, \xi) \tag{10.25}
\end{equation*}
$$

In particular, we have $\mathrm{RT}_{\mathcal{G}^{\prime}}(S, \alpha)=\mathrm{RT}_{\mathcal{G}^{\prime}}^{(\varepsilon)}(S, \alpha)=\emptyset$.
Let $\left\{j_{1}, \ldots, j_{\ell}\right\}$ be the set of all $i \in[n]$ such that $\xi\left(w_{i}\right) \neq \alpha$. We denote the set $\left\{j_{1}, \ldots, j_{\ell}\right\}$ by iter $(\widetilde{w})$.
We define the mapping

$$
\Phi_{\widetilde{w}}: \operatorname{RT}_{\mathcal{G}}(\xi[\alpha] \widetilde{w}) \times \operatorname{RT}_{\mathcal{G}^{\prime}}\left(S,\left.\xi\right|_{w_{j_{1}}}\right) \times \ldots \times \mathrm{RT}_{\mathcal{G}^{\prime}}\left(S,\left.\xi\right|_{w_{j_{\ell}}}\right) \rightarrow \mathrm{RT}_{\mathcal{G}^{\prime}}^{\widetilde{w}}(S, \xi)
$$

as follows. Let $d \in \operatorname{RT}_{\mathcal{G}}\left(\xi[\alpha]_{\widetilde{w}}\right)$ and $d_{1} \in \operatorname{RT}_{\mathcal{G}^{\prime}}\left(S,\left.\xi\right|_{w_{j_{1}}}\right), \ldots, d_{\ell} \in \operatorname{RT}_{\mathcal{G}^{\prime}}\left(S,\left.\xi\right|_{w_{j_{\ell}}}\right)$. We define the tree

$$
d^{\prime}=d\left[\left(B_{1} \rightarrow S\right)\left(d_{1}\right)\right]_{w_{j_{1}}} \cdots\left[\left(B_{\ell} \rightarrow S\right)\left(d_{\ell}\right)\right]_{w_{j_{\ell}}}
$$

where, for each $i \in[\ell], B_{i}$ is the nonterminal on the left-hand side of the $\alpha$-rule $d\left(w_{j_{i}}\right)$. (Since $\operatorname{pos}\left(\xi[\alpha]_{\widetilde{w}}\right)=$ $\operatorname{pos}(d)$ and $\xi[\alpha]_{\widetilde{w}}\left(w_{j_{i}}\right)=\alpha$, the rule $d\left(w_{j_{i}}\right)$ is an $\alpha$-rule.) It is obvious that $y d^{\prime} \in \operatorname{RT}_{\mathcal{G}^{\prime}}^{\widetilde{w}}(S, \xi)$. Now we define $\Phi_{\widetilde{w}}\left(d, d_{1}, \ldots, d_{\ell}\right)=d^{\prime}$.

Also it is easy to see that $\Phi_{\widetilde{w}}$ is a bijection. Moreover,

$$
\begin{equation*}
\mathrm{wt}_{\mathcal{G}^{\prime}}\left(\Phi_{\widetilde{w}}\left(d, d_{1}, \ldots, d_{\ell}\right)\right)=\mathrm{wt}_{\mathcal{G}}(d) \otimes \bigotimes_{i \in[\ell]} \mathrm{wt}_{\mathcal{G}^{\prime}}\left(d_{i}\right) \tag{10.26}
\end{equation*}
$$

by the definitions of $R^{\prime}$ and $\mathrm{wt}_{\mathcal{G}^{\prime}}$ and by the fact that B is commutative.
By induction on $\left(\mathrm{T}_{\Sigma}, \prec_{\Sigma}^{+}\right)$, we prove that the following statement holds:

$$
\begin{equation*}
\text { For every } \xi \in \mathrm{T}_{\Sigma} \text {, we have: } \llbracket \mathcal{G}^{\prime} \rrbracket(\xi)=\llbracket \mathcal{G} \rrbracket_{\alpha}^{*}(\xi) \tag{10.27}
\end{equation*}
$$

(We recall that $\mathcal{G}$ is finite-derivational or B is $\sigma$-complete).
I.B.: Let $\xi \in \Sigma^{(0)}$. If $\xi \neq \alpha$, then

$$
\llbracket \mathcal{G}^{\prime} \rrbracket(\xi)=\sum_{d^{\prime} \in \mathrm{RT}_{\mathcal{G}^{\prime}}(\xi)}^{\oplus} \mathrm{wt}_{\mathcal{G}^{\prime}}\left(d^{\prime}\right)=\sum_{d^{\prime} \in \mathrm{RT}_{\mathcal{G}^{\prime}}(S, \xi)}^{\oplus} \mathrm{wt}_{\mathcal{G}^{\prime}}\left(d^{\prime}\right)=\sum_{d \in \mathrm{RT}_{\mathcal{G}}(\xi)}^{\oplus} \mathrm{wt}_{\mathcal{G}}(d)=\llbracket \mathcal{G} \rrbracket(\xi)=\llbracket \mathcal{G} \rrbracket_{\alpha}^{*}(\xi)
$$

where the last equality holds by (10.23).
Now let $\xi=\alpha$. Then we can calculate as follows:

$$
\llbracket \mathcal{G}^{\prime} \rrbracket(\alpha)=\mathrm{wt}_{\mathcal{G}^{\prime}}\left(\left(S_{0} \rightarrow \alpha\right)\right)=\mathbb{1}=\llbracket \mathcal{G} \rrbracket_{\alpha}^{*}(\alpha),
$$

where the first equality holds by (10.24) and the last equality holds by (10.22).
I.S.: Let $\xi \in\left(\mathrm{T}_{\Sigma} \backslash \Sigma^{(0)}\right)$. We can calculate as follows.

$$
\text { (because } \Phi_{\widetilde{w}} \text { is bijective and by (10.26), where iter }(\widetilde{w})=\left\{j_{1}, \ldots, j_{\ell}\right\} \text { ) }
$$

$$
=\bigoplus_{\widetilde{w}=\left(w_{1}, \ldots, w_{n}\right) \in \operatorname{cut}_{\alpha}(\xi)}\left(\sum_{d \in \mathrm{RT}_{\mathcal{G}}\left(\xi[\alpha]_{\widetilde{w}}\right)}^{\oplus} \mathrm{wt}_{\mathcal{G}}(d)\right) \otimes \bigotimes_{i \in[\ell]}\left(\sum_{d_{i} \in \mathrm{RT}_{\mathcal{G}^{\prime}}\left(S,\left.\xi\right|_{w_{j_{i}}}\right)}^{\oplus} \mathrm{wt}_{\mathcal{G}^{\prime}}\left(d_{i}\right)\right) \quad \text { (by distributivity) }
$$

$$
=\bigoplus_{\widetilde{w}=\left(w_{1}, \ldots, w_{n}\right) \in \operatorname{cut}_{\alpha}(\xi)} \llbracket \mathcal{G} \rrbracket\left(\xi[\alpha]_{\widetilde{w}}\right) \otimes \bigotimes_{i \in[\ell]}\left(\sum_{d_{i} \in \mathrm{RT}_{\mathcal{G}^{\prime}}\left(S, \xi \mid w_{j_{i}}\right)}^{\oplus} \mathrm{wt}_{\mathcal{G}^{\prime}}\left(d_{i}\right)\right)
$$

$$
\left.=\bigoplus_{\widetilde{w}=\left(w_{1}, \ldots, w_{n}\right) \in \operatorname{cut}_{\alpha}(\xi)} \llbracket \mathcal{G} \rrbracket\left(\xi[\alpha]_{\widetilde{w}}\right) \otimes \bigotimes_{i \in[\ell]}\left(\sum_{d_{i} \in \mathrm{RT}_{\mathcal{G}^{\prime}}\left(\left.\xi\right|_{w_{j_{i}}}\right)}^{\oplus} \mathrm{wt}_{\mathcal{G}^{\prime}}\left(d_{i}\right)\right) \quad \text { (by (10.24)} \text { because }\left.\xi\right|_{w_{j_{i}}} \neq \alpha\right)
$$

$$
=\bigoplus_{\widetilde{w}=\left(w_{1}, \ldots, w_{n}\right) \in \operatorname{cut}_{\alpha}(\xi)} \llbracket \mathcal{G} \rrbracket\left(\xi[\alpha]_{\widetilde{w}}\right) \otimes \bigotimes_{i \in[\ell]} \llbracket \mathcal{G}^{\prime} \rrbracket\left(\left.\xi\right|_{w_{j_{i}}}\right)
$$

$$
=\bigoplus_{\widetilde{w}=\left(w_{1}, \ldots, w_{n}\right) \in \operatorname{cut}_{\alpha}(\xi)} \llbracket \mathcal{G} \rrbracket\left(\xi[\alpha]_{\widetilde{w}}\right) \otimes \llbracket \mathcal{G}^{\prime} \rrbracket\left(\left.\xi\right|_{w_{1}}\right) \otimes \ldots \otimes \llbracket \mathcal{G}^{\prime} \rrbracket\left(\left.\xi\right|_{w_{n}}\right)
$$

$$
\text { (because for each } i \in[n] \backslash \operatorname{iter}(\widetilde{w}) \text { we have }\left.\xi\right|_{w_{i}}=\alpha \text { and } \llbracket \mathcal{G}^{\prime} \rrbracket(\alpha)=\mathbb{1} \text { ) }
$$

$$
\left.=\bigoplus_{\substack{\widetilde{w}=\left(w_{1}, \ldots, w_{n}\right) \in \operatorname{cut}_{\alpha}(\xi): \\ \widetilde{w} \neq(\varepsilon)}} \llbracket \mathcal{G} \rrbracket\left(\xi[\alpha]_{\widetilde{w}}\right) \otimes \llbracket \mathcal{G}^{\prime} \rrbracket\left(\left.\xi\right|_{w_{1}}\right) \otimes \ldots \otimes \llbracket \mathcal{G}^{\prime} \rrbracket\left(\left.\xi\right|_{w_{n}}\right) \quad \text { (because } \llbracket \mathcal{G} \rrbracket\left(\xi[\alpha]_{(\varepsilon)}\right)=\llbracket \mathcal{G} \rrbracket(\alpha)=\mathbb{O}\right)
$$

$$
=\bigoplus_{\widetilde{w}=\left(w_{1}, \ldots, w_{n}\right) \in \operatorname{cut}_{\alpha}(\xi):} \llbracket \mathcal{G} \rrbracket\left(\xi[\alpha]_{\widetilde{w}}\right) \otimes \llbracket(\varepsilon) \in \mathbb{G} \rrbracket_{\alpha}^{*}\left(\left.\xi\right|_{w_{1}}\right) \otimes \ldots \otimes \llbracket \mathcal{G} \rrbracket_{\alpha}^{*}\left(\left.\xi\right|_{w_{n}}\right)
$$

(by I.H.: since $\widetilde{w} \neq(\varepsilon)$, we have $\left.\xi\right|_{w_{i}} \prec_{\Sigma}^{+} \xi$ for every $i \in[n]$ )

$$
\begin{aligned}
& \llbracket \mathcal{G}^{\prime} \rrbracket(\xi)=\sum_{d^{\prime} \in \mathrm{RT}_{\mathcal{G}^{\prime}}(\xi)}^{\oplus} \mathrm{wt}_{\mathcal{G}^{\prime}}\left(d^{\prime}\right) \\
& =\bigoplus_{\widetilde{w} \in \operatorname{cut}_{\alpha}(\xi)} \sum_{d^{\prime} \in \mathrm{RT}_{\mathcal{G}^{\prime}}^{\tilde{w}}(S, \xi)}^{\oplus} \mathrm{wt}_{\mathcal{G}^{\prime}}\left(d^{\prime}\right) \\
& =\bigoplus_{\widetilde{w}=\left(w_{1}, \ldots, w_{n}\right) \in \operatorname{cut}_{\alpha}(\xi)} \sum_{\substack{d \in \operatorname{RT}_{\mathcal{G}}(\xi[\alpha] \tilde{w}),(\forall i \in[\ell]): d_{i} \in \mathrm{RT}_{\mathcal{G}^{\prime}}\left(S,\left.\xi\right|_{w_{j_{i}}}\right)}}^{\oplus}\left(\mathrm{wt}_{\mathcal{G}}(d) \otimes \bigotimes_{i \in[\ell]} \mathrm{wt}_{\mathcal{G}^{\prime}}\left(d_{i}\right)\right)
\end{aligned}
$$

$$
\begin{array}{lr}
=\bigoplus_{\tilde{w}=\left(w_{1}, \ldots, w_{n}\right) \in \operatorname{cut}_{\alpha}(\xi)} \llbracket \mathcal{G} \rrbracket\left(\xi[\alpha]_{\widetilde{w}}\right) \otimes \llbracket \mathcal{G} \rrbracket_{\alpha}^{*}\left(\left.\xi\right|_{w_{1}}\right) \otimes \ldots \otimes \llbracket \mathcal{G} \rrbracket_{\alpha}^{*}\left(\left.\xi\right|_{w_{n}}\right) & \text { (because } \llbracket \mathcal{G} \rrbracket(\alpha)=0) \\
=\left(\llbracket \mathcal{G} \rrbracket \circ_{\alpha} \llbracket \mathcal{G} \rrbracket_{\alpha}^{*}\right)(\xi) & \\
=\left(\llbracket \mathcal{G} \rrbracket \circ_{\alpha} \llbracket \mathcal{G} \rrbracket_{\alpha}^{*} \oplus \chi(\{\alpha\})\right)(\xi) & \\
=\llbracket \mathcal{G} \rrbracket_{\alpha}^{*}(\xi) . & \text { (because } \xi \neq \alpha) \\
\text { (by Lemma 10.7.3) }
\end{array}
$$

This finishes the proof of (10.27). Thus $\llbracket \mathcal{G}^{\prime} \rrbracket=\llbracket \mathcal{G} \rrbracket_{\alpha}^{*}$.

Proof of (2): Let $\mathcal{G}$ be finite-derivational. Then, by Lemma 9.2.1(3), we can even construct an equivalent chain-free wrtg. Consequently, the definition of $\mathcal{G}^{\prime}$, as given in the proof of (1), is constructive. Finally, by analysing the construction of $\mathcal{G}^{\prime}$, we obtain that $\mathcal{G}^{\prime}$ is finite-derivational.

Next we give an example of the concepts which appear in the proof of Theorem 10.7.4.
Example 10.7.5. Let $\Sigma=\left\{\sigma^{(2)}, \alpha^{(0)}\right\}$ and $\mathcal{G}$ be the ( $\Sigma$, Nat)-wrtg which has the rules

$$
\begin{aligned}
& S \rightarrow \sigma(S, A): 2 \\
& S \rightarrow \beta: 2 \\
& A \rightarrow \alpha: 2
\end{aligned}
$$

and the weight of each rule is 2 (indicated by ": 2 " after the rule). If we apply the construction in the proof of Theorem 10.7.4 then we obtain the $(\Sigma, \mathrm{Nat})$-wrtg $\mathcal{G}^{\prime}$ with the rules and weights

$$
\begin{array}{lr}
S \rightarrow \sigma(S, A): 2 & S_{0} \rightarrow \alpha: 1 \\
S \rightarrow \beta: 2 & A \rightarrow S: 2 \\
A \rightarrow \alpha: 2 . &
\end{array}
$$

Now let us consider the tree $\xi \in \mathrm{T}_{\Sigma}$, the cut $\widetilde{w}=(12,2)$ in $\operatorname{cut}_{\alpha}(\xi)$, and tree $\xi[\alpha] \widetilde{w}$ in Figure 10.5 , By definition, we have $\operatorname{iter}(\widetilde{w})=\{12\}$. In the upper part of Figure 10.6, there is an example of a tree $d^{\prime} \in \operatorname{RT}_{\mathcal{G}^{\prime}}^{\widetilde{w}}(S, \xi)$. Moreover, for the trees $d \in \operatorname{RT}_{\mathcal{G}}\left(\xi[\alpha]_{\widetilde{w}}\right)$ and $d_{1} \in \operatorname{RT}_{\mathcal{G}^{\prime}}\left(S,\left.\xi\right|_{12}\right)$ in the lower part of Figure 10.6, we have

$$
d^{\prime}=\Phi_{\widetilde{w}}\left(d, d_{1}\right) \quad \text { and } \quad \mathrm{wt}_{\mathcal{G}^{\prime}}\left(d^{\prime}\right)=\mathrm{wt}_{\mathcal{G}}(d) \cdot \mathrm{wt}_{\mathcal{G}^{\prime}}\left(d_{1}\right)
$$

where $\mathrm{wt}_{\mathcal{G}^{\prime}}\left(d^{\prime}\right)=2^{8}, \mathrm{wt}_{\mathcal{G}}(d)=2^{5}$, and $\mathrm{wt}_{\mathcal{G}^{\prime}}\left(d_{1}\right)=2^{3}$.
Finally, as a consequence of Theorem 10.7 .4 , we show that $\operatorname{Rec}(\Sigma, B)$ is closed under Kleene-stars.

Corollary 10.7.6. (cf. DPV05, Lm. 6.7]) Let $\Sigma$ be a ranked alphabet, $\alpha \in \Sigma^{(0)}$, and B be a commutative semiring. Let $\mathcal{A}$ be a $(\Sigma, \mathrm{B})$-wta such that $\llbracket \mathcal{A} \rrbracket$ is $\alpha$-proper. Then we can construct a $(\Sigma, \mathrm{B})$-wta $\mathcal{A}^{\prime}$ such that $\llbracket \mathcal{A}^{\prime} \rrbracket=\llbracket \mathcal{A} \rrbracket_{\alpha}^{*}$. Thus, in particular, if B is a commutative semiring, then the set $\operatorname{Rec}(\Sigma, \mathrm{B})$ is closed under Kleene-stars.

Proof. By Lemma 9.2 .6 we can construct $(\Sigma, \mathrm{B})-\operatorname{wrtg} \mathcal{G}$ such that $\mathcal{G}$ is in tree automata form and $\llbracket \mathcal{A} \rrbracket=$ $\llbracket \mathcal{G} \rrbracket$. Then, in particular, $\mathcal{G}$ is finite-derivational. By Theorem 10.7.4 we can construct a finite-derivational $(\Sigma, \mathrm{B})$-wrtg $\mathcal{G}^{\prime}$ such that $\llbracket \mathcal{G}^{\prime} \rrbracket=\llbracket \mathcal{G} \rrbracket_{\alpha}^{*}$. Then, by Lemma 9.2 .8 , we can construct a $(\Sigma, \mathrm{B})$-wta $\mathcal{A}^{\prime}$ such that $\llbracket \mathcal{G}^{\prime} \rrbracket=\llbracket \mathcal{A}^{\prime} \rrbracket$.

### 10.8 Closure under yield-intersection with weighted recognizable languages

Bar-Hillel, Perles, and Shamir [BPS61, Thm. 8.1] proved that the intersection of any context-free language and any recognizable language is again a context-free language. This theorem is very valuable,


Figure 10.5: Example of a tree $\xi \in \mathrm{T}_{\Sigma}$, a cut $\widetilde{w} \in \operatorname{cut}_{\alpha}(\xi)$, and a tree $\xi[\alpha]_{\widetilde{w}}$, cf. Example 10.7 .5 ,


Figure 10.6: Example of trees $d^{\prime} \in \operatorname{RT}_{\mathcal{G}^{\prime}}^{\widetilde{\mathcal{W}^{\prime}}}(S, \xi), d \in \operatorname{RT}_{\mathcal{G}}\left(\xi[\alpha]_{\widetilde{w}}\right)$, and $d_{1} \in \mathrm{RT}_{\mathcal{G}^{\prime}}\left(S,\left.\xi\right|_{12}\right)$, cf. Figure 10.5 and Example 10.7.5
in particular, in natural language processing, where one wishes to have a finite description of the set of all derivations of a string given a context-free grammar MS09, MS10. Here we extend this result by replacing (a) the context-free language by a recognizable weighted tree language $r$, (b) the recognizable language by an r-recognizable weighted language $L$, and (c) the intersection by the Hadamard product. Roughly speaking, we prove that the Hadamard product of $r$ and $L \circ$ yield is a recognizable weighted tree language if $r$ and $L$ take their weights in a commutative semiring (cf. Theorem 10.8.2). We refer to MS09, MS10] for even more extended results.

A set $\mathcal{L}$ of B -weighted tree languages is closed under yield-intersection with weighted recognizable languages if for every ( $\Sigma, \mathrm{B}$ )-weighted tree language $r$ in $\mathcal{L}, \Gamma \subseteq \Sigma^{(0)}$, and r-recognizable ( $\Gamma$, B)-weighted language $L: \Gamma^{*} \rightarrow B$, the $(\Sigma, \mathrm{B})$-weighted tree language $r \otimes\left(L \circ\right.$ yield $\left._{\Gamma}\right)$ is in $\mathcal{L}$.

Since we have already proved that the set $\operatorname{Rec}^{\text {run }}(\Sigma, B)$ is closed under Hadamard product if $B$ is a commutative semiring, it remains to prove that $L \circ$ yield $_{\Gamma}$ is a recognizable weighted tree language. Indeed, this even holds for commutative strong bimonoids. Our construction is based on the key idea of the proof of BPS61, Thm. 8.1], cf. also MS09, Thm. 2]. (We also refer to Example 3.2.7)

Lemma 10.8.1. Let $B$ be commutative, $\Gamma \subseteq \Sigma^{(0)}$, and $\mathcal{B}$ be a $(\Gamma, B)$-wsa. Then we can construct a $(\Sigma, \mathrm{B})$-wta $\mathcal{A}$ such that $\llbracket \mathcal{A} \rrbracket^{\text {run }}=\llbracket \mathcal{B} \rrbracket^{\text {run }} \circ$ yield $_{\Gamma}$.

Proof. Let $\mathcal{B}=(P, \lambda, \mu, \gamma)$. Intuitively, we construct $\mathcal{A}$ such that it guesses, at each $\Gamma$-labeled leaf of a given input $\Sigma$-tree $\xi$, a transition of $\mathcal{B}$ and, at each $\left(\Sigma^{(0)} \backslash \Gamma\right)$-labeled leaf of $\xi$, a pair $(p, p)$ for some $p \in P$. Then, while moving up towards the root of $\xi$, the wta $\mathcal{A}$ checks whether subruns of $\mathcal{B}$ can be composed to larger runs of $\mathcal{B}$. The first state and the last state of such a subrun are coded as a pair and forms a state of $\mathcal{A}$.

Formally, we construct the ( $\Sigma, \mathrm{B})$-wta $\mathcal{A}=(Q, \delta, F)$ as follows.

- $Q=P \times P$,
- we let $F_{\left(p, p^{\prime}\right)}=\lambda(p) \otimes \gamma\left(p^{\prime}\right)$ for every $\left(p, p^{\prime}\right) \in Q$,
- for every $\alpha \in \Gamma$ and $\left(p, p^{\prime}\right) \in Q$ we let

$$
\delta_{0}\left(\varepsilon, \alpha,\left(p, p^{\prime}\right)\right)=\mu\left(p, \alpha, p^{\prime}\right)
$$

and for every $\alpha \in \Sigma^{(0)} \backslash \Gamma$ and $\left(p, p^{\prime}\right) \in Q$ we let

$$
\delta_{0}\left(\varepsilon, \alpha,\left(p, p^{\prime}\right)\right)=\left\{\begin{array}{lc}
\mathbb{1} & \text { if } p=p^{\prime} \\
\mathbb{0} & \text { otherwise }
\end{array}\right.
$$

and for every $k \in \mathbb{N}_{+}, \sigma \in \Sigma^{(k)}$, and $\left(p_{1}, p_{1}^{\prime}\right),\left(p_{2}, p_{2}^{\prime}\right), \ldots,\left(p_{k}, p_{k}^{\prime}\right),\left(p, p^{\prime}\right) \in Q$ we let

$$
\begin{gathered}
\delta_{k}\left(\left(p_{1}, p_{1}^{\prime}\right) \cdots\left(p_{k}, p_{k}^{\prime}\right), \sigma,\left(p, p^{\prime}\right)\right)= \\
\begin{cases}\mathbb{1} & \text { if } p=p_{1}, p_{i}^{\prime}=p_{i+1} \text { for each } i \in[k-1], \text { and } p_{k}^{\prime}=p^{\prime} \\
\mathbb{0} & \text { otherwise. }\end{cases}
\end{gathered}
$$

This finishes the construction of $\mathcal{A}$.

Next we prove that $\llbracket \mathcal{A} \rrbracket^{\text {run }}=\llbracket \mathcal{B} \rrbracket^{\text {run }} \circ$ yield $_{\Gamma}$, for which we need some auxiliary tools and statements. Let $\xi \in \mathrm{T}_{\Sigma}$ and $\rho \in P^{\mid \text {yield }}(\xi) \mid+1$. Then there exist $n \in \mathbb{N}$ and $p_{1}, \ldots, p_{n} \in P$ such that $\rho=p_{0} \cdots p_{n}$. Let

$$
u_{0}, w_{1}, u_{1}, \ldots, u_{n-1}, w_{n}, u_{n}
$$

be the list of elements of $\operatorname{pos}_{\Sigma^{(0)}}(\xi)$ ordered by the lexicographic order $<_{\text {lex }}$ on $\operatorname{pos}(\xi)$, where $\operatorname{pos}_{\Gamma}(\xi)=$ $\left\{w_{1}, \ldots, w_{n}\right\}$ and, for each $i \in[0, n]$, the $u_{i}$ is the corresponding list of elements in $\operatorname{pos}_{\left(\Sigma^{(0)} \backslash \Gamma\right)}(\xi)$.

We define the binary relation $\prec$ on $\operatorname{pos}(\xi)$ by letting $w_{1} \prec w_{2}$ if there is an $i \in \mathbb{N}$ such that $w_{1}=w_{2} i$ . Obviously, $\prec$ is well-founded and $\min _{\prec}(\operatorname{pos}(\xi))=\operatorname{pos}_{\Sigma^{(0)}}(\xi)$, i.e., it is the set of leaves of $\xi$.


Figure 10.7: Value of $\varphi_{\xi, \rho}(w)$ if $w \in u_{i}$ for some $i \in[0, n]$.


Figure 10.8: Value of $\varphi_{\xi, \rho}(w)$ if $w=w_{i}$ for some $i \in[n]$.

Then we define the mapping

$$
\varphi_{\xi, \rho}: \operatorname{pos}(\xi) \rightarrow Q
$$

by induction on $(\operatorname{pos}(\xi), \prec)$ for each $w \in \operatorname{pos}(\xi)$ as follows.

$$
\varphi_{\xi, \rho}(w)= \begin{cases}\left(p_{i}, p_{i}\right) & \text { if } w \in u_{i} \text { for some } i \in[0, n] \\ \left(p_{i-1}, p_{i}\right) & \text { if } w=w_{i} \text { for some } i \in[n] \\ \left(\varphi_{\xi, \rho}(w 1)_{1}, \varphi_{\xi, \rho}(w k)_{2}\right) & \text { otherwise },\end{cases}
$$

where $w \in u_{i}$ abbreviates that $w$ is an element in the list $u_{i}$ and $k=\operatorname{rk}_{\xi}(w)$. Moreover, $\varphi_{\xi, \rho}(w 1)_{1}$ denotes the first component of $\varphi_{\xi, \rho}(w 1)$, and $\varphi_{\xi, \rho}(w k)_{2}$ denotes the second component of $\varphi_{\xi, \rho}(w k)$. In the particular case that $n=0$, we have $\varphi_{\xi, \rho}(w)=\left(p_{0}, p_{0}\right)$ for each $w \in \operatorname{pos}(\xi)$. Moreover, $\varphi_{\xi, \rho}(\varepsilon)=\left(p_{0}, p_{n}\right)$.

We demonstrate the definition of $\rho$ in Figures 10.7, 10.8, and 10.9 where $\alpha_{i}=\xi\left(w_{i}\right)$ for each $i \in[n]$ and hence yield ${ }_{\Gamma}(\xi)=\alpha_{1} \cdots \alpha_{n}$.

In fact, $\varphi_{\xi, \rho}$ is a run of $\mathcal{A}$ on $\xi$, i.e., $\varphi_{\xi, \rho} \in \mathrm{R}_{\mathcal{A}}(\xi)$. Thus, if $\xi=\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)$, then for each $l \in[k]$, the subrun $\left.\left(\varphi_{\xi, \rho}\right)\right|_{l}$ is defined and we have

$$
\begin{equation*}
\left.\left(\varphi_{\xi, \rho}\right)\right|_{l}=\varphi_{\xi_{l},\left.\rho\right|_{\left(i_{i}, j_{l}\right)}}, \tag{10.28}
\end{equation*}
$$

where $\left.\rho\right|_{\left(i_{l}, j_{l}\right)}$ is the subsequence $p_{i_{l}} \cdots p_{j_{l}}$ of $\rho$ determined by the indices

$$
i_{l}=\max \left(m \in[n] \mid w_{m} \in \bigcup_{\kappa=1}^{l-1} \operatorname{pos}_{\Gamma}\left(\xi_{\kappa}\right)\right) \text { and } j_{l}=\max \left(m \in[n] \mid w_{m} \in \bigcup_{\kappa=1}^{l} \operatorname{pos}_{\Gamma}\left(\xi_{\kappa}\right)\right)
$$



Figure 10.9: Value of $\varphi_{\xi, \rho}(w)$ if $w$ is not a leaf of $\xi$.
and the definition $\max (\emptyset)=0$. Thus $i_{1}=0, j_{l}=i_{l+1}$ for each $l \in[k-1], j_{k}=n$, and if $\operatorname{pos}_{\Gamma}\left(\xi_{l}\right)=\emptyset$, then $i_{l}=j_{l}$. (For instance, let $\xi=\sigma\left(\xi_{1}, \xi_{2}, \xi_{3}\right), \operatorname{pos}_{\Gamma}\left(\xi_{1}\right)=\left\{w_{1}, w_{2}\right\}$ with $w_{1}<_{\text {lex }} w_{2}, \operatorname{pos}_{\Gamma}\left(\xi_{2}\right)=\emptyset$, and $\operatorname{pos}_{\Gamma}\left(\xi_{3}\right)=\left\{w_{3}\right\}$, and let $\rho=p_{0} p_{1} p_{2} p_{3}$. Then $\left(i_{1}, j_{1}\right)=(0,2),\left(i_{2}, j_{2}\right)=(2,2)$, and $\left(i_{3}, j_{3}\right)=(2,3)$ and thus the corresponding subsequences are $p_{0} p_{1} p_{2}, p_{2}$, and $p_{2} p_{3}$.)

This finishes the preparations.

Now, by induction on $\mathrm{T}_{\Sigma}$, we prove that the following statement holds:

$$
\begin{equation*}
\text { For every } \xi \in \mathrm{T}_{\Sigma} \text { and } \rho \in P^{\mid \text {yield }_{\Gamma}(\xi) \mid+1} \text { we have } \mathrm{wt}_{\mathcal{B}}^{-}\left(\operatorname{yield}_{\Gamma}(\xi), \rho\right)=\mathrm{wt}_{\mathcal{A}}\left(\xi, \varphi_{\xi, \rho}\right) \text {. } \tag{10.29}
\end{equation*}
$$

I.B.: Let $\xi \in \Sigma^{(0)}$. We distinguish two cases.

Case (a): Let $\xi \in \Sigma^{(0)} \backslash \Gamma$. Now let $\rho \in P^{\mid \text {yield }}{ }_{\Gamma}(\xi) \mid+1$. Since $\left|\operatorname{yield}_{\Gamma}(\xi)\right|+1=1$, there exists a $p \in P$ such that $\rho=p$. Then $\varphi_{\xi, \rho}(\varepsilon)=(p, p)$. Moreover, due to the construction of $\delta$, we have

$$
\operatorname{wt}_{\mathcal{A}}\left(\xi, \varphi_{\xi, \rho}\right)=\delta_{0}(\varepsilon, \alpha,(p, p))=\mathbb{1}
$$

Since $\operatorname{wt}_{\mathcal{B}}^{-}(\varepsilon, p)=\mathbb{1}$ by definition, we obtain (10.29).
Case (b): Let $\xi \in \Gamma$. Now let $\rho \in P^{\mid \text {yield }_{\Gamma}(\xi) \mid+1}$. Since $\left|\operatorname{yield}_{\Gamma}(\xi)\right|+1=2$, there exist $p, p^{\prime} \in P$ such that $\rho=p p^{\prime}$. Then $\varphi_{\xi, \rho}(\varepsilon)=\left(p, p^{\prime}\right)$. Moreover, due to the construction of $\delta$, we have

$$
\mathrm{wt}_{\mathcal{A}}\left(\xi, \varphi_{\xi, \rho}\right)=\delta_{0}\left(\varepsilon, \alpha,\left(p, p^{\prime}\right)\right)=\mu\left(p, \alpha, p^{\prime}\right)
$$

Since $\operatorname{wt}_{\mathcal{B}}^{-}(\varepsilon, p)=\mu\left(p, \alpha, p^{\prime}\right)$ by definition, we obtain (10.29).
I.S.: Let $\xi=\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)$ with $k \in \mathbb{N}_{+}$. Then we can calculate as follows:

$$
\begin{align*}
& \mathrm{wt}_{\mathcal{B}}^{-}\left(\operatorname{yield}_{\Gamma}(\xi), \rho\right)=\bigotimes_{l \in[k]} \mathrm{wt}_{\mathcal{\mathcal { B }}}^{-}\left(\operatorname{yield}_{\Gamma}\left(\xi_{l}\right),\left.\rho\right|_{\left(i_{l}, j_{l}\right)}\right) \\
= & \bigotimes_{l \in[k]} \mathrm{wt}_{\mathcal{A}}\left(\xi_{l}, \rho_{\left.\xi_{l},\left.\rho\right|_{\left(i_{l}, j_{l}\right)}\right)}\right)  \tag{byI.H.}\\
= & \left(\bigotimes_{l \in[k]} \mathrm{wt}_{\mathcal{A}}\left(\xi_{l},\left.\left(\varphi_{\xi, \rho}\right)\right|_{l}\right)\right) \otimes \delta_{k}\left(\varphi_{\xi, \rho}(1) \cdots \varphi_{\xi, \rho}(k), \sigma, \varphi_{\xi, \rho}(\varepsilon)\right) \quad \text { (by I.H.) } \\
= & \operatorname{wt}_{\mathcal{A}}\left(\xi, \varphi_{\xi, \rho}\right) .
\end{align*}
$$

This finishes the proof of (10.29).

Let $\xi \in \mathrm{T}_{\Sigma}$ and assume that yield ${ }_{\Gamma}(\xi)$ has length $n$. For each $\rho_{\mathcal{A}} \in \mathrm{R}_{\mathcal{A}}(\xi) \backslash\left\{\varphi_{\xi, \rho} \mid \rho \in P^{n+1}\right\}$ there exists a $w \in \operatorname{pos}(\xi)$ such that, assuming that $\operatorname{rk}_{\Sigma}(\xi(w))=k$, we have

- $\rho_{\mathcal{A}}(w)_{1} \neq \rho_{\mathcal{A}}(w 1)_{1}$ or $\rho_{\mathcal{A}}(w)_{2} \neq \rho_{\mathcal{A}}(w k)_{2}$ or
- there exists an $i \in[k-1]$ such that $\rho_{\mathcal{A}}(w i)_{2} \neq \rho_{\mathcal{A}}(w(i+1))_{1}$ or
- $k=0, \xi(w) \notin \Gamma$, and $\rho_{\mathcal{A}}(w)_{1} \neq \rho_{\mathcal{A}}(w)_{2}$.

In this case, by definition of $\delta_{k}$, we have $\delta_{k}\left(\rho_{\mathcal{A}}(w 1) \cdots \rho_{\mathcal{A}}(w k), \xi(w), \rho_{\mathcal{A}}(w)\right)=\mathbb{D}$ and thus $\mathrm{wt}_{\mathcal{A}}\left(\xi, \rho_{\mathcal{A}}\right)=\mathbb{0}$. Thus we have

$$
\begin{equation*}
\operatorname{wt}_{\mathcal{A}}\left(\xi, \rho_{\mathcal{A}}\right)=\mathbb{O} \text { for each } \rho_{\mathcal{A}} \in \mathrm{R}_{\mathcal{A}}(\xi) \backslash\left\{\varphi_{\xi, \rho} \mid \rho \in P^{n+1}\right\} \tag{10.30}
\end{equation*}
$$

For each $\rho \in P^{n+1}$ we denote the first state of $\rho$ and the last state of $\rho$ by $\operatorname{fst}(\rho)$ and $\operatorname{lst}(\rho)$, respectively. Then:

$$
\begin{aligned}
& \llbracket \mathcal{A} \rrbracket^{\mathrm{run}}(\xi)=\bigoplus_{\rho_{\mathcal{A}} \in \mathrm{R}_{\mathcal{A}}(\xi)} \mathrm{wt}_{\mathcal{A}}\left(\xi, \rho_{\mathcal{A}}\right) \otimes F_{\rho_{\mathcal{A}}(\varepsilon)} \\
& =\bigoplus_{\rho \in P^{n+1}} \operatorname{wt}_{\mathcal{A}}\left(\xi, \varphi_{\xi, \rho}\right) \otimes F_{(\mathrm{fst}(\rho), \operatorname{lst}(\rho))} \quad\left(\text { by } 10.30 \text { and the fact that } \varphi_{\xi, \rho}(\varepsilon)=(\operatorname{fst}(\rho), \operatorname{lst}(\rho))\right) \\
& \left.=\bigoplus_{\rho \in P^{n+1}} \mathrm{wt}_{\mathcal{B}}^{-}\left(\operatorname{yield}_{\Gamma}(\xi), \rho\right) \otimes F_{(\mathrm{fst}(\rho), \operatorname{lst}(\rho))} \quad \quad \text { (by (10.29) }\right) \\
& =\bigoplus_{\left(p, p^{\prime}\right) \in Q^{\prime}} \bigoplus_{\substack{\rho \in P^{n+1}: \\
\mathrm{fst}(\rho)=p, \operatorname{lst}(\rho)=p^{\prime}}} \mathrm{wt}_{\mathcal{B}}^{-}\left(\operatorname{yield}_{\Gamma}(\xi), \rho\right) \otimes F_{\left(p, p^{\prime}\right)} \\
& =\bigoplus_{\left(p, p^{\prime}\right) \in Q^{\prime}} \bigoplus_{\rho \in P^{n+1} ;} \mathrm{wt}_{\mathcal{B}}^{-}\left(\operatorname{yield}_{\Gamma}(\xi), \rho\right) \otimes \lambda(p) \otimes \gamma\left(p^{\prime}\right) \quad \quad \text { (by construction) } \\
& =\bigoplus_{\left(p, p^{\prime}\right) \in Q^{\prime}} \bigoplus_{\rho \in P^{n+1}:} \quad \mathrm{wt}_{\mathcal{B}}\left(\operatorname{yield}_{\Gamma}(\xi), \rho\right) \quad \quad \text { (by commutativity) } \\
& =\llbracket \mathcal{B} \rrbracket^{\text {run }}\left(\operatorname{yield}_{\Gamma}(\xi)\right) \text {. }
\end{aligned}
$$

The next theorem was proved in MS09, Thm. 2] via a direct construction. We show a modular proof by exploiting Lemma 10.8 .1 and closure under Hadamard product (where the latter requires that B is a commutative semiring).

Theorem 10.8.2. MS09, Thm. 2] Let $\Sigma$ be a ranked alphabet, $\Gamma \subseteq \Sigma^{(0)}$, and $\mathrm{B}=(B, \oplus, \otimes, \mathbb{0}, \mathbb{1})$ be a commutative semiring. For every $(\Sigma, \mathrm{B})-$ wta $\mathcal{A}$ and every $(\Gamma, \mathrm{B})$-wsa $\mathcal{B}$, we can construct a $(\Sigma, \mathrm{B})-$ wta $\mathcal{A}^{\prime}$ such that

$$
\llbracket \mathcal{A}^{\prime} \rrbracket=\llbracket \mathcal{A} \rrbracket \otimes\left(\llbracket \mathcal{B} \rrbracket \circ \operatorname{yield}_{\Gamma}\right) .
$$

Thus, in particular, the set $\operatorname{Rec}(\Sigma, \mathrm{B})$ is closed under yield-intersection.

Proof. By Corollary 5.3 .3 and by convention on page 121 we have $\llbracket \mathcal{A} \rrbracket=\llbracket \mathcal{A} \rrbracket^{\text {run }}$ and $\llbracket \mathcal{A}^{\prime} \rrbracket=\llbracket \mathcal{A}^{\prime} \rrbracket^{\text {run }}$. Moreover, by Corollary 5.3 .4 and by convention on page 122 we have $\llbracket \mathcal{B} \rrbracket=\llbracket \mathcal{B} \rrbracket{ }^{\text {run }}$. By Lemma 10.8.1 , we can construct a $(\Sigma, B)$-wta $\mathcal{C}$ such that $\llbracket \mathcal{C} \rrbracket=\llbracket \mathcal{B} \rrbracket \circ$ yield ${ }_{\Gamma}$. Then, by Theorem 10.4.1(1), we can construct a $(\Sigma, \mathrm{B})$-wta $\mathcal{A}^{\prime}$ such that $\llbracket \mathcal{A}^{\prime} \rrbracket=\llbracket \mathcal{A} \rrbracket \otimes \llbracket \mathcal{C} \rrbracket$.

Finally, we verify that Theorem 10.8 .2 generalizes the classical result of Bar-Hillel, Perles, and Shamir [BPS61, Thm. 8.1]. We achieve this by proving that the latter is equivalent to Theorem 10.8.2 for the case that $B$ is the Boolean semiring.

Corollary 10.8.3. Let $G$ be a $\Gamma$-cfg and $A$ be a $\Gamma$-fsa. Then we can construct a $\Gamma$-cfg $G^{\prime}$ such that $\mathrm{L}\left(G^{\prime}\right)=\mathrm{L}(G) \cap \mathrm{L}(A)$.

Proof. By Corollary 8.3.4 we can construct a ranked alphabet $\Sigma$ with $\Gamma \subseteq \Sigma^{(0)}$ and a $\Sigma$-fta $A^{\prime}$ such that $\mathrm{L}(G)=$ yield $_{\Gamma}\left(\mathrm{L}\left(A^{\prime}\right)\right)$. By Corollary 3.4.2, we can construct a $\left(\Sigma\right.$, Boole)-wta $\mathcal{A}^{\prime}$ such that $\mathrm{L}(G)=$
$\operatorname{yield}_{\Gamma}\left(\operatorname{supp}\left(\llbracket \mathcal{A}^{\prime} \rrbracket\right)\right)$. Moreover, by Observation 3.3.4 we can construct a ( $\Gamma$, Boole)-wsa $\mathcal{A}$ such that $\mathrm{L}(A)=\operatorname{supp}\left(\llbracket \mathcal{A} \rrbracket^{\text {run }}\right)$. Then we can calculate as follows.

$$
\begin{aligned}
\mathrm{L}(G) \cap \mathrm{L}(A) & =\operatorname{yield}_{\Gamma}\left(\operatorname{supp}\left(\llbracket \mathcal{A}^{\prime} \rrbracket\right)\right) \cap \operatorname{supp}\left(\llbracket \mathcal{A} \rrbracket^{\text {run }}\right) \\
& =\operatorname{supp}\left(\chi\left(\operatorname{yield}_{\Gamma}\right)\left(\llbracket \mathcal{A}^{\prime} \rrbracket\right)\right) \cap \operatorname{supp}\left(\llbracket \mathcal{A} \rrbracket^{\text {run }}\right) \\
& =\operatorname{supp}\left(\chi\left(\operatorname{yield}_{\Gamma}\right)\left(\llbracket \mathcal{A}^{\prime} \rrbracket\right) \wedge \llbracket \mathcal{A} \rrbracket^{\text {run }}\right) \\
& ={ }^{(*)} \operatorname{supp}\left(\chi\left(\operatorname{yield}_{\Gamma}\right)\left(\llbracket \mathcal{A}^{\prime} \rrbracket \wedge\left(\llbracket \mathcal{A} \rrbracket^{\text {run }} \circ \text { yield }_{\Gamma}\right)\right)\right)
\end{aligned}
$$

where at $(*)$ we have used the following subcalculation for each $w \in \Gamma^{*}$ :

$$
\begin{aligned}
\left(\chi\left(\operatorname{yield}_{\Gamma}\right)\left(\llbracket \mathcal{A}^{\prime} \rrbracket\right) \wedge \llbracket \mathcal{A} \rrbracket^{\mathrm{run}}\right)(w)= & \chi\left(\operatorname{yield}_{\Gamma}\right)\left(\llbracket \mathcal{A}^{\prime} \rrbracket\right)(w) \wedge \llbracket \mathcal{A} \rrbracket^{\text {run }}(w) \\
= & \left(\sum_{\xi \in \operatorname{yield}_{\Gamma}^{-1}(w)}^{\vee} \llbracket \mathcal{A}^{\prime} \rrbracket(\xi)\right) \wedge \llbracket \mathcal{A} \rrbracket^{\text {run }}(w) \\
= & \sum_{\xi \in \operatorname{yield}_{\Gamma}^{-1}(w)}^{\vee}\left(\llbracket \mathcal{A}^{\prime} \rrbracket(\xi) \wedge \llbracket \mathcal{A} \rrbracket^{\text {run }}(w)\right) \quad \text { (by distributivity) } \\
= & \sum_{\xi \in \operatorname{yield}_{\Gamma}^{-1}(w)}^{\vee}\left(\llbracket \mathcal{A}^{\prime} \rrbracket(\xi) \wedge \llbracket \mathcal{A} \rrbracket^{\text {run }}\left(\operatorname{yield}_{\Gamma}(\xi)\right)\right) \\
= & \sum_{\xi \in \operatorname{yield}_{\Gamma}^{-1}(w)}^{\vee}\left(\llbracket \mathcal{A}^{\prime} \rrbracket \wedge \llbracket \mathcal{A} \rrbracket^{\text {run }} \circ \operatorname{yield}_{\Gamma}\right)(\xi) \\
& =\chi\left(\operatorname{yield}_{\Gamma}\right)\left(\llbracket \mathcal{A}^{\prime} \rrbracket \wedge \llbracket \mathcal{A} \rrbracket^{\text {run }} \circ \operatorname{yield}_{\Gamma}\right)(w)
\end{aligned}
$$

By Theorem 10.8 .2 , we can construct a $\left(\Sigma\right.$, Boole)-wta $\mathcal{B}$ such that $\llbracket \mathcal{B} \rrbracket^{\text {run }}=\llbracket \mathcal{A}^{\prime} \rrbracket \wedge\left(\llbracket \mathcal{A} \rrbracket^{\text {run }} \circ\right.$ yield $\left.{ }_{\Gamma}\right)$. Hence we can continue with:

$$
\begin{aligned}
\operatorname{supp}\left(\chi\left(\operatorname{yield}_{\Gamma}\right)\left(\llbracket \mathcal{A}^{\prime} \rrbracket \wedge\left(\llbracket \mathcal{A} \rrbracket^{\text {run }} \circ \operatorname{yield}_{\Gamma}\right)\right)\right) & =\operatorname{supp}\left(\chi\left(\operatorname{yield}_{\Gamma}\right)\left(\llbracket \mathcal{B} \rrbracket^{\text {run }}\right)\right) \\
& =\operatorname{yield}_{\Gamma}\left(\operatorname{supp}\left(\llbracket \mathcal{B} \rrbracket^{\text {run }}\right)\right)
\end{aligned}
$$

(by (2.31))
By Corollary 3.4.2, we can construct a $\Sigma$-fta $C$ such that $\operatorname{supp}\left(\llbracket \mathcal{B} \rrbracket^{\text {run }}\right)=\mathrm{L}(C)$. Moreover, by Corollary 8.3.4. we can construct a $\Gamma$-cfg $G^{\prime}$ such that yield ${ }_{\Gamma}(\mathrm{L}(C))=\mathrm{L}\left(G^{\prime}\right)$.

### 10.9 Closure under strong bimonoid homomorphisms

A set $\mathcal{L}$ of weighted tree languages is closed under strong bimonoid homomorphisms if for every ( $\Sigma, \mathrm{B}$ )weighted tree language $r \in \mathcal{L}$, strong bimonoid $C=(C,+, \times, 0,1)$, and strong bimonoid homomorphism $f: B \rightarrow C$, the $(\Sigma, \mathrm{C})$-weighted tree language $f \circ r$ is in $\mathcal{L}$.

Let $\mathcal{A}=(Q, \delta, F)$ be a $(\Sigma, \mathrm{B})$-wta, C be a strong bimonoid, and $f: B \rightarrow C$ be a strong bimonoid homomorphism. Then we define the $f$-image of $\mathcal{A}$, denoted by $f(\mathcal{A})$, to be the $(\Sigma, \mathrm{C})$-wta $\left(Q, \delta^{\prime}, F^{\prime}\right)$ by defining $\left(\delta^{\prime}\right)_{k}=f \circ \delta_{k}$ for each $k \in \mathbb{N}$, and $F^{\prime}=f \circ F$.

Note that, if $\mathcal{A}$ is bu deterministic (or crisp deterministic), then so is $f(\mathcal{A})$. Moreover, if $\mathcal{A}$ is total and $f^{-1}(0)=\{\mathbb{O}\}$, then $f(\mathcal{A})$ is total. The condition $f^{-1}(0)=\{\mathbb{O}\}$ cannot be dropped, which can be seen as follows. Let $\mathcal{A}=(Q, \delta, F)$ be a total ( $\Sigma$, Nat)-wta (where Nat is the semiring of natural numbers) such that there exist $k \in \mathbb{N}, \sigma \in \Sigma^{(k)}, w \in Q^{k}$, and $q \in Q$ with $\delta_{k}(w, \sigma, q)=4$ and for each $p \in Q \backslash\{q\}$ we have $\delta_{k}(w, \sigma, p)=0$. We consider the ring $\operatorname{Intmod} 4=\left(\{0,1,2,3\},+{ }_{4}, \cdot_{4}, 0,1\right)$ of natural numbers modulo 4. Finally, we consider the canonical semiring homomorphism $f: \mathbb{N} \rightarrow\{0,1,2,3\}$. Since $f(0)=f(4)=0$, in $f(\mathcal{A})$ we have $\left(\delta^{\prime}\right)_{k}(w, \sigma, p)=0$ for every $p \in Q$. Hence $f(\mathcal{A})$ is not total.

The following observation is obvious but useful.

Observation 10.9.1. Let $\mathcal{A}=(Q, \delta, F)$ be a $(\Sigma, \mathrm{B})$-wta, $\mathrm{C}=(C,+, \times, 0,1)$ be a strong bimonoid, and $f: B \rightarrow C$ be a strong bimonoid homomorphism such that, for each $b \in B$, the element $f(b)$ in $C$ can be constructed. Then we can construct the ( $\Sigma, \mathrm{C})$-wta $f(\mathcal{A})$.

Now we can prove the main result of this section.
Lemma 10.9.2. Let $\mathcal{A}=(Q, \delta, F)$ be a ( $\Sigma, \mathrm{B})$-wta, $(C,+, \times, 0,1)$ be a strong bimonoid, and $f: B \rightarrow C$ be a strong bimonoid homomorphism. Then $f \circ \llbracket \mathcal{A} \rrbracket^{\text {init }}=\llbracket f(\mathcal{A}) \rrbracket^{\text {init }}$ and $f \circ \llbracket \mathcal{A} \rrbracket^{\text {run }}=\llbracket f(\mathcal{A}) \rrbracket^{\text {run }}$.

Proof. First we show that $f \circ \llbracket \mathcal{A} \rrbracket^{\text {init }}=\llbracket f(\mathcal{A}) \rrbracket^{\text {init }}$. For this, we define the mapping $\widetilde{f}: B^{Q} \rightarrow C^{Q}$ for every $v \in B^{Q}$ and $q \in Q$ by $\widetilde{f}(v)_{q}=f\left(v_{q}\right)$. We prove that $\widetilde{f}$ is a $\Sigma$-algebra homomorphism from the vector algebra $\mathrm{V}(\mathcal{A})=\left(B^{Q}, \delta_{\mathcal{A}}\right)$ to the vector algebra $\mathrm{V}(f(\mathcal{A}))=\left(C^{Q}, \delta_{f(\mathcal{A})}^{\prime}\right)$. For this, let $k \in \mathbb{N}, \sigma \in \Sigma^{(k)}$, and $v_{1}, \ldots, v_{k} \in B^{Q}$. Then we can calculate as follows.

$$
\begin{aligned}
& \widetilde{f}\left(\delta_{\mathcal{A}}(\sigma)\left(v_{1}, \ldots, v_{k}\right)\right)_{q}=f\left(\delta_{\mathcal{A}}(\sigma)\left(v_{1}, \ldots, v_{k}\right)_{q}\right) \\
& =f\left(\bigoplus_{q_{1} \cdots q_{k} \in Q^{k}}\left(\bigotimes_{i \in[k]}\left(v_{i}\right)_{q_{i}}\right) \otimes \delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right)\right) \\
& ={\underset{q_{1} \cdots q_{k} \in Q^{k}}{ }\left(\underset{i \in[k]}{X} f\left(\left(v_{i}\right)_{q_{i}}\right)\right) \times f\left(\delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right)\right), ~\left({ }^{\prime}\right)} \\
& \text { (because } f \text { is a strong bimonoid homomorphism) }
\end{aligned}
$$

$$
\begin{aligned}
& =\delta_{f(\mathcal{A})}^{\prime}(\sigma)\left(\widetilde{f}\left(v_{1}\right), \ldots, \widetilde{f}\left(v_{k}\right)\right)_{q} .
\end{aligned}
$$

Hence $\tilde{f}$ is a $\Sigma$-algebra homomorphism. By Theorem 2.6.3, $\tilde{f} \circ \mathrm{~h}_{\mathcal{A}}$ is a $\Sigma$-algebra homomorphism from the $\Sigma$-term algebra $\left(\mathrm{T}_{\Sigma}, \theta_{\Sigma}\right)$ to the vector algebra $\mathrm{V}(f(\mathcal{A}))=\left(C^{Q}, \delta_{f(\mathcal{A})}^{\prime}\right)$. Since ${\underset{\sim}{f}}_{f(\mathcal{A})}$ is also a $\Sigma$-algebra homomorphism of the same type, it follows from Theorem 2.6.5 that $\mathrm{h}_{f(\mathcal{A})}=\tilde{f} \circ \mathrm{~h}_{\mathcal{A}}$. Hence, for every $\xi \in \mathrm{T}_{\Sigma}$ and $q \in Q$, we have $\mathrm{h}_{f(\mathcal{A})}(\xi)_{q}=f\left(\mathrm{~h}_{\mathcal{A}}(\xi)_{q}\right)$.

Then for each $\xi \in \mathrm{T}_{\Sigma}$ :

$$
\begin{aligned}
\left(f \circ \llbracket \mathcal{A} \rrbracket^{\text {init }}\right)(\xi) & =f\left(\llbracket \mathcal{A} \rrbracket^{\text {init }}(\xi)\right)=f\left(\bigoplus_{q \in Q} \mathrm{~h}_{\mathcal{A}}(\xi)_{q} \otimes F_{q}\right) \\
& ={\underset{q \in Q}{ } f\left(\mathrm{~h}_{\mathcal{A}}(\xi)_{q}\right) \times f\left(F_{q}\right)={\underset{q \in Q}{ }} \mathrm{~h}_{f(\mathcal{A})}(\xi)_{q} \times F_{q}^{\prime}=\llbracket f(\mathcal{A}) \rrbracket^{\mathrm{init}}(\xi) .}^{\mathrm{q}} .
\end{aligned}
$$

Now we show that $f \circ \llbracket \mathcal{A} \rrbracket^{\text {run }}=\llbracket f(\mathcal{A}) \rrbracket^{\text {run }}$. For this, by induction on $\mathrm{T}_{\Sigma}$, we prove that the following statement holds:

$$
\begin{equation*}
\text { For every } \xi \in \mathrm{T}_{\Sigma} \text { and } \rho \in \mathrm{R}_{\mathcal{A}}(\xi), \text { we have: } \mathrm{wt}_{f(\mathcal{A})}(\xi, \rho)=f\left(\mathrm{wt}_{\mathcal{A}}(\xi, \rho)\right) \tag{10.31}
\end{equation*}
$$

Let $\xi=\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)$. Then

$$
\begin{aligned}
\mathrm{wt}_{f(\mathcal{A})}(\xi, \rho) & =\left(\underset{i \in[k]}{X} \mathrm{wt}_{f(\mathcal{A})}\left(\xi_{i},\left.\rho\right|_{i}\right)\right) \times \delta_{k}^{\prime}(\rho(1) \cdots \rho(k), \sigma, \rho(\varepsilon)) \\
& =\left(\underset{i \in[k]}{X} f\left(\mathrm{wt}_{\mathcal{A}}\left(\xi_{i},\left.\rho\right|_{i}\right)\right)\right) \times f\left(\delta_{k}(\rho(1) \cdots \rho(k), \sigma, \rho(\varepsilon))\right)
\end{aligned}
$$

(by I.H. and construction, recall that $\left.\rho\right|_{i} \in \mathrm{R}_{\mathcal{A}}\left(\xi_{i}\right)$ for $1 \leq i \leq k$ )
$=f\left(\left(\bigotimes_{i \in[k]} \mathrm{wt}_{\mathcal{A}}\left(\xi_{i},\left.\rho\right|_{i}\right)\right) \otimes \delta_{k}(\rho(1) \cdots \rho(k), \sigma, \rho(\varepsilon))\right)$
(by the fact that $f$ is a strong bimonoid homomorphisms)

$$
=f\left(\mathrm{wt}_{\mathcal{A}}(\xi, \rho)\right)
$$

This proves (10.31). Then for each $\xi \in \mathrm{T}_{\Sigma}$ :

$$
\begin{aligned}
& \left(f \circ \llbracket \mathcal{A} \rrbracket^{\mathrm{run}}\right)(\xi)=f\left(\llbracket \mathcal{A} \rrbracket^{\mathrm{run}}(\xi)\right)=f\left(\bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}}(\xi)} \mathrm{wt}_{\mathcal{A}}(\xi, \rho) \otimes F_{\rho(\varepsilon)}\right)
\end{aligned}
$$

where the third equation uses again that $h$ is a strong bimonoid homomorphism and the fourth one uses the fact that $\mathrm{R}_{\mathcal{A}}(\xi)=\mathrm{R}_{f(\mathcal{A})}(\xi)$.

The next theorem generalizes $\left.\mathrm{BMS}^{+} 06, \mathrm{Lm} .3\right]$ and [V09, Thm. 3.9] from semirings to strong bimonoids. It uses the following abbreviation. Let $\mathcal{C}$ be a set of $(\Sigma, B)$-weighted tree languages, $\mathrm{C}=$ $(C,+, \times, 0,1)$ be a strong bimonoid and $f: B \rightarrow C$ be a strong bimonoid homomorphism. We define $f \circ \mathcal{C}=\{f \circ r \mid r \in \mathcal{C}\}$. Obviously, $f \circ \mathcal{C}$ is a set of $(\Sigma, \mathrm{C})$-weighted tree languages.

Theorem 10.9.3. Let $\Sigma$ be a ranked alphabet, $\mathrm{B}=(B, \oplus, \otimes, \mathbb{0}, \mathbb{1})$ and $\mathrm{C}=(C,+, \times, 0,1)$ be strong bimonoids, and $f: B \rightarrow C$ be a strong bimonoid homomorphism. Then the following three statements hold.
(1) $f \circ \operatorname{Rec}^{\text {init }}(\Sigma, \mathrm{B}) \subseteq \operatorname{Rec}^{\text {init }}(\Sigma, \mathrm{C})$ and $f \circ \operatorname{Rec}^{\text {run }}(\Sigma, \mathrm{B}) \subseteq \operatorname{Rec}^{\mathrm{run}}(\Sigma, \mathrm{C})$.
(2) If $f$ is surjective, then $f \circ \operatorname{Rec}^{\text {init }}(\Sigma, \mathrm{B})=\operatorname{Rec}^{\text {init }}(\Sigma, \mathrm{C})$ and $f \circ \operatorname{Rec}^{\text {run }}(\Sigma, \mathrm{B})=\operatorname{Rec}^{\text {run }}(\Sigma, \mathrm{C})$.
(3) Statements (1) and (2) also hold for the subsets of $\operatorname{Rec}^{\text {init }}(\Sigma, \mathrm{B}), \operatorname{Rec}^{\mathrm{init}}(\Sigma, \mathrm{C}), \operatorname{Rec}^{\text {run }}(\Sigma, \mathrm{B})$, and $\operatorname{Rec}{ }^{\mathrm{run}}(\Sigma, \mathrm{C})$ which are recognizable by bu deterministic wta. The same holds if we replace bu deterministic by crisp deterministic.
Thus, in particular, the sets $\operatorname{Rec}^{\mathrm{init}}\left(\Sigma,,_{-}\right), \operatorname{Rec}^{\mathrm{run}}\left(\Sigma,_{-}\right)$and $\operatorname{Rec}\left(\Sigma,{ }_{-}\right)$are closed under strong bimonoid homomorphisms.

Proof. Proof of (1): Let $\mathcal{A}$ be a $(\Sigma, \mathrm{B})$-wta such that $r=\llbracket \mathcal{A} \rrbracket^{\text {init }}$. By Lemma 10.9 .2 we have that $f \circ \llbracket \mathcal{A} \rrbracket^{\text {init }}=\llbracket f(\mathcal{A}) \rrbracket^{\text {init }}$. Hence $f \circ r \in \operatorname{Rec}^{\text {init }}(\Sigma, \mathrm{C})$. By a similar argument we can prove that $r \in$ $\operatorname{Rec}^{\text {run }}(\Sigma, \mathrm{B})$ implies $f \circ r \in \operatorname{Rec}^{\text {run }}(\Sigma, \mathrm{C})$.

Proof of (2): By (1) we have $f \circ \operatorname{Rec}^{\text {init }}(\Sigma, \mathrm{B}) \subseteq \operatorname{Rec}^{\text {init }}(\Sigma, \mathrm{C})$ and $f \circ \operatorname{Rec}^{\text {run }}(\Sigma, \mathrm{B}) \subseteq \operatorname{Rec}^{\text {run }}(\Sigma, \mathrm{C})$.
Next we assume that $f$ is surjective. Then we show that also the other inclusions hold. For this, let $\mathcal{A}^{\prime}=\left(Q, \delta^{\prime}, F^{\prime}\right)$ be an arbitrary $(\Sigma, \mathrm{C})$-wta. We construct the $(\Sigma, \mathrm{B})$-wta $\mathcal{A}=(Q, \delta, F)$ such that

- for every $k \in \mathbb{N}, \sigma \in \Sigma^{(k)}, w \in Q^{k}$, and $q \in Q$ we let $\delta_{k}(w, \sigma, q)=b$ where $b$ is determined as follows: if $\delta_{k}^{\prime}(w, \sigma, q)=0$, then we let $b=\mathbb{0}$, if $\delta_{k}^{\prime}(w, \sigma, q)=1$, then we let $b=\mathbb{1}$, and if $\delta_{k}^{\prime}(w, \sigma, q) \notin\{0,1\}$, then we let $b$ be an arbitrary element in $f^{-1}\left(\delta_{k}^{\prime}(w, \sigma, q)\right)$, and
- for each $q \in Q$, we let $F_{q}=b$, for some $b \in f^{-1}\left(F_{q}^{\prime}\right)$.

In both items such values $b$ exist because $f$ is surjective. We have $\mathcal{A}^{\prime}=f(\mathcal{A})$ and by Lemma 10.9.2 we conclude $f \circ \llbracket \mathcal{A} \rrbracket^{\text {init }}=\llbracket \mathcal{A}^{\prime} \rrbracket^{\text {init }}$ and $f \circ \llbracket \mathcal{A} \rrbracket^{\text {run }}=\llbracket \mathcal{A}^{\prime} \rrbracket^{\text {run }}$. Hence the other inclusions also hold.

Proof of (3): This statement holds by (1) and (2) and by the fact that the constructions of the $f$-image $f(\mathcal{A})$ and of $\mathcal{A}^{\prime}$ (in the proof of (2)) preserve bu determinism and crisp determinism.

### 10.10 Closure under tree relabelings

In this section we will prove that $\operatorname{Rec}^{\text {run }}\left(\__{-}, \mathrm{B}\right)$ is closed under tree relabelings. We recall that a $(\Sigma, \Delta)$-tree relabeling is a family $\tau=\left(\tau_{k} \mid k \in \mathbb{N}\right)$ such that $\tau_{k}(\sigma) \subseteq \Delta^{(k)}$ for each $\sigma \in \Sigma^{(k)}$. A tree relabeling $\tau$ is non-overlapping if $\tau_{k}(\sigma) \cap \tau_{k}\left(\sigma^{\prime}\right)=\emptyset$ for every $k \in \mathbb{N}$ and $\sigma, \sigma^{\prime} \in \Sigma^{(k)}$ with $\sigma \neq \sigma^{\prime}$. Its extension is the
mapping $\tau: \mathrm{T}_{\Sigma} \rightarrow \mathcal{P}_{\text {fin }}\left(\mathrm{T}_{\Delta}\right)$, which can be also considered as binary relation $\tau \subseteq \mathrm{T}_{\Sigma} \times \mathrm{T}_{\Delta}$. Then its characteristic mapping $\chi(\tau): \mathrm{T}_{\Sigma} \times \mathrm{T}_{\Delta} \rightarrow B$ is supp-i-finite.

Since $\chi(\tau)$ is supp-i-finite, for each $r: \mathrm{T}_{\Sigma} \rightarrow B$, the application $\chi(\tau)(r)$ is defined by (2.28) and by (2.29) and (2.30), for each $\zeta \in \mathrm{T}_{\Delta}$, we have

$$
\chi(\tau)(r)(\zeta)=\bigoplus_{\xi \in \tau^{-1}(\zeta)} r(\xi)
$$

A set $\mathcal{L}$ of B -weighted tree languages is closed under tree relabelings if for every $(\Sigma, \mathrm{B})$-weighted tree language $r \in \mathcal{L}$ and every $(\Sigma, \Delta)$-tree relabeling $\tau$, the $(\Delta, \mathrm{B})$-weighted tree language $\chi(\tau)(r)$ is in $\mathcal{L}$.

Theorem 10.10.1. (cf. SVF09, Lm. 6]) Let $\Sigma$ and $\Delta$ be ranked alphabets, B be a strong bimonoid, and $\mathcal{A}$ be $a(\Sigma, \mathrm{~B})-w t a$. Moreover, let $\tau$ be $a(\Sigma, \Delta)$-tree relabeling. Then we can construct $a(\Delta, \mathrm{~B})-w t a \mathcal{B}$ such that $\llbracket \mathcal{B} \rrbracket^{\mathrm{run}}=\chi(\tau)\left(\llbracket \mathcal{A} \rrbracket^{\mathrm{run}}\right)$. If, moreover, $\mathcal{A}$ is bu deterministic (or: crisp deterministic) and $\tau$ is non-overlapping, then $\mathcal{B}$ is bu deterministic (and crisp deterministic, respectively). Thus, in particular, the set $\operatorname{Rec}^{\text {run }}(-, \mathrm{B})$ is closed under tree relabelings.

Proof. Let $\mathcal{A}=(Q, \delta, F)$ be a $(\Sigma, \mathrm{B})$-wta and $\tau=\left(\tau_{k} \mid k \in \mathbb{N}\right)$ be a $(\Sigma, \Delta)$-tree relabeling.
We construct the $(\Delta, \mathrm{B})$-wta $\mathcal{B}=\left(Q^{\prime}, \delta^{\prime}, F^{\prime}\right)$ where

- $Q^{\prime}=Q \times \Sigma$
- for every $k \in \mathbb{N}, \gamma \in \Delta^{(k)},(q, \sigma),\left(q_{1}, \sigma_{1}\right), \ldots,\left(q_{k}, \sigma_{k}\right) \in Q \times \Sigma$,

$$
\left(\delta^{\prime}\right)_{k}\left(\left(q_{1}, \sigma_{1}\right) \cdots\left(q_{k}, \sigma_{k}\right), \gamma,(q, \sigma)\right)= \begin{cases}\delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right) & \text { if } \gamma \in \tau_{k}(\sigma) \\ 0 & \text { otherwise }\end{cases}
$$

- $\left(F^{\prime}\right)_{(q, \gamma)}=F_{q}$ for each $(q, \gamma) \in Q^{\prime}$.

In general, the above construction does not preserve bu determinism. For instance, let $a, b \in B$ with $a \neq \mathbb{O} \neq b$, and let $\mathcal{A}$ be a bu deterministic ( $\Sigma, \mathrm{B}$ )-wta with transition mapping $\delta_{1}$ such that $\delta_{1}\left(q, \sigma_{1}, p\right)=a$ and $\delta_{1}\left(q, \sigma_{2}, p^{\prime}\right)=b$. Moreover, let $\tau_{1}\left(\sigma_{1}\right)=\tau_{1}\left(\sigma_{2}\right)=\{\gamma\}$ and $\kappa \in \Sigma$. Then $\left(\delta_{1}^{\prime}\right)\left((q, \kappa), \gamma,\left(p, \sigma_{1}\right)\right)=a$ and $\left(\delta_{1}^{\prime}\right)\left((q, \kappa), \gamma,\left(p^{\prime}, \sigma_{2}\right)\right)=b$ and hence $\mathcal{B}$ is not bu deterministic.

However, if $\tau$ is non-overlapping, then this phenomenon cannot occur. Thus, if $\mathcal{A}$ is bu deterministic and $\tau$ is non-overlapping, then $\mathcal{B}$ is bu deterministic. Moreover, if $\mathcal{A}$ is crisp deterministic and $\tau$ is non-overlapping, then $\mathcal{B}$ is crisp deterministic.

Next we prove that $\llbracket \mathcal{B} \rrbracket^{\text {run }}=\chi(\tau)\left(\llbracket \mathcal{A} \rrbracket^{\text {run }}\right)$. Let $\xi \in \mathrm{T}_{\Sigma}$ and $\zeta \in \tau(\xi)$. We define the mapping $\varphi_{\xi, \zeta}: \mathrm{R}_{\mathcal{A}}(\xi) \rightarrow \mathrm{R}_{\mathcal{B}}(\zeta)$ for every $\rho \in \mathrm{R}_{\mathcal{A}}(\xi)$ and $w \in \operatorname{pos}(\xi)$ by $\varphi_{\xi, \zeta}(\rho)(w)=(\rho(w), \xi(w))$. Obviously, $\varphi_{\xi, \zeta}$ is injective. Moreover, we define the mapping $\varphi_{\xi, \zeta}^{\prime}: \mathrm{R}_{\mathcal{A}}(\xi) \rightarrow \operatorname{im}\left(\varphi_{\xi, \zeta}\right)$ by $\varphi_{\xi, \zeta}^{\prime}(\rho)=\varphi_{\xi, \zeta}(\rho)$ for each $\rho \in \mathrm{R}_{\mathcal{A}}(\xi)$. Clearly, $\varphi_{\xi, \zeta}^{\prime}$ is bijective.

The following statements are easy to see.

$$
\begin{align*}
& \text { For every } \zeta \in \mathrm{T}_{\Delta}, \xi \in \tau^{-1}(\zeta) \text {, and } \rho \in \mathrm{R}_{\mathcal{A}}(\xi): \operatorname{wt}_{\mathcal{A}}(\xi, \rho)=\operatorname{wt}_{\mathcal{B}}\left(\zeta, \varphi_{\xi, \zeta}^{\prime}(\rho)\right)  \tag{10.32}\\
& \qquad \text { For every } \zeta \in \mathrm{T}_{\Delta} \text { and } \rho^{\prime} \in \mathrm{R}_{\mathcal{B}}(\zeta) \backslash \bigcup_{\xi \in \tau^{-1}(\zeta)} \operatorname{im}\left(\varphi_{\xi, \zeta}^{\prime}\right): \operatorname{wt}_{\mathcal{B}}\left(\zeta, \rho^{\prime}\right)=\mathbb{0} \tag{10.33}
\end{align*}
$$

For every $\zeta \in \mathrm{T}_{\Delta}$ and $\xi_{1}, \xi_{2} \in \tau^{-1}(\zeta)$ : if $\xi_{1} \neq \xi_{2}$, then $\operatorname{im}\left(\varphi_{\xi_{1}, \zeta}^{\prime}\right) \cap \operatorname{im}\left(\varphi_{\xi_{2}, \zeta}^{\prime}\right)=\emptyset$.
Let $\zeta \in \mathrm{T}_{\Delta}$. We can calculate as follows.

$$
\llbracket \mathcal{B} \rrbracket^{\mathrm{run}}(\zeta)=\bigoplus_{\rho^{\prime} \in \mathrm{R}_{\mathcal{B}}(\zeta)} \mathrm{wt}_{\mathcal{B}}\left(\zeta, \rho^{\prime}\right) \otimes\left(F^{\prime}\right)_{\rho^{\prime}(\varepsilon)}
$$

$$
\begin{align*}
& =\bigoplus_{\rho^{\prime} \in \cup_{\xi \in \tau^{-1}(\zeta)} \operatorname{im}\left(\varphi_{\xi, \zeta}^{\prime}\right)} \mathrm{wt}_{\mathcal{B}}\left(\zeta, \rho^{\prime}\right) \otimes\left(F^{\prime}\right)_{\rho^{\prime}(\varepsilon)}  \tag{10.33}\\
& =\bigoplus_{\xi \in \tau^{-1}(\zeta)} \bigoplus_{\rho^{\prime} \in \operatorname{im}\left(\varphi_{\xi, \zeta}^{\prime}\right)} \mathrm{wt}_{\mathcal{B}}\left(\zeta, \rho^{\prime}\right) \otimes\left(F^{\prime}\right)_{\rho^{\prime}(\varepsilon)} \\
& =\bigoplus_{\xi \in \tau^{-1}(\zeta)} \bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}}(\xi)} \mathrm{wt}_{\mathcal{B}}\left(\zeta, \varphi_{\xi, \zeta}^{\prime}(\rho)\right) \otimes F_{\rho(\varepsilon)} \\
& =\bigoplus_{\xi \in \tau^{-1}(\zeta)} \bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}}(\xi)} \mathrm{wt}_{\mathcal{A}}(\xi, \rho) \otimes F_{\rho(\varepsilon)} \\
& =\bigoplus_{\xi \in \tau^{-1}(\zeta)} \llbracket \mathcal{A} \rrbracket^{\mathrm{run}}(\xi)=\chi(\tau)\left(\llbracket \mathcal{A} \rrbracket^{\mathrm{run}}\right)(\zeta) . \\
& \text { (by (10.33)) } \\
& \text { (because } \left.\varphi_{\xi, \zeta}^{\prime} \text { is bijective) }\right) \\
& \text { (by (10.32)) }
\end{align*}
$$

As a consequence of Theorem 10.10 .1 we reobtain the well-known closure of recognizable tree languages under tree relabelings.

Corollary 10.10.2. (cf. Eng75b, Thm. 3.48]) Let $A$ be a $\Sigma$-fta and $\tau$ a ( $\Sigma, \Delta$ )-tree relabeling. Then we can construct a $\Delta$-fta $B$ such that $\mathrm{L}(B)=\tau(\mathrm{L}(A))$.

Proof. By Corollary $3.4 .2(\mathrm{~A}) \Rightarrow(\mathrm{B})$, we can construct a $(\Sigma$, Boole $)$-wta $\mathcal{A}$ such that $\mathrm{L}(A)=\operatorname{supp}(\llbracket \mathcal{A} \rrbracket)$. Then we have

$$
\tau(\mathrm{L}(A))=\tau(\operatorname{supp}(\llbracket \mathcal{A} \rrbracket))=\operatorname{supp}(\chi(\tau)(\llbracket \mathcal{A} \rrbracket))
$$

where the last equality is due to (2.31). By Theorem 10.10 .1 we can construct a ( $\Delta$, Boole)-wta $\mathcal{B}$ such that $\llbracket \mathcal{B} \rrbracket=\chi(\tau)(\llbracket \mathcal{A} \rrbracket)$. Finally, by Corollary $3.4 .2(B) \Rightarrow(\mathrm{A})$, we can construct a $\Delta$-fta $B$ such that $\mathrm{L}(B)=\operatorname{supp}(\llbracket \mathcal{B} \rrbracket)$. Hence $\mathrm{L}(B)=\tau(\mathrm{L}(A))$.

### 10.11 Closure under linear and nondeleting tree homomorphisms

In this section we will prove that $\operatorname{Rec}^{\text {run }}\left(\begin{array}{c}, ~ B)\end{array}\right.$ is closed under linear, nondeleting, and productive tree homomorphism. We will even prove a more general closure of regular weighted tree languages under such tree homomorphisms.

Let $h=\left(h_{k} \mid k \in \mathbb{N}\right)$ be a $(\Sigma, \Delta)$-tree homomorphism. We can view its extension $h: \mathrm{T}_{\Sigma} \rightarrow \mathrm{T}_{\Delta}$ as binary relation $h \subseteq \mathrm{~T}_{\Sigma} \times \mathrm{T}_{\Delta}$; then its characteristic mapping has type $\chi(h): \mathrm{T}_{\Sigma} \times \mathrm{T}_{\Delta} \rightarrow B$. We say that $h$ is supp- $i$-finite if $\chi(h): \mathrm{T}_{\Sigma} \times \mathrm{T}_{\Delta} \rightarrow B$ is supp-i-finite, i.e., for each $\zeta \in \mathrm{T}_{\Delta}$, the set $h^{-1}(\zeta)$ is finite.

In general, a tree homomorphism $h$ is not supp-i-finite. For instance, if $\Sigma=\left\{\gamma^{(1)}, \alpha^{(0)}\right\}, \Delta=\left\{\beta^{(0)}\right\}$, and $h(\gamma)=x_{1}$ and $h(\alpha)=\beta$, then $h^{-1}(\beta)=\left\{\gamma^{n}(\alpha) \mid n \in \mathbb{N}\right\}$ is infinite. However, if $h$ is productive and nondeleting, then $h$ is supp-i-finite.

Now let $h$ be a $(\Sigma, \Delta)$-tree homomorphism and $r: \mathrm{T}_{\Sigma} \rightarrow B$ such that $h$ is supp-i-finite or B is $\sigma$ complete. Then the application of $\chi(h): \mathrm{T}_{\Sigma} \times \mathrm{T}_{\Delta} \rightarrow B$ to $r$ is defined by (2.28), and by (2.30), for each $\zeta \in \mathrm{T}_{\Delta}$, we have

$$
\chi(h)(r)(\zeta)=\sum_{\xi \in h^{-1}(\zeta)}^{\oplus} r(\xi)
$$

Let us denote by $\operatorname{Hom}(\Sigma, \Delta)$ the set of all tree homomorphisms from $\Sigma$ to $\Delta$ and let $\mathcal{C}$ be a subset of $\operatorname{Hom}(\Sigma, \Delta)$ such that each $h \in \mathcal{C}$ is supp-i-finite. A set $\mathcal{L}$ of B -weighted tree languages is closed under tree homomorphisms in $\mathcal{C}$ if for every ( $\Sigma, \mathrm{B}$ )-weighted tree language $r \in \mathcal{L}$ and every $(\Sigma, \Delta)$-tree homomorphism $h$ in $\mathcal{C}$, the ( $\Delta, \mathrm{B}$ )-weighted tree language $\chi(h)(r)$ is in $\mathcal{C}$.

The next theorem can be compared to [FMV11, Thm. 5.3] where the closure of the set of recognizable B-weighted tree languages under linear and nondeleting weighted extended tree transformations was
proved if B is a $\sigma$-complete and commutative semiring. We recall that linear and nondeleting tree homomorphisms are particular linear and nondeleting weighted extended tree transducers. (We also refer to Boz99, Prop. 25].) This result is weaker than what one might expect when looking at Eng75b, Thm. 3.65] for the unweighted case. There, it was proved that the set of recognizable tree languages is closed under linear tree homomorphisms (which need not be nondeleting). The underlying construction can be reformulated easily to Boole-weighted tree languages. But we do not know whether it is possible to generalize the latter to B-weighted tree languages because it is not clear how to handle the weights of deleted subtrees.

Theorem 10.11.1. Let $\mathcal{G}$ be $a(\Sigma, \mathrm{~B})$-wrtg such that $\mathcal{G}$ is finite-derivational or B is $\sigma$-complete. Moreover, let $h$ be a linear and nondeleting tree homomorphism from $\Sigma$ to $\Delta$ such that $h$ is productive or B is $\sigma$ complete. Then we can construct $a(\Delta, B)-w r t g \mathcal{G}^{\prime}$ such that (a) if $\mathcal{G}$ is finite-derivational (chain-free) and $h$ is productive, then $\mathcal{G}^{\prime}$ is finite-derivational (chain-free) and (b) $\llbracket \mathcal{G}^{\prime} \rrbracket=\chi(h)(\llbracket \mathcal{G} \rrbracket)$.

Proof. Let $\mathcal{G}=(N, S, R, w t)$ and $h=\left(h_{k} \mid k \in \mathbb{N}\right)$. By Lemma 9.2.2 we can assume that $\mathcal{G}$ is alphabetic. We construct the $(\Delta, \mathrm{B})$-wrtg $\mathcal{G}^{\prime}=\left(N^{\prime}, S^{\prime}, R^{\prime}, w t^{\prime}\right)$, where

- $N^{\prime}=N \times \Sigma$,
- $S^{\prime}=S \times \Sigma$, and
- $R^{\prime}$ is the smallest set of rules satisfying the following conditions:
- for each rule $r=\left(A \rightarrow \sigma\left(A_{1}, \ldots, A_{k}\right)\right)$ in $R$ and for every $\sigma_{1}, \ldots, \sigma_{k} \in \Sigma$, the rule $r^{\prime}=$ $\left((A, \sigma) \rightarrow h_{k}(\sigma)\left[\left(A_{1}, \sigma_{1}\right), \ldots,\left(A_{k}, \sigma_{k}\right)\right]\right)$ is in $R^{\prime}$ and $w t^{\prime}\left(r^{\prime}\right)=w t(r)$, and
- for each rule $r=(A \rightarrow B)$ in $R$ and each $\sigma \in \Sigma$, the rule $r^{\prime}=((A, \sigma) \rightarrow(B, \sigma))$ in $R^{\prime}$ and $w t^{\prime}\left(r^{\prime}\right)=w t(r)$.
Since $h$ is linear and nondeleting, the tree $h_{k}(\sigma)$ is a context for each $\sigma \in \Sigma^{(k)}$ and $k \in \mathbb{N}$. Hence, for each rule of $\mathcal{G}^{\prime}$ of the form $(A, \sigma) \rightarrow h_{k}(\sigma)\left[\left(A_{1}, \sigma_{1}\right), \ldots,\left(A_{k}, \sigma_{k}\right)\right]$, each nonterminal $\left(A_{i}, \sigma_{i}\right)$ occurs exactly once in the right-hand side of that rule. Moreover, if $\mathcal{G}$ is chain-free and $h$ is productive, then $\mathcal{G}^{\prime}$ is chain-free.

We define the mapping

$$
\varphi: \mathrm{RT}_{\mathcal{G}}\left(N, \mathrm{~T}_{\Sigma}\right) \rightarrow \mathrm{RT}_{\mathcal{G}^{\prime}}\left(N^{\prime}, \mathrm{T}_{\Delta}\right)
$$

by induction on the well-founded set $\left(\mathrm{RT}_{\mathcal{G}}\left(N, \mathrm{~T}_{\Sigma}\right), \prec\right)$ where $\prec=\prec_{R} \cap\left(\mathrm{RT}_{\mathcal{G}}\left(N, \mathrm{~T}_{\Sigma}\right) \times \mathrm{RT}_{\mathcal{G}}\left(N, \mathrm{~T}_{\Sigma}\right)\right)$. Then $\min _{\prec}\left(\mathrm{RT}_{\mathcal{G}}\left(N, \mathrm{~T}_{\Sigma}\right)\right)$ is the set of all rules of $R$ with right-hand side in $\Sigma^{(0)}$.

Let $d \in \mathrm{RT}_{\mathcal{G}}\left(N, \mathrm{~T}_{\Sigma}\right)$. We can distinguish the following two cases
Case (a): Let $d=r\left(d_{1}, \ldots, d_{k}\right)$ for some rule $r=\left(A \rightarrow \sigma\left(A_{1}, \ldots, A_{k}\right)\right)$ and $d_{i} \in \mathrm{RT}_{\mathcal{G}}\left(N, \mathrm{~T}_{\Sigma}\right)$ for each $i \in[k]$. Then we let $\varphi(d)=r^{\prime}\left(d_{1}^{\prime}, \ldots, d_{k}^{\prime}\right)$, where $r^{\prime}=\left((A, \sigma) \rightarrow h_{k}(\sigma)\left[\left(A_{1}, \sigma_{1}\right), \ldots,\left(A_{k}, \sigma_{k}\right)\right]\right)$ with $\sigma_{i}=\pi\left(d_{i}\right)(\varepsilon)$ and $d_{i}^{\prime}=\varphi\left(d_{i}\right)$ for each $i \in[k]$.

Case (b): Let $d=r\left(d_{1}\right)$ for some rule $r=(A \rightarrow B)$ in $R$ and $d_{1} \in \operatorname{RT}_{\mathcal{G}}\left(N, \mathrm{~T}_{\Sigma}\right)$. Then we let $\varphi\left(\overline{d)=r^{\prime}\left(d_{1}^{\prime}\right.}\right)$, where $r^{\prime}=((A, \sigma) \rightarrow(B, \sigma))$ and $d_{1}^{\prime}=\varphi\left(d_{1}\right)$.

Due to the construction of $\mathcal{G}^{\prime}$,

$$
\text { for every } A \in N \text { and } \xi \in \mathrm{T}_{\Sigma}, \text { we have } \varphi\left(\mathrm{RT}_{\mathcal{G}}(A, \xi)\right)=\mathrm{RT}_{\mathcal{G}^{\prime}}((A, \xi(\varepsilon)), h(\xi))
$$

and

$$
\begin{equation*}
\text { for every }(A, \sigma) \in N^{\prime} \text { and } \zeta \in \mathrm{T}_{\Delta} \text {, we have } \mathrm{RT}_{\mathcal{G}^{\prime}}((A, \sigma), \zeta)=\bigcup_{\substack{\xi \in h^{-1}(\zeta): \\ \xi(\varepsilon)=\sigma}} \varphi\left(\mathrm{RT}_{\mathcal{G}}(A, \xi)\right) \tag{10.35}
\end{equation*}
$$

It follows that, if $h$ is productive (and thus it is supp-i-finite) and $\mathcal{G}$ is finite-derivational, then $\mathcal{G}^{\prime}$ is finite-derivational. It is easy to see that

$$
\begin{equation*}
\varphi \text { is a bijection and } \mathrm{wt}_{\mathcal{G}}(d)=\mathrm{wt}_{\mathcal{G}^{\prime}}(\varphi(d)) \text { for each } d \in \mathrm{RT}_{\mathcal{G}}\left(N, \mathrm{~T}_{\Sigma}\right) . \tag{10.36}
\end{equation*}
$$

Moreover,

$$
\begin{equation*}
\text { the family }\left(\varphi\left(\operatorname{RT}_{\mathcal{G}}(A, \xi)\right) \mid \xi \in h^{-1}(\zeta), \xi(\varepsilon)=\sigma\right) \text { is a partitioning of } \mathrm{RT}_{\mathcal{G}^{\prime}}((A, \sigma), \zeta) \tag{10.37}
\end{equation*}
$$

because each $d \in \varphi\left(\operatorname{RT}_{\mathcal{G}}(A, \xi)\right)$ encodes $\xi$, thus $\xi \neq \xi^{\prime}$ implies $\varphi\left(\operatorname{RT}_{\mathcal{G}}(A, \xi)\right) \cap \varphi\left(\operatorname{RT}_{\mathcal{G}}\left(A, \xi^{\prime}\right)\right)=\emptyset$. Hence the union on right-hand side of (10.35) is a disjoint union.

Now let $\zeta \in \mathrm{T}_{\Delta}$. Then we can compute as follows

$$
\begin{aligned}
\llbracket \mathcal{G}^{\prime} \rrbracket(\zeta) & =\sum_{d^{\prime} \in \mathrm{RT}_{\mathcal{G}^{\prime}}(\zeta)}^{\oplus} \mathrm{wt}_{\mathcal{G}^{\prime}}\left(d^{\prime}\right)=\bigoplus_{(A, \sigma) \in S^{\prime}} \sum_{d^{\prime} \in \mathrm{RT}_{\mathcal{G}^{\prime}}((A, \sigma), \zeta)}^{\oplus} \mathrm{wt}_{\mathcal{G}^{\prime}}\left(d^{\prime}\right) \\
& ={ }^{(*)} \bigoplus_{(A, \sigma) \in S^{\prime}} \sum_{\substack{\xi \in h^{-1}(\zeta), \xi(\varepsilon)=\sigma \\
d \in \mathrm{RT}_{\mathcal{G}}(A, \xi)}}^{\oplus} \mathrm{wt}_{\mathcal{G}}(d)=\bigoplus_{\xi \in h^{-1}(\zeta)} \sum_{\substack{A \in S \\
d \in \mathrm{RT}_{\mathcal{G}}(A, \xi)}}^{\oplus} \mathrm{wt}_{\mathcal{G}}(d) \\
& =\bigoplus_{\xi \in h^{-1}(\zeta)} \llbracket \mathcal{G} \rrbracket(\xi)=\chi(h)(\llbracket \mathcal{G} \rrbracket)(\zeta),
\end{aligned}
$$

where $(*)$ follows from (10.36) and (10.37).

Corollary 10.11.2. Let $\Sigma$ and $\Delta$ be ranked alphabets, B be a strong bimonoid, and $\mathcal{A}$ be a ( $\Sigma, \mathrm{B})$-wta. Moreover, let $h$ be a linear, nondeleting, and productive tree homomorphism from $\Sigma$ to $\Delta$. Then we can construct a $(\Delta, \mathrm{B})$-wta $\mathcal{A}^{\prime}$ such that $\llbracket \mathcal{A}^{\prime} \rrbracket^{\text {run }}=\chi(h)\left(\llbracket \mathcal{A} \rrbracket^{\text {run }}\right)$. Thus, in particular, the set $\operatorname{Rec}^{\text {run }}(-, \mathrm{B})$ is closed under linear, nondeleting, and productive tree homomorphisms.

Proof. By Lemma 9.2 .6 we can construct $(\Sigma, \mathrm{B})$-wrtg $\mathcal{G}$ such that $\mathcal{G}$ is in tree automata form and $\llbracket \mathcal{A} \rrbracket^{\text {run }}=$ $\llbracket \mathcal{G} \rrbracket$. In particular, $\mathcal{G}$ is chain-free. By Theorem 10.11 .1 we can construct a chain-free $(\Sigma, \mathrm{B})$-wrtg $\mathcal{G}^{\prime}$ such that $\llbracket \mathcal{G}^{\prime} \rrbracket=\chi(h)(\llbracket \mathcal{G} \rrbracket)$. Finally, by Lemma $9.2 .8(\mathrm{a})$, we can construct a $(\Sigma, \mathrm{B})$-wta $\mathcal{A}^{\prime}$ such that $\llbracket \mathcal{G}^{\prime} \rrbracket=\llbracket \mathcal{A}^{\prime} \rrbracket^{\text {run }}$.

### 10.12 Closure under inverse of linear tree homomorphisms

Let $h$ be a $(\Sigma, \Delta)$-tree homomorphism and $r: \mathrm{T}_{\Delta} \rightarrow B$. Since the extension $h: \mathrm{T}_{\Sigma} \rightarrow \mathrm{T}_{\Delta}$ is a particular binary relation $h \subseteq \mathrm{~T}_{\Sigma} \times \mathrm{T}_{\Delta}$, its inverse $h^{-1} \subseteq \mathrm{~T}_{\Delta} \times \mathrm{T}_{\Sigma}$ is well defined. Moreover, $\chi\left(h^{-1}\right): \mathrm{T}_{\Delta} \times \mathrm{T}_{\Sigma} \rightarrow B$ is supp-i-finite, and hence $r$ is $\chi\left(h^{-1}\right)$-summable. Thus, the application of $\chi\left(h^{-1}\right)$ to $r$ is defined by (2.28), and by (2.30) we obtain, for each $\xi \in \mathrm{T}_{\Sigma}$ :

$$
\chi\left(h^{-1}\right)(r)(\xi)=\sum_{\zeta \in h(\xi)}^{\oplus} r(\zeta)=r(h(\xi))
$$

Again let us denote by $\operatorname{Hom}(\Sigma, \Delta)$ the set of all tree homomorphisms from $\Sigma$ to $\Delta$. Let $\mathcal{C}$ be a subset of the set $\operatorname{Hom}(\Sigma, \Delta)$. A set $\mathcal{L}$ of B -weighted tree languages is closed under inverse of tree homomorphisms in $\mathcal{C}$ if for every $(\Delta, \mathrm{B})$-weighted tree language $r \in \mathcal{L}$ and every $(\Sigma, \Delta)$-tree homomorphism $h$ in $\mathcal{C}$, the $(\Sigma, \mathrm{B})$-weighted tree language $\chi\left(h^{-1}\right)(r)=r \circ h$ is in $\mathcal{L}$.

In this section we show that, if $B$ is a commutative semiring, then $\operatorname{Rec}(-, B)$ is closed under the inverse of linear tree homomorphisms.

Let $\mathcal{A}=(Q, \delta, F)$ be a $(\Sigma, \mathrm{B})$-wta. We wish to extend the $\Sigma$-algebra homomorphism $\mathrm{h}_{\mathcal{A}}: \mathrm{T}_{\Sigma} \rightarrow B^{Q}$ to a mapping which can also process trees with variables in $Z_{n}$ (cf. [FV15, p. 85]). Formally, let $n \in \mathbb{N}$ and $q_{1}, \ldots, q_{n} \in Q$. We define the mapping $f: Z_{n} \rightarrow B^{Q}$ such that, for each $z_{i} \in Z_{n}$ and $q \in Q$, we let


Figure 10.10: The value of $\mathrm{h}_{\mathcal{A}}\left(\sigma\left(\sigma\left(\alpha, z_{2}\right), \sigma\left(z_{3}, z_{1}\right)\right)\left[\xi_{1}, \xi_{2}, \xi_{3}\right]\right)$ where on the right-hand side of the equality only one summand (for $q_{1} q_{2} q_{3}=p_{1} p_{2} p_{3}$ in $Q^{3}$ ) is shown.
$f\left(z_{i}\right)_{q}=\mathbb{1}$ if $q=q_{i}$, and $\mathbb{O}$ otherwise. Since the $\Sigma$-term algebra $\left(\mathrm{T}_{\Sigma}\left(Z_{n}\right), \theta_{\Sigma}\right)$ over $Z_{n}$ is freely generated by $Z_{n}$ over the set of all $\Sigma$-algebras (cf. Theorem 2.9.3), there exists a unique extension of $f$ to a $\Sigma$-algebra homomorphism from $\left(\mathrm{T}_{\Sigma}\left(Z_{n}\right), \theta_{\Sigma}\right)$ to the vector algebra $\mathrm{V}(\mathcal{A})=\left(B^{Q}, \delta_{\mathcal{A}}\right)$. We denote this $\Sigma$-algebra homomorphism by $\mathrm{h}_{\mathcal{A}}^{q_{1} \cdots q_{n}}$. Thus, in particular, for every $\sigma \in \Sigma^{(k)}$ with $k \in \mathbb{N}$ and $\xi_{1}, \ldots, \xi_{k} \in \mathrm{~T}_{\Sigma}\left(Z_{n}\right)$, we have

$$
\mathrm{h}_{\mathcal{A}}^{q_{1} \cdots q_{n}}\left(\sigma\left(\xi, \ldots, \xi_{k}\right)\right)_{q}=\bigoplus_{p_{1} \cdots p_{k} \in Q^{k}}\left(\bigotimes_{i \in[k]} \mathrm{h}_{\mathcal{A}}^{q_{1} \cdots q_{n}}\left(\xi_{i}\right)_{p_{i}}\right) \otimes \delta_{k}\left(p_{1} \cdots p_{k}, \sigma, q\right)
$$

Obviously, $\mathrm{h}_{\mathcal{A}}^{\varepsilon}=\mathrm{h}_{\mathcal{A}}$.
In the next lemma, for every $\zeta \in \mathrm{T}_{\Sigma}\left(Z_{n}\right)$ which is linear in $Z_{n}$ and trees $\xi_{1}, \ldots, \xi_{n} \in \mathrm{~T}_{\Sigma}$, we wish to express the value $\mathrm{h}_{\mathcal{A}}\left(\zeta\left[\xi_{1}, \ldots, \xi_{n}\right]\right)_{q} \in B$ by means of the homomorphic images of $\xi_{1}, \ldots, \xi_{n}$ (cf. Figure 10.10). For this we need some technical tools to identify variables in subterms of $\zeta$.

Let $\left(m_{1}, \ldots, m_{l}\right)$ be the sequence of indices of variables occurring in $\zeta$ in a left-to-right order; we denote this sequence by $\operatorname{seq}(\zeta)$. Obviously, $l \leq n$ and $m_{j} \leq n$ for each $j \in[l]$. For instance, if $\zeta=\sigma\left(\sigma\left(\alpha, z_{2}\right), \sigma\left(z_{3}, z_{1}\right)\right)$, then $\operatorname{seq}(\zeta)=(2,3,1)$. We note that, if $\zeta=\sigma\left(\zeta_{1}, \ldots, \zeta_{k}\right)$, then $\operatorname{seq}(\zeta)=\operatorname{seq}\left(\zeta_{1}\right) \cdots \operatorname{seq}\left(\zeta_{k}\right)$ (by dropping intermediate occurrences of ")(").

Moreover, we denote by $\operatorname{lin}(\zeta)$ the tree obtained from $\zeta$ by replacing its variables by $z_{1}, \ldots, z_{l}$ in a left-to-right order. Hence $\operatorname{seq}(\operatorname{lin}(\zeta))=(1, \ldots, l)$. For instance, $\operatorname{lin}\left(\sigma\left(\sigma\left(\alpha, z_{2}\right), \sigma\left(z_{3}, z_{1}\right)\right)\right)=$ $\sigma\left(\sigma\left(\alpha, z_{1}\right), \sigma\left(z_{2}, z_{3}\right)\right)$.

Lemma 10.12.1. (cf. FV15, Lm. 5.4]) Let B be a semiring and $\mathcal{A}=(Q, \delta, F)$ a ( $\Sigma, \mathrm{B})$-wta. Moreover, let $n \in \mathbb{N}$. Then, for every $\zeta \in \mathrm{T}_{\Sigma}\left(Z_{n}\right)$ which $\zeta$ is linear in $Z_{n}, \xi_{1}, \ldots, \xi_{n} \in \mathrm{~T}_{\Sigma}$, and $q \in Q$, we have

$$
\mathrm{h}_{\mathcal{A}}\left(\zeta\left[\xi_{1}, \ldots, \xi_{n}\right]\right)_{q}=\bigoplus_{q_{1} \cdots q_{l} \in Q^{l}}\left(\bigotimes_{j \in[l]} \mathrm{h}_{\mathcal{A}}\left(\xi_{m_{j}}\right)_{q_{j}}\right) \otimes \mathrm{h}_{\mathcal{A}}^{q_{1} \cdots q_{l}}(\operatorname{lin}(\zeta))_{q}
$$

where $\operatorname{seq}(\zeta)=\left(m_{1}, \ldots, m_{l}\right)$.

Proof. We prove the statement by induction on $\mathrm{T}_{\Sigma}\left(Z_{n}\right)$. For this, let $\zeta \in \mathrm{T}_{\Sigma}\left(Z_{n}\right)$ such that $\zeta$ is linear in $Z_{n}$.
I.B.: Let $\zeta=z_{j}$ for some $j \in[n]$. Then $\operatorname{seq}(\zeta)=(j)$ and

$$
\mathrm{h}_{\mathcal{A}}\left(z_{j}\left[\xi_{1}, \ldots, \xi_{n}\right]\right)_{q}=\mathrm{h}_{\mathcal{A}}\left(\xi_{j}\right)_{q}=\bigoplus_{q_{1} \in Q} \mathrm{~h}_{\mathcal{A}}\left(\xi_{j}\right)_{q_{1}} \otimes \mathrm{~h}_{\mathcal{A}}^{q_{1}}\left(z_{j}\right)_{q}
$$

Let $\zeta=\sigma$ for some $\sigma \in \Sigma^{(0)}$. The proof of this case is the proof of case $k=0$ of the I.S.
I.S.: Let $\zeta \in \mathrm{T}_{\Sigma}\left(Z_{n}\right) \backslash Z_{n}$. Then $\zeta=\sigma\left(\zeta_{1}, \ldots, \zeta_{k}\right)$ for some $k \in \mathbb{N}, \sigma \in \Sigma^{(k)}$, and $\zeta_{1}, \ldots, \zeta_{k} \in \mathrm{~T}_{\Sigma}\left(Z_{n}\right)$. For each $j \in[k]$, we have that $\zeta_{j} \in \mathrm{~T}_{\Sigma}\left(Z_{n}\right)$ and $\zeta_{j}$ is linear in $Z_{n}$. Moreover, let $\operatorname{seq}(\zeta)=\left(m_{1}, \ldots, m_{l}\right)$ and let $\operatorname{seq}\left(\zeta_{i}\right)=\left(n_{1}^{i}, \ldots, n_{\kappa_{i}}^{i}\right)$ for each $i \in[k]$. Hence

$$
\begin{equation*}
l=\sum_{j=1}^{k} \kappa_{j} \text { and } \quad\left(m_{1}, \ldots, m_{l}\right)=\left(n_{1}^{1}, \ldots, n_{\kappa_{1}}^{1}, \ldots, n_{1}^{l}, \ldots, n_{\kappa_{l}}^{l}\right) \tag{10.38}
\end{equation*}
$$

Then

$$
\begin{aligned}
& \mathrm{h}_{\mathcal{A}}\left(\zeta\left[\xi_{1}, \ldots, \xi_{n}\right]\right)_{q} \\
& =\bigoplus_{p_{1} \cdots p_{k} \in Q^{k}}\left[\bigotimes_{i \in[k]} \mathrm{h}_{\mathcal{A}}\left(\zeta_{i}\left[\xi_{1}, \ldots, \xi_{n}\right]\right)_{p_{i}}\right] \otimes \delta_{k}\left(p_{1} \cdots p_{k}, \sigma, q\right) \\
& =\bigoplus_{p_{1} \cdots p_{k} \in Q^{k}}\left[\bigotimes_{i \in[k]}\left(\bigoplus_{q_{1}^{i} \cdots q_{\kappa_{i}}^{i} \in Q^{\kappa_{i}}}\left(\bigotimes_{j \in\left[\kappa_{i}\right]} \mathrm{h}_{\mathcal{A}}\left(\xi_{n_{j}^{i}}\right)_{q_{j}^{i}}\right) \otimes \mathrm{h}_{\mathcal{A}}^{q_{1}^{i} \cdots q_{\kappa_{i}}^{i}}\left(\operatorname{lin}\left(\zeta_{i}\right)\right)_{p_{i}}\right)\right] \otimes \delta_{k}\left(p_{1} \cdots p_{k}, \sigma, q\right) \quad \text { (by I.H.) } \\
& =\bigoplus_{p_{1} \cdots p_{k} \in Q^{k}} \bigoplus_{q_{1}^{1} \cdots q_{\kappa_{1}}^{1} \in Q^{\kappa_{1}}} \cdots \bigoplus_{q_{1}^{k} \cdots q_{\kappa_{k}}^{k} \in Q^{\kappa_{k}}} \\
& \bigotimes_{i \in[k]}\left(\left(\bigotimes_{j \in\left[\kappa_{i}\right]} \mathrm{h}_{\mathcal{A}}\left(\xi_{n_{j}^{i}}\right)_{q_{j}^{i}}\right) \otimes \mathrm{h}_{\mathcal{A}}^{q_{1}^{i} \cdots q_{\kappa_{i}}^{i}}\left(\operatorname{lin}\left(\zeta_{i}\right)\right)_{p_{i}}\right) \otimes \delta_{k}\left(p_{1} \cdots p_{k}, \sigma, q\right) \quad \quad \text { (by right-distributivity) } \\
& =\bigoplus_{q_{1}^{1} \cdots q_{\kappa_{1}}^{1} \in Q^{\kappa_{1}}} \cdots \bigoplus_{q_{1}^{k} \cdots q_{\kappa_{k}}^{k} \in Q^{\kappa_{k}}}\left(\bigotimes_{i \in[k]} \bigotimes_{j \in\left[\kappa_{i}\right]} \mathrm{h}_{\mathcal{A}}\left(\xi_{n_{j}^{i}}\right)_{q_{j}^{i}}\right) \otimes \\
& \bigoplus_{p_{1} \cdots p_{k} \in Q^{k}}\left(\bigotimes_{i \in[k]} \mathrm{h}_{\mathcal{A}}^{q_{1}^{i} \cdots q_{\kappa_{i}}^{i}}\left(\operatorname{lin}\left(\zeta_{i}\right)\right)_{p_{i}}\right) \otimes \delta_{k}\left(p_{1} \cdots p_{k}, \sigma, q\right) \quad \text { (by left-distributivity) } \\
& =\bigoplus_{q_{1} \cdots q_{l} \in Q^{l}}\left(\bigotimes_{j \in[l]} \mathrm{h}_{\mathcal{A}}\left(\xi_{m_{j}}\right)_{q_{j}}\right) \otimes \mathrm{h}_{\mathcal{A}}^{q_{1} \cdots q_{l}}(\operatorname{lin}(\zeta))_{q}
\end{aligned}
$$

where the last equality uses (10.38) and the fact that

$$
\bigoplus_{p_{1} \cdots p_{k} \in Q^{k}}\left(\bigotimes_{i \in[k]} \mathrm{h}_{\mathcal{A}}^{q_{1}^{i} \cdots q_{\kappa_{i}}^{i}}\left(\operatorname{lin}\left(\zeta_{i}\right)\right)_{p_{i}}\right) \otimes \delta_{k}\left(p_{1} \cdots p_{k}, \sigma, q\right)=\mathrm{h}_{\mathcal{A}}^{q^{1} \cdots q^{l}}(\operatorname{lin}(\zeta))_{q}
$$

where $q^{j}=q_{1}^{j} \cdots q_{\kappa_{j}}^{j}$.
In the next theorem we apply Lemma 10.12 .1 to the case where $\zeta=h_{k}(\sigma)$ for some linear tree homomorphism $h=\left(h_{k} \mid k \in \mathbb{N}\right)$. It is helpful to view $\operatorname{seq}\left(h_{k}(\sigma)\right)$ as a set of indices; then we can write something like $i \notin \operatorname{seq}\left(h_{k}(\sigma)\right)$ with the obvious meaning.

The next theorem can be compared to [FMV11, Thm. 5.1] where the closure of the set of recognizable B-weighted tree languages under inverse of linear weighted extended tree transformations was proved if B is a $\sigma$-complete and commutative semiring. We recall that linear tree homomorphisms are particular linear weighted extended tree transducers.

Theorem 10.12.2. Let $\Sigma$ and $\Delta$ be ranked alphabets, B be a commutative semiring, and $\mathcal{A}$ be $a(\Delta, \mathrm{~B})$ wta. Moreover, let $h$ be a linear $(\Sigma, \Delta)$-tree homomorphism. Then we can construct a $(\Sigma, \mathrm{B})$-wta $\mathcal{B}$ such that $\llbracket \mathcal{B} \rrbracket=\chi\left(h^{-1}\right)(\llbracket \mathcal{A} \rrbracket)$. Thus, in particular, the set $\operatorname{Rec}\left({ }_{-}, \mathrm{B}\right)$ is closed under the inverse of linear tree homomorphisms.

Proof. Let $\mathcal{A}=(Q, \delta, F)$ and $h=\left(h_{k} \mid k \in \mathbb{N}\right)$. For the construction of $\mathcal{B}$ we use a technique from Eng75a, Thm. 2.8]: it adds a new state $e$ which takes care of those subtrees which are deleted by the tree homomorphism $h$ (the latter corresponding to the given linear top-down tree transducer in Eng75a, Thm. 2.8]).

Formally, we construct $\mathcal{B}=\left(Q^{\prime}, \delta^{\prime}, F^{\prime}\right)$ such that $Q^{\prime}=Q \cup\{e\}$ and for every $k \in \mathbb{N}, \sigma \in \Sigma^{(k)}$, $q_{1}, \ldots, q_{k}, q \in Q^{\prime}$ we let

$$
\left(\delta^{\prime}\right)_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right)= \begin{cases}\mathrm{h}_{\mathcal{A}}^{w}\left(\operatorname{lin}\left(h_{k}(\sigma)\right)\right)_{q} & \text { if } q \in Q \text { and }(\forall i \in[k]): q_{i}=e \text { iff } i \notin \operatorname{seq}\left(h_{k}(\sigma)\right) \\ \mathbb{1} & \text { if } q=q_{1}=\ldots=q_{k}=e \\ \mathbb{O} & \text { otherwise }\end{cases}
$$

where $w=q_{m_{1}} \cdots q_{m_{l}}$ if $\operatorname{seq}\left(h_{k}(\sigma)\right)=\left(m_{1}, \ldots, m_{l}\right)$. Moreover, we let $\left(F^{\prime}\right)_{q}=F_{q}$ for each $q \in Q$, and $\left(F^{\prime}\right)_{e}=\mathbb{0}$.

The following property is easy to see.

$$
\begin{equation*}
\text { For every } \xi \in \mathrm{T}_{\Sigma} \text { we have: } \mathrm{h}_{\mathcal{B}}(\xi)_{e}=\mathbb{1} \tag{10.39}
\end{equation*}
$$

By induction on $\mathrm{T}_{\Sigma}$, we prove that the following statement holds:

$$
\begin{equation*}
\text { For every } \xi \in \mathrm{T}_{\Sigma} \text { and } q \in Q \text { we have: } \mathrm{h}_{\mathcal{B}}(\xi)_{q}=\mathrm{h}_{\mathcal{A}}(h(\xi))_{q} . \tag{10.40}
\end{equation*}
$$

Let $\xi=\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)$. We assume that $\operatorname{seq}\left(h_{k}(\sigma)\right)=\left(m_{1}, \ldots, m_{l}\right)$. Then we can compute as follows.

$$
\begin{aligned}
& \mathrm{h}_{\mathcal{B}}\left(\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)\right)_{q}=\bigoplus_{q_{1} \cdots q_{k} \in\left(Q^{\prime}\right)^{k}}\left(\bigotimes_{i \in[k]} \mathrm{h}_{\mathcal{B}}\left(\xi_{i}\right)_{q_{i}}\right) \otimes\left(\delta^{\prime}\right)_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right) \\
& =\bigoplus_{q_{1} \cdots q_{k} \in\left(Q^{\prime}\right)^{k}}\left(\bigotimes_{i \in[k]}\left\{\begin{array}{ll}
\mathrm{h}_{\mathcal{A}}\left(h\left(\xi_{i}\right)\right)_{q_{i}} & \text { if } q_{i} \in Q \\
\mathbb{1} & \text { otherwise }
\end{array}\right) \otimes\left(\delta^{\prime}\right)_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right)\right. \\
& \text { (by I.H. and (10.39) ) } \\
& =\bigoplus_{\substack{q_{1} \cdots q_{k} \in\left(Q^{\prime}\right)^{k}: \\
q_{i}=e \stackrel{\text { eseq }}{\Longleftrightarrow}}}\left(\bigotimes_{i \in[k]}\left\{\begin{array}{ll}
\mathrm{h}_{\mathcal{A}}\left(h\left(\xi_{i}\right)\right)_{q_{i}} & \text { if } q_{i} \in Q \\
\mathbb{1} & \text { otherwise }
\end{array}\right) \otimes \mathrm{h}_{\mathcal{A}}^{w}\left(\operatorname{lin}\left(h_{k}(\sigma)\right)\right)_{q}\right. \\
& \text { (by construction where } w=q_{m_{1}} \cdots q_{m_{l}} \text { ) } \\
& =\bigoplus_{q_{1} \cdots q_{l} \in Q^{l}}\left(\bigotimes_{j \in[l]} \mathrm{h}_{\mathcal{A}}\left(h\left(\xi_{m_{j}}\right)\right)_{q_{j}}\right) \otimes \mathrm{h}_{\mathcal{A}}^{q_{1} \cdots q_{l}}\left(\operatorname{lin}\left(h_{k}(\sigma)\right)\right)_{q} \quad \text { (by the commutativity of B) } \\
& =\mathrm{h}_{\mathcal{A}}\left(h_{k}(\sigma)\left[h\left(\xi_{1}\right), \ldots, h\left(\xi_{k}\right)\right]\right)_{q} \quad \quad \text { (by Lemma 10.12.1) } \\
& =\mathrm{h}_{\mathcal{A}}\left(h\left(\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)\right)_{q} .\right.
\end{aligned}
$$

Then, for each $\xi \in \mathrm{T}_{\Sigma}$ we can compute as follows.

$$
\begin{align*}
\llbracket \mathcal{B} \rrbracket(\xi) & =\bigoplus_{q \in Q^{\prime}} \mathrm{h}_{\mathcal{B}}(\xi)_{q} \otimes\left(F^{\prime}\right)_{q}=\bigoplus_{q \in Q} \mathrm{~h}_{\mathcal{B}}(\xi)_{q} \otimes F_{q} \\
& =\bigoplus_{q \in Q} \mathrm{~h}_{\mathcal{A}}(h(\xi))_{q} \otimes F_{q}  \tag{10.40}\\
& =\llbracket \mathcal{A} \rrbracket(h(\xi))=\chi\left(h^{-1}\right)(\llbracket \mathcal{A} \rrbracket)(\xi) .
\end{align*}
$$

We note that in [GS84, Thm. 2.4.18] (also cf. Eng75b, Cor. 4.60]) it was proved that the set of recognizable tree languages is closed under inverse of arbitrary tree homomorphisms. The same is true for crisp deterministically recognizable weighted tree languages over Unitlnt ${ }_{u, i}$ BLB10, Prop. 6] (taking Theorem $4.3 .5(\mathrm{~A}) \Leftrightarrow(\mathrm{B})$ into account). However, in the arbitrary weighted case, the restriction to linear tree homomorphisms is important. In fact, we can give a (nonlinear) tree homomorphism $h$ and a bu deterministic wta $\mathcal{A}$ over the semiring Nat such that $\chi\left(h^{-1}\right)(\llbracket \mathcal{A} \rrbracket)$ is not recognizable.
Example 10.12.3. FMV11, Ex. 5.1] Let $\Sigma=\left\{\gamma^{(1)}, \alpha^{(0)}\right\}$ and $\Delta=\left\{\sigma^{(2)}, \alpha^{(0)}\right\}$. For each $n \in \mathbb{N}$, let $\zeta_{n}$ be the fully balanced tree of height $n$ over $\Delta$, which is defined by $\zeta_{0}=\alpha$ and $\zeta_{n}=\sigma\left(\zeta_{n-1}, \zeta_{n-1}\right)$ for every $n \geq 1$. We consider the $(\Sigma, \Delta)$-tree homomorphism with $h_{1}(\gamma)=\sigma\left(z_{1}, z_{1}\right)$ and $h_{0}(\alpha)=\alpha$. It is clear that $h\left(\gamma^{n}(\alpha)\right)=\zeta_{n}$ for each $n \in \mathbb{N}$ where $\gamma^{n}(\alpha)$ abbreviates the tree $\gamma(\gamma(\cdots \gamma(\alpha) \cdots))$ containing $n$ times the symbol $\gamma$.

Moreover, we consider the bu deterministic ( $\Delta$, Nat)-wta $\mathcal{A}=(\{q\}, \delta, F)$, where $\delta_{2}(q q, \sigma, q)=1$, $\delta_{0}(\varepsilon, \alpha, q)=2$, and $F_{q}=1$. For each $\zeta \in \mathrm{T}_{\Delta}$, we have $\llbracket \mathcal{A} \rrbracket(\zeta)=2^{n}$, where $n$ is the number of occurrences of $\alpha$ in $\zeta$. Consequently, we obtain that

$$
\chi\left(h^{-1}\right)(\llbracket \mathcal{A} \rrbracket)\left(\gamma^{n}(\alpha)\right)=\llbracket \mathscr{A} \rrbracket\left(h\left(\gamma^{n}(\alpha)\right)\right)=\llbracket \mathcal{A} \rrbracket\left(\zeta_{n}\right)=2^{2^{n}}
$$

for each $n \in \mathbb{N}$.
Now we can easily show by contradiction that the weighted tree language $\chi\left(h^{-1}\right)(\llbracket \mathcal{A} \rrbracket)$ is not recognizable. For this, we assume there exists a $\left(\Delta\right.$, Nat)-wta $\mathcal{B}=\left(Q, \delta^{\prime}, F^{\prime}\right)$ such that $\llbracket \mathcal{B} \rrbracket=\chi\left(h^{-1}\right)(\llbracket \mathcal{A} \rrbracket)$.

By Lemma 3.5.6, there exists a $K \in \mathbb{N}$ such that, for each $n \in \mathbb{N}$, we have $\llbracket \mathcal{B} \rrbracket\left(\gamma^{n}(\alpha)\right) \leq K^{n+1}$. Then, for each $n \in \mathbb{N}$, we have $2^{2^{n}}=\chi\left(h^{-1}\right)(\llbracket \mathcal{A} \rrbracket)\left(\gamma^{n}(\alpha)\right)=\llbracket \mathcal{B} \rrbracket\left(\gamma^{n}(\alpha)\right) \leq K^{n+1}$. But this is not true. Hence, there does not exist a $(\Delta, N a t)$-wta $\mathcal{B}=\left(Q, \delta^{\prime}, F^{\prime}\right)$ such that $\llbracket \mathcal{B} \rrbracket=\chi\left(h^{-1}\right)(\llbracket \mathcal{A} \rrbracket)$.

### 10.13 Closure under weighted projective bimorphisms

In this section, we let $\Psi$ be a ranked alphabet. We define the concept of $(\Sigma, \Psi, \mathrm{B})$-weighted projective bimorphism as special weighted regular tree grammar; each such grammar $\mathcal{H}$ computes a $(\Sigma, \Psi, \mathrm{B})$-weighted tree transformation, denoted by $\llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}$; we also call $\llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}$ a $(\Sigma, \Psi, \mathrm{B})$-weighted projective bimorphism. Intuitively, a weighted projective bimorphism is a tree relabeling in which, additionally, unary input symbols can be deleted without producing output and unary output symbols can be produced without consuming an input symbol.

Then we prove that, roughly speaking, the sets $\operatorname{Reg}(,, B)$ and $\operatorname{Rec}(-, B)$ are closed under weighted projective bimorphisms if B is a commutative semiring. Given an alphabetic $(\Sigma, \mathrm{B})$-wrtg $\mathcal{G}$ and a $(\Sigma, \Psi, \mathrm{B})$ weighted projective bimorphism $\mathcal{H}$, we will proceed in four steps as follows:

1. we split the semantics of $\mathcal{G}$ into an $([R], \Sigma, \mathrm{B})$-weighted projective bimorphism $\mathcal{H}_{\mathcal{G}}$ and a characteristic mapping $\chi\left(\mathrm{T}_{[R]}\right)$ such that $\llbracket \mathcal{G} \rrbracket=\llbracket \mathcal{H}_{\mathcal{G}} \rrbracket^{\text {tt }}\left(\chi\left(\mathrm{T}_{[R]}\right)\right)$, where $R$ is the set of rules of $\mathcal{G}$ viewed as ranked alphabet and $[R]$ is the skeleton alphabet of $R$ (cf. Lemma 10.13.6),
2. we prove that weighted projective bimorphisms are closed under composition (cf. Theorem 10.13.7); thus, in particular, we can construct an $([R], \Psi, \mathrm{B})$-weighted projective bimorphism $\mathcal{H}^{\prime}$ such that $\llbracket \mathcal{H}^{\prime} \rrbracket^{\mathrm{tt}}=\llbracket \mathcal{H} \mathcal{G}_{\mathcal{G}} \rrbracket^{\mathrm{tt}} ; \llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}$,
3. we merge $\mathcal{H}^{\prime}$ and the characteristic mapping $\chi\left(\mathrm{T}_{[R]}\right)$ into a $(\Psi, \mathrm{B})$-wrtg $\mathcal{G}^{\prime}$ such that $\llbracket \mathcal{G}^{\prime} \rrbracket=$ $\llbracket \mathcal{H}^{\prime} \rrbracket^{\text {tt }}\left(\chi\left(\mathrm{T}_{[R]}\right)\right)$ (cf. Lemma 10.13.8), and
4. finally, we deduce the mentioned closure results for $\operatorname{Reg}(-, B)$ and $\operatorname{Rec}(-, B)$ (cf. Theorem 10.13 .9 and Corollary 10.13.10).
On first glance, this split-compose-merge procedure looks too complicated for the purpose of just showing the closure of recognizable weighted tree languages under weighted projective bimorphisms. However, in Chapter 15 we will need the result of the second step (i.e., closure of weighted projective bimorphisms under composition), and we want to benefit from this result already here in our current setting.

### 10.13.1 Weighted projective bimorphisms

We define the ranked alphabet $[\Sigma \Psi]$ such that, for each $k \in \mathbb{N}$, we let $[\Sigma \Psi]^{(k)}=\Sigma^{(k)} \times \Psi^{(k)}$ if $k \neq 1$, and we let $[\Sigma \Psi]^{(1)}=\left(\left(\Sigma^{(1)} \cup\{\varepsilon\}\right) \times\left(\Psi^{(1)} \cup\{\varepsilon\}\right)\right) \backslash\{(\varepsilon, \varepsilon)\}$.

Moreover, we define the tree homomorphism $\pi_{1}: \mathrm{T}_{[\Sigma \Psi]} \rightarrow \mathrm{T}_{\Sigma}$ and the tree homomorphism $\pi_{2}$ : $\mathrm{T}_{[\Sigma \Psi]} \rightarrow \mathrm{T}_{\Psi}$ which, intuitively, project to the first component of a symbol $[\sigma, \psi]$ and to the second component, respectively. Formally, the tree homomorphism $\pi_{1}$ is induced by the family $\left(\left(\pi_{1}\right)_{k} \mid k \in \mathbb{N}\right)$ of mappings $\left(\pi_{1}\right)_{k}:[\Sigma \Psi]^{(k)} \rightarrow \mathrm{T}_{\Sigma}\left(X_{k}\right)$ such that, for every $k \in \mathbb{N}$ and $[\sigma, \psi] \in[\Sigma \Psi]^{(k)}$, we let

$$
\left(\pi_{1}\right)_{k}([\sigma, \psi])= \begin{cases}\sigma\left(x_{1}, \ldots, x_{k}\right) & \text { if } \sigma \neq \varepsilon \\ x_{1} & \text { otherwise }\end{cases}
$$

We recall that $\sigma=\varepsilon$ is possible only in case $k=1$. Analogously, we define the family $\left(\left(\pi_{2}\right)_{k} \mid k \in \mathbb{N}\right)$ of mappings $\left(\pi_{2}\right)_{k}:[\Sigma \Psi]^{(k)} \rightarrow \mathrm{T}_{\Psi}\left(X_{k}\right)$ which induces the tree homomorphism $\pi_{2}$. Obviously, $\pi_{1}$ and $\pi_{2}$ are linear and nondeleting; moreover, for each $[\sigma, \psi] \in[\Sigma \Psi]^{(k)}$, we have height $\left(\pi_{1}([\sigma, \psi])\right) \leq 1$ and $\operatorname{height}\left(\pi_{2}([\sigma, \psi])\right) \leq 1$.

A B-weighted projective bimorphism over $\Sigma$ and $\Psi$ (for short: $(\Sigma, \Psi, \mathrm{B})$-wpb, or just: wpb) is an alphabetic $([\Sigma \Psi], \mathrm{B})-$ wrtg $\mathcal{H}=(N, S, R, w t)$. The alphabets $\Sigma$ and $\Psi$ are called input alphabet and output alphabet of $\mathcal{H}$, respectively; correspondingly, trees in $\mathrm{T}_{\Sigma}$ and $\mathrm{T}_{\Psi}$ are called input trees and output trees of $\mathcal{H}$, respectively.

Obviously, a $(\Sigma, \Psi, \mathrm{B})$-wpb $\mathcal{H}$ can contain two types of rules:

1. $A \rightarrow[\sigma, \psi]\left(A_{1}, \ldots, A_{k}\right)$ with $k \in \mathbb{N}$ and $[\sigma, \psi] \in[\Sigma, \Psi]^{(k)}$, and
2. $A \rightarrow B$ (chain rule).

Let us consider the first type of rules. We call $\sigma$ the input of $r$ and $\psi$ the output of $r$, and we denote them by $\operatorname{inp}(r)$ and out $(r)$, respectively. Moreover, we give the names shown in the second column of the following table to such rules depending on whether they read a symbol from the input or not and whether they write a symbol to the output or not; in the last column we show the denotation for the set of all rules of a particular type.

| $A \rightarrow[\sigma, \psi]\left(A_{1}, \ldots, A_{k}\right)$ in $R$ | type of the rule <br> r= read, w=write | denotation for the set <br> of all rules of that type |
| :---: | :---: | :---: |
| $\sigma \neq \varepsilon$ and $\psi \neq \varepsilon$ | rw-rule | $R^{\text {rw }}$ |
| $\sigma=\varepsilon$ and $\psi \neq \varepsilon$ | $\varepsilon \mathrm{w}$-rule | $R^{\varepsilon \mathrm{w}}$ |
| $\sigma \neq \varepsilon$ and $\psi=\varepsilon$ | r - rule | $R^{\mathrm{r} \varepsilon}$ |

Moreover, we let $R^{-\mathrm{w}}=R^{\mathrm{rw}} \cup R^{\varepsilon \mathrm{w}}$ and $R^{\mathrm{r}-}=R^{\mathrm{rw}} \cup R^{\mathrm{r} \varepsilon}$. By viewing $R$ as a ranked alphabet, both $\varepsilon w$-rules and $r \varepsilon$-rules have rank 1, i.e., $R^{\varepsilon w} \cup R^{r \varepsilon} \subseteq R^{(1)}$, and each chain rule is in $R^{(1)}$. Finally, we note that, since each wpb is a particular wrtg (and in its turn, each wrtg is a particular wcfg; and in its turn, each wcfg is a cfg equipped with a weight mapping, cf. Figure (1.1), we have the concept of rule tree of $\mathcal{H}$ available, as well as the abbreviation $\mathrm{RT}_{\mathcal{H}}\left(N^{\prime}, \eta\right)$ for each $N^{\prime} \subseteq N$ and $\eta \in \mathrm{T}_{[\Sigma \Psi]}$, cf. page [55],

Now we define the weighted tree transformation computed by a $(\Sigma, \Psi, \mathrm{B})$-wpb $\mathcal{H}=(N, S, R, w t)$. For this, let $N^{\prime} \subseteq N, \xi \in \mathrm{~T}_{\Sigma}$, and $\zeta \in \mathrm{T}_{\Psi}$. We define

$$
\begin{equation*}
\operatorname{RT}_{\mathcal{H}}\left(N^{\prime}, \xi, \zeta\right)=\bigcup_{\substack{\eta \in \mathrm{T}_{(\Sigma \Psi 1]}: \\ \pi_{1}(\eta)=\xi, \pi_{2}(\eta)=\zeta}} \mathrm{RT}_{\mathcal{H}}\left(N^{\prime}, \eta\right) \tag{10.41}
\end{equation*}
$$

and we define $\mathrm{RT}_{\mathcal{H}}(\xi, \zeta)=\mathrm{RT}_{\mathcal{H}}(S, \xi, \zeta)$. Then the family

$$
\begin{equation*}
\left(\mathrm{RT}_{\mathcal{H}}\left(N^{\prime}, \eta\right) \mid \eta \in \mathrm{T}_{[\Sigma \Psi]}: \pi_{1}(\eta)=\xi, \pi_{2}(\eta)=\zeta\right) \tag{10.42}
\end{equation*}
$$

is a partitioning of the set $\mathrm{RT}_{\mathcal{H}}\left(N^{\prime}, \xi, \zeta\right)$ because (10.41) holds and $\eta \neq \eta^{\prime}$ implies that $\mathrm{RT}_{\mathcal{H}}\left(N^{\prime}, \eta\right) \cap$ $\mathrm{RT}_{\mathcal{H}}\left(N^{\prime}, \eta^{\prime}\right)=\emptyset$. Moreover, the set $\left\{\eta \in \mathrm{T}_{[\Sigma \Psi]} \mid \pi_{1}(\eta)=\xi, \pi_{2}(\eta)=\zeta\right\}$ is finite, hence (10.42) is a


Figure 10.11: An illustration of the relationships between the mappings $\pi_{\mathcal{H}}, \pi_{1}$, and $\pi_{2}$.
partitioning into finitely many subsets. Due to this fact, $\mathcal{H}$ is finite-derivational if and only if, for every $\xi \in \mathrm{T}_{\Sigma}$ and $\zeta \in \mathrm{T}_{\Psi}$, the set $\mathrm{RT}_{\mathcal{H}}(\xi, \zeta)$ is finite.

We call $\mathcal{H}$

- finite-input if for every $\zeta \in \mathrm{T}_{\Psi}$, the set $\left\{\xi \in \mathrm{T}_{\Sigma} \mid \mathrm{RT}_{\mathcal{H}}(\xi, \zeta) \neq \emptyset\right\}$ is finite;
- finite-output if for every $\xi \in \mathrm{T}_{\Sigma}$, the set $\left\{\zeta \in \mathrm{T}_{\Psi} \mid \mathrm{RT}_{\mathcal{H}}(\xi, \zeta) \neq \emptyset\right\}$ is finite.

If $R=R^{-\mathrm{w}}$, then $\mathcal{H}$ is finite-input, and if $R=R^{\mathrm{r}}$, then $\mathcal{H}$ is finite-output.
If $\mathcal{H}$ is finite-derivational or B is $\sigma$-complete, then we define the weighted projective bimorphism computed by $\mathcal{H}$ to be the weighted tree transformation $\llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}: \mathrm{T}_{\Sigma} \times \mathrm{T}_{\Psi} \rightarrow B$ such that, for every $\xi \in \mathrm{T}_{\Sigma}$ and $\zeta \in \mathrm{T}_{\Psi}$, we have

$$
\llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}(\xi, \zeta)=\sum_{d \in \mathrm{RT}_{\mathcal{H}}(\xi, \zeta)}^{\oplus} \mathrm{wt}_{\mathcal{H}}(d)
$$

where $\mathrm{wt}_{\mathcal{H}}: \mathrm{T}_{R} \rightarrow B$ is the weight mapping defined in (8.1) (viewing $\mathcal{H}$ as wcfg).
It is easy to see that, if $\mathcal{H}$ is finite-input, then $\llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}$ is supp-i-finite, and if $\mathcal{H}$ is finite-output, then $\llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}$ is supp-o-finite.

Let $\tau: \mathrm{T}_{\Sigma} \times \mathrm{T}_{\Psi} \rightarrow B$ be a weighted tree transformation. We say that $\tau$ is a $(\Sigma, \Psi, \mathrm{B})$-weighted projective bimorphism (or just: weighted projective bimorphism) if there exists a $(\Sigma, \Psi, \mathrm{B})$-wpb $\mathcal{H}$ such that $\mathcal{H}$ is finite-derivational if B is not $\sigma$-complete, and $\llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}=\tau$.

A set $\mathcal{L}$ of B -weighted tree languages is closed under weighted projective bimorphisms if for every ( $\Sigma, \mathrm{B}$ )-weighted tree language $r \in \mathcal{L}$ and every $(\Sigma, \Psi, \mathrm{B})$-bimorphism $\mathcal{H}$, the ( $\Psi, \mathrm{B})$-weighted tree language $\llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}(r)$ is in $\mathcal{L}$.

In Figure 10.11 we illustrate the relationships between the mappings $\pi_{\mathcal{H}}, \pi_{1}$, and $\pi_{2}$.
Up to syntactic modifications, the concept of $(\Sigma, \Psi, \mathrm{B})$-wpb is the same as the concept of B -weighted generalized relabeling tree transducers over $\Sigma$ and $\Psi$ (for short: $(\Sigma, \Psi, \mathrm{B})$-transducer) as it was defined in FV22b. The syntactic modifications are shown in the following table where we assume that $\sigma \in \Sigma^{(k)}$ and $\psi \in \Psi^{(k)}$ :

| $k \in \mathbb{N}$ | rule of a $(\Sigma, \Psi, \mathrm{B})$-wpb | rule of a $(\Sigma, \Psi, \mathrm{B})$-transducer |
| :--- | :--- | :--- |
| $k \geq 0$ | $A \rightarrow[\sigma, \psi]\left(A_{1}, \ldots, A_{k}\right)$ | $\left(A, \sigma\left(x_{1}, \ldots, x_{k}\right), A_{1} \cdots A_{k}, \psi\left(x_{1}, \ldots, x_{k}\right)\right)$ |
| $k=1$ | $A \rightarrow[\sigma, \varepsilon]\left(A_{1}\right)$ | $\left(A, \sigma\left(x_{1}\right), A_{1}, x_{1}\right)$ |
| $k=1$ | $A \rightarrow[\varepsilon, \psi]\left(A_{1}\right)$ | $\left(A, x_{1}, A_{1}, \psi\left(x_{1}\right)\right)$ |
| $k=1$ | $A \rightarrow A_{1}$ | $\left(A, x_{1}, A_{1}, x_{1}\right)$ |

In FV22b] we have assumed that B is a commutative and $\sigma$-complete semiring. Here we are more liberal and do not require per se that B is $\sigma$-complete (and commutative). As a consequence, we have to be more careful with the definedness of the semantics $\llbracket \mathcal{H} \rrbracket^{\text {tt }}$ of a weighted projective bimorphism $\mathcal{H}$.

Also, we refer the reader to Theorem 15.2 .1 where the weighted tree transformation $\llbracket \mathcal{H} \rrbracket^{\text {tt }}$ is decomposed into the inverse of a simple tree homomorphism, followed by the Hadamard product with the recognizable weighted tree language $\llbracket \mathcal{H} \rrbracket^{\text {wrt }}$, followed by a simple tree homomorphism; this decomposition connects our definition of wpb with the classical definition of bimorphisms in AD76, AD82. For details we refer to Section 15.2 ,

Finally, we note that weighted projective bimorphisms are particular linear and nondeleting weighted extended tree transducers [FMV11 in which each rule processes at most one input symbol and generates at most one output symbol. For a comparison to linear nondeleting recognizable tree transducers Kui99c] we refer to [FMV11.
Example 10.13.1. We consider the ranked alphabets $\Sigma=\left\{\sigma^{(2)}, \gamma^{(1)}, \alpha^{(0)}\right\}$ and $\Psi=\left\{\sigma^{(2)}, \gamma^{\prime(1)}, \alpha^{(0)}\right\}$. As weight algebra we consider the semiring $\operatorname{Nat}=(\mathbb{N},+, \cdot, 0,1)$ of natural numbers.

We wish to define a $(\Sigma, \Psi, N a t)-$ wpb $\mathcal{H}$ such that, for each $(\xi, \zeta) \in \operatorname{supp}\left(\llbracket \mathcal{H} \rrbracket^{\text {tt }}\right)$, there exists a tree $\kappa \in \mathrm{T}_{\left\{\sigma^{(2)}, \alpha^{(0)}\right\}}$ (called kernel of $(\xi, \zeta)$ ) and (a) $\xi$ is obtained from $\kappa$ by inserting above an arbitrary position an arbitrary number of $\gamma$-labeled positions and (b) $\zeta$ is obtained from $\kappa$ by inserting above an arbitrary position an arbitrary number of $\gamma^{\prime}$-labeled positions; moreover, the insertions of $\gamma$ into $\xi$ and of $\gamma^{\prime}$ into $\zeta$ can happen interleaved and in an arbitrary ordering. Then, we define $\mathcal{H}$ such that $\llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}(\xi, \zeta)$ is the number of all possible orderings of insertions to obtain the pair $(\xi, \zeta)$ from $\kappa$ by this insertion method.

For this we let $\mathcal{H}=(N, S, R, w t)$ with $N=S=\{A\}$ and we let $R$ contain the following rules:

$$
\begin{array}{lll}
A \rightarrow[\sigma, \sigma](A, A) & A \rightarrow[\alpha, \alpha] & (\text { for the generation of } \kappa) \\
A \rightarrow[\gamma, \varepsilon](A) & A \rightarrow\left[\varepsilon, \gamma^{\prime}\right](A) & \left(\text { for inserting } \gamma \text { and } \gamma^{\prime}\right)
\end{array}
$$

We let $w t(r)=1$ for each $r \in R$. Obviously, $\llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}(\xi, \zeta)=\left|\mathrm{RT}_{\mathcal{H}}(\xi, \zeta)\right|$ for every $\xi \in \mathrm{T}_{\Sigma}$ and $\zeta \in \mathrm{T}_{\Psi}$.
For instance, let $\xi=\sigma(\gamma(\alpha), \gamma \gamma(\alpha))$ and $\zeta=\gamma^{\prime}\left(\sigma\left(\gamma^{\prime}(\alpha), \gamma^{\prime}(\alpha)\right)\right)$. The kernel of $(\xi, \zeta)$ is the tree $\kappa=\sigma(\alpha, \alpha)$. Then (a) above position $\varepsilon$ of $\kappa, \mathcal{H}$ inserts no $\gamma($ for $\xi)$ and one $\gamma^{\prime}$ (for $\zeta$ ) and there exists one (trivial) ordering of this insertion, (b) above position 1 of $\kappa, \mathcal{H}$ inserts one $\gamma$ (for $\xi$ ) and one $\gamma^{\prime}$ (for $\zeta)$ and there are two orderings of these insertions, and (c) above position 2 of $\kappa, \mathcal{H}$ inserts two $\gamma \mathrm{s}$ (for $\xi$ ) and one $\gamma^{\prime}$ (for $\zeta$ ) and there are three orderings of these insertions. Multiplying up the number of all such orderings yields $\llbracket \mathcal{H} \rrbracket^{\text {tt }}(\xi, \zeta)=2 \cdot 3=6$. Figure 10.12 illustrates two derivations $d_{1}, d_{2} \in \mathrm{RT}_{\mathcal{H}}(\xi, \zeta)$, the [ $\Sigma \Psi$ ]-trees $\eta_{1}=\pi\left(d_{1}\right)$ and $\eta_{2}=\pi\left(d_{2}\right)$, and the trees $\xi, \zeta$ and $\kappa$. The brackets indicate the regions where reordering is possible without changing $\xi$ and $\zeta$.

Example 10.13.2. We have noted that, if $\mathcal{H}$ is finite-input, then $\llbracket \mathcal{H} \rrbracket^{\text {tt }}$ is supp-i-finite. In this example we show that the reverse direction does not hold, even if $\mathcal{H}$ is chain-free. That means, there exists a chain-free wpb $\mathcal{H}$ such that $\llbracket \mathcal{H} \rrbracket^{\text {tt }}$ is supp-i-finite and $\mathcal{H}$ is not finite-input. We let $\Sigma=\left\{\gamma^{(1)}, \alpha^{(0)}\right\}$ and $\Psi=\left\{\beta^{(0)}\right\}$. We consider the ring Intmod4 $=\left(\{0,1,2,3\},{ }_{4}, \cdot 4,0,1\right)$ as defined in Example 2.6.9 (5) and the chain-free $(\Sigma, \Psi$, Intmod4)-wpb $\mathcal{H}$ with set $\{A, B\}$ of nonterminals, $A$ as only initial nonterminal, and the rules:

$$
r_{1}=(A \rightarrow[\gamma, \varepsilon](B)) r_{2}=(B \rightarrow[\gamma, \varepsilon](A)) r_{3}=(A \rightarrow[\alpha, \beta])
$$

and $w t\left(r_{1}\right)=w t\left(r_{2}\right)=2$ and $w t\left(r_{3}\right)=1$. Obviously, $\mathrm{T}_{\Psi}=\{\beta\}$ and the set $\left\{\xi \in \mathrm{T}_{\Sigma} \mid \llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}(\xi, \beta) \neq 0\right\}=$ $\{\alpha\}$ is finite. Hence $\llbracket \mathcal{H} \rrbracket^{\text {tt }}$ is supp-i-finite. Moreover, $\mathcal{H}$ is not finite-input, because $\left\{\xi \in \mathrm{T}_{\Sigma} \mid \mathrm{RT}_{\mathcal{H}}(\xi, \beta) \neq\right.$ $\emptyset\}=\left\{\gamma^{n} \alpha \mid n \in \mathbb{N}\right\}$ is infinite.


Figure 10.12: Illustration of two rule trees $d_{1}, d_{2} \in \mathrm{RT}_{\mathcal{H}}(\xi, \zeta)$, the $[\Sigma \Psi]$-trees $\eta_{1}=\pi\left(d_{1}\right)$ and $\eta_{2}=\pi\left(d_{2}\right)$, and the trees $\xi=\sigma(\gamma(\alpha), \gamma \gamma(\alpha)), \zeta=\gamma^{\prime}\left(\sigma\left(\gamma^{\prime}(\alpha), \gamma^{\prime}(\alpha)\right)\right)$, and $\kappa=\sigma(\alpha, \alpha)$ (cf. Example 10.13.1).

By definition, each wbp $\mathcal{H}$ is a particular wrtg. Thus $\mathcal{H}$ has two semantics: the weighted tree language $\llbracket \mathcal{H} \rrbracket$ and the weighted tree transformation $\llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}$. The next lemma shows that, under some additional conditions, the equality of the weighted tree languages of two wpb implies the equality of their weighted tree transformations.

Lemma 10.13.3. Let $\mathcal{H}$ and $\mathcal{H}^{\prime}$ be $(\Sigma, \Psi, \mathrm{B})$-wpb such that both $\mathcal{H}$ and $\mathcal{H}^{\prime}$ are finite-derivational or B is $\sigma$-complete. If $\llbracket \mathcal{H} \rrbracket=\llbracket \mathcal{H}^{\prime} \rrbracket$, then $\llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}=\llbracket \mathcal{H}^{\prime} \rrbracket^{\text {tt }}$.

Proof. Let us assume that $\llbracket \mathcal{H} \rrbracket=\llbracket \mathcal{H}^{\prime} \rrbracket$ and let $\xi \in \mathrm{T}_{\Sigma}$ and $\zeta \in \mathrm{T}_{\Psi}$. Then

$$
\begin{aligned}
& \llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}(\xi, \zeta)=\sum_{d \in \mathrm{RT}_{\mathcal{H}}(\xi, \zeta)}^{\oplus} \mathrm{wt}_{\mathcal{H}}(d)=\bigoplus_{\substack{\eta \in \mathrm{T}_{[\Sigma \Psi]}: \\
\pi_{1}(\eta)=\xi, \pi_{2}(\eta)=\zeta}} \sum_{d \in \mathrm{RT}_{\mathcal{H}}(\eta)}^{\oplus} \mathrm{wt}_{\mathcal{H}}(d)=\bigoplus_{\substack{\eta \in \mathrm{T}_{[\Sigma \Psi]}: \\
\pi_{1}(\eta)=\xi, \pi_{2}(\eta)=\zeta}} \llbracket \mathcal{H} \rrbracket(\eta)= \\
& \bigoplus_{\substack{\eta \in \mathrm{T}_{[\Sigma \Psi]}: \\
\pi_{1}(\eta)=\xi, \pi_{2}(\eta)=\zeta}} \llbracket \mathcal{H}^{\prime} \rrbracket(\eta)=\bigoplus_{\substack{\eta \in \mathrm{T}_{[\Sigma \Psi]}: \\
\pi_{1}(\eta)=\xi, \pi_{2}(\eta)=\zeta}} \sum_{d \in \mathrm{RT}_{\mathcal{H}^{\prime}}(\eta)}^{\oplus} \mathrm{wt}_{\mathcal{H}^{\prime}}(d)=\sum_{d \in \mathrm{RT}_{\mathcal{H}^{\prime}}(\xi, \zeta)}^{\oplus} \mathrm{wt}_{\mathcal{H}^{\prime}}(d)=\llbracket \mathcal{H}^{\prime} \rrbracket^{\mathrm{tt}}(\xi, \zeta),
\end{aligned}
$$

where the second and the sixth equality hold because (10.42) is a partitioning (of $\mathrm{RT}_{\mathcal{H}}(\xi, \zeta)$ and $\mathrm{RT}_{\mathcal{H}^{\prime}}(\xi, \zeta)$, respectively), the fourth one holds by our assumption, and all other ones hold by definition.

The next example shows that the reverse of Lemma 10.13 .3 does not hold.
Example 10.13.4. We consider the ranked alphabets $\Sigma=\left\{\gamma^{(1)}, \alpha^{(0)}\right\}$ and $\Psi=\left\{\gamma^{\prime(1)}, \alpha^{(0)}\right\}$. As weight algebra we consider the semiring Boole $=(\mathbb{B}, \vee, \wedge, 0,1)$.

We consider the two $\left(\Sigma, \Psi\right.$, Boole)-wpb $\mathcal{H}=(N, S, R, w t)$ and $\mathcal{H}^{\prime}=\left(N, S, R^{\prime}, w t^{\prime}\right)$ with $N=\{A, B, C\}$, $S=\{A\}$, and the rules shown in the following table.

| rules of $\mathcal{H}:$ | rules of $\mathcal{H}^{\prime}:$ |
| :--- | :--- |
| $A \rightarrow[\gamma, \varepsilon](B)$ | $A \rightarrow\left[\varepsilon, \gamma^{\prime}\right](B)$ |
| $B \rightarrow\left[\varepsilon, \gamma^{\prime}\right](C)$ | $B \rightarrow[\gamma, \varepsilon](C)$ |
| $C \rightarrow[\alpha, \alpha]$ | $C \rightarrow[\alpha, \alpha]$ |

and each rule has weight 1 . Thus, $\mathcal{H}$ and $\mathcal{H}^{\prime}$ differ in the order in which they read the input symbol $\gamma$ and write the output symbol $\gamma^{\prime}$. For every $\xi \in \mathrm{T}_{\Sigma}$ and $\zeta \in \mathrm{T}_{\Psi}$ we have

$$
\llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}(\xi, \zeta)=1 \quad \text { iff } \quad \xi=\gamma(\alpha) \text { and } \zeta=\gamma^{\prime}(\alpha) \quad \text { iff } \quad \llbracket \mathcal{H}^{\prime} \rrbracket^{\mathrm{tt}}(\xi, \zeta)=1
$$

Thus $\llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}=\llbracket \mathcal{H}^{\prime} \rrbracket^{\mathrm{tt}}$. However,

$$
\operatorname{supp}(\llbracket \mathcal{H} \rrbracket)=\left\{[\gamma, \varepsilon]\left[\varepsilon, \gamma^{\prime}\right]([\alpha, \alpha])\right\} \quad \text { and } \operatorname{supp}(\llbracket \mathcal{H} \rrbracket)=\left\{\left[\varepsilon, \gamma^{\prime}\right][\gamma, \varepsilon]([\alpha, \alpha])\right\}
$$

and thus $\llbracket \mathcal{H} \rrbracket \neq \llbracket \mathcal{H}^{\prime} \rrbracket$.
The next lemma shows that, under certain conditions, we can transform a wpb into an equivalent chain-free wpb (with respect to the computed weighted tree transformations). This lemma is based on the corresponding lemma for wrtg (cf. Lemma 9.2.1) which, in its turn, is based on the corresponding lemma for wcfg (cf. Theorem 8.2.6).

Lemma 10.13.5. (cf. FV22b, Lm. 4.1]) Let B be a semiring and $\mathcal{H}$ be a $(\Sigma, \Psi, \mathrm{B})$-wpb such that $\mathcal{H}$ is finite-derivational or B is $\sigma$-complete. Then the following three statements hold.
(1) There exists a chain-free $(\Sigma, \Psi, \mathrm{B})$-wpb $\mathcal{H}^{\prime}$ such that $\llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}=\llbracket \mathcal{H}^{\prime} \rrbracket^{\mathrm{tt}}$.
(2) If $\mathcal{H}$ is finite-derivational, then we can construct a chain-free $(\Sigma, \Psi, \mathrm{B})$-wpb $\mathcal{H}^{\prime}$ such that $\llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}=$ $\llbracket \mathcal{H}^{\prime} \rrbracket^{\mathrm{tt}}$.
(3) In both statements (1) and (2) the following holds: if $\mathcal{H}$ is finite-input (resp. finite-output), then $\mathcal{H}^{\prime}$ is finite-input (resp. finite-output).

Proof. Proof of (1): By Lemma $9.2 .1(3)$, there exists an alphabetic and chain-free ( $[\Sigma \Psi], \mathrm{B})$-wrtg $\mathcal{H}^{\prime}$ such that $\llbracket \mathcal{H} \rrbracket=\llbracket \mathcal{H}^{\prime} \rrbracket$. Then, by definition of wpb, $\mathcal{H}^{\prime}$ is a chain-free $(\Sigma, \Psi, \mathrm{B})$-wpb, and by Lemma 10.13.3 we have $\llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}=\llbracket \mathcal{H}^{\prime} \rrbracket^{\mathrm{tt}}$.

Proof of (2): Now let $\mathcal{H}$ be finite-derivational. Then, Lemma 9.2.1 (3) is constructive and hence we can construct a chain-free $(\Sigma, \Psi, \mathrm{B})$-wpb $\mathcal{H}^{\prime}$ such that $\llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}=\llbracket \mathcal{H}^{\prime} \rrbracket^{\mathrm{tt}}$.

Proof of (3): By analysing the proof of Theorem 8.2 .6 (on which the proof of Lemma 9.2.1(3) is based), it is easy to see that, if $\mathcal{H}$ is finite-input (resp. finite-output), then $\mathcal{H}^{\prime}$ is finite-input (resp. finite-output).

### 10.13.2 Split

Next we characterize the weighted tree language $\llbracket \mathcal{G} \rrbracket$ of a wrtg $\mathcal{G}$ in terms of the image of a characteristic mapping under a weighted projective bimorphism. For this we introduce the concept of skeleton alphabet as follows. We define the skeleton alphabet of $\Sigma$, denoted by [ $\Sigma$ ], to be the ranked alphabet with

$$
[\Sigma]^{(k)}= \begin{cases}\{[k]\} & \text { if } \Sigma^{(k)} \neq \emptyset \\ \emptyset & \text { otherwise }\end{cases}
$$

for each $k \in \mathbb{N}$ (i.e., $[\Sigma]^{(k)}$ is a singleton or empty and $\left.[\Sigma]^{(0)}=\{[0]\}\right)$.
Let $\mathcal{G}=(N, S, R, w t)$ be an arbitrary $(\Sigma, \mathrm{B})$-wrtg. We define a deterministic tree relabeling $\beta$ which, intuitively, maps each occurrence of rule $r$ of $\mathcal{G}$ in a tree $d \in \mathrm{~T}_{R}$ to the symbol of the skeleton alphabet which corresponds to the rank of $r$. Formally, we define $\beta=\left(\beta_{k} \mid k \in \mathbb{N}\right)$ with $\beta_{k}: R^{(k)} \rightarrow[R]^{(k)}$ by

$$
\beta_{k}(r)=[k] \text { for each } r \in R^{(k)} .
$$

For each $d \in \mathrm{~T}_{\mathrm{R}}$ we call $\beta(d)$ the skeleton of $d$. We note that, for every $\xi \in \mathrm{T}_{\Sigma}$, the $\mathrm{T}_{[R]}$-indexed family $\left(\beta^{-1}(b) \cap \mathrm{RT}_{\mathcal{G}}(\xi) \mid b \in \mathrm{~T}_{[R]}\right)$ is a partitioning of $\mathrm{RT}_{\mathcal{G}}(\xi)$. Moreover, for every $\xi \in \mathrm{T}_{\Sigma}$ and $b \in \mathrm{~T}_{[R]}$, the set $\beta^{-1}(b) \cap \mathrm{RT}_{\mathcal{G}}(\xi)$ is finite.

Lemma 10.13.6. (cf. FV22b, Lm. 4.3] for the trivial storage type) Let $\mathcal{G}=(N, S, R, w t)$ be an alphabetic $(\Sigma, \mathrm{B})$-wrtg such that $\mathcal{G}$ is finite-derivational or B is $\sigma$-complete. Then we can construct a chain-free $([R], \Sigma, \mathrm{B})$-wpb $\mathcal{H}$ such that (a) $\mathcal{H}$ is finite-output, (b) $\mathcal{H}$ is finite-input if $\mathcal{G}$ is finite-derivational, and $(\mathrm{c}) \llbracket \mathcal{G} \rrbracket=\llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}\left(\chi\left(\mathrm{T}_{[R]}\right)\right)$.

Proof. We construct the chain-free $([R], \Sigma, \mathrm{B})$-wpb $\mathcal{H}=\left(N, S, R^{\prime}, w t^{\prime}\right)$ as follows.

- For every $r=\left(A \rightarrow \sigma\left(A_{1}, \ldots, A_{k}\right)\right)$ in $R$,
the rule $r^{\prime}=\left(A \rightarrow[[k], \sigma]\left(A_{1}, \ldots, A_{k}\right)\right)$ is in $R^{\prime}$ and $w t^{\prime}\left(r^{\prime}\right)=w t(r)$.
- For every $r=(A \rightarrow B)$ in $R$, the rule $r^{\prime}=(A \rightarrow[[1], \varepsilon](B))$ is in $R^{\prime}$ and $w t^{\prime}\left(r^{\prime}\right)=w t(r)$.

We continue with introducing some useful mappings. We define the deterministic $\left(R^{\prime}, R\right)$-tree relabel$\operatorname{ing} \varphi=\left(\varphi_{k} \mid k \in \mathbb{N}\right)$ for every $k \in \mathbb{N}$ and $r^{\prime} \in R^{\prime(k)}$ by

$$
\varphi_{k}\left(r^{\prime}\right)= \begin{cases}A \rightarrow \sigma\left(A_{1}, \ldots, A_{k}\right) & \text { if } r^{\prime}=\left(A \rightarrow[[k], \sigma]\left(A_{1}, \ldots, A_{k}\right)\right) \\ A \rightarrow B & \text { if } r^{\prime}=(A \rightarrow[[1], \varepsilon](B))\end{cases}
$$

where in the second case $k=1$. Clearly, the deterministic tree relabeling $\varphi: \mathrm{T}_{R^{\prime}} \rightarrow \mathrm{T}_{R}$ is bijective. Moreover,

$$
\begin{equation*}
\text { for each } d \in \operatorname{RT}_{\mathcal{H}}\left(N, \mathrm{~T}_{[[R] \Sigma]}\right) \text {, we have: } \mathrm{wt}_{\mathcal{H}}(d)=\mathrm{wt}_{\mathcal{G}}(\varphi(d)) \tag{10.43}
\end{equation*}
$$

This can be proved easily by induction on $\left(\mathrm{RT}_{\mathcal{H}}\left(N, \mathrm{~T}_{[[R] \Sigma]}\right), \prec\right)$ where

$$
\prec=\prec_{R^{\prime}} \cap\left(\operatorname{RT}_{\mathcal{H}}\left(N, \mathrm{~T}_{[[R] \Sigma]}\right) \times \mathrm{RT}_{\mathcal{H}}\left(N, \mathrm{~T}_{[[R] \Sigma]}\right)\right) .
$$

Obviously, $\prec$ is well-founded and $\min _{\prec}\left(\operatorname{RT}_{\mathcal{H}}\left(N, \mathrm{~T}_{[[R] \Sigma]}\right)\right)=\left(R^{\prime}\right)^{(0)}$.
Also, for every $b \in \mathrm{~T}_{[R]}$ and $\xi \in \mathrm{T}_{\Sigma}$, we have $\varphi\left(\mathrm{RT}_{\mathcal{H}}(b, \xi)\right)=\beta^{-1}(b) \cap \mathrm{RT}_{\mathcal{G}}(\xi)$. Thus, for every $b \in \mathrm{~T}_{[R]}$ and $\xi \in \mathrm{T}_{\Sigma}$, the mapping $\varphi_{b, \xi}: \mathrm{RT}_{\mathcal{H}}(b, \xi) \rightarrow \beta^{-1}(b) \cap \mathrm{RT}_{\mathcal{G}}(\xi)$ defined by $\varphi_{b, \xi}=\left.\varphi\right|_{\mathrm{RT}} ^{\mathcal{H}(b, \xi)}$ is bijective. Figure 10.13 illustrates the connection between the mappings $\beta$ and $\varphi$.

Now we prove (a), (b), and (c). Property (a) holds, because $R^{\prime}=\left(R^{\prime}\right)^{\mathrm{r}-}$ and thus $\mathcal{H}$ is finite-output (and hence, $\llbracket \mathcal{H} \rrbracket^{\text {tt }}$ is supp-o-finite).

To see Property (b), we assume that $\mathcal{G}$ is finite-derivational and let $\xi \in \mathrm{T}_{\Sigma}$. Then the set $\mathrm{RT}_{\mathcal{G}}(\xi)$ is finite. Since the family $\left(\beta^{-1}(b) \cap \mathrm{RT}_{\mathcal{G}}(\xi) \mid b \in \mathrm{~T}_{[R]}\right)$ is a partitioning of the finite set $\mathrm{RT}_{\mathcal{G}}(\xi)$, the set $\{b \in$ $\left.\mathrm{T}_{[R]} \mid \beta^{-1}(b) \cap \mathrm{RT}_{\mathcal{G}}(\xi) \neq \emptyset\right\}$ is also finite. Since $\varphi_{b, \xi}$ is bijective, also the set $\left\{b \in \mathrm{~T}_{[R]} \mid \mathrm{RT}_{\mathcal{H}}(b, \xi) \neq \emptyset\right\}$ is finite. It means that $\mathcal{H}$ is finite-input, which proves Property (b).

Now we turn to the proof of Property (c). Obviously, $\mathcal{H}$ is chain-free and hence finite-derivational; thus, $\llbracket \mathcal{H} \rrbracket^{\text {tt }}$ is well defined. Moreover, if $\mathcal{G}$ is finite-derivational, then $\mathcal{H}$ is finite-input (due to Property (b)), and thus $\llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}$ is supp-i-finite. Hence $\llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}\left(\chi\left(\mathrm{T}_{[R]}\right)\right)$ is well defined.

Now we prove that $\llbracket \mathcal{G} \rrbracket=\llbracket \mathcal{H} \rrbracket^{\text {tt }}\left(\chi\left(\mathrm{T}_{[R]}\right)\right)$. For this, let $\xi \in \mathrm{T}_{\Sigma}$. Then

$$
\left(\llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}\left(\chi\left(\mathrm{~T}_{[R]}\right)\right)\right)(\xi)=\sum_{b \in \mathrm{~T}_{[R]}}^{\oplus} \chi\left(\mathrm{T}_{[R]}\right)(b) \otimes \llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}(b, \xi)
$$

(because $\llbracket \mathcal{H} \rrbracket^{\text {tt }}$ is supp-i-finite if B is not $\sigma$-complete)


Figure 10.13: An illustration of the mappings $\beta, \pi$, and $\varphi$.
$=\sum_{b \in \mathrm{~T}_{[R]}}^{\oplus} \llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}(b, \xi)$
$=\sum_{b \in \mathrm{~T}_{[R]}}^{\oplus} \bigoplus_{d^{\prime} \in \mathrm{RT}_{\mathcal{H}}(b, \xi)} \operatorname{wt}_{\mathcal{H}}\left(d^{\prime}\right) \quad \quad$ (because $\mathcal{H}$ is finite-derivational)
$=\sum_{b \in \mathrm{~T}_{[R]}}^{\oplus} \bigoplus_{d^{\prime} \in \mathrm{RT}_{\mathcal{H}}(b, \xi)} \mathrm{wt}_{\mathcal{G}}\left(\varphi_{b, \xi}\left(d^{\prime}\right)\right)$
(by (10.43) and definition of $\varphi_{b, \xi}$ )
$=\sum_{b \in \mathrm{~T}_{[R]}}^{\oplus} \bigoplus_{d \in \beta^{-1}(b) \cap \mathrm{RT}_{\mathcal{G}}(\xi)} \mathrm{wt}_{\mathcal{G}}(d)$
(since $\varphi_{b, \xi}$ is bijective)
$=\sum_{d \in \mathrm{RT}_{\mathcal{G}}(\xi)}^{\oplus} \mathrm{wt}_{\mathcal{G}}(d) \quad \quad$ (because $\left(\beta^{-1}(b) \cap \mathrm{RT}_{\mathcal{G}}(\xi) \mid b \in \mathrm{~T}_{[R]}\right)$ is a partitioning of $\mathrm{RT}_{\mathcal{G}}(\xi)$ )
$=\llbracket \mathcal{G} \rrbracket(\xi) . \quad$ (because $\mathcal{G}$ is finite-derivational or B is $\sigma$-complete)

### 10.13.3 Closure of weighted projective bimorphisms under composition

In this subsection we prove that weighted projective bimorphisms are closed under composition. For the proof, we introduce a notation. For every ( $\Sigma, \Psi, \mathrm{B}$ )-bimorphism $\mathcal{H}=\left(N, S, R\right.$, wt) and $N^{\prime} \subseteq N$, we let

$$
\operatorname{RT}_{\mathcal{H}}\left(N^{\prime}\right)=\bigcup_{\eta \in \mathrm{T}_{[\Sigma \Psi]}} \operatorname{RT}_{\mathcal{H}}\left(N^{\prime}, \eta\right)
$$

Theorem 10.13.7. FV22b, Thm. 4.2] Let $\Sigma, \Psi$, and $\Delta$ be ranked alphabets and B be a commutative semiring. Let $\mathcal{H}_{1}$ be $a(\Sigma, \Psi, \mathrm{~B})-w p b$ and $\mathcal{H}_{2}$ be $a(\Psi, \Delta, \mathrm{~B})-w p b$ such that both $\mathcal{H}_{1}$ and $\mathcal{H}_{2}$ are finitederivational or B is $\sigma$-complete. Moreover, let $\llbracket \mathcal{H}_{1} \rrbracket^{\mathrm{tt}}$ be supp-o-finite or $\llbracket \mathcal{H}_{2} \rrbracket^{\mathrm{tt}}$ be supp-i-finite or B is $\sigma$-complete. Then the following three statements hold.
(1) There exists a $(\Sigma, \Delta, \mathrm{B})$-wpb $\mathcal{H}$ such that $\llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}=\llbracket \mathcal{H}_{1} \rrbracket^{\mathrm{tt}} ; \llbracket \mathcal{H}_{2} \rrbracket^{\mathrm{tt}}$.
(2) If $\mathcal{H}_{1}$ and $\mathcal{H}_{2}$ are finite-derivational, then we can construct a finite-derivational $(\Sigma, \Delta, \mathrm{B})$-wpb $\mathcal{H}$ such that $\llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}=\llbracket \mathcal{H}_{1} \rrbracket^{\mathrm{tt}} ; \llbracket \mathcal{H}_{2} \rrbracket^{\mathrm{tt}}$.
(3) In both statements (1) and (2) the following holds. If $\mathcal{H}_{1}$ and $\mathcal{H}_{2}$ are finite-input, then $\mathcal{H}$ is finiteinput.

Proof. Proof of (1). Let $\mathcal{H}_{1}=\left(N_{1}, S_{1}, R_{1}, w t_{1}\right)$ and $\mathcal{H}_{2}=\left(N_{2}, S_{2}, R_{2}, w t_{2}\right)$. Since $\mathcal{H}_{1}$ and $\mathcal{H}_{2}$ are finitederivational or B is $\sigma$-complete, the weighted tree transformations $\llbracket \mathcal{H}_{1} \rrbracket^{\mathrm{tt}}$ and $\llbracket \mathcal{H}_{2} \rrbracket^{\text {tt }}$ are well defined. Moreover, since $\llbracket \mathcal{H}_{1} \rrbracket^{\text {tt }}$ is supp-o-finite or $\llbracket \mathcal{H}_{2} \rrbracket^{\text {tt }}$ is supp-i-finite or B is $\sigma$-complete, the weighted tree transformation $\llbracket \mathcal{H}_{1} \rrbracket^{\mathrm{tt}} ; \llbracket \mathcal{H}_{2} \rrbracket^{\mathrm{tt}}$ is well defined. By Lemma $10.13 .5(1)$ we can assume that $\mathcal{H}_{1}$ and $\mathcal{H}_{2}$ are chain-free.

We define the $(\Sigma, \Delta, \mathrm{B})$-wpb $\mathcal{H}=(N, S, R, w t)$ by using a modification of the usual product construction for tree transducers Eng75b, Bak79] and for weighted tree transducers in EFV02]. We have to use a modification for the following reason. Before producing an output symbol $\psi \in \Psi$, the wpb $\mathcal{H}_{1}$ can execute a sequence $s_{1}$ of r $\varepsilon$-rules on some input tree $\xi \in \mathrm{T}_{\Sigma}$. Also, before reading $\psi$, the wpb $\mathcal{H}_{2}$ can execute a sequence $s_{2}$ of $\varepsilon$ w-rules, thereby producing a part of some output tree $\zeta \in \mathrm{T}_{\Delta}$. If we combine each rule in the sequence $s_{1}$ with each rule in the sequence $s_{2}$ into one rule for $\mathcal{H}$, then we introduce a number of artificial orderings for the applications of the rules in $s_{1}$ and $s_{2}$. As a consequence, for one single summand in $\left(\llbracket \mathcal{H}_{1} \rrbracket^{\mathrm{tt}} ; \llbracket \mathcal{H}_{2} \rrbracket^{\mathrm{tt}}\right)(\xi, \zeta)$, there will be multiple corresponding summands in $\llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}(\xi, \zeta)$. Thus, in order to establish a weight balance between $\llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}(\xi, \zeta)$ and $\left(\llbracket \mathcal{H}_{1} \rrbracket^{\mathrm{tt}} ; \llbracket \mathcal{H}_{2} \rrbracket^{\mathrm{tt}}\right)(\xi, \zeta)$, we have to forbid artificial orderings in the rule trees of $\mathcal{H}$.

For this purpose, we introduce auxiliary control nonterminals as follows. Before $\mathcal{H}$ simulates the combination of a rule $r_{1}=\left(A_{1} \rightarrow[\sigma, \psi]\left(A_{11}, \ldots, A_{1 k}\right)\right)$ in $R_{1}{ }^{-\mathrm{w}}$ and of a rule $r_{2}=\left(A_{2} \rightarrow[\psi, \delta]\left(A_{21}, \ldots, A_{2 k}\right)\right)$ in $R_{2}^{\mathrm{r}}$-, it first simulates rules in $R_{1}^{\mathrm{r} \varepsilon}$ (using states of the form $\left(r_{1}, r_{2}\right)$ ), and second it simulates rules in $R_{2}^{\varepsilon w}$ (using states of the form $\left\langle r_{1}, r_{2}\right\rangle$ ).

Formally, we define $\mathcal{H}$ as follows. We let

$$
\begin{aligned}
N= & \left\{\left(r_{1}, r_{2}\right) \mid r_{1} \in R_{1}, r_{2} \in R_{2}\right\} \cup\left\{\left\langle r_{1}, r_{2}\right\rangle \mid r_{1} \in R_{1}^{-\mathrm{w}}, r_{2} \in R_{2}\right\} \\
& \cup\left\{\left(r_{1}, r_{2}\right) \mid\left(r_{1}, r_{2}\right) \in R_{1}^{-\mathrm{w}} \times R_{2}^{\mathrm{r}-}, \operatorname{out}\left(r_{1}\right)=\operatorname{inp}\left(r_{2}\right)\right\} \text { and } \\
S= & \left\{\left(r_{1}, r_{2}\right) \in Q \mid \operatorname{lhs}\left(r_{1}\right) \in S_{1}, \operatorname{lhs}\left(r_{2}\right) \in S_{2}\right\} .
\end{aligned}
$$

The set $R$ and the weight mapping wt are defined by the following five cases.

1. For every $r_{1}=\left(A_{1} \rightarrow[\sigma, \varepsilon]\left(A_{1}^{\prime}\right)\right)$ in $R_{1}^{\mathrm{r} \varepsilon}, r_{1}^{\prime} \in R_{1}$ with $\operatorname{lhs}\left(r_{1}^{\prime}\right)=A_{1}^{\prime}$, and $r_{2} \in R_{2}$, the rule

$$
r=\left(\left(r_{1}, r_{2}\right) \rightarrow[\sigma, \varepsilon]\left(\left(r_{1}^{\prime}, r_{2}\right)\right)\right)
$$

is in $R$ and $w t(r)=w t_{1}\left(r_{1}\right)$.
2. For every $r_{1} \in R_{1}^{-\mathrm{w}}$ and $r_{2} \in R_{2}$, the rule

$$
\left.r=\left(\Delta r_{1}, r_{2}\right) \rightarrow\left\langle r_{1}, r_{2}\right\rangle\right)
$$

is in $R$ and $w t(r)=\mathbb{1}$.
3. For every $r_{1} \in R_{1}^{-\mathrm{w}}, r_{2}=\left(A_{2} \rightarrow[\varepsilon, \psi]\left(A_{2}^{\prime}\right)\right)$ in $R_{2}^{\varepsilon \mathrm{w}}$, and $r_{2}^{\prime} \in R_{2}$ with $\operatorname{lhs}\left(r_{2}^{\prime}\right)=A_{2}^{\prime}$, the rule

$$
r=\left(\left\langle r_{1}, r_{2}\right\rangle \rightarrow[\varepsilon, \psi]\left(\left\langle r_{1}, r_{2}^{\prime}\right\rangle\right)\right)
$$

is in $R$ and $w t(r)=w t_{2}\left(r_{2}\right)$.
4. For every $r_{1} \in R_{1}^{-\mathrm{w}}$ and $r_{2} \in R_{2}^{\mathrm{r}}-$, the rule

$$
r=\left(\left\langle r_{1}, r_{2}\right\rangle \rightarrow\left(r_{1}, r_{2}\right)\right)
$$

is in $R$ and $w t(r)=\mathbb{1}$.
5. For every $r_{1}=\left(A_{1} \rightarrow[\sigma, \psi]\left(A_{11}, \ldots, A_{1 k}\right)\right)$ in $R_{\overline{1}}{ }^{\mathrm{w}}$,
$r_{2}=\left(A_{2} \rightarrow[\psi, \delta]\left(A_{21}, \ldots, A_{2 k}\right)\right)$ in $R_{2}^{\mathrm{r}-}$,
$r_{11}, \ldots, r_{1 k} \in R_{1}$ with $\operatorname{lhs}\left(r_{1 j}\right)=A_{1 j}(j \in[k])$, and
$r_{21}, \ldots, r_{2 k} \in R_{2}$ with $\operatorname{lhs}\left(r_{2 j}\right)=A_{2 j}(j \in[k])$, the rule

$$
r=\left(\left(r_{1}, r_{2}\right) \rightarrow[\sigma, \delta]\left(\left(r_{11}, r_{21}\right), \ldots,\left(r_{1 k}, r_{2 k}\right)\right)\right)
$$

is in $R$ and $w t(r)=w t_{1}\left(r_{1}\right) \otimes w t_{2}\left(r_{2}\right)$.
Clearly, each rule $r$ produced in Cases 1-4 has rank 1. Next, we mention that, for each $d \in$ $\operatorname{RT}_{\mathcal{H}}\left(\left(r_{1}, r_{2}\right), \mathrm{T}_{[\Sigma \Psi]}\right)$ with $r_{i} \in R_{i}$ for $i=\{1,2\}$ and each leaf $w \in \operatorname{pos}(d)$ with $w=i_{1} \ldots i_{m}$ for some $m \geq 1$, the sequence of cases which has produced $d\left(i_{1}\right) \ldots d\left(i_{m}\right) \in R^{*}$, is contained in $\left(1^{\ell} 23^{u} 45\right)^{+}$for some $\ell, u \in \mathbb{N}$ (see Figure 10.14).


Figure 10.14: A rule tree $d$ of $\mathcal{H}$.
We also note that $\mathcal{H}$ is not chain-free due to rules of type 2 and 4 . However, it is easy to see that, if $\mathcal{H}_{1}$ and $\mathcal{H}_{2}$ are finite-derivational, then $\mathcal{H}$ is finite-derivational, because rules of type 2 and 4 cannot occur arbitrarily often in rule trees. Hence $\llbracket \mathcal{H} \rrbracket^{\text {tt }}$ is well defined.

For the rest of the proof we need some preparation. We define a mapping $h$ which retrieves from a rule tree in $\operatorname{RT}_{\mathcal{H}}\left(N, \mathrm{~T}_{[\Sigma \Delta]}\right)$ the intermediate tree in $\mathrm{T}_{\Psi}$. Formally, we define the mapping $h: \operatorname{RT}_{\mathcal{H}}\left(N, \mathrm{~T}_{[\Sigma \Delta]}\right) \rightarrow \mathrm{T}_{\Psi}$ by induction on $\left(\mathrm{RT}_{\mathcal{H}}\left(N, \mathrm{~T}_{[\Sigma \Delta]}\right), \prec\right)$ where

$$
\prec=\prec_{R} \cap\left(\operatorname{RT}_{\mathcal{H}}\left(N, \mathrm{~T}_{[\Sigma \Delta]}\right) \times \mathrm{RT}_{\mathcal{H}}\left(N, \mathrm{~T}_{[\Sigma \Delta]}\right)\right) .
$$

Obviously, $\prec$ is well-founded and $\min _{\prec}\left(\mathrm{RT}_{\mathcal{H}}\left(N, \mathrm{~T}_{[\Sigma \Delta]}\right)=R^{(0)}\right.$.

Let $d=r\left(d_{1}, \ldots, d_{k}\right)$ be in $\operatorname{RT}_{\mathcal{H}}\left(N, \mathrm{~T}_{[\Sigma \Delta]}\right)$ with $r \in R$ and $d_{1}, \ldots, d_{k} \in \operatorname{RT}_{\mathcal{H}}\left(N, \mathrm{~T}_{[\Sigma \Delta]}\right)$. Then we let

$$
h\left(r\left(d_{1}, \ldots, d_{k}\right)\right)= \begin{cases}h\left(d_{1}\right) & \text { if } r \text { is obtained by Cases } 1,2,3, \text { or } 4 \\ \psi\left(h\left(d_{1}\right), \ldots, h\left(d_{k}\right)\right) & \text { if } r \text { is obtained by Case } 5 \\ & \operatorname{lhs}(r)=\left(r_{1}, r_{2}\right) \text { for some } r_{1} \text { and } r_{2} \\ & \text { and } \psi=\operatorname{out}\left(r_{1}\right)\end{cases}
$$

We define the set $N_{0.0}=\left\{\left(r_{1}, r_{2}\right) \mid r_{1} \in R_{1}, r_{2} \in R_{2}\right\}$. Moreover, we define the mapping

$$
\varphi: \mathrm{RT}_{\mathcal{H}}\left(N_{0 \cdot D}, \mathrm{~T}_{[\Sigma \Delta]}\right) \rightarrow \mathrm{RT}_{\mathcal{H}_{1}}\left(N_{1}, \mathrm{~T}_{\Sigma}\right) \times \mathrm{RT}_{\mathcal{H}_{2}}\left(N_{2}, \mathrm{~T}_{\Psi}\right)
$$

by induction on $\left(\operatorname{RT}_{\mathcal{H}}\left(N_{0.1}, \mathrm{~T}_{[\Sigma \Delta]}\right), \prec\right)$ where

$$
\prec=\prec_{R}^{+} \cap\left(\operatorname{RT}_{\mathcal{H}}\left(N_{0 . D}, \mathrm{~T}_{[\Sigma \Delta]}\right) \times \operatorname{RT}_{\mathcal{H}}\left(N_{0 . D}, \mathrm{~T}_{[\Sigma \Delta]}\right)\right)
$$

as follows. Obviously, $\prec$ is well-founded and $\min _{\prec}\left(\mathrm{RT}_{\mathcal{H}}\left(N_{0 . D}, \mathrm{~T}_{[\Sigma \Delta]}\right)\right)=\left\{(A \rightarrow \xi) \in R \mid A \in N_{0 . \rho}, \xi \in\right.$ $\left.[\Sigma \Delta]^{(0)}\right\}$.)

Let $d \in \operatorname{RT}_{\mathcal{H}}\left(N_{0 \cdot D}, \mathrm{~T}_{[\Sigma \Delta]}\right)$. Then there exist $r_{1} \in R_{1}, r_{2} \in R_{2}, \xi \in \mathrm{~T}_{\Sigma}$, and $\zeta \in \mathrm{T}_{\Delta}$ such that $\left.d \in \operatorname{RT}_{\mathcal{H}}\left(\Delta r_{1}, r_{2}\right\rangle, \xi, \zeta\right)$.

Due to the definition of rules of $\mathcal{H}$, the rule tree $d$ has the form shown in Figure 10.14 for some $\ell, u \in \mathbb{N}$, rules $r_{i}^{j}$, symbols $\sigma_{j}$ and $\psi_{j}$ (each of appropriate type which can be read off easily from the definition of $R)$, as well as rule $\left(r_{1}^{\ell+1}, r_{2}^{u+1}\right) \rightarrow[\sigma, \delta]\left(\left(r_{11}, r_{21}\right), \ldots,\left(r_{1 k}, r_{2 k}\right)\right)$ in $R$, and $d_{1}, \ldots, d_{k} \in \operatorname{RT}_{\mathcal{H}}\left(N_{0 . \mid}, \mathrm{T}_{[\Sigma \Delta]}\right)$; moreover, $r_{1}=r_{1}^{1}$ and $r_{2}=r_{2}^{1}$.

Then

- there exists a $\psi \in \Psi^{(k)}$ such that $h(d)=\psi\left(h\left(d_{1}\right), \ldots, h\left(d_{k}\right)\right)$,
- $\xi=\sigma\left(\pi_{1}\left(d_{1}\right), \ldots, \pi\left(d_{k}\right)\right)$,
- $\zeta=\delta\left(\pi_{2}\left(d_{1}\right), \ldots, \pi_{2}\left(d_{k}\right)\right)$, and
- $d_{i} \in \mathrm{RT}_{\mathcal{H}}\left(\left(r_{1 i}, r_{2 i}\right), \pi_{1}\left(d_{i}\right), \pi_{2}\left(d_{i}\right)\right)$ for each $i \in[k]$.

We define $\varphi(d)=\left(t_{1}, t_{2}\right)$, where $t_{1}$ and $t_{2}$ are shown in Figure 10.15 and where for each $i \in[k]$ we let $\left(t_{1 i}, t_{2 i}\right)=\varphi\left(d_{i}\right)$.

Next we introduce a kind of typing for elements of $\mathrm{RT}_{\mathcal{H}}\left(N_{0 . D}, \mathrm{~T}_{[\Sigma \Delta]}\right)$ and for elements of $\mathrm{RT}_{\mathcal{H}_{1}}\left(N_{1}, \mathrm{~T}_{\Sigma}\right) \times \mathrm{RT}_{\mathcal{H}_{2}}\left(N_{2}, \mathrm{~T}_{\Psi}\right)$. Formally, let $r_{1} \in R_{1}, r_{2} \in R_{2}, \xi \in \mathrm{~T}_{\Sigma}, \theta \in \mathrm{T}_{\Psi}$, and $\zeta \in \mathrm{T}_{\Delta}$. We define the following two sets:

$$
\begin{aligned}
\mathrm{RT}_{\mathcal{H}}\left(\left(r_{1}, r_{2}\right), \xi, \theta, \zeta\right) & =\left\{d \in \mathrm{RT}_{\mathcal{H}}\left(\left(r_{1}, r_{2}\right), \xi, \zeta\right) \mid h(d)=\theta\right\} \\
\operatorname{RT}_{\mathcal{H}_{1} \times \mathcal{H}_{2}}\left(r_{1}, r_{2}, \xi, \theta, \zeta\right) & =\left\{\left(t_{1}, t_{2}\right) \in \mathrm{RT}_{\mathcal{H}_{1}}\left(N_{1}, \xi, \theta\right) \times \mathrm{RT}_{\mathcal{H}_{2}}\left(N_{2}, \theta, \zeta\right) \mid t_{1}(\varepsilon)=r_{1}, t_{2}(\varepsilon)=r_{2}\right\}
\end{aligned}
$$

Then the families

$$
\begin{array}{r}
\left(\mathrm{RT}_{\mathcal{H}}\left(\left(r_{1}, r_{2}\right), \xi, \theta, \zeta\right) \mid r_{1}, r_{2}, \xi, \theta, \zeta \text { as above }\right) \\
\left(\mathrm{RT}_{\mathcal{H}_{1} \times \mathcal{H}_{2}}\left(r_{1}, r_{2}, \xi, \theta, \zeta\right) \mid r_{1}, r_{2}, \xi, \theta, \zeta \text { as above }\right)
\end{array}
$$

are partitionings of $\mathrm{RT}_{\mathcal{H}}\left(N_{0 \cdot D}, \mathrm{~T}_{[\Sigma \Delta]}\right)$ and of $\mathrm{RT}_{\mathcal{H}_{1}}\left(N_{1}, \mathrm{~T}_{\Sigma}\right) \times \mathrm{RT}_{\mathcal{H}_{2}}\left(N_{2}, \mathrm{~T}_{\Psi}\right)$, respectively. One might say that an element $d \in \mathrm{RT}_{\mathcal{H}}\left(\left(r_{1}, r_{2}\right), \xi, \theta, \zeta\right)$ and an element $\left(t_{1}, t_{2}\right) \in \mathrm{RT}_{\mathcal{H}_{1} \times \mathcal{H}_{2}}\left(r_{1}, r_{2}, \xi, \theta, \zeta\right)$ have type $\left(r_{1}, r_{2}, \xi, \theta, \zeta\right)$.

It is easy to see that $\varphi\left(\mathrm{RT}_{\mathcal{H}}\left(\left(r_{1}, r_{2}\right), \xi, \theta, \zeta\right)\right) \subseteq \mathrm{RT}_{\mathcal{H}_{1} \times \mathcal{H}_{2}}\left(r_{1}, r_{2}, \xi, \theta, \zeta\right)$. Thus, intuitively, $\varphi$ is type preserving. Based on this property, we define the mapping

$$
\left.\varphi_{r_{1}, r_{2}, \xi, \theta, \zeta}: \mathrm{RT}_{\mathcal{H}}\left(0 r_{1}, r_{2}\right), \xi, \theta, \zeta\right) \rightarrow \mathrm{RT}_{\mathcal{H}_{1} \times \mathcal{H}_{2}}\left(r_{1}, r_{2}, \xi, \theta, \zeta\right)
$$

for each $\left.d \in \mathrm{RT}_{\mathcal{H}}\left(\| r_{1}, r_{2}\right), \xi, \theta, \zeta\right)$ by $\varphi_{r_{1}, r_{2}, \xi, \theta, \zeta}(d)=\varphi(d)$.


Figure 10.15: A pair $\left(t_{1}, t_{2}\right)$ of rule trees of $\mathcal{H}_{1}$ and $\mathcal{H}_{2}$, respectively, where $\operatorname{lhs}\left(t_{1 j}(\varepsilon)\right)=A_{1 j}$ and $\operatorname{lhs}\left(t_{2 j}(\varepsilon)\right)=A_{2 j}$ for $j \in[k]$.

We show that each mapping $\varphi_{r_{1}, r_{2}, \xi, \theta, \zeta}$ is injective. For this, let $d$ and $d^{\prime}$ be different trees with $\varphi_{r_{1}, r_{2}, \xi, \theta, \zeta}(d)=\left(t_{1}, t_{2}\right)$ and $\varphi_{r_{1}, r_{2}, \xi, \theta, \zeta}\left(d^{\prime}\right)=\left(t_{1}^{\prime}, t_{2}^{\prime}\right)$. Then there exists a position $w \in \operatorname{pos}(d) \cap \operatorname{pos}\left(d^{\prime}\right)$ with $d(w) \neq d^{\prime}(w)$. Since the nonterminals of $\mathcal{H}$ encode rules of $\mathcal{H}_{1}$ and $\mathcal{H}_{2}$, this yields that the rules in $t_{1}$ and $t_{1}^{\prime}$ or the rules in $t_{2}$ and $t_{2}^{\prime}$ corresponding to $d(w)$ and $d^{\prime}(w)$, respectively, are different. Hence $\left(t_{1}, t_{2}\right) \neq\left(t_{1}^{\prime}, t_{2}^{\prime}\right)$, i.e., $\varphi_{r_{1}, r_{2}, \xi, \theta, \zeta}$ is injective.

Now let $\left(t_{1}, t_{2}\right) \in \operatorname{RT}_{\mathcal{H}_{1} \times \mathcal{H}_{2}}\left(r_{1}, r_{2}, \xi, \theta, \zeta\right)$. Since each $t_{1} \in \operatorname{RT}_{\mathcal{H}_{1}}\left(N_{1}\right)$ and each $t_{2} \in \operatorname{RT}_{\mathcal{H}_{2}}\left(N_{2}\right)$ have the form as shown in Figure 10.15, we can combine $t_{1}$ and $t_{2}$ into a $\left.d \in \mathrm{RT}_{\mathcal{H}}\left(0 r_{1}, r_{2}\right), \xi, \theta, \zeta\right)$ shown in Figure 10.14. For this $d$ we have $\varphi_{r_{1}, r_{2}, \xi, \theta, \zeta}(d)=\left(t_{1}, t_{2}\right)$, hence each mapping $\varphi_{r_{1}, r_{2}, \xi, \theta, \zeta}$ is also surjective.

Moreover, by analysing the weights of the rules shown in Figure 10.14 with those shown in Figure 10.15 and taking the commutativity of B into account, we easily obtain that

$$
\mathrm{wt}_{\mathcal{H}}(d)=\mathrm{wt}_{\mathcal{H}_{1}}\left(t_{1}\right) \otimes \mathrm{wt}_{\mathcal{H}_{2}}\left(t_{2}\right)
$$

for each $d \in \operatorname{RT}_{\mathcal{H}}\left(\left(r_{1}, r_{2}\right), \xi, \theta, \zeta\right)$, where $\varphi_{r_{1}, r_{2}, \xi, \theta, \zeta}(d)=\left(t_{1}, t_{2}\right)$. Thus, each $\varphi_{r_{1}, r_{2}, \xi, \theta, \zeta}$ is a weight preserving bijection.

Now we can finish the proof of (a) as follows. For every $\xi \in \mathrm{T}_{\Sigma}$ and $\zeta \in \mathrm{T}_{\Delta}$, we have

$$
\begin{aligned}
& \left(\llbracket \mathcal{H}_{1} \rrbracket^{\mathrm{tt}} ; \llbracket \mathcal{H}_{2} \rrbracket^{\mathrm{tt}}\right)(\xi, \zeta)=\sum_{\theta \in \mathrm{T}_{\Psi}}{ }^{\oplus} \llbracket \mathcal{H}_{1} \rrbracket^{\mathrm{tt}}(\xi, \theta) \otimes \llbracket \mathcal{H}_{2} \rrbracket^{\mathrm{tt}}(\theta, \zeta) \\
& =\sum_{\theta \in \mathrm{T}_{\Psi}}^{\oplus} \sum_{t_{1} \in \mathrm{RT}_{\mathcal{H}_{1}}(\xi, \theta)}^{\oplus} \sum_{t_{2} \in \mathrm{RT}_{\mathcal{H}_{2}}(\theta, \zeta)}^{\oplus} \mathrm{wt}_{\mathcal{H}_{1}}\left(t_{1}\right) \otimes \mathrm{wt}_{\mathcal{H}_{2}}\left(t_{2}\right) \quad \text { (by distributivity) }
\end{aligned}
$$

$$
\begin{aligned}
& =\sum_{\theta \in \mathrm{T}_{\Psi}}^{\oplus} \bigoplus_{\substack{r_{1} \in R_{1}: \\
\operatorname{lhs}\left(r_{1}\right) \in S_{1} \operatorname{lins}\left(r_{2}\right) \in S_{2}}} \bigoplus_{\substack{r_{2} \in R_{2}: \\
\left(t_{1}, t_{2}\right) \in \mathrm{RT}_{\mathcal{H}_{1} \times \mathcal{H}_{2}}\left(r_{1}, r_{2}, \xi, \theta, \zeta\right)}}^{\oplus} \mathrm{wt}_{\mathcal{H}_{1}}\left(t_{1}\right) \otimes \mathrm{wt}_{\mathcal{H}_{2}}\left(t_{2}\right) \\
& =\sum_{\theta \in \mathrm{T}_{\Psi}}^{\oplus} \bigoplus_{\substack{r_{1} \in R_{1}: \\
\operatorname{lhs}\left(r_{1}\right) \in S_{1}}} \bigoplus_{\substack{r_{2} \in R_{2}: \\
\operatorname{lhs}\left(r_{2}\right) \in S_{2}}} \sum_{d \in \operatorname{RT} \mathrm{~T}_{\mathcal{H}}\left(\left(r_{1}, r_{2}\right), \xi, \theta, \zeta\right)}^{\oplus} \operatorname{wt}_{\mathcal{H}}(d) \\
& \text { (because } \varphi_{r_{1}, r_{2}, \xi, \theta, \zeta} \text { is a weight preserving bijection ) } \\
& \text { ( from } \mathrm{RT}_{\mathcal{H}}\left(\left(r_{1}, r_{2}\right), \xi, \theta, \zeta\right) \text { to } \mathrm{RT}_{\mathcal{H}_{1} \times \mathcal{H}_{2}}\left(r_{1}, r_{2}, \xi, \theta, \zeta\right) \text { ) } \\
& =\sum_{\theta \in \mathrm{T}_{\Psi}}^{\oplus} \bigoplus_{\begin{array}{c}
r_{1} \in R_{1}: \\
\operatorname{lhs}\left(r_{1}\right) \in S_{1}
\end{array} \bigoplus_{\substack{r_{2} \in R_{2}: \\
\operatorname{lhs}\left(r_{2}\right) \in S_{2}}} \sum_{\begin{array}{c}
d \in \operatorname{RT\mathcal {H}}(N, \xi, S): \\
h(d)=\theta \\
\operatorname{lhs}(d(\varepsilon))=\left(r_{1}, r_{2}\right)
\end{array}}^{\oplus} \operatorname{wt}_{\mathcal{H}}(d)}
\end{aligned}
$$

$$
=\sum_{\theta \in \mathrm{T}_{\Psi}}^{\oplus} \sum_{\substack{d \in \mathrm{RT}_{\mathcal{H}}(\xi, \zeta): \\ h(d)=\theta}}^{\oplus} \mathrm{wt}_{\mathcal{H}}(d)=\sum_{d \in \mathrm{RT}_{\mathcal{H}}(\xi, \zeta)}^{\oplus} \mathrm{wt}_{\mathcal{H}}(d)=\llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}(\xi, \zeta)
$$

Proof of (2): Now let $\mathcal{H}_{1}$ and $\mathcal{H}_{2}$ be finite-derivational. Then, by Lemma 10.13.5(2), we can even construct equivalent chain-free wpb. Consequently, the definition of $\mathcal{H}$, as it is given in the proof of (1), is constructive and $\mathcal{H}$ is finite-derivational. Moreover, by the proof of (1), we have $\llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}=\llbracket \mathcal{H}_{1} \rrbracket^{\mathrm{tt}} ; \llbracket \mathcal{H}_{2} \rrbracket^{\mathrm{tt}}$.

Proof of (3). For this we assume that $\mathcal{H}_{1}$ and $\mathcal{H}_{2}$ are finite-input. Let $\zeta \in \mathrm{T}_{\Delta}$. Since $\mathcal{H}_{1}$ and $\mathcal{H}_{2}$ are finite-input, the set $A=\left\{\theta \in \mathrm{T}_{\Psi} \mid \operatorname{RT}_{\mathcal{H}_{2}}(\theta, \zeta) \neq \emptyset\right\}$ is finite, and for every $\theta \in A$ the set $B=\left\{\xi \in \mathrm{T}_{\Sigma} \mid\right.$ $\left.\mathrm{RT}_{\mathcal{H}_{1}}(\xi, \theta) \neq \emptyset\right\}$ is finite. Hence the set $C=\left\{\xi \in \mathrm{T}_{\Sigma} \mid\left(\exists \theta \in \mathrm{T}_{\Psi}\right): \mathrm{RT}_{\mathcal{H}_{1}}(\xi, \theta) \neq \emptyset \wedge \mathrm{RT}_{\mathcal{H}_{2}}(\theta, \zeta) \neq \emptyset\right\}$ is finite.

Next we show that $\left\{\xi \in \mathrm{T}_{\Sigma} \mid \operatorname{RT}_{\mathcal{H}}(\xi, \zeta) \neq \emptyset\right\} \subseteq C$. For this, let $\xi \in \mathrm{T}_{\Sigma}$ and $d \in \mathrm{RT}_{\mathcal{H}}(\xi, \zeta)$. Then there exists $\left\langle r_{1}, r_{2}\right) \in S$ such that $d \in \mathrm{RT}_{\mathcal{H}}\left(\left\langle r_{1}, r_{2}\right), \xi, h(d), \zeta\right)$. Since $\varphi_{r_{1}, r_{2}, \xi, h(d), \zeta}$ is bijective, there exist $t_{1} \in \mathrm{RT}_{\mathcal{H}_{1}}\left(N_{1}, \xi, h(d)\right)$ and $t_{2} \in \mathrm{RT}_{\mathcal{H}_{2}}\left(N_{2}, h(d), \zeta\right)$. Since $\operatorname{lhs}\left(r_{i}\right) \in S_{i}$ for $i \in\{1,2\}$, we have $t_{1} \in \operatorname{RT}_{\mathcal{H}_{1}}(\xi, h(d))$ and $t_{2} \in \operatorname{RT}_{\mathcal{H}_{2}}(h(d), \zeta)$. Hence for $h(d) \in \mathrm{T}_{\Psi}$ we have $\mathrm{RT}_{\mathcal{H}_{1}}(\xi, h(d)) \neq \emptyset$ and $\mathrm{RT}_{\mathcal{H}_{2}}(h(d), \zeta) \neq \emptyset$, which means that $\xi \in C$.

Since $\left\{\xi \in \mathrm{T}_{\Sigma} \mid \operatorname{RT}_{\mathcal{H}}(\xi, \zeta) \neq \emptyset\right\} \subseteq C$ and $C$ is finite, also $\left\{\xi \in \mathrm{T}_{\Sigma} \mid \operatorname{RT}_{\mathcal{H}}(\xi, \zeta) \neq \emptyset\right\}$ is finite. Hence $\mathcal{H}$ is finite-input.

### 10.13.4 Merge

Here we express the application of a weighted projective bimorphism to a characteristic mapping in terms of a weighted regular tree grammar. Since this process is a kind of inverse to the split in Subsection 10.13.2. we call it merge. We recall that $[\Delta]$ denote the skeleton alphabet of $\Delta$, defined on page 228 ,

Lemma 10.13.8. (cf. [FV19, Lm. 4.4]) Let $\mathcal{H}$ be a chain-free ([ $\Delta], \Psi, \mathrm{B}$ )-wpb such that $\mathcal{H}$ is finiteinput or B is $\sigma$-complete. Then we can construct an alphabetic $(\Psi, \mathrm{B})-\operatorname{wrtg} \mathcal{G}$ such that (a) $\mathcal{G}$ is finitederivational if $\mathcal{H}$ is finite-input and (b) $\llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}\left(\chi\left(\mathrm{T}_{[\Delta]}\right)\right)=\llbracket \mathcal{G} \rrbracket$.

Proof. Let $\mathcal{H}=(N, S, R, w t)$. We recall that $\mathcal{H}$ is a chain-free and alphabetic $([[\Delta] \Psi], \mathrm{B})$-wrtg. We construct the $(\Psi, \mathrm{B})-\operatorname{wrtg} \mathcal{G}=\left(N^{\prime}, S^{\prime}, R^{\prime}, w t^{\prime}\right)$ as follows. We let $N^{\prime}=R$ and $S^{\prime}=\{r \in R \mid \operatorname{lhs}(r) \in S\}$. The set $R^{\prime}$ of rules and the weight mapping $w t^{\prime}$ are defined as follows.

1. For every $r=\left(A \rightarrow[[k], \psi]\left(A_{1}, \ldots, A_{k}\right)\right)$ in $R$ and every $r_{1}, \ldots, r_{k} \in R$ with $\operatorname{lhs}\left(r_{i}\right)=A_{i}$ for each $i \in[k]$, the rule $r^{\prime}=\left(r \rightarrow \psi\left(r_{1}, \ldots, r_{k}\right)\right)$ is in $R^{\prime}$ and $w t^{\prime}\left(r^{\prime}\right)=w t(r)$.
2. For every $r=\left(A \rightarrow[[1], \varepsilon]\left(A_{1}\right)\right)$ in $R$ and every $r_{1} \in R$ with $\operatorname{lhs}\left(r_{1}\right)=A_{1}$, the rule $r^{\prime}=\left(r \rightarrow r_{1}\right)$ is in $R^{\prime}$ and $w t^{\prime}\left(r^{\prime}\right)=w t(r)$.
3. For every $r=\left(A \rightarrow[\varepsilon, \psi]\left(A_{1}\right)\right)$ in $R$ and every $r_{1} \in R$ with $\operatorname{lhs}\left(r_{1}\right)=A_{1}$, the rule $r^{\prime}=\left(r \rightarrow \psi\left(r_{1}\right)\right)$ is in $R^{\prime}$ and $w t^{\prime}\left(r^{\prime}\right)=w t(r)$.
Obviously, $\mathcal{G}$ is alphabetic.
Before we prove Properties (a) and (b) we need some preparations. We define the mapping $\beta^{\prime}: \mathrm{T}_{R^{\prime}} \rightarrow$ $\mathrm{T}_{[\Delta]}$ by induction on $\mathrm{T}_{R^{\prime}}$ for each $r^{\prime}\left(d_{1}^{\prime}, \ldots, d_{k}^{\prime}\right) \in \mathrm{T}_{R^{\prime}}$ as follows:

$$
\beta^{\prime}\left(r^{\prime}\left(d_{1}^{\prime}, \ldots, d_{k}^{\prime}\right)\right)= \begin{cases}{[k]\left(\beta^{\prime}\left(d_{1}^{\prime}\right), \ldots, \beta^{\prime}\left(d_{k}^{\prime}\right)\right)} & \text { if } r^{\prime} \text { is defined by } 1 . \\ {[1]\left(\beta^{\prime}\left(d_{1}^{\prime}\right)\right)} & \text { if } r^{\prime} \text { is defined by } 2 . \\ \beta^{\prime}\left(d_{1}^{\prime}\right) & \text { if } r^{\prime} \text { is defined by } 3 .\end{cases}
$$

In other words, if the terminal symbol in the right-hand side of $\operatorname{lhs}\left(r^{\prime}\right)$ has the form $[[k], \psi]$ or $[[1], \varepsilon]$, then $\beta^{\prime}$ outputs the terminal symbol $[k]$ and [1], respectively; if the terminal symbol in the right-hand side of $\operatorname{lhs}\left(r^{\prime}\right)$ has the form $[\varepsilon, \psi]$, then $\beta^{\prime}$ does not output any terminal symbol.


Figure 10.16: Example trees $b \in \mathrm{~T}_{[\Delta]}$ (left) and $\xi \in \mathrm{T}_{\Psi}$ (right).


Figure 10.17: Example trees $d \in \mathrm{RT}_{\mathcal{H}}(N, b, \xi)$ (left) with $b \in \mathrm{~T}_{[\Delta]}$ and $\xi \in \mathrm{T}_{\Psi}$ from Fig. 10.16, and $\varphi_{b, \xi}(d) \in \mathrm{RT}_{\mathcal{G}}\left(N^{\prime}, \xi\right)$ (right). Moreover, $\beta^{\prime}\left(\varphi_{b, \xi}(d)\right)=b$ and $\pi\left(\varphi_{b, \xi}(d)\right)=\xi$.
(This $\beta^{\prime}$ is a modification of the mapping $\beta$ defined on page 229, For instance, if $\beta^{\prime}$ is applied to the right tree in Figure 10.17, then the left tree in Figure 10.16 is obtained.

Lastly, we define the relation

$$
\prec=\prec_{R} \cap\left(\mathrm{RT}_{\mathcal{H}}\left(N, \mathrm{~T}_{[[\Delta] \Psi]}\right) \times \mathrm{RT}_{\mathcal{H}}\left(N, \mathrm{~T}_{[[\Delta] \Psi]}\right)\right) .
$$

It is easy to see that $\prec$ is well-founded and $\min _{\prec}\left(\operatorname{RT}_{\mathcal{H}}\left(N, \mathrm{~T}_{[[\Delta] \Psi]}\right)\right)$ is the set of terminal rules of $\mathcal{H}$.
Then, by induction on $\left(\operatorname{RT}_{\mathcal{H}}\left(N, \mathrm{~T}_{[[\Delta] \Psi]}\right), \prec\right)$, we define the mapping

$$
\varphi: \mathrm{RT}_{\mathcal{H}}\left(N, \mathrm{~T}_{[[\Delta] \Psi]}\right) \rightarrow \mathrm{RT}_{\mathcal{G}}\left(N^{\prime}, \mathrm{T}_{\Psi}\right)
$$

as follows.
Let $d \in \operatorname{RT}_{\mathcal{H}}\left(N, \mathrm{~T}_{[[\Delta] \Psi]}\right)$. Then there exists $r=\left(A \rightarrow[\kappa, \lambda]\left(A_{1}, \ldots, A_{k}\right)\right)$ in $R$ and $d_{1}, \ldots, d_{k} \in$ $\mathrm{RT}_{\mathcal{H}}\left(N, \mathrm{~T}_{[[\Delta] \Psi]}\right)$ such that $d=r\left(d_{1}, \ldots, d_{k}\right)$.

- If $\kappa=[k]$ for some $k \neq 1$ and $\lambda \in \Psi$, then we define

$$
\varphi\left(r\left(d_{1}, \ldots, d_{k}\right)\right)=\left(r \rightarrow \lambda\left(d_{1}(\varepsilon), \ldots, d_{k}(\varepsilon)\right)\right)\left(\varphi\left(d_{1}\right), \ldots, \varphi\left(d_{k}\right)\right)
$$

- If $\kappa=[1]$ and $\lambda=\varepsilon$, then $k=1$ and we define

$$
\varphi\left(r\left(d_{1}\right)\right)=\left(r \rightarrow d_{1}(\varepsilon)\right)\left(\varphi\left(d_{1}\right)\right)
$$

- If $\kappa=\varepsilon$ and $\lambda \in \Psi$, then $k=1$ and we define

$$
\varphi\left(r\left(d_{1}\right)\right)=\left(r \rightarrow \psi\left(d_{1}(\varepsilon)\right)\right)\left(\varphi\left(d_{1}\right)\right)
$$

The following is easy to see:

$$
\text { for every } b \in \mathrm{~T}_{[\Delta]} \text { and } \xi \in \mathrm{T}_{\Psi} \text {, we have } \varphi\left(\mathrm{RT}_{\mathcal{H}}(N, b, \xi)\right) \subseteq \mathrm{RT}_{\mathcal{G}}\left(N^{\prime}, \xi\right) \cap\left(\beta^{\prime}\right)^{-1}(b) .
$$

Thus, for every $b \in \mathrm{~T}_{[\Delta]}$ and $\xi \in \mathrm{T}_{\Psi}$, we can define the mapping

$$
\varphi_{b, \xi}: \operatorname{RT}_{\mathcal{H}}(N, b, \xi) \rightarrow \mathrm{RT}_{\mathcal{G}}\left(N^{\prime}, \xi\right) \cap\left(\beta^{\prime}\right)^{-1}(b)
$$

by letting $\varphi_{b, \xi}(d)=\varphi(d)$ for each $d \in \operatorname{RT}_{\mathcal{H}}(N, b, \xi)$.
Next we prove that $\varphi_{b, \xi}$ is bijective for every $b \in \mathrm{~T}_{[\Delta]}$ and $\xi \in \mathrm{T}_{\Psi}$. For this proof let us abbreviate $\varphi_{b, \xi}, \operatorname{RT}_{\mathcal{H}}(N, b, \xi)$, and $\mathrm{RT}_{\mathcal{G}}\left(N^{\prime}, \xi\right) \cap\left(\beta^{\prime}\right)^{-1}(b)$ by $\varphi, \mathrm{RT}_{\mathcal{H}}$, and $\mathrm{RT}_{\mathcal{G}}$, respectively. To illustrate our arguments we show examples of $b \in \mathrm{~T}_{[\Delta]}$ and $\xi \in \mathrm{T}_{\Psi}$ in Figure 10.16, and examples of $d \in \mathrm{RT}_{\mathcal{H}}$ and $\varphi(d) \in \mathrm{RT}_{\mathcal{G}}$ in Figure10.17. First we observe that $\varphi$ is shape preserving, i.e., for every $d \in \mathrm{RT}_{\mathcal{H}}$, we have $\operatorname{pos}(\varphi(d))=\operatorname{pos}(d)$ (see Figure 10.17). Now let $d_{1}, d_{2} \in \mathrm{RT}_{\mathcal{H}}$ such that $d_{1} \neq d_{2}$. If $\operatorname{pos}\left(d_{1}\right) \neq \operatorname{pos}\left(d_{2}\right)$, then $\varphi\left(d_{1}\right) \neq \varphi\left(d_{2}\right)$ because $\varphi$ is shape preserving. If $\operatorname{pos}\left(d_{1}\right)=\operatorname{pos}\left(d_{2}\right)$, then there exists a $w \in \operatorname{pos}\left(d_{1}\right)$ such that $d_{1}(w) \neq d_{2}(w)$. But then $\varphi\left(d_{1}\right)(w) \neq \varphi\left(d_{2}\right)(w)$ because the left-hand side nonterminals in $\varphi\left(d_{1}\right)(w)$ and $\varphi\left(d_{2}\right)(w)$ are $d_{1}(w)$ and $d_{2}(w)$, respectively (see Figure 10.17). This means that $\varphi$ is injective. Moreover, $\varphi$ is surjective because, given a $d \in \mathrm{RT}_{\mathcal{G}}$, we can easily reobtain a $d \in \mathrm{RT}_{\mathcal{H}}$ such that $\varphi(d)=d$. In fact, $d$ can be constructed by replacing, at each position $u$ of $d$, the rule $d(u)$ of $\mathcal{G}$ by the rule $\operatorname{lhs}(d(u))$ of $\mathcal{H}$ (see again Figure 10.17).

Obviously, it also holds that $\varphi_{b, \xi}$ is weight preserving, i.e., $\mathrm{wt}_{\mathcal{H}}(d)=\operatorname{wt}_{\mathcal{G}}\left(\varphi_{b, \xi}(d)\right)$ for every $d \in$ $\mathrm{RT}_{\mathcal{H}}(N, b, \xi)$. Then, for every $b \in \mathrm{~T}_{[\Delta]}$ and $\xi \in \mathrm{T}_{\Psi}$, there exists a weight preserving bijection $\bar{\varphi}_{b, \xi}$ : $\operatorname{RT}_{\mathcal{H}}(b, \xi) \rightarrow \operatorname{RT}_{\mathcal{G}}(\xi) \cap\left(\beta^{\prime}\right)^{-1}(b)$.

Now we prove property (a). Assume that $\mathcal{H}$ is finite-input and let $\xi \in \mathrm{T}_{\Psi}$. Then the set $\{b \in$ $\left.\mathrm{T}_{[\Delta]} \mid \mathrm{RT}_{\mathcal{H}}(b, \xi) \neq \emptyset\right\}$ is finite. Since $\mathcal{H}$ is chain-free, it is also finite-derivational, i.e., the set $\mathrm{RT}_{\mathcal{H}}(b, \xi)$ is finite for every $b \in \mathrm{~T}_{[\Delta]}$. Since $\varphi_{b, \xi}$ is bijective, we obtain that the set $\mathrm{RT}_{\mathcal{G}}(\xi) \cap\left(\beta^{\prime}\right)^{-1}(b)$ is finite for each $b \in \mathrm{~T}_{[\Delta]}$ and thus the set $\left\{b \in \mathrm{~T}_{[\Delta]} \mid \operatorname{RT}_{\mathcal{G}}(\xi) \cap\left(\beta^{\prime}\right)^{-1}(b) \neq \emptyset\right\}$ is also finite. Moreover, the family $\left(\operatorname{RT}_{\mathcal{G}}(\xi) \cap\left(\beta^{\prime}\right)^{-1}(b) \mid b \in \mathrm{~T}_{[\Delta]}\right)$ is a partitioning of $\mathrm{RT}_{\mathcal{G}}(\xi)$. Thus, $\mathrm{RT}_{\mathcal{G}}(\xi)$ is finite and hence $\mathcal{G}$ is finite-derivational. This proves (a).

Finally we prove Property $(\mathrm{b}): \llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}\left(\chi\left(\mathrm{T}_{[\Delta]}\right)\right)=\llbracket \mathcal{G} \rrbracket$. For this, let $\xi \in \mathrm{T}_{\Psi}$. Then

$$
\begin{aligned}
& \left(\llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}\left(\chi\left(\mathrm{~T}_{[\Delta]}\right)\right)\right)(\xi)=\sum_{b \in \mathrm{~T}_{[\Delta]}}^{\oplus} \chi\left(\mathrm{T}_{[\Delta]}\right)(b) \otimes \llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}(b, \xi) \\
& \text { (by (2.34) because } \llbracket \mathcal{H} \rrbracket \text { is finite-input or } \mathrm{B} \text { is } \sigma \text {-complete) } \\
& =\sum_{b \in \mathrm{~T}_{[\Delta]}}^{\oplus} \llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}(b, \xi) \\
& =\sum_{b \in \mathrm{~T}_{[\Delta]}}^{\oplus} \bigoplus_{d \in \mathrm{RT}_{\mathcal{H}}(b, \xi)} \mathrm{wt}_{\mathcal{H}}(d) \quad \text { (because } \mathcal{H} \text { is chain-free and hence finite-derivational) } \\
& =\sum_{b \in \mathrm{~T}_{[\Delta]}}^{\oplus} \bigoplus_{d \in \mathrm{RT}_{\mathcal{G}}(\xi) \cap\left(\beta^{\prime}\right)^{-1}(b)} \mathrm{wt}_{\mathcal{G}}(d) \quad \text { (because } \bar{\varphi}_{b, \xi} \text { is bijective and weight preserving) } \\
& =\sum_{d \in \mathrm{RT}_{\mathcal{G}}(\xi)}^{\oplus} \mathrm{wt}_{\mathcal{G}}(d)=\llbracket \mathcal{G} \rrbracket(\xi) . \quad \text { (because } \mathcal{G} \text { is finite-derivational if } \llbracket \mathcal{H} \rrbracket \text { is finite-input) }
\end{aligned}
$$

### 10.13.5 Closure result for wrtg and wta

Finally we can prove the closure results for the sets $\operatorname{Reg}\left(\_, B\right)$ and $\operatorname{Rec}\left(\_, B\right)$.

Theorem 10.13.9. (cf. [FV22b, Thm. 6.3]) Let B be a commutative semiring. Let $\mathcal{G}$ be a ( $\Sigma, \mathrm{B}$ )-wrtg and $\mathcal{H}$ be $a(\Sigma, \Psi, \mathrm{~B})$-wpb such that (both $\mathcal{G}$ and $\mathcal{H}$ are finite-derivational and $\mathcal{H}$ is finite-input) or B is $\sigma$-complete. Then the following two statements hold.
(1) Then there exists a $(\Psi, \mathrm{B})-w r t g \mathcal{G}^{\prime}$ such that (a) $\mathcal{G}^{\prime}$ is finite-derivational if $\mathcal{G}$ is finite-derivational and $\mathcal{H}$ is finite-input and (b) $\llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}(\llbracket \mathcal{G} \rrbracket)=\llbracket \mathcal{G}^{\prime} \rrbracket$.
(2) If $\mathcal{H}$ is finite-derivational, then we can construct a ( $\Psi, \mathrm{B})$-wrtg $\mathcal{G}^{\prime}$ such that (a) $\mathcal{G}^{\prime}$ is finitederivational if $\mathcal{G}$ is finite-derivational and $\mathcal{H}$ is finite-input and (b) $\llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}(\llbracket \mathcal{G} \rrbracket)=\llbracket \mathcal{G}^{\prime} \rrbracket$.

Proof. Proof of (1): Let $\mathcal{G}=(N, S, R, \mathrm{wt})$. By Lemma 9.2.2, we may assume that $\mathcal{G}$ is alphabetic. Then, by Lemma 10.13 .6 , we can construct a chain-free $([R], \Sigma, \mathrm{B})$-wpb $\mathcal{H}_{\mathcal{G}}$ such that $\mathcal{H}_{\mathcal{G}}$ is finite-output, $\mathcal{H}_{\mathcal{G}}$ is finite-input if $\mathcal{G}$ is finite-derivational, and $\llbracket \mathcal{G} \rrbracket=\llbracket \mathcal{H}_{\mathcal{G}} \rrbracket^{\mathrm{tt}}\left(\chi\left(\mathrm{T}_{[R]}\right)\right)$. In particular, since $\mathcal{H}_{\mathcal{G}}$ is chain-free, it is also finite-derivational. Then

$$
\llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}(\llbracket \mathcal{G} \rrbracket)=\llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}\left(\llbracket \mathcal{H}_{\mathcal{G}} \rrbracket^{\mathrm{tt}}\left(\chi\left(\mathrm{~T}_{[R]}\right)\right)\right)=\left(\llbracket \mathcal{H}_{\mathcal{G}} \rrbracket^{\mathrm{tt}} ; \llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}\right)\left(\chi\left(\mathrm{T}_{[R]}\right)\right),
$$

where the last equality is due to Observation 2.10 .3 (recall that $B$ is a semiring). Any of the two conditions that (i) $\llbracket \mathcal{H}_{\mathcal{G}} \rrbracket^{\mathrm{tt}}$ is supp-o-finite and (ii) $\llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}$ is supp-i-finite or B is $\sigma$-complete, assures that by Theorem 10.13.7(1) there exists an $([R], \Psi, \mathrm{B})$-wpb $\mathcal{H}^{\prime}$ such that $\llbracket \mathcal{H}^{\prime} \rrbracket^{\mathrm{tt}}=\llbracket \mathcal{H}_{\mathcal{G}} \rrbracket^{\mathrm{tt}} ; \llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}$. If $\mathcal{H}$ and $\mathcal{H}_{\mathcal{G}}$ are finiteinput, then by Theorem $10.13 .7(3)$, also $\mathcal{H}^{\prime}$ is finite-input. By Lemma 10.13 .5 (1) we can assume that $\mathcal{H}^{\prime}$ is chain-free and finite-input.

Finally, by Lemma 10.13 .8 , if $\mathcal{H}^{\prime}$ is given, then we can construct a $(\Psi, \mathrm{B})-$ wrtg $\mathcal{G}^{\prime}$ such that $\mathcal{G}^{\prime}$ is finite-derivational $\mathcal{H}^{\prime}$ is finite-input and $\llbracket \mathcal{H}^{\prime} \rrbracket^{\mathrm{tt}}\left(\chi\left(\mathrm{T}_{[R]}\right)\right)=\llbracket \mathcal{G}^{\prime} \rrbracket$.

Proof of (2): Now assume that $\mathcal{H}$ is finite-derivational. Then we follow the proof of (1) and, instead of Theorem 10.13.7(1) and Lemma 10.13.5(1), we apply Theorem 10.13.7(2) and Lemma 10.13.5(2), respectively. Hence, we can even construct $\mathcal{H}^{\prime}$. Thus we can also construct $\mathcal{G}^{\prime}$ with the mentioned properties.

Corollary 10.13.10. Let $\Sigma$ and $\Psi$ be ranked alphabets, B be a commutative semiring, and $\mathcal{A}$ be a ( $\Sigma, \mathrm{B})$ wta. Moreover, let $\mathcal{H}$ be a $(\Sigma, \Psi, \mathrm{B})$-wpb such that (a) $\mathcal{H}$ is finite-derivational and finite-input or (b) B is $\sigma$-complete. Then we can construct a $(\Psi, \mathrm{B})$-wta $\mathcal{A}^{\prime}$ such that $\llbracket \mathcal{H} \rrbracket^{\text {tt }}(\llbracket \mathcal{A} \rrbracket)=\llbracket \mathcal{A}^{\prime} \rrbracket$. Thus, in particular, if $B$ is a commutative semiring, then $\operatorname{Rec}(-, B)$ is closed under weighted projective bimorphisms.

Proof. By Lemma 9.2 .6 we can construct a $(\Sigma, \mathrm{B})-$ wrtg $\mathcal{G}$ such that $\mathcal{G}$ is in tree automata form and $\llbracket \mathcal{A} \rrbracket=\llbracket \mathcal{G} \rrbracket$. Then, in particular, $\mathcal{G}$ is finite-derivational. By Theorem 10.13 .9 we can construct a finitederivational $(\Psi, \mathrm{B})-$ wrtg $\mathcal{G}^{\prime}$ such that $\llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}(\llbracket \mathcal{G} \rrbracket)=\llbracket \mathcal{G}^{\prime} \rrbracket$. Finally, by Lemma 9.2.8, we can construct a $(\Sigma, \mathrm{B})$-wta $\mathcal{A}^{\prime}$ such that $\llbracket \mathcal{G}^{\prime} \rrbracket=\llbracket \mathcal{A}^{\prime} \rrbracket$.

### 10.13.6 Closure under Hadamard product: an alternative proof

We have proved that the set of recognizable ( $\Sigma, \mathrm{B}$ )-weighted tree languages is closed under Hadamard product if $B$ is a commutative semiring (cf. Theorem 10.4.1). This closure also follows from the closure of this set under weighted projective bimorphisms.

Corollary 10.13.11. (cf. Theorem 10.4.1) Let B be a commutative semiring. Moreover, let $\mathcal{A}_{1}$ and $\mathcal{A}_{2}$ be two $(\Sigma, \mathrm{B})$-wta. Then we can construct a $(\Sigma, \mathrm{B})$-wta $\mathcal{B}$ such that $\llbracket \mathcal{B} \rrbracket=\llbracket \mathcal{A}_{1} \rrbracket \otimes \llbracket \mathcal{A}_{2} \rrbracket$.

Proof. Let $\mathcal{A}_{2}=(Q, \delta, F)$. By Theorem 7.3.1 we can assume that $\mathcal{A}_{2}$ is root weight normalized and $\operatorname{supp}(F)=\left\{q_{f}\right\}$. We construct the $(\Sigma, \Sigma, \mathrm{B})-\mathrm{wpb} \mathcal{H}=(N, S, R, w t)$ as follows.

- $N=Q$ and $S=\left\{q_{f}\right\}$,
- For every $k \in \mathbb{N}, \sigma \in \Sigma^{(k)}, q \in Q$, and $q_{1} \cdots q_{k} \in Q^{k}$, the rule $r=\left(q \rightarrow[\sigma, \sigma]\left(q_{1}, \ldots, q_{k}\right)\right)$ is in $R$ and $w t(r)=\delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right)$.
We note that $\mathcal{H}$ is in tree automata form and finite-input. Then it is easy to see that $\llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}=\overline{\llbracket \mathcal{A}_{2} \rrbracket}$, i.e., $\llbracket \mathcal{H} \rrbracket^{\text {tt }}$ is the diagonalization of $\llbracket \mathcal{A}_{2} \rrbracket($ cf. (2.32) $)$. Then by Equation (2.33) we have $\llbracket \mathcal{H} \rrbracket^{\text {tt }}\left(\llbracket \mathcal{A}_{1} \rrbracket\right)=\llbracket \mathcal{A}_{1} \rrbracket \otimes$ $\llbracket \mathcal{A}_{2} \rrbracket$ and thus by Corollary 10.13 .10 we can construct a $(\Sigma, \mathrm{B})$-wta $\mathcal{B}$ such that $\llbracket \mathcal{B} \rrbracket=\llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}\left(\llbracket \mathcal{A}_{1} \rrbracket\right)$.


### 10.13.7 Closure under yield-intersection with weighted recognizable languages: an alternative proof

For every $(\Sigma, \mathrm{B})$-wta $\mathcal{A}, \Gamma \subseteq \Sigma^{(0)}$, and $(\Gamma, \mathrm{B})$-wsa $\mathcal{B}$, we have constructed a $(\Sigma, \mathrm{B})$-wta $\mathcal{A}^{\prime}$ such that $\llbracket \mathcal{A}^{\prime} \rrbracket=\llbracket \mathcal{A} \rrbracket \otimes\left(\llbracket \mathcal{B} \rrbracket \circ\right.$ yield $\left._{\Gamma}\right)$, if B is a commutative semiring (cf. Theorem 10.8.2). Here we give an alternative proof which uses the fact that the set of recognizable $(\Sigma, B)$-weighted tree languages is closed under weighted projective bimorphisms.

Corollary 10.13.12. (cf. Theorem 10.8.2) Let $B$ be a commutative semiring and $\Gamma \subseteq \Sigma^{(0)}$. For every $(\Sigma, \mathrm{B})$-wta $\mathcal{A}$ and every wsa $\mathcal{B}$ over $\Gamma$ and B , we can construct a $(\Sigma, \mathrm{B})$-wta $\mathcal{A}^{\prime}$ such that $\llbracket \mathcal{A}^{\prime} \rrbracket=$ $\llbracket \mathcal{A} \rrbracket \otimes\left(\llbracket \mathcal{B} \rrbracket \circ\right.$ yield $\left.{ }_{\Gamma}\right)$. Thus, in particular, the $\operatorname{set} \operatorname{Rec}^{\text {run }}(\Sigma, \mathrm{B})$ is closed under yield-intersection.

Proof. Let $\mathcal{B}=(P, \lambda, \mu, \gamma)$ be a $\left(\Sigma^{(0)}, \mathrm{B}\right)$-wsa.
The idea is to construct a $(\Sigma, \Sigma, \mathrm{B})$-wpb $\mathcal{H}$ such that, for every $\xi, \zeta \in \mathrm{T}_{\Sigma}$ we have

$$
\llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}(\xi, \zeta)= \begin{cases}\llbracket \mathcal{B} \rrbracket\left(\operatorname{yield}_{\Gamma}(\zeta)\right) & \text { if } \xi=\zeta  \tag{10.44}\\ \mathbb{O} & \text { otherwise }\end{cases}
$$

We construct $\mathcal{H}=(N, S, R, w t)$ as follows.

- $N=(P \times P) \cup\{S\}$ where $S$ is a new symbol, and
- $R$ and $w t$ are defined as follows:
- for each $\left(p, p^{\prime}\right) \in N$, the rule $r=\left(S \rightarrow\left(p, p^{\prime}\right)\right)$ is in $R$ and we let $w t(r)=\lambda(p) \otimes \gamma\left(p^{\prime}\right)$,
- for each $\alpha \in \Gamma$ and $\left(p, p^{\prime}\right) \in P \times P$ we let $r=\left(\left(p, p^{\prime}\right) \rightarrow[\alpha, \alpha]\right)$ be a rule in $R$ with $w t(r)=$ $\mu\left(p, \alpha, p^{\prime}\right)$,
- for each $\alpha \in \Sigma^{(0)} \backslash \Gamma$ and $p \in P$ we let $r=((p, p) \rightarrow[\alpha, \alpha])$ be a rule in $R$ with $w t(r)=\mathbb{1}$, and
- for every $k \in \mathbb{N}_{+}, \sigma \in \Sigma^{(k)}$, and $\left(p_{1}, p_{1}^{\prime}\right),\left(p_{2}, p_{2}^{\prime}\right), \ldots,\left(p_{k}, p_{k}^{\prime}\right),\left(p, p^{\prime}\right) \in N$ we let the rule $r=\left(\left(p, p^{\prime}\right) \rightarrow[\sigma, \sigma]\left(\left(p_{1}, p_{1}^{\prime}\right), \ldots,\left(p_{k}, p_{k}^{\prime}\right)\right)\right)$ be in $R$ with

$$
w t(r)= \begin{cases}\mathbb{1} & \text { if } p=p_{1}, p_{i}^{\prime}=p_{i+1} \text { for each } i \in[k-1], \text { and } p_{k}^{\prime}=p^{\prime} \\ \mathbb{D} & \text { otherwise }\end{cases}
$$

Then $\mathcal{H}$ is finite-derivational and finite-input, and hence $\llbracket \mathcal{H} \rrbracket^{\text {tt }}$ is supp-i-finite. Moreover, apart from the chain rules of the form $S \rightarrow\left(p, p^{\prime}\right)$, the wpb $\mathcal{H}$ closely corresponds to the wta $\mathcal{A}$ constructed in the proof of Lemma 10.8.1. Indeed, there exists a bijection between $\mathrm{RT}_{\mathcal{H}}(\xi, \xi)$ and $\mathrm{R}_{\mathcal{A}}(\xi)$, which is defined in a similar way as the bijection in the proof of Lemma 9.2.5.

It is quite obvious that $(10.44)$ holds. Since $\llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}$ is supp-i-finite, $\llbracket \mathcal{A} \rrbracket$ is $\llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}$-summable. Hence $\llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}(\llbracket \mathcal{A} \rrbracket)$ is defined. Then, for each $\zeta \in \mathrm{T}_{\Sigma}$, we have

$$
\begin{array}{rlr}
\llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}(\llbracket \mathcal{A} \rrbracket)(\zeta) & =\sum_{\xi \in \mathrm{T}_{\Sigma}}^{\oplus} \llbracket \mathcal{A} \rrbracket(\xi) \otimes \llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}(\xi, \zeta) \\
& =\llbracket \mathcal{A} \rrbracket(\zeta) \otimes \llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}(\zeta, \zeta) \quad \\
& =\left(\llbracket \mathcal{A} \rrbracket \otimes\left(\llbracket \mathcal{B} \rrbracket \circ \text { yield }{ }_{\Gamma}\right)\right)(\zeta) \quad \text { (by the second case of (10.44)) }
\end{array}
$$

By Corollary 10.13 .10 we can construct a $(\Sigma, \mathrm{B})$-wta $\mathcal{A}^{\prime}$ such that $\llbracket \mathcal{A}^{\prime} \rrbracket=\llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}(\llbracket \mathcal{A} \rrbracket)$.

### 10.13.8 Closure under tree relabelings: an alternative proof

Finally, we show that each tree relabeling can be computed by a particular weighted projective bimorphism. Then, for the case that $B$ is a commutative semiring, the closure of $\operatorname{Rec}(-, B)$ under tree relabelings (cf. Theorem 10.10.1) can be reobtained as corollary of Corollary 10.13.10. However, we recall that Theorem 10.10.1 holds for arbitrary strong bimonoids and not only for commutative semirings.

Observation 10.13.13. Let $\tau$ be a $(\Sigma, \Delta)$-tree relabeling and $r: \mathrm{T}_{\Sigma} \rightarrow B$. Then we can construct a chain-free, finite-input, and finite-output $(\Sigma, \Delta, \mathrm{B})$-wpb $\mathcal{H}$ such that $\tau(r)=\llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}(r)$.

Proof. Let $\tau=\left(\tau_{k} \mid k \in \mathbb{N}\right)$. We construct the ( $\left.\Sigma, \Delta, \mathrm{B}\right)$-wpb $\mathcal{H}=\left(\left\{S_{0}\right\}, S_{0}, R, \mathrm{wt}\right)$ as follows. For every $k \in \mathbb{N}, \sigma \in \Sigma^{(k)}$, and $\gamma \in \tau_{k}(\sigma)$, the set $R$ contains the rule $r=\left(S_{0} \rightarrow[\sigma, \gamma]\left(S_{0}, \ldots, S_{0}\right)\right)$ with $k$ occurrences of $S_{0}$ and $\mathrm{wt}(r)=\mathbb{1}$. Clearly, $\mathcal{H}$ has the desired properties. In particular, $\llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}$ is supp-ifinite.

Then for every $\xi \in \mathrm{T}_{\Sigma}$ and $\zeta \in \mathrm{T}_{\Delta}: \llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}(\xi, \zeta) \in\{\mathbb{O}, \mathbb{1}\}$, and $\llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}(\xi, \zeta)=\mathbb{1}$ if and only if $\zeta \in \tau(\xi)$. Now let $\zeta \in \mathrm{T}_{\Delta}$. Then we can calculate as follows:

$$
\tau(r)(\zeta)=\bigoplus_{\xi \in \tau^{-1}(\zeta)} r(\xi)=\bigoplus_{\xi \in \mathrm{T}_{\Sigma}} r(\xi) \otimes \llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}(\xi, \zeta)=\left(\llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}(r)\right)(\zeta)
$$

Hence $\tau(r)=\llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}(r)$.
The following corollary follows from Corollary 10.13 .10 and Observation 10.13 .13
Corollary 10.13.14. Let B be a commutative semiring and $\mathcal{A}$ be a $(\Sigma, \mathrm{B})$-wta. Moreover, let $\tau$ be a $(\Sigma, \Delta)$-tree relabeling. Then we can construct a $(\Delta, \mathrm{B})$-wta $\mathcal{A}^{\prime}$ such that $\tau(\llbracket \mathcal{A} \rrbracket)=\llbracket \mathcal{A}^{\prime} \rrbracket$ (cf. Theorem 10.10.1). Thus, in particular, if $B$ is a commutative semiring, then the set $\operatorname{Rec}(-, B)$ is closed under tree relabelings.

### 10.14 Summary of some of the closure properties

In the three tables of Figure 10.18 we summarize some of the closure properties of the set of weighted tree languages recognized by wta. Each entry refers to the theorem or corollary where the precise formulation of the closure property can be found; in the figure we only have indicated the additionally required properties of the strong bimonoid. For some of the operations in the first table and the third table, we could only prove the corresponding closure property for the case that $B$ is a semiring; for a few other operations we did not need distributivity from both sides. In the summary, we did not consider the case of bu deterministic and crisp deterministic wta.

|  | $\operatorname{Rec}^{\text {run }}(\Sigma, \mathrm{B})$ | $\operatorname{Rec}^{\text {init }}(\Sigma, \mathrm{B})$ | $\operatorname{Rec}(\Sigma, \mathrm{B})$ |
| :---: | :---: | :---: | :---: |
| sum | Thm. 10.1.1 | Thm. 10.1.1 | Thm. 10.1.1 |
| scalar multiplications ... from left <br> ... from right | $\begin{aligned} & \text { Thm. } 10.2 .1{ }^{(1)} \\ & \text { (left-distr.) } \\ & \text { Thm. } 10.2 .1{ }^{2} \text { ) } \\ & \text { (right-distr.) } \end{aligned}$ | $\begin{gathered} \text { Thm. } 10.2 .1 \text { (1) } \\ \text { (left-distr.) } \\ \text { Thm. } 10.2 .1{ }^{(2)} \\ \text { (right-distr.) } \end{gathered}$ | $\begin{aligned} & \text { Thm. } 10.2 .1 \text { (1) } \\ & \text { Thm. } 10.2 .1 \text { (2) } \end{aligned}$ |
| Hadamard product |  |  | Thm. $\frac{\text { 10.4.1 also Cor. } 10.13 .11}{\text { (commutative) }}$ |
| top-concatenations |  |  | Cor. 10.5.2 |
| tree concatenations |  |  | $\begin{gathered} \text { Cor. } 10.6 .2 \\ \text { (commutative) } \end{gathered}$ |
| Kleene stars |  |  | $\begin{aligned} & \text { Cor. } 10.7 .6 \\ & \text { (commutative) } \end{aligned}$ |
| yield-intersection |  |  | $\frac{\text { Thm. 10.8.2 also Cor. } 10.13 .12}{\text { (commutative) }}$ |


|  | $\operatorname{Rec}^{\mathrm{run}}(\Sigma,-)$ | $\operatorname{Rec}^{\mathrm{init}}(\Sigma,)$, | $\operatorname{Rec}(\Sigma,-)$ |
| :--- | :---: | :---: | :---: |
| strong bimonoid hom. | Thm. 10.9 .3$]$ | Thm. 10.9 .3$]$ | Thm. 10.9 .3$]$ |


|  | $\operatorname{Rec}^{\text {run }}(-, \mathrm{B})$ | $\operatorname{Rec}^{\text {1nit }}(-, \mathrm{B})$ | $\operatorname{Rec}(-, \mathrm{B})$ |
| :--- | :---: | :---: | :---: |
| tree relabeling | Thm. 10.10 .1 |  | Thm. 10.10.1 also Cor. 10.13.14 |
| linear, nondeleting, and <br> productive tree hom. | Cor. 10.11 .2 |  | Cor. 10.11.2 |
| inverse of linear tree hom. |  |  | Thm. 10.12.2 <br> (commutative) |
| weighted projective bimorphisms <br> (finite-derivational, finite-input <br> or B $\sigma$-complete) |  |  | Cor. 10.13.10 <br> (commutative) |

Figure 10.18: A summary of some closure properties of some sets of weighted tree languages recognized by wta, where "distr." and "hom." abbreviate distributive and homomorphisms, respectively, and in the third column $B$ is a semiring.

## Chapter 11

## Characterizations by weighted local systems

In this chapter we show two characterization theorems for wta. The first one is Theorem 11.2.6 it is based on a decomposition theorem of wta which is due to [Fül15. The latter result generalizes the fact that each recognizable tree language is the image of a local tree language under a deterministic tree relabeling [Tha67, Prop. 2] (cf. [GS84, Thm. 2.9.5] and Eng75b, Cor. 3.59(i) $\Rightarrow$ (ii)]).

The second characterization theorem is Theorem 11.3.1, which follows the idea of decomposing a bottom-up tree transducer. In Eng75a, Thm. 3.5] it was proved that each bottom-up tree transducer can be decomposed into a relabeling, followed by the intersection with a local tree language, followed by a tree homomorphism. Also the reverse composition result holds Eng75a, p.220]. In Theorem[11.3.1(A) $\Rightarrow$ (B) we decompose the run semantics $\llbracket \mathcal{A} \rrbracket^{\text {run }}$ of a $(\Sigma, \mathrm{B})$-wta $\mathcal{A}$ into an inverse deterministic tree relabeling (corresponding to the above relabeling), followed by the intersection with a local tree language, followed by a homomorphism which interprets trees in some evaluation algebra with carrier set $B$ (corresponding to the above tree homomorphism). Similar decompositions were proved for weighted tree automata over multioperator monoids [SVF09, Thm. 1].

Before proving the decomposition theorems we recall the definitions of local systems, local tree languages, and weighted local system.

### 11.1 Local tree languages and weighted local systems

We consider the ranked alphabet $\operatorname{Fork}(\Sigma)$ where $\operatorname{Fork}(\Sigma)^{(k)}=\Sigma^{k} \times \Sigma^{(k)}$ for each $k \in \mathbb{N}$. Each element of $\operatorname{Fork}(\Sigma)^{(k)}$ has the form $\left(\sigma_{1} \cdots \sigma_{k}, \sigma\right)$ with $\sigma_{1}, \ldots, \sigma_{k} \in \Sigma$ and $\sigma \in \Sigma^{(k)}$, and it is called $k$-fork or just fork.

A $\Sigma$-local system [GS84, Sec. 2.9] is a pair $(K, H)$ where $K \subseteq \operatorname{Fork}(\Sigma)$ is a set of forks and $H \subseteq \Sigma$. The tree language generated by $(K, H)$, denoted by $\mathrm{L}(K, H)$, is defined as follows. First, we let

$$
\mathrm{L}(K)=\left\{\xi \in \mathrm{T}_{\Sigma} \mid(\xi(w 1) \cdots \xi(w k), \xi(w)) \in K \text { for each } w \in \operatorname{pos}(\xi), \text { where } k=\operatorname{rk}(\xi(w))\right\}
$$

Second, we define

$$
\mathrm{L}(K, H)=\{\xi \in \mathrm{L}(K) \mid \xi(\varepsilon) \in H\}
$$

For the particular $\Sigma$-local system $(\operatorname{Fork}(\Sigma), \Sigma)$, we have $L(\operatorname{Fork}(\Sigma))=\mathrm{L}(\operatorname{Fork}(\Sigma), \Sigma)=\mathrm{T}_{\Sigma}$. Let $L \subseteq \mathrm{~T}_{\Sigma}$. We call $L$ a local tree language if there exists a $\Sigma$-local system $(K, H)$ such that $L=\mathrm{L}(K, H)$.

A $(\Sigma, \mathrm{B})$-weighted local system (for short: $(\Sigma, \mathrm{B})$-wls) [Fül15] is a tuple $\mathcal{S}=(g, F)$ such that $g=\left(g_{k} \mid\right.$ $k \in \mathbb{N}$ ) is an $\mathbb{N}$-indexed family of mappings $g_{k}: \operatorname{Fork}(\Sigma)^{(k)} \rightarrow B$ and $F: \Sigma \rightarrow B$. We say that $\mathcal{S}$ has identity root weights if $\operatorname{im}(F) \subseteq\{\mathbb{0}, \mathbb{1}\}$.

We define the mapping $g^{\prime}: \mathrm{T}_{\Sigma} \rightarrow B$ by induction on $\mathrm{T}_{\Sigma}$. For every $\xi=\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)$, we let

$$
g^{\prime}(\xi)=g^{\prime}\left(\xi_{1}\right) \otimes \cdots \otimes g^{\prime}\left(\xi_{k}\right) \otimes g_{k}\left(\xi_{1}(\varepsilon) \cdots \xi_{k}(\varepsilon), \sigma\right)
$$

In the following we drop the prime from $g^{\prime}$ and simply write $g$ for $g^{\prime}$. The ( $\left.\Sigma, \mathrm{B}\right)$-weighted tree language determined by $\mathcal{S}$, denoted by $\llbracket \mathcal{S} \rrbracket$, is the mapping $\llbracket \mathcal{S} \rrbracket: \mathrm{T}_{\Sigma} \rightarrow B$ defined for each $\xi \in \mathrm{T}_{\Sigma}$ by

$$
\llbracket \mathcal{S} \rrbracket(\xi)=g(\xi) \otimes F(\xi(\varepsilon))
$$

Since the value $g\left(\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)\right) \in B$ depends on the root labels of $\xi_{1}, \ldots, \xi_{k}$, in general there does not exist a $\Sigma$-algebra $(B, \lambda)$ such that $g$ is the unique $\Sigma$-algebra homomorphism from ( $\left.\mathrm{T}_{\Sigma}, \theta_{\Sigma}\right)$ to $(B, \lambda)$.

Example 11.1.1. We consider the ranked alphabet $\Sigma=\left\{\sigma^{(2)}, \gamma^{(1)}, \alpha^{(0)}\right\}$ and the weighted tree language $\#{ }_{\sigma(., \alpha)}: \mathrm{T}_{\Sigma} \rightarrow \mathbb{N}$ defined in Example 3.2.11, where $\#_{\sigma(., \alpha)}(\xi)$ is the number of occurrences of the pattern $\sigma(., \alpha)$ in $\xi$ for each $\xi \in \mathrm{T}_{\Sigma}$.

We define the $\left(\Sigma\right.$, Nat $\left._{\text {max },+}\right)$-wls $\mathcal{S}=(g, F)$ such that

- for every $\theta_{1}, \theta_{2} \in \Sigma$ we let $g_{2}\left(\theta_{1} \theta_{2}, \sigma\right)=1$ if $\theta_{1} \theta_{2} \in\{\sigma \alpha, \gamma \alpha, \alpha \alpha\}$ and 0 otherwise; and we let $g_{1}\left(\theta_{1}, \gamma\right)=0$ and $g_{0}(\varepsilon, \alpha)=0$, and
- $F(\sigma)=F(\gamma)=F(\alpha)=0$.

It should be clear that $\llbracket \mathcal{S} \rrbracket(\xi)=\#_{\sigma(., \alpha)}(\xi)$ for each $\xi \in \mathrm{T}_{\Sigma}$.
Let $(K, H)$ be a $\Sigma$-local system and $\kappa=\left(\kappa_{k} \mid k \in \mathbb{N}\right)$ an $\mathbb{N}$-indexed family of mappings $\kappa_{k}: \Sigma^{(k)} \rightarrow B$. We interpret the trees in $\mathrm{L}(K, H)$ by the unique $\Sigma$-algebra homomorphism $\mathrm{h}_{\mathrm{M}(\Sigma, \kappa)}: \mathrm{T}_{\Sigma} \rightarrow B$, where $\mathrm{M}(\Sigma, \kappa)$ is the $(\Sigma, \kappa)$-evaluation algebra defined in Section 2.9. Then we obtain the following ( $\Sigma, \mathrm{B}$ )weighted tree language:

$$
\begin{aligned}
\left(\chi(\mathrm{L}(K, H)) \otimes \mathrm{h}_{\mathrm{M}(\Sigma, \kappa)}\right): \mathrm{T}_{\Sigma} & \rightarrow B \\
\xi & \mapsto \begin{cases}\mathrm{~h}_{\mathrm{M}(\Sigma, \kappa)}(\xi) & \text { if } \xi \in \mathrm{L}(K, H) \\
\mathbb{0} & \text { otherwise }\end{cases}
\end{aligned}
$$

In the next lemma we prove that the weighted tree language $\chi(\mathrm{L}(K, H)) \otimes \mathrm{h}_{\mathrm{M}(\Sigma, \kappa)}$ can be computed by a ( $\Sigma, \mathrm{B})$-wls.

Lemma 11.1.2. Let $(K, H)$ be a $\Sigma$-local system and $\kappa=\left(\kappa_{k} \mid k \in \mathbb{N}\right)$ a family of mappings $\kappa_{k}: \Sigma^{(k)} \rightarrow B$. We can construct a $(\Sigma, \mathrm{B})$-wls $\mathcal{S}$ which has identity root weights such that $\llbracket \mathcal{S} \rrbracket=\chi(\mathrm{L}(K, H)) \otimes \mathrm{h}_{\mathrm{M}(\Sigma, \kappa)}$.

Proof. We define the $(\Sigma, \mathrm{B})$-wls $\mathcal{S}=(g, F)$ as follows. For every $k \in \mathbb{N}$ and $\sigma_{1}, \ldots, \sigma_{k} \in \Sigma$, and $\sigma \in \Sigma^{(k)}$, we have

$$
g_{k}\left(\sigma_{1} \cdots \sigma_{k}, \sigma\right)=\left\{\begin{array}{ll}
\kappa_{k}(\sigma) & \text { if }\left(\sigma_{1} \cdots \sigma_{k}, \sigma\right) \in K \\
\mathbb{0} & \text { otherwise }
\end{array} \quad \text { and } \quad F(\sigma)= \begin{cases}\mathbb{1} & \text { if } \sigma \in H \\
\mathbb{0} & \text { otherwise }\end{cases}\right.
$$

By induction on $\mathrm{T}_{\Sigma}$, we prove that the following statement holds:

$$
\text { For each } \xi \in \mathrm{T}_{\Sigma}, \text { we have } g(\xi)=\left\{\begin{array}{lc}
\mathrm{h}_{\mathrm{M}(\Sigma, \kappa)}(\xi) & \text { if } \xi \in \mathrm{L}(K)  \tag{11.1}\\
\mathbb{0} & \text { otherwise }
\end{array}\right.
$$

Let $\xi=\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)$. Then

$$
g(\xi)=\left(\bigotimes_{i \in[k]} g\left(\xi_{i}\right)\right) \otimes g_{k}\left(\xi_{1}(\varepsilon) \cdots \xi_{k}(\varepsilon), \sigma\right)
$$

$$
\begin{aligned}
& = \begin{cases}\left(\otimes_{i \in[k]} \mathrm{h}_{\mathrm{M}(\Sigma, \kappa)}\left(\xi_{i}\right)\right) \otimes g_{k}\left(\xi_{1}(\varepsilon) \cdots \xi_{k}(\varepsilon), \sigma\right) & \text { if }(\forall i \in[k]): \xi_{i} \in \mathrm{~L}(K) \\
\mathbb{O} & \text { otherwise }\end{cases} \\
& =\left\{\begin{array}{ll}
\left(\otimes_{i \in[k]} \mathrm{h}_{\mathrm{M}(\Sigma, \kappa)}\left(\xi_{i}\right)\right) \otimes \kappa_{k}(\sigma) & \text { if }(\forall i \in[k]): \xi_{i} \in \mathrm{~L}(K) \\
0 & \text { and }\left(\xi_{1}(\varepsilon) \cdots \xi_{k}(\varepsilon), \sigma\right) \in K \quad \text { otherwise }
\end{array} \quad\right. \text { (by I.H.) } \\
& = \begin{cases}\mathrm{h}_{\mathrm{M}(\Sigma, \kappa)}(\xi) & \text { if } \xi \in \mathrm{L}(K) \\
0 & \text { otherwise } .\end{cases}
\end{aligned}
$$

This proves (11.1). Now let $\xi \in \mathrm{T}_{\Sigma}$. Then

$$
\llbracket \mathcal{S} \rrbracket(\xi)=g(\xi) \otimes F(\xi(\varepsilon))=\left\{\begin{array}{ll}
g(\xi) & \text { if } \xi \in \mathrm{L}(K) \\
0 & \text { and } \xi(\varepsilon) \in H \\
0 & \text { otherwise }
\end{array}= \begin{cases}\mathrm{h}_{\mathrm{M}(\Sigma, \kappa)}(\xi) & \text { if } \xi \in \mathrm{L}(K, H) \\
0 & \text { otherwise }\end{cases}\right.
$$

where the last equality is due to (11.1).
It seems that the inverse of Lemma 11.1.2 does not hold, because mappings of the form $g_{k}: \operatorname{Fork}(\Sigma)^{(k)} \rightarrow B$ cannot be coded by mappings of the form $\kappa_{k}: \Sigma^{(k)} \rightarrow B$.

In the next lemma, we prove that, for each ( $\Sigma$, Boole)-wls $\mathcal{S}=(g, F)$, the support of $\llbracket \mathcal{S} \rrbracket$ is a local tree language. As preparation, we define the support local system of $\mathcal{S}$, denoted by $\operatorname{supp}(\mathcal{S})$, to be the $\Sigma$-local system $(K, H)$, where $K=\bigcup_{k \in \mathbb{N}} \operatorname{supp}\left(g_{k}\right)$ and $H=\operatorname{supp}(F)$.
Lemma 11.1.3. Let $\mathcal{S}=(g, F)$ be a $(\Sigma$, Boole)-wls. Then $\operatorname{supp}(\llbracket \mathcal{S} \rrbracket)=\mathrm{L}(\operatorname{supp}(\mathcal{S}))$.
Proof. Let $\operatorname{supp}(\mathcal{S})=(K, H)$. By induction on $\mathrm{T}_{\Sigma}$, we prove that the following statement holds:
For each $\xi \in \mathrm{T}_{\Sigma}$, we have $\xi \in \operatorname{supp}(g)$ if and only if $\xi \in \mathrm{L}(K)$.
For this, let $\xi=\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)$. Then

$$
\begin{aligned}
\xi \in \operatorname{supp}(g) & \Longleftrightarrow g\left(\xi_{1}\right) \wedge \cdots \wedge g\left(\xi_{k}\right) \wedge g_{k}\left(\xi_{1}(\varepsilon) \cdots \xi_{k}(\varepsilon), \sigma\right)=1 \\
& \Longleftrightarrow(\forall i \in[k]): \xi_{i} \in \operatorname{supp}\left(g_{i}\right) \text { and } g_{k}\left(\xi_{1}(\varepsilon) \cdots \xi_{k}(\varepsilon), \sigma\right)=1 \\
& \Longleftrightarrow(\forall i \in[k]): \xi_{i} \in \mathrm{~L}(K) \text { and }\left(\xi_{1}(\varepsilon) \cdots \xi_{k}(\varepsilon), \sigma\right) \in K
\end{aligned}
$$

(by I.H. and the definition of $K$ )
$\Longleftrightarrow \xi \in \mathrm{L}(K)$.
This proves (11.2). Then for each $\xi \in \mathrm{T}_{\Sigma}$ we have

$$
\xi \in \operatorname{supp}(\llbracket \mathcal{S} \rrbracket) \Longleftrightarrow g(\xi)=1 \text { and } F(\xi(\varepsilon))=1 \Longleftrightarrow \xi \in \mathrm{~L}(K) \text { and } \xi(\varepsilon) \in H \Longleftrightarrow \xi \in \mathrm{~L}(K, H) .
$$

The next theorem states that, intuitively, $\Sigma$-local systems and ( $\Sigma$, Boole)-wls are essentially the same.
Theorem 11.1.4. Let $L \subseteq \mathrm{~T}_{\Sigma}$ be a tree language. The following two statements are equivalent.
(A) We can construct a $\Sigma$-local system $(K, H)$ such that $L=\mathrm{L}(K, H)$.
(B) We can construct a $(\Sigma$, Boole)-wls $\mathcal{S}$ such that $L=\operatorname{supp}(\llbracket \mathcal{S} \rrbracket)$.

Proof. Proof of $(\mathrm{A}) \Rightarrow(\mathrm{B})$ : Let $\kappa=\left(\kappa_{k} \mid k \in \mathbb{N}\right)$ be a family of mappings $\kappa_{k}: \Sigma^{(k)} \rightarrow \mathbb{B}$ such that $\kappa_{k}(\sigma)=1$ for every $k \in \mathbb{N}$ and $\sigma \in \Sigma^{(k)}$. Then

$$
L=\mathrm{L}(K, H)=\operatorname{supp}(\chi(\mathrm{L}(K, H)))=\operatorname{supp}\left(\chi(\mathrm{L}(K, H)) \otimes \mathrm{h}_{\mathrm{M}(\Sigma, \kappa)}\right) .
$$

By Lemma $\llbracket 1.1 .2$ we can construct a $\left(\Sigma\right.$, Boole)-wls $\mathcal{S}$ such that $\chi(L(K, H)) \otimes \mathrm{h}_{M(\Sigma, \kappa)}=\llbracket \mathcal{S} \rrbracket$. Thus $L=\operatorname{supp}(\llbracket \mathcal{S} \rrbracket)$.

Proof of $(\mathrm{B}) \Rightarrow(\mathrm{A})$ : This follows from Lemma 11.1.3

### 11.2 Characterization by weighted local system and deterministic relabeling

In this section we prove the first characterization theorem for recognizable weighted tree languages. Roughly speaking, it says that each recognizable ( $\Sigma, \mathrm{B}$ )-weighted tree language can be represented as the application of some deterministic tree relabeling to the weighted rule tree language of some wrtg in tree automata form (cf. Theorem 11.2.6).

As preparation, we recall the well known fact that the rule tree language of a context-free grammar is a local tree language (cf. Tha67, Prop. 1], also cf. Eng75b, Thm. 3.57] and [GS84, Thm. 3.2.9]).

Lemma 11.2.1. Let $G$ be a $\Gamma$-cfg with rule set $R$. We can construct an $R$-local system $(K, H)$ such that $\mathrm{L}(K, H)=\mathrm{RT}_{G}$.

Proof. Let $G=(N, S, R)$. We construct the $R$-local system $(K, H)$ as follows.
We construct the set $K \subseteq \operatorname{Fork}(R)$ such that, for every $r=\left(A \rightarrow u_{0} A_{1} u_{1} \cdots A_{k} u_{k}\right)$ in $R$ and every $r_{1}=\left(A_{1} \rightarrow \alpha_{1}\right), \ldots, r_{k}=\left(A_{k} \rightarrow \alpha_{k}\right)$ in $R$, the tuple $\left(r_{1} \cdots r_{k}, r\right)$ is in $K$.

Moreover, we let $H=\{r \mid \operatorname{lhs}(r) \in S\}$. It is obvious that $\mathrm{L}(K, H)=\mathrm{RT}_{G}$.

We note that, due to Eng75b, Thm. 3.57], the inverse of Lemma 11.2.1 does not hold. Next we generalize Lemma 11.2.1 to the weighted case.

Lemma 11.2.2. Let $\mathcal{G}$ be a $(\Gamma, \mathrm{B})$-wcfg with rule set $R$. The following two statements hold.
(1) We can construct an $R$-local system $(K, H)$ such that $\llbracket \mathcal{G} \rrbracket^{\mathrm{wrt}}=\chi(\mathrm{L}(K, H)) \otimes \mathrm{wt}_{\mathcal{G}}$.
(2) We can construct an $(R, \mathrm{~B})$-wls $\mathcal{S}$ with identity root weights such that $\llbracket \mathcal{G} \rrbracket^{\text {wrt }}=\llbracket \mathcal{S} \rrbracket$.

Proof. Let $\mathcal{G}=(N, S, R, w t)$ and $G$ be the $\Gamma$-cfg underlying $\mathcal{G}$. We construct the $R$-local system $(K, H)$ as in Lemma 11.2.1 and then, for $(K, H)$ and $w t$, the $(R, \mathrm{~B})$-wls $\mathcal{S}$ as in Lemma 11.1.2. Then we obtain

$$
\begin{array}{rlr}
\llbracket \mathcal{G} \rrbracket^{\mathrm{wrt}} & =\chi\left(\mathrm{RT}_{\mathcal{G}}\right) \otimes \mathrm{wt}_{\mathcal{G}} & \text { (by definition and because } \left.\mathrm{wt}_{\mathcal{G}} \otimes \chi\left(\mathrm{RT}_{\mathcal{G}}\right)=\chi\left(\mathrm{RT}_{\mathcal{G}}\right) \otimes \mathrm{wt}_{\mathcal{G}}\right) \\
& =\chi\left(\mathrm{RT}_{G}\right) \otimes \mathrm{wt}_{\mathcal{G}} & \\
& =\chi(\mathrm{L}(K, H)) \otimes \mathrm{wt}_{\mathcal{G}} & \text { (by Lemma 11.2.1) } \\
& =\chi(\mathrm{L}(K, H)) \otimes \mathrm{h}_{\mathrm{M}(R, w t)} & \text { (we recall that } \mathrm{h}_{\mathrm{M}(R, w t)}=\mathrm{wt}_{\mathcal{G}} \text { holds by our convention) } \\
& =\llbracket \mathcal{S} \rrbracket . & \text { (by Lemma 11.1.2) }
\end{array}
$$

It is known that each local tree language is recognizable GS84, Thm. 2.9.4]. The following result generalizes this to the weighted case (where [Fül15, Lm. 1] requires that $B$ is a semiring).

Lemma 11.2.3. Fül15, Lm. 1] For each ( $\Sigma, \mathrm{B}$ )-wls $\mathcal{S}$, we can construct a bu deterministic ( $\Sigma, \mathrm{B}$ )-wta such that $\llbracket \mathcal{A} \rrbracket=\llbracket \mathcal{S} \rrbracket$.

Proof. Let $\mathcal{S}=(g, F)$. We construct the $(\Sigma, \mathrm{B})$-wta $\mathcal{A}=\left(Q, \delta, F_{\mathcal{A}}\right)$ as follows:

- $Q=\{\bar{\sigma} \mid \sigma \in \Sigma\}$,
- for every $k \in \mathbb{N}, \sigma_{1} \ldots \sigma_{k} \in \Sigma^{k}, \sigma \in \Sigma^{(k)}$, and $\omega \in \Sigma$,

$$
\delta_{k}\left(\overline{\sigma_{1}} \ldots \overline{\sigma_{k}}, \sigma, \bar{\omega}\right)= \begin{cases}g_{k}\left(\sigma_{1} \ldots \sigma_{k}, \sigma\right) & \text { if } \omega=\sigma \\ \mathbb{O} & \text { otherwise }\end{cases}
$$

- $F_{\mathcal{A}}(\bar{\sigma})=F(\sigma)$ for every $\sigma \in \Sigma$.

It is clear that $\mathcal{A}$ is bu deterministic. Next, by induction on $\mathrm{T}_{\Sigma}$, we prove that the following statement holds:

$$
\text { For every } \xi \in T_{\Sigma} \text { and } \omega \in \Sigma, \text { we have } \mathrm{h}_{\mathcal{A}}(\xi)_{\bar{\omega}}= \begin{cases}g(\xi) & \text { if } \omega=\xi(\varepsilon)  \tag{11.3}\\ \mathbb{0} & \text { otherwise }\end{cases}
$$

Let $\xi=\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)$. Then

$$
\begin{aligned}
& \mathrm{h}_{\mathcal{A}}\left(\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)\right)_{\bar{\omega}} \\
= & \bigoplus_{\overline{\sigma_{1}} \ldots \overline{\sigma_{k}} \in Q^{k}} \mathrm{~h}_{\mathcal{A}}\left(\xi_{1}\right)_{\overline{\sigma_{1}}} \otimes \ldots \otimes \mathrm{~h}_{\mathcal{A}}\left(\xi_{k}\right)_{\overline{\sigma_{k}}} \otimes \delta_{k}\left(\overline{\sigma_{1}} \ldots \overline{\sigma_{k}}, \sigma, \bar{\omega}\right) \\
= & \left.\mathrm{h}_{\mathcal{A}}\left(\xi_{1}\right)_{\overline{\xi_{1}(\varepsilon)}} \otimes \ldots \otimes \mathrm{h}_{\mathcal{A}}\left(\xi_{k}\right)_{\overline{\xi_{k}(\varepsilon)}} \otimes \delta_{k} \overline{\xi_{1}(\varepsilon)} \cdots \overline{\xi_{k}(\varepsilon)}, \sigma, \bar{\omega}\right)
\end{aligned}
$$

$$
\text { (because for each } \overline{\sigma_{1}} \ldots \overline{\sigma_{k}} \in Q^{k} \text { with } \overline{\sigma_{1}} \ldots \overline{\sigma_{k}} \neq \overline{\xi_{1}(\varepsilon)} \cdots \overline{\xi_{k}(\varepsilon)}
$$

$$
\text { there exists an } i \in[k] \text { such that } \mathrm{h}_{\mathcal{A}}\left(\xi_{i}\right)_{\overline{\sigma_{i}}}=\mathbb{O} \text { by I.H.) }
$$

$$
= \begin{cases}g\left(\xi_{1}\right) \otimes \ldots \otimes g\left(\xi_{k}\right) \otimes g_{k}\left(\xi_{1}(\varepsilon) \ldots \xi_{k}(\varepsilon), \sigma\right) & \text { if } \omega=\sigma  \tag{byI.H.}\\ 0 & \text { otherwise }\end{cases}
$$

$$
= \begin{cases}g\left(\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)\right) & \text { if } \omega=\sigma \\ \mathbb{0} & \text { otherwise }\end{cases}
$$

This proves (11.3).
Finally, let $\xi \in T_{\Sigma}$. Then we get

$$
\llbracket \mathcal{A} \rrbracket(\xi)=\bigoplus_{\bar{\omega} \in Q} \mathrm{~h}_{\mathcal{A}}(\xi)_{\bar{\omega}} \otimes F_{\mathcal{A}}(\bar{\omega})=g(\xi) \otimes F_{\mathcal{A}}(\overline{\xi(\varepsilon)})=g(\xi) \otimes F(\xi(\varepsilon))=\llbracket \mathcal{S} \rrbracket(\xi)
$$

where the second equality follows from (11.3) and the other ones from the corresponding definitions.
For example, if we apply the construction in Lemma 11.2 .3 to the ( $\Sigma$, Nat $_{\text {max },+}$ )-wls $\mathcal{S}$ of Example 11.1.1, then we obtain the bu deterministic ( $\Sigma, \operatorname{Nat}_{\mathrm{max},+}$ )-wta $\mathcal{A}$ of Example 3.2.12,

Next we verify that Lemma 11.2 .3 is a generalization of GS84, Thm. 2.9.4]. We achieve this by proving that the latter result is equivalent to Lemma 11.2 .3 for the case that B is the semiring Boole.

Corollary 11.2.4. For each $\Sigma$-local system $(K, H)$, we can construct a bu deterministic $\Sigma$-fta $A$ such that $\mathrm{L}(K, H)=\mathrm{L}(A)$. Thus, in particular, each local tree language is recognizable.

Proof. Let $(K, H)$ be a $\Sigma$-local system. We construct the $\left(\Sigma\right.$, Boole)-wls $\mathcal{S}=(g, F)$ by letting $g_{k}=$ $\chi\left(K \cap \operatorname{Fork}(\Sigma)^{(k)}\right)$ for each $k \in \mathbb{N}$, and $F=\chi(H)$. Then $(K, H)=\operatorname{supp}(\mathcal{S})$ and, by Lemma 11.1.3, we have $\mathrm{L}(K, H)=\operatorname{supp}(\llbracket \mathcal{S} \rrbracket)$. By Lemma 11.2.3, we can construct a bu deterministic ( $\Sigma$, Boole)-wta $\mathcal{A}$ such that $\mathrm{L}(K, H)=\operatorname{supp}(\llbracket \mathcal{A} \rrbracket)$. By Corollary 3.4.2, we can construct a $\Sigma$-fta $A$ such that $\mathrm{L}(K, H)=\mathrm{L}(A)$.

We can easily demonstrate that bu deterministic wta are more powerful than weighted local systems. For instance, let $\Sigma=\left\{\gamma^{(1)}, \alpha^{(0)}\right\}$ and consider the ( $\Sigma$, Boole)-weighted tree language $r$ defined by $r(\gamma(\gamma(\alpha)))=1$ and $r(\xi)=0$ for every other $\xi \in \mathrm{T}_{\Sigma}$. It is easy to show that $r \in \operatorname{bud}-\operatorname{Rec}(\Sigma$, Boole) and there does not exist a $(\Sigma$, Boole)-wls $\mathcal{S}$ such that $\llbracket \mathcal{S} \rrbracket=r$. To see the latter, we assume that there exists a $(\Sigma$, Boole)-wls $\mathcal{S}=(g, F)$ such that $\llbracket \mathcal{S} \rrbracket=r$. Thus, using $\xi$ as abbreviation for $\gamma(\gamma(\alpha))$, we have $\llbracket \mathcal{S} \rrbracket(\xi)=g(\xi) \wedge F(\gamma)=1$. Hence $g(\xi)=F(\gamma)=1$, and thus, in particular, $g_{1}(\gamma, \gamma)=g_{1}(\alpha, \gamma)=g_{0}(\varepsilon, \alpha)=1$. But this means that, for each $n \in \mathbb{N}_{+}$, we have $g\left(\gamma^{n}(\alpha)\right)=1$ and hence $\llbracket \mathcal{S} \rrbracket\left(\gamma^{n}(\alpha)\right)=1$. This is a contradiction to $\llbracket \mathcal{S} \rrbracket=r$.

It is also known that each recognizable tree language is the image of a rule tree language of some context-free grammar under a deterministic tree relabeling (cf. Tha67, Prop. 2], Eng75b, Thm. 3.58]). The next lemma generalizes this to the weighted case.

Lemma 11.2.5. Let $\mathcal{A}$ be a $(\Sigma, \mathrm{B})$-wta. We can construct a $(\Sigma, \mathrm{B})$-wrtg $\mathcal{G}$ in tree automata form with rule set $R$ and a deterministic $(R, \Sigma)$-tree relabeling $\tau$ such that $\llbracket \mathcal{A} \rrbracket^{\text {run }}=\chi(\tau)\left(\llbracket \mathcal{G} \rrbracket^{\mathrm{wrt}}\right)$.

Proof. By Lemma 9.2 .6 we can construct a $(\Sigma, \mathrm{B})$-wrtg $\mathcal{G}$ in tree automata form such that $\llbracket \mathcal{A} \rrbracket^{\text {run }}=\llbracket \mathcal{G} \rrbracket$. Let $R$ be the set of rules of $\mathcal{G}$. As argued in Section 9.1 the projection $\pi_{\mathcal{G}}: \mathrm{T}_{R} \rightarrow \mathrm{~T}_{\Sigma}$ is determined by the $(R, \Sigma)$-tree homomorphism $\pi_{\mathcal{G}}=\left(\left(\pi_{\mathcal{G}}\right)_{k} \mid k \in \mathbb{N}\right)$ defined, for every $k \in \mathbb{N}$ and $r \in R^{(k)}$ of the form $r=$ $\left(A \rightarrow \sigma\left(A_{1}, \ldots, A_{k}\right)\right)$, by $\left(\pi_{\mathcal{G}}\right)_{k}(r)=\sigma\left(z_{1}, \ldots, z_{k}\right)$ (note that $\mathcal{G}$ is in tree automata form). We construct the deterministic $(R, \Sigma)$-tree relabeling $\tau=\left(\tau_{k} \mid k \in \mathbb{N}\right)$ such that, for each rule $A \rightarrow \sigma\left(A_{1}, \ldots, A_{k}\right)$, we let

$$
\tau_{k}\left(A \rightarrow \sigma\left(A_{1}, \ldots, A_{k}\right)\right)=\sigma
$$

Obviously, the mappings $\pi_{\mathcal{G}}: \mathrm{T}_{R} \rightarrow \mathrm{~T}_{\Sigma}$ and $\tau: \mathrm{T}_{R} \rightarrow \mathrm{~T}_{\Sigma}$ are equal. Then, by definition, we obtain $\llbracket \mathcal{G} \rrbracket=\chi\left(\pi_{\mathcal{G}}\right)\left(\llbracket \mathcal{G} \rrbracket^{\mathrm{wrt}}\right)=\chi(\tau)\left(\llbracket \mathcal{G} \rrbracket^{\mathrm{wrt}}\right)$.

Now we can prove the first characterization theorem.

Theorem 11.2.6. Let $\Sigma$ be a ranked alphabet, B be a strong bimonoid, and $r: \mathrm{T}_{\Sigma} \rightarrow B$. Then the following three statements are equivalent.
(A) We can construct a $(\Sigma, \mathrm{B})-w t a \mathcal{A}$ such that $r=\llbracket \mathcal{A} \rrbracket^{\mathrm{run}}$.
(B) We can construct

- a ranked alphabet $R$,
- a deterministic $(R, \Sigma)$-tree relabeling $\tau$, and
- $a(\Sigma, \mathrm{~B})-w r t g \mathcal{G}$ in tree automata form with rule set $R$
such that $r=\chi(\tau)\left(\llbracket \mathcal{G} \rrbracket^{\mathrm{wrt}}\right)$.
(C) We can construct
- a ranked alphabet $R$,
- a deterministic $(R, \Sigma)$-tree relabeling $\tau$, and
- an ( $R, \mathrm{~B}$ )-wls $\mathcal{S}$ with identity root weights
such that $r=\chi(\tau)(\llbracket \mathcal{S} \rrbracket)$.

Proof. Proof of $(\mathrm{A}) \Rightarrow(\mathrm{B})$ : It follows from Lemma 11.2 .5 ,
Proof of $(\mathrm{B}) \Rightarrow(\mathrm{C})$ : It follows from the fact that each $(\Sigma, \mathrm{B})-$ wrtg is a $\left(\Sigma^{\Xi}, \mathrm{B}\right)-$ wcfg and by Lemma 11.2.2(2).

Proof of $(\mathrm{C}) \Rightarrow(\mathrm{A})$ : It follows from Lemma 11.2 .3 and Theorem 10.10 .1 (closure of $\operatorname{Rec}^{\text {run }}\left(\_, B\right)$ under tree relabelings).

Next we verify that Theorem $11.2 .6(\mathrm{~A}) \Rightarrow(\mathrm{C})$ generalizes GS84, Thm. 2.9.5]. We achieve this by proving that the latter result is equivalent to Theorem $11.2 .6(\mathrm{~A}) \Rightarrow(\mathrm{C})$ for the case that B is the semiring Boole.

Corollary 11.2.7. Let $L \subseteq \mathrm{~T}_{\Sigma}$ be recognizable. We can construct a ranked alphabet $R$, a deterministic $(R, \Sigma)$-tree relabeling $\tau$, and an $R$-local system $(K, H)$ such that $L=\tau(\mathrm{L}(K, H))$.

Proof. Let $A$ be a $\Sigma$-fta such that $L=\mathrm{L}(A)$. By Corollary 3.4.2 (A) $\Rightarrow(\mathrm{B})$, we can construct a ( $\Sigma$, Boole)wta $\mathcal{A}$ such that $L=\operatorname{supp}(\llbracket \mathcal{A} \rrbracket)$. By Theorem $11.2 .6(\mathrm{~A}) \Rightarrow(\mathrm{C})$, we can construct a ranked alphabet $R$, a deterministic $(R, \Sigma)$-tree relabeling $\tau$, and an ( $R$, Boole)-wls $\mathcal{S}$ (which has identity root weights by definition) such that $L=\operatorname{supp}(\chi(\tau)(\llbracket \mathcal{S} \rrbracket))$. By (2.31), we obtain that $L=\tau(\operatorname{supp}(\llbracket \mathcal{S} \rrbracket))$. By Theorem 11.1.4 $(\mathrm{B}) \Rightarrow(\mathrm{A})$, we can construct a $\Sigma$-local system $(K, H)$ such that $L=\tau(\mathrm{L}(K, H))$.

If one traces back the constructions involved in the proof of Theorem 11.2 .6 ( A$) \Rightarrow(\mathrm{C})$ and composes them together, then one can eventually find out how a given wta is decomposed into a wls and a deterministic tree relabeling. Here we compose these constructions and show the resulting direct construction (where we start from a wta with identity root weights).

Construction 11.2.8. Let $\mathcal{A}=(Q, \delta, F)$ be a $(\Sigma, \mathrm{B})$-wta with identity root weights. Then we can construct a ranked alphabet $R$, a deterministic $(R, \Sigma)$-tree relabeling $\tau$, and an ( $R$, B)-wls $\mathcal{S}$ such that $\llbracket \mathcal{A} \rrbracket^{\text {run }}=\chi(\tau)(\llbracket \mathcal{S} \rrbracket)$ as follows.

- For each $k \in \mathbb{N}$, we let $R^{(k)}=\left\{q \rightarrow \sigma\left(q_{1}, \ldots, q_{k}\right) \mid \sigma \in \Sigma^{(k)}, q, q_{1}, \ldots, q_{k} \in Q\right\}$,
- $\tau=\left(\tau_{k} \mid k \in \mathbb{N}\right)$ such that, for every $k \in \mathbb{N}$ and $q \rightarrow \sigma\left(q_{1}, \ldots, q_{k}\right)$ in $R^{(k)}$, we let $\tau_{k}\left(q \rightarrow \sigma\left(q_{1}, \ldots, q_{k}\right)\right)=\sigma$, and
- $\mathcal{S}=\left(g, F^{\prime}\right)$ such that $g=\left(g_{k} \mid k \in \mathbb{N}\right)$ and for every $k \in \mathbb{N}, r_{1}, \ldots, r_{k}$ in $R$, and $r \in R^{(k)}$, we let

$$
g_{k}\left(r_{1} \cdots r_{k}, r\right)= \begin{cases}\delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right) & \text { if } r=\left(q \rightarrow \sigma\left(q_{1}, \ldots, q_{k}\right)\right) \text { and } \\ & \text { for each } i \in[k] \text { we have } \operatorname{lhs}\left(r_{i}\right)=q_{i} \\ 0 & \text { otherwise }\end{cases}
$$

and, for each $r$ in $R$, we let $F^{\prime}(r)=F(\operatorname{lhs}(r))$.
In fact, we could even give an arbitrary $(\Sigma, \mathrm{B})$-wta as input to Construction 11.2 .8 (i.e., which does not necessarily have identity root weights) and still we would obtain that $\llbracket \mathcal{A} \rrbracket^{\text {run }}=\chi(\tau)(\llbracket \mathcal{S} \rrbracket)$.

### 11.3 Characterization by local systems and evaluation algebras

In the next theorem we prove the second characterization result for recognizable weighted tree languages. In particular, (B) and $(C)$ only differ in the way in which the $\Theta$-tree language is defined: in (B) it is generated by a $\Theta$-local system and in $(\mathrm{C})$ it is recognized by a bu deterministic $\Theta$-fta.

Theorem 11.3.1. Let $\Sigma$ be a ranked alphabet, B be a strong bimonoid, and $r: \mathrm{T}_{\Sigma} \rightarrow B$. Then the following three statements are equivalent.
(A) We can construct $a(\Sigma, \mathrm{~B})-w t a \mathcal{A}$ such that $r=\llbracket \mathcal{A} \rrbracket^{\text {run }}$.
(B) We can construct

- a ranked alphabet $\Theta$,
- a deterministic $(\Theta, \Sigma)$-tree relabeling $\tau$,
- a $\Theta$-local system $(K, H)$, and
- a family $\kappa=\left(\kappa_{k} \mid k \in \mathbb{N}\right)$ of mappings $\kappa_{k}: \Theta^{(k)} \rightarrow B$
such that, for each $\xi \in \mathrm{T}_{\Sigma}$, the following holds: $r(\xi)=\mathrm{h}_{\mathrm{M}(\Theta, \kappa)}\left(\tau^{-1}(\xi) \cap \mathrm{L}(K, H)\right)$.
(C) We can construct
- a ranked alphabet $\Theta$,
- a deterministic $(\Theta, \Sigma)$-tree relabeling $\tau$,
- a bu deterministic $\Theta$-fta $A$, and
- a family $\kappa=\left(\kappa_{k} \mid k \in \mathbb{N}\right)$ of mappings $\kappa_{k}: \Theta^{(k)} \rightarrow B$
such that, for each $\xi \in \mathrm{T}_{\Sigma}$, the following holds: $r(\xi)=\mathrm{h}_{\mathrm{M}(\Theta, \kappa)}\left(\tau^{-1}(\xi) \cap \mathrm{L}(A)\right)$.

Proof. Proof of $(\mathrm{A}) \Rightarrow(\mathrm{B})$ : Let $\mathcal{A}$ be a $(\Sigma, \mathrm{B})$-wta. By Theorem $11.2 .6(\mathrm{~A}) \Rightarrow(\mathrm{B})$, we construct a ranked alphabet $R$, a deterministic $(R, \Sigma)$-tree relabeling $\tau$, and a $(\Sigma, \mathrm{B})$-wrtg $\mathcal{G}$ in tree automata form with rule set $R$ such that $\llbracket \mathcal{A} \rrbracket^{\text {run }}=\chi(\tau)\left(\llbracket \mathcal{G} \rrbracket^{\text {wrt }}\right)$. Let $w t$ be the weight assignment of $\mathcal{G}$.

Then, as in Lemma 11.2.2(1), we can construct an $R$-local system ( $K, H$ ) such that

$$
\llbracket \mathcal{A} \rrbracket^{\mathrm{run}}=\chi(\tau)\left(\llbracket \mathcal{G} \rrbracket^{\mathrm{wrt}}\right)=\chi(\tau)\left(\chi(\mathrm{L}(K, H)) \otimes \mathrm{h}_{\mathrm{M}(R, w t)}\right)
$$

Let $\xi \in \mathrm{T}_{\Sigma}$. By applying Observation 2.10 .1 (choosing $\Sigma, \Delta, L$, and $r$ to be $R, \Sigma, \mathrm{~L}(K, H)$, and $\mathrm{h}_{\mathrm{M}(\Sigma, w t)}$, respectively), we obtain that

$$
\llbracket \mathcal{A} \rrbracket^{\mathrm{run}}(\xi)=\mathrm{h}_{\mathrm{M}(R, w t)}\left(\tau^{-1}(\xi) \cap \mathrm{L}(K, H)\right) .
$$

By choosing $\Theta$ and $\kappa$ to be $R$ and $\left(\left.w t\right|_{R^{(k)}} \mid k \in \mathbb{N}\right.$ ), respectively, we have proved (B).
Proof of $(\mathrm{B}) \Rightarrow(\mathrm{C})$ : This follows from Corollary 11.2 .4 which states that, for each $\Theta$-local system ( $K, H$ ), a bu deterministic $\Theta$-fta $A$ can be constructed such that $\mathrm{L}(A)=\mathrm{L}(K, H)$.

Proof of $(\mathrm{C}) \Rightarrow(\mathrm{A})$ : Let $\Theta$ be a ranked alphabet, $\tau=\left(\tau_{k} \mid k \in \mathbb{N}\right)$ be a deterministic $(\Theta, \Sigma)$-tree relabeling, $A=(Q, \delta, F)$ be a bu deterministic $\Theta$-fta, and a family $\kappa=\left(\kappa_{k} \mid k \in \mathbb{N}\right)$ of mappings $\kappa_{k}: \Theta^{(k)} \rightarrow B$. By Theorem 4.3.6 we can assume that $A$ is total and bu deterministic. Then, for each $\zeta \in \mathrm{L}(A)$, we denote by $\rho_{\zeta}$ the unique valid run of $A$ on $\zeta$. We will abbreviate $\mathrm{h}_{\mathrm{M}(\Theta, \kappa)}$ by h .

We construct the ( $\Sigma, \mathrm{B})$-wta $\mathcal{A}$ by coding the preimage of $\tau$ into the states of $\mathcal{A}$. We let $\mathcal{A}=\left(Q^{\prime}, \delta^{\prime}, F^{\prime}\right)$ with $Q^{\prime}=Q \times \Theta$ and, for each $(q, \theta) \in Q^{\prime}$ we let $F_{(q, \theta)}^{\prime}=\mathbb{1}$ if $q \in F$, and $F_{(q, \theta)}^{\prime}=\mathbb{0}$ otherwise. Moreover, for every $k \in \mathbb{N}, \sigma \in \Sigma^{(k)}$, and $\left(q_{1}, \theta_{1}\right), \ldots,\left(q_{k}, \theta_{k}\right),(q, \theta) \in Q^{\prime}$ we define

$$
\left(\delta^{\prime}\right)_{k}\left(\left(q_{1}, \theta_{1}\right) \cdots\left(q_{k}, \theta_{k}\right), \sigma,(q, \theta)\right)= \begin{cases}\kappa_{k}(\theta) & \text { if }\left(q_{1} \cdots q_{k}, \theta, q\right) \in \delta_{k} \text { and } \tau_{k}(\theta)=\sigma \\ 0 & \text { otherwise } .\end{cases}
$$

We let $\mathrm{R}_{\mathcal{A}}=\bigcup_{\xi \in \mathrm{T}_{\Sigma}} \mathrm{R}_{\mathcal{A}}(\xi)$, and we define the mapping $\bar{\tau}: \mathrm{R}_{\mathcal{A}} \rightarrow \mathrm{T}_{\Sigma}$ such that, for every $\xi \in \mathrm{T}_{\Sigma}$ and $\rho \in \mathrm{R}_{\mathcal{A}}(\xi)$, we let $\bar{\tau}(\rho)$ be the $\Sigma$-tree mapping

$$
t: \operatorname{pos}(\xi) \rightarrow \Sigma \text { with } t(w)=\tau_{k}\left(\rho(w)_{2}\right) \text { for each } w \in \operatorname{pos}(\xi) \text { where } k=\operatorname{rk}_{\Theta}\left(\rho(w)_{2}\right)
$$

We note that $t$ uniquely determines an element in $\mathrm{T}_{\Sigma}$ (cf. Section [2.9). Moreover, there may exist $\xi \in \mathrm{T}_{\Sigma}$ and $\rho \in \mathrm{R}_{\mathcal{A}}(\xi)$ such that $\bar{\tau}(\rho) \neq \xi$; however, the weight of such a run $\rho$ is $\mathbb{0}$. Formally,

$$
\begin{equation*}
\text { for each } \rho \in \mathrm{R}_{\mathcal{A}}(\xi): \text { if } \bar{\tau}(\rho) \neq \xi \text {, then } \operatorname{wt}(\xi, \rho)=0 . \tag{11.4}
\end{equation*}
$$

For each $\xi \in \mathrm{T}_{\Sigma}$, we abbreviate by $\mathrm{R}_{\mathcal{A}}(F \times \Theta, \xi)$ the set $\bigcup_{q \in F, \theta \in \Theta} \mathrm{R}_{\mathcal{A}}((q, \theta), \xi)$ and by $\mathrm{R}_{\mathcal{A}}(F \times \Theta)$ the set $\bigcup_{\xi \in \mathrm{T}_{\Sigma}} \mathrm{R}_{\mathcal{A}}(F \times \Theta, \xi)$. Then we define the mapping

$$
\varphi: \mathrm{L}(A) \rightarrow \mathrm{R}_{\mathcal{A}}(F \times \Theta)
$$

for each $\zeta \in \mathrm{L}(A)$ and $w \in \operatorname{pos}(\xi)$ by $\varphi(\zeta)(w)=\left(\rho_{\zeta}(w), \zeta(w)\right)$. (We recall that $\rho_{\zeta}$ is the unique valid run of $A$ on $\zeta \in \mathrm{L}(A)$.) It is easy to see that $\varphi$ is a bijection. Moreover, $\varphi(\zeta) \in \mathrm{R}_{\mathcal{A}}(F \times \Theta, \tau(\zeta))$.

By induction on $\mathrm{T}_{\Theta}$, we prove that the following statement holds:

$$
\begin{equation*}
\text { For every } \zeta \in \mathrm{L}(A) \text {, we have } \mathrm{h}(\zeta)=\mathrm{wt}(\tau(\zeta), \varphi(\zeta)) \tag{11.5}
\end{equation*}
$$

cf. Figure 11.1, where $\mathrm{R}_{A}^{\mathrm{v}}(q, \zeta)$ denotes the set of valid $q$-runs on $\zeta$.
Let $\zeta=\theta\left(\zeta_{1}, \ldots, \zeta_{k}\right)$. Then

$$
\begin{align*}
\mathrm{h}\left(\theta\left(\zeta_{1}, \ldots, \zeta_{k}\right)\right) & =\mathrm{h}\left(\zeta_{1}\right) \otimes \ldots \otimes \mathrm{h}\left(\zeta_{k}\right) \otimes \kappa_{k}(\theta) \\
& =\left(\bigotimes_{i \in[k]} \mathrm{wt}\left(\tau\left(\zeta_{i}\right), \varphi\left(\zeta_{i}\right)\right)\right) \otimes \kappa_{k}(\theta)  \tag{byI.H.}\\
& =\left(\bigotimes_{i \in[k]} \mathrm{wt}\left(\tau\left(\zeta_{i}\right), \varphi\left(\zeta_{i}\right)\right)\right) \otimes\left(\delta^{\prime}\right)_{k}\left(\left(\rho_{\zeta_{1}}(\varepsilon), \zeta_{1}(\varepsilon)\right) \cdots\left(\rho_{\zeta_{k}}(\varepsilon), \zeta_{k}(\varepsilon)\right), \tau_{k}(\theta),\left(\rho_{\zeta}(\varepsilon), \zeta(\varepsilon)\right)\right)
\end{align*}
$$

(by definition of $\delta^{\prime}$ )

$$
=\left(\bigotimes_{i \in[k]} \operatorname{wt}\left(\tau\left(\zeta_{i}\right),\left.\varphi(\zeta)\right|_{i}\right)\right) \otimes\left(\delta^{\prime}\right)_{k}\left(\varphi\left(\zeta_{1}\right)(\varepsilon) \cdots \varphi\left(\zeta_{k}\right)(\varepsilon), \tau_{k}(\theta), \varphi(\zeta)(\varepsilon)\right)
$$

(by definition of $\varphi$ )


Figure 11.1: An illustration of (11.5) : $\mathrm{h}(\zeta)=\operatorname{wt}(\tau(\zeta), \varphi(\zeta))$, where $\tau(\zeta)=\xi$.

$$
\begin{aligned}
& =\left(\bigotimes_{i \in[k]} \operatorname{wt}\left(\tau\left(\zeta_{i}\right),\left.\varphi(\zeta)\right|_{i}\right)\right) \otimes\left(\delta^{\prime}\right)_{k}\left(\varphi(\zeta)(1) \cdots \varphi(\zeta)(k), \tau_{k}(\theta), \varphi(\zeta)(\varepsilon)\right) \\
& =\operatorname{wt}\left(\tau_{k}(\theta)\left(\tau\left(\zeta_{1}\right), \ldots, \tau\left(\zeta_{k}\right)\right), \varphi(\zeta)\right) \quad \quad \text { (by definition of wt) } \\
& =\operatorname{wt}\left(\tau\left(\theta\left(\zeta_{1}, \ldots, \zeta_{k}\right)\right), \varphi(\zeta)\right)
\end{aligned}
$$

This proves (11.5). Next we prove that $\llbracket \mathcal{A} \rrbracket^{\mathrm{run}}(\xi)=\mathrm{h}\left(\tau^{-1}(\xi) \cap \mathrm{L}(A)\right)$ for each $\xi \in \mathrm{T}_{\Sigma}$.

$$
\begin{array}{rlrl}
\mathrm{h}\left(\tau^{-1}(\xi) \cap \mathrm{L}(A)\right) & =\bigoplus_{\zeta \in \tau^{-1}(\xi) \cap \mathrm{L}(A)} \mathrm{h}(\zeta) & \text { (by definition) } \\
& =\bigoplus_{\zeta \in \tau^{-1}(\xi) \cap \mathrm{L}(A)} \mathrm{wt}(\xi, \varphi(\zeta)) \\
& =\bigoplus_{q \in F, \theta \in \Theta} \bigoplus_{\substack{ \\
\rho \in \mathrm{R}_{\mathcal{A}}((q, \theta), \xi): \\
\bar{\tau}(\rho)=\xi}} \mathrm{wt}(\xi, \rho) \\
& =\bigoplus_{q \in F, \theta \in \Theta} \bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}}((q, \theta), \xi)} \mathrm{wt}(\xi, \rho) \\
& =\bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}}(\xi)} \mathrm{wt}(\xi, \rho) \otimes F_{\rho(\xi)}^{\prime}=\llbracket \mathcal{A} \rrbracket^{\mathrm{run}}(\xi) & \text { (because } \varphi \text { is bijective) } \\
\end{array}
$$

As we have seen, the proof of Theorem $11.3 .1(A) \Rightarrow(B)$ is based on Theorem $11.2 .6(A) \Rightarrow(B)$. Since Observation 2.10.1 is symmetric, one might wonder whether the proof of Theorem 11.2.6 (A) $\Rightarrow(B)$ could be based on Theorem $11.3 .1(\mathrm{~A}) \Rightarrow(\mathrm{B})$. However, this is not possible because there exist local tree languages which are not rule tree languages, see our remark after Lemma 11.2.1.

## Chapter 12

## Rational operations and Kleene's theorem

In Chapter 10 we have proved that, in particular, for every commutative semiring $B$, the $\operatorname{set} \operatorname{Rec}(\Sigma, B)$ of recognizable $(\Sigma, B)$-weighted tree languages is closed under the rational operations, i.e., sum, tree concatenations, and Kleene-stars.

In Theorem 12.2.3, we will show that each recognizable weighted tree language can be expressed by polynomial weighted tree language and rational operations (cf. [DPV05, Thm. 5.2]). This generalizes the analysis part of Kleene's result Kle56] (recognizable implies rational), and the corresponding result for tree languages [TW68, Thm. 9] (cf. Eng75b, Thm. 3.43] and GS84, Thm. 2.5.7]) and weighted string languages (cf. Sch61]). We recall that, in the tree case, extra symbols for tree concatenation were supplied; we will deal with this issue in detail below.

Taking together the closure results from Chapter 10 and the analysis result from this chapter, we obtain Kleene's result for recognizable weighted tree languages in Theorem 12.1 .2 (cf. AB87, Thm. 2.3] and [DPV05, Thm. 7.1]). It generalizes the corresponding results for the unweighted case (cf. TW68, Thm. 8], Eng75b, Thm. 3.43], and [GS84, Thm. 2.5.8]).

Kleene's result has been extended both, to structures different from finite trees and to algebras different from semirings, e.g., formal tree series over additive $B$ - $\Sigma$-algebras for a commutative, $\sigma$-complete semiring B [Boz99, p.28] (also cf. Kui98, Thm. 6.14]), formal power series in partially commuting variables (traces) and semirings DG99, infinite strings and max-plus semiring DK03, DK06, strings and lattice-ordered monoids LP05, pictures and commutative semirings BG05, Mäu05, Mäu07, strings and Conway semirings ÉK, Sect. 1.3], trees over infinite ranked alphabets and continuous and commutative semirings ÉK03, Thm. 4.3], ÉK, Thm. 6.4.3], trees and $\sigma$-complete distributive lattices ÉL07, trees and distributive multioperator monoids (which satisfy some closure properties) [FMV09], trees and treevaluation monoids GFD19, and weighted tree automata with storage over commutative, $\sigma$-complete semirings [FV19, DFV21].

In this chapter we recall from DPV05 the proof of the equivalence of recognizable weighted tree languages and rational weighted tree languages for commutative semirings.

In the rest of this chapter, $\mathrm{B}=(B, \oplus, \otimes, \mathbb{O}, \mathbb{1})$ denotes an arbitrary commutative semiring.

### 12.1 Rational weighted tree languages

Here we formally define the concept of rational operations and rational weighted tree languages.
An operation on the set of $(\Sigma, \mathrm{B})$-weighted tree languages is a rational operation if it is the sum, a tree
concatenation, or a Kleene-star. The set of rational ( $\Sigma, \mathrm{B}$ )-weighted tree languages, denoted by Rat( $\Sigma, \mathrm{B}$ ), is the smallest set of $(\Sigma, \mathrm{B})$-weighted tree languages which contains each polynomial $(\Sigma, \mathrm{B})$-weighted tree language and is closed under the rational operations. We call each weighted tree language in Rat( $\Sigma, \mathrm{B}$ ) rational.

For our main result, we will have to prove, in particular, that the semantics $\llbracket \mathcal{A} \rrbracket$ of a $(\Sigma, \mathrm{B})$-wta $\mathcal{A}=(Q, \delta, F)$ is rational. We follow the idea of [TW68, Sect. 3] where extra symbols for the tree concatenations are used, and these extra symbols are the states of $\mathcal{A}$. In order to have such extra nullary symbols available for tree concatenations and not to change the type of $\llbracket \mathcal{A} \rrbracket$ (which is $\mathrm{T}_{\Sigma} \rightarrow B$ and not $\mathrm{T}_{\Sigma \cup Q} \rightarrow B$ ) we use the concept of 0-extension of ranked alphabet and $\mathbb{0}$-extension of a weighted tree language (cf. [FV19, Def. 1] and [DFV21, Def. 2]).

Formally, a 0-extension of $\Sigma$ is a ranked alphabet $\Theta$ such that $\Sigma \subseteq \Theta, \operatorname{rk}_{\Theta}(\sigma)=\operatorname{rk}_{\Sigma}(\sigma)$ for each $\sigma \in \Sigma$, and $\operatorname{rk}_{\Theta}(\sigma)=0$ for each $\sigma \in \Theta \backslash \Sigma$. If $\Theta$ is a 0 -extension of $\Sigma$, then we write $\Theta \geq_{0} \Sigma$.

Now let $\Theta \geq_{0} \Sigma$ and $r: \mathrm{T}_{\Sigma} \rightarrow B$. The $\mathbb{D}$-extension of $r$ to $\mathrm{T}_{\Theta}$, denoted by $r \upharpoonright^{\Theta, 0}$, is the weighted tree language $r \upharpoonright^{\Theta, \varnothing}: \mathrm{T}_{\Theta} \rightarrow B$ such that
(a) $\left.\left(r \Gamma^{\Theta, 0}\right)\right|_{\mathrm{T}_{\Sigma}}=r$ and
(b) $\left(\left.r\right|^{\Theta, 0}\right)(\xi)=\mathbb{0}$ for every $\xi \in \mathrm{T}_{\Theta} \backslash \mathrm{T}_{\Sigma}$.

These concepts have the following transitivity property.
Observation 12.1.1. Let $\Theta$ and $\Delta$ be ranked alphabets such that $\Delta \geq_{0} \Theta$ and $\Theta \geq_{0} \Sigma$. Then $\Delta \geq_{0} \Sigma$. Moreover, let $r: \mathrm{T}_{\Sigma} \rightarrow B$. Then $\left(r r^{\Theta, 0}\right) \upharpoonright^{\Delta, 0}=r r^{\Delta, 0}$.

The set of extended rational ( $\Sigma, \mathrm{B}$ )-weighted tree languages, denoted by $\operatorname{Rat}(\Sigma, \mathrm{B})^{\mathrm{ext}}$, contains each $(\Sigma, \mathrm{B})$-weighted tree language $r$ such that $r r^{\Theta, 0} \in \operatorname{Rat}(\Theta, \mathrm{~B})$ for some 0 -extension $\Theta$ of $\Sigma$.

Similarly, the set of extended recognizable $(\Sigma, \mathrm{B})$-weighted tree languages, denoted by $\operatorname{Rec}(\Sigma, \mathrm{B})^{\mathrm{ext}}$, contains each $(\Sigma, \mathrm{B})$-weighted tree language $r$ such that $r r^{\Theta, 0} \in \operatorname{Rec}(\Theta, \mathrm{~B})$ for some 0 -extension $\Theta$ of $\Sigma$.

In the next two sections, we will prove that $\operatorname{Rec}(\Sigma, B) \subseteq \operatorname{Rat}(\Sigma, B)^{\operatorname{ext}}$ (cf. Theorem 12.2.3) and $\operatorname{Rat}(\Sigma, \mathrm{B}) \subseteq \operatorname{Rec}(\Sigma, \mathrm{B})$ (cf. Theorem 12.3.1). These theorems imply the following main result of this chapter.

Theorem 12.1.2. Let $\Sigma$ be a ranked alphabet and B be a commutative semiring. Then $\operatorname{Rec}(\Sigma, \mathrm{B})^{\mathrm{ext}}=$ $\operatorname{Rat}(\Sigma, B)^{\text {ext }}$ 。

Proof. $\operatorname{Rec}(\Sigma, \mathrm{B})^{\text {ext }} \subseteq \operatorname{Rat}(\Sigma, \mathrm{B})^{\text {ext }}$ : Let $r \in \operatorname{Rec}(\Sigma, \mathrm{~B})^{\text {ext }}$. Then $\left.r\right|^{\Theta, 0} \in \operatorname{Rec}(\Theta, \mathrm{~B})$ for some 0 -extension $\Theta$ of $\Sigma$. By Theorem 12.2 .3 (with $\Sigma=\Theta$ ), we have that $r \upharpoonright^{\Theta, 0} \in \operatorname{Rat}^{\text {ext }}(\Theta, \mathrm{B})$. Hence $\left(r r^{\Theta, 0}\right) r^{\Delta, 0} \in \operatorname{Rat}(\Delta, \mathrm{~B})$ for some 0 -extension $\Delta$ of $\Theta$. Since $\Delta \geq_{0} \Theta$ and $\Theta \geq_{0} \Sigma$, we obtain by Observation 12.1.1 that $\left(r \upharpoonright^{\Theta, 0}\right) \upharpoonright^{\Delta, 0}=r \upharpoonright^{\Delta, 0}$. Hence $r r^{\Delta, 0} \in \operatorname{Rat}(\Delta, \mathrm{~B})$. Since $\Delta \geq_{0} \Sigma$, we obtain $r \in \operatorname{Rat}(\Sigma, \mathrm{~B})^{\text {ext }}$.
$\underline{\operatorname{Rat}(\Sigma, \mathrm{B})^{\text {ext }} \subseteq \operatorname{Rec}(\Sigma, \mathrm{B})^{\text {ext }} \text { : Let } r \in \operatorname{Rat}(\Sigma, \mathrm{~B})^{\text {ext }} \text {. Then } r{ }^{\Theta, 0} \in \operatorname{Rat}(\Theta, \mathrm{~B}) \text { for some } 0 \text {-extension } \Theta \text { of }, ~(\Theta)}$ $\Sigma$. By Theorem 12.3 .1 (with $\Sigma=\Theta$ ), we have that $r r^{\Theta, 0} \in \operatorname{Rec}(\Theta, \mathrm{~B})$. Hence we have $r \in \operatorname{Rec}(\Sigma, \mathrm{~B})^{\text {ext }}$.

We note that Theorem 12.1.2 is similar to DPV05, Thm. 7.1], but slightly different in the following way. Theorem 7.1 of [DPV05] says that $\mathrm{B}^{\text {rec }}\left\langle\left\langle\mathrm{T}_{\Sigma}\left(Q_{\infty}\right)\right\rangle\right\rangle=\mathrm{B}^{\text {rat }}\left\langle\left\langle\mathrm{T}_{\Sigma}\left(Q_{\infty}\right)\right\rangle\right\rangle$, where

$$
\begin{array}{ll}
\mathrm{B}^{\mathrm{rec}}\left\langle\left\langle\mathrm{~T}_{\Sigma}\left(Q_{\infty}\right)\right\rangle\right\rangle \text { is defined to be } \bigcup\left(\mathrm{B}^{\mathrm{rec}}\left\langle\left\langle\mathrm{~T}_{\Theta}\right\rangle\right\rangle \mid \text { ranked alphabet } \Theta \text { such that } \Theta \geq_{0} \Sigma\right) \text { and } \\
\mathrm{B}^{\mathrm{rat}}\left\langle\left\langle\mathrm{~T}_{\Sigma}\left(Q_{\infty}\right)\right\rangle\right\rangle \quad \text { is defined to be } \bigcup\left(\mathrm{B}^{\mathrm{rat}}\left\langle\left\langle\mathrm{~T}_{\Theta}\right\rangle\right\rangle \mid \text { ranked alphabet } \Theta \text { such that } \Theta \geq_{0} \Sigma\right) \text { and }
\end{array}
$$

$\mathrm{B}^{\text {rat }}\left\langle\left\langle\mathrm{T}_{\Theta}\right\rangle\right\rangle$ is the smallest set of $(\Theta, \mathrm{B})$-weighted tree languages which is closed under scalar multiplication, top-concatenations, and rational operations. Moreover, $\mathrm{B}^{\mathrm{rec}}\left\langle\left\langle\mathrm{T}_{\Theta}\right\rangle\right\rangle$ is just another denotation of the set $\operatorname{Rec}(\Theta, B)$.

However, this setting yields the following type conflict when trying to prove that $\mathrm{B}^{\text {rec }}\left\langle\left\langle\mathrm{T}_{\Theta}\right\rangle\right\rangle \subseteq$ $\mathrm{B}^{\text {rat }}\left\langle\left\langle\mathrm{T}_{\Sigma}\left(Q_{\infty}\right)\right\rangle\right\rangle$ for some 0 -extension $\Theta$ of $\Sigma$. For each $(\Theta, \mathrm{B})$-wta $\mathcal{A}=(Q, \delta, F)$, its semantics has the type $\llbracket \mathcal{A} \rrbracket: \mathrm{T}_{\Theta} \rightarrow B$. In DPV05, Thm. 5.2] it is proved that there exists an $r \in \mathrm{~B}^{\mathrm{rat}}\left\langle\left\langle\mathrm{T}_{\Theta \cup Q}\right\rangle\right\rangle$ (where $\Theta \cup Q$ is a 0 -extension of $\Theta$ ) such that, for each $\xi \in \mathrm{T}_{\Theta}$, the equality $r(\xi)=\llbracket \mathcal{A} \rrbracket(\xi)$ holds and, for each $\xi \in \mathrm{T}_{\Theta \cup Q} \backslash \mathrm{~T}_{\Theta}$, we have $r(\xi)=\mathbb{0}$. Nevertheless, by the definition of $\mathrm{B}^{\mathrm{rat}}\left\langle\left\langle\mathrm{T}_{\Theta \cup Q}\right\rangle\right\rangle, r$ has the type $r: \mathrm{T}_{\Theta \cup Q} \rightarrow B$ which is different from the type of $\llbracket \mathcal{A} \rrbracket$. Hence $\llbracket \mathcal{A} \rrbracket \neq r$ and, in fact, there does not exist an $r \in \mathrm{~B}^{\mathrm{rat}}\left\langle\left\langle\mathrm{T}_{\Theta \cup Q}\right\rangle\right\rangle$ such that $\llbracket \mathcal{A} \rrbracket=r$.

We solve this type conflict by the concepts of 0 -extension and $\mathbb{0}$-extension. Such a type conflict does not occur in the proof of the analysis theorem [TW68, Thm. 9] because (unweighted) tree languages are not mappings.

In Dro22 the natural question was posed whether $\operatorname{Rec}(\Sigma, B) \backslash \operatorname{Rat}(\Sigma, B) \neq \emptyset$, i.e., whether the extra nullary symbols used in $\operatorname{Rat}(\Sigma, \mathrm{B})^{\text {ext }}$ are really necessary (or just comfortable to use). We claim that, for the ranked alphabet $\Sigma=\left\{\sigma^{(2)}, \sigma^{\prime(2)}, \alpha^{(0)}\right\}$, the weighted tree language $\chi_{\text {Boole }}(L(K, H)) \in \operatorname{Rec}(\Sigma$, Boole $) \backslash$ $\operatorname{Rat}\left(\Sigma\right.$, Boole) where $(K, H)$ is the $\Sigma$-local system with $K=\left\{\left(\sigma \sigma^{\prime}, \sigma\right),(\alpha \alpha, \sigma),\left(\alpha \alpha, \sigma^{\prime}\right)\right\}$ and $H=\{\sigma\}$.

Finally, we note that there were investigations to overcome the difference between string concatenation and tree concatenation (cf. the discussion at the beginning of Section 10.6) by employing forests and forest concatenation Str09, Dör19, Dör21. However, since we want to deal with trees, we will not report on these investigations.

### 12.2 From recognizable to rational

In this section we prove Theorem 12.2 .3 , i.e., for each $(\Sigma, B)$-recognizable weighted tree language $r$ there exists a 0 -extension $\Theta$ of $\Sigma$ such that $\left.r\right|^{\Theta, 0} \in \operatorname{Rat}(\Theta, B)$.

Intuitively, this result shows that the semantics of a wta can be computed in a dynamic programming style. This computation is organized in the same way as the computation of the transitive closure of the edge relation of graphs in War62, of the all-pairs shortest-path problem in [Flo62, and of the algebraic path problem for idempotent $\sigma$-complete semirings AHU74, Alg. 5.5]. We follow the lines in DPV05 (which was crucially inspired by Eng03).

As preparation, we extend the concept of run for trees which may contain states of a wta. If for a position $w$ of a tree $\xi$ the symbol $\xi(w)$ is a state, then the extended run assigns the state $\xi(w)$ to that position. Formally, let $\mathcal{A}=(Q, \delta, F)$ be a $(\Sigma, \mathrm{B})$-wta and $\xi \in \mathrm{T}_{\Sigma}(Q)$. A run of $\mathcal{A}$ on $\xi$ is a mapping $\rho: \operatorname{pos}(\xi) \rightarrow Q$ such that $\rho(w)=\xi(w)$ for each $w \in \operatorname{pos}_{Q}(\xi)$. In the same way as for runs on trees in $\mathrm{T}_{\Sigma}$, we define the restriction $\left.\rho\right|_{i}$ for a run of $\mathcal{A}$ on $\xi \in \mathrm{T}_{\Sigma}(Q)$ and $i \in[\operatorname{rk}(\xi(\varepsilon))]$. Moreover, we denote the set of all runs of $\mathcal{A}$ on $\xi$ by $\mathrm{R}_{\mathcal{A}}(\xi)$.

Next we define, for each $\xi \in \mathrm{T}_{\Sigma}(Q)$, the weight of each run in $\mathrm{R}_{\mathcal{A}}(\xi)$. For this we extend the definition of $\mathrm{wt}_{\mathcal{A}}: \mathrm{TR} \rightarrow B$ given in (3.1). Formally, we let $\mathrm{TR}_{Q}=\left\{(\xi, \rho) \mid \xi \in \mathrm{T}_{\Sigma}(Q), \rho \in \mathrm{R}_{\mathcal{A}}(\xi)\right\}$ and we define the binary relation $\prec$ on $\mathrm{TR}_{Q}$ by

$$
\prec=\left\{\left(\left(\left.\xi\right|_{i},\left.\rho\right|_{i}\right),(\xi, \rho)\right) \mid(\xi, \rho) \in \operatorname{TR}_{Q}, i \in[\operatorname{rk}(\xi(\varepsilon)]\} .\right.
$$

Then $\prec$ is well-founded and

$$
\min _{\prec}\left(\operatorname{TR}_{Q}\right)=\left\{(\alpha, \rho) \mid \alpha \in \Sigma^{(0)}, \rho:\{\varepsilon\} \rightarrow Q\right\} \cup\{(q, \rho) \mid q \in Q, \rho=\{(\varepsilon, q)\}\}
$$

We define the mapping

$$
\mathrm{wt}_{\mathcal{A}}^{\prime}: \mathrm{TR}_{Q} \rightarrow B
$$

by induction on $\left(\mathrm{TR}_{Q}, \prec\right)$ as follows. Let $(\xi, \rho) \in \operatorname{TR}_{Q}$. If $\xi \in Q$ (and hence $\left.\rho=\{(\varepsilon, \xi)\}\right)$, then we let $\mathrm{wt}_{\mathcal{A}}^{\prime}(\xi, \rho)=\mathbb{1}$. If $\xi=\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)$, then we let

$$
\begin{equation*}
\mathrm{wt}_{\mathcal{A}}^{\prime}(\xi, \rho)=\left(\bigotimes_{i \in[k]} \mathrm{wt}_{\mathcal{A}}^{\prime}\left(\left.\xi\right|_{i},\left.\rho\right|_{i}\right)\right) \otimes \delta_{k}(\rho(1) \cdots \rho(k), \sigma, \rho(\varepsilon)) \tag{12.1}
\end{equation*}
$$

where $k$ and $\sigma$ abbreviate $\operatorname{rk}(\xi(\varepsilon))$ and $\xi(\varepsilon)$, respectively. We call $\mathrm{wt}_{\mathcal{A}}^{\prime}(\xi, \rho)$ the weight of $\rho$ (by $\mathcal{A}$ on $\xi$ ). If there is no confusion, then we drop the index $\mathcal{A}$ from $\mathrm{wt}_{\mathcal{A}}^{\prime}$ and write just $\mathrm{wt}^{\prime}(\xi, \rho)$ for the weight of $\rho$. Since, for each $\xi \in \mathrm{T}_{\Sigma}$, we have $\mathrm{wt}^{\prime}(\xi, \rho)=\mathrm{wt}(\xi, \rho)$, we drop the prime from $\mathrm{wt}^{\prime}$ and simply write wt.

Next we define restrictions on (generalized) runs in the way that we restrict the set of states which may occur at positions different from the root and different from $Q$-labeled positions. Formally, let $q \in Q$, $P \subseteq Q$, and $\xi \in \mathrm{T}_{\Sigma}(Q)$. A $q$-run of $\mathcal{A}$ on $\xi$ using $P$ is a run $\rho: \operatorname{pos}(\xi) \rightarrow Q$ such that

- $\rho(\varepsilon)=q$,
- $\rho(w) \in P$ for each $w \in \operatorname{pos}(\xi) \backslash\left(\{\varepsilon\} \cup \operatorname{pos}_{Q}(\xi)\right)$.

The set of all $q$-runs of $\mathcal{A}$ on $\xi$ using $P$ is denoted by $\mathrm{R}_{\mathcal{A}}^{P}(q, \xi)$. We denote the set $\bigcup_{q \in \mathrm{Q}} \mathrm{R}_{\mathcal{A}}^{P}(q, \xi)$ by $\mathrm{R}_{\mathcal{A}}^{P}(\xi)$.

Next we recall a particular weighted tree language Eng03, DPV05 with a slight modification. We consider the 0-extension $\Sigma \cup Q$ of $\Sigma$, i.e., $(\Sigma \cup Q)^{(0)}=\Sigma^{(0)} \cup Q$. For every $P \subseteq Q$ and $q \in Q$, we define the weighted tree language $S_{\mathcal{A}}(P, q): \mathrm{T}_{\Sigma}(Q) \rightarrow B$ for each $\xi \in \mathrm{T}_{\Sigma}(Q)$ by

$$
S_{\mathcal{A}}(P, q)(\xi)= \begin{cases}\bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}}^{P}(q, \xi)} \mathrm{wt}(\xi, \rho) & \text { if } \xi \in \mathrm{T}_{\Sigma}(Q) \backslash Q \\ \mathbb{O} & \text { otherwise }\end{cases}
$$

Then, for each $q^{\prime} \in Q$, the weighted tree language $S_{\mathcal{A}}(P, q)$ is $q^{\prime}$-proper. We recall that in Eng03, DPV05] the language $S_{\mathcal{A}}\left(Q^{\prime}, P, q\right)$ was defined where $Q^{\prime} \subseteq Q$. But since we will only use $S_{\mathcal{A}}(Q, P, q)$, we refrain from the first parameter and keep it fixed with value $Q$.

Next we prove that each $S_{\mathcal{A}}(P, q)$ is a rational weighted tree language. For this, we will have to decompose runs of $\mathcal{A}$ at particular positions. Formally, let $P \subseteq Q, q \in Q$, and $p \in Q \backslash P$. Let $\xi \in \mathrm{T}_{\Sigma}(Q) \backslash Q$. We define the mapping

$$
\varphi: \mathrm{R}_{\mathcal{A}}^{P \cup\{p\}}(q, \xi) \rightarrow \mathrm{U}^{P \cup\{p\}}(q, \xi)
$$

where

$$
\begin{aligned}
\mathrm{U}^{P \cup\{p\}}(q, \xi)=\left\{\left(\widetilde{v}, \rho^{\prime}, \rho_{1}, \ldots, \rho_{n}\right) \mid\right. & n \in \mathbb{N}, \widetilde{v}=\left(v_{1}, \ldots, v_{n}\right) \text { in } \operatorname{cut}_{p}(\xi) \backslash\{(\varepsilon)\}, \rho^{\prime} \in \mathrm{R}_{\mathcal{A}}^{P}(q, \xi[p] \widetilde{v}), \\
& \text { and } \left.\left.\xi\right|_{v_{i}} \notin Q \backslash\{p\} \text { and } \rho_{i} \in \mathrm{R}_{\mathcal{A}}^{P \cup\{p\}}\left(p,\left.\xi\right|_{v_{i}}\right) \text { for each } i \in[n]\right\}
\end{aligned}
$$

and for every $\rho \in \mathrm{R}_{\mathcal{A}}^{P \cup\{p\}}(q, \xi)$ we define

$$
\varphi(\rho)=\left(\left(v_{1}, \ldots, v_{n}\right), \rho^{\prime}, \rho_{1}, \ldots, \rho_{n}\right)
$$

such that the following conditions hold (cf. Figure 12.1):

1. $\left\{w \in \rho^{-1}(p) \backslash\{\varepsilon\} \mid(\forall v \in \operatorname{prefix}(w) \backslash\{w\}): \rho(v) \neq p\right\}=\left\{v_{1}, \ldots, v_{n}\right\}$,
2. $\rho^{\prime}: \operatorname{pos}\left(\xi[p]_{\tilde{v}}\right) \rightarrow Q$ is such that $\rho^{\prime}=\left.\rho\right|_{\operatorname{pos}(\xi[p] \tilde{v})}$, and
3. for each $i \in[n]$, we have $\rho_{i}=\left.\rho\right|_{v_{i}}$.

By the first condition we have $\left.\xi\right|_{v_{i}} \notin Q \backslash\{p\}$, because otherwise $\rho\left(v_{i}\right) \in Q \backslash\{p\}$, and hence $v_{i} \notin \rho^{-1}(p)$.
Obviously, $\varphi$ is bijective and, if $\varphi(\rho)=\left(\left(v_{1} \ldots, v_{n}\right), \rho^{\prime}, \rho_{1}, \ldots, \rho_{n}\right)$, then

$$
\begin{equation*}
\mathrm{wt}(\xi, \rho)=\mathrm{wt}\left(\xi[p]_{\tilde{v}}, \rho^{\prime}\right) \otimes \bigotimes_{i \in[n]} \mathrm{wt}\left(\left.\xi\right|_{v_{i}}, \rho_{i}\right) \tag{12.2}
\end{equation*}
$$

We note that (12.2) uses (12.1), Observation 3.1.1) and the assumption that $B$ is commutative.
The next lemma shows how the weighted tree languages in the family

$$
\left(S_{\mathcal{A}}(P, q) \mid P \subseteq Q, q \in Q\right)
$$

can be computed in a dynamic programming style (cf. Figure 12.2) ).


Figure 12.1: An example of $\rho \in \mathbf{R}_{\mathcal{A}}^{P \cup\{p\}}(q, \xi)$ and $\varphi(\rho)$ with $Q=\left\{p, q_{1}, q_{2}, q\right\}, P=\left\{q_{1}\right\}$.


Figure 12.2: An illustration of the equality $S_{\mathcal{A}}(P \cup\{p\}, q)=S_{\mathcal{A}}(P, q) \circ_{p} S_{\mathcal{A}}(P, p)_{p}^{*}$, where $S_{\mathcal{A}}$ is abbreviated by $S$.

Lemma 12.2.1. (cf. Eng03, Lm. 12] and DPV05, Lm. 5.1]) For every $P \subseteq Q, p \in Q \backslash P$, and $q \in Q$, we have

$$
S_{\mathcal{A}}(P \cup\{p\}, q)=S_{\mathcal{A}}(P, q) \circ_{p} S_{\mathcal{A}}(P, p)_{p}^{*}
$$

Proof. Let $P \subseteq Q$ and $p \in Q \backslash P$. By induction on $\mathrm{T}_{\Sigma}(Q)$, we prove that the following statement holds:
For every $\xi \in \mathrm{T}_{\Sigma}(Q)$ and $q \in Q$, we have $S_{\mathcal{A}}(P \cup\{p\}, q)(\xi)=\left(S_{\mathcal{A}}(P, q) \circ_{p} S_{\mathcal{A}}(P, p)_{p}^{*}\right)(\xi)$.
I.B.: Let $\xi \in Q$. Then $S_{\mathcal{A}}(P \cup\{p\}, q)(\xi)=\mathbb{O}=\left(S_{\mathcal{A}}(P, q) \circ_{p} S_{\mathcal{A}}(P, p)_{p}^{*}\right)(\xi)$, because $S_{\mathcal{A}}(P, q)$ is $q^{\prime}$-proper for each $q^{\prime} \in Q$. The case that $\xi \in \Sigma^{(0)}$ is covered in the I.S.
I.S.: Let $\xi=\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)$. For each $\widetilde{v}=\left(v_{1}, \ldots, v_{n}\right)$ in $\operatorname{cut}_{p}(\xi)$, we denote $v_{i}$ also by $\widetilde{v}_{i}$.

$$
\text { (by definition of } \mathrm{U}^{P \cup\{p\}}(q, \xi) \text { ) }
$$

$$
\left.=\bigoplus_{\substack{\widetilde{v}=\left(v_{1}, \ldots, v_{n}\right) \in \operatorname{cut}_{p}(\xi) \backslash\{(\varepsilon)\}: \\\left(\forall[p] \tilde{v} \notin Q \\(\forall i \in[n]): \xi| |_{v_{i}} \notin Q \backslash\{p\}\right.}}\left(\bigoplus_{\substack{\rho^{\prime} \in \mathrm{R}_{\mathcal{A}}^{P}\left(q, \xi[p]_{\tilde{v}}\right)}} \mathrm{wt}\left(\xi[p]_{\tilde{v}}, \rho^{\prime}\right)\right) \otimes \bigotimes_{i \in[n]} \bigoplus_{\rho_{i} \in \mathrm{R}_{\mathcal{A}}^{P \cup\{p\}}\left(p,\left.\xi\right|_{v_{i}}\right)} \mathrm{wt}\left(\left.\xi\right|_{v_{i}}, \rho_{i}\right)\right)
$$

$$
\begin{aligned}
& S_{\mathcal{A}}(P \cup\{p\}, q)(\xi) \\
& =\bigoplus_{\rho \in \mathrm{R}^{P \cup\{p\}}(q, \xi)} \mathrm{wt}(\xi, \rho) \quad \quad \text { (by definition and the fact that } \xi \notin Q \text { ) } \\
& =\bigoplus_{\left(\widetilde{v}, \rho^{\prime}, \rho_{1}, \ldots, \rho_{n}\right) \in \mathrm{U}^{P \cup\{p\}}(q, \xi)} \mathrm{wt}\left(\xi[p] \widetilde{v}, \rho^{\prime}\right) \otimes \bigotimes_{i \in[n]} \mathrm{wt}\left(\left.\xi\right|_{\tilde{v}_{i}}, \rho_{i}\right) \quad \text { (because } \varphi \text { is a weight preserving bijection) } \\
& =\underset{\substack{\tilde{v}=\left(v_{1}, \ldots, v_{n}\right) \in \operatorname{cut}_{p}(\xi) \backslash\{(\varepsilon)\}: \\
\xi[p] \tilde{v} \notin Q}}{\bigoplus_{\rho^{\prime} \in \mathrm{R}_{\mathcal{A}}^{P}(q, \xi[p] \tilde{v})} \bigoplus_{\rho_{1} \in \mathrm{R}_{\mathcal{A}}^{P \cup\{p\}}} \bigoplus_{\left(p,\left.\xi\right|_{v_{1}}\right)} \ldots \rho_{\rho_{n} \in \mathrm{R}_{\mathcal{A}}^{P \cup\{p\}}\left(p,\left.\xi\right|_{v_{n}}\right)}} \operatorname{wt}\left(\xi[p] \widetilde{v}, \rho^{\prime}\right) \otimes \bigotimes_{i \in[n]} \mathrm{wt}\left(\left.\xi\right|_{v_{i}}, \rho_{i}\right) \\
& (\forall i \in[n]): \xi \mid v_{i} \notin Q \backslash\{p\}
\end{aligned}
$$

$$
=\bigoplus_{\substack{\tilde{v}=\left(v_{1}, \ldots, v_{n}\right) \in \operatorname{cut}_{p}(\xi) \backslash\{(\varepsilon)\}: \\ \xi\left[p \left[\tilde{v} \notin Q,(\forall i \in[n]):\left.\xi\right|_{v_{i}} \notin Q \backslash\{p\}\right.\right.}} S_{\mathcal{A}}(P, q)\left(\xi[p]_{\tilde{v}}\right) \otimes \bigotimes_{i \in[n]}\left(\left(S_{\mathcal{A}}(P \cup\{p\}, p) \oplus \mathbb{1} . p\right)\left(\left.\xi\right|_{v_{i}}\right)\right.
$$

In the previous equality we have used the following argument (for the second factor up to the $(n+1)$-st factor). If, for some $i \in[n]$, we have $\left.\xi\right|_{v_{i}}=p$, then $\mathrm{R}_{\mathcal{A}}^{P \cup\{p\}}\left(p,\left.\xi\right|_{v_{i}}\right)$ contains exactly one run, say $\rho$, and $\rho(\varepsilon)=p$. Then $\bigoplus_{\rho_{i} \in \mathrm{R}_{\mathcal{A}}^{P \cup\{p\}}\left(p,\left.\xi\right|_{v_{i}}\right)} \mathrm{wt}\left(\left.\xi\right|_{v_{i}}, \rho_{i}\right)=\mathbb{1}$. On the other hand, then we have $S_{\mathcal{A}}(P \cup\{p\}, p)\left(\left.\xi\right|_{v_{i}}\right)=$ 0 . Thus, we have to add $\mathbb{1}$. .

Then we can continue as follows:

The previous equality uses, for each $i \in[n]$, the following facts. If $\left.\xi\right|_{v_{i}} \notin Q$, then we have used the I.H. with $q=p$. This is possible because $\left.\xi\right|_{v_{i}}$ is a subtree of $\xi$ due to the condition $\xi[p]_{\left(v_{1}, \ldots, v_{n}\right)} \neq p$. If $\left.\xi\right|_{v_{i}}=p$, then we have used that

$$
\left(\left(S_{\mathcal{A}}(P \cup\{p\}, p) \oplus \mathbb{1} . p\right)\left(\left.\xi\right|_{v_{i}}\right)=\mathbb{1}=\left(\left(S_{\mathcal{A}}(P, p) \circ_{p} S_{\mathcal{A}}(P, p)_{p}^{*}\right) \oplus \mathbb{1} . p\right)\left(\left.\xi\right|_{v_{i}}\right)\right.
$$

Then we can continue as follows:

$$
\begin{aligned}
& \bigoplus_{\substack{\left.\left(v_{1}, \ldots, v_{n}\right) \in \operatorname{cut}_{p}(\xi) \backslash\{(\xi)\}: \\
\xi[p]\right]_{\left(v_{1}, \ldots, v_{n}\right) \notin Q}^{(\forall i \in[n]): \xi v_{i} \notin Q \backslash\{p\}}}} S_{\mathcal{A}}(P, q)\left(\xi[p]_{\left(v_{1}, \ldots, v_{n}\right)}\right) \otimes \bigotimes_{i \in[n]}\left(\left(S_{\mathcal{A}}(P, p) \circ_{p} S_{\mathcal{A}}(P, p)_{p}^{*}\right) \oplus \mathbb{1} . p\right)\left(\left.\xi\right|_{v_{i}}\right) \\
& =\bigoplus_{\substack{\left(v_{1}, \ldots, v_{n}\right) \in \operatorname{cut}_{p}(\xi) \backslash\{(\varepsilon)\}: \\
\xi[p]\left(v_{1}, \ldots, v_{n}\right) \notin Q,(\forall i \in[n]):\left.\xi\right|_{v_{i} \notin Q \backslash\{p\}}}} S_{\mathcal{A}}(P, q)\left(\xi[p]_{\left(v_{1}, \ldots, v_{n}\right)}\right) \otimes \bigotimes_{i \in[n]} S_{\mathcal{A}}(P, p)_{p}^{*}\left(\left.\xi\right|_{v_{i}}\right) \quad \quad \text { (by Lemma 10.7.3) } \\
& =\bigoplus_{\substack{\left(v_{1}, \ldots, v_{n}\right) \in \operatorname{cut}_{p}(\xi)}} S_{\mathcal{A}}(P, q)\left(\xi[p]_{\left(v_{1}, \ldots, v_{n}\right)}\right) \otimes \bigotimes_{i \in[n]} S_{\mathcal{A}}(P, p)_{p}^{*}\left(\left.\xi\right|_{v_{i}}\right)
\end{aligned}
$$

$$
\text { (by definition of } S_{\mathcal{A}}(P, q) \text { and because } S_{\mathcal{A}}(P, p)_{p}^{*}(q)=\mathbb{0} \text { for each } q \in Q \backslash\{p\} \text { ) }
$$

$$
=\left(S_{\mathcal{A}}(P, q) \circ_{p} S_{\mathcal{A}}(P, p)_{p}^{*}\right)(\xi)
$$

(by definition of $\circ_{p}$ )

With the help of Lemma 12.2 .1 we can show that weighted languages of the form $S_{\mathcal{A}}(P, q)$ are rational.
Lemma 12.2.2. For every $P \subseteq Q$ and $q \in Q$ we have $S_{\mathcal{A}}(P, q) \in \operatorname{Rat}(\Sigma \cup Q, \mathrm{~B})$.
Proof. We prove the statement by induction on $(\mathcal{P}(Q), \prec)$, where for every $P, P^{\prime} \in \mathcal{P}(Q)$ we define $P^{\prime} \prec P$ if there exists a $p \in Q \backslash P^{\prime}$ such that $P=P^{\prime} \cup\{p\}$. Obviously, $\prec$ is well-founded and $\min _{\prec}(\mathcal{P}(Q))=\{\emptyset\}$.
I.B.: Let $P=\emptyset$. For every $k \in \mathbb{N}, \sigma \in \Sigma^{(k)}, q, q_{1}, \ldots, q_{k} \in Q$, we define the run $\rho_{q_{1} \cdots q_{k}, q}^{\sigma}$ : $\operatorname{pos}\left(\sigma\left(q_{1}, \ldots, q_{k}\right)\right) \rightarrow Q$ by $\rho_{q_{1} \cdots q_{k}, q}^{\sigma}(\varepsilon)=q$ and $\rho_{q_{1} \cdots q_{k}, q}^{\sigma}(i)=q_{i}$ for each $i \in[k]$. Then for each $\xi \in \mathrm{T}_{\Sigma}(Q)$ we have

$$
\mathrm{R}_{\mathcal{A}}^{\emptyset}(q, \xi)= \begin{cases}\left\{\rho_{q_{1} \cdots q_{k}, q}^{\sigma}\right\} & \text { if } \xi=\sigma\left(q_{1}, \ldots, q_{k}\right) \\ \emptyset & \text { otherwise }\end{cases}
$$

$$
\begin{aligned}
& \bigoplus_{\left(v_{1}, \ldots, v_{n}\right) \in \operatorname{cut}_{p}(\xi) \backslash\{(\varepsilon)\}:} S_{\mathcal{A}}(P, q)\left(\xi[p]_{\left(v_{1}, \ldots, v_{n}\right)}\right) \otimes \bigotimes_{i \in[n]}\left(\left(S_{\mathcal{A}}(P \cup\{p\}, p) \oplus \mathbb{1} . p\right)\left(\left.\xi\right|_{v_{i}}\right)\right. \\
& \begin{array}{c}
\xi[p]\left(v_{1}, \ldots, v_{n}\right) \notin Q, \\
(\forall i \in[n]): \mid v_{i} \notin Q \backslash\{p\}
\end{array}
\end{aligned}
$$

Let $\Sigma(Q)=\left\{\sigma\left(q_{1}, \ldots, q_{k}\right) \mid k \in \mathbb{N}, \sigma \in \Sigma^{(k)}, q_{1}, \ldots, q_{k} \in Q\right\}$. Then

$$
S_{\mathcal{A}}(\emptyset, q)=\bigoplus_{\sigma\left(q_{1}, \ldots, q_{k}\right) \in \Sigma(Q)} \delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right) \cdot \sigma\left(q_{1}, \ldots, q_{k}\right)
$$

Since $S_{\mathcal{A}}(\emptyset, q)$ is polynomial, we have that $S_{\mathcal{A}}(\emptyset, q)$ is rational.
I.S.: Let $P=P^{\prime} \cup\{p\}$ for some $p \in Q \backslash P^{\prime}$. For the induction step, we assume that $S_{\mathcal{A}}\left(P^{\prime}, q\right)$ is rational for each $q \in Q$. By Lemma 12.2 .1 we have $S_{\mathcal{A}}\left(P^{\prime} \cup\{p\}, q\right)=S_{\mathcal{A}}\left(P^{\prime}, q\right) \circ_{p} S_{\mathcal{A}}\left(P^{\prime}, p\right)_{p}^{*}$. Thus, by I.H. and the definition of rational weighted tree languages, we obtain that $S_{\mathcal{A}}\left(P^{\prime} \cup\{p\}, q\right)$ is rational.

Theorem 12.2.3. (cf. DPV05, Thm. 5.2]) $\operatorname{Rec}(\Sigma, \mathrm{B}) \subseteq \operatorname{Rat}(\Sigma, \mathrm{B})^{\mathrm{ext}}$.
Proof. Let $\mathcal{A}=(Q, \delta, F)$ be a $(\Sigma, \mathrm{B})$-wta with $Q=\left\{q_{1}, \ldots, q_{n}\right\}$. By Theorem 7.3.1 we can assume that $\mathcal{A}$ is root weight normalized. Let $q_{f} \in Q$ such that $\operatorname{supp}(F)=\left\{q_{f}\right\}$ and $F\left(q_{f}\right)=\mathbb{1}$.

We will prove that $\llbracket \mathcal{A} \rrbracket \in \operatorname{Rat}(\Sigma, \mathrm{B})^{\text {ext }}$. For this we want to employ Lemma 12.2 .2 (for $P=Q$ and $\left.q=q_{f}\right)$. We observe that $S_{\mathcal{A}}\left(Q, q_{f}\right)(\xi)=\llbracket \mathcal{A} \rrbracket(\xi)$ for each $\xi \in \mathrm{T}_{\Sigma}$. Since $\operatorname{supp}\left(S_{\mathcal{A}}\left(Q, q_{f}\right)\right)$ may contain trees in $\mathrm{T}_{\Sigma}(Q) \backslash \mathrm{T}_{\Sigma}$, the weighted tree language $S_{\mathcal{A}}\left(Q, q_{f}\right)$ may not be the $\mathbb{0}$-extension of $\llbracket \mathcal{A} \rrbracket$ to $\mathrm{T}_{\Sigma \cup Q}$. In order to make it so, we annihilate those trees from $\operatorname{supp}\left(S_{\mathcal{A}}\left(Q, q_{f}\right)\right)$.

Therefore, we define $r: \mathrm{T}_{\Sigma}(Q) \rightarrow B$ for each $\xi \in \mathrm{T}_{\Sigma}(Q)$ by

$$
r(\xi)=\left(\cdots\left(S_{\mathcal{A}}\left(Q, q_{f}\right) \circ_{q_{1}} \widetilde{\mathbb{D}}\right) \cdots \circ_{q_{n}} \widetilde{\mathbb{D}}\right)(\xi)
$$

It is obvious that, for each $\xi \in \mathrm{T}_{\Sigma}(Q)$, we have

$$
r(\xi)= \begin{cases}\llbracket \mathcal{A} \rrbracket(\xi) & \text { if } \xi \in \mathrm{T}_{\Sigma} \\ 0 & \text { otherwise }\end{cases}
$$

i.e., $r=\llbracket \mathcal{A} \rrbracket \upharpoonright^{\Sigma \cup Q, D}$.

It remains to prove that $r \in \operatorname{Rat}(\Sigma \cup Q, \mathrm{~B})$. By Lemma 12.2 .2 we have that $S_{\mathcal{A}}\left(Q, q_{f}\right) \in \operatorname{Rat}(\Sigma \cup Q, \mathrm{~B})$. By definition, $\widetilde{\mathbb{O}}$ is a polynomial, hence $\widetilde{\mathbb{O}} \in \operatorname{Rat}(\Sigma \cup Q, \mathrm{~B})$. Since $\operatorname{Rat}(\Sigma \cup Q, \mathrm{~B})$ is closed under $q_{i^{-}}$ concatenation, we obtain that $r \in \operatorname{Rat}(\Sigma \cup Q, \mathrm{~B})$. Hence $\llbracket \mathcal{A} \rrbracket \in \operatorname{Rat}(\Sigma, \mathrm{B})^{\mathrm{ext}}$.

We finish this section by an informal comparison. If we analyse the proofs of Theorem 12.2.3 and of Lemmas 12.2 .1 and 12.2 .2 then we realize that, for each wta $\mathcal{A}$, we can represent the weighted tree language $\llbracket \mathcal{A} \rrbracket$ in terms of polynomials and the operations tree concatenation and Kleene-stars; in particular, the summation $\oplus$ is not needed. This is different from the situation for (unweighted) string languages. In the following we explain the reason of this difference.

In the string case, assuming that the set $Q$ of states of some finite-state string automaton $A$ is $\{1, \ldots, n\}$, the dynamic programming equality which we use in the analysis of the behaviour of $A$ is

$$
\begin{equation*}
L_{i, j}^{(k+1)}=L_{i, j}^{(k)} \cup L_{i, k+1}^{(k)}\left(L_{k+1, k+1}^{(k)}\right)^{*} L_{k+1, j}^{(k)} \tag{12.3}
\end{equation*}
$$

Har78, p. 58], where $i, j$, and $k$ are states; and $L_{i, j}^{(k)}$ is the set of strings which lead $A$ from state $i$ to state $j$ and, intermediately, may only visit states in $\{1, \ldots, k\}$.

Generalizing this scenario to the (unweighted) tree case means to turn the input string $90^{\circ}$ counterclockwise, i.e., the starting state is at the leaf, and to extend this monadic tree into a "real" tree by allowing arbitrary ranks. This implies that the computation does not only start at one point (leaf) in one state $i$ but at each leaf, and hence $i$ must be replaced by a whole set $P$ of possible starting states. Then (12.3) turns into

$$
\begin{equation*}
L_{P, j}^{(k+1)}=L_{P, j}^{(k)} \cup L_{P \cup\{k+1\}, j}^{(k)} \circ_{k+1}\left(L_{P \cup\{k+1\}, k+1}^{(k)}\right)_{k+1}^{*} \circ_{k+1} L_{P, k+1}^{(k)} \tag{12.4}
\end{equation*}
$$

|  | $\operatorname{Pol}(\Sigma, \mathrm{B})$ | scalar multipl. | top-concat. | rational operations |
| :---: | :---: | :---: | :---: | :---: |
| $\mathrm{B}^{\text {rat }}\left\langle\left\langle\mathrm{T}_{\Sigma}\right\rangle\right\rangle$ DPV05]: |  | closed | closed | closed |
| $\mathrm{B}_{+ \text {Pol }}^{\text {rat }}\left\langle\left\langle\mathrm{T}_{\Sigma}\right\rangle\right\rangle$ DPV05: | includes | closed | closed | closed |
| $\operatorname{Rat}(\Sigma, \mathrm{B})$ : | includes |  |  | closed |
| $\mathrm{B}^{\text {rat }}\left\langle\left\langle\mathrm{T}_{\Sigma}\left(Q_{\infty}\right)\right\rangle\right\rangle$ DPV05 | $=\bigcup\left(\mathrm{B}^{\mathrm{rat}}\left\langle\left\langle\mathrm{~T}_{\Theta}\right\rangle\right\rangle \mid \text { ranked alphabet } \Theta \text { such that } \Theta \geq_{0} \Sigma\right)$ |  |  |  |
| $\mathrm{B}^{\text {rat }}\left\langle\left\langle\mathrm{T}_{\Sigma}\right\rangle\right\rangle^{\text {ext }}$ | $=\left\{r: \mathrm{T}_{\Sigma} \rightarrow B\left\|\left(\exists \Theta \geq_{0} \Sigma\right): r\right\|^{\Theta, 0} \in \mathrm{~B}^{\mathrm{rat}}\left\langle\left\langle\mathrm{~T}_{\Theta}\right\rangle\right\rangle\right\}$ |  |  |  |
| $\operatorname{Rat}(\Sigma, \mathrm{B})^{\text {ext }}$ | $=\left\{r: \mathrm{T}_{\Sigma} \rightarrow B\left\|\left(\exists \Theta \geq{ }_{0} \Sigma\right): r\right\|^{\Theta, \varnothing} \in \operatorname{Rat}(\Theta, \mathrm{B})\right\}$ |  |  |  |

Figure 12.3: An overview of the definition of several sets of rational weighted tree language. The denotation $\mathrm{B}_{+ \text {Pol }}^{\text {rat }}\left\langle\left\langle\mathrm{T}_{\Sigma}\right\rangle\right\rangle$ does not occur in DPV05], but we invented this here for the sake of brevity.
where we assume that $k+1 \notin P$ (and by keeping in mind that the order of arguments of tree concatenation is reversed with respect to that order in string concatenation). This corresponds to (*) on page 78 of GS84. Now we can realize that the first part of the union, i.e., the set $L_{P, j}^{(k)}$, is a subset of $L_{P \cup\{k+1\}, j}^{(k)}$ and, since the state $k+1$ does not occur in trees in $L_{P, j}^{(k)}$, we obtain that

$$
L_{P, j}^{(k)} \subseteq L_{P \cup\{k+1\}, j}^{(k)} \circ_{k+1}\left(L_{P \cup\{k+1\}, k+1}^{(k)}\right)_{k+1}^{*} \circ_{k+1} L_{P, k+1}^{(k)} .
$$

Hence (12.4) turns into

$$
\begin{equation*}
L_{P, j}^{(k+1)}=L_{P \cup\{k+1\}, j}^{(k)} \circ_{k+1}\left(L_{P \cup\{k+1\}, k+1}^{(k)}\right)_{k+1}^{*} \circ_{k+1} L_{P, k+1}^{(k)} \tag{12.5}
\end{equation*}
$$

and the union disappeared. Then (12.5) corresponds to the equality in Eng75b, p. 21] where we have to identify the set $\{1, \ldots, k\}$ of states with the set $Q$ of nonterminals, the state $k+1$ with the nonterminal B , and the state $j$ with the nonterminal $A$. Hence, we can represent each recognizable tree language in terms of finite tree languages and the operations tree concatenation and Kleene-stars; in particular, the union is not needed. And we generalized this scenario to recognizable weighted tree languages.

### 12.3 From rational to recognizable

Here we prove Theorem 12.3 .1 i.e., $\operatorname{Rat}(\Sigma, \mathrm{B}) \subseteq \operatorname{Rec}(\Sigma, \mathrm{B})$, or in words: each rational $(\Sigma, \mathrm{B})$-weighted tree language is recognizable.

Theorem 12.3.1. $\operatorname{Rat}(\Sigma, \mathrm{B}) \subseteq \operatorname{Rec}(\Sigma, \mathrm{B})$.
Proof. Let $r$ be a polynomial $(\Sigma, \mathrm{B})$-weighted tree language. Then there exist $n \in \mathbb{N}_{+}, b_{1}, \ldots, b_{n} \in B$, and $\xi_{1}, \ldots, \xi_{k} \in \mathrm{~T}_{\Sigma}$ such that $r=b_{1} \cdot \xi_{1} \oplus \ldots \oplus b_{n} \cdot \xi_{n}$. Since for each $i \in[n]$ the singleton $\left\{\xi_{i}\right\}$ is a recognizable $\Sigma$-tree language, the weighted tree language $r$ is a recognizable step mapping. By Theorem 10.3.1, there exists a crisp deterministic $(\Sigma, \mathrm{B})$-wta $\mathcal{A}$ such that $\llbracket \mathcal{A} \rrbracket=r$. Hence $r \in \operatorname{Rec}(\Sigma, \mathrm{~B})$.

By Theorem 10.1.1 and Corollaries 10.6 .2 and 10.7.6, the set $\operatorname{Rec}(\Sigma, \mathrm{B})$ is closed under the rational operations. Since $\operatorname{Rat}(\Sigma, B)$ is the smallest set which contains each polynomial ( $\Sigma, \mathrm{B}$ )-weighted tree language and is closed under the rational operations, the statement follows.

### 12.4 Alternative definition of rational weighted tree languages

In Figure 12.3 we give an overview of the relevant sets of rational weighted tree languages where the sets $\mathrm{B}_{+ \text {Pol }}^{\text {rat }}\left\langle\left\langle\mathrm{T}_{\Sigma}\right\rangle\right\rangle$ and $\mathrm{B}^{\text {rat }}\left\langle\left\langle\mathrm{T}_{\Sigma}\right\rangle\right\rangle{ }^{\text {ext }}$ will be defined below.

In this section we will prove the following: including polynomial weighted tree languages into $\mathrm{B}^{\text {rat }}\left\langle\left\langle\mathrm{T}_{\Sigma}\right\rangle\right\rangle$ (yielding the set $\mathrm{B}_{+\mathrm{Pol}}^{\mathrm{rat}}\left\langle\left\langle\mathrm{T}_{\Sigma}\right\rangle\right\rangle$ ) does not enrich the set $\mathrm{B}^{\mathrm{rat}}\left\langle\left\langle\mathrm{T}_{\Sigma}\right\rangle\right\rangle$ (cf. Observation 12.4 .2 and DPV05, Obs. 3.19]). Moreover, we prove that the extensions of the sets $\mathrm{B}^{\text {rat }}\left\langle\left\langle\mathrm{T}_{\Sigma}\right\rangle\right\rangle$ and $\operatorname{Rat}(\Sigma, \mathrm{B})$ are equal when using our concept of $\mathbb{0}$-extension (cf. Theorem 12.4.7).

As preparation, we characterize the set $\operatorname{Pol}(\Sigma, \mathrm{B})$ of polynomial $(\Sigma, \mathrm{B})$-weighted tree languages in terms of closure properties.

Lemma 12.4.1. The set $\operatorname{Pol}(\Sigma, \mathrm{B})$ is the smallest set of $(\Sigma, \mathrm{B})$-weighted tree languages which is closed under sum, scalar multiplications, and top-concatenations. Hence, in particular, $\operatorname{Pol}(\Sigma, \mathrm{B}) \subseteq \mathrm{B}^{\mathrm{rat}}\left\langle\left\langle\mathrm{T}_{\Sigma}\right\rangle\right\rangle$.

Proof. For convenience, we denote by $\mathcal{C}$ the smallest set of $(\Sigma, \mathrm{B})$-weighted tree languages which is closed under sum, scalar multiplications, and top-concatenations.

First we prove that each polynomial $(\Sigma, \mathrm{B})$-weighted tree language is in $\mathcal{C}$. As preparation we prove by induction on $\mathrm{T}_{\Sigma}$ that the monomial $\mathbb{1} . \xi$ is in $\mathcal{C}$ for each $\xi \in \mathrm{T}_{\Sigma}$. Let $\xi=\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)$ and assume that $\mathbb{1} . \xi_{i}$ is in $\operatorname{Pol}(\Sigma, \mathrm{B})$ for each $i \in[k]$. Obviously,

$$
\mathbb{1} . \xi=\operatorname{top}_{\sigma}\left(\mathbb{1} \cdot \xi_{1}, \ldots, \mathbb{1} . \xi_{k}\right)
$$

and hence $\mathbb{1} . \xi$ is in $\mathcal{C}$ because $\mathcal{C}$ is closed under top-concatenations.
Now let $r: \mathrm{T}_{\Sigma} \rightarrow B$ be polynomial. We distinguish two cases.
Case (a): Let $\operatorname{supp}(r)=\emptyset$. Let $\alpha \in \Sigma^{(0)}$ (recall that $\Sigma^{(0)} \neq \emptyset$ by our general assumption on ranked alphabets). Then $r=\mathbb{O} \otimes \mathbb{1} . \alpha$, and since $\mathbb{1} . \alpha$ is in $\mathcal{C}$ and $\mathcal{C}$ is closed under scalar multiplications, we have that $r$ is in $\mathcal{C}$.
 $\mathbb{1} . \xi_{i}$ is in $\mathcal{C}$. Since $\mathcal{C}$ is closed under scalar multiplications and sum, we have that $r$ is in $\mathcal{C}$.

Next we prove that each $r \in \mathcal{C}$ is polynomial, i.e., $\operatorname{supp}(r)$ is finite. This is easy to see, because the application of top-concatenations, scalar multiplications, and sum preserve the property of finite support.

Thus $\operatorname{Pol}(\Sigma, \mathrm{B})=\mathcal{C}$. The inclusion $\operatorname{Pol}(\Sigma, \mathrm{B}) \subseteq \mathrm{B}^{\text {rat }}\left\langle\left\langle\mathrm{T}_{\Sigma}\right\rangle\right\rangle$ follows from the definition of $\mathrm{B}^{\mathrm{rat}}\left\langle\left\langle\mathrm{T}_{\Sigma}\right\rangle\right\rangle$.
We denote by $\mathrm{B}_{+ \text {Pol }}^{\text {rat }}\left\langle\left\langle\mathrm{T}_{\Sigma}\right\rangle\right\rangle$ the smallest set of $(\Sigma, \mathrm{B})$-weighted tree languages which contains $\operatorname{Pol}(\Sigma, \mathrm{B})$ and is closed under scalar multiplications, top-concatenations, and the rational operations.

Observation 12.4.2. DPV05, Obs. 3.19] $\mathrm{B}^{\text {rat }}\left\langle\left\langle\mathrm{T}_{\Sigma}\right\rangle\right\rangle=\mathrm{B}_{+ \text {Pol }}^{\text {rat }}\left\langle\left\langle\mathrm{T}_{\Sigma}\right\rangle\right\rangle$.
Proof. The inclusion $\mathrm{B}^{\mathrm{rat}}\left\langle\left\langle\mathrm{T}_{\Sigma}\right\rangle\right\rangle \subseteq \mathrm{B}_{+ \text {Pol }}^{\text {rat }}\left\langle\left\langle\mathrm{T}_{\Sigma}\right\rangle\right\rangle$ is obvious. The inclusion $\mathrm{B}_{+\mathrm{Pol}}^{\mathrm{rat}}\left\langle\left\langle\mathrm{T}_{\Sigma}\right\rangle\right\rangle \subseteq \mathrm{B}^{\mathrm{rat}}\left\langle\left\langle\mathrm{T}_{\Sigma}\right\rangle\right\rangle$ holds because $\operatorname{Pol}(\Sigma, \mathrm{B}) \subseteq \mathrm{B}^{\text {rat }}\left\langle\left\langle\mathrm{T}_{\Sigma}\right\rangle\right\rangle$ (by Lemma 12.4.1), the fact that $\mathrm{B}^{\text {rat }}\left\langle\left\langle\mathrm{T}_{\Sigma}\right\rangle\right\rangle$ is closed under scalar multiplications, top-concatenations, and the rational operations, and the fact that $\mathrm{B}_{+ \text {Pol }}^{\text {rat }}\left\langle\left\langle\mathrm{T}_{\Sigma}\right\rangle\right\rangle$ is the smallest such set.

Next we prove that the extensions of the sets $\mathrm{B}^{\mathrm{rat}}\left\langle\left\langle\mathrm{T}_{\Sigma}\right\rangle\right\rangle$ and $\operatorname{Rat}(\Sigma, \mathrm{B})$ (using our concepts of $\mathbb{0}$ extension) are equal.

The set of extended DPV-rational $(\Sigma, \mathrm{B})$-weighted tree languages, denoted by $\mathrm{B}^{\text {rat }}\left\langle\left\langle\mathrm{T}_{\Sigma}\right\rangle\right\rangle^{\text {ext }}$, contains each ( $\Sigma, \mathrm{B}$ )-weighted tree language $r$ such that $r\rangle^{\Theta, 0} \in \mathrm{~B}^{\text {rat }}\left\langle\left\langle\mathrm{T}_{\Theta}\right\rangle\right\rangle$ for some 0 -extension $\Theta$ of $\Sigma$.

Thus, we wish to show that $\operatorname{Rat}(\Sigma, \mathrm{B})^{\text {ext }}=\mathrm{B}^{\mathrm{rat}}\left\langle\left\langle\mathrm{T}_{\Sigma}\right\rangle\right\rangle^{\text {ext }}$. Intuitively, this means that, for the extended sets, closure under scalar multiplications and top-concatenations can be traded for polynomials, and vice versa. We prove the two directions in separate lemmas.

Lemma 12.4.3. $\operatorname{Rat}(\Sigma, \mathrm{B})^{\text {ext }} \subseteq \mathrm{B}^{\mathrm{rat}}\left\langle\left\langle\mathrm{T}_{\Sigma}\right\rangle\right\rangle^{\text {ext }}$.
Proof. Let $r \in \operatorname{Rat}(\Sigma, \mathrm{~B})^{\text {ext }}$. Hence $r r^{\Theta, 0} \in \operatorname{Rat}(\Theta, \mathrm{~B})$ for some 0 -extension $\Theta$ of $\Sigma$.
 Hence $\mathrm{B}^{\text {rat }}\left\langle\left\langle\mathrm{T}_{\Theta}\right\rangle\right\rangle$ is a set of $(\Theta, B)$-weighted tree languages which contains $\operatorname{Pol}(\Theta, B)$ and is closed under
the rational operations. Since $\operatorname{Rat}(\Theta, B)$ is the smallest such set, we obtain $\operatorname{Rat}(\Theta, B) \subseteq \mathrm{B}^{\text {rat }}\left\langle\left\langle\mathrm{T}_{\Theta}\right\rangle\right\rangle$. Hence $\left.r\right|^{\Theta, 0} \in \mathrm{~B}^{\mathrm{rat}}\left\langle\left\langle\mathrm{T}_{\Theta}\right\rangle\right\rangle$. Thus, by definition, we obtain that $r \in \mathrm{~B}^{\mathrm{rat}}\left\langle\left\langle\mathrm{T}_{\Sigma}\right\rangle\right\rangle^{\mathrm{ext}}$.

For the other direction, i.e., $\mathrm{B}^{\mathrm{rat}}\left\langle\left\langle\mathrm{T}_{\Sigma}\right\rangle\right\rangle^{\mathrm{ext}} \subseteq \operatorname{Rat}(\Sigma, \mathrm{B})^{\text {ext }}$, we first prove an auxiliary technical lemma.
Lemma 12.4.4. Let $\Delta$ be a 0 -extension of $\Sigma$ and let $r: \mathrm{T}_{\Sigma} \rightarrow B$. If $r \in \operatorname{Rat}(\Sigma, \mathrm{~B})$, then $r r^{\Delta, 0} \in$ $\operatorname{Rat}(\Delta, B)$.

Proof. First, we characterize $\operatorname{Rat}(\Sigma, \mathrm{B})$ by means of Theorem 2.6.17 such that we can prove a statement on $\operatorname{Rat}(\Sigma, \mathrm{B})$ by induction on $\mathbb{N}$. For this, we consider the $\sigma$-complete lattice $\left(\mathcal{P}\left(B^{\mathrm{T}_{\Sigma}}\right), \subseteq\right)$ and the mapping $f: \mathcal{P}\left(B^{T_{\Sigma}}\right) \rightarrow \mathcal{P}\left(B^{T_{\Sigma}}\right)$ defined for each $C \in \mathcal{P}\left(B^{T_{\Sigma}}\right)$ by

$$
\begin{aligned}
f(C)= & C \cup \operatorname{Pol}(\Sigma, \mathrm{~B}) \cup\left\{r_{1} \oplus r_{2} \mid r_{1}, r_{2} \in C\right\} \cup\left\{r_{1} \circ_{\alpha} r_{2} \mid r_{1}, r_{2} \in C, \alpha \in \Sigma^{(0)}\right\} \cup \\
& \left\{(r)_{\alpha}^{*} \mid r \in C, \alpha \in \Sigma^{(0)}, \text { and } r \text { is } \alpha \text {-proper }\right\}
\end{aligned}
$$

The mapping $f$ is continuous and, by definition,

$$
\operatorname{Rat}(\Sigma, \mathrm{B})=\bigcap\left(C \mid C \in \mathcal{P}\left(B^{\mathrm{T}_{\Sigma}}\right) \text { such that } f(C) \subseteq C\right)
$$

Then, by Theorem 2.6.17, we have

$$
\begin{equation*}
\operatorname{Rat}(\Sigma, \mathrm{B})=\bigcup\left(f^{n}(\emptyset) \mid n \in \mathbb{N}\right) \tag{12.6}
\end{equation*}
$$

Now, by induction on $\mathbb{N}$, we prove that the following statement holds:

$$
\begin{equation*}
\text { For every } n \in \mathbb{N} \text { and } r: \mathrm{T}_{\Sigma} \rightarrow B: \text { if } r \in f^{n}(\emptyset), \text { then } r r^{\Delta, 0} \in \operatorname{Rat}(\Delta, \mathrm{~B}) . \tag{12.7}
\end{equation*}
$$

I.B.: Let $n=0$. Since $f^{n}(\emptyset)=f^{0}(\emptyset)=\emptyset$, statement (12.7) trivially holds.
I.S.: Let $n=n^{\prime}+1$ for some $n^{\prime} \in \mathbb{N}$. Moreover, let $r \in f^{n}(\emptyset)$. We assume that (12.7) holds for each $r^{\prime} \in f^{n^{\prime}}(\emptyset)$. We proceed by case analysis.

Case (a): Let $r \in f^{n^{\prime}}(\emptyset)$. Then our statement holds by I.H.
Case (b): Let $r \in \operatorname{Pol}(\Sigma, \mathrm{~B})$. Then obviously $r^{\Delta, 0}$ is a $(\Delta, \mathrm{B})$-polynomial, hence $r r^{\Delta, 0} \in \operatorname{Rat}(\Delta, \mathrm{~B})$.
Case (c): Let $r=r_{1} \oplus r_{2}$ for some ( $\left.\Sigma, \mathrm{B}\right)$-weighted tree languages $r_{1}$ and $r_{2}$ in $f^{n^{\prime}}(\emptyset)$. By I.H., we have $r_{1} \upharpoonright^{\Delta, 0} \in \operatorname{Rat}(\Delta, \mathrm{~B})$ and $r_{2} \upharpoonright^{\Delta, 0} \in \operatorname{Rat}(\Delta, \mathrm{~B})$. Obviously, $r \upharpoonright^{\Delta, 0}=r_{1} \upharpoonright^{\Delta, 0} \oplus r_{2} \upharpoonright^{\Delta, 0}$. Since $\operatorname{Rat}(\Delta, \mathrm{B})$ is closed under sum, we have that $r r^{\Delta, 0} \in \operatorname{Rat}(\Delta, B)$.

Cases (d) and (e): Let $r=r_{1} \circ_{\alpha} r_{2}$ and $r=\left(r_{1}\right)_{\alpha}^{*}$, respectively, for some $r_{1}$ and $r_{2}$ in $f^{n^{\prime}}(\emptyset)$ and $\alpha \in \Sigma$. The proofs of the cases are similar to the proof of Case (c).

This finishes the proof of (12.7). The statement of the lemma follows from (12.7) and (12.6).
In the proof of $\mathrm{B}^{\mathrm{rat}}\left\langle\left\langle\mathrm{T}_{\Sigma}\right\rangle\right\rangle^{\text {ext }} \subseteq \operatorname{Rat}(\Sigma, \mathrm{B})^{\text {ext }}$ we will use the fact that, for each 0 -extension $\Theta$ of $\Sigma$, the set $\operatorname{Rat}(\Theta, B)^{\text {ext }}$ is closed under scalar-multiplications, top-concatenations, and the rational operations. One way to prove this is to use $\operatorname{Rat}(\Theta, B)^{\text {ext }} \subseteq \operatorname{Rec}(\Theta, B)^{\text {ext }}$ (by Theorem 12.1.2), to exploit closure properties of $\operatorname{Rec}(\Theta, B)^{\text {ext }}$, and to use $\operatorname{Rec}(\Theta, B)^{\text {ext }} \subseteq \operatorname{Rat}(\Theta, B)^{\text {ext }}$ (again by Theorem 12.1.2). But here we show an alternative proof, in which we stay inside the area of rational weighted tree languages.

Lemma 12.4.5. Let $\Theta$ be an arbitrary ranked alphabet. Then $\operatorname{Rat}(\Theta, B)^{\text {ext }}$ is closed under (a) scalarmultiplications, (b) top-concatenations, and the (c) rational operations.

Proof. (a) Here we prove that $\operatorname{Rat}(\Theta, \mathrm{B})^{\text {ext }}$ is closed under scalar-multiplications. Let $r \in \operatorname{Rat}(\Theta, \mathrm{~B})^{\text {ext }}$ and $b \in B$. We have to prove that $b \otimes r \in \operatorname{Rat}(\Theta, \mathrm{~B})^{\text {ext }}$.

By our assumption, $r r^{\Delta, 0} \in \operatorname{Rat}(\Delta, B)$ for some 0 -extension $\Delta$ of $\Theta$. Let $\alpha \in \Delta^{(0)}$ be an arbitrary nullary symbol (recall that $\Delta^{(0)} \neq \emptyset$ by definition of ranked alphabets.) We define the ( $\Delta, \mathrm{B}$ )-weighted tree language $s$ by

$$
s=(b . \alpha) \circ_{\alpha}\left(r \upharpoonright^{\Delta, 0}\right)
$$

Since $b . \alpha \in \operatorname{Rat}(\Delta, B)$ and $\operatorname{Rat}(\Delta, B)$ is closed under tree concatenations, we have that $s \in \operatorname{Rat}(\Delta, B)$. Moreover, for each $\xi \in \mathrm{T}_{\Delta}$, we have

$$
s(\xi)=b \otimes\left(r \upharpoonright^{\Delta, 0}(\xi)\right)
$$

Hence $s=b \otimes r \upharpoonright^{\Delta, 0}$, and since $b \otimes r \Gamma^{\Delta, 0}=(b \otimes r) \upharpoonright^{\Delta, 0}$ we obtain that $b \otimes r \in \operatorname{Rat}(\Theta, \mathrm{~B})^{\text {ext }}$.
(b) We prove that $\operatorname{Rat}(\Theta, \mathrm{B})^{\text {ext }}$ is closed under top-concatenations. Let $k \in \mathbb{N}, \sigma \in \Theta^{(k)}$, and $r_{1}, \ldots, r_{k}$ be $(\Theta, B)$-weighted tree languages in $\operatorname{Rat}(\Theta, \mathrm{B})^{\text {ext }}$. We have to prove that top $\left(r_{1}, \ldots, r_{k}\right) \in \operatorname{Rat}(\Theta, \mathrm{B})^{\text {ext }}$.

For each $i \in[k]$ let $r_{i} \upharpoonright^{\Delta_{i}, \mathrm{D}} \in \operatorname{Rat}\left(\Delta_{i}, \mathrm{~B}\right)$ for some 0 -extension $\Delta_{i}$ of $\Theta$. Without loss of generality we can assume that $\left(\Delta_{i} \backslash \Theta\right) \cap\left(\Delta_{j} \backslash \Theta\right)=\emptyset$ for every $i, j \in[k]$ with $i \neq j$. We define the ranked alphabet $\Delta=\bigcup_{i \in[k]} \Delta_{i}$. Obviously, $\Delta \geq_{0} \Delta_{i}$ for each $i \in[k]$. By Observation 12.1.1 we have that $\left(\left.r_{i}\right|^{\Delta_{i}, \mathbb{D}}\right) \upharpoonright^{\Delta, 0}=\left.r_{i}\right|^{\Delta, 0}$ and hence, by Lemma 12.4.4, we have that $\left.r_{i}\right|^{\Delta, 0} \in \operatorname{Rat}(\Delta, \mathrm{~B})$.

Let $P=\left\{p_{1}, \ldots, p_{k}\right\}$ be a set disjoint with $\Delta$. We view $\Delta \cup P$ as 0 -extension of $\Delta$. We define the $(\Delta \cup P, \mathrm{~B})$-weighted tree language $r$ by

$$
r=\left(\ldots\left(\mathbb{1} . \sigma\left(p_{1}, \ldots, p_{k}\right) \circ_{p_{1}} r_{1} \upharpoonright^{\Delta \cup P, 0}\right) \ldots\right) \circ_{p_{k}} r_{k} \upharpoonright^{\Delta \cup P, 0} .
$$

Since $\mathbb{1} . \sigma\left(p_{1}, \ldots, p_{k}\right) \in \operatorname{Rat}(\Delta \cup P, \mathrm{~B})$ and $r_{i} \upharpoonright^{\Delta \cup P, 0} \in \operatorname{Rat}(\Delta \cup P, \mathrm{~B})$ (for each $i \in[k]$ by Lemma 12.4.4) and $\operatorname{Rat}(\Delta \cup P, \mathrm{~B})$ is closed under tree concatenations, we have that $r \in \operatorname{Rat}(\Delta \cup P, \mathrm{~B})$.

We will show that, for each $\xi \in \mathrm{T}_{\Delta \cup P}$, we have

$$
r(\xi)= \begin{cases}\operatorname{top}_{\sigma}\left(r_{1}, \ldots, r_{k}\right)(\xi) & \text { if } \xi=\sigma\left(\xi_{1}, \ldots, \xi_{k}\right) \text { for some } \xi_{1}, \ldots, \xi_{k} \in \mathrm{~T}_{\Theta}  \tag{12.8}\\ 0 & \text { otherwise }\end{cases}
$$

Hence $r=\operatorname{top}_{\sigma}\left(r_{1}, \ldots, r_{k}\right) \upharpoonright^{\Delta \cup P, \varnothing}$. Since $\Delta \cup P \geq_{0} \Theta$, we obtain that $\operatorname{top}_{\sigma}\left(r_{1}, \ldots, r_{k}\right) \in \operatorname{Rat}(\Theta, \mathrm{B})^{\text {ext }}$.
For the proof of (12.8), for each $i \in[0, k]$ we define

$$
s_{i}=\left(\ldots\left(\mathbb{1} . \sigma\left(p_{1}, \ldots, p_{k}\right) \circ_{p_{1}} r_{1} \upharpoonright\right) \ldots\right) \circ_{p_{i}} r_{i} \upharpoonright
$$

where $r_{j} \upharpoonright$ is an abbreviation of $r_{j} \upharpoonright \Delta \cup P, 0$ for each $j \in[i]$. (We note that $s_{0}=\mathbb{1} . \sigma\left(p_{1}, \ldots, p_{k}\right)$.) Then, by induction on $([0, k],<)$, we prove that the following statement holds:

For each $i \in[0, k]$ and $\xi \in \mathrm{T}_{\Delta \cup P}$, we have

$$
s_{i}(\xi)= \begin{cases}\bigotimes_{j \in[i]} r_{j}\left(\xi_{j}\right) & \text { if } \xi=\sigma\left(\xi_{1}, \ldots, \xi_{i}, p_{i+1}, \ldots, p_{k}\right) \text { for some } \xi_{1}, \ldots, \xi_{i} \in \mathrm{~T}_{\Theta}  \tag{12.9}\\ \mathbb{O} & \text { otherwise }\end{cases}
$$

Let $\xi \in \mathrm{T}_{\Delta \cup P}$.
I.B.: Let $i=0$. The statement is obvious because $s_{0}=\mathbb{1} . \sigma\left(p_{1}, \ldots, p_{k}\right)$ and the product of an $\emptyset$-indexed family over $B$ is defined to be $\mathbb{1}$.
I.S.: Let $i=i^{\prime}+1$ : Then we have

$$
\begin{equation*}
s_{i}(\xi)=\left(s_{i^{\prime}} \circ_{p_{i}} r_{i} \upharpoonright(\xi)=\bigoplus_{\left(w_{1}, \ldots, w_{n}\right) \in \operatorname{cut}_{p_{i}}(\xi)} s_{i^{\prime}}\left(\xi\left[p_{i}\right]_{\left(w_{1}, \ldots, w_{n}\right)}\right) \otimes r_{i} \upharpoonright\left(\left.\xi\right|_{w_{1}}\right) \otimes \ldots \otimes r_{i} \upharpoonright\left(\left.\xi\right|_{w_{n}}\right)\right. \tag{12.10}
\end{equation*}
$$

By the I.H. we obtain

$$
s_{i^{\prime}}\left(\xi\left[p_{i}\right]_{\left(w_{1}, \ldots, w_{n}\right)}\right)= \begin{cases}\bigotimes_{j \in\left[i^{\prime}\right]} r_{j}\left(\xi_{j}\right) & \text { if } \xi\left[p_{i}\right]_{\left(w_{1}, \ldots, w_{n}\right)}=\sigma\left(\xi_{1}, \ldots, \xi_{i^{\prime}}, p_{i}, \ldots, p_{k}\right) \\ & \text { for some } \xi_{1}, \ldots, \xi_{i^{\prime}} \in \mathrm{T}_{\Theta} \\ 0 & \text { otherwise }\end{cases}
$$

Now we observe that the condition $\xi\left[p_{i}\right]_{\left(w_{1}, \ldots, w_{n}\right)}=\sigma\left(\xi_{1}, \ldots, \xi_{i^{\prime}}, p_{i}, \ldots, p_{k}\right)$ for some $\xi_{1}, \ldots, \xi_{i^{\prime}} \in \mathrm{T}_{\Theta}$ implies that $n=1, w_{1}=i$ and there exists $\xi_{i} \in \mathrm{~T}_{\Delta \cup P}$ such that $\xi=\sigma\left(\xi_{1}, \ldots, \xi_{i^{\prime}}, \xi_{i}, p_{i}, \ldots, p_{k}\right)$.

Hence (12.10) continues as

$$
= \begin{cases}\left(\otimes_{j \in\left[i^{\prime}\right]} r_{j}\left(\xi_{j}\right)\right) \otimes r_{i}\left\lceil\left(\left.\xi\right|_{i}\right)\right. & \text { if } \xi=\sigma\left(\xi_{1}, \ldots, \xi_{i^{\prime}}, \xi_{i}, p_{i+1}, \ldots, p_{k}\right) \\ 0 & \text { for some } \xi_{1}, \ldots, \xi_{i^{\prime}} \in \mathrm{T}_{\Theta} \text { and } \xi_{i} \in \mathrm{~T}_{\Delta \cup P} \\ \text { otherwise }\end{cases}
$$

Since $\xi=\sigma\left(\xi_{1}, \ldots, \xi_{i^{\prime}}, \xi_{i}, p_{i+1}, \ldots, p_{k}\right)$ implies that $\left.\xi\right|_{i}=\xi_{i}$, and $r_{i} \upharpoonright$ is the $\mathbb{0}$-extension of $r_{i}$ to $\mathrm{T}_{\Delta \cup P}$, we obtain

$$
= \begin{cases}\bigotimes_{j \in[i]} r_{j}\left(\xi_{j}\right) & \text { if } \xi=\sigma\left(\xi_{1}, \ldots, \xi_{i^{\prime}}, \xi_{i}, p_{i+1}, \ldots, p_{k}\right) \\ & \text { for some } \xi_{1}, \ldots, \xi_{i} \in \mathrm{~T}_{\Theta} \\ 0 & \text { otherwise }\end{cases}
$$

This proves (12.9). From this latter, in case $i=k$ and using the definition of $\operatorname{top}_{\sigma}\left(r_{1}, \ldots, r_{k}\right)$ we obtain (12.8).
(c) We prove that $\operatorname{Rat}(\Theta, B)^{\text {ext }}$ is closed under the rational operations, i.e., sum, tree concatenations, and Kleene-stars.

Closure under sum: Let $r_{1}, r_{2} \in \operatorname{Rat}(\Theta, \mathrm{~B})^{\text {ext }}$. We show that $r_{1} \oplus r_{2} \in \operatorname{Rat}(\Theta, \mathrm{~B})^{\text {ext }}$.
Let $r_{1} \upharpoonright^{\Delta_{1}, 0} \in \operatorname{Rat}\left(\Delta_{1}, \mathrm{~B}\right)$ for some 0 -extension $\Delta_{1}$ of $\Theta$. Also let $r_{2} \upharpoonright^{\Delta_{2}, 0} \in \operatorname{Rat}\left(\Delta_{2}, \mathrm{~B}\right)$ for some $0-$ extension $\Delta_{2}$ of $\Theta$. Without loss of generality we can assume that $\left(\Delta_{1} \backslash \Theta\right) \cap\left(\Delta_{2} \backslash \Theta\right)=\emptyset$. We define the ranked alphabet $\Delta=\Delta_{1} \cup \Delta_{2}$. Let $i \in[2]$. Obviously, $\Delta \geq_{0} \Delta_{i}$, and hence $\Delta \geq_{0} \Theta$. By Observation 12.1.1, we have that $\left(\left.r_{i}\right|^{\Delta_{i}, 0}\right) \Gamma^{\Delta, 0}=r_{i} \upharpoonright^{\Delta, 0}$ and hence, by Lemma 12.4.4, we have that $r_{i} \upharpoonright^{\Delta, 0} \in \operatorname{Rat}(\Delta, \mathrm{~B})$.

Since $\operatorname{Rat}(\Delta, \mathrm{B})$ is closed under sum, we obtain that $r_{1} \upharpoonright^{\Delta, 0} \oplus r_{2} \upharpoonright^{\Delta, 0} \in \operatorname{Rat}(\Delta, \mathrm{~B})$. Since $\left(r_{1} \oplus r_{2}\right) \upharpoonright^{\Delta, 0}$ $(\xi)=r_{1} \upharpoonright^{\Delta, 0}(\xi) \oplus r_{2} \upharpoonright^{\Delta, 0}(\xi)$ for each $\xi \in \mathrm{T}_{\Delta}$, we have $\left(r_{1} \oplus r_{2}\right) \upharpoonright^{\Delta, 0} \in \operatorname{Rat}(\Delta, \mathrm{~B})$. Hence $r_{1} \oplus r_{2} \in$ $\operatorname{Rat}(\Theta, B)^{\text {ext }}$.

Closure under tree concatenations: Let $r_{1}, r_{2} \in \operatorname{Rat}(\Theta, \mathrm{~B})^{\text {ext }}$ and $\alpha \in \Theta^{(0)}$. We show that $r_{1} \circ_{\alpha} r_{2} \in$ $\operatorname{Rat}(\Theta, B)^{\text {ext }}$. In the same way as above we have that $r_{1} \upharpoonright^{\Delta, 0} \in \operatorname{Rat}(\Delta, B)$ and $r_{2} \upharpoonright^{\Delta, 0} \in \operatorname{Rat}(\Delta, B)$ for some 0 -extension $\Delta$ of $\Theta$.

Since $\operatorname{Rat}(\Delta, B)$ is closed under tree concatenations, we obtain that $r_{1} \upharpoonright^{\Delta, 0}{ }_{\alpha} r_{2} r^{\Delta, 0} \in \operatorname{Rat}(\Delta, B)$. Since $\left(r_{1} \circ_{\alpha} r_{2}\right) \upharpoonright^{\Delta, 0}(\xi)=r_{1} \upharpoonright^{\Delta, 0}(\xi) \circ_{\alpha} r_{2} \upharpoonright^{\Delta, 0}(\xi)$ for each $\xi \in \mathrm{T}_{\Delta}$, we have $\left(r_{1} \circ_{\alpha} r_{2}\right) \upharpoonright^{\Delta, 0} \in \operatorname{Rat}(\Delta$, B$)$. Hence $r_{1} \circ_{\alpha} r_{2} \in \operatorname{Rat}(\Theta, \mathrm{~B})^{\text {ext }}$.

Closure under Kleene-stars: Let $\alpha \in \Theta^{(0)}$ and $r \in \operatorname{Rat}(\Theta, \mathrm{~B})^{\text {ext }}$ be an $\alpha$-proper weighted tree language. We show that $r_{\alpha}^{*} \in \operatorname{Rat}(\Theta, \mathrm{~B})^{\text {ext }}$. By definition, we have that $r{ }^{\Delta, 0} \in \operatorname{Rat}(\Delta, \mathrm{~B})$ for some 0 -extension $\Delta$ of $\Theta$.

Since $\operatorname{Rat}(\Delta, B)$ is closed under Kleene stars, we obtain that $\left(r^{\Delta, 0}\right)_{\alpha}^{*} \in \operatorname{Rat}(\Delta, B)$. Since $\left(r_{\alpha}^{*}\right) \upharpoonright^{\Delta, 0}$ $(\xi)=\left(r \upharpoonright^{\Delta, 0}\right)_{\alpha}^{*}(\xi)$ for each $\xi \in \mathrm{T}_{\Delta}$, we have $\left(r_{\alpha}^{*}\right) r^{\Delta, 0} \in \operatorname{Rat}(\Delta, \mathrm{~B})$. Hence $r_{\alpha}^{*} \in \operatorname{Rat}(\Theta, \mathrm{~B})^{\text {ext }}$.

Lemma 12.4.6. $\mathrm{B}^{\mathrm{rat}}\left\langle\left\langle\mathrm{T}_{\Sigma}\right\rangle\right\rangle^{\mathrm{ext}} \subseteq \operatorname{Rat}(\Sigma, \mathrm{B})^{\text {ext }}$.
Proof. Let $r \in \mathrm{~B}^{\mathrm{rat}}\left\langle\left\langle\mathrm{T}_{\Sigma}\right\rangle\right\rangle^{\text {ext }}$. Hence $\left.r\right\rangle^{\Theta, 0} \in \mathrm{~B}^{\mathrm{rat}}\left\langle\left\langle\mathrm{T}_{\Theta}\right\rangle\right\rangle$ for some 0-extension $\Theta$ of $\Sigma$.
By Lemma 12.4.5 the set $\operatorname{Rat}(\Theta, B)^{\text {ext }}$ is closed under (a) scalar-multiplications, (b) topconcatenations, and the (c) rational operations. Since $\mathrm{B}^{\text {rat }}\left\langle\left\langle\mathrm{T}_{\Theta}\right\rangle\right\rangle$ is the smallest set which has these closure properties, we obtain that $\mathrm{B}^{\mathrm{rat}}\left\langle\left\langle\mathrm{T}_{\Theta}\right\rangle\right\rangle \subseteq \operatorname{Rat}(\Theta, \mathrm{B})^{\text {ext }}$, and hence $r{ }^{\Theta, 0} \in \operatorname{Rat}(\Theta, \mathrm{~B})^{\text {ext }}$. Thus $\left(r \upharpoonright^{\Theta, 0}\right) \upharpoonright^{\Delta, 0} \in \operatorname{Rat}(\Delta, B)$ for some 0 -extension $\Delta$ of $\Theta$. Since $\Delta \geq_{0} \Theta$ and $\Theta \geq_{0} \Sigma$ and $\left(r \upharpoonright^{\Theta, 0}\right) \upharpoonright^{\Delta, 0}=r r^{\Delta, 0}$ by Observation 12.1.1, we obtain that $r r^{\Delta, 0} \in \operatorname{Rat}(\Delta, \mathrm{~B})$. Thus, by definition, we have $r \in \operatorname{Rat}(\Sigma, \mathrm{~B})^{\text {ext }}$.

Then the next theorem follows from Lemmas 12.4 .3 and 12.4.6.

Theorem 12.4.7. Let $\Sigma$ be a ranked alphabet and B be a commutative semiring. Then $\operatorname{Rat}(\Sigma, B)^{\text {ext }}=$ $\mathrm{B}^{\mathrm{rat}}\left\langle\left\langle\mathrm{T}_{\Sigma}\right\rangle\right\rangle^{\mathrm{ext}}$.

## Chapter 13

## Elementary operations and Médvédjév's theorem

In his paper Méd56] (cf. also Bli65]), Médvédjév proved a characterization of the set of regular string languages as the smallest set of string languages which contains some elementary sets and is closed under some elementary operations. He considered this as an alternative to Kleene's characterization in terms of finite languages and the rational operations. This characterization has been generalized to the tree case in [GS84, Thm. 2.8.6] and to the weighted case for semirings in Her17] and Her20b, Ch. 5]. In this chapter we will report on the generalization to the weighted tree case following Her20b, Ch. 5], and we will go slightly beyond this.

We will prove the main theorem for semirings. However, some of the lemmas also hold for arbitrary strong bimonoids because distributivity is not needed. We note that we will use these lemmas in Chapter 19 (for bounded lattices).

### 13.1 Representable weighted tree languages

In this section we define the generalizations of the mentioned elementary sets and elementary operations to the weighted tree case. First we define the set of B-representations. It is a family of sets $\operatorname{Rep} \operatorname{Ex}(\Sigma, B)$ of expressions where $\Sigma$ is an arbitrary ranked alphabet; this provides the type of these expressions.

Formally, the family of B-representations, denoted by $\operatorname{Rep} \operatorname{Ex}(B)$, is the family

$$
\operatorname{RepEx}(\mathrm{B})=(\operatorname{RepEx}(\Sigma, \mathrm{B}) \mid \Sigma \text { ranked alphabet })
$$

defined as the smallest family $R=(R(\Sigma) \mid \Sigma$ ranked alphabet) of expressions such that Conditions (1)-(6) hold. (We recall that $\Delta$ is an arbitrary ranked alphabet.)
(1) For every $\sigma \in \Sigma$ and $b \in B$, the expression $\mathrm{RT}_{\Sigma, \sigma, b}$ is in $R(\Sigma)$.
(2) For every $k \in \mathbb{N}_{+}$, element $\widetilde{\gamma}=\left(\gamma_{1}, \ldots, \gamma_{k}\right)$ of $\Sigma^{k}$, and $b \in B$, the expression $\operatorname{NXT}_{\Sigma, \widetilde{\gamma}, b}$ is in $R(\Sigma)$.
(3) If $e_{1}, e_{2} \in R(\Sigma)$, then $e_{1}+e_{2} \in R(\Sigma)$.
(4) If $e_{1}, e_{2} \in R(\Sigma)$, then $e_{1} \times e_{2} \in R(\Sigma)$.
(5) If $e \in R(\Delta)$ and $\tau$ is a $(\Delta, \Sigma)$-tree relabeling, then $\tau(e) \in R(\Sigma)$.
(6) If $e \in R(\Sigma)$, then $\operatorname{REST}(e) \in R(\Sigma)$.

We note that in Case (5) the type of the representation changes from $\Delta$ to $\Sigma$ according to the type of the tree relabeling. Moreover, in Her17] and Her20b, Ch. 5], only deterministic tree relabelings are allowed. But since the set of recognizable weighted tree languages is closed under arbitrary tree relabelings (cf. Theorem 10.10.1), we can prove the main theorem also for a larger set of representations.

Informally, we might understand the set $\operatorname{RepEx}(B)$ as the set of expressions generated by the following EBNF; it uses the infinite family ( $e_{\Sigma} \mid \Sigma$ ranked alphabet) of nonterminals, and for each nonterminal $e_{\Sigma}$ it has the following rule:

$$
\begin{equation*}
e_{\Sigma}::=\mathrm{RT}_{\Sigma, \sigma, b}\left|\operatorname{NXT}_{\Sigma, \widetilde{\gamma}, b}\right| e_{\Sigma}+e_{\Sigma}\left|e_{\Sigma} \times e_{\Sigma}\right| \tau\left(e_{\Delta}\right) \mid \operatorname{REST}\left(e_{\Sigma}\right) \tag{13.1}
\end{equation*}
$$

where $\sigma \in \Sigma, \widetilde{\gamma} \in \Sigma^{k}$ for some $k \in \mathbb{N}_{+}, b \in B$, and $\tau$ is a ( $\Delta, \Sigma$ )-tree relabeling.
In order to perform inductive proofs or to define objects by induction, we will consider the well-founded set

$$
(\operatorname{Rep} \operatorname{Ex}(\mathrm{B}), \prec)
$$

where $\prec$ is the binary relation on $\operatorname{Rep} \operatorname{Ex}(\mathrm{B})$ defined as follows. For every $e_{1}, e_{2} \in \operatorname{Rep} \operatorname{Ex}(\mathrm{~B})$ we let $e_{1} \prec e_{2}$ if $e_{1}$ is a direct subexpression of $e_{2}$ in the sense of (13.1). The relation $\prec$ is well-founded, and $\min _{\prec}(\operatorname{Rep} \operatorname{Ex}(\mathrm{B}))$ is the set of all representations of the form $\mathrm{RT}_{\Sigma, \sigma, b}$ or $\mathrm{NXT}_{\Sigma, \widetilde{\gamma}, b}$.

Next we define the semantics of $B$-expressions by induction on $(\operatorname{RepEx}(B), \prec)$. In particular, for each $e \in \operatorname{Rep} \operatorname{Ex}(\Sigma, \mathrm{~B})$, the semantics of $e$ is a weighted tree language $\llbracket e \rrbracket: \mathrm{T}_{\Sigma} \rightarrow B$.
(1) Let $\sigma \in \Sigma$ and $b \in B$. We define $\llbracket \mathrm{RT}_{\Sigma, \sigma, b} \rrbracket: \mathrm{T}_{\Sigma} \rightarrow B$ by

$$
\llbracket \mathrm{RT}_{\Sigma, \sigma, b} \rrbracket=b \otimes \chi\left(L_{\Sigma, \sigma}\right)
$$

where $L_{\Sigma, \sigma}=\left\{\xi \in \mathrm{T}_{\Sigma} \mid \xi(\varepsilon)=\sigma\right\}$. That is, for every $\xi \in \mathrm{T}_{\Sigma}$, we have

$$
\llbracket \mathrm{RT}_{\Sigma, \sigma, b} \rrbracket(\xi)= \begin{cases}b & \text { if } \xi \in L_{\Sigma, \sigma} \\ 0 & \text { otherwise }\end{cases}
$$

A weighted tree language of the form $\llbracket \mathrm{RT}_{\Sigma, \sigma, b} \rrbracket$ is called $(\Sigma, \sigma, b)$-root mapping or simply root mapping.
(2) Let $k \in \mathbb{N}_{+}, \widetilde{\gamma}=\left(\gamma_{1}, \ldots, \gamma_{k}\right)$ be an element of $\Sigma^{k}$, and $b \in B$. We define $\llbracket \mathrm{NXT}_{\Sigma, \widetilde{\gamma}, b} \rrbracket: \mathrm{T}_{\Sigma} \rightarrow B$ by

$$
\mathrm{NXT}_{\Sigma, \widetilde{\gamma}, b}=b \otimes \chi\left(L_{\Sigma, \tilde{\gamma}}\right)
$$

where $L_{\Sigma, \tilde{\gamma}}=\left\{\xi \in \mathrm{T}_{\Sigma} \mid \operatorname{rk}(\xi(\varepsilon))=k \wedge(\forall i \in[k]): \xi(i)=\gamma_{i}\right\}$. That is, for every $\xi \in \mathrm{T}_{\Sigma}$, we have

$$
\llbracket \mathrm{NXT}_{\Sigma, \widetilde{\gamma}, b} \rrbracket(\xi)= \begin{cases}b & \text { if } \xi \in L_{\Sigma, \tilde{\gamma}} \\ 0 & \text { otherwise }\end{cases}
$$

In particular, if $\Sigma^{(k)}=\emptyset$, then $L_{\Sigma, \widetilde{\gamma}}=\emptyset$ and thus $\llbracket \mathrm{NXT}_{\Sigma, \widetilde{\gamma}, b} \rrbracket=\widetilde{\mathbb{0}}$. A weighted tree language of the form $\llbracket \mathrm{NXT}_{\Sigma, \widetilde{\gamma}, b} \rrbracket$ is called $(\Sigma, \widetilde{\gamma}, b)$-next mapping or simply next mapping.
(3) Let $e_{1}, e_{2} \in \operatorname{RepEx}(\Sigma, \mathrm{~B})$. We define $\llbracket e_{1}+e_{2} \rrbracket: \mathrm{T}_{\Sigma} \rightarrow B$ by $\llbracket e_{1}+e_{2} \rrbracket=\llbracket e_{1} \rrbracket \oplus \llbracket e_{2} \rrbracket$.
(4) Let $e_{1}, e_{2} \in \operatorname{Rep} \operatorname{Ex}(\Sigma, \mathrm{~B})$. We define $\llbracket e_{1} \times e_{2} \rrbracket: \mathrm{T}_{\Sigma} \rightarrow B$ by $\llbracket e_{1} \times e_{2} \rrbracket=\llbracket e_{1} \rrbracket \otimes \llbracket e_{2} \rrbracket$.
(5) Let $e \in \operatorname{Rep} \operatorname{Ex}(\Delta, \mathrm{~B})$ and $\tau$ is a $(\Delta, \Sigma)$-tree relabeling. We define $\llbracket \tau(e) \rrbracket: \mathrm{T}_{\Sigma} \rightarrow B$ by $\llbracket \tau(e) \rrbracket=$ $\chi(\tau)(\llbracket e \rrbracket)$.
(6) Let $e \in \operatorname{RepEx}(\Sigma, \mathrm{~B})$. We define $\llbracket \operatorname{REST}(e) \rrbracket: \mathrm{T}_{\Sigma} \rightarrow B$ for each $\xi \in \mathrm{T}_{\Sigma}$ by

$$
\llbracket \operatorname{REST}(e) \rrbracket(\xi)=\bigotimes_{\substack{w \in \operatorname{pos}(\xi) \\ \text { in } \leq \mathrm{dp} \text { order }}} \llbracket e \rrbracket\left(\left.\xi\right|_{w}\right)
$$

i.e., the factors of the product are ordered by depth-first post-order $<_{d p}$ on $\operatorname{pos}(\xi)$. The weighted tree language $\llbracket \operatorname{REST}(e) \rrbracket$ is called the restriction of $\llbracket e \rrbracket$.

We extend the binary operation + on the set of B-expressions in the natural way to an operation with a finite set $I$ of arguments $e_{i}$ with $i \in I$, and we denote this expression by $+_{i \in I} e_{i}$. The weighted tree language $\llbracket+_{i \in I} e_{i} \rrbracket$ is well defined, because $\oplus$ is associative and commutative.

Let $r: \mathrm{T}_{\Sigma} \rightarrow B$ be a weighted tree language. We say that $r$ is elementary if $r$ is a root mapping or a next mapping. Since each of the $\Sigma$-tree languages $L_{\Sigma, \sigma}$ and $L_{\Sigma, \widetilde{\gamma}}$ is recognizable, each elementary ( $\Sigma, \mathrm{B}$ )-weighted tree language is a ( $\Sigma, \mathrm{B}$ )-recognizable one-step mapping. The operations sum, Hadamard product, tree relabeling, and restriction (cf. (3)-(6), respectively) are called elementary operations.

A weighted tree language $r: \mathrm{T}_{\Sigma} \rightarrow B$ is representable if there exists an $e \in \operatorname{RepEx}(\Sigma, \mathrm{~B})$ such that $r=\llbracket e \rrbracket$. The set of all representable ( $\Sigma, \mathrm{B}$ )-weighted tree languages is denoted by $\operatorname{Rep}(\Sigma, \mathrm{B})$. Moreover, $\operatorname{Rep}(-, B)$ denotes the set of all representable $(\Sigma, B)$-weighted tree languages for some ranked alphabet $\Sigma$. Hence, $\operatorname{Rep}\left(\_, B\right)$ is the smallest set of B-weighted tree languages which contains the elementary weighted tree languages and is closed under the elementary operations.

Next we give an example of a recognizable ( $\Sigma, \mathrm{Nat}_{\mathrm{max},+}$ )-weighted tree language which is representable.
Example 13.1.1. We consider the ranked alphabet $\Sigma=\left\{\sigma^{(2)}, \gamma^{(1)}, \alpha^{(0)}\right\}$, the arctic semiring Nat max,$+=$ $\left(\mathbb{N}_{-\infty}, \max ,+,-\infty, 0\right)$, and the weighted tree language $\#_{\sigma(., \alpha)}: \mathrm{T}_{\Sigma} \rightarrow \mathbb{N}$ defined in Example 3.2.11 We recall that, for each $\xi \in \mathrm{T}_{\Sigma}$, the value $\#_{\sigma(., \alpha)}(\xi)$ is the number of occurrences of the pattern $\sigma(., \alpha)$ in $\xi$. In Example 3.2 .12 we gave a bu deterministic $\left(\Sigma\right.$, Nat $\left._{\text {max },+}\right)$-wta $\mathcal{A}$ such that $\llbracket \mathcal{A} \rrbracket(\xi)=\#_{\sigma(., \alpha)}(\xi)$ for each $\xi \in \mathrm{T}_{\Sigma}$.

Now we give an Nat $_{\text {max },+}$-representation $e$ such that $\llbracket e \rrbracket=\llbracket \mathcal{A} \rrbracket$. We define

$$
e=\operatorname{REST}\left(\left(\text { 十 }_{\kappa \in \Sigma} \operatorname{NXT}_{\Sigma, \kappa \alpha, 0}\right) \times \mathrm{RT}_{\Sigma, \sigma, 1}+\mathcal{1}_{\kappa \in \Sigma} \mathrm{RT}_{\Sigma, \kappa, 0}\right)
$$

where we have assumed that $\times$ has higher binding priority than + , and we have abbreviated $(\kappa, \alpha)$ by $\kappa \alpha$.

First, we observe that, for each $\xi \in \mathrm{T}_{\Sigma}$, we have $\llbracket \underset{\kappa \in \Sigma}{+} \mathrm{RT}_{\Sigma, \kappa, 0} \rrbracket(\xi)=0$. Second, by using the abbreviation

$$
e^{\prime}=\left(\boldsymbol{H}_{\kappa \in \Sigma} \mathrm{NXT}_{\Sigma, \kappa \alpha, 0}\right) \times \mathrm{RT}_{\Sigma, \sigma, 1}
$$

we prove by case analysis that, for each $\xi \in \mathrm{T}_{\Sigma}$,

$$
\llbracket e^{\prime} \rrbracket(\xi)= \begin{cases}1 & \text { if } \xi=\sigma\left(\xi^{\prime}, \alpha\right) \text { for some } \xi^{\prime} \in \mathrm{T}_{\Sigma} \\ -\infty & \text { otherwise }\end{cases}
$$

Case (a): Let $\xi(\varepsilon)=\alpha$. Then

$$
\begin{aligned}
\llbracket e^{\prime} \rrbracket(\xi) & =\llbracket 十_{\kappa \in \Sigma} \mathrm{NXT}_{\Sigma, \kappa \alpha, 0} \rrbracket(\xi)+\llbracket \mathrm{RT}_{\Sigma, \sigma, 1} \rrbracket(\xi) & \\
& =\llbracket \boldsymbol{W}_{\kappa \in \Sigma} \mathrm{NXT}_{\Sigma, \kappa \alpha, 0} \rrbracket(\xi)+-\infty & \quad \text { (because } \xi(\varepsilon)=\alpha) \\
& =-\infty &
\end{aligned}
$$

Case (b): Let $\xi(\varepsilon)=\gamma$. In a similar way as in Case (a), we can calculate that $\llbracket e^{\prime} \rrbracket(\xi)=-\infty$.
$\overline{\text { Case (c): }}$ Let $\xi(\varepsilon)=\sigma$ and $\xi(2) \neq \alpha$. Then

$$
\begin{aligned}
\llbracket e^{\prime} \rrbracket(\xi) & =\max \left(\llbracket \mathrm{NXT}_{\Sigma, \sigma \alpha, 0} \rrbracket(\xi), \llbracket \mathrm{NXT}_{\Sigma, \gamma \alpha, 0} \rrbracket(\xi), \llbracket \mathrm{NXT}_{\Sigma, \alpha \alpha, 0} \rrbracket(\xi)\right)+\llbracket \mathrm{RT}_{\Sigma, \sigma, 1} \rrbracket(\xi) \\
& =\max (-\infty,-\infty,-\infty)+1=-\infty
\end{aligned}
$$

Case (d): Let $\xi(\varepsilon)=\sigma, \xi(2)=\alpha$, and $\xi(1)=\sigma$. Then

$$
\begin{aligned}
\llbracket e^{\prime} \rrbracket(\xi) & =\max \left(\llbracket \mathrm{NXT}_{\Sigma, \sigma \alpha, 0} \rrbracket(\xi), \llbracket \mathrm{NXT}_{\Sigma, \gamma \alpha, 0} \rrbracket(\xi), \llbracket \mathrm{NXT}_{\Sigma, \alpha \alpha, 0} \rrbracket(\xi)\right)+\llbracket \mathrm{RT}_{\Sigma, \sigma, 1} \rrbracket(\xi) \\
& =\max (0,-\infty,-\infty)+1=0+1=1
\end{aligned}
$$



$$
\begin{aligned}
\llbracket e^{\prime} \rrbracket(\xi) & =\max \left(\llbracket \mathrm{NXT}_{\Sigma, \sigma \alpha, 0} \rrbracket(\xi), \llbracket \mathrm{NXT}_{\Sigma, \gamma \alpha, 0} \rrbracket(\xi), \llbracket \mathrm{NXT}_{\Sigma, \alpha \alpha, 0} \rrbracket(\xi)\right)+\llbracket \mathrm{RT}_{\Sigma, \sigma, 1} \rrbracket(\xi) \\
& =\max (-\infty, 0,-\infty)+1=0+1=1
\end{aligned}
$$



$$
\begin{aligned}
\llbracket e^{\prime} \rrbracket(\xi) & =\max \left(\llbracket \mathrm{NXT}_{\Sigma, \sigma \alpha, 0} \rrbracket(\xi), \llbracket \mathrm{NXT}_{\Sigma, \gamma \alpha, 0} \rrbracket(\xi), \llbracket \mathrm{NXT}_{\Sigma, \alpha \alpha, 0} \rrbracket(\xi)\right)+\llbracket \mathrm{RT}_{\Sigma, \sigma, 1} \rrbracket(\xi) \\
& =\max (-\infty,-\infty, 0)+1=0+1=1
\end{aligned}
$$

Hence, for each $\xi \in \mathrm{T}_{\Sigma}$, we have

$$
\llbracket e^{\prime}+\mathcal{1}_{\kappa \in \Sigma} \operatorname{RT}_{\Sigma, \kappa, 0} \rrbracket(\xi)= \begin{cases}1 & \text { if } \xi=\sigma\left(\xi^{\prime}, \alpha\right) \text { for some } \xi^{\prime} \in \mathrm{T}_{\Sigma}  \tag{13.2}\\ 0 & \text { otherwise }\end{cases}
$$

Finally, for each $\xi \in \mathrm{T}_{\Sigma}$, we can calculate as follows:

$$
\begin{aligned}
\llbracket e \rrbracket(\xi) & =\llbracket \operatorname{REST}\left(e^{\prime}+\mathcal{T}_{\kappa \in \Sigma} \mathrm{RT}_{\Sigma, \kappa, 0}\right) \rrbracket(\xi)={\underset{w \in \operatorname{pos}(\xi)}{ } \llbracket e^{\prime}+\mathcal{T}_{\kappa \in \Sigma} \mathrm{RT}_{\Sigma, \kappa, 0} \rrbracket\left(\left.\xi\right|_{w}\right)}={\underset{\substack{w \in \operatorname{pos}(\xi): \\
\xi(w)=\sigma, \xi(w 2)=\alpha}}{ } 1}=\#_{\sigma(., \alpha)}(\xi)
\end{aligned}
$$

It turns out that each recognizable $(\Sigma, \mathrm{B})$-weighted tree language is representable, but not vice versa as the next example shows.
Example 13.1.2. Her20b, Ex. 5.2.1] We consider the string ranked alphabet $\Sigma=\left\{\gamma^{(1)}, \alpha^{(0)}\right\}$ and the semiring Nat $=(\mathbb{N},+, \cdot, 0,1)$ of natural numbers. Moreover, we let $r: \mathrm{T}_{\Sigma} \rightarrow \mathbb{N}$ such that $r\left(\gamma^{n}(\alpha)\right)=$ $2^{(n+1)^{2}}$ for each $n \in \mathbb{N}$.

We show that there does not exist a $(\Sigma, N$ at $)$-wta $\mathcal{A}$ such that $\llbracket \mathcal{A} \rrbracket=r$. For this, we assume that there exists such a $(\Sigma, \mathbb{N})$-wta. Then, by Lemma 3.5.6, there exists $K \in \mathbb{N}$ such that, for each $n \in \mathbb{N}$, we have $\llbracket \mathcal{A} \rrbracket\left(\gamma^{n}(\alpha)\right)=\llbracket e \rrbracket\left(\gamma^{n}(\alpha)\right)=2^{(n+1)^{2}} \leq K^{n+1}$. However, this is a contradiction, because if $n$ is big enough, then $2^{(n+1)^{2}}>K^{n+1}$.

Next we consider the $\mathbb{N}$-representation

$$
\left.e=\operatorname{REST}\left(\mathrm{RT}_{\Sigma, \gamma, 2}+\operatorname{RT}_{\Sigma, \alpha, 2}\right) \times \operatorname{REST}\left(\operatorname{REST}^{2} \mathrm{RT}_{\Sigma, \gamma, 4}+\mathrm{RT}_{\Sigma, \alpha, 1}\right)\right)
$$

Let $n \in \mathbb{N}$. Then we can calculate as follows (where we denote by $\Pi$ the extension of the binary operation - to finite numbers of arguments):

$$
\begin{aligned}
& \llbracket \operatorname{REST}\left(\mathrm{RT}_{\Sigma, \gamma, 2}+\mathrm{RT}_{\Sigma, \alpha, 2}\right) \rrbracket\left(\gamma^{n}(\alpha)\right) \\
= & \prod_{w \in \operatorname{pos}(\xi)} \llbracket \mathrm{RT}_{\Sigma, \gamma, 2}+\mathrm{RT}_{\Sigma, \alpha, 2} \rrbracket\left(\left.\gamma^{n}(\alpha)\right|_{w}\right) \quad \text { (note that } \cdot \text { is commutative) } \\
= & \llbracket \mathrm{RT}_{\Sigma, \gamma, 2}+\mathrm{RT}_{\Sigma, \alpha, 2} \rrbracket(\alpha) \cdot \prod_{w \in \operatorname{pos}(\xi) \backslash\left\{1^{n}\right\}} \llbracket \mathrm{RT}_{\Sigma, \gamma, 2}+\mathrm{RT}_{\Sigma, \alpha, 2} \rrbracket\left(\left.\gamma^{n}(\alpha)\right|_{w}\right) \\
= & \llbracket \mathrm{RT}_{\Sigma, \alpha, 2} \rrbracket(\alpha) \cdot \prod_{w \in \operatorname{pos}(\xi) \backslash\left\{1^{n}\right\}}^{\llbracket \mathrm{RT}_{\Sigma, \gamma, 2} \rrbracket\left(\left.\gamma^{n}(\alpha)\right|_{w}\right)} \\
= & 2 \cdot \prod_{w \in \operatorname{pos}(\xi) \backslash\left\{1^{n}\right\}} 2=2^{n+1} .
\end{aligned}
$$

In a similar way we can calculate that $\llbracket \operatorname{REST}\left(\operatorname{RT}_{\Sigma, \gamma, 4}+\operatorname{RT}_{\Sigma, \alpha, 1}\right) \rrbracket\left(\gamma^{n}(\alpha)\right)=4^{n}$. Using Gaussian sum, we have

$$
\begin{aligned}
\operatorname{REST}\left(\operatorname{REST}\left(\operatorname{RT}_{\Sigma, \gamma, 4}+\operatorname{RT}_{\Sigma, \alpha, 1}\right)\right)\left(\gamma^{n}(\alpha)\right) & =\prod_{w \in \operatorname{pos}(\xi)} \operatorname{REST}\left(\operatorname{RT}_{\Sigma, \gamma, 4}+\operatorname{RT}_{\Sigma, \alpha, 1}\right)\left(\left.\gamma^{n}(\alpha)\right|_{w}\right) \\
& =1 \cdot 4^{1} \cdot 4^{2} \cdot \ldots \cdot 4^{n}=4^{\frac{n^{2}+n}{2}}=2^{n^{2}+n}
\end{aligned}
$$

Finally, $\llbracket e \rrbracket\left(\gamma^{n}(\alpha)\right)=2^{n+1} \cdot 2^{n^{2}+n}=2^{(n+1)^{2}}$.
Hence, for a characterization of $\operatorname{Rec}(\Sigma, B)$ in terms of representations, we have to restrict the set $\operatorname{Rep}(\Sigma, \mathrm{B})$ (cf. Her20b Sec. 5.2]). We define the restriction in a purely syntactic way (in contrast to Her17, Her20b). In fact, we define two restrictions: restricted representations and $\times$-restricted representations. Intuitively,

- in a restricted representation, (a) REST may not occur nested and (b) each tree relabeling occurring in a subexpression of the form $\operatorname{REST}(e)$ must be non-overlapping, and
- a $\times$-restricted representation is a restricted representation such that, in each subexpression of the form $e_{1} \times e_{2}$, there exists at least one $i \in\{1,2\}$ such that $e_{i}$ does not contain REST and each tree relabeling in $e_{i}$ is non-overlapping.
Formally, let $\operatorname{RepEx}^{r}(\mathrm{~B})$ be the subset of $\operatorname{Rep} \operatorname{Ex}(\mathrm{B})$ generated by the following EBNF; it uses the infinite family $\left(e_{\Sigma}^{r} \mid \Sigma\right.$ ranked alphabet $) \cup\left(f_{\Sigma}^{r} \mid \Sigma\right.$ ranked alphabet) of nonterminals, and for every nonterminals $e_{\Sigma}^{r}$ and $f_{\Sigma}^{r}$ it has the following rules:

$$
\begin{align*}
& e_{\Sigma}^{r}::=\operatorname{RT}_{\Sigma, \sigma, b}\left|\operatorname{NXT}_{\Sigma, \widetilde{\gamma}, b}\right| e_{\Sigma}^{r}+e_{\Sigma}^{r}\left|e_{\Sigma}^{r} \times e_{\Sigma}^{r}\right| \tau\left(e_{\Delta}^{r}\right) \mid \operatorname{REST}\left(f_{\Sigma}^{r}\right) \\
& f_{\Sigma}^{r}::=\operatorname{RT}_{\Sigma, \sigma, b}\left|\operatorname{NXT}_{\Sigma, \widetilde{\gamma}, b}\right| f_{\Sigma}^{r}+f_{\Sigma}^{r}\left|f_{\Sigma}^{r} \times f_{\Sigma}^{r}\right| \tau^{\prime}\left(f_{\Delta}^{r}\right) \tag{13.3}
\end{align*}
$$

where $\sigma \in \Sigma, \widetilde{\gamma} \in \Sigma^{k}$ for some $k \in \mathbb{N}_{+}, b \in B, \tau$ is a $(\Delta, \Sigma)$-tree relabeling, and $\tau^{\prime}$ is a non-overlapping $(\Delta, \Sigma)$-tree relabeling. We call each element of $\operatorname{RepEx}^{r}(\mathrm{~B})$ a restricted representation.

Let $\operatorname{RepEx}{ }^{\times r}(\mathrm{~B})$ be the subset of $\operatorname{RepEx}^{r}(\mathrm{~B})$ generated by the following EBNF; it uses the infinite family ( $e_{\Sigma}^{\times r} \mid \Sigma$ ranked alphabet $) \cup\left(f_{\Sigma}^{r} \mid \Sigma\right.$ ranked alphabet) of nonterminals, and for every nonterminals $e_{\Sigma}^{\times r}$ and $f_{\Sigma}^{r}$ it has the following rules:

$$
\begin{align*}
e_{\Sigma}^{\times r}:: & =\mathrm{RT}_{\Sigma, \sigma, b}\left|\mathrm{NXT}_{\Sigma, \tilde{\gamma}, b}\right| e_{\Sigma}^{\times r}+e_{\Sigma}^{\times r}\left|e_{\Sigma}^{\times r} \times f_{\Sigma}^{r}\right| f_{\Sigma}^{r} \times e_{\Sigma}^{\times r}\left|f_{\Sigma}^{r} \times f_{\Sigma}^{r}\right| \tau\left(e_{\Delta}^{\times r}\right) \mid \operatorname{REST}\left(f_{\Sigma}^{r}\right)  \tag{13.4}\\
f_{\Sigma}^{r}:: & =\mathrm{RT}_{\Sigma, \sigma, b}\left|\operatorname{NXT}_{\Sigma, \widetilde{\gamma}, b}\right| f_{\Sigma}^{r}+f_{\Sigma}^{r}\left|f_{\Sigma}^{r} \times f_{\Sigma}^{r}\right| \tau^{\prime}\left(f_{\Delta}^{r}\right)
\end{align*}
$$

where $\sigma \in \Sigma, \widetilde{\gamma} \in \Sigma^{k}$ for some $k \in \mathbb{N}_{+}, b \in B, \tau$ is a $(\Delta, \Sigma)$-tree relabeling, and $\tau^{\prime}$ is a non-overlapping $(\Delta, \Sigma)$-tree relabeling. We call each element of $\operatorname{RepEx}^{\times r}(\mathrm{~B})$ a $\times$-restricted representation. We note that the rules for the nonterminals $f_{\Sigma}^{r}$ in (13.3) and (13.4) are identical. Clearly, $\operatorname{RepEx}{ }^{\times r}(\mathrm{~B}) \subset \operatorname{RepEx}^{r}(\mathrm{~B}) \subset$ $\operatorname{RepEx}(\mathrm{B})$.

A weighted tree language $r: \mathrm{T}_{\Sigma} \rightarrow B$ is restricted representable (and $\times$-restricted representable) if there exists an $e \in \operatorname{RepEx}^{r}(\Sigma, \mathrm{~B})$ (and an $e \in \operatorname{RepEx}^{\times r}(\Sigma, \mathrm{~B})$, respectively) such that $r=\llbracket e \rrbracket$.
Corollary 13.1.3. Let $e \in{\operatorname{Rep} \operatorname{Ex}^{r}}^{r}(\Sigma, \mathrm{~B})$ be generated by nonterminals of the form $f_{\Sigma}^{r}$ using the rules in (13.3). Then we can construct $n \in \mathbb{N}_{+}, b_{1}, \ldots, b_{n} \in B$, and $\Sigma$-fta $A_{1}, \ldots, A_{n}$ such that $\llbracket e \rrbracket=$ $\bigoplus_{i \in[n]} b_{i} \otimes \chi\left(\mathrm{~L}\left(A_{i}\right)\right)$. Thus, in particular, $\llbracket e \rrbracket$ is a recognizable step mapping.

Proof. We prove the statement by induction on $\left(\operatorname{RepEx}(\mathrm{B})^{\prime}, \prec^{\prime}\right)$, where $\operatorname{RepEx}(\mathrm{B})^{\prime}$ is the set of all representations which can be generated by nonterminals of the form $f_{\Sigma}^{r}$ using the rules in (13.3), and $\prec^{\prime}=\left.\prec\right|_{\operatorname{RepEx}(B)^{\prime} \times \operatorname{RepEx}(\mathrm{B})^{\prime} .}$
I.B.: For each $e \in \operatorname{RepEx}(\mathrm{~B})^{\prime}$ of the form $e=\mathrm{RT}_{\Sigma, \sigma, b}$ or $e=\mathrm{NXT}_{\Sigma, \widetilde{\gamma}, b}$, we can easily construct the $\Sigma$-fta $A_{\Sigma, \sigma}$ and $A_{\Sigma, \tilde{\gamma}}$ such that $\mathrm{L}\left(A_{\Sigma, \sigma}\right)=L_{\Sigma, \sigma}$ and $\mathrm{L}\left(A_{\Sigma, \tilde{\gamma}}\right)=L_{\Sigma, \tilde{\gamma}}$. Then $\llbracket \mathrm{RT}_{\Sigma, \sigma, b} \rrbracket=b \otimes \chi\left(\mathrm{~L}\left(A_{\Sigma, \sigma}\right)\right)$ and $\llbracket \mathrm{NXT}_{\Sigma, \widetilde{\gamma}, b} \rrbracket=b \otimes \chi\left(\mathrm{~L}\left(A_{\Sigma, \tilde{\gamma}}\right)\right)$.
I.S.: It follows from Theorem $10.3 .1(\mathrm{~B}) \Rightarrow(\mathrm{A})$ and

- Theorem 10.1.1(2) (for the sum),
- Theorem 10.4.1 (3) (for the Hadamard product), and
- Theorem 10.10.1 (for the non-overlapping tree relabeling).

For instance, the $\mathrm{Nat}_{\text {max },+}$-representation

$$
e=\operatorname{REST}\left(\left(\text { 十 }_{\kappa \in \Sigma} \operatorname{NXT}_{\Sigma, \kappa \alpha, 0}\right) \times \mathrm{RT}_{\Sigma, \sigma, 1}+\mathcal{F}_{\kappa \in \Sigma} \mathrm{RT}_{\Sigma, \kappa, 0}\right)
$$

shown in Example 13.1.1 is $\times$-restricted.
Now we can formulate the main result of this chapter.

Theorem 13.1.4. (cf. Her17, Thm. 6] and Her20b, Thm. 5.2.2]) Let $\Sigma$ be a ranked alphabet, B be a semiring, and $r: \mathrm{T}_{\Sigma} \rightarrow B$. Then the following two statements are equivalent.
(A) We can construct a $(\Sigma, \mathrm{B})-w t a \mathcal{A}$ such that $\llbracket \mathcal{A} \rrbracket=r$.
(B) We can construct an $e \in \operatorname{RepEx}^{\times r}(\Sigma, \mathrm{~B})$ such that $\llbracket e \rrbracket=r$.

Moreover, if B is commutative, then $(A)$ and ( $B$ ) are equivalent to:
(C) We can construct an $e \in \operatorname{RepEx}^{r}(\Sigma, \mathrm{~B})$ such that $\llbracket e \rrbracket=r$.

The proof of this theorem follows from Lemmas 13.2 .2 and 13.3 .2 , which will be shown in the next two sections.

We note that in Her17] and Her20a, Sect. 5.3], a comparison between representable weighted tree languages and weighted tree languages defined by the weighted MSO-logic of [DG05, DG07, DG09] has been started.

### 13.2 From recognizable to restricted representable

Since each wta can be decomposed into a deterministic tree relabeling and a weighted local system (cf. Theorem 11.2.6) and the set $\operatorname{Rep}(-, B)$ is closed under tree relabelings (by definition), we first prove that each weighted tree language determined by a weighted local system is $\times$-restricted representable.

Lemma 13.2.1. Let $\mathcal{S}$ be a $(\Sigma, \mathrm{B})$-wls. Then we can construct an $e \in \operatorname{RepEx}{ }^{\times r}(\Sigma, \mathrm{~B})$ such that $\llbracket e \rrbracket=\llbracket \mathcal{S} \rrbracket$.
Proof. Let $\mathcal{S}=(g, F)$ be a $(\Sigma, \mathrm{B})$-wls. The idea for the construction of $e \in \operatorname{RepEx}^{\times r}(\Sigma, \mathrm{~B})$ is to simulate the weight $g_{k}\left(\left(\sigma_{1} \cdots \sigma_{k}, \sigma\right)\right)$ of the fork $\left(\sigma_{1} \cdots \sigma_{k}, \sigma\right)$ by the representation $\mathrm{NXT}_{\Sigma,\left(\sigma_{1}, \ldots, \sigma_{k}\right), \mathbb{1}} \times$ $\mathrm{RT}_{\Sigma, \sigma, g_{k}\left(\sigma_{1} \cdots \sigma_{k}, \sigma\right)}$ if $k \geq 1$, and by the representation $\mathrm{RT}_{\Sigma, \sigma, g_{0}(\varepsilon, \sigma)}$ if $k=0$.

Formally, we define the B-representation $e=\operatorname{REST}\left(e_{1}+e_{2}\right) \times e_{3}$ where

$$
\begin{gathered}
e_{1}=\mathcal{F}_{\sigma \in \Sigma^{(0)}} \mathrm{RT}_{\Sigma, \sigma, g_{0}(\varepsilon, \sigma)} \quad \text { and } \quad e_{2}=\mathcal{W}_{\substack{k \geq 1, \sigma \in \Sigma^{(k)}, \sigma_{1}, \ldots, \sigma_{k} \in \Sigma^{\prime}}}\left(\mathrm{NXT}_{\Sigma,\left(\sigma_{1}, \ldots, \sigma_{k}\right), \mathbb{1}} \times \mathrm{RT}_{\Sigma, \sigma, g_{k}\left(\sigma_{1} \cdots \sigma_{k}, \sigma\right)}\right) \\
\text { and } e_{3}=\mathcal{W}_{\sigma \in \Sigma} \mathrm{RT}_{\Sigma, \sigma, F(\sigma)}
\end{gathered}
$$

It is easy to see that $e$ is $\times$-restricted. It remains to show that $\llbracket \mathcal{S} \rrbracket=\llbracket e \rrbracket$. Obviously,

$$
\begin{equation*}
\text { for every } \xi \in \mathrm{T}_{\Sigma} \text {, we have } \llbracket e_{3} \rrbracket(\xi)=F(\xi(\varepsilon)) \text {. } \tag{13.5}
\end{equation*}
$$

Moreover,
for every $\xi \in \mathrm{T}_{\Sigma}$ and $w \in \operatorname{pos}(\xi)$, we have

$$
\begin{equation*}
\llbracket e_{1}+e_{2} \rrbracket\left(\left.\xi\right|_{w}\right)=g_{\mathrm{rk}(\xi(w))}(\xi(w 1) \cdots \xi(w \operatorname{rk}(\xi(w))), \xi(w)) \tag{13.6}
\end{equation*}
$$

Then, for each $\xi \in \mathrm{T}_{\Sigma}$, we can calculate as follows.

$$
\begin{aligned}
\llbracket \operatorname{REST}\left(e_{1}+e_{2}\right) \times e_{3} \rrbracket(\xi)= & \left(\bigotimes_{\substack{w \in \operatorname{pos}(\xi), \\
\text { in } \leq \operatorname{dp} \text { order }}} \llbracket e_{1}+e_{2} \rrbracket\left(\left.\xi\right|_{w}\right)\right) \otimes \llbracket e_{3} \rrbracket(\xi) \\
= & \left(\bigotimes_{\substack{w \in \operatorname{pos}(\xi), \\
\text { in } \leq \operatorname{dp} \text { order }\\
}} g_{\mathrm{rk}(\xi(w))}(\xi(w 1) \cdots \xi(w \operatorname{rk}(\xi(w))), \xi(w))\right) \otimes \llbracket e_{3} \rrbracket(\xi) \quad(\text { by (13.6) }) \\
= & g(\xi) \otimes F(\xi(\varepsilon)) \quad \quad \text { (by definition of } g \text { and (13.5)) } \\
= & \llbracket \mathcal{S} \rrbracket(\xi) .
\end{aligned}
$$

Lemma 13.2.2. (cf. Her17, Lm. 12] and Her20b, Thm. 5.2.8]) Let $\mathcal{A}$ be a ( $\Sigma, \mathrm{B}$ )-wta. Then we can construct an $e \in \operatorname{RepEx}^{\times r}(\Sigma, \mathrm{~B})$ such that $\llbracket \mathcal{A} \rrbracket^{\mathrm{run}}=\llbracket e \rrbracket$.

Proof. By Theorem 11.2.6, we can construct a ranked alphabet $\Theta$, a deterministic $(\Theta, \Sigma)$-tree relabeling $\tau$, and a $(\Theta, \mathrm{B})$-weighted local system $\mathcal{S}$ with unit root weights such that $\llbracket \mathcal{A} \rrbracket^{\text {run }}=\chi(\tau)(\llbracket \mathcal{S} \rrbracket)$. By Lemma 13.2.1, we can construct an $e^{\prime} \in \operatorname{RepEx}^{\times r}(\Sigma, \mathrm{~B})$ such that $\llbracket e^{\prime} \rrbracket=\llbracket \mathcal{S} \rrbracket$. Since $e^{\prime}$ is a $\times$-restricted representation, also $\tau\left(e^{\prime}\right)$ is a $\times$-restricted representation. Moreover, $\llbracket \mathcal{A} \rrbracket^{\text {run }}=\chi(\tau)(\llbracket \mathcal{S} \rrbracket)=\chi(\tau)\left(\llbracket e^{\prime} \rrbracket\right)=$ $\llbracket \tau\left(e^{\prime}\right) \rrbracket$. Thus we can choose $e=\tau\left(e^{\prime}\right)$.

### 13.3 From restricted representable to recognizable

Here we prove that each $\times$-restricted representable weighted tree language is recognizable; moreover, if B is commutative, then each restricted representable weighted tree language is recognizable (cf. Lemma 13.3.2). As to be expected, the proof is by induction on the structure of representations.

We start with a lemma which shows the effect of the restriction of recognizable step mappings. One might compare the construction of this lemma with the one of [DV06, Lm. 5.5] (based in [DG07, Lm. 4.4]) in which the following is proved for a formula $\varphi$ of weighted monadic second order logic: if $\llbracket \varphi \rrbracket$ is a recognizable step mapping, then $\llbracket \forall x . \varphi \rrbracket$ is recognizable (cf. Lemma 14.4.16).

Lemma 13.3.1. (cf. Her17, Lm. 10] and Her20b, Lm. 5.2.5]) Let $e \in \operatorname{RepEx}(\Sigma, \mathrm{~B})$ such that there exists a crisp deterministic $(\Sigma, \mathrm{B})$-wta with $\llbracket \mathcal{A} \rrbracket=\llbracket e \rrbracket$. We can construct a $(\Sigma, \mathrm{B})$-wta $\mathcal{B}$ such that $\llbracket \mathcal{B} \rrbracket^{\text {run }}=\llbracket \operatorname{REST}(e) \rrbracket$.

Proof. By Lemma 4.3.1, for each $\xi \in \mathrm{T}_{\Sigma}$, there exists $q \in Q$ such that $Q_{\neq \mathcal{A}}^{\mathrm{R}_{\mathcal{A}}(\xi)}=\{q\}$ and there exists exactly one $\rho \in \mathrm{R}_{\mathcal{A}}(q, \xi)$ such that $\mathrm{wt}_{\mathcal{A}}(\xi, \rho) \neq \mathbb{O}$ and for this $\rho$ we have $\mathrm{wt}_{\mathcal{A}}(\xi, \rho)=\mathbb{1}$. We denote this unique run by $\rho_{\xi}$. Thus,
for every $\xi \in \mathrm{T}_{\Sigma}$ and $w \in \operatorname{pos}(\xi)$, we have

$$
\begin{equation*}
\delta_{\operatorname{rk}(\xi(w))}\left(\rho_{\xi}(w 1) \cdots \rho_{\xi}(w \operatorname{rk}(\xi(w))), \xi(w), \rho_{\xi}(w)\right)=\mathbb{1} \text { and } \mathrm{wt}_{\mathcal{A}}\left(\left.\xi\right|_{w},\left.\left(\rho_{\xi}\right)\right|_{w}\right)=\mathbb{1} \tag{13.7}
\end{equation*}
$$

Hence,

$$
\begin{equation*}
\text { for every } \xi \in \mathrm{T}_{\Sigma} \text { and } w \in \operatorname{pos}(\xi) \text {, we have } \llbracket \mathcal{A} \rrbracket\left(\left.\xi\right|_{w}\right)=F_{\rho_{\xi}(w)} \tag{13.8}
\end{equation*}
$$

We construct the $(\Sigma, \mathrm{B})$-wta $\mathcal{B}$ such that $\llbracket \mathcal{B} \rrbracket^{\text {run }}=\llbracket \operatorname{REST}(e) \rrbracket$ using the following idea. On a given input tree $\xi$, the wta $\mathcal{B}$ simulates the state behaviour of $\mathcal{A}$ and, at each position $w$ of $\xi$, the weight of the transition $t$ which $\mathcal{A}$ applies at $w$, is the product of $\delta_{k}(t) \in\{\mathbb{O}, \mathbb{1}\}($ where $k=\operatorname{rk}(\xi(w)))$ and the root weight of the target state of $t$. Formally, we let $\mathcal{B}=\left(Q, \delta^{\prime}, F^{\prime}\right)$ with $\left(F^{\prime}\right)_{q}=\mathbb{1}$ for each $q \in Q$, and for every $k \in \mathbb{N}, \sigma \in \Sigma^{(k)}$, and $q, q_{1}, \ldots, q_{k} \in Q$ we let

$$
\left(\delta^{\prime}\right)_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right)=\delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right) \otimes F_{q}
$$

We recall that $\delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right) \in\{\mathbb{O}, \mathbb{1}\}$. Obviously, $\mathcal{B}$ is bu deterministic, but it is not crisp deterministic, and not even total (because B need not be zero-divisor free). Also, for each $\xi \in \mathrm{T}_{\Sigma}$, we have $\mathrm{R}_{\mathcal{A}}(\xi)=\mathrm{R}_{\mathcal{B}}(\xi)$ and

$$
\begin{equation*}
\text { for each } \rho \in \mathrm{R}_{\mathcal{B}}(\xi): \text { if } \rho \neq \rho_{\xi}, \text { then } \operatorname{wt}_{\mathcal{B}}(\xi, \rho)=\mathbb{0} \tag{13.9}
\end{equation*}
$$

Then, for each $\xi \in \mathrm{T}_{\Sigma}$, we can calculate as follows:

$$
\begin{aligned}
& \llbracket \mathcal{B} \rrbracket^{\mathrm{run}}(\xi)=\bigoplus_{\rho \in \mathrm{R}_{\mathcal{B}}(\xi)} \mathrm{wt}_{\mathcal{B}}(\xi, \rho) \otimes\left(F^{\prime}\right)_{\rho(\varepsilon)} \\
& =\mathrm{wt}_{\mathcal{B}}\left(\xi, \rho_{\xi}\right) \quad \text { (by (13.91)) } \\
& =\bigotimes_{\substack{w \in \operatorname{pos}(\xi) \\
\text { in } \leq \operatorname{dap} \text { order }}}\left(\delta^{\prime}\right)_{\operatorname{rk}(\xi(w))}\left(\rho_{\xi}(w 1) \cdots \rho_{\xi}(w \operatorname{rk}(\xi(w))), \xi(w), \rho_{\xi}(w)\right) \quad \text { (by Observation 3.1.1) } \\
& =\bigotimes_{\substack{w \in \operatorname{pos}(\xi) \\
\text { in } \leq \operatorname{dord} \text { order }}} \delta_{\mathrm{rk}(\xi(w))}\left(\rho_{\xi}(w 1) \cdots \rho_{\xi}(w \operatorname{rk}(\xi(w))), \xi(w), \rho_{\xi}(w)\right) \otimes F_{\rho_{\xi}(w)} \\
& =\bigotimes_{\substack{w \in \operatorname{pos}(\xi) \\
\text { in } \leq \text { dp order }}} F_{\rho_{\xi}(w)} \\
& \text { (by (13.7)) } \\
& =\bigotimes_{\substack{w \in \operatorname{pos}(\xi) \\
\text { in } \leq \operatorname{dp} \text { order }}} \llbracket \mathcal{A} \rrbracket\left(\left.\xi\right|_{w}\right) \\
& =\bigotimes_{\substack{w \in \text { pos }(\xi) \\
\text { in } \leq \text { dp order }}} \llbracket e \rrbracket\left(\left.\xi\right|_{w}\right)=\llbracket \operatorname{REST}(e) \rrbracket(\xi) .
\end{aligned}
$$

In the next lemma, we will need distributivity, because the Hadamard product is one of the elementary operations, and distributivity is needed in order to guarantee closure of the set of recognizable weighted tree languages under Hadamard product.
Lemma 13.3.2. (cf. Her17, Lm. 11] and Her20b, Thm. 5.2.7]) Let B be a semiring and $r: \mathrm{T}_{\Sigma} \rightarrow B$. Then the following two statements hold.
(1) For each $e \in \operatorname{Rep} \operatorname{Ex}^{\times r}(\Sigma, \mathrm{~B})$, we can construct a $(\Sigma, \mathrm{B})$-wta $\mathcal{A}$ such that $\llbracket \mathcal{A} \rrbracket=\llbracket e \rrbracket$.
(2) Let B be commutative. For each $e \in \operatorname{RepEx}^{r}(\Sigma, \mathrm{~B})$, we can construct a $(\Sigma, \mathrm{B})$-wta $\mathcal{A}$ such that $\llbracket \mathcal{A} \rrbracket=\llbracket e \rrbracket$.

Proof. Proof of (1): We prove the statement by induction on the well-founded set $\left(\operatorname{RepEx}^{\times r}(\mathrm{~B}), \prec^{\prime}\right)$ which is defined in an obvious way as restriction of $(\operatorname{RepEx}(B), \prec)$.
I.B.: Let $e=\operatorname{RT}_{\Sigma, \sigma, b}$ or $e=\operatorname{NXT}_{\Sigma, \tilde{\gamma}, b}$. Then the statement follows from Corollary 13.1.3 and Theorem $10.3 .1(\mathrm{~B}) \Rightarrow(\mathrm{A})$.
I.S.: We distinguish four cases.

Case (a): Let $e=e_{1}+e_{2}$ for some $e_{1}, e_{2} \in \operatorname{RepEx}{ }^{\times r}(\Sigma, \mathrm{~B})$, and let $\mathcal{A}_{1}$ and $\mathcal{A}_{2}$ be ( $\left.\Sigma, \mathrm{B}\right)$-wta such that $\llbracket e_{1} \rrbracket=\llbracket \mathcal{A}_{1} \rrbracket$ and $\llbracket e_{2} \rrbracket=\llbracket \mathcal{A}_{2} \rrbracket$. Then by Theorem 10.1.1(1) we can construct a $(\Sigma, \mathrm{B})$-wta $\mathcal{A}$ such that $\llbracket \mathcal{A} \rrbracket=\llbracket \mathcal{A}_{1} \rrbracket \oplus \llbracket \mathcal{A}_{2} \rrbracket=\llbracket e_{1} \rrbracket \oplus \llbracket e_{2} \rrbracket=\llbracket e_{1}+e_{2} \rrbracket=\llbracket e \rrbracket$.

Case (b): Let $e=e_{1} \times e_{2}$ for some $e_{1}, e_{2} \in \operatorname{RepEx}^{\times r}(\Sigma, \mathrm{~B})$. We distinguish three cases.
Case (b1): Let $\mathcal{A}_{1}$ be a $(\Sigma, \mathrm{B})$-wta such that $\llbracket \mathcal{A}_{1} \rrbracket=\llbracket e_{1} \rrbracket$ and let $e_{2}$ be generated by nonterminals of the form $f_{\Sigma}^{r}$ using the rules in (13.3).

By Corollary 13.1.3, we can construct $n \in \mathbb{N}_{+}, b_{1}, \ldots, b_{n} \in B$, and $\Sigma$-fta $A_{1}, \ldots, A_{n}$ such that $\llbracket e_{2} \rrbracket=\bigoplus_{i \in[n]} b_{i} \otimes \chi\left(\mathrm{~L}\left(A_{i}\right)\right)$. By Theorem $10.3 .1(\mathrm{~B}) \Rightarrow(\mathrm{A})$, we can construct a crisp deterministic ( $\left.\Sigma, \mathrm{B}\right)$ wta $\mathcal{A}_{2}$ such that $\llbracket \mathcal{A}_{2} \rrbracket=\llbracket e_{2} \rrbracket$. Since B is right-distributive, by Theorem 10.4.3(1), we can construct a $(\Sigma, \mathrm{B})$-wta $\mathcal{A}$ such that $\llbracket \mathcal{A} \rrbracket=\llbracket \mathcal{A}_{1} \rrbracket \otimes \llbracket \mathcal{A}_{2} \rrbracket$. Then $\llbracket \mathcal{A} \rrbracket=\llbracket \mathcal{A}_{1} \rrbracket \otimes \llbracket \mathcal{A}_{2} \rrbracket=\llbracket e_{1} \rrbracket \otimes \llbracket e_{2} \rrbracket=\llbracket e_{1} \times e_{2} \rrbracket=\llbracket e \rrbracket$.

Case (b2): Let $e_{1}$ be generated by nonterminals of the form $f_{\Sigma}^{r}$ using the rules in (13.3) and let $\mathcal{A}_{2}$ be a $(\Sigma, \mathrm{B})$-wta such that $\llbracket \mathcal{A}_{2} \rrbracket=\llbracket e_{2} \rrbracket$.

As for $e_{2}$ in Case (b1), we can construct a crisp deterministic ( $\Sigma, \mathrm{B}$ )-wta $\mathcal{A}_{1}$ such that $\llbracket \mathcal{A}_{1} \rrbracket=\llbracket e_{1} \rrbracket$. Then, by using Theorem 10.4.3(4) instead of Theorem 10.4.3(1), we can finish the proof in the same way as in Case (b1).

Case (b3): Let $e_{1}$ and $e_{2}$ be generated by nonterminals of the form $f_{\Sigma}^{r}$ using the rules in (13.3). As indicated in the previous cases, we can construct crisp deterministic $(\Sigma, \mathrm{B})$-wta $\mathcal{A}_{1}$ and $\mathcal{A}_{2}$ such that $\llbracket e_{1} \rrbracket=\llbracket \mathcal{A}_{1} \rrbracket$ and $\llbracket e_{2} \rrbracket=\llbracket \mathcal{A}_{2} \rrbracket$. By Theorem 10.4.1(3), we can construct a crisp deterministic ( $\Sigma$, B)-wta $\mathcal{A}$ such that $\llbracket \mathcal{A} \rrbracket=\llbracket \mathcal{A}_{1} \rrbracket \otimes \llbracket \mathcal{A}_{2} \rrbracket$.

Case (c): Let $e=\tau\left(e^{\prime}\right)$ for some $e^{\prime} \in \operatorname{RepEx}^{\times r}(\Delta, \mathrm{~B})$ and $(\Delta, \Sigma)$-tree relabeling. Moreover, let $\mathcal{A}^{\prime}$ be
 $\llbracket \mathcal{A} \rrbracket=\chi(\tau)\left(\llbracket \mathcal{A}^{\prime} \rrbracket\right)$. Then $\llbracket \mathcal{A} \rrbracket=\chi(\tau)\left(\llbracket \mathcal{A}^{\prime} \rrbracket\right)=\chi(\tau)\left(\llbracket e^{\prime} \rrbracket\right)=\llbracket \tau\left(e^{\prime}\right) \rrbracket=\llbracket e \rrbracket$.

Case (d): Let $e=\operatorname{REST}\left(e^{\prime}\right)$ for some $e^{\prime} \in \operatorname{RepEx}{ }^{\times r}(\Sigma, \mathrm{~B})$ which is generated by nonterminals of the form $f_{\Sigma}^{r}$ using the rules in (13.3). By Corollary 13.1 .3 and Theorem $10.3 .1(B) \Rightarrow(A)$, we can construct a crisp deterministic $(\Sigma, \mathrm{B})$-wta $\mathcal{A}=(Q, \delta, F)$ such that $\llbracket \mathcal{A} \rrbracket=\llbracket e \rrbracket$. Then by Lemma 13.3.1 we can construct a $(\Sigma, \mathrm{B})$-wta $\mathcal{B}$ such that $\llbracket \mathcal{B} \rrbracket=\llbracket \operatorname{REST}\left(e^{\prime}\right) \rrbracket$.

Proof of (2): The proof of this statement proceeds in the same way as the proof of (1) except that, in the case that $e=e_{1} \times e_{2}$, we use Theorem 10.4.1(1) (which needs commutativity) instead of Theorems 10.4.3(1) and 10.4.1 (3) for closure under Hadamard product.

## Chapter 14

## Weighted MSO-logic and Büchi-Elgot-Trakhtenbrot's theorem

The Büchi-Elgot-Trakhtenbrot's theorem (B-E-T's theorem¹ [Büc60, Elg61, Tra61, cf. also Str94, states that regular string languages and string languages definable in monadic second-order logic coincide. In this chapter we prove the corresponding B-E-T's theorem for the set $\operatorname{Rec}(\Sigma, B)$ of recognizable ( $\Sigma, \mathrm{B}$ )weighted tree languages for an arbitrary strong bimonoid B (cf. Theorem 14.3.1). The theorem states that, for every ( $\Sigma, \mathrm{B}$ )-weighted tree language $r$ the following equivalence holds: $r$ is r-recognizable if and only if $r$ is definable by a sentence of $\operatorname{MSO}(\Sigma, \mathrm{B})$-logic.

The first approach for a weighted version of B-E-T's theorem appeared in DG05, DG07, DG09, where they generalized from the unweighted version (i.e., B-weighted string languages) to the weighted version (i.e., B-weighted string languages where $B$ is an arbitrary semiring). Then the interpretation of a formula $\varphi$ on a string $w$ does not deliver the Boolean-valued answer to the question whether $w$ is a model of $\varphi$, but it delivers an arbitrary element in B as truth value. A first problem in the design of the weighted logic was to give a semantics to the negation of a formula, because in a semiring (or strong bimonoid) it is not clear whether a value has a kind of complement. The solution published in DG05, DG07, DG09] can be understood as follows: start from the usual MSO-logic with atomic formulas (i.e., tests $\mathrm{P}_{a}(x)$ on labels, next relation $x \leq y$, and membership $x \in X$ ), negation, disjunction, first-order existential quantification, and second-order existential quantification; turn each formula into a semantically equivalent formula in which the negations are applied to atomic formulas only; this can be achieved by introducing conjunction, first-order universal quantification, and second-order universal quantification as part of the logic and by replacing

- $\neg(\varphi \vee \psi)$ by $\neg \varphi \wedge \neg \psi$,
- $\neg \exists x . \varphi$ by $\forall x . \neg \varphi$, and
- $\neg \exists X . \varphi$ by $\forall X . \neg \varphi$.

As a part of the solution, the truth value of each atomic formula can only be $\mathbb{D}$ or $\mathbb{1}$ (i.e., the unit elements of B ); thus, its truth value can be complemented. Moreover, each $b \in B$ is added as new atomic formula, and this has the semantics $b$. Disjunction and existential quantification are semantically expressed by the summation of the semiring B , and conjunction and universal quantification are expressed by the multiplication of B. Overall, the following EBNF shows the syntax of the weighted MSO-logic (where now $\varphi$ is a nonterminal and not a formula):

$$
\begin{align*}
\varphi::= & b\left|\mathrm{P}_{a}(x)\right| x \leq y|x \in X| \neg \mathrm{P}_{a}(x)|\neg(x \leq y)| \neg(x \in X) \mid  \tag{14.1}\\
& \varphi \vee \varphi|\varphi \wedge \varphi| \exists x . \varphi|\forall x . \varphi| \exists X . \varphi \mid \forall X . \varphi .
\end{align*}
$$

[^15]In fact, the first-order fragment of (14.1) is a subset of the fuzzy predicate calculus [Wec78, Sec. 1.2.2]. Since Wechler uses fuzzy algebras [Wec78, Def. 1.2] as weight algebras, atomic formulas can have any truth value (not only $\mathbb{O}$ and $\mathbb{1}$ ) and the negation can be applied to any formula. The interpretation of the propositional connectives and quantifiers as it is defined in DG05, DG07, DG09 is essentially the same as in the fuzzy predicate calculus Wec78, Def. 1.8 and 1.10].

It turned out that the weighted MSO-logic generated by (14.1) is too strong for B-E-T's theorem. Hence, in DG05, DG07, DG09 restrictions are defined which, roughly speaking, amount to (a) having a kind of commutativity of the semiring multiplication (for exchanging values which appear in conjunctions), (b) restricting the application of first-order universal quantification to formulas of which the semantics is a recognizable step mapping, and (c) dropping second-order universal quantification. Indeed, this restricted weighted MSO-logic characterizes recognizable weighted string languages [DG09, Thm. 4.7]. In the proof (at least) two technical difficulties were mastered.
(A) The proof of "recognizable implies definable" follows the well-known idea of expressing runs of an automaton by formulas. Since the resulting formulas just check structural properties, i.e., have semantic value $\mathbb{O}$ or $\mathbb{1}$, it was important to disambiguate subformulas which involve disjunction or existential quantification, because the underlying semiring is not necessarily additively idempotent.
(B) Since conjunction and first-order universal quantification are part of the weighted logic, preservation of recognizability under these constructs had to be proven. The main problem showed up with first-order universal quantification, because there exists a non recognizable weighted language which is definable by a formula of the form $\forall x . \varphi$ in that weighted logic [DG05, Ex. 3.4]. Thus, appropriate restrictions on the form of the body formula $\varphi$ had to be found (cf. restriction (b) above), and then a new automaton construction had to be invented. This problem did not arise in classical unweighted MSO logic, because in classical logic universal quantification can be expressed using existential quantification and negations.

This approach to weighted MSO-logic has been generalized to a number of structures (different from strings) and weight algebras (different from semirings), e.g. for Mazurkiewicz traces and semirings Mei06, ranked and unranked trees and semirings DV06, DV11b, finite and infinite strings with discounting and semirings DR07, nested words and semirings Mat08, Mat10, timed words and semirings Qua09, pictures and semirings [Fic11, strings and infinite strings and valuation monoids DM12], ranked und unranked trees and tree-valuation monoids DGMM11, DHV15. In BGMZ14 (see also [BGMZ10]) an approach different to [DG09 was taken: there, not the logic was restricted but the automaton model was extended. Indeed, the set of weighted string languages recognizable by pebble two-way weighted automata is equal to the set of weighted string languages definable by, roughly speaking, a first-order fragment of the unrestricted weighted MSO-logic of DG09 enriched by a transitive closure operator. In [FV15] semiring-weighted tree automata were characterized by a weighted transitive closure logic. For a survey we refer to GM18. The disambiguation reported in (A) above is cumbersome. This has been overcome in BGMZ10, FSV12, BGMZ14 by allowing explicitly (Boolean-valued) MSO-formulas as guards of weighted formulas. Finally, we mention that, in Li08b, B-E-T's theorem has been proved for mv-algebra.

In this chapter we follow the approach of [FSV12] in which B-E-T's theorem was proved for recognizable weighted tree languages over absorptive multioperator monoids [FSV12, Thm. 4.1]. In particular, each strong bimonoid (and hence, each semiring) is an absorptive multioperator monoid. In the MSOlogic proposed in that paper, formulas of classical MSO-logic on trees can be used to guard weighted formulas. Moreover, a universal quantification over some recognizable step mapping (as it appears in the restricted MSO-logic of [DG05] is represented by a guarded atomic formula where the guard simulates the step languages and the atomic formula is interpreted as a unique algebra homomorphism (cf. Lemma 14.4.12). For semirings, each formula of the restricted weighted MSO-logic of DV06 (which is a straightforward generalization of [DG09] can be transformed syntactically into an equivalent formula of the logic in [FSV12] (cf. FSV12, Lm. 5.10]), and vice versa (cf. FSV12, Lm. 5.12]). We note that the approach of FSV12 has been generalized in FHV18] to weighted tree grammars with storage over complete M-monoids, and B-E-T's theorem was proved in [FHV18, Thm. 7.4] (for the string case cf.
[VDH16, Thm. 8]).
The goal of this chapter is to report on B-E-T's theorem [FSV12, Thm. 4.1] for the particular case of strong bimonoids (cf. Theorem 14.3.1). Moreover, we prove that extending the logic by weighted conjunction and weighted first-order universal quantification on recognizable step formulas does not increase its expressive power if the underlying weight algebra is a commutative semiring (cf. Theorem 14.4.11). Finally, in Section 14.5 we show a strong relationship between a decomposition result of wta (proved in Chapter (11) and the fact that, for weighted tree languages, r-recognizable implies definable (cf. Theorems 14.5 .2 and 14.5.4).

### 14.1 Monadic second-order logic

We recall the monadic second-order logic on trees (cf. TW68, Don70, GS97]) and make several formal definitions, constructions, and proofs explicit.

As first-order variables we use small letters from the end of the Latin alphabet, e.g., $x, x_{1}, x_{2}, \ldots, y, z$, and as second-order variables we use capital letters, like $X, X_{1}, X_{2}, \ldots, Y, Z$. The set of monadic secondorder formulas over $\Sigma$, denoted by $\operatorname{MSO}(\Sigma)$, is the set of all expressions generated by the following EBNF with nonterminal $\varphi$ :

$$
\begin{equation*}
\varphi::=\operatorname{label}_{\sigma}(x)\left|\operatorname{edge}_{i}(x, y)\right| x \in X|\neg \varphi|(\varphi \vee \varphi)|\exists x . \varphi| \exists X . \varphi \tag{14.2}
\end{equation*}
$$

where $\sigma \in \Sigma$ and $i \in[\operatorname{maxrk}(\Sigma)]$.
In order to perform inductive proofs or to define objects for $\operatorname{MSO}(\Sigma)$ by induction, we will consider the well-founded set

$$
\left(\operatorname{MSO}(\Sigma), \prec_{\mathrm{MSO}(\Sigma)}\right)
$$

where $\prec_{\mathrm{MSO}(\Sigma)}$ is the binary relation on $\operatorname{MSO}(\Sigma)$ defined as follows. For every $\varphi_{1}, \varphi_{2} \in \operatorname{MSO}(\Sigma)$, we let $\varphi_{1} \prec_{\mathrm{MSO}(\Sigma)} \varphi_{2}$ if $\varphi_{1}$ is a direct subformula of $\varphi_{2}$ in the sense of (14.2). Then $\prec_{\mathrm{MSO}(\Sigma)}$ is well-founded and $\min _{\prec_{\text {MSO }(\Sigma)}}(\operatorname{MSO}(\Sigma))$ is the set of formulas of the form $\operatorname{label}_{\sigma}(x)$, edge $_{i}(x, y)$, and $x \in X$.

Next we prepare the definition of the semantics of $\operatorname{MSO}(\Sigma)$-formulas. Let $\mathcal{V}$ be a finite set of variables; we abbreviate by $\mathcal{V}^{(1)}$ and $\mathcal{V}^{(2)}$ the set of first-order variables in $\mathcal{V}$ and the set of second-order variables in $\mathcal{V}$, respectively. Let $\xi \in \mathrm{T}_{\Sigma}$. A $\mathcal{V}$-assignment for $\xi$ is a mapping $\eta$ with domain $\mathcal{V}$ which maps each first-order variable in $\mathcal{V}$ to a position of $\xi$ and each second-order variable to a subset of $\operatorname{pos}(\xi)$. By $\Phi_{\mathcal{V}, \xi}$ we denote the set of all $\mathcal{V}$-assignments for $\xi$. Let $\eta \in \Phi_{\mathcal{V}, \xi}, x$ be a first-order variable, and $w \in \operatorname{pos}(\xi)$. By $\eta[x \mapsto w]$ we denote the $(\mathcal{V} \cup\{x\})$-assignment for $\xi$ that agrees with $\eta$ on $\mathcal{V} \backslash\{x\}$ and that satisfies $\eta[x \mapsto w](x)=w$. Similarly, if $X$ is a second-order variable and $W \subseteq \operatorname{pos}(\xi)$, then $\eta[X \mapsto W]$ denotes the $(\mathcal{V} \cup\{X\})$-assignment for $\xi$ that agrees with $\eta$ on $\mathcal{V} \backslash\{X\}$ and that satisfies $\eta[X \mapsto W](X)=W$.

Since a $\Sigma$-fta recognizes $\Sigma$-trees and not pairs $(\xi, \eta) \in \mathrm{T}_{\Sigma} \times \Phi_{\mathcal{V}, \xi}$, we code a $\mathcal{V}$-assignment for $\xi$ into the labels of $\xi$ in the usual way. Formally, we define the ranked alphabet $\Sigma_{\mathcal{V}}$ by letting

$$
\left(\Sigma_{\mathcal{V}}\right)^{(k)}=\Sigma^{(k)} \times \mathcal{P}(\mathcal{V}) \text { for each } k \in \mathbb{N}
$$

Instead of $\left(\Sigma_{\mathcal{V}}\right)^{(k)}$ we will write $\Sigma_{\mathcal{V}}^{(k)}$. We identify the sets $\Sigma$ and $\Sigma_{\emptyset}$. A tree $\zeta \in \mathrm{T}_{\Sigma_{\mathcal{V}}}$ is called valid if for each $x \in \mathcal{V}^{(1)}$ there exists a unique $w \in \operatorname{pos}(\zeta)$ such that $x$ is contained in the second component of $\zeta(w)$. We denote by $\mathrm{T}_{\Sigma_{\mathcal{V}}}^{\mathrm{v}}$ the set of all valid trees in $\mathrm{T}_{\Sigma_{\mathcal{V}}}$.

The two sets $\left\{(\xi, \eta) \mid \xi \in \mathrm{T}_{\Sigma}, \eta \in \Phi_{\mathcal{V}, \xi}\right\}$ and $\mathrm{T}_{\Sigma \mathcal{V}}^{v}$ are in a one-to-one correspondence via $(\xi, \eta) \mapsto \zeta$, where $\operatorname{pos}(\zeta)=\operatorname{pos}(\xi)$ and

$$
\zeta(w)=\left(\xi(w),\left\{x \in \mathcal{V}^{(1)} \mid w=\eta(x)\right\} \cup\left\{X \in \mathcal{V}^{(2)} \mid w \in \eta(X)\right\}\right)
$$

In the sequel, we will not distinguish between the sets $\left\{(\xi, \eta) \mid \xi \in \mathrm{T}_{\Sigma}, \eta \in \Phi_{\mathcal{V}, \xi}\right\}$ and $\mathrm{T}_{\Sigma_{\mathcal{V}}}^{\mathrm{v}}$. In particular, sometimes we call pairs $(\xi, \eta)$ in the first set also trees.

Let $(\xi, \eta) \in \mathrm{T}_{\Sigma \nu}^{\mathrm{v}}, x$ be a first-order variable, and $w \in \operatorname{pos}(\xi)$. By $(\xi, \eta)[x \mapsto w]$ we denote the valid tree $(\xi, \eta[x \mapsto w])$ over $\Sigma_{\mathcal{V} \cup\{x\}}$. Similarly, if $X$ is a second-order variable and $W \subseteq \operatorname{pos}(\zeta)$, then $(\xi, \eta)[X \mapsto W]$ denotes the valid tree $(\xi, \eta[X \mapsto W])$ over $\Sigma_{\mathcal{V} \cup\{X\}}$. Moreover, let $\mathcal{U} \subseteq \mathcal{V}$. Then $\left.\eta\right|_{\mathcal{U}}$ is a $\mathcal{U}$-assignment and we denote the tree $(\xi, \eta \mid \mathcal{U})$ also by $\left.(\xi, \eta)\right|_{\mathcal{U}}$.

For the definition of the semantics of $\operatorname{MSO}(\Sigma)$-formulas, we need the notion of free variable; for later purpose, we also define the notion of bound variables. Formally, for each $\varphi \in \operatorname{MSO}(\Sigma)$, we define the set Free $(\varphi)$ of free variables of $\varphi$ and the set $\operatorname{Bound}(\varphi)$ of bound variables of $\varphi$ by induction on (MSO $\left.(\Sigma), \prec_{\mathrm{MSO}(\Sigma)}\right)$ as follows:

- Free $\left(\operatorname{label}_{\sigma}(x)\right)=\{x\}$ and $\operatorname{Bound}\left(\operatorname{label}_{\sigma}(x)\right)=\emptyset$,
- Free $\left(\operatorname{edge}_{i}(x, y)\right)=\{x, y\}$ and Bound $\left(\operatorname{edge}_{i}(x, y)\right)=\emptyset$,
- Free $(x \in X)=\{x, X\}$ and $\operatorname{Bound}(x \in X)=\emptyset$,
- $\operatorname{Free}(\neg \varphi)=\operatorname{Free}(\varphi)$ and $\operatorname{Bound}(\neg \varphi)=\operatorname{Bound}(\varphi)$,
- $\operatorname{Free}\left(\varphi_{1} \vee \varphi_{2}\right)=\operatorname{Free}\left(\varphi_{1}\right) \cup \operatorname{Free}\left(\varphi_{2}\right)$ and $\operatorname{Bound}\left(\varphi_{1} \vee \varphi_{2}\right)=\operatorname{Bound}\left(\varphi_{1}\right) \cup \operatorname{Bound}\left(\varphi_{2}\right)$, and
- $\operatorname{Free}(\exists x . \varphi)=\operatorname{Free}(\varphi) \backslash\{x\}$ and $\operatorname{Free}(\exists X . \varphi)=\operatorname{Free}(\varphi) \backslash\{X\}$ and $\operatorname{Bound}(\exists x . \varphi)=\operatorname{Bound}(\varphi) \cup\{x\}$ and $\operatorname{Bound}(\exists X . \varphi)=\operatorname{Bound}(\varphi) \cup\{X\}$.
If a formula $\varphi$ has the free variables, say, $x, y$, and $X$ and no others, then we also write $\varphi(x, y, X)$. As usual, we abbreviate formulas like $\exists x . \exists y . \exists z . \varphi$ by $\exists x, y, z . \varphi$. A formula $\varphi \in \operatorname{MSO}(\Sigma)$ is called $\operatorname{MSO}(\Sigma)$ sentence (or just: sentence) if $\operatorname{Free}(\varphi)=\emptyset$.

Let $\varphi \in \operatorname{MSO}(\Sigma)$ and $\mathcal{V}$ be a finite set such that $\operatorname{Free}(\varphi) \subseteq \mathcal{V}$. For every $\xi \in \mathrm{T}_{\Sigma}$ and $\eta \in \Phi_{\mathcal{V}, \xi}$, the relation " $(\xi, \eta)$ satisfies $\varphi$ ", denoted by $(\xi, \eta) \models \varphi$, is defined by induction on $\left(\operatorname{MSO}(\Sigma), \prec_{\mathrm{MSO}(\Sigma)}\right)$ as follows.

$$
\begin{aligned}
& (\xi, \eta) \models \operatorname{label}_{\sigma}(x) \quad \text { iff } \quad \sigma=\xi(\eta(x)) \\
& (\xi, \eta) \models \operatorname{edge}_{i}(x, y) \quad \text { iff } \quad \eta(y)=\eta(x) . i \\
& (\xi, \eta) \models(x \in X) \quad \text { iff } \quad \eta(x) \in \eta(X) \\
& (\xi, \eta) \models\left(\varphi_{1} \vee \varphi_{2}\right) \quad \text { iff } \quad(\xi, \eta) \models \varphi_{1} \text { or }(\xi, \eta) \models \varphi_{2} \\
& (\xi, \eta) \models(\neg \varphi) \quad \text { iff } \quad(\xi, \eta) \models \varphi \text { is not true } \\
& (\xi, \eta) \models(\exists x . \varphi) \quad \text { iff } \quad \text { there exists a } w \in \operatorname{pos}(\xi) \text { such that }(\xi, \eta)[x \rightarrow w] \models \varphi \\
& (\xi, \eta) \models(\exists X . \varphi) \quad \text { iff } \quad \text { there exists a set } W \subseteq \operatorname{pos}(\xi) \text { such that }(\xi, \eta)[X \rightarrow W] \models \varphi \text {. }
\end{aligned}
$$

We denote by $\mathrm{L}_{\mathcal{V}}(\varphi)$ the set $\left\{(\xi, \eta) \in \mathrm{T}_{\Sigma_{\mathcal{V}}}^{\mathrm{v}} \mid(\xi, \eta) \models \varphi\right\}$, and we will simply write $\mathrm{L}(\varphi)$ instead of $\mathrm{L}_{\text {Free }(\varphi)}(\varphi)$. A tree language $L \subseteq \mathrm{~T}_{\Sigma}$ is called $M S O$ definable if there exists a sentence $\varphi \in \operatorname{MSO}(\Sigma)$ such that $L=\mathrm{L}(\varphi)$.

For every formula $\varphi$, finite set $\mathcal{V}$ of variables with $\operatorname{Free}(\varphi) \subseteq \mathcal{V}$, and tree $\zeta$, the membership $\zeta \in \mathrm{L}_{\mathcal{V}}(\varphi)$ is independent from the values assigned to variables outside of Free $(\varphi)$. This is formally expressed in the following consistency lemma.

Lemma 14.1.1. Let $\varphi \in \operatorname{MSO}(\Sigma)$ and $\mathcal{V}$ be a finite set of variables containing Free $(\varphi)$. Then for every $(\xi, \eta) \in \mathrm{T}_{\Sigma \mathcal{V}}^{\mathrm{v}}$, the following equivalence holds: $(\xi, \eta) \in \mathrm{L}_{\mathcal{V}}(\varphi)$ iff $\left(\xi,\left.\eta\right|_{\text {Free }(\varphi)}\right) \in \mathrm{L}(\varphi)$.

Proof. We prove the statement by induction on $\left(\operatorname{MSO}(\Sigma), \prec_{\operatorname{MSO}(\Sigma)}\right)$. Let $(\xi, \eta) \in \mathrm{T}_{\Sigma_{\nu}}^{\mathrm{v}}$.
I.B.: For the induction base we distinguish three cases.

Case (a): Let $\varphi=\operatorname{label}_{\sigma}(x)$. Then $\operatorname{Free}(\varphi)=\{x\}$ and we have

$$
(\xi, \eta) \models \varphi \text { iff } \sigma=\xi(\eta(x)) \text { iff } \sigma=\xi\left(\left.\eta\right|_{\{x\}}(x)\right) \text { iff }\left(\xi,\left.\eta\right|_{\{x\}}\right) \models \varphi \text { iff }\left(\xi,\left.\eta\right|_{\text {Free }(\varphi)}\right) \models \varphi
$$

$\underline{\text { Case (b): Let } \varphi=\operatorname{edge}_{i}(x, y) . \text { Then } \operatorname{Free}(\varphi)=\{x, y\} \text { and we have }}$
$(\xi, \eta) \models \varphi$ iff $\eta(y)=\eta(x) . i$ iff $\left.\eta\right|_{\{x, y\}}(y)=\left.\eta\right|_{\{x, y\}}(x) . i$ iff $\quad\left(\xi,\left.\eta\right|_{\{x, y\}}\right) \models \varphi$ iff $\left(\xi,\left.\eta\right|_{\text {Free }(\varphi)}\right) \models \varphi$.
Case (c): Let $\varphi=(x \in X)$. Then $\operatorname{Free}(\varphi)=\{x, X\}$ and the proof of this case is very similar to the proof of Case (b).
I.S.: For the induction step we distinguish four cases.


$$
\begin{align*}
(\xi, \eta) \models \varphi & \text { iff }(\xi, \eta) \models \varphi_{1} \text { or }(\xi, \eta) \models \varphi_{2} \\
& \text { iff }\left(\xi,\left.\eta\right|_{\text {Free }\left(\varphi_{1}\right)}\right) \models \varphi_{1} \text { or }\left(\xi,\left.\eta\right|_{\text {Free }\left(\varphi_{2}\right)}\right) \models \varphi_{2}  \tag{byI.H.}\\
& \text { iff }\left(\xi,\left.\eta\right|_{\text {Free }(\varphi)}\right) \models \varphi_{1} \text { or }\left(\xi,\left.\eta\right|_{\operatorname{Free}(\varphi)}\right) \models \varphi_{2}  \tag{byI.H.}\\
& \text { iff }\left(\xi,\left.\eta\right|_{\operatorname{Free}(\varphi)}\right) \models \varphi .
\end{align*}
$$

Case (b): Let $\varphi=\neg \varphi^{\prime}$. This is similar to Case (a).


$$
\begin{array}{rlr}
(\xi, \eta) \models \varphi & \text { iff } \quad(\exists w \in \operatorname{pos}(\xi)):(\xi, \eta[x \rightarrow w]) \models \varphi^{\prime} \\
& \text { iff }(\exists w \in \operatorname{pos}(\xi)):\left(\xi,\left.(\eta[x \rightarrow w])\right|_{\text {Free }\left(\varphi^{\prime}\right)}\right) \models \varphi^{\prime} \\
& \text { iff } \quad(\exists w \in \operatorname{pos}(\xi)):\left(\xi,\left.\left(\left(\left.\eta\right|_{\operatorname{Free}\left(\varphi^{\prime}\right)}\right)[x \rightarrow w]\right)\right|_{\text {Free }\left(\varphi^{\prime}\right)}\right) \models \varphi^{\prime} \\
& \text { (because } \left.\left.(\eta[x \rightarrow w])\right|_{\text {Free }\left(\varphi^{\prime}\right)}=\left.\left(\left(\left.\eta\right|_{\text {Free }\left(\varphi^{\prime}\right)}\right)[x \rightarrow w]\right)\right|_{\text {Free }\left(\varphi^{\prime}\right)}\right) \\
& \text { iff }(\exists w \in \operatorname{pos}(\xi)):\left(\xi,\left(\left.\eta\right|_{\operatorname{Free}\left(\varphi^{\prime}\right)}\right)[x \rightarrow w]\right) \models \varphi^{\prime} & \text { (by I.H.) }) \\
& \text { iff }(\exists w \in \operatorname{pos}(\xi)):\left(\xi,\left(\left.\eta\right|_{\operatorname{Free}(\varphi)}\right)[x \rightarrow w]\right) \models \varphi^{\prime} & \text { (by (*)) }  \tag{*}\\
& \text { iff }\left(\xi,\left.\eta\right|_{\operatorname{Free}(\varphi)}\right) \models \varphi .
\end{array}
$$

where at $\left(^{*}\right)$ we have used that either (i) $x \notin \operatorname{Free}\left(\varphi^{\prime}\right)$ and thus Free $\left(\varphi^{\prime}\right)=\operatorname{Free}(\varphi)$ or (ii) $x \in \operatorname{Free}\left(\varphi^{\prime}\right)$ and thus $\operatorname{Free}\left(\varphi^{\prime}\right)=\operatorname{Free}(\varphi) \cup\{x\}$. In both cases we have $\left(\left.\eta\right|_{\text {Free }\left(\varphi^{\prime}\right)}\right)[x \rightarrow w]=\left(\left.\eta\right|_{\text {Free }(\varphi)}\right)[x \rightarrow w]$.

We will use the following macros:

$$
\begin{aligned}
\varphi \wedge \psi & :=\neg(\neg \varphi \vee \neg \psi) \\
\varphi \rightarrow \psi & :=\neg \varphi \vee \psi \\
\varphi \leftrightarrow \psi: & :(\varphi \rightarrow \psi) \wedge(\psi \rightarrow \varphi) \\
\forall x \cdot \varphi & :=\neg(\exists x \cdot \neg \varphi) \\
\forall X \cdot \varphi & :=\neg(\exists X \cdot \neg \varphi) \\
\bigvee_{i \in I} \varphi_{i} & :=\varphi_{i_{1}} \vee \cdots \vee \varphi_{i_{n}} \text { and } \\
\bigwedge_{i \in I} \varphi_{i} & :=\varphi_{i_{1}} \wedge \cdots \wedge \varphi_{i_{n}} \text { for each finite family }\left(\varphi_{i} \mid i \in I\right) \text { with } I=\left\{i_{1}, \ldots, i_{n}\right\} \\
\operatorname{label}_{\Delta}(x) & :=\bigvee_{\sigma \in \Delta} \operatorname{label}_{\sigma}(x) \text { for each } \Delta \subseteq \Sigma \\
\operatorname{edge}(x, y) & :=\bigvee_{i \in[\operatorname{maxrk}(\Sigma)]} \quad \operatorname{edge}_{i}(x, y) \\
(x=y) & :=\forall X \cdot(x \in X) \leftrightarrow(y \in X) \\
\operatorname{root}(x) & :=\neg \exists y \cdot \operatorname{edge}(y, x)
\end{aligned}
$$

Since disjunction and conjunction are associative, the placement of parentheses in the expressions $\varphi_{i_{1}} \vee$ $\cdots \vee \varphi_{i_{n}}$ and $\varphi_{i_{1}} \wedge \cdots \wedge \varphi_{i_{n}}$ is irrelevant and hence not shown.

Example 14.1.2. (cf. e.g. CE12, Sec. 1.3.1]) We consider the following formulas in $\operatorname{MSO}(\Sigma)$ :

$$
\begin{aligned}
& \operatorname{closed}(X)=\forall z_{1} \cdot \forall z_{2} \cdot\left(\operatorname{edge}\left(z_{1}, z_{2}\right) \wedge\left(z_{1} \in X\right)\right) \rightarrow\left(z_{2} \in X\right) \\
& \operatorname{path}(x, y)=\forall X \cdot(\operatorname{closed}(X) \wedge(x \in X)) \rightarrow(y \in X)
\end{aligned}
$$

We claim that

$$
\begin{equation*}
\mathrm{L}_{\{x, y\}}(\operatorname{path}(x, y))=\left\{(\xi,\{(x, u),(y, v)\}) \in \mathrm{T}_{\Sigma_{\{x, y\}}} \mid u, v \in \operatorname{pos}(\xi) \text { such that } u \in \operatorname{prefix}(v)\right\} \tag{14.3}
\end{equation*}
$$

(a) First we prove the inclusion from left to right. Let $\xi \in \mathrm{T}_{\Sigma}, u, v \in \operatorname{pos}(\xi)$, and $\eta=\{(x, u),(y, v)\}$ such that $(\xi, \eta) \in \mathrm{L}_{\{x, y\}}(\operatorname{path}(x, y))$. Then $(\xi, \eta) \in \mathrm{T}_{\Sigma_{\{x, y\}}}^{\mathrm{v}}$ and $(\xi, \eta) \models \operatorname{path}(x, y)$.

Moreover, we have:

$$
\begin{equation*}
\text { for each set } U \subseteq \operatorname{pos}(\xi): \text { if }(\xi, \eta \cup\{(X, U)\}) \models \operatorname{closed}(X) \text { and } u \in U \text {, then } v \in U \text {. } \tag{14.4}
\end{equation*}
$$

Moreover, we have $\left(\xi, \eta \cup\left\{\left(X, \operatorname{pos}\left(\left.\xi\right|_{u}\right)\right\}\right) \models \operatorname{closed}(X)\right.$ and $u \in \operatorname{pos}\left(\left.\xi\right|_{u}\right)$. Hence, by (14.4) with $U=$ $\operatorname{pos}\left(\left.\xi\right|_{u}\right)$, we have that $v \in \operatorname{pos}\left(\left.\xi\right|_{u}\right)$. Thus $u \in \operatorname{prefix}(v)$.
(b) Next we prove the inclusion from right to left. For this, let $\xi \in \mathrm{T}_{\Sigma}, u, v \in \operatorname{pos}(\xi)$, and $\eta=$ $\{(x, u),(y, v)\}$ such that $(\xi, \eta) \in \mathrm{T}_{\Sigma_{\{x, y\}}^{\mathrm{V}}}$ and $u \in \operatorname{prefix}(v)$. Thus there exist $n \in \mathbb{N}, j_{1}, \ldots, j_{n} \in \mathbb{N}_{+}$such that $v=u j_{1} \cdots j_{n}$. Now let $U \subseteq \operatorname{pos}(\xi)$ such that $(\xi,\{(X, U)\}) \models \operatorname{closed}(X)$ and $u \in U$.

First, by induction on $([0, n],<)$, we prove that the following statement holds:

$$
\begin{equation*}
\text { For each } \ell \in[0, n] \text {, we have } u j_{1} \cdots j_{\ell} \in U \tag{14.5}
\end{equation*}
$$

I.B.: Let $\ell=0$. The statement trivially holds, because $u \in U$.
I.S.: Let $\ell=\ell^{\prime}+1$ for some $\ell^{\prime} \in \mathbb{N}$. We assume that (14.5) holds for $\ell^{\prime} \in[0, n-1]$. By I.H., we have $u j_{1} \cdots j_{\ell^{\prime}} \in U$. Since $(\xi,\{(X, U)\}) \models \operatorname{closed}(X)$ and

$$
\left(\xi,\left\{(X, U),\left(z_{1}, u j_{1} \cdots j_{\ell^{\prime}}\right),\left(z_{2}, u j_{1} \cdots j_{\ell^{\prime}} j_{\ell}\right)\right\}\right) \models\left(\operatorname{edge}\left(z_{1}, z_{2}\right) \wedge z_{1} \in X\right)
$$

we have that

$$
\left(\xi,\left\{(X, U),\left(z_{1}, u j_{1} \cdots j_{\ell^{\prime}}\right),\left(z_{2}, u j_{1} \cdots j_{\ell^{\prime}} j_{\ell}\right)\right\}\right) \models\left(z_{2} \in X\right)
$$

and hence $u j_{1} \cdots j_{\ell^{\prime}} j_{\ell} \in U$. This proves (14.5).
Then (14.5) implies (by choosing $\ell=n$ ) that $u j_{1} \cdots j_{n} \in U$, and hence $v \in U$. Thus $(\xi, \eta) \models \operatorname{path}(x, y)$ and hence $(\xi, \eta) \in \mathrm{L}_{\{x, y\}}(\operatorname{path}(x, y))$. This finishes the proof of (14.3).

Finally, we recall B-E-T's theorem for recognizable tree languages. It follows from Lemmas 14.1.4 and 14.1.9.

Theorem 14.1.3. TW68, Don70]: Let $L \subseteq \mathrm{~T}_{\Sigma}$. Then the following two statements are equivalent.
(A) We can construct an fta $A$ over $\Sigma$ such that $\mathrm{L}(A)=L$.
(B) We can construct a sentence $\varphi \in \operatorname{MSO}(\Sigma)$ such that $\mathrm{L}(\varphi)=L$.

Lemma 14.1.4. For each $\Sigma$-fta $A$, we can construct a sentence $\varphi \in \operatorname{MSO}(\Sigma)$ such that $\mathrm{L}(A)=\mathrm{L}(\varphi)$.
Proof. Let $A=(Q, \delta, F)$ be a $\Sigma$-fta. Without loss of generality we can assume that there exists an $n \in \mathbb{N}$ such that $Q=[n]$. We let $\mathcal{U}=\left\{X_{1}, \ldots, X_{n}\right\}$.

Now we construct a sentence $\varphi \in \operatorname{MSO}(\Sigma)$ such that $\mathrm{L}(\varphi)=\mathrm{L}(A)$. The following relationship will be the key for this construction. Let $\xi \in \mathrm{T}_{\Sigma}$. We can relate
(a) mappings from $\operatorname{pos}(\xi)$ into $Q$ (i.e., runs of $A$ on $\xi$ ) and
(b) $Q$-indexed partitionings over $\mathcal{P}(\operatorname{pos}(\xi))$
in the following way. Let $\rho: \operatorname{pos}(\xi) \rightarrow Q$ and $P=\left(P_{q} \mid q \in Q\right)$ be a $Q$-indexed partitioning over $\mathcal{P}(\operatorname{pos}(\xi))$. Then $\rho$ and $P$ are related iff $P_{q}=\rho^{-1}(q)$ for every $q \in Q$. Obviously, this relationship is a one-to-one correspondence.

We define the sentence $\varphi \in \operatorname{MSO}(\Sigma)$ by

$$
\varphi=\exists X_{1} \ldots \exists X_{n} \cdot \varphi_{\mathrm{part}} \wedge \varphi_{\mathrm{valid}} \wedge \varphi_{\mathrm{final}}
$$

where

- $\varphi_{\text {part }}=\forall x . \bigvee_{q \in Q}\left(\left(x \in X_{q}\right) \wedge \bigwedge_{\substack{p \in Q: \\ p \neq q}} \neg\left(x \in X_{p}\right)\right)$,
- $\varphi_{\text {valid }}=\varphi_{\text {valid }, \neq 0} \wedge \varphi_{\text {valid },=0}$ with

$$
\begin{aligned}
\varphi_{\text {valid }, \neq 0}=\forall x . & \bigwedge_{k \in \mathbb{N}_{+}, \sigma \in \Sigma^{(k)}} \operatorname{label}_{\sigma}(x) \rightarrow \\
\forall y_{1}, \ldots, y_{k} \cdot & \left(\operatorname{edge}_{1}\left(x, y_{1}\right) \wedge \ldots \wedge \operatorname{edge}_{k}\left(x, y_{k}\right) \rightarrow\right. \\
& \left.\underset{\substack{q, q_{1}, \ldots, q_{k} \in Q: \\
\left(q_{1} \ldots q_{k}, \sigma, q\right) \in \delta_{k}}}{\bigvee}\left(\left(y_{1} \in X_{q_{1}}\right) \wedge \ldots \wedge\left(y_{k} \in X_{q_{k}}\right) \wedge\left(x \in X_{q}\right)\right)\right) \text { and } \\
\varphi_{\text {valid },=0}=\forall x \cdot \bigwedge_{\alpha \in \Sigma^{(0)}} & \operatorname{label}_{\alpha}(x) \rightarrow\left(\underset{\substack{q \in Q: \\
(\varepsilon, \alpha, q) \in \delta_{0}}}{\bigvee}\left(x \in X_{q}\right)\right), \quad \text { and }
\end{aligned}
$$

- $\varphi_{\text {final }}=\forall x .\left(\operatorname{root}(x) \rightarrow \bigvee_{q \in F}\left(x \in X_{q}\right)\right)$.

For every $\xi \in \mathrm{T}_{\Sigma}$ and $\eta \in \Phi_{\mathcal{U}, \xi}$, the following statement is obvious.
$(\xi, \eta) \models \varphi_{\text {part }} \wedge \varphi_{\text {valid }} \wedge \varphi_{\text {final }}$ iff the $Q$-indexed family $\Psi=\left(\eta\left(X_{q}\right) \mid q \in Q\right)$ is a partitioning of $\operatorname{pos}(\xi)$ and the run related to $\Psi$ is accepting .

Finally, we can calculate as follows:

$$
\begin{aligned}
\xi \in \mathrm{L}(\varphi) \text { iff } & \text { there exists a } \eta \in \Phi_{\mathcal{U}, \xi} \text { such that } \\
& \text { the } Q \text {-indexed family } \Psi=\left(\eta\left(X_{q}\right) \mid q \in Q\right) \text { is a partitioning of } \operatorname{pos}(\xi) \\
& \text { and the run related to } \Psi \text { is accepting } \\
\text { iff } & \mathrm{R}_{\mathcal{A}}^{\mathrm{a}}(\xi) \neq \emptyset \\
\text { iff } & \xi \in \mathrm{L}(A) .
\end{aligned}
$$

Next we will prove that definable implies recognizable. For this we need a few preparations. First, we observe that, for each finite set $\mathcal{V}$ of variables, the set $T_{\Sigma_{\mathcal{V}}}^{v}$ is a recognizable $\Sigma_{\mathcal{V}}$-tree language.
Lemma 14.1.5. Let $\Sigma$ be a ranked alphabet and $\mathcal{V}$ be a finite set of first-order variables. Then we can construct a total and bu deterministic $\Sigma_{\mathcal{V}}$-fta $A$ such that $\mathrm{L}(A)=\mathrm{T}_{\Sigma_{\mathcal{V}}}^{\mathrm{v}}$.

Proof. Let $\mathcal{V}=\left\{x_{1}, \ldots, x_{m}\right\}$. Obviously, $\mathrm{T}_{\Sigma_{\mathcal{V}}}^{\mathrm{v}}=\bigcap_{j \in[m]} L_{j}$ where

$$
L_{j}=\left\{\xi \in \mathrm{T}_{\Sigma_{\mathcal{V}}} \mid x_{j} \text { occurs exactly once in } \xi\right\}
$$

First, for each $j \in[m]$, we construct a total and bu deterministic $\Sigma_{\mathcal{V}}$-fta $B_{j}=(Q, \delta, F)$ such that $L_{j}=\mathrm{L}\left(B_{j}\right)$. The idea for this is to count the number of occurrences of $x_{j}$ up to 2 while traversing the tree from the leaves towards the root. If $x_{j}$ is encountered twice, then the tree automaton reaches a non-final state which is propagated towards the root.

Formally, we let $Q=\{0,1,2\}$ and $F=\{1\}$. For every $k \in \mathbb{N},(\sigma, U) \in \Sigma_{\mathcal{V}}^{(k)}$, and $q_{1}, \ldots, q_{k} \in Q$ we let

$$
\delta_{k}\left(q_{1} \cdots q_{k},(\sigma, U)\right)= \begin{cases}2 & \text { if }\left(+_{i \in[k]} q_{i} \geq 2\right) \vee\left(\left(+_{i \in[k]} q_{i}=1\right) \wedge\left(x_{j} \in U\right)\right) \\ 1 & \text { if }\left(\left(十_{i \in[k]} q_{i}=1\right) \wedge\left(x_{j} \notin U\right)\right) \vee\left(\left(+_{i \in[k]} q_{i}=0\right) \wedge\left(x_{j} \in U\right)\right) \\ 0 & \text { otherwise },\end{cases}
$$

where + denotes the extension of + to finite sums in the monoid $(\mathbb{N},+, 0)$. It is clear that $\mathrm{L}(A)=L_{j}$.
Then, by Theorems 2.13 .2 and 2.13.3, we can construct a total and bu deterministic $\Sigma_{\mathcal{V}}$-fta $A$ such that $\mathrm{L}(A)=\bigcap_{j \in[m]} \mathrm{L}\left(B_{j}\right)$. Then

$$
\mathrm{L}(A)=\bigcap_{j \in[m]} \mathrm{L}\left(B_{j}\right)=\bigcap_{j \in[m]} L_{j}=\mathrm{T}_{\Sigma \mathcal{V}}^{\mathrm{v}}
$$

As second preparation we prove that two particular tree languages are recognizable.
Lemma 14.1.6. Let $\Delta$ be a ranked alphabet, $\Gamma \subseteq \Delta$, and $L_{\Gamma}=\left\{\xi \in \mathrm{T}_{\Delta} \mid(\exists w \in \operatorname{pos}(\xi)): \xi(w) \in \Gamma\right\}$. Then we can construct a total and bu deterministic $\Delta$-fta $A$ such that $\mathrm{L}(A)=L_{\Gamma}$.

Proof. We construct the $\Delta$-fta $A=(Q, \delta, F)$ with $Q=\{0,1\}$ (and each state is viewed as natural number), $F=\{1\}$, and for every $k \geq 0, \sigma \in \Delta^{(k)}, q_{1}, \ldots, q_{k} \in Q$ we define

$$
\delta_{k}\left(q_{1} \cdots q_{k}, \sigma\right)= \begin{cases}1 & \text { if }\left(\underset{i \in[k]}{+} q_{i} \geq 1\right) \vee(\sigma \in \Gamma) \\ 0 & \text { otherwise }\end{cases}
$$

It is obvious that $A$ is total and deterministic, and that $\mathrm{L}(A)=L_{\Gamma}$.
Lemma 14.1.7. Let $\Delta$ be a ranked alphabet, $\Gamma_{1}, \Gamma_{2} \subseteq \Delta, j \in \mathbb{N}$ with $j \geq 1$, and $L_{\Gamma_{1}, \Gamma_{2}, j}=\left\{\xi \in \mathrm{T}_{\Delta} \mid\right.$ $\left.\left(\exists w_{1}, w_{2} \in \operatorname{pos}(\xi)\right): w_{2}=w_{1} j, \xi\left(w_{1}\right) \in \Gamma_{1}, \xi\left(w_{2}\right) \in \Gamma_{2}\right\}$. Then we can construct a $\Delta$-fta $A$ such that $\mathrm{L}(A)=L_{\Gamma_{1}, \Gamma_{2}, j}$.

Proof. We construct the $\Delta$-fta $A=(Q, \delta, F)$ with $Q=\{0,1,2\}, F=\{2\}$, and for every $k \geq 0$, we define $\delta_{k}$ to be the smallest set $\delta_{k}^{\prime}$ such that $\left(q_{1} \cdots q_{k}, \sigma, q\right) \in \delta_{k}^{\prime}$ iff (at least) one of the following four conditions is satisfied:

- $q=0$,
- $\sigma \in \Gamma_{2}$ and $q=1$,
- $j \leq k, q_{j}=1, \sigma \in \Gamma_{1}$, and $q=2$, or
- $(\exists l \in[k]): q_{l}=2$ and $q=2$.

It is obvious that $L(\mathcal{A})=L_{\Gamma_{1}, \Gamma_{2}, j}$.
We will prove that definable implies recognizable. As preparation, we prove a consistency lemma for fta.

Lemma 14.1.8. Let $\varphi \in \operatorname{MSO}(\Sigma)$ with $\operatorname{Free}(\varphi)=\mathcal{V}$, and $A$ be a $\Sigma_{\mathcal{V}}$-fta such that $\mathrm{L}(A)=\mathrm{L}(\varphi)$. Moreover, let $V$ be a first-order variable or a second-order variable. Then we can construct a $\Sigma_{\mathcal{V} \cup\{V\}}$-fta $A^{\prime}$ such that $\mathrm{L}\left(A^{\prime}\right)=\mathrm{L}_{\mathcal{V} \cup\{V\}}(\varphi)$.

Proof. Case (a): Let $V \in \mathcal{V}$. Then we can choose $A^{\prime}=A$ and we are ready.
Case (b): Let $V \notin \mathcal{V}$. Moreover, let $A=(Q, \delta, F)$. First we construct the $\Sigma_{\mathcal{V} \cup\{V\}}$-fta $B=\left(Q, \delta^{\prime}, F\right)$ such that, for every $k \in \mathbb{N},(\sigma, \mathcal{W}) \in\left(\Sigma_{\mathcal{V} \cup\{V\}}\right)^{(k)}$, and $q_{1}, \ldots, q_{k}, q \in Q$ we define

$$
\delta_{k}^{\prime}\left(q_{1} \cdots q_{k},(\sigma, \mathcal{W}), q\right)=\delta_{k}\left(q_{1} \cdots q_{k},(\sigma, \mathcal{W} \backslash\{V\}), q\right)
$$

Obviously, for each $\xi \in \mathrm{T}_{\Sigma_{\mathcal{V} \cup \mathcal{V}\}}}$, we have $\operatorname{pos}(\xi)=\operatorname{pos}(\xi \mid \mathcal{V})$ and $\mathrm{R}_{B}^{\mathrm{a}}(\xi)=\mathrm{R}_{A}^{\mathrm{a}}(\xi \mid \mathcal{V})$. Hence, we have

$$
\begin{equation*}
\xi \in \mathrm{L}(B) \quad \text { iff } \quad \mathrm{R}_{B}^{\mathrm{a}}(\xi) \neq \emptyset \quad \text { iff } \quad \mathrm{R}_{A}^{\mathrm{a}}(\xi \mid \mathcal{V}) \neq \emptyset \quad \text { iff } \quad \xi \mid \mathcal{V} \in \mathrm{L}(A) \tag{14.6}
\end{equation*}
$$

Next we show that

$$
\begin{equation*}
\mathrm{L}(B) \cap \mathrm{T}_{\Sigma_{\mathcal{V} \cup\{V\}}^{\mathrm{v}}}=\mathrm{L}_{\mathcal{V} \cup\{V\}}(\varphi) \tag{14.7}
\end{equation*}
$$

For this, let $\xi \in \mathrm{T}_{\Sigma_{\mathcal{V} \cup\{V\}}}$. Then we have

$$
\begin{aligned}
\xi \in \mathrm{L}(B) \cap \mathrm{T}_{\Sigma_{\mathcal{V} \cup V \mathcal{V}}} & \text { iff }(\xi \mid \mathcal{V} \in \mathrm{L}(A)) \wedge\left(\xi \in \mathrm{T}_{\Sigma_{\mathcal{V} \cup\{V\}}}^{\mathrm{v}}\right) \\
& \text { iff }(\xi \mid \mathcal{V} \in \mathrm{L}(\varphi)) \wedge\left(\xi \in \mathrm{T}_{\Sigma_{\mathcal{V} \cup V \mathcal{}}}\right) \\
& \text { iff } \xi \in \mathrm{L}_{\mathcal{V} \cup\{V\}}(\varphi)
\end{aligned}
$$

Now, by Lemma 14.1.5, we can construct a $\Sigma_{\mathcal{V} \cup\{V\}}$-fta $C$ such that $\mathrm{L}(C)=\mathrm{T}_{\Sigma_{\mathcal{V} \cup\{V\}}}$. By Theorem 2.13.3, we can construct and $\Sigma_{\mathcal{V} \cup\{V\}}$-fta $A^{\prime}$ with $\mathrm{L}\left(A^{\prime}\right)=\mathrm{L}(B) \cap \mathrm{L}(C)$. Then, by (14.7) we have $\mathrm{L}\left(A^{\prime}\right)=\mathrm{L}_{\mathcal{V} \cup\{V\}}(\varphi)$.

Now we can prove that definable implies recognizable.
Lemma 14.1.9. For every finite set $\mathcal{V}$ of variables and formula $\varphi \in \operatorname{MSO}(\Sigma)$ with $\operatorname{Free}(\varphi) \subseteq \mathcal{V}$, we can construct a $\Sigma_{\mathcal{V}}$-fta $A$ such that $\mathrm{L}(A)=\mathrm{L}_{\mathcal{V}}(\varphi)$.

Proof. First, by induction on the well-founded set $\left(\mathrm{MSO}(\Sigma), \prec_{\mathrm{MSO}(\Sigma)}\right)$, we prove the following.
For each $\varphi \in \operatorname{MSO}(\Sigma)$, we can construct a $\Sigma_{\text {Free }(\varphi)}$-fta $B$ such that $\mathrm{L}(B)=\mathrm{L}(\varphi)$.
I.B.: For the induction base we distinguish three cases.

Case (a): Let $\varphi=\operatorname{label}_{\sigma}(x)$. Then $\operatorname{Free}(\varphi)=\{x\}$. Moreover, let $\Gamma=\{(\sigma,\{x\})\}$. Clearly L $(\varphi)=$ $L_{\Gamma} \overline{\cap T_{\Sigma_{\{x\}}}^{\mathrm{v}}}$. By Lemma 14.1.6 we can construct a $\Sigma_{\{x\}}$-fta $C$ such that $\mathrm{L}(C)=L_{\Gamma}$. By Lemma 14.1.5 we can construct a $\Sigma_{\mathcal{V}}$-fta $D$ such that $\mathrm{L}(D)=\mathrm{T}_{\Sigma_{\{x\}}}^{\mathrm{v}}$. By Theorem 2.13.3, we can construct a $\Sigma_{\{x\}}$-fta $B$ such that $\mathrm{L}(B)=\mathrm{L}(C) \cap \mathrm{L}(D)$.

Case (b): Let $\varphi=\operatorname{edge}_{j}(x, y)$. Then $\operatorname{Free}(\varphi)=\{x, y\}$. Moreover, let $\Gamma_{1}=\{(\sigma,\{x\}) \mid \sigma \in \Sigma\}$ and $\Gamma_{2}=\{(\sigma,\{y\}) \mid \sigma \in \Sigma\}$. Clearly $\mathrm{L}(\varphi)=L_{\Gamma_{1}, \Gamma_{2}, j} \cap \mathrm{~T}_{\Sigma_{\{x, y\}}}^{\mathrm{v}}$. By Lemma 14.1.7, we can construct a $\Sigma_{\{x, y\}}$-fta $C$ such that $\mathrm{L}(C)=L_{\Gamma_{1}, \Gamma_{2}, j}$. Then we proceed as in Case (a).

Case (c): Let $\varphi=(x \in X)$. Then $\operatorname{Free}(\varphi)=\{x, X\}$. Moreover, let $\Gamma=\{(\sigma,\{x, X\}) \mid \sigma \in \Sigma\}$. Clearly $\mathrm{L}(\varphi)=L_{\Gamma} \cap \mathrm{T}_{\Sigma_{\{x, X\}}^{\mathrm{v}}}$. Then we proceed as in Case (a).
I.S.: For the induction step we distinguish four cases.
 a $\Sigma_{\text {Free }(\varphi) \text {-fta }} D$ such that $\mathrm{L}(D)=\mathrm{T}_{\Sigma_{\text {Free }(\varphi)}}$. By I.H. we can construct a $\Sigma_{\text {Free }(\varphi)}$-fta $C$ such that $\mathrm{L}(C)=\mathrm{L}(\psi)$. By Theorem 2.13.3, we can construct a $\Sigma_{\text {Free }(\varphi)}$-fta $B$ such that $\mathrm{L}(B)=\mathrm{L}(D) \backslash \mathrm{L}(C)$.

Case (b): Let $\varphi=\psi_{1} \vee \psi_{2}$. We note that $\mathrm{L}(\varphi)=\mathrm{L}_{\text {Free }(\varphi)}\left(\psi_{1}\right) \cup \mathrm{L}_{\text {Free }(\varphi)}\left(\psi_{2}\right)$. By I.H. we can construct $\Sigma_{\text {Free }\left(\psi_{i}\right)}$-fta $C_{i}$ such that $\mathrm{L}\left(C_{i}\right)=\mathrm{L}\left(\psi_{i}\right)$ for $i \in\{1,2\}$. Then, by Lemma 14.1.8 we can construct $\Sigma_{\text {Free }(\varphi)^{-}}$ fta $D_{i}$ such that $\mathrm{L}\left(D_{i}\right)=\mathrm{L}_{\text {Free }(\varphi)}\left(\psi_{i}\right)$ for $i \in\{1,2\}$. Lastly, by Theorem [2.13.3, we can construct a $\Sigma_{\text {Free }(\varphi)}$-fta $B$ such that $\mathrm{L}(B)=\mathrm{L}\left(D_{1}\right) \cup \mathrm{L}\left(D_{2}\right)$.

Case (c): Let $\varphi=(\exists x \cdot \psi)$. By I.H. we can construct a $\Sigma_{\text {Free }(\psi)}$-fta $C$ such that $\mathrm{L}(C)=\mathrm{L}(\psi)$. Then, by Lemma 14.1.8, we can construct a $\Sigma_{\mathrm{Free}(\psi) \cup\{x\}}-\mathrm{fta} C^{\prime}$ such that $\mathrm{L}\left(C^{\prime}\right)=\mathrm{L}_{\mathrm{Free}(\psi) \cup\{x\}}(\psi)$. By Theorem 2.13.2, we can construct a total and bu deterministic $\Sigma_{\text {Free }(\psi) \cup\{x\}}$-fta $D=(Q, \delta, F)$ such that $\mathrm{L}(D)=$ $\mathrm{L}_{\text {Free }(\psi) \cup\{x\}}(\psi)$.

Lastly, we construct the $\Sigma_{\mathrm{Free}(\varphi)}$-fta $B=\left(Q^{\prime}, \delta^{\prime}, F^{\prime}\right)$ such that $Q^{\prime}=Q \times\{0,1\}$ (where each state is viewed as natural number), $F^{\prime}=F \times\{1\}$, and for each $k \in \mathbb{N}$, we have

$$
\begin{aligned}
\delta_{k}^{\prime} & =\left\{\left(\left(q_{1}, 0\right) \cdots\left(q_{k}, 0\right),(\sigma, U),(q, 0)\right) \mid(\sigma, U) \in \Sigma_{\operatorname{Free}(\varphi)}^{(k)},\left(q_{1} \cdots q_{k},(\sigma, U), q\right) \in \delta_{k}\right\} \\
& \cup\left\{\left(\left(q_{1}, 0\right) \cdots\left(q_{k}, 0\right),(\sigma, U),(q, 1)\right) \mid(\sigma, U) \in \Sigma_{\operatorname{Free}(\varphi)}^{(k)},\left(q_{1} \cdots q_{k},(\sigma, U \cup\{x\}), q\right) \in \delta_{k}\right\} \\
& \cup\left\{\left(\left(q_{1}, p_{1}\right) \cdots\left(q_{k}, p_{k}\right),(\sigma, U),(q, 1)\right) \mid(\sigma, U) \in \Sigma_{\operatorname{Free}(\varphi)}^{(k)},\left(q_{1} \cdots q_{k},(\sigma, U), q\right) \in \delta_{k},+_{i \in[k]} p_{i}=1\right\} .
\end{aligned}
$$

Now we prove that $\mathrm{L}(\varphi)=\mathrm{L}(B)$. For this, let $\zeta=(\xi, \eta)$ be in $\mathrm{T}_{\Sigma_{\mathrm{Free}(\varphi)}}$. For each $\rho: \operatorname{pos}(\xi) \rightarrow Q^{\prime}$, we define $\pi_{1}(\rho): \operatorname{pos}(\xi) \rightarrow Q$ and $\pi_{2}(\rho): \operatorname{pos}(\xi) \rightarrow\{0,1\}$ such that, for each $w \in \operatorname{pos}(\xi)$, we let $\pi_{1}(\rho)(w)$ and $\pi_{2}(\rho)(w)$ be the first component of $\rho(w)$ and the second component of $\rho(w)$, respectively.

$$
\begin{aligned}
& \quad \zeta \in \mathrm{L}(\varphi) \\
& \text { iff } \zeta \models(\exists x \cdot \psi) \\
& \text { iff }(\exists w \in \operatorname{pos}(\xi)):(\xi, \eta[x \mapsto w]) \models \psi \\
& \text { iff }(\exists w \in \operatorname{pos}(\xi)):(\xi, \eta[x \mapsto w]) \in \mathrm{L}(D) \\
& \text { iff }\left(\exists w \in \operatorname{pos}(\xi), \rho_{1} \in \mathrm{R}_{D}^{\mathrm{v}}((\xi, \eta[x \mapsto w]))\right): \rho_{1}(\varepsilon) \in F \\
& \text { iff }\left(\exists w \in \operatorname{pos}(\xi), \rho \in \mathrm{R}_{B}^{\mathrm{v}}(\zeta)\right): \pi_{1}(\rho) \in \mathrm{R}_{D}^{\mathrm{v}}((\xi, \eta[x \mapsto w])) \wedge \pi_{1}(\rho)(\varepsilon) \in F \wedge \\
& \quad(\forall v \in \operatorname{prefix}(w)): \pi_{2}(\rho)(v)=1 \wedge(\forall v \in \operatorname{pos}(\xi) \backslash \operatorname{prefix}(w)): \pi_{2}(\rho)(v)=0 \\
& \text { iff }\left(\exists \rho \in \mathrm{R}_{B}^{\mathrm{v}}(\zeta)\right): \rho(\varepsilon) \in F^{\prime} \\
& \text { iff } \zeta \in \mathrm{L}(B) \text {. }
\end{aligned}
$$

We can also give an alternative proof as follows (using closure under tree relabelings). We observe that $\mathrm{L}(\varphi)=\tau\left(\mathrm{L}_{\mathrm{Free}(\psi) \cup\{x\}}(\psi)\right)$ where $\tau=\left(\tau_{k} \mid k \in \mathbb{N}\right)$ is the deterministic $\left(\Sigma_{\mathrm{Free}(\psi) \cup\{x\}}, \Sigma_{\mathrm{Free}(\varphi)}\right)$-tree relabeling defined, for each $k \in \mathbb{N}$ and $(\sigma, U) \in \Sigma_{\text {Free }(\psi) \cup\{x\}}^{(k)}$ by $\tau((\sigma, U))=(\sigma, U \cap \operatorname{Free}(\varphi))$. This can be seen as follows:

$$
\begin{aligned}
&(\xi, \rho) \in \mathrm{L}(\varphi) \text { iff }(\xi, \rho) \models \varphi \\
& \text { iff }(\exists w \in \operatorname{pos}(\xi)):(\xi, \rho[x \rightarrow w]) \models \psi \\
& \text { iff }\left(\exists \zeta^{\prime} \in \mathrm{T}_{\Sigma_{\mathrm{Free}(\psi) \cup\{x\}}}\right): \tau\left(\zeta^{\prime}\right)=(\xi, \rho) \text { and } \zeta^{\prime} \models \psi \\
& \text { iff }\left(\exists \zeta^{\prime} \in \mathrm{L}_{\operatorname{Free}(\psi) \cup\{x\}}(\psi)\right): \tau\left(\zeta^{\prime}\right)=(\xi, \rho) \\
& \text { iff }(\xi, \rho) \in \tau\left(\mathrm{L}_{\mathrm{Free}(\psi) \cup\{x\}}(\psi)\right) .
\end{aligned}
$$

By I.H. we can construct a $\Sigma_{\text {Free }(\psi)}$-fta $C$ such that $\mathrm{L}(C)=\mathrm{L}(\psi)$. Hence, by Lemma 14.1.8, we can also construct a $\Sigma_{\text {Free }(\psi) \cup\{x\}}$-fta $D$ such that $\mathrm{L}(D)=\mathrm{L}_{\text {Free }(\psi) \cup\{x\}}(\psi)$. Then, by the fact that $\mathrm{L}(\varphi)=$ $\tau\left(\mathrm{L}_{\mathrm{Free}(\psi) \cup\{x\}}(\psi)\right)$ and by Theorem 10.10.1 we can construct a $\Sigma_{\text {Free }(\varphi)}$-fta $B$ such that $\mathrm{L}(B)=\mathrm{L}(\varphi)$.

Case (d): Let $\varphi=(\exists X . \psi)$. By I.H. we can construct a $\Sigma_{\text {Free }(\psi)}$-fta $C$ such that $\mathrm{L}(C)=\mathrm{L}(\psi)$. Then, by Lemma 14.1.8, we can construct a $\Sigma_{\text {Free }(\psi) \cup\{X\}}$-fta $C^{\prime}=(Q, \delta, F)$ such that $\mathrm{L}\left(C^{\prime}\right)=\mathrm{L}_{\mathrm{Free}(\psi) \cup\{X\}}(\psi)$.

Lastly, we construct the $\Sigma_{\text {Free }(\varphi)-\mathrm{fta}} B=\left(Q, \delta^{\prime}, F\right)$ such that, for every $k \in \mathbb{N}$, we let

$$
\begin{aligned}
\delta_{k}^{\prime}=\left\{\left(q_{1} \cdots q_{k},(\sigma, U), q\right) \mid\right. & (\sigma, U) \in \Sigma_{\operatorname{Free}(\varphi)}^{(k)} \text { and } \\
& \left(\left(q_{1} \cdots q_{k},(\sigma, U), q\right) \in \delta_{k} \text { or }\left(q_{1} \cdots q_{k},(\sigma, U \cup\{X\}), q\right) \in \delta_{k}\right\}
\end{aligned}
$$

Then it is easy to see that $\mathrm{L}(B)=\mathrm{L}(\varphi)$.
Also here we can give an alternative proof as follows. Now we observe that $\mathrm{L}(\varphi)=\tau\left(\mathrm{L}_{\mathrm{Free}(\psi) \cup\{X\}}(\psi)\right)$ where $\tau=\left(\tau_{k} \mid k \in \mathbb{N}\right)$ is the $\left(\Sigma_{\operatorname{Free}(\psi) \cup\{X\}}, \Sigma_{\operatorname{Free}(\varphi)}\right)$-tree relabeling defined, for every $k \in \mathbb{N}$ and $(\sigma, U) \in \Sigma_{\operatorname{Free}(\psi) \cup\{X\}}^{(k)}$ by $\tau((\sigma, U))=(\sigma, U \cap \operatorname{Free}(\varphi))$. Then the proof can be finished similarly as in Case (c).

This finishes the proof of (14.8). Finally, by applying Lemma 14.1 .8 to the $\mathrm{fta} B$ an appropriate number of times, we obtain the desired $\Sigma_{\mathcal{V}}$-fta $A$ with $\mathrm{L}(A)=\mathrm{L}_{\mathcal{V}}(\varphi)$.

### 14.2 Adding weights

Now we add weights to $\operatorname{MSO}(\Sigma)$ in the following way. As atomic formulas of the weighted logic $\mathrm{MSO}(\Sigma, \mathrm{B})$ we use formulas of the form $\mathrm{H}(\kappa)$, where $\kappa$ is an $\mathbb{N}$-indexed family of mappings $\kappa_{k}:\left(\Sigma_{\mathcal{U}}\right)^{(k)} \rightarrow B$. This
introduces the weights into the logic. Roughly speaking, the semantics of $\mathrm{H}(\kappa)$ is the unique $\Sigma$-algebra homomorphism $\mathrm{h}_{\mathrm{M}(\Sigma, \kappa)}$ from the $\Sigma_{\mathcal{U}}$-term algebra to the ( $\Sigma, \kappa$ )-evaluation algebra (cf. Section 2.9 and Equation (2.27) . In the setting of the weighted logics of DG05, the formula $\mathrm{H}(\kappa)$ can be simulated by a weighted first-order universal quantification over a recognizable step formula (cf. Lemma 14.4.9).

We define the set of MSO formulas over $\Sigma$ and B , denoted by $\operatorname{MSO}(\Sigma, \mathrm{B})$, by the following EBNF with nonterminal $e$ :

$$
\begin{equation*}
e::=\mathrm{H}(\kappa)|(\varphi \triangleright e)|(e+e)\left|+_{x} e\right|+_{x} e \tag{14.9}
\end{equation*}
$$

where

- there exists a finite set $\mathcal{U}$ of variables such that $\kappa=\left(\kappa_{k} \mid k \in \mathbb{N}\right)$ is an $\mathbb{N}$-indexed family of mappings $\kappa_{k}:\left(\Sigma_{\mathcal{U}}\right)^{(k)} \rightarrow B$,
- $\varphi \in \operatorname{MSO}(\Sigma)$.

We will drop parentheses whenever no confusion arises. We call formulas of the form $\mathrm{H}(\kappa)$ atomic formulas, formulas of the form $\varphi \triangleright e$ guarded formulas, and formulas of the form $+_{x} e$ and $+_{x} e$ the weighted first-order existential quantification (of e) and weighted second-order existential quantification (of e), respectively.

As for $\operatorname{MSO}(\Sigma)$-formulas in Section 14.1 , in order to perform inductive proofs or to define objects by induction, we will consider the well-founded set

$$
\left(\mathrm{MSO}(\Sigma, \mathrm{~B}), \prec_{\mathrm{MSO}(\Sigma, \mathrm{~B})}\right)
$$

where $\prec_{\mathrm{MSO}(\Sigma, \mathrm{B})}$ is the binary relation on $\operatorname{MSO}(\Sigma, \mathrm{B})$ defined as follows. For every $e_{1}, e_{2} \in \operatorname{MSO}(\Sigma, \mathrm{~B})$, we let $e_{1} \prec_{\operatorname{MSO}(\Sigma, \mathrm{B})} e_{2}$ if $e_{1}$ is a direct subformula of $e_{2}$ in the sense of (14.9). Then $\prec_{\mathrm{MSO}(\Sigma, \mathrm{B})}$ is well-founded and $\min _{\prec_{\mathrm{MSO}(\Sigma, \mathrm{B})}}(\operatorname{MSO}(\Sigma, \mathrm{B}))$ is the set of formulas of the form $\mathrm{H}(\kappa)$.

Next we define the semantics of MSO formulas over $\Sigma$ and $B$. For this purpose, we need the notion of free variable; for later purpose, we also define the notion of bound variables. Formally, for each $e \in \operatorname{MSO}(\Sigma, \mathrm{~B})$, we define the set Free $(e)$ of free variables of $e$ and the set $\operatorname{Bound}(e)$ of bound variables of $e$ by induction on $\left(\operatorname{MSO}(\Sigma, \mathrm{B}), \prec_{\mathrm{MSO}}(\Sigma, \mathrm{B})\right)$ as follows:

- If $\mathcal{U}$ is a finite set of variables and $\kappa=\left(\kappa_{k} \mid n \in \mathbb{N}\right)$ with $\kappa_{k}:\left(\Sigma_{\mathcal{U}}\right)^{(k)} \rightarrow B$, then $\operatorname{Free}(\mathrm{H}(\kappa))=\mathcal{U}$ and $\operatorname{Bound}(\mathrm{H}(\kappa))=\emptyset$,
- $\operatorname{Free}(\varphi \triangleright e)=\operatorname{Free}(\varphi) \cup \operatorname{Free}(e)$ and $\operatorname{Bound}(\varphi \triangleright e)=\operatorname{Bound}(\varphi) \cup \operatorname{Bound}(e)$,
- Free $\left(e_{1}+e_{2}\right)=\operatorname{Fr} e \mathrm{e}\left(e_{1}\right) \cup \operatorname{Free}\left(e_{2}\right)$ and $\operatorname{Bound}\left(e_{1}+e_{2}\right)=\operatorname{Bound}\left(e_{1}\right) \cup \operatorname{Bound}\left(e_{2}\right)$, and
- Free $\left(+_{x} e\right)=\operatorname{Free}(e) \backslash\{x\}$ and Free $\left(+_{X} e\right)=\operatorname{Free}(e) \backslash\{X\}$ and $\operatorname{Bound}\left(+_{x} e\right)=\operatorname{Bound}(e) \cup\{x\}$ and $\operatorname{Bound}\left(+_{X} e\right)=\operatorname{Bound}(e) \cup\{X\}$.
If Free $(e)=\emptyset$, then we call $e$ a sentence.
Since the semantics of an atomic formula uses the concept of evaluation algebra, we briefly recall it from Section 2.9, For a given $\mathbb{N}$-indexed family $\kappa=\left(\kappa_{k} \mid k \in \mathbb{N}\right)$ with $\kappa_{k}: \Sigma^{(k)} \rightarrow B$, the ( $\left.\Sigma, \mathrm{B}\right)$-evaluation algebra $(\mathrm{B}, \bar{\kappa})$ with $\bar{\kappa}(\sigma)\left(b_{1}, \ldots, b_{k}\right)=b_{1} \otimes \cdots \otimes b_{k} \otimes \kappa_{k}(\sigma)$, is denoted by $\mathrm{M}(\Sigma, \kappa)$, and the $\Sigma$-algebra homomorphism from the $\Sigma$-term algebra to $M(\Sigma, \kappa)$ is denoted by $h_{M(\Sigma, \kappa)}$. We will employ this concept for various ranked alphabets when defining the semantics of atomic formulas.

In the sequel we will abbreviate $\mathrm{h}_{\mathrm{M}(\Sigma, \kappa)}$ by $\mathrm{h}_{\kappa}$.
Now let $\mathcal{U}$ and $\mathcal{V}$ be finite sets of variables such that $\mathcal{U} \subseteq \mathcal{V}$. Moreover, let $\kappa=\left(\kappa_{k} \mid k \in \mathbb{N}\right)$ with $\kappa_{k}$ : $\Sigma_{\mathcal{U}}^{(k)} \rightarrow B$. We define the $\mathbb{N}$-indexed family $\kappa[\mathcal{U} \rightsquigarrow \mathcal{V}]=\left(\kappa[\mathcal{U} \rightsquigarrow \mathcal{V}]_{k} \mid k \in \mathbb{N}\right)$ with $\kappa[\mathcal{U} \rightsquigarrow \mathcal{V}]_{k}: \Sigma_{\mathcal{V}}^{(k)} \rightarrow B$ such that, for every $\sigma \in \Sigma^{(k)}$ and $\mathcal{W} \subseteq \mathcal{V}$, we have

$$
\kappa[\mathcal{U} \rightsquigarrow \mathcal{V}]_{k}(\sigma, \mathcal{W})=\kappa_{k}(\sigma, \mathcal{U} \cap \mathcal{W})
$$

Let $e \in \operatorname{MSO}(\Sigma, \mathrm{~B})$ and $\mathcal{V}$ be a finite set of variables such that $\mathcal{V} \supseteq$ Free $(e)$. We define the semantics of e with respect to $\mathcal{V}$, denoted by $\llbracket e \rrbracket_{\mathcal{V}}$, to be a $\left(\Sigma_{\mathcal{V}}, \mathrm{B}\right)$-weighted tree language. In other words, for each
$e \in \operatorname{MSO}(\Sigma, \mathrm{~B})$, we define the family $\left(\llbracket e \rrbracket_{\mathcal{V}} \mid \mathcal{V} \supseteq \operatorname{Free}(\mathrm{e}), \mathcal{V}\right.$ finite $)$. Since $\llbracket e \rrbracket_{\mathcal{V}}$ depends on the semantics of the subformulas of $e$ (each with respect to its own set of variables), we define the family

$$
\left(\left(\llbracket e \rrbracket_{\mathcal{V}} \mid \mathcal{V} \supseteq \operatorname{Free}(\mathrm{e}), \mathcal{V} \text { finite }\right) \mid e \in \operatorname{MSO}(\Sigma, \mathrm{~B})\right)
$$

by induction on $\left(\operatorname{MSO}(\Sigma, \mathrm{B}), \prec_{\mathrm{MSO}(\Sigma, \mathrm{B})}\right)$ as follows.

- Let $\mathcal{U}$ be a finite set of variables and $\kappa=\left(\kappa_{k} \mid k \in \mathbb{N}\right)$ be an $\mathbb{N}$-indexed family with $\kappa_{k}:\left(\Sigma_{\mathcal{U}}\right)^{(k)} \rightarrow B$. Let $\mathcal{V} \supseteq \operatorname{Free}(\mathrm{H}(\kappa))$ be a finite set of variables. Then, for each $\zeta \in \mathrm{T}_{\Sigma_{\mathcal{V}}}$, we define

$$
\llbracket H(\kappa) \rrbracket \mathcal{V}(\zeta)= \begin{cases}\mathrm{h}_{\kappa[\mathcal{U} \rightsquigarrow \mathcal{V}]}(\zeta) & \text { if } \zeta \in \mathrm{T}_{\Sigma_{\mathcal{V}}}^{\mathrm{v}} \\ 0 & \text { otherwise }\end{cases}
$$

- Let $\varphi \in \operatorname{MSO}(\Sigma)$ and $e \in \operatorname{MSO}(\Sigma, \mathrm{~B})$. Let $\mathcal{V} \supseteq \operatorname{Free}(\varphi \triangleright e)$ be a finite set of variables. Then, for each $\zeta \in \mathrm{T}_{\Sigma_{\mathcal{V}}}$, we define

$$
\llbracket \varphi \triangleright e \rrbracket \mathcal{V}(\zeta)= \begin{cases}\llbracket e \rrbracket \mathcal{V}(\zeta) & \text { if } \zeta \in \mathrm{L}_{\mathcal{V}}(\varphi) \\ \mathbb{0} & \text { otherwise }\end{cases}
$$

- Let $e_{1}, e_{2} \in \operatorname{MSO}(\Sigma, \mathrm{~B})$. Let $\mathcal{V} \supseteq \operatorname{Free}\left(e_{1}+e_{2}\right)$ be a finite set of variables. Then, for each $\zeta \in \mathrm{T}_{\Sigma_{\mathcal{V}}}$, we define

$$
\llbracket e_{1}+e_{2} \rrbracket \mathcal{V}(\zeta)= \begin{cases}\left(\llbracket e_{1} \rrbracket \mathcal{V}(\zeta) \oplus \llbracket e_{2} \rrbracket \mathcal{V}(\zeta)\right. & \text { if } \zeta \in \mathrm{T}_{\Sigma \mathcal{V}}^{\mathrm{v}} \\ \mathbb{O} & \text { otherwise }\end{cases}
$$

- Let $x$ be a first-order variable and $e \in \operatorname{MSO}(\Sigma, \mathrm{~B})$. Let $\mathcal{V} \supseteq \operatorname{Free}\left(+_{x} e\right)$ be a finite set of variables. Then, for each $\zeta \in \mathrm{T}_{\Sigma_{\mathcal{V}}}$, we define

$$
\llbracket+_{x} e \rrbracket \mathcal{V}(\zeta)= \begin{cases}\bigoplus_{w \in \operatorname{pos}(\zeta)} \llbracket e \rrbracket \mathcal{V} \cup\{x\} \\ 0 & \zeta[x \mapsto w]) \\ \text { if } \zeta \in \mathrm{T}_{\Sigma \mathcal{V}}^{\mathrm{v}} \\ \text { otherwise }\end{cases}
$$

- Let $X$ be a second-order variable and $e \in \operatorname{MSO}(\Sigma, \mathrm{~B})$. Let $\mathcal{V} \supseteq \operatorname{Free}\left(+_{X} e\right)$ be a finite set of variables. Then, for each $\zeta \in \mathrm{T}_{\Sigma_{\mathcal{V}}}$, we define

$$
\llbracket+_{X} e \rrbracket \mathcal{V}(\zeta)= \begin{cases}\bigoplus_{W \subseteq \operatorname{pos}(\zeta)} \llbracket e \rrbracket \mathcal{V} \cup\{X\} \\ 0 & (\zeta[X \mapsto W]) \\ \text { if } \zeta \in \mathrm{T}_{\Sigma_{\mathcal{V}}}^{\mathrm{v}} \\ \text { otherwise }\end{cases}
$$

By definition, it is clear that the following holds:

$$
\begin{align*}
& \text { for every finite set } \mathcal{V} \text { of variables, } \varphi \in \operatorname{MSO}(\Sigma) \text {, and } e \in \operatorname{MSO}(\Sigma, \mathrm{~B}) \\
& \text { such that } \mathcal{V} \supseteq \operatorname{Free}(\varphi \triangleright e) \text {, we have } \llbracket \varphi \triangleright e \rrbracket \mathcal{V}=\chi\left(\operatorname{L}_{\mathcal{V}}(\varphi)\right) \otimes \llbracket e \rrbracket \mathcal{V} \text {. } \tag{14.10}
\end{align*}
$$

We abbreviate $\llbracket e \rrbracket_{\text {Free(e) }}$ by $\llbracket e \rrbracket$. We say that a $(\Sigma, \mathrm{B})$-weighted tree language $r$ is definable by a formula in $\operatorname{MSO}(\Sigma, \mathrm{B})$-logic (or: definable) if there exists a sentence $e \in \operatorname{MSO}(\Sigma, \mathrm{~B})$ with $\llbracket e \rrbracket=r$.

Next we show a consistency lemma for formulas in $\operatorname{MSO}(\Sigma, \mathrm{B})$.
Lemma 14.2.1. (cf. [FSV12, Lm. 3.8]) Let $e \in \operatorname{MSO}(\Sigma, \mathrm{~B})$ and let $\mathcal{V}$ and $\mathcal{W}$ be finite sets of variables with $\operatorname{Free}(e) \subseteq \mathcal{W} \subseteq \mathcal{V}$. Then, for each $(\xi, \eta) \in \mathrm{T}_{\Sigma_{\mathcal{V}}}$, we have $\llbracket e \rrbracket_{\mathcal{V}}(\xi, \eta)=\llbracket e \rrbracket \mathcal{W}(\xi, \eta \mid \mathcal{W})$.

Proof. We prove the statement by induction on $\left(\operatorname{MSO}(\Sigma, \mathrm{B}), \prec_{\mathrm{MSO}}(\Sigma, \mathrm{B})\right)$.
I.B.: Let $e=\mathrm{H}(\kappa)$. Let $\mathcal{U}$ be a finite set of variables and $\kappa=\left(\kappa_{k} \mid k \in \mathbb{N}\right)$ an $\mathbb{N}$-indexed family of mappings $\kappa_{k}:\left(\Sigma_{\mathcal{U}}\right)^{(k)} \rightarrow B$. Moreover, let $\mathcal{V}$ and $\mathcal{W}$ be finite sets of variables with $\mathcal{U} \subseteq \mathcal{W} \subseteq \mathcal{V}$ and $(\xi, \eta) \in \mathrm{T}_{\Sigma_{\mathcal{V}}}^{\mathrm{v}}$. Then

$$
\llbracket \mathrm{H}(\kappa) \rrbracket \mathcal{V}(\xi, \eta)=\mathrm{h}_{\kappa[\mathcal{U} \rightsquigarrow \mathcal{V}]}(\xi, \eta)=^{(*)} \mathrm{h}_{\kappa[\mathcal{U} \rightsquigarrow \mathcal{W}]}\left(\xi,\left.\eta\right|_{\mathcal{W}}\right)=\llbracket \mathrm{H}(\kappa) \rrbracket_{\mathcal{W}}\left(\xi,\left.\eta\right|_{\mathcal{W}}\right)
$$

where the equality $(*)$ can be proved as follows. Let $\zeta \in \mathrm{T}_{\Sigma_{\mathcal{V}}}^{\mathrm{v}}$ and $\zeta^{\prime} \in \mathrm{T}_{\Sigma_{\mathcal{W}}}^{\mathrm{v}}$ be the trees which correspond to $(\xi, \eta)$ and $(\xi, \eta \mid \mathcal{W})$, respectively. We have to show that

$$
\begin{equation*}
\mathrm{h}_{\kappa[\mathcal{U} \rightsquigarrow \mathcal{V}]}(\zeta)=\mathrm{h}_{\kappa[\mathcal{U} \rightsquigarrow \mathcal{W}]}\left(\zeta^{\prime}\right) . \tag{14.11}
\end{equation*}
$$

Obviously, $\zeta^{\prime}$ can be obtained from $\zeta$ by replacing each symbol $\left(\sigma, \mathcal{V}^{\prime}\right)$ in $\zeta$ by ( $\sigma, \mathcal{W} \cap \mathcal{V}^{\prime}$ ). By 2.27, $\mathrm{h}_{\kappa[\mathcal{U} \rightsquigarrow \mathcal{V}]}(\zeta)$ is the product of the weights of the symbols of $\zeta$, where the factors are ordered according to the depth-first post-order of the positions. The product $\mathrm{h}_{\kappa[\mathcal{U} \leadsto \mathcal{W}]}\left(\zeta^{\prime}\right)$ is defined analogously. Since $\operatorname{pos}(\zeta)=\operatorname{pos}\left(\zeta^{\prime}\right)$, there exists a bijection between the lists of the factors of the two products. Moreover, the factors which correspond to each other are equal, because for every $k \in \mathbb{N}, \sigma \in \Sigma^{(k)}$, and $\mathcal{V}^{\prime} \subseteq \mathcal{V}$, by definition we have

$$
\kappa[\mathcal{U} \rightsquigarrow \mathcal{V}]_{k}\left(\sigma, \mathcal{V}^{\prime}\right)=\kappa_{k}\left(\sigma, \mathcal{U} \cap \mathcal{V}^{\prime}\right)=\kappa_{k}\left(\sigma, \mathcal{U} \cap\left(\mathcal{W} \cap \mathcal{V}^{\prime}\right)\right)=\kappa[\mathcal{U} \rightsquigarrow \mathcal{W}]_{k}\left(\sigma, \mathcal{W} \cap \mathcal{V}^{\prime}\right)
$$

This proves (14.11) and hence $(*)$.
I.S.: For each form of $e$, the induction step is straightforward, hence we do not show the details. We note that in case $e=\left(\varphi \triangleright e^{\prime}\right)$ we use Lemma 14.1.1.

Example 14.2.2. Let $b \in B$ and $x$ be a first-order variable.

1. We define the family $\kappa(b)=\left(\kappa(b)_{k} \mid k \in \mathbb{N}\right)$ of mappings $\kappa(b)_{k}: \Sigma_{\{x\}}^{(k)} \rightarrow B$ such that, for each $(\sigma, U) \in \Sigma_{\{x\}}^{(k)}$, we let

$$
\kappa(b)_{k}((\sigma, U))= \begin{cases}b & \text { if } U=\{x\} \\ \mathbb{1} & \text { otherwise }\end{cases}
$$

Then,

$$
\begin{equation*}
\text { for every } \xi \in \mathrm{T}_{\Sigma} \text { and } w \in \operatorname{pos}(\xi) \text {, we have } \llbracket \mathrm{H}(\kappa(b)) \rrbracket_{\{x\}}(\xi,[x \mapsto w])=b \tag{14.12}
\end{equation*}
$$

This can be seen as follows:

$$
\begin{align*}
\llbracket \mathrm{H}(\kappa(b)) \rrbracket_{\{x\}}(\xi,[x \mapsto w])= & \mathrm{h}_{\kappa(b)[\{x\} \rightsquigarrow\{x\}]}(\xi,[x \mapsto w])=\mathrm{h}_{\kappa(b)}(\xi,[x \mapsto w]) \\
& =\bigotimes_{\substack{v \in \operatorname{pos}(\xi) \\
\text { in } \leq \mathrm{dpp}^{\text {order }}}} \kappa(b)_{\operatorname{rk}(\xi(v))}((\xi,[x \mapsto w])(v))  \tag{2.27}\\
& =\underbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}_{n-1} \otimes b \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}=b,
\end{align*}
$$

where we assume that $w$ occurs at the $n$-th position in the $\leq_{d p}$ order of $\operatorname{pos}(\xi)$.
2. Let us consider the $\operatorname{MSO}(\Sigma, \mathrm{B})$-sentence $\langle b\rangle$ defined by

$$
\langle b\rangle=\mathcal{F}_{x}(\operatorname{root}(x) \triangleright \mathrm{H}(\kappa(b)))
$$

where the family $\kappa(b)$ is defined in the first part of this example. It is clear that $\mathrm{L}_{\{x\}}(\operatorname{root}(x))=$ $\left\{(\xi,[x \mapsto \varepsilon]) \mid \xi \in \mathrm{T}_{\Sigma}\right\}$. Then, for each $\xi \in \mathrm{T}_{\Sigma}$, we have

$$
\begin{aligned}
\llbracket\langle b\rangle \rrbracket(\xi)= & \llbracket \mathcal{W}_{x}(\operatorname{root}(x) \triangleright \mathrm{H}(\kappa(b))) \rrbracket(\xi)=\bigoplus_{w \in \operatorname{pos}(\xi)} \llbracket \operatorname{root}(x) \triangleright \mathrm{H}(\kappa(b)) \rrbracket_{\{x\}}(\xi,[x \mapsto w]) \\
= & \left.\bigoplus_{w \in \operatorname{pos}(\xi)} \chi\left(\mathrm{L}_{\{x\}}(\operatorname{root}(x))\right)(\xi,[x \mapsto w]) \otimes \llbracket \mathrm{H}(\kappa(b)) \rrbracket_{\{x\}}(\xi,[x \mapsto w]) \quad \text { (by (14.10)}\right) \\
= & \bigoplus_{\substack{w \in \operatorname{pos}(\xi): \\
(\xi,[x \mapsto w]) \in \mathrm{L}_{\{x\}}(\operatorname{root}(x))}} \llbracket \mathrm{H}(\kappa(b)) \rrbracket_{\{x\}}(\xi,[x \mapsto w])=\llbracket \mathrm{H}(\kappa(b)) \rrbracket_{\{x\}}(\xi,[x \mapsto \varepsilon])
\end{aligned}
$$

$$
=b
$$

(by (14.12))
Hence $\llbracket\langle b\rangle \rrbracket=\widetilde{b}$.
3. Now let $\varphi$ be an $\operatorname{MSO}(\Sigma)$-formula. We consider the $\operatorname{MSO}(\Sigma, \mathrm{B})$-formula $\varphi \triangleright\langle b\rangle$. Then, for each finite set $\mathcal{V}$ of variables with $\operatorname{Free}(\varphi) \subseteq \mathcal{V}$ and $\xi \in \mathrm{T}_{\Sigma_{\mathcal{V}}}$, we have

$$
\begin{aligned}
\llbracket \varphi \triangleright\langle b\rangle \rrbracket \mathcal{V}(\xi) & =\chi\left(L_{\mathcal{V}}(\varphi)\right)(\xi) \otimes \llbracket\langle b\rangle \rrbracket \mathcal{V}(\xi) \\
& =\llbracket\langle b\rangle \rrbracket \mathcal{V}(\xi) \otimes \chi(\mathrm{L} \mathcal{V}(\varphi))(\xi) \\
& =\llbracket\langle b\rangle \rrbracket(\xi) \otimes \chi\left(\mathrm{L}_{\mathcal{V}}(\varphi)\right)(\xi) \\
& =\left(b \otimes \chi\left(\mathrm{~L}_{\mathcal{V}}(\varphi)\right)\right)(\xi) .
\end{aligned}
$$

(by (14.10))
(because $\chi(\mathrm{L}(\varphi))(\xi) \in\{\mathbb{0}, \mathbb{1}\}$ )
(by Lemma 14.2.1)
(because $\llbracket\langle b\rangle \rrbracket(\xi)=b$ )
Hence $\llbracket \varphi \triangleright\langle b\rangle \rrbracket \mathcal{V}=b \otimes \chi\left(\mathrm{~L}_{\mathcal{V}}(\varphi)\right)$.
Now we will define particular $\operatorname{MSO}(\Sigma, B)$-formulas called recognizable step formulas. We show that the semantics of recognizable step sentences are recognizable step mappings, and vice versa, each recognizable step mapping is the semantics of such a recognizable step sentence.

A $(\Sigma, \mathrm{B})$-recognizable step formula is a formula in $\operatorname{MSO}(\Sigma, \mathrm{B})$ of the form

$$
\begin{equation*}
\left(\varphi_{1} \triangleright\left\langle b_{1}\right\rangle\right)+\ldots+\left(\varphi_{n} \triangleright\left\langle b_{n}\right\rangle\right) \tag{14.13}
\end{equation*}
$$

where $\varphi_{i}$ is an $\operatorname{MSO}(\Sigma)$-formula and $\left\langle b_{i}\right\rangle$ is an $\operatorname{MSO}(\Sigma, \mathrm{B})$-sentence defined in Example 14.2.2(2) for each $i \in[n]$. If, in addition, $\varphi_{1}, \ldots, \varphi_{n}$ are sentences, then (14.13) is called a ( $\Sigma, \mathrm{B}$ )-recognizable step sentence.

By Example14.2.2(3), for each finite set $\mathcal{V}$ of variables such that $\mathcal{V} \supseteq \bigcup_{i \in[n]} \operatorname{Free}\left(\varphi_{i}\right)$, we have

$$
\begin{equation*}
\llbracket\left(\varphi_{1} \triangleright\left\langle b_{1}\right\rangle\right)+\ldots+\left(\varphi_{n} \triangleright\left\langle b_{n}\right\rangle\right) \rrbracket \mathcal{V}=\bigoplus_{i \in[n]} b_{i} \otimes \chi\left(\operatorname{L\mathcal {V}}\left(\varphi_{i}\right)\right) \tag{14.14}
\end{equation*}
$$

Lemma 14.2.3. Let $r: \mathrm{T}_{\Sigma} \rightarrow B$ be a weighted tree language, $n \in \mathbb{N}_{+}$, and $b_{1}, \ldots, b_{n} \in B$. Then the following two statements are equivalent.
(A) We can construct $\Sigma$-fta $A_{1}, \ldots, A_{n}$ such that $r=\bigoplus_{i \in[n]} b_{i} \otimes \chi\left(\mathrm{~L}\left(A_{i}\right)\right)$.
(B) We can construct sentences $\varphi_{1}, \ldots, \varphi_{n}$ in $\operatorname{MSO}(\Sigma)$ such that, for each $i \in[n]$, we have $r=\llbracket\left(\varphi_{1} \triangleright\left\langle b_{1}\right\rangle\right)+\ldots+\left(\varphi_{n} \triangleright\left\langle b_{n}\right\rangle\right) \rrbracket$.

Proof. The proof follows from Theorem $14.1 .3(\mathrm{~A}) \Leftrightarrow(\mathrm{B})$ and (14.14).
Example 14.2.4. In this example, the weight algebra is the semiring $\operatorname{Nat}=(\mathbb{N},+, \cdot, 0,1)$.

1. We consider the $\operatorname{MSO}(\Sigma, \mathrm{Nat})$-sentence

$$
e=\mathcal{F}_{x}(\varphi(x) \triangleright \mathrm{H}(\kappa(1))),
$$

where $\varphi(x)=\left(\operatorname{label}_{\sigma}(x) \vee \neg \operatorname{label}_{\sigma}(x)\right)$ for some $\sigma \in \Sigma$, and the $\mathbb{N}$-indexed family $\kappa(1)$ of mappings is defined in Example 14.2.2
Obviously, $\mathrm{L}_{\{x\}}(\varphi(x))=\mathrm{T}_{\Sigma_{\{x\}}}^{\mathrm{v}}$. Then, for each $\xi \in \mathrm{T}_{\Sigma}$, we have

$$
\begin{aligned}
& \llbracket e \rrbracket(\xi)=\llbracket \text { 耳 }_{x}(\varphi(x) \triangleright \mathrm{H}(\kappa(1))) \rrbracket(\xi)=\underset{w \in \operatorname{pos}(\xi)}{+} \llbracket \varphi(x) \triangleright \mathrm{H}(\kappa(1)) \rrbracket_{\{x\}}(\xi,[x \mapsto w]) \\
& \left.=\operatorname{H}_{w \in \operatorname{pos}(\xi)} \chi\left(\mathrm{L}_{\{x\}}(\varphi(x))\right)(\xi,[x \mapsto w]) \cdot \llbracket \mathrm{H}(\kappa(1)) \rrbracket_{\{x\}}(\xi,[x \mapsto w]) \quad \text { (by (14.10) }\right) \\
& =\operatorname{f}_{w \in \operatorname{pos}(\xi)} \chi\left(\mathrm{T}_{\Sigma_{\{x\}}}^{\mathrm{v}}\right)(\xi,[x \mapsto w]) \cdot \llbracket \mathrm{H}(\kappa(1)) \rrbracket_{\{x\}}(\xi,[x \mapsto w])
\end{aligned}
$$

$$
\begin{array}{ll}
={\underset{w \in \operatorname{pos}(\xi)}{ }}\left[\mathrm{H}(\kappa(1)) \rrbracket_{\{x\}}(\xi,[x \mapsto w])\right. & \text { (because } \left.(\xi,[x \mapsto w]) \in \mathrm{T}_{\Sigma_{\{x\}}^{\mathrm{v}}}\right) \\
={\underset{w \in \operatorname{pos}(\xi)}{ } 1} 1 & \text { (by (14.121) }) \\
=\operatorname{size}(\xi)
\end{array}
$$

Hence $\llbracket e \rrbracket=$ size.
2. Now let $\Sigma=\left\{\sigma^{(2)}, \gamma^{(1)}, \alpha^{(0)}\right\}$ and consider the $\operatorname{MSO}(\Sigma$, Nat $)$-sentence

$$
e=\boldsymbol{T}_{x}(\varphi(x) \triangleright \mathrm{H}(\kappa(1))),
$$

where $\varphi(x)=\left(\operatorname{label}_{\sigma}(x) \wedge \forall y .\left(\operatorname{edge}_{2}(x, y) \rightarrow \operatorname{label}_{\alpha}(y)\right)\right)$. It is easy to see that

$$
\mathrm{L}_{\{x\}}(\varphi(x))=\left\{(\xi,[x \mapsto w]) \mid \xi \in \mathrm{T}_{\Sigma}, w \in \operatorname{pos}(\xi), \xi(w)=\sigma \text { and } \xi(w 2)=\alpha\right\} .
$$

Then for each $\xi \in \mathrm{T}_{\Sigma}$, we have

$$
\begin{aligned}
& =\underset{\substack{w \in \operatorname{pos}(\xi): \\
\xi(w)=\sigma, \xi(w)=\alpha}}{ } \llbracket \mathrm{H}(\kappa(1)) \rrbracket_{\{x\}}(\xi,[x \mapsto w]) \\
& =\underset{\substack{w \in \operatorname{pos}(\xi): \\
\xi(w)=\sigma, \xi(w)=\alpha}}{ } 1 \\
& =\#_{\sigma(., \alpha)}(\xi) . \quad \text { (for } \#_{\sigma(., \alpha)} \text {, see Example 3.2.11) }
\end{aligned}
$$

Hence $\llbracket e \rrbracket=\#_{\sigma(., \alpha)}$.
Example 14.2.5. Here we show that the weighted tree language height: $\mathrm{T}_{\Sigma} \rightarrow \mathbb{N}$ is definable over the arctic semiring $\operatorname{Nat}_{\max ,+}=\left(\mathrm{N}_{-\infty}, \max ,+,-\infty, 0\right)$. We let $\varphi$ be the following $\operatorname{MSO}(\Sigma)$-formula with Free $(\varphi)=\{x, X\}$ :

$$
\begin{aligned}
\varphi(x, X) & =\operatorname{leaf}(x) \wedge \operatorname{Path}(X, x), \text { where } \\
\operatorname{leaf}(x) & =\neg \exists y \cdot \operatorname{edge}(x, y), \\
\operatorname{Path}(X, x) & =\forall y \cdot(y \in X) \leftrightarrow \operatorname{path}(y, x),
\end{aligned}
$$

and path $(y, x)$ is the formula defined in Example 14.1.2
The following is obvious:

$$
\begin{equation*}
\mathrm{L}_{\{x, X\}}(\varphi)=\left\{(\xi,[x \mapsto w, X \mapsto \operatorname{prefix}(w)]) \mid \xi \in \mathrm{T}_{\Sigma}, w \in \operatorname{pos}(\xi), \xi(w) \in \Sigma^{(0)}\right\} \tag{14.15}
\end{equation*}
$$

Then we consider the $\operatorname{MSO}\left(\Sigma, \mathrm{Nat}_{\text {max },+}\right)$-formula

$$
e_{\text {height }}=\mathcal{T}_{x} \mathcal{T}_{X}(\varphi(x, X) \triangleright \mathrm{H}(\kappa))
$$

where $\kappa=\left(\kappa_{k} \mid k \in \mathbb{N}\right)$ and $\kappa_{k}:\left(\Sigma_{\{x, X\}}\right)^{(k)} \rightarrow B$ defined, for each $(\sigma, U) \in\left(\Sigma_{\{x, X\}}\right)^{(k)}$, by

$$
\kappa_{k}((\sigma, U))= \begin{cases}1 & \text { if } X \in U \text { and } k \geq 1 \\ 0 & \text { otherwise }\end{cases}
$$

It is clear that, for every $(\xi,[x \mapsto w, X \mapsto \operatorname{prefix}(w)]) \in \mathrm{L}_{\{x, X\}}(\varphi)$, we have

$$
\begin{equation*}
\llbracket \mathrm{H}(\kappa) \rrbracket_{\{x, X\}}(\xi,[x \mapsto w, X \mapsto \operatorname{prefix}(w)])=|w| . \tag{14.16}
\end{equation*}
$$

Then, for each $\xi \in \mathrm{T}_{\Sigma}$, we have

$$
\begin{align*}
\llbracket e_{\text {height }} \rrbracket(\xi) & =\llbracket \mathcal{F}_{x} \text { 十 }_{X}(\varphi(x, X) \triangleright \mathrm{H}(\kappa)) \rrbracket(\xi) \\
& =\max \left(\max \left(\llbracket \varphi(x, X) \triangleright \mathrm{H}(\kappa) \rrbracket_{\{x, X\}}(\xi[x \mapsto w, X \mapsto W]) \mid W \subseteq \operatorname{pos}(\xi)\right) \mid w \in \operatorname{pos}(\xi)\right) \\
& =\max \left(\llbracket \varphi(x, X) \triangleright \mathrm{H}(\kappa) \rrbracket_{\{x, X\}}(\xi[x \mapsto w, X \mapsto W]) \mid W \subseteq \operatorname{pos}(\xi), w \in \operatorname{pos}(\xi)\right) \\
& =\max \left(\llbracket \mathrm{H}(\kappa) \rrbracket_{\{x, X\}}\left(\xi[x \mapsto w, X \mapsto \operatorname{prefix}(w) \rrbracket) \mid w \in \operatorname{pos}(\xi), \xi(w) \in \Sigma^{(0)}\right) \quad(\text { by (14.15) })\right. \\
& =\max \left(|w| \mid w \in \operatorname{pos}(\xi), \xi(w) \in \Sigma^{(0)}\right)  \tag{14.16}\\
& =\operatorname{height}(\xi) .
\end{align*}
$$

We finish this section with two easy properties. Roughly speaking, they say that weighted existential quantification can be expressed by deterministic tree relabelings, where in the first-order case we have to take care of preserving validity.

Lemma 14.2.6. Let $e \in \operatorname{MSO}(\Sigma, \mathrm{~B})$.
(1) Let $\mathcal{V}=\operatorname{Free}\left(+_{x} e\right)$. Then $\llbracket+_{x} e \rrbracket=\chi(\tau)\left(\llbracket e \rrbracket_{\mathcal{V} \cup\{x\}}\right) \otimes \chi\left(\mathrm{T}_{\Sigma_{\mathcal{V}}}^{\mathrm{v}}\right)$, where $\tau=\left(\tau_{k} \mid k \in \mathbb{N}\right)$ is the deterministic $\left(\Sigma_{\mathcal{V} \cup\{x\}}, \Sigma_{\mathcal{V}}\right)$-tree relabeling such that, for every $k \in \mathbb{N}, \sigma \in \Sigma^{(k)}$, and $\mathcal{W} \subseteq \mathcal{V} \cup\{x\}$, we let $\tau_{k}((\sigma, \mathcal{W}))=\{(\sigma, \mathcal{W} \backslash\{x\})\}$.
(2) Let $\mathcal{V}=\operatorname{Free}\left(+_{X} e\right)$. Then $\llbracket+_{X} e \rrbracket=\chi(\tau)\left(\llbracket e \rrbracket_{\mathcal{V} \cup\{X\}}\right)$, where $\tau=\left(\tau_{k} \mid k \in \mathbb{N}\right)$ is the deterministic $\left(\Sigma_{\mathcal{V} \cup\{X\}}, \Sigma_{\mathcal{V}}\right)$-tree relabeling such that, for every $k \in \mathbb{N}, \sigma \in \Sigma^{(k)}$, and $\mathcal{W} \subseteq \mathcal{V} \cup\{X\}$, we let $\tau_{k}((\sigma, \mathcal{W}))=\{(\sigma, \mathcal{W} \backslash\{X\})\}$.

Proof. Proof of (1): Let $\mathcal{V}=\operatorname{Free}\left(+_{x} e\right)$ and $\xi \in \mathrm{T}_{\Sigma_{\mathcal{V}}}$. We note that $x \notin \mathcal{V}$. We distinguish the following two cases.

Case (a): Let $\xi \notin \mathrm{T}_{\Sigma_{\mathcal{V}}}^{\mathrm{v}}$. Then $\llbracket+_{x} e \rrbracket(\xi)=\mathbb{0}=\chi(\tau)(\llbracket e \rrbracket \mathcal{V} \cup\{x\})(\xi) \otimes \chi\left(\mathrm{T}_{\Sigma_{\mathcal{V}}}^{\mathrm{v}}\right)(\xi)$.
Case (b): Let $\xi \in \mathrm{T}_{\Sigma_{\mathcal{V}}}^{\mathrm{v}}$. Then we can calculate as follows:

$$
\begin{aligned}
& \llbracket+_{x} e \rrbracket(\xi)=\bigoplus_{w \in \operatorname{pos}(\xi)} \llbracket e \rrbracket_{\mathcal{V} \cup\{x\}}(\xi[x \mapsto w]) \\
& =\bigoplus_{\zeta \in\{\xi[x \mapsto w] \mid w \in \operatorname{pos}(\xi)\}} \llbracket e \rrbracket_{\mathcal{V} \cup\{x\}}(\zeta) \quad \text { (because } \xi[x \mapsto w] \neq \xi\left[x \mapsto w^{\prime}\right] \text { for } w \neq w^{\prime} \text { ) } \\
& =\bigoplus_{\left.\zeta \in \tau^{-1}(\xi) \cap T_{\mathcal{V}_{\mathcal{V} \cup\{x\}}} \llbracket e \rrbracket_{\mathcal{V} \cup\{x\}}(\zeta) \quad \quad \text { (by definition of } \tau\right)} \\
& \left.=\bigoplus_{\zeta \in \tau^{-1}(\xi)} \llbracket e \rrbracket_{\mathcal{V} \cup\{x\}}(\zeta) \quad \text { (because } \llbracket e \rrbracket_{\mathcal{V} \cup\{x\}}(\zeta)=\mathbb{0} \text { for } \zeta \in \tau^{-1}(\xi) \backslash T_{\Sigma_{\mathcal{V} \cup\{x\}}^{\mathrm{v}}}\right) \\
& =\chi(\tau)\left(\llbracket e \rrbracket_{\mathcal{V} \cup\{x\}}\right)(\xi) \\
& =\left(\chi(\tau)\left(\llbracket \mathbb{e}_{\mathcal{V} \cup\{x\}}\right) \otimes \chi\left(T_{\Sigma_{\mathcal{V}}}^{v}\right)\right)(\xi) . \quad\left(\text { because } \chi\left(T_{\Sigma_{\nu}}^{v}\right)(\xi)=\mathbb{1}\right)
\end{aligned}
$$

Proof of (2): Let $\mathcal{V}=\operatorname{Free}\left(+_{X} e\right)$ and $\xi \in \mathrm{T}_{\Sigma_{\mathcal{V}}}$. We note that $X \notin \mathcal{V}$. Since $\mathcal{U}^{(1)}=\mathcal{V}^{(1)}$ (i.e., the set of first-order variables of $\mathcal{U}$ is the same as the set of first-order variables of $\mathcal{V}$ ), we have

$$
\begin{equation*}
\text { for every } \xi \in \mathrm{T}_{\Sigma_{\mathcal{V}}} \text { and } \zeta \in \tau^{-1}(\xi): \xi \in \mathrm{T}_{\Sigma_{\mathcal{V}}}^{\mathrm{v}} \text { iff } \zeta \in \mathrm{T}_{\Sigma_{\mathcal{V} \cup\{X\}}^{\mathrm{v}}} \tag{14.17}
\end{equation*}
$$

We distinguish the following two cases.

Case (a): Let $\xi \notin \mathrm{T}_{\Sigma_{\mathcal{V}}}^{\mathrm{v}}$. Then $\llbracket+_{X} e \rrbracket(\xi)=\mathbb{O}=\chi(\tau)\left(\llbracket e \rrbracket_{\mathcal{V} \cup\{x\}}\right)(\xi)$ where the second equality holds by (14.17).
$\underline{\text { Case (b): Let } \xi \in \mathrm{T}_{\Sigma_{\mathcal{V}}}^{\mathrm{v}} \text {. Then we can calculate as follows: }}$

$$
\left.\begin{array}{rl}
\llbracket \mathcal{T}_{X} e \rrbracket(\xi) & =\bigoplus_{W \subseteq \operatorname{pos}(\xi)} \llbracket e \rrbracket_{\mathcal{V} \cup\{X\}}(\xi[X \mapsto W]) \\
& \left.=\bigoplus_{\zeta \in\{\xi[X \mapsto W] \mid W \subseteq \operatorname{pos}(\xi)\}} \llbracket e \rrbracket_{\mathcal{V} \cup\{X\}}(\zeta) \quad \text { (because } \xi[X \mapsto W] \neq \xi\left[X \mapsto W^{\prime}\right] \text { for } W \neq W^{\prime}\right) \\
& =\bigoplus_{\zeta \in \tau^{-1}(\xi)} \llbracket e \rrbracket_{\mathcal{V} \cup\{X\}}(\zeta) \\
& =\chi(\tau)\left(\llbracket e \rrbracket_{\mathcal{V} \cup\{X\}}\right)(\xi) \tag{2.30}
\end{array} \quad \text { (by definition of } \tau\right)
$$

### 14.3 The main result

The main theorem of this chapter will be the following B-E-T theorem for weighted tree languages over strong bimonoids.

Theorem 14.3.1. Let $\Sigma$ be a ranked alphabet, B be a strong bimonoid, and $r: \mathrm{T}_{\Sigma} \rightarrow B$. Then the following two statements are equivalent.
(A) We can construct a $(\Sigma, \mathrm{B})-w t a \mathcal{A}$ such that $r=\llbracket \mathcal{A} \rrbracket^{\text {run }}$.
(B) We can construct a sentence $e \in \operatorname{MSO}(\Sigma, \mathrm{~B})$ such that $r=\llbracket e \rrbracket$.

This theorem follows from Theorems 14.3 .2 and 14.3 .8 , which we will prove in the next two subsections.

### 14.3.1 From recognizable to definable

We prove that, for each $(\Sigma, \mathrm{B})$-wta $\mathcal{A}$, we can construct a sentence $e \in \operatorname{MSO}(\Sigma, \mathrm{~B})$ such that the run semantics $\llbracket \mathcal{A} \rrbracket^{\text {run }}$ is equal to $\llbracket e \rrbracket$. In the usual way, we use second-order quantifications to label positions of the given tree $\xi \in \mathrm{T}_{\Sigma}$ by transitions of $\mathcal{A}$; by means of a formula of $\operatorname{MSO}(\Sigma)$-logic we can check whether this labeling corresponds to a run on $\xi$. Finally, we define a mapping $\kappa$ which translates each transition into its weight.

Theorem 14.3.2. FSV12, Lm. 4.2] For each $(\Sigma, \mathrm{B})-w t a \mathcal{A}$, we can construct a sentence $e \in \operatorname{MSO}(\Sigma, \mathrm{~B})$ such that $\llbracket e \rrbracket=\llbracket \mathcal{A} \rrbracket^{\text {run }}$.

Proof. Let $\mathcal{A}=(Q, \delta, F)$ be a $(\Sigma, \mathrm{B})$-wta. By Theorem 7.3.1 we can assume that $\mathcal{A}$ is root weight normalized. Thus $\operatorname{supp}(F)$ contains exactly one element, say, $q_{f}$, and $F_{q_{f}}=\mathbb{1}$.

We define the set

$$
\mathcal{U}=\bigcup\left(Q^{k} \times Q \mid k \in \mathbb{N} \text { such that } \mathrm{rk}^{-1}(k) \neq \emptyset\right)
$$

Let $n \in \mathbb{N}_{+}$be the cardinality of $\mathcal{U}$. Then we choose an arbitrary bijection $\nu: \mathcal{U} \rightarrow\left\{X_{1}, \ldots, X_{n}\right\}$ and fix it for the rest of the proof. In the sequel, we will not distinguish between a state behaviour $\left(q_{1} \cdots q_{k}, q\right) \in \mathcal{U}$ and the second-order variable $\nu\left(\left(q_{1} \cdots q_{k}, q\right)\right)$. Thus, it is legitimate to consider each $\left(q_{1} \cdots q_{k}, q\right) \in \mathcal{U}$ as a second-order variable. Then, in particular, $\mathcal{U}=\mathcal{U}^{(2)}$ and hence $\mathrm{T}_{\Sigma_{\mathcal{U}}}=\mathrm{T}_{\Sigma_{\mathcal{U}}}^{\mathrm{v}}$.

Moreover, for every $\xi \in \mathrm{T}_{\Sigma}$ and $\rho \in \mathrm{R}_{\mathcal{A}}(\xi)$, we define the $\mathcal{U}$-assignment $\eta_{\xi, \rho} \in \Phi_{\mathcal{U}, \xi}$ as follows. For each $\left(q_{1} \cdots q_{k}, q\right) \in \mathcal{U}$, we let

$$
\eta_{\xi, \rho}\left(\left(q_{1} \cdots q_{k}, q\right)\right)=\left\{w \in \operatorname{pos}(\xi) \mid \rho(w)=q, \operatorname{rk}(\xi(w))=k,(\forall i \in[k]): \rho(w i)=q_{i}\right\}
$$

Now we define the formula $\varphi \in \operatorname{MSO}(\Sigma)$ where $\operatorname{Free}(\varphi)=\mathcal{U}$ with the following intention:

$$
\begin{equation*}
\text { The mapping } f: \mathrm{R}_{\mathcal{A}}\left(q_{f}, \xi\right) \rightarrow\left\{\eta \in \Phi_{\mathcal{U}, \xi} \mid(\xi, \eta) \in \mathrm{L}_{\mathcal{U}}(\varphi)\right\} \text { which is } \tag{14.18}
\end{equation*}
$$ defined for each $\rho \in \mathrm{R}_{\mathcal{A}}\left(q_{f}, \xi\right)$ by $f(\rho)=\eta_{\xi, \rho}$, is bijective.

For this, we let

$$
\begin{equation*}
\varphi=\varphi_{\mathrm{part}} \wedge \varphi_{\mathrm{run}} \wedge \varphi_{\mathrm{suc}} \tag{14.19}
\end{equation*}
$$

where

- $\varphi_{\text {part }}$ checks whether the family $(\eta(X) \mid X \in \mathcal{U})$ forms a partitioning of $\operatorname{pos}(\xi)$, i.e., whether each position of $\xi$ is assigned to exactly one transition,
- $\varphi_{\text {run }}$ checks whether $\eta$ codes a run in $\mathrm{R}_{\mathcal{A}}(\xi)$, and
- $\varphi_{\text {suc }}$ checks whether the target state of the transition associated to the root of $\xi$ is $q_{f}$.

Formally, we define

$$
\begin{aligned}
\varphi_{\text {part }} & =\forall x \cdot\left(\bigvee_{X \in \mathcal{U}}\left(x \in X \wedge \bigwedge_{Y \in \mathcal{U} \backslash\{X\}} \neg(x \in Y)\right)\right) \\
\varphi_{\text {run }} & =\forall x . \bigwedge_{\left(q_{1} \cdots q_{k}, q\right) \in \mathcal{U}}\left[\left(x \in\left(q_{1} \cdots q_{k}, q\right)\right) \rightarrow \bigwedge_{i \in[k]}\left(\forall y \cdot \operatorname{edge}_{i}(x, y) \rightarrow \bigvee_{\left(q_{1}^{\prime} \cdots q_{\ell}^{\prime}, q_{i}\right) \in \mathcal{U}} y \in\left(q_{1}^{\prime} \cdots q_{\ell}^{\prime}, q_{i}\right)\right)\right] \\
\varphi_{\text {suc }} & =\forall x \cdot \operatorname{root}(x) \rightarrow \bigvee_{\left(q_{1} \cdots q_{k}, q_{f}\right) \in \mathcal{U}} x \in\left(q_{1} \cdots q_{k}, q_{f}\right)
\end{aligned}
$$

It is easy to check that (14.18) holds.
Now we define the $\mathbb{N}$-indexed family $\left(\kappa_{k} \mid k \in \mathbb{N}\right)$ with $\kappa_{k}: \Sigma_{\mathcal{U}}^{(k)} \rightarrow B$ which supplies the weights. For every $k \in \mathbb{N}, \sigma \in \Sigma^{(k)}$, and $\mathcal{W} \subseteq \mathcal{U}$ we let

$$
\kappa_{k}((\sigma, \mathcal{W}))= \begin{cases}\delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right) & \text { if } \mathcal{W}=\left\{\left(q_{1} \cdots q_{k}, q\right)\right\}  \tag{14.20}\\ \mathbb{O} & \text { otherwise }\end{cases}
$$

Let $\xi \in \mathrm{T}_{\Sigma}$ and $\rho \in \mathrm{R}_{\mathcal{A}}(\xi)$ be arbitrary but fixed. We consider $\left(\xi, \eta_{\xi, \rho}\right)$ to be an element of $\mathrm{T}_{\Sigma_{\mathcal{U}}}^{\mathrm{v}}$ (by the identification which we discussed in Section 14.1).

For the inductive proof of the next statement, we define the well-founded set

$$
(\operatorname{pos}(\xi), \prec)
$$

where the binary relation $\prec$ on $\operatorname{pos}(\xi)$ is defined as follows. For every $w_{1}, w_{2} \in \operatorname{pos}(\xi)$, we let $w_{1} \prec w_{2}$ if there exists $i \in \mathbb{N}$ such that $w_{1}=w_{2} i$. Obviously, $\prec$ is well-founded and $\min _{\prec}(\operatorname{pos}(\xi))=\operatorname{pos}_{\Sigma^{(0)}}(\xi)$, i.e., it is the set of leaves of $\xi$ (cf. the proof of Lemma 10.8.1). Then we can prove the next statement by induction on $(\operatorname{pos}(\xi), \prec)$.

$$
\begin{equation*}
\text { For every } w \in \operatorname{pos}(\xi): \mathrm{h}_{\kappa}\left(\left.\left(\xi, \eta_{\xi, \rho}\right)\right|_{w}\right)=\mathrm{wt}_{\mathcal{A}}\left(\left.\xi\right|_{w},\left.\rho\right|_{w}\right) . \tag{14.21}
\end{equation*}
$$

For this, let $w \in \operatorname{pos}(\xi)$ with $\sigma=\xi(w)$, and $k=\operatorname{rk}_{\Sigma}(\sigma)$. Then we have

$$
\left(\xi, \eta_{\xi, \rho}\right)(w)=\left(\sigma,\left\{X \in \mathcal{U} \mid w \in \eta_{\xi, \rho}(X)\right\}\right)=(\sigma,\{(\rho(w 1) \cdots \rho(w k), \rho(w))\})
$$

Using $t$ as abbreviation for $(\rho(w 1) \cdots \rho(w k), \rho(w))$ we obtain

$$
\begin{equation*}
\mathrm{h}_{\kappa}\left(\left.\left(\xi, \eta_{\xi, \rho}\right)\right|_{w}\right)=\left(\bigotimes_{i \in[k]} \mathrm{h}_{\kappa}\left(\left.\left(\xi, \eta_{\xi, \rho}\right)\right|_{w i}\right)\right) \otimes \kappa_{k}(t) \tag{2.26}
\end{equation*}
$$

$$
\begin{array}{lr}
\left.=\left(\bigotimes_{i \in[k]} \mathrm{wt}_{\mathcal{A}}\left(\left.\xi\right|_{w i},\left.\rho\right|_{w i}\right)\right)\right) \otimes \delta_{k}(t) & \text { (by I.H. and the definition of } \kappa \text { ) } \\
\left.=\mathrm{wt}_{\mathcal{A}}\left(\left.\xi\right|_{w},\left.\rho\right|_{w}\right)\right) . & \text { (by Observation 3.1.1) }
\end{array}
$$

Now we define the sentence $e \in \operatorname{MSO}(\Sigma, \mathrm{~B})$ and prove that it simulates $\mathcal{A}$ (we recall that we do not distinguish between elements of $\mathcal{U}$ and elements of $\left.\left\{X_{1}, \ldots, X_{n}\right\}\right)$ :

$$
\begin{equation*}
e=\mathcal{F}_{X_{1}} \cdots \mathcal{F}_{X_{n}}(\varphi \triangleright \mathrm{H}(\kappa)) . \tag{14.22}
\end{equation*}
$$

Then, for each $\xi \in \mathrm{T}_{\Sigma}$, we have:

$$
\begin{align*}
& \llbracket e \rrbracket(\xi)=\bigoplus_{W_{1}, \ldots, W_{n} \subseteq \operatorname{pos}(\xi)} \llbracket \varphi \triangleright \mathrm{H}(\kappa) \rrbracket \mathcal{u}\left(\xi\left[X_{1} \mapsto W_{1}, \ldots, X_{n} \rightarrow W_{n}\right]\right) \\
& =\bigoplus_{\eta \in \Phi_{\mathcal{U}, \xi}} \llbracket \varphi \triangleright \mathrm{H}(\kappa) \rrbracket_{\mathcal{U}}(\xi, \eta)=\bigoplus_{\substack{\eta \in \Phi_{\mathcal{U}, \xi}: \\
(\xi, \eta) \in \mathrm{L}_{\mathcal{U}}(\varphi)}} \llbracket \mathrm{H}(\kappa) \rrbracket_{\mathcal{U}}(\xi, \eta) \\
& =\bigoplus_{\eta \in \Phi_{\mathcal{U}, \xi:}} \mathrm{h}_{\kappa}((\xi, \eta)) \quad \text { (because } \kappa[\mathcal{U} \rightsquigarrow \mathcal{U}]=\kappa \text { ) } \\
& (\xi, \eta) \in \mathrm{L} \dot{u}(\varphi) \\
& =\bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}}\left(q_{f}, \xi\right)} \mathrm{h}_{\kappa}\left(\left(\xi, \eta_{\xi, \rho}\right)\right)  \tag{14.18}\\
& \left.=\bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}}\left(q_{f}, \xi\right)} \mathrm{wt}_{\mathcal{A}}(\xi, \rho) \quad \quad \text { (by (14.21) for } w=\varepsilon\right) \\
& =\llbracket \mathcal{A} \rrbracket^{\text {run }}(\xi) .
\end{align*}
$$

Example 14.3.3. We illustrate the definition of the $\operatorname{MSO}(\Sigma, \mathrm{B})$-formula (14.22) in the proof of Theorem 14.3.2. For this we consider the root weight normalized ( $\left.\Sigma, \mathrm{Nat}_{\mathrm{max},+}\right)$-wta $\mathcal{A}=(Q, \delta, F)$ from Example 3.2.4 which i-recognizes the $\left(\Sigma, \mathrm{Nat}_{\text {max },+}\right)$-weighted tree language height. We recall that $\Sigma=\left\{\sigma^{(2)}, \alpha^{(0)}\right\}$, $\mathrm{Nat}_{\text {max },+}=\left(\mathbb{N}_{-\infty}, \max ,+,-\infty, 0\right), Q=\{\mathrm{h}, 0\}, \delta_{0}(\varepsilon, \alpha, \mathrm{~h})=\delta_{0}(\varepsilon, \alpha, 0)=0$ and for every $q_{1}, q_{2}, q \in Q$,

$$
\delta_{2}\left(q_{1} q_{2}, \sigma, q\right)= \begin{cases}1 & \text { if } q_{1} q_{2} q \in\{\mathrm{~h} 0 \mathrm{~h}, 0 \mathrm{hh}\} \\ 0 & \text { if } q_{1} q_{2} q=000 \\ -\infty & \text { otherwise }\end{cases}
$$

and $F_{\mathrm{h}}=0$ and $F_{0}=-\infty$.
The set $\mathcal{U}$ of second-order variables is the following:

$$
\begin{aligned}
\mathcal{U}=\{ & (\varepsilon, \alpha, \mathrm{h}),(\varepsilon, \alpha, 0) \\
& (00, \sigma, 0),(0 \mathrm{~h}, \sigma, 0),(\mathrm{h} 0, \sigma, 0),(\mathrm{hh}, \sigma, 0) \\
& (00, \sigma, \mathrm{~h}),(0 \mathrm{~h}, \sigma, \mathrm{~h}),(\mathrm{h} 0, \sigma, \mathrm{~h}),(\mathrm{hh}, \sigma, \mathrm{~h})\} .
\end{aligned}
$$

Let $\varphi \in \operatorname{MSO}(\Sigma)$ be the instance of the MSO formula (14.19) for the particular ( $\Sigma, \operatorname{Nat}_{\text {max, }}$ )-wta $\mathcal{A}$ above. We will not give further details of $\varphi$. Instead, in Figure 14.2 we show the tree $\left(\xi, \eta_{2}\right) \in \mathrm{T}_{\Sigma_{\mathcal{U}}}^{\mathrm{v}}$ which satisfies $\varphi$, where $\xi=\sigma(\sigma(\alpha, \alpha), \alpha)$ and the assignment $\eta_{2}$ is defined in Figure 14.1

Next we instantiate the family of (14.20) to the particular ( $\Sigma$, $\mathrm{Nat}_{\text {max },+}$ )-wta $\mathcal{A}$ above. This provides the following $\mathbb{N}$-indexed family $\kappa=\left(\kappa_{k} \mid k \in \mathbb{N}\right)$ with $\kappa_{k}:\left(\Sigma_{\mathcal{U}}\right)^{(k)} \rightarrow \mathbb{N}_{-\infty}$ :

$$
\begin{aligned}
& \kappa_{0}\left((\alpha,\{(\varepsilon, \alpha, h)\})=\delta_{0}(\varepsilon, \alpha, \mathrm{~h})=0\right. \\
& \kappa_{0}\left((\alpha,\{(\varepsilon, \alpha, 0)\})=\delta_{0}(\varepsilon, \alpha, 0)=0\right.
\end{aligned}
$$

| $X$ | $\eta_{1}(X)$ | $\eta_{2}(X)$ | $\eta_{3}(X)$ |
| ---: | :---: | :---: | :---: |
| $(\varepsilon, \alpha, \mathrm{h})$ | $\{11\}$ | $\{12\}$ | $\{2\}$ |
| $(\varepsilon, \alpha, 0)$ | $\{12,2\}$ | $\{11,2\}$ | $\{11,12\}$ |
| $(00, \sigma, 0)$ | $\emptyset$ | $\emptyset$ | $\{1\}$ |
| $(0 \mathrm{~h}, \sigma, 0)$ | $\emptyset$ | $\emptyset$ | $\emptyset$ |
| $(\mathrm{h} 0, \sigma, 0)$ | $\emptyset$ | $\emptyset$ | $\emptyset$ |
| $(\mathrm{hh}, \sigma, 0)$ | $\emptyset$ | $\emptyset$ | $\emptyset$ |
| $(00, \sigma, \mathrm{~h})$ | $\emptyset$ | $\emptyset$ | $\emptyset$ |
| $(0 \mathrm{~h}, \sigma, \mathrm{~h})$ | $\emptyset$ | $\{1\}$ | $\{\varepsilon\}$ |
| $(\mathrm{h} 0, \sigma, \mathrm{~h})$ | $\{\varepsilon, 1\}$ | $\{\varepsilon\}$ | $\emptyset$ |
| $(\mathrm{hh}, \sigma, \mathrm{h})$ | $\emptyset$ | $\emptyset$ | $\emptyset$ |

Figure 14.1: The $\mathcal{U}$-assignments $\eta_{1}, \eta_{2}$, and $\eta_{3}$ for $\xi$.

$$
\begin{gathered}
\kappa_{2}\left((\sigma,\{(\mathrm{~h} 0, \sigma, \mathrm{~h})\})=\delta_{2}(\mathrm{~h} 0, \sigma, \mathrm{~h})=1\right. \\
\kappa_{2}\left((\sigma,\{(0 \mathrm{~h}, \sigma, \mathrm{~h})\})=\delta_{2}(0 \mathrm{~h}, \sigma, \mathrm{~h})=1\right. \\
\kappa_{2}\left((\sigma,\{(00, \sigma, 0)\})=\delta_{2}(00, \sigma, 0)=0\right.
\end{gathered}
$$

and for each other argument, $\kappa_{0}$ and $\kappa_{2}$ yield $-\infty$. Then we instantiate the formula (14.22) to be the $\operatorname{MSO}\left(\Sigma, \mathrm{Nat}_{\mathrm{max},+}\right)$-formula $e$ is defined by

$$
e=\Psi_{(\varepsilon, \alpha, \mathrm{h})} 十_{(\varepsilon, \alpha, 0)} 十_{(00, \sigma, 0)} \cdots \mathcal{F}_{(\mathrm{hh}, \sigma, \mathrm{~h})} \varphi \triangleright \mathrm{H}(\kappa)
$$

Next we compute $\llbracket e \rrbracket(\xi)$ for the tree $\xi=\sigma(\sigma(\alpha, \alpha), \alpha)$. First, we note that

$$
\left\{(\xi, \eta) \in \mathrm{T}_{\Sigma_{\mathcal{U}}} \mid(\xi, \eta) \models \varphi\right\}=\left\{\left(\xi, \eta_{1}\right),\left(\xi, \eta_{2}\right),\left(\xi, \eta_{3}\right)\right\}
$$

where the $\mathcal{U}$-assignments $\eta_{1}, \eta_{2}$, and $\eta_{3}$ for $\xi$ are defined in the table shown in Figure 14.1 (where $X \in \mathcal{U}$ ).
By applying the unique $\Sigma_{\mathcal{U}}$-algebra homomorphism $\mathrm{h}_{\kappa}$ to $\left(\xi, \eta_{2}\right)$, we obtain

$$
\begin{aligned}
\mathrm{h}_{\kappa}\left(\left(\xi, \eta_{2}\right)\right) & =\mathrm{h}_{\kappa}\left(\left.\left(\xi, \eta_{2}\right)\right|_{1}\right)+\mathrm{h}_{\kappa}\left(\left.\left(\xi, \eta_{2}\right)\right|_{2}\right)+\kappa_{2}((\sigma,\{(\mathrm{~h} 0, \sigma, h)\})) \\
& =\mathrm{h}_{\kappa}\left(\left.\left(\xi, \eta_{2}\right)\right|_{1}\right)+\mathrm{h}_{\kappa}\left(\left.\left(\xi, \eta_{2}\right)\right|_{2}\right)+1 \\
& =\left(\mathrm{h}_{\kappa}\left(\left.\left(\xi, \eta_{2}\right)\right|_{11}\right)+\mathrm{h}_{\kappa}\left(\left.\left(\xi, \eta_{2}\right)\right|_{12}\right)+\kappa_{2}((\sigma,\{(0 \mathrm{~h}, \sigma, h)\}))\right)+\mathrm{h}_{\kappa}\left(\left.\left(\xi, \eta_{2}\right)\right|_{2}\right)+1 \\
& =\left(\mathrm{h}_{\kappa}\left(\left.\left(\xi, \eta_{2}\right)\right|_{11}\right)+\mathrm{h}_{\kappa}\left(\left.\left(\xi, \eta_{2}\right)\right|_{12}\right)+1\right)+\mathrm{h}_{\kappa}\left(\left.\left(\xi, \eta_{2}\right)\right|_{2}\right)+1 \\
& =\left(\kappa _ { 0 } \left((\alpha,\{(\varepsilon, \alpha, 0)\})+\kappa_{0}((\alpha,\{(\varepsilon, \alpha, \mathrm{~h})\})+1)+\kappa_{0}((\alpha,\{(\varepsilon, \alpha, 0)\})+1\right.\right. \\
& =(0+0+1)+0+1=2 .
\end{aligned}
$$

By similar computations we obtain $\mathrm{h}_{\kappa}\left(\left(\xi, \eta_{1}\right)\right)=2$ and $\mathrm{h}_{\kappa}\left(\left(\xi, \eta_{3}\right)\right)=1$. Finally,

$$
\begin{aligned}
\llbracket e \rrbracket(\xi) & =\max \left(\llbracket \mathrm{H}(\kappa) \rrbracket \mathcal{U}\left(\xi, \eta_{1}\right), \llbracket \mathrm{H}(\kappa) \rrbracket \mathcal{U}\left(\xi, \eta_{2}\right), \llbracket \mathrm{H}(\kappa) \rrbracket \mathcal{U}\left(\xi, \eta_{3}\right)\right) \\
& =\max \left(\mathrm{h}_{\kappa}\left(\left(\xi, \eta_{1}\right)\right), \mathrm{h}_{\kappa}\left(\left(\xi, \eta_{2}\right)\right), \mathrm{h}_{\kappa}\left(\left(\xi, \eta_{3}\right)\right)\right) \\
& =\max (2,2,1)=2=\operatorname{height}(\xi) .
\end{aligned}
$$

### 14.3.2 From definable to recognizable

By induction on $\left(\operatorname{MSO}(\Sigma, \mathrm{B}), \prec_{\operatorname{MSO}(\Sigma, \mathrm{B})}\right)$, we prove that, for each formula $e \in \operatorname{MSO}(\Sigma, \mathrm{~B})$, we can construct a $\left(\Sigma_{\text {Free }(e)}, \mathrm{B}\right)$-wta $\mathcal{A}$ such that $\llbracket \mathcal{A} \rrbracket^{\text {run }}=\llbracket e \rrbracket$.

For this we make some preparation. As first preparation, we prove a consistency lemma for wta.


Figure 14.2: The $\Sigma_{\mathcal{U}}$-tree $\left(\xi, \eta_{2}\right)$; it satisfies $\varphi$.

Lemma 14.3.4. Let $e \in \operatorname{MSO}(\Sigma, \mathrm{~B})$ with $\operatorname{Free}(e)=\mathcal{V}$, and $\mathcal{A}$ be a $\left(\Sigma_{\mathcal{V}}, \mathrm{B}\right)$-wta such that $\llbracket \mathcal{A} \rrbracket^{\text {run }}=\llbracket e \rrbracket$. Moreover, let $V$ be a first-order variable or a second-order variable. Then we can construct a $\left(\Sigma_{\mathcal{V} \cup\{V\}}, B\right)$ wta $\mathcal{A}^{\prime}$ such that $\llbracket \mathcal{A}^{\prime} \rrbracket^{\text {run }}=\llbracket e \rrbracket_{\mathcal{V} \cup\{V\}}$.

Proof. Case (a): Let $V \in \mathcal{V}$. Then we can choose $\mathcal{A}^{\prime}=\mathcal{A}$ and we are ready.
Case (b): Let $V \notin \mathcal{V}$. Moreover, let $\mathcal{A}=(Q, \delta, F)$. First we construct the $\left(\Sigma_{\mathcal{V} \cup\{V\}}, \mathrm{B}\right)$-wta $\mathcal{B}=$


$$
\delta_{k}^{\prime}\left(q_{1} \cdots q_{k},(\sigma, \mathcal{W}), q\right)=\delta_{k}\left(q_{1} \cdots q_{k},(\sigma, \mathcal{W} \backslash\{V\}), q\right)
$$

Obviously, for each $\xi \in \mathrm{T}_{\Sigma_{\mathcal{V} \cup \mathcal{V}\}}}$, we have $\operatorname{pos}(\xi)=\operatorname{pos}(\xi \mid \mathcal{V})$ and $\mathrm{R}_{\mathcal{B}}(\xi)=\mathrm{R}_{\mathcal{A}}(\xi \mid \mathcal{V})$. Moreover, for each $\rho \in \mathrm{R}_{\mathcal{B}}(\xi)$, we have $\operatorname{wt}_{\mathcal{B}}(\xi, \rho)=\mathrm{wt}_{\mathcal{A}}(\xi \mid \mathcal{V}, \rho)$. Then

$$
\llbracket \mathcal{B} \rrbracket^{\mathrm{run}}(\xi)=\bigoplus_{\rho \in \mathrm{R}_{\mathcal{B}}(\xi)} \mathrm{wt}_{\mathcal{B}}(\xi, \rho) \otimes F_{\rho(\varepsilon)}=\bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}}(\xi \mid \mathcal{V})} \mathrm{wt}_{\mathcal{A}}(\xi \mid \mathcal{V}, \rho) \otimes F_{\rho(\varepsilon)}=\llbracket \mathcal{A} \rrbracket^{\mathrm{run}}(\xi \mid \mathcal{V})
$$

Now we prove the following statement.

$$
\begin{equation*}
\text { For each } \xi \in \mathrm{T}_{\Sigma_{\mathcal{V} \cup\{V\}}} \text { we have } \llbracket \mathcal{B} \rrbracket^{\text {run }}(\xi) \otimes \chi\left(\mathrm{T}_{\Sigma_{\mathcal{V} \cup\{V\}}^{\mathrm{v}}}\right)(\xi)=\llbracket e \rrbracket_{\mathcal{V} \cup\{V\}}(\xi) \tag{14.23}
\end{equation*}
$$

For this, let $\xi \in \mathrm{T}_{\Sigma_{\mathcal{V} \cup V Y\}}}$.
Case (i): Let $\xi$ not be valid. Then both sides of (14.23) evaluate to $\mathbb{0}$.
Case (ii): Let $\xi$ be valid. Then we obtain

$$
\begin{array}{rlrl}
\llbracket \mathcal{B} \rrbracket^{\mathrm{run}}(\xi) \otimes \chi\left(\mathrm{T}_{\Sigma_{\mathcal{V} \cup\{V\}}^{\mathrm{v}}}\right)(\xi) & =\llbracket \mathcal{B} \rrbracket^{\mathrm{run}}(\xi) & & \text { (because } \xi \text { is valid) } \\
& =\llbracket \mathcal{A} \rrbracket^{\mathrm{run}}(\xi \mid \mathcal{V}) & \\
& =\llbracket e \rrbracket(\xi \mid \mathcal{V}) & \\
& =\llbracket e \rrbracket_{\mathcal{V} \cup\{V\}}(\xi) & \text { (by Lemma 14.2.1) }
\end{array}
$$

This finishes the proof of (14.23).
By Lemma 14.1.5, we can construct a $\Sigma_{\mathcal{V} \cup\{V\}}$-fta $A$ such that $\mathrm{L}(A)=\mathrm{T}_{\Sigma_{\mathcal{V} \cup\{V\}}}$. Finally, by Theorem 10.4.3. we can construct a $\left(\Sigma_{\mathcal{V} \cup\{V\}}, B\right)$-wta $\mathcal{A}^{\prime}$ such that $\llbracket \mathcal{A}^{\prime} \rrbracket^{\text {run }}=\llbracket \mathcal{B} \rrbracket^{\text {run }} \otimes \chi\left(\mathrm{T}_{\Sigma_{\mathcal{V} \cup\{V\}}^{\mathrm{v}}}\right)$. Hence $\llbracket \mathcal{A}^{\prime} \rrbracket^{\text {run }}=$ $\llbracket e \rrbracket_{\mathcal{V} \cup\{V\}}$.

Now we proceed by transforming atomic formulas, guarded formulas, and weighted existential quantifications into a wta. (For formulas of the form $e_{1}+e_{2}$ we will exploit a closure result.)

Lemma 14.3.5. FSV12, Lm. 4.5] For every finite sets $\mathcal{V}$ of variables and family $\kappa=\left(\kappa_{k} \mid k \in \mathbb{N}\right)$ of mappings $\kappa_{k}:\left(\Sigma_{\mathcal{V}}\right)^{(k)} \rightarrow B$, we can construct a bu deterministic $\left(\Sigma_{\mathcal{V}}, \mathrm{B}\right)$-wta $\mathcal{B}$ such that $\llbracket \mathrm{H}(\kappa) \rrbracket=\llbracket \mathcal{B} \rrbracket$.

Proof. We recall that $\operatorname{Free}(H(\kappa))=\mathcal{V}$. It is easy to see that $\llbracket H(\kappa) \rrbracket=\mathrm{h}_{\kappa} \otimes \chi\left(\mathrm{T}_{\Sigma_{\nu}}^{v}\right)$. In Example 3.2.17we have constructed a bu deterministic $\left(\Sigma_{\mathcal{V}}, \mathrm{B}\right)$-wta $\mathcal{A}$ such that $\llbracket \mathcal{A} \rrbracket=\mathrm{h}_{\kappa}$. Then, by Lemma 14.1.5 and by Theorem 10.4 .3 (2), we can construct a bu deterministic $\left(\Sigma_{\mathcal{V}}, B\right)$-wta $\mathcal{B}$ such that $\llbracket \mathcal{B} \rrbracket=\mathrm{h}_{\kappa} \otimes \chi\left(\mathrm{T}_{\Sigma_{\mathcal{V}}}^{\mathrm{v}}\right)$.

In the next lemma we prove that adding a guard to a formula preserves r-recognizability.
Lemma 14.3.6. (cf. [FSV12, Lm. 4.10]) Let $\varphi \in \operatorname{MSO}(\Sigma)$ and $e \in \operatorname{MSO}(\Sigma, \mathrm{~B})$. We let $\mathcal{U}=\operatorname{Free}(\varphi)$ and $\mathcal{V}=\operatorname{Free}(e)$. If there exists a $\left(\Sigma_{\mathcal{V}}, \mathrm{B}\right)$-wta $\mathcal{A}$ such that $\llbracket \mathcal{A} \rrbracket^{\text {run }}=\llbracket e \rrbracket$, then we can construct a $\left(\Sigma_{\mathcal{U} \cup \mathcal{V}}, B\right)$-wta $\mathcal{B}$ such that $\llbracket \mathcal{B} \rrbracket^{\text {run }}=\llbracket \varphi \triangleright e \rrbracket$.

Proof. Starting with $\mathcal{A}$, we can apply Lemma 14.3 .4 an appropriate number of times; thereby we can construct a $\left(\Sigma_{\mathcal{U} \cup \mathcal{V}}, B\right)$-wta $\mathcal{A}^{\prime}$ such that $\llbracket \mathcal{A}^{\prime} \rrbracket^{\text {run }}=\llbracket e \rrbracket_{\mathcal{U} \cup \mathcal{V}}$. Moreover, by Lemma 14.1.9 we can construct a $\Sigma_{\mathcal{U} \cup \mathcal{V}}$-fta $D$ which recognizes $\mathrm{L}_{\mathcal{U} \cup \mathcal{V}}(\varphi)$, i.e., $\mathrm{L}(D)=\mathrm{L}_{\mathcal{U} \cup \mathcal{V}}(\varphi)$. Then, by Theorem 10.4.3(2), we can construct a $\left(\Sigma_{\mathcal{U} \cup \mathcal{V}}, \mathrm{B}\right)$-wta $\mathcal{B}$ such that $\llbracket \mathcal{B} \rrbracket^{\text {run }}=\chi(\mathrm{L}(D)) \otimes \llbracket \mathcal{A}^{\prime} \rrbracket^{\text {run }}$. Finally, we can calculate as follows.

$$
\begin{aligned}
\llbracket \varphi \triangleright e \rrbracket & =\chi(\mathrm{L} \mathcal{U} \cup \mathcal{V}(\varphi)) \otimes \llbracket e \rrbracket_{\mathcal{U} \cup \mathcal{V}} \\
& =\chi(\mathrm{L}(D)) \otimes \llbracket \mathcal{A}^{\prime} \rrbracket^{\text {run }}=\llbracket \mathcal{B} \rrbracket^{\text {run }}
\end{aligned}
$$

The next lemma shows that weighted existential quantification preserves r-recognizability.
Lemma 14.3.7. [FSV12, Lm. 4.9] Let $e \in \operatorname{MSO}(\Sigma, \mathrm{~B})$. Then the following two statements hold.
(1) If $\mathcal{V}=\operatorname{Free}\left(+_{x} e\right), \mathcal{U}=\operatorname{Free}(e)$, and there exists a $\left(\Sigma_{\mathcal{U}}, \mathrm{B}\right)$-wta $\mathcal{A}$ with $\llbracket \mathcal{A} \rrbracket^{\text {run }}=\llbracket e \rrbracket$, then we can construct a $\left(\Sigma_{\mathcal{V}}, \mathrm{B}\right)$-wta $\mathcal{B}$ such that $\llbracket \mathcal{B} \rrbracket^{\text {run }}=\llbracket+_{x} e \rrbracket$.
(2) If $\mathcal{V}=\operatorname{Free}\left(+_{X} e\right), \mathcal{U}=\operatorname{Free}(e)$, and there exists a $\left(\Sigma_{\mathcal{U}}, \mathrm{B}\right)$-wta $\mathcal{A}$ with $\llbracket \mathcal{A} \rrbracket^{\text {run }}=\llbracket e \rrbracket$, then we can construct a $\left(\Sigma_{\mathcal{V}}, \mathrm{B}\right)$-wta $\mathcal{B}$ such that $\llbracket \mathcal{B} \rrbracket^{\text {run }}=\llbracket+_{X} e \rrbracket$.

Proof. Proof of (1). Assume that $\mathcal{V}=\operatorname{Free}\left(+_{x} e\right), \mathcal{U}=\operatorname{Free}(e)$, and there exists a $\left(\Sigma_{\mathcal{U}}, \mathrm{B}\right)$-wta $\mathcal{A}$ such that $\llbracket \mathcal{A} \rrbracket^{\text {run }}=\llbracket e \rrbracket$. We note that $\mathcal{V}=\mathcal{U} \backslash\{x\}$, thus $x \notin \mathcal{V}$.

We define the deterministic $\left(\Sigma_{\mathcal{V} \cup\{x\}}, \Sigma_{\mathcal{V}}\right)$-tree relabeling $\tau=\left(\tau_{k} \mid k \in \mathbb{N}\right)$ such that, for every $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $\mathcal{W} \subseteq \mathcal{V} \cup\{x\}$ we let $\tau_{k}((\sigma, \mathcal{W}))=\{(\sigma, \mathcal{W} \backslash\{x\})\}$.

Then, starting from the $\left(\Sigma_{\mathcal{U}}, \mathrm{B}\right)$-wta $\mathcal{A}$, by Lemma 14.3.4, we can construct a $\left(\Sigma_{\mathcal{U} \cup\{x\}}, \mathrm{B}\right)$-wta $\mathcal{A}^{\prime}$ such that $\llbracket \mathcal{A}^{\prime} \rrbracket^{\text {run }}=\llbracket e \rrbracket_{\mathcal{U} \cup\{x\}}$. Since $\mathcal{U} \cup\{x\}=(\mathcal{U} \backslash\{x\}) \cup\{x\}=\mathcal{V} \cup\{x\}$, we obtain that $\llbracket \mathcal{A}^{\prime} \rrbracket^{\text {run }}=\llbracket e \rrbracket \mathcal{V} \cup\{x\}$. By using $\mathcal{A}^{\prime}$ and Theorem 10.10.1 we can construct a $\left(\Sigma_{\mathcal{V}}, \mathrm{B}\right)$-wta which r-recognizes $\chi(\tau)(\llbracket e \rrbracket \mathcal{V} \cup\{x\})$. By Lemma 14.1.5, we can construct a $\Sigma_{\mathcal{V}}$-fta which recognizes $\mathrm{T}_{\Sigma_{\mathcal{V}}}$. By Theorem 10.4.3(2), we can construct a $\left(\Sigma_{\mathcal{V}}, \mathrm{B}\right)$-wta $\mathcal{B}$ such that $\llbracket \mathcal{B} \rrbracket^{\text {run }}=\chi(\tau)\left(\llbracket e \rrbracket_{\mathcal{V} \cup\{x\}}\right) \otimes \chi\left(\mathrm{T}_{\Sigma_{\mathcal{V}}}^{\mathrm{v}}\right)$. Finally, by Lemma 14.2.6(1), we obtain $\llbracket \mathcal{B} \rrbracket^{\mathrm{run}}=\llbracket+_{x} e \rrbracket$.

Proof of (2). Assume that $\mathcal{V}=\operatorname{Free}\left(+_{X} e\right), \mathcal{U}=\operatorname{Free}(e)$, and there exists a $\left(\Sigma_{\mathcal{U}}, \mathrm{B}\right)$-wta $\mathcal{A}$ such that $\llbracket \mathcal{A} \rrbracket^{\text {run }}=\llbracket e \rrbracket$. We note that $\mathcal{V}=\mathcal{U} \backslash\{X\}$, this $X \notin \mathcal{V}$.

We define the $\left(\Sigma_{\mathcal{V} \cup\{X\}}, \Sigma_{\mathcal{V}}\right)$-tree relabeling $\tau=\left(\tau_{k} \mid k \in \mathbb{N}\right)$ such that, for every $k \in \mathbb{N}, \sigma \in \Sigma^{(k)}$, and $\mathcal{W} \subseteq \mathcal{V} \cup\{X\}$ we let $\tau_{k}((\sigma, \mathcal{W}))=\{(\sigma, \mathcal{W} \backslash\{X\})\}$.

Similarly as above, starting from the $\left(\Sigma_{\mathcal{U}}, \mathrm{B}\right)$-wta $\mathcal{A}$, by Lemma 14.3 .4 we can construct a $\left(\Sigma_{\mathcal{U} \cup\{X\}}, \mathrm{B}\right)$ wta $\mathcal{A}^{\prime}$ such that $\llbracket \mathcal{A}^{\prime} \rrbracket^{\text {run }}=\llbracket e \rrbracket_{\mathcal{U} \cup\{X\}}$. Since $\mathcal{U} \cup\{X\}=(\mathcal{U} \backslash\{X\}) \cup\{X\}=\mathcal{V} \cup\{X\}$, we obtain that $\llbracket \mathcal{A}^{\prime} \rrbracket^{\text {run }}=\llbracket e \rrbracket_{\mathcal{V} \cup\{X\}}$. By using $\mathcal{A}^{\prime}$ and Theorem 10.10.1, we can construct a $\left(\Sigma_{\mathcal{V}}, \mathrm{B}\right)$-wta $\mathcal{B}$ such that $\llbracket \mathcal{B} \rrbracket^{\text {run }}=\chi(\tau)\left(\llbracket e \rrbracket_{\mathcal{V} \cup\{X\}}\right)$. Finally, by Lemma $14.2 .6(2)$, we obtain $\llbracket \mathcal{B} \rrbracket^{\text {run }}=\llbracket+_{X} e \rrbracket$.

Now we can prove that definability implies recognizability.
Theorem 14.3.8. For every $e \in \operatorname{MSO}(\Sigma, \mathrm{~B})$ and $\mathcal{V}=\operatorname{Free}(e)$, we can construct a $\left(\Sigma_{\mathcal{V}}, \mathrm{B}\right)$-wta $\mathcal{B}$ such that $\llbracket \mathcal{B} \rrbracket^{\mathrm{run}}=\llbracket e \rrbracket$.

Proof. By induction on $\left(\operatorname{MSO}(\Sigma, \mathrm{B}), \prec_{\mathrm{MSO}(\Sigma, \mathrm{B})}\right)$, we prove the statement of the theorem.
I.B.: Let $e=\mathrm{H}(\kappa)$ where $\kappa: \Sigma_{\mathcal{V}} \rightarrow B$ for some finite set $\mathcal{V}$ of variables. By Lemma 14.3.5, we can construct a bu deterministic $\left(\Sigma_{\mathcal{V}}, \mathrm{B}\right)$-wta $\mathcal{B}$ such that $\llbracket \mathcal{B} \rrbracket=\llbracket \mathrm{H}(\kappa) \rrbracket$.
I.S.: Case (a): Let $e=\left(\varphi \triangleright e^{\prime}\right)$ and $\mathcal{V}=\operatorname{Free}\left(\varphi \triangleright e^{\prime}\right)$. By I.H., we can construct a $\left(\Sigma_{\text {Free }\left(e^{\prime}\right)}, \mathrm{B}\right)$-wta $\mathcal{A}$ such that $\llbracket \mathcal{A} \rrbracket^{\text {run }}=\llbracket e^{\prime} \rrbracket$. By Lemma 14.3 .6 , we can construct a $\left(\Sigma_{\mathcal{V}}, \mathrm{B}\right)$-wta $\mathcal{B}$ such that $\llbracket \mathcal{B} \rrbracket^{\text {run }}=\llbracket e \rrbracket$.

Case (b): Let $e=e_{1}+e_{2}$ and let $\mathcal{V}=\operatorname{Free}\left(e_{1}+e_{2}\right)$. By I.H., we can construct a $\left(\Sigma_{\text {Free }\left(e_{1}\right)}\right.$, B$)$-wta
 can apply Lemma 14.3 .4 an appropriate number of times; thereby we can construct a $\left(\Sigma_{\mathcal{V}}, \mathrm{B}\right)$-wta $\mathcal{A}_{1}^{\prime}$ such that $\llbracket \mathcal{A}_{1}^{\prime} \rrbracket^{\text {run }}=\llbracket e_{1} \rrbracket \mathcal{V}$. Similarly we can construct a $\left(\Sigma_{\mathcal{V}}, \mathrm{B}\right)-$ wta $\mathcal{A}_{2}^{\prime}$ such that $\llbracket \mathcal{A}_{2}^{\prime} \rrbracket^{\text {run }}=\llbracket e_{2} \rrbracket \mathcal{V}$. By Theorem 10.1 .1 we can construct a $\left(\Sigma_{\mathcal{V}}, \mathrm{B}\right)$-wta $\mathcal{B}$ such that $\llbracket \mathcal{B} \rrbracket^{\text {run }}=\llbracket \mathcal{A}_{1}^{\prime} \rrbracket^{\text {run }}+\llbracket \mathcal{A}_{2}^{\prime} \rrbracket^{\text {run }}$. Then $\llbracket \mathcal{B} \rrbracket^{\text {run }}=\llbracket \mathcal{A}_{1}^{\prime} \rrbracket^{\text {run }}+\llbracket \mathcal{A}_{2}^{\prime} \rrbracket^{\text {run }}=\llbracket e_{1} \rrbracket \mathcal{V}+\llbracket e_{2} \rrbracket \mathcal{V}=\llbracket e_{1}+e_{2} \rrbracket \mathcal{V}$.

Case (c): Let $e=+_{x} e^{\prime}$ and let $\mathcal{V}=\operatorname{Free}\left(+_{x} e^{\prime}\right)$. By I.H., we can construct a $\left(\Sigma_{\text {Free }\left(e^{\prime}\right)}\right.$, B)-wta $\mathcal{A}$ such that $\llbracket \mathcal{A} \rrbracket^{\text {run }}=\llbracket e^{\prime} \rrbracket$. By Lemma $14.3 .7(1)$, we can construct a $\left(\Sigma_{\mathcal{V}}, \mathrm{B}\right)$-wta $\mathcal{B}$ such that $\llbracket \mathcal{B} \rrbracket^{\text {run }}=\llbracket e \rrbracket$.

Case (d): Let $e=+_{X} e^{\prime}$ and let $\mathcal{V}=\operatorname{Free}\left(+_{X} e^{\prime}\right)$. We can finish the proof as in Case (d), except that we use Lemma 14.3.7(2).

### 14.4 Adding weighted conjunction and weighted universal quantification

In this section we extend $\operatorname{MSO}(\Sigma, B)$ by weighted conjunction, weighted first-order universal quantification, and weighted second-order universal quantification DG05, DG07, DG09; the resulting logic is denoted by $\mathrm{MSO}^{\text {ext }}(\Sigma, \mathrm{B})$. Since $\mathrm{MSO}^{\text {ext }}(\Sigma, \mathrm{B})$ is more powerful than run recognizability, we identify a fragment of $\mathrm{MSO}^{\text {ext }}(\Sigma, \mathrm{B})$ which characterizes the set of run recognizable $(\Sigma, \mathrm{B})$-weighted tree languages, if $B$ is a commutative semiring (cf. Theorem 14.4.11). Roughly speaking, in that fragment of $\mathrm{MSO}^{\text {ext }}(\Sigma, \mathrm{B})$, (a) the weighted first-order universal quantification may only be applied to a recognizable step formula, and (b) the weighted second-order universal quantification is forbidden. The latter restriction corresponds to the restriction (c) of the weighted MSO-logic in DG05, DG07, DG09, which is discussed in the preface of this chapter. Our restriction (a) is more severe than the corresponding restriction (b) there, because it forbids nesting of weighted first-order universal quantification. In Section 19.8 we will prove that, for each bounded lattice B , the full extended weighted logic $\mathrm{MSO}^{\text {ext }}(\Sigma, \mathrm{B})$ characterizes the set of run recognizable ( $\Sigma, \mathrm{B}$ )-weighted tree languages (cf. Theorem 19.8.4).

In this section, we assume that B is commutative.

### 14.4.1 The extended weighted MSO-logic $\mathrm{MSO}^{\text {ext }}(\Sigma, \mathrm{B})$

We define the set of extended MSO formulas over $\Sigma$ and B , denoted by $\mathrm{MSO}^{\mathrm{ext}}(\Sigma, \mathrm{B})$, by the following EBNF with nonterminal $e$ :

$$
\begin{equation*}
e::=\mathrm{H}(\kappa)|(\varphi \triangleright e)|(e+e)|(e \times e)|+_{x} e\left|+_{X} e\right| X_{x} e \mid X_{X} e \tag{14.24}
\end{equation*}
$$

where

- there exists a finite set $\mathcal{U}$ of variables such that $\kappa=\left(\kappa_{k} \mid k \in \mathbb{N}\right)$ is an $\mathbb{N}$-indexed family of mappings $\kappa_{k}:\left(\Sigma_{\mathcal{U}}\right)^{(k)} \rightarrow B$,
- $\varphi \in \operatorname{MSO}(\Sigma)$.

Formulas of the form $\left(e_{1} \times e_{2}\right), X_{x} e$, and $X_{X} e$ are called weighted conjunction, weighted firstorder universal quantification, and weighted second-order universal quantification, respectively. Obviously, $\operatorname{MSO}(\Sigma, \mathrm{B}) \subset \mathrm{MSO}^{\mathrm{ext}}(\Sigma, \mathrm{B})$.

For each $e \in \operatorname{MSO}^{\text {ext }}(\Sigma, \mathrm{B})$, the set Free $(e)$ of free variables of $e$ and set Bound $(e)$ of bound variables of $e$ are defined in the same way as for $\operatorname{MSO}(\Sigma, \mathrm{B})$-formulas and additionally we have:

- $\operatorname{Free}\left(e_{1} \times e_{2}\right)=\operatorname{Free}\left(e_{1}\right) \cup \operatorname{Free}\left(e_{2}\right)$ and $\operatorname{Bound}\left(e_{1} \times e_{2}\right)=\operatorname{Bound}\left(e_{1}\right) \cup \operatorname{Bound}\left(e_{2}\right)$,
- Free $\left(X_{x} e\right)=\operatorname{Free}(e) \backslash\{x\}$ and $\operatorname{Bound}\left(X_{x} e\right)=\operatorname{Bound}(e) \cup\{x\}$, and
- Free $\left(X_{X} e\right)=\operatorname{Free}(e) \backslash\{X\}$ and $\operatorname{Bound}\left(X_{X} e\right)=\operatorname{Bound}(e) \cup\{X\}$.

As in Section 14.2, in order to perform inductive proofs or to define objects by induction, we will consider the well-founded set

$$
\left(\mathrm{MSO}^{\mathrm{ext}}(\Sigma, \mathrm{~B}), \prec_{\mathrm{MSO}^{\mathrm{ext}}(\Sigma, \mathrm{~B})}\right)
$$

where $\prec_{\text {MSO }}{ }^{\text {ext }}(\Sigma, \mathrm{B})$ is the binary relation on $\operatorname{MSO}^{\text {ext }}(\Sigma, \mathrm{B})$ defined as follows. Already in the definition and also in the rest of this chapter, we abbreviate $\prec_{\mathrm{MSO}^{\text {ext }}(\Sigma, \mathrm{B})}$ by $\prec_{\mathrm{MSO}^{\text {ext }}}$. For every $e_{1}, e_{2} \in \mathrm{MSO}^{\text {ext }}(\Sigma, \mathrm{B})$, we let $e_{1} \prec_{\mathrm{MSO}}{ }^{\text {ext }} e_{2}$ if $e_{1}$ is a direct subformula of $e_{2}$ in the sense of (14.24). Then $\prec_{\text {MSO }}{ }^{\text {ext }}$ is well-founded, and $\min _{\prec_{\text {MSO }}^{\text {ext }}}\left(\operatorname{MSO}^{\text {ext }}(\Sigma, B)\right)$ is the set of all formulas of the form $\mathrm{H}(\kappa)$.

By induction on $\left(\mathrm{MSO}^{\text {ext }}(\Sigma, \mathrm{B}), \prec_{\text {MSO }}{ }^{\text {ext }}\right)$, we define the $\mathrm{MSO}^{\text {ext }}(\Sigma, \mathrm{B})$-indexed family

$$
\left((\llbracket e \rrbracket \mathcal{V} \mid \mathcal{V} \supseteq \text { Free }(\mathrm{e}), \mathcal{V} \text { finite }) \mid e \in \mathrm{MSO}^{\mathrm{ext}}(\Sigma, B)\right)
$$

as follows, where $\llbracket e \rrbracket_{\mathcal{V}}$ is a $\left(\Sigma_{\mathcal{V}}, \mathrm{B}\right)$-weighted tree language which is called the semantics of $e$ with respect to $\mathcal{V}$.

- Let $e=\mathrm{H}(\kappa), e=\left(\varphi \triangleright e^{\prime}\right), e=e_{1}+e_{2}, e=+_{x} e^{\prime}$, or $e=+_{X} e^{\prime}$. Let $\mathcal{V} \supseteq$ Free $(e)$ be a finite set of variables. The definition of $\llbracket e \rrbracket \mathcal{V}$ is the same as in Section 14.2
- Let $e_{1}, e_{2} \in \operatorname{MSO}^{\text {ext }}(\Sigma, \mathrm{B})$. Let $\mathcal{V} \supseteq \operatorname{Free}\left(e_{1} \times e_{2}\right)$ be a finite set of variables. Then, for each $\zeta \in \mathrm{T}_{\Sigma_{\mathcal{V}}}$, we define

$$
\llbracket e_{1} \times e_{2} \rrbracket \mathcal{V}(\zeta)= \begin{cases}\left(\llbracket e_{1} \rrbracket \mathcal{V}(\zeta) \otimes \llbracket e_{2} \rrbracket \mathcal{V}(\zeta)\right. & \text { if } \zeta \in \mathrm{T}_{\Sigma \mathcal{V}}^{\mathrm{v}} \\ \mathbb{O} & \text { otherwise }\end{cases}
$$

- Let $x$ be a first-order variable and $e \in \operatorname{MSO}^{\text {ext }}(\Sigma, \mathrm{B})$. Let $\mathcal{V} \supseteq$ Free $\left(X_{x} e\right)$ be a finite set of variables. Then, for each $\zeta \in \mathrm{T}_{\Sigma_{\mathcal{V}}}$, we define

$$
\llbracket X_{x} e \rrbracket_{\mathcal{V}}(\zeta)= \begin{cases}\bigotimes_{w \in \operatorname{pos}(\zeta)} \llbracket e \rrbracket_{\mathcal{V} \cup\{x\}}(\zeta[x \mapsto w]) & \text { if } \zeta \in \mathrm{T}_{\Sigma \mathcal{V}}^{\mathrm{v}} \\ 0 & \text { otherwise }\end{cases}
$$

- Let $X$ be a second-order variable and $e \in \operatorname{MSO}^{\text {ext }}(\Sigma, \mathrm{B})$. Let $\mathcal{V} \supseteq \operatorname{Free}\left(X_{X} e\right)$ be a finite set of variables. Then, for each $\zeta \in \mathrm{T}_{\Sigma_{\mathcal{V}}}$, we define

$$
\llbracket X_{X} e \rrbracket \mathcal{V}(\zeta)= \begin{cases}\bigotimes_{W \subseteq \operatorname{pos}(\zeta)} \llbracket \rrbracket_{\mathcal{V} \cup\{X\}}(\zeta[X \mapsto W]) & \text { if } \zeta \in \mathrm{T}_{\Sigma_{\mathcal{V}}}^{\mathrm{v}} \\ \mathbb{O} & \text { otherwise }\end{cases}
$$

We note that the definitions of $\llbracket X_{x} e \rrbracket \mathcal{V}$ and $\llbracket X_{X} e \rrbracket \mathcal{V}$ use the fact that $\otimes$ is commutative (cf. page 21).
Next we show that the extension of an assignment does not change the semantics of extended MSO formulas.

Lemma 14.4.1. (cf. Lemma 14.2.1) Let $e \in \operatorname{MSO}^{\text {ext }}(\Sigma, \mathrm{B})$ and let $\mathcal{V}$ and $\mathcal{W}$ be finite sets of variables with $\operatorname{Free}(e) \subseteq \mathcal{W} \subseteq \mathcal{V}$. Then for every $(\xi, \eta) \in \mathrm{T}_{\Sigma_{\mathcal{V}}}^{\mathbf{v}}$, we have $\llbracket e \rrbracket_{\mathcal{V}}(\xi, \eta)=\llbracket e \rrbracket \mathcal{W}(\xi, \eta \mid \mathcal{W})$.

Proof. We prove the statement by induction on $\left(\mathrm{MSO}^{\text {ext }}(\Sigma, \mathrm{B}), \prec_{\mathrm{MSO}}{ }^{\text {ext }}\right)$. The proof is very similar to that of Lemma 14.2.1 and thus it is dropped.

We will prove the main theorem of this section (cf. Theorem 14.4.11) by replacing subformulas of a formula in $\mathrm{MSO}^{\text {ext }}(\Sigma, \mathrm{B})$ by equivalent formulas in $\operatorname{MSO}(\Sigma, \mathrm{B})$. Next we will formalize this replacement.

Let $\varphi$ be an $\operatorname{MSO}(\Sigma)$-formula and $V$ and $W$ be two first-order variables or two second-order variables. Intuitively, we let $\varphi[V / W]$ be the $\operatorname{MSO}(\Sigma)$-formula obtained from $\varphi$ by replacing each free occurrence
of $V$ by $W$ (similar to $\alpha$-conversion in $\lambda$-calculus). Formally, we define the $\operatorname{MSO}(\Sigma)$-formula $\varphi[V / W]$ by induction on $\left(\operatorname{MSO}(\Sigma), \prec_{\mathrm{MSO}(\Sigma)}\right)(\mathrm{cf}$. (14.2)) as follows:

$$
\begin{aligned}
& \operatorname{label}_{\sigma}(x)[V / W]=\left\{\begin{array}{lll}
\operatorname{label}_{\sigma}(W) & \text { if } x=V \\
\operatorname{label}_{\sigma}(x) & \text { otherwise }
\end{array} \quad \operatorname{edge}_{i}(x, y)[V / W]= \begin{cases}\operatorname{edge}_{i}(W, y) & \text { if } x=V, y \neq V \\
\operatorname{edge}_{i}(x, W) & \text { if } y=V, x \neq V \\
\operatorname{edge}_{i}(W, W) & \text { if } x=y=V \\
\operatorname{edge}_{i}(x, y) & \text { otherwise }\end{cases} \right. \\
& (x \in X)[V / W]= \begin{cases}(W \in X) & \text { if } x=V \\
(x \in W) & \text { if } X=V \\
(x \in X) & \text { otherwise }\end{cases} \\
& (\neg \varphi)[V / W]=\neg(\varphi[V / W]) \quad\left(\varphi_{1} \vee \varphi_{2}\right)[V / W]=\varphi_{1}[V / W] \vee \varphi_{2}[V / W]
\end{aligned}
$$

Next by induction on $\left(\operatorname{MSO}^{\text {ext }}(\Sigma, \mathrm{B}), \prec_{\mathrm{MSO}^{\text {ext }}}\right)$ we define $e[V / W]$ for each $\mathrm{MSO}^{\text {ext }}(\Sigma, \mathrm{B})$-formula $e$ and first-order or second order variables $V$ and $W$.


$$
\mathrm{H}(\kappa)[V / W]= \begin{cases}\mathrm{H}(\kappa) & \text { if } V \notin \mathcal{U} \\ \mathrm{H}\left(\kappa^{\prime}\right) & \text { otherwise }\end{cases}
$$

where, for each $k \in \mathbb{N}$, the mapping $\kappa_{k}^{\prime}: \Sigma_{\mathcal{V}}^{(k)} \rightarrow B$ is defined as follows: $\mathcal{V}=(\mathcal{U} \backslash\{V\}) \cup\{W\}$ and for every $(\sigma, \mathcal{W}) \in \Sigma_{\mathcal{V}}^{(k)}$, we have

$$
\kappa_{k}^{\prime}((\sigma, \mathcal{W}))= \begin{cases}\kappa_{k}((\sigma, \mathcal{W})) & \text { if } W \notin \mathcal{W} \\ \kappa_{k}((\sigma,(\mathcal{W} \backslash\{W\}) \cup\{V\})) & \text { otherwise }\end{cases}
$$

Case (b): Let $e=\left(\varphi \triangleright e^{\prime}\right)$. Then $e[V / W]=\left(\varphi[V / W] \triangleright e^{\prime}[V / W]\right)$.
The definition of $e[V / W]$ for the other cases of $e$ is left to the reader.

We observe that renaming a bound variable in a formula does not change the semantics of that formula.
Observation 14.4.2. The following two statements hold.
(1) Let $\exists V . \varphi$ be an $\operatorname{MSO}(\Sigma)$-formula, where $V$ is a variable, and $\mathcal{V}$ be a finite set of variables with Free $(\exists V \cdot \varphi) \subseteq \mathcal{V}$. Moreover, let $W$ be a variable of the same type as $V$ with $W \notin \mathcal{V}$. Then $\mathrm{L}_{\mathcal{V}}(\exists V \cdot \varphi)=\mathrm{L}_{\mathcal{V}}(\exists W \cdot \varphi[V / W])$.
(2) Let $Q_{V} e$ be an $\operatorname{MSO}^{\text {ext }}(\Sigma, \mathrm{B})$-formula for some $Q \in\{+, X\}$, where $V$ is a variable, and $\mathcal{V}$ be a finite set of variables with Free $\left(Q_{V} e\right) \subseteq \mathcal{V}$. Moreover, let $W$ be a variable of the same type as $V$ with $W \notin \mathcal{V}$. Then $\llbracket Q_{V} e \rrbracket \mathcal{V}=\llbracket Q_{W} e[V / W] \rrbracket_{\mathcal{V}}$.

Moreover, we formalize three variants of the observation that, for each formula, the replacement of a subformula by an equivalent formula does not change the semantics of the original formula.
Observation 14.4.3. Let $\varphi \in \operatorname{MSO}(\Sigma)$ and let $\mathcal{V} \supseteq \operatorname{Free}(\varphi)$ be a finite set of variables. Moreover, let $\psi$ be a subformula of $\varphi$. Also let $\psi^{\prime} \in \operatorname{MSO}(\Sigma)$ such that $\operatorname{Free}\left(\psi^{\prime}\right)=\operatorname{Free}(\psi)$ and $\mathrm{L}\left(\psi^{\prime}\right)=\mathrm{L}(\psi)$. Then $\mathrm{L}_{\mathcal{V}}(\varphi)=\mathrm{L}_{\mathcal{V}}\left(\varphi\left[\psi / \psi^{\prime}\right]\right)$, where $\varphi\left[\psi / \psi^{\prime}\right]$ is the formula in $\operatorname{MSO}(\Sigma)$ obtained from $\varphi$ by replacing each occurrence of the subformula $\psi$ by $\psi^{\prime}$.

Observation 14.4.4. Let $e \in \operatorname{MSO}^{\text {ext }}(\Sigma, \mathrm{B})$ and let $\mathcal{V} \supseteq \operatorname{Free}(e)$ be a finite set of variables. Moreover, let $\psi$ be an $\operatorname{MSO}(\Sigma)$-subformula of $e$. Also let $\psi^{\prime} \in \operatorname{MSO}(\Sigma)$ such that $\operatorname{Free}\left(\psi^{\prime}\right)=\operatorname{Free}(\psi)$ and $\mathrm{L}\left(\psi^{\prime}\right)=\mathrm{L}(\psi)$. Then $\llbracket e \rrbracket \mathcal{V}=\llbracket e\left[\psi / \psi^{\prime}\right] \rrbracket \mathcal{V}$, where $e\left[\psi / \psi^{\prime}\right]$ is the formula in $\operatorname{MSO}^{\text {ext }}(\Sigma, \mathrm{B})$ obtained from $e$ by replacing each occurrence of the subformula $\psi$ by $\psi^{\prime}$.
Observation 14.4.5. Let $e \in \operatorname{MSO}^{\mathrm{ext}}(\Sigma, \mathrm{B})$ and let $\mathcal{V} \supseteq$ Free $(e)$ be a finite set of variables. Moreover, let $t$ be an $\operatorname{MSO}^{\text {ext }}(\Sigma, \mathrm{B})$ subformula of $e$. Also let $t^{\prime} \in \mathrm{MSO}^{\text {ext }}(\Sigma, \mathrm{B})$ such that $\operatorname{Free}\left(t^{\prime}\right)=\operatorname{Free}(t)$ and $\llbracket t^{\prime} \rrbracket=\llbracket t \rrbracket$. Then $\llbracket e \rrbracket \mathcal{v}=\llbracket e\left[t / t^{\prime}\right] \rrbracket \mathcal{v}$, where $e\left[t / t^{\prime}\right]$ is the formula in $\mathrm{MSO}^{\text {ext }}(\Sigma, \mathrm{B})$ obtained from $e$ by replacing each occurrence of the subformula $t$ by $t^{\prime}$.

Let $e$ be an $\operatorname{MSO}^{\text {ext }}(\Sigma, \mathrm{B})$-formula. We say that $e$ is in normal form if $\operatorname{Free}(e) \cap \operatorname{Bound}(e)=\emptyset$.
Next we show that each $\mathrm{MSO}^{\text {ext }}(\Sigma, \mathrm{B})$-formula can be transformed into an equivalent one which is in normal form (cf. rectified formula in Gal87] and Sch89]).
Lemma 14.4.6. Let $e$ be an $\mathrm{MSO}^{\text {ext }}(\Sigma, \mathrm{B})$-formula and $\mathcal{V}$ be a finite set of variables with Free $(e) \subseteq \mathcal{V}$. We can construct an $\operatorname{MSO}^{\text {ext }}(\Sigma, \mathrm{B})$-formula $e^{\prime}$ such that $e^{\prime}$ is in normal form, Free $(e)=\operatorname{Free}\left(e^{\prime}\right)$, and $\llbracket e \rrbracket \mathcal{V}=\llbracket e^{\prime} \rrbracket \mathcal{V}$.

Proof. We obtain $e^{\prime}$ by performing an iteration on the following steps (a), (b), and (c).
(a) If $e$ is already in normal form, then we are ready. Otherwise, let us perform one of the following steps (b) or (c).
(b) Choose a subformula of $e$ of the form $\varphi=\exists V . \psi$ such that $V \in \operatorname{Free}(e) \cap \operatorname{Bound}(e)$ and $\operatorname{Free}(e) \cap$ $\operatorname{Bound}(\psi)=\emptyset$. Then take a variable $W$ of the same type as $V$ with $W \notin \operatorname{Free}(e) \cup \operatorname{Bound}(e)$, and let $\varphi^{\prime}=\exists W \cdot \psi[V / W]$. Note that $\operatorname{Free}\left(\varphi^{\prime}\right)=\operatorname{Free}(\varphi)$ and by Observation 14.4.2 $(1)$, we have $\mathrm{L}\left(\varphi^{\prime}\right)=\mathrm{L}(\varphi)$. Hence, by Observation 14.4.4 we have $\llbracket e \rrbracket \mathcal{V}=\llbracket e\left[\varphi / \varphi^{\prime}\right] \rrbracket \mathcal{V}$. Then continue at (a) with $e\left[\varphi / \varphi^{\prime}\right]$ instead of $e$.
(c) Choose a subformula of $e$ of the form $t=Q_{V} u$ for some $Q \in\{十, X\}$ such that $V \in$ Free $(e) \cap$ $\operatorname{Bound}(e)$ and $\operatorname{Free}(e) \cap \operatorname{Bound}(u)=\emptyset$. Then take a variable $W$ of the same type as $V$ with $W \notin$ Free $(e) \cup \operatorname{Bound}(e)$, and let $t^{\prime}=Q_{W} u[V / W]$. Note that Free $\left(t^{\prime}\right)=\operatorname{Free}(t)$ and by Observation 14.4.2(2), we have $\llbracket t^{\prime} \rrbracket=\llbracket t \rrbracket$. Hence, by Observation 14.4 .5 we have $\llbracket e \rrbracket v=\llbracket e\left[t / t^{\prime}\right\rfloor \rrbracket \mathcal{v}$. Then continue at (a) with $e\left[t / t^{\prime}\right]$ instead of $e$.

The next example shows the usefulness of weighted conjunction and weighted universal first-order quantification as specification tool.
Example 14.4.7. In this example we show that, for each ( $\Sigma, \mathrm{B}$ )-weighted local system $\mathcal{S}$ (cf. Section 11.1) we can construct a sentence $e_{\mathcal{S}} \in \operatorname{MSO}^{\mathrm{ext}}(\Sigma, \mathrm{B})$ such that $(\mathrm{a}) \llbracket e_{\mathcal{S}} \rrbracket=\llbracket \mathcal{S} \rrbracket$ and (b) $e_{\mathcal{S}}$ does not contain weighted second-order universal or existential quantification. For the definition of the macro $\langle b\rangle$ (with $b \in B$ ) we refer to Example 14.2.2, In an obvious way, we extend the syntax of $\mathrm{MSO}^{\text {ext }}(\Sigma, \mathrm{B})$ by allowing that + can combine finitely many formulas (and not just two). Hence, for each finite family ( $e_{i} \mid i \in I$ ) of formulas $e_{i} \in \mathrm{MSO}^{\text {ext }}(\Sigma, \mathrm{B})$ we may write

$$
\underset{i \in I}{\nmid} e_{i}
$$

This formula has the obvious semantics.
Let $\mathcal{S}=(g, F)$. We construct the sentence $e_{\mathcal{S}}=e_{1} \times e_{2}$ in $\mathrm{MSO}^{\mathrm{ext}}(\Sigma, \mathrm{B})$ where the intuition behind $e_{1}$ and $e_{2}$ is that they simulate $g$ and $F$, respectively. Formally, we let

$$
\begin{aligned}
& e_{1}=X_{x}\left(\underset{\substack{k \in \operatorname{maxrk}(\Sigma), f \in \operatorname{Fork}(\Sigma)^{(k)}}}{ }\left(\varphi_{f}(x) \triangleright\langle g(f)\rangle\right)\right) \\
& e_{2}=\boldsymbol{F}_{x}\left(\underset{\sigma \in \Sigma}{ }\left(\operatorname{root}(x) \wedge \operatorname{label}_{\sigma}(x) \triangleright\langle F(\sigma)\rangle\right)\right)
\end{aligned}
$$

and, for each fork $\left(\sigma_{1} \cdots \sigma_{k}, \sigma\right) \in \operatorname{Fork}(\Sigma)^{(k)}$, the formula $\varphi_{\left(\sigma_{1} \cdots \sigma_{k}, \sigma\right)}(x)$ in $\operatorname{MSO}\left(\Sigma_{\{x\}}\right)$ defined by

$$
\varphi_{\left(\sigma_{1} \cdots \sigma_{k}, \sigma\right)}(x)=\operatorname{label}_{\sigma}(x) \wedge \forall y . \bigwedge_{i \in[k]}\left(\operatorname{edge}_{i}(x, y) \rightarrow \operatorname{label}_{\sigma_{i}}(y)\right)
$$

Clearly, the subformulas
of $e_{1}$ and $e_{2}$, respectively, are recognizable step formulas.
Now we consider the semantics of $e_{\mathcal{S}}$. The following is easy to see:

$$
\text { For every } \xi \in \mathrm{T}_{\Sigma} \text { and } w \in \operatorname{pos}(\xi) \text {, we have that }
$$

$$
\begin{align*}
& (\xi,[x \mapsto w]) \models \varphi_{\left(\sigma_{1} \cdots \sigma_{k}, \sigma\right)}(x) \quad \text { iff }  \tag{14.25}\\
& \left(\xi(w)=\sigma \text { and } \xi(w i)=\sigma_{i} \text { for each } i \in[\operatorname{rk}(\xi(w))]\right)
\end{align*}
$$

Since for every $\xi \in \mathrm{T}_{\Sigma}$ and $w \in \operatorname{pos}(\xi)$, there exists exactly one fork $f$ which occurs in $\xi$ at $w$, we can also see the following easily (using (14.25)).

For every $\xi \in \mathrm{T}_{\Sigma}$ and $w \in \operatorname{pos}(\xi)$, we have that

$$
\begin{equation*}
\llbracket \underset{\substack{k \in \operatorname{maxrk}(\Sigma), f \in \operatorname{Fork}(\Sigma)^{(k)}}}{ }\left(\varphi_{f}(x) \triangleright\langle g(f)\rangle\right) \rrbracket(\xi,[x \mapsto w])=g((\xi(w 1) \cdots \xi(w \operatorname{rk}(\xi(w))), \xi(w))) \tag{14.26}
\end{equation*}
$$

Now let $\xi \in \mathrm{T}_{\Sigma}$. Then we can calculate as follows.

$$
\begin{align*}
\llbracket e_{1} \rrbracket(\xi) & =\llbracket X_{x} \cdot\left({\left.\underset{\substack{k \in \operatorname{maxrk}(\Sigma), f \in \operatorname{Fork}(\Sigma)^{(k)}}}{ }\left(\varphi_{f}(x) \triangleright\langle g(f)\rangle\right)\right) \rrbracket(\xi)}=\bigotimes_{w \in \operatorname{pos}(\xi)} \llbracket{\left.\underset{\substack{k \in \operatorname{maxrk}(\Sigma), f \in \operatorname{Fork}(\Sigma)^{(k)}}}{ }\left(\varphi_{f}(x) \triangleright\langle g(f)\rangle\right)\right) \rrbracket_{\{x\}}(\xi,[x \mapsto w])}=\bigotimes_{w \in \operatorname{pos}(\xi)} g((\xi(w 1) \cdots \xi(w \operatorname{rk}(\xi(w))), \xi(w)))\right. \\
& =g(\xi)
\end{align*}
$$

Moreover, we can calculate as follows.

$$
\begin{aligned}
\llbracket e_{2} \rrbracket(\xi) & =\llbracket+_{x} \cdot\left(\underset{\sigma \in \Sigma}{ }\left(\operatorname{root}(x) \wedge \operatorname{label}_{\sigma}(x) \triangleright\langle F(\sigma)\rangle\right)\right) \rrbracket(\xi) \\
& =\bigoplus_{w \in \operatorname{pos}(\xi)} \llbracket{\underset{\sigma}{* \in \Sigma}}\left(\operatorname{root}(x) \wedge \operatorname{label}_{\sigma}(x) \triangleright\langle F(\sigma)\rangle\right) \rrbracket_{\{x\}}(\xi,[x \mapsto w]) \\
& =\llbracket\langle F(\xi(\varepsilon))\rangle \rrbracket \quad\left(\operatorname{because}(\xi,[x \mapsto w]) \models\left(\operatorname{root}(x) \wedge \operatorname{label}_{\sigma}(x)\right) \text { iff } w=\varepsilon \text { and } \sigma=\xi(w)\right) \\
& =F(\xi(\varepsilon)) .
\end{aligned}
$$

Hence $\llbracket e \rrbracket(\xi)=\llbracket e_{1} \times e_{2} \rrbracket(\xi)=\llbracket e_{1} \rrbracket(\xi) \otimes \llbracket e_{2} \rrbracket(\xi)=g(\xi) \otimes F(\xi(\varepsilon))=\llbracket \mathcal{S} \rrbracket(\xi)$.

### 14.4.2 The carefully extended weighted MSO-logic $\mathrm{MSO}^{\text {cext }}(\Sigma, \mathrm{B})$

Similar to the situation of the weighted MSO formulas defined in DV06] (also cf. [DG05, Ex. 3.4]), there exists an extended MSO formula $e \in \operatorname{MSO}^{\text {ext }}(\Sigma$, Nat) such that $\llbracket e \rrbracket$ is not recognizable. In fact, we can give an atomic formula such that its weighted first-order universal quantification is not recognizable. The same holds for the weighted second-order universal quantification.

Example 14.4.8. Let $\Sigma=\left\{\gamma^{(1)}, \alpha^{(0)}\right\}$. We consider the semiring Nat of natural numbers and the formula $e \in \mathrm{MSO}^{\text {ext }}(\Sigma$, Nat $)$, where

$$
e=X_{x} \mathrm{H}(\kappa)
$$

and $\kappa=\left(\kappa_{k} \mid k \in \mathbb{N}\right)$ with $\kappa_{k}: \Sigma_{\{x\}}^{(k)} \rightarrow \mathbb{N}$ and $\kappa_{k}((\sigma, \mathcal{U}))=2$ for each $(\sigma, \mathcal{U}) \in \Sigma_{\{x\}}^{(k)}$.
Then, for each $n \in \mathbb{N}$, we have (using $\Pi$ as notation for the generalization of $\cdot$ to a finite number of arguments)

$$
\begin{aligned}
\llbracket X_{x} \mathrm{H}(\kappa) \rrbracket\left(\gamma^{n}(\alpha)\right) & =\prod_{w \in \operatorname{pos}\left(\gamma^{n}(\alpha)\right)} \llbracket \mathrm{H}(\kappa) \rrbracket_{\{x\}}\left(\gamma^{n}(\alpha)[x \mapsto w]\right) \\
& =\prod_{w \in \operatorname{pos}\left(\gamma^{n}(\alpha)\right)} \mathrm{h}_{\kappa}\left(\gamma^{n}(\alpha)[x \mapsto w]\right)=\prod_{w \in \operatorname{pos}\left(\gamma^{n}(\alpha)\right)} 2^{n+1}=2^{(n+1)^{2}} .
\end{aligned}
$$

In Example 13.1 .2 we have shown that there does not exist a $(\Sigma, N a t)$-wta $\mathcal{A}$ such that $\llbracket \mathcal{A} \rrbracket=\llbracket e \rrbracket$.
Also weighted second-order universal quantification grows too fast for being recognizable; it grows even faster than weighted first-order universal quantification. To see this, we consider the formula $f \in$ $\mathrm{MSO}^{\text {ext }}(\Sigma$, Nat $)$, where

$$
f=X_{X} \mathrm{H}(\kappa)
$$

and $\kappa=\left(\kappa_{k} \mid k \in \mathbb{N}\right)$ with $\kappa_{k}: \Sigma_{\{X\}}^{(k)} \rightarrow \mathbb{N}$ and $\kappa_{k}((\sigma, \mathcal{U}))=2$ for each $(\sigma, \mathcal{U}) \in \Sigma_{\{X\}}^{(k)}$.
Then, for each $n \in \mathbb{N}$, we have

$$
\llbracket X_{X} \mathrm{H}(\kappa) \rrbracket\left(\gamma^{n}(\alpha)\right)=\prod_{W \subseteq \operatorname{pos}\left(\gamma^{n}(\alpha)\right)} \llbracket \mathrm{H}(\kappa) \rrbracket_{\{X\}}\left(\gamma^{n}(\alpha)[X \mapsto W]\right)=\prod_{W \subseteq \operatorname{pos}\left(\gamma^{n}(\alpha)\right)} 2^{n+1}=2^{(n+1) \cdot 2^{n+1}}
$$

As in Example 13.1 .2 , we can prove that there does not exist a $(\Sigma$, Nat)-wta $\mathcal{A}$ such that $\llbracket \mathcal{A} \rrbracket=\llbracket f \rrbracket$.
In order to decrease the computational power, we define the above mentioned fragment of $\mathrm{MSO}^{\text {ext }}(\Sigma, \mathrm{B})$. Formally, we define the set of carefully extended MSO formulas over $\Sigma$ and B , denoted by $\mathrm{MSO}^{\text {cext }}(\Sigma, \mathrm{B})$, to be the set of all formulas $e \in \mathrm{MSO}^{\text {ext }}(\Sigma, \mathrm{B})$ such that the following two conditions hold:

- if $e$ contains a subformula $X_{x} e^{\prime}$, then $e^{\prime}$ is a recognizable step formula, i.e., a formula in $\operatorname{MSO}(\Sigma, \mathrm{B})$ of the form $\left(\varphi_{1} \triangleright\left\langle b_{1}\right\rangle\right)+\ldots+\left(\varphi_{n} \triangleright\left\langle b_{n}\right\rangle\right)$ as specified in (14.13) and
- $e$ does not have a subformula of the form $X_{X} e^{\prime}$.

Thus, in particular, weighted first-order universal quantification cannot be nested. Moreover, obviously, $\operatorname{MSO}(\Sigma, \mathrm{B}) \subset \mathrm{MSO}^{\text {cext }}(\Sigma, \mathrm{B}) \subset \mathrm{MSO}^{\mathrm{ext}}(\Sigma, \mathrm{B})$.

As contribution to the comparison of $\operatorname{MSO}(\Sigma, \mathrm{B})$ and the weighted MSO-logics of DG05, DG07, DG09, we prove that each atomic formula of the form $\mathrm{H}(\kappa)$ can be expressed by a weighted first-order quantification over a recognizable step formula.

Lemma 14.4.9. (cf. FSV12, Lm. 5.12]) Let $\mathcal{U}$ be a finite set of variables and $\kappa=\left(\kappa_{k} \mid k \in \mathbb{N}\right)$ be an $\mathbb{N}$-indexed family with $\kappa_{k}: \Sigma_{\mathcal{U}}^{(k)} \rightarrow B$. Moreover, let $x \notin \mathcal{U}$. We can construct a recognizable step formula $e$ such that Free $(e)=\mathcal{U} \cup\{x\}$ and $\llbracket H(\kappa) \rrbracket=\llbracket X_{x} e \rrbracket$.

Proof. Let $\Sigma_{\mathcal{U}}=\left\{\left(\sigma_{1}, U_{1}\right), \ldots,\left(\sigma_{n}, U_{n}\right)\right\}$. We define the recognizable step formula

$$
e=\left(\varphi_{\left(\sigma_{1}, U_{1}\right)}\left(\{x\} \cup U_{1}\right) \triangleright\left\langle b_{\left(\sigma_{1}, U_{1}\right)}\right\rangle\right)+\ldots+\left(\varphi_{\left(\sigma_{n}, U_{n}\right)}\left(\{x\} \cup U_{n}\right) \triangleright\left\langle b_{\left(\sigma_{n}, U_{n}\right)}\right\rangle\right)
$$

where, for each $i \in[n]$, we let

- $b_{\left(\sigma_{i}, U_{i}\right)}=\kappa_{\mathrm{rk}\left(\left(\sigma_{i}, U_{i}\right)\right.}\left(\left(\sigma_{i}, U_{i}\right)\right)$ and
- $\varphi_{\left(\sigma_{i}, U_{i}\right)}\left(\{x\} \cup U_{i}\right)$ is the $\operatorname{MSO}(\Sigma)$-formula

$$
\operatorname{label}_{\sigma_{i}}(x) \wedge\left(\bigwedge_{y \in U_{i}}(x=y)\right) \wedge\left(\bigwedge_{y \in \mathcal{U} \backslash U_{i}} \neg(x=y)\right) \wedge\left(\bigwedge_{X \in U_{i}}(x \in X)\right) \wedge\left(\bigwedge_{X \in \mathcal{U} \backslash U_{i}} \neg(x \in X)\right)
$$

Obviously, for every $\zeta \in \mathrm{T}_{\Sigma \mathcal{U}}^{\mathrm{v}}, w \in \operatorname{pos}(\zeta)$, and $\left(\sigma_{i}, U_{i}\right) \in \mathcal{U}$, we have

$$
\begin{equation*}
\zeta[x \mapsto w] \in \mathrm{L}_{\mathcal{U}}\left(\varphi_{\left(\sigma_{i}, U_{i}\right)}\right) \quad \text { iff } \quad \zeta(w)=\left(\sigma_{i}, U_{i}\right) \tag{14.27}
\end{equation*}
$$

and hence $\left(\mathrm{L}_{\mathcal{U}}\left(\varphi_{\left(\sigma_{i}, U_{i}\right)}\right) \mid i \in[n]\right)$ is a partitioning of $\mathrm{T}_{\Sigma_{\mathcal{U} \cup\{x\}}^{\mathrm{v}}}$. Thus we have

$$
\llbracket \varphi_{\left(\sigma_{i}, U_{i}\right)}\left(\{x\} \cup U_{i}\right) \triangleright\left\langle b_{\left(\sigma_{i}, U_{i}\right)}\right\rangle \rrbracket(\zeta[x \mapsto w])= \begin{cases}b_{\left(\sigma_{i}, U_{i}\right)} & \text { if } \zeta(w)=\left(\sigma_{i}, U_{i}\right)  \tag{14.28}\\ \mathbb{O} & \text { otherwise } .\end{cases}
$$

Let $\zeta \in \mathrm{T}_{\Sigma_{\mathcal{U}}}$. If $\zeta$ is not valid, then $\llbracket H(\kappa) \rrbracket(\zeta)=\mathbb{0}=\llbracket X_{x} e \rrbracket(\zeta)$. Now let $\zeta$ be valid. Then we have

$$
\begin{array}{rlr}
\llbracket X_{x} e \rrbracket(\zeta) & =\bigotimes_{w \in \operatorname{pos}(\zeta)} \llbracket e \rrbracket_{\mathcal{U} \cup\{x\}}(\zeta[x \mapsto w]) \\
& =\bigotimes_{w \in \operatorname{pos}(\zeta)} b_{\zeta(w)}  \tag{14.28}\\
& =\bigotimes_{w \in \operatorname{pos}(\zeta)} \kappa_{\operatorname{rk}(\zeta(w))}(\zeta(w)) \\
& =\llbracket \mathrm{H}(\kappa) \rrbracket(\zeta) & \quad \text { (by (14.28)) } \\
\end{array}
$$

In Lemma 14.4.12 we show how formulas of the form $X_{x} e$, where $e$ is a recognizable step formula, can be simulated by a formulas in $\operatorname{MSO}(\Sigma, \mathrm{B})$ (cf. $t$ in (14.29). Here we prove that atomic formulas alone are not powerful enough to simulate recognizable step formulas.

Lemma 14.4.10. For the ranked alphabet $\Sigma=\left\{\alpha^{(2)}, \gamma^{(1)}, \alpha^{(0)}\right\}$ we can construct a recognizable step sentence $e$ over $\Sigma$ and Nat such that $\llbracket e \rrbracket \neq \llbracket H(\kappa) \rrbracket$ for any family $\kappa=\left(\kappa_{k} \mid k \in \mathbb{N}\right)$ of mappings $\kappa_{k}: \Sigma^{(k)} \rightarrow \mathbb{N}$.

Proof. By Theorem $10.3 .1(\mathrm{~A}) \Rightarrow(\mathrm{B})$ and Lemma $14.2 .3(\mathrm{~A}) \Rightarrow(\mathrm{B})$, it is sufficient to construct a crisp deterministic $(\Sigma, N a t)$-wta $\mathcal{A}$ such that $\llbracket \mathcal{A} \rrbracket \neq \llbracket H(\kappa) \rrbracket$ for any family $\kappa=\left(\kappa_{k} \mid k \in \mathbb{N}\right)$ of mappings $\kappa_{k}: \Sigma^{(k)} \rightarrow \mathbb{N}$.

We consider the crisp deterministic ( $\Sigma, \mathrm{Nat})$-wta $\mathcal{A}$ constructed in Example 3.2.16. There it was shown that $\llbracket \mathcal{A} \rrbracket=$ twothree, where the weighted tree language twothree : $\mathrm{T}_{\Sigma} \rightarrow \mathbb{N}$ is defined by

$$
\text { twothree }(\xi)= \begin{cases}2 & \text { if }|\operatorname{pos}(\xi)| \text { is even } \\ 3 & \text { otherwise }\end{cases}
$$

for each $\xi \in \mathrm{T}_{\Sigma}$.

We show by contradiction that $\mathcal{A}$ has the desired property. For this, we assume that there is a family $\kappa=\left(\kappa_{k} \mid k \in \mathbb{N}\right)$ of mappings $\kappa_{k}: \Sigma^{(k)} \rightarrow \mathbb{N}$ with $\llbracket \mathrm{H}(\kappa) \rrbracket=\llbracket \mathcal{A} \rrbracket$. By definition,

$$
\llbracket H(\kappa) \rrbracket\left(\gamma^{n}(\alpha)\right)=\kappa_{0}(\alpha) \cdot\left(\kappa_{1}(\gamma)\right)^{n}
$$

for each $n \in \mathbb{N}$. Since $\operatorname{im}(\llbracket \mathcal{A} \rrbracket)=\{2,3\}$, i.e., a finite set, it follows that $\kappa_{1}(\gamma)=0$ or $\kappa_{1}(\gamma)=1$. In the first case $\llbracket \mathrm{H}(\kappa) \rrbracket\left(\gamma^{n}(\alpha)\right)=0$ for each $n \in \mathbb{N}_{+}$. In the second case $\llbracket \mathrm{H}(\kappa) \rrbracket\left(\gamma^{n}(\alpha)\right)=\kappa_{0}(\alpha)$ for each $n \in \mathbb{N}$. Thus, in both cases, we have $\llbracket \mathrm{H}(\kappa) \rrbracket\left(\gamma^{2}(\alpha)\right)=\llbracket \mathrm{H}(\kappa) \rrbracket(\gamma(\alpha))$. This contradicts $\llbracket \mathrm{H}(\kappa) \rrbracket=\llbracket \mathcal{A} \rrbracket$, because $\llbracket \mathcal{A} \rrbracket\left(\gamma^{2}(\alpha)\right)=2 \neq 3=\llbracket \mathcal{A} \rrbracket(\gamma(\alpha))$.

### 14.4.3 The main result for $\operatorname{MSO}^{\text {cext }}(\Sigma, \mathrm{B})$

In this subsection we prove that the two weighted logics $\operatorname{MSO}(\Sigma, \mathrm{B})$ and $\mathrm{MSO}^{\text {cext }}(\Sigma, \mathrm{B})$ are equivalent (cf. Theorem 14.4.11). This theorem follows directly from Lemmas 14.4.12 and Lemma 14.4.15 For each bounded lattice B , we will prove that even the two weighted $\operatorname{logics} \operatorname{MSO}(\Sigma, \mathrm{B})$ and $\mathrm{MSO}^{\text {ext }}(\Sigma, \mathrm{B})$ are equivalent (cf. Theorem 19.8.4).

Theorem 14.4.11. Let $\Sigma$ be a ranked alphabet and B be a commutative semiring. Moreover, let $e \in$ $\mathrm{MSO}^{\text {cext }}(\Sigma, \mathrm{B})$ and $\mathcal{V} \supseteq$ Free $(e)$ be a finite set of variables. Then we can construct an $\mathrm{MSO}(\Sigma, \mathrm{B})$-formula $f$ such that $\operatorname{Free}(e)=\operatorname{Free}(f)$ and $\llbracket e \rrbracket \mathcal{V}=\llbracket f \rrbracket \mathcal{V}$.

In the next lemma we get rid of weighted first-order universal quantification. We follow the proof of [FSV12, Lm. 5.10]; in its turn, the construction in the proof of the latter lemma was inspired by the construction in the proof of [G09, Lm. 5.4].

Lemma 14.4.12. (cf. FSV12, Lm. 5.10]) Let $e \in \operatorname{MSO}^{\text {cext }}(\Sigma, \mathrm{B})$ and $\mathcal{V} \supseteq$ Free $(e)$ be a finite set of variables. Then we can construct an $\operatorname{MSO}^{\text {cext }}(\Sigma, \mathrm{B})$-formula $f$ such that (a) Free $(e)=\operatorname{Free}(f),(\mathrm{b}) \llbracket e \rrbracket \nu \mathcal{V}=$ $\llbracket f \rrbracket \mathcal{V}$, and (c) $f$ does not contain a subformula of the form $X_{x} e^{\prime}$.

Proof. If $e$ does not contain a subformula of the form $X_{x} e^{\prime}$, then we can choose $f$ to be $e$ and we are ready. Otherwise, we choose an occurrence of a weighted first-order universal quantification in $e$. More precisely, let $X_{x} e^{\prime}$ be an occurrence of a subformula of $e$. By definition of $\operatorname{MSO}^{\text {cext }}(\Sigma, \mathrm{B})$, the subformula $e^{\prime}$ is a recognizable step formula, i.e., there exist $n \in \mathbb{N}_{+}, b_{1}, \ldots, b_{n} \in B$, and $\varphi_{1}, \ldots, \varphi_{n} \in \operatorname{MSO}(\Sigma)$ such that

$$
e^{\prime}=\left(\varphi_{1} \triangleright\left\langle b_{1}\right\rangle\right)+\ldots+\left(\varphi_{n} \triangleright\left\langle b_{n}\right\rangle\right) .
$$

Let $\operatorname{Free}\left(X_{x} e^{\prime}\right)=\mathcal{V}^{\prime}$. Then $\operatorname{Free}\left(e^{\prime}\right) \subseteq \mathcal{V}^{\prime} \cup\{x\}$.
Next we define the formula $t$ by which we will replace the selected occurrence of the subformula $X_{x} e^{\prime}$ later. Intuitively, the $\operatorname{MSO}(\Sigma)$-formula $\left(\bigwedge_{i \in[n]} \forall x .\left(\left(x \in X_{i}\right) \leftrightarrow \varphi_{i}\right)\right)$ is a part of $t$ and guarantees that, for each $i \in[n]$, the set of positions assigned to $X_{i}$ is exactly the set of positions for which $\varphi_{i}$ is true (cf. (14.30)). Formally, we let

$$
\begin{equation*}
t=\mathcal{F}_{X_{1}} \cdots \mathcal{F}_{X_{n}}\left(\left(\bigwedge_{i \in[n]} \forall x .\left(\left(x \in X_{i}\right) \leftrightarrow \varphi_{i}\right)\right) \triangleright \mathrm{H}(\kappa)\right) \tag{14.29}
\end{equation*}
$$

be the formula in $\operatorname{MSO}(\Sigma, \mathrm{B})$ where

- $X_{1}, \ldots, X_{n}$ are second-order variables which do not occur in Free $(e) \cup \operatorname{Bound}(e)$ and
- $\kappa$ is the $\mathbb{N}$-indexed family $\left(\kappa_{k} \mid k \in \mathbb{N}\right)$ of mappings $\kappa_{k}: \Sigma_{\mathcal{U}}^{(k)} \rightarrow B$, where $\mathcal{U}=\mathcal{V}^{\prime} \cup\left\{X_{1}, \ldots, X_{n}\right\}$, such that, for each $(\sigma, \mathcal{W}) \in \Sigma_{\mathcal{U}}^{(k)}$, we define

$$
\kappa_{k}((\sigma, \mathcal{W}))=\bigoplus_{i \in[n]: X_{i} \in \mathcal{W}} b_{i}
$$

On purpose, we use the variable $x$, which might occur in $\varphi_{i}$, also as variable for the universal quantification; thereby we "synchronize" the sequence $X_{1}, \ldots, X_{n}$ with the formulas $\varphi_{1}, \ldots, \varphi_{n}$. Obviously, Free $(t)=\mathcal{V}^{\prime}$ and hence $\operatorname{Free}\left(X_{x} e^{\prime}\right)=\operatorname{Free}(t)$.

Next we analyse the semantics of the $\operatorname{MSO}(\Sigma)$-subformula $\left(\bigwedge_{i \in[n]} \forall x .\left(\left(x \in X_{i}\right) \leftrightarrow \varphi_{i}\right)\right)$. Let $(\xi, \mu) \in$ $\mathrm{T}_{\Sigma_{\mathcal{V}^{\prime}}}^{\mathrm{v}}$ and let $V_{1}^{\prime}, \ldots, V_{n}^{\prime} \subseteq \operatorname{pos}(\xi)$. Obviously, the following equivalence holds:

$$
\begin{align*}
& \left(\xi, \mu\left[X_{1} \mapsto V_{1}^{\prime}, \ldots, X_{n} \mapsto V_{n}^{\prime}\right]\right) \models\left(\bigwedge_{i \in[n]} \forall x .\left(\left(x \in X_{i}\right) \leftrightarrow \varphi_{i}\right)\right) \text { iff }  \tag{14.30}\\
& (\forall i \in[n]): V_{i}^{\prime}=\left\{w \in \operatorname{pos}(\xi) \mid(\xi, \mu[x \mapsto w]) \in \mathrm{L}_{\mathcal{V}^{\prime} \cup\{x\}}\left(\varphi_{i}\right)\right\} .
\end{align*}
$$

For every $i \in[n]$, we abbreviate the set $\left\{w \in \operatorname{pos}(\xi) \mid(\xi, \mu[x \mapsto w]) \in \operatorname{L}_{\mathcal{V}^{\prime} \cup\{x\}}\left(\varphi_{i}\right)\right\}$ by $W(\xi, \mu, i)$.
Next we prove that

$$
\begin{equation*}
\llbracket t \rrbracket=\llbracket X_{x} e^{\prime} \rrbracket \tag{14.31}
\end{equation*}
$$

For this, let $(\xi, \mu) \in \mathrm{T}_{\Sigma_{\mathcal{V}^{\prime}}}$. If $(\xi, \mu)$ is not valid, then both sides of (14.31) evaluate to $\mathbb{0}$. Now let $(\xi, \mu) \in \mathrm{T}_{\Sigma_{\mathcal{V}^{\prime}}}^{\mathrm{v}}$. Then we obtain

$$
\begin{aligned}
\llbracket t \rrbracket(\xi, \mu) & =\bigoplus_{V_{1}^{\prime}, \ldots, V_{n}^{\prime} \subseteq \operatorname{pos}(\xi)}\left(\left(\bigwedge_{i \in[n]} \forall x .\left(\left(x \in X_{i}\right) \leftrightarrow \varphi_{i}\right)\right) \triangleright \mathrm{H}(\kappa)\right)\left(\xi, \mu\left[X_{1} \mapsto V_{1}^{\prime}, \ldots, X_{n} \mapsto V_{n}^{\prime}\right]\right) \\
& =\llbracket \mathrm{H}(\kappa) \rrbracket \mathcal{u}\left(\xi, \mu\left[X_{1} \mapsto W(\xi, \mu, 1), \ldots, X_{n} \mapsto W(\xi, \mu, n)\right]\right) \\
& =\mathrm{h}_{\kappa[\mathcal{U} \rightsquigarrow \mathcal{U}]}\left(\left(\xi, \mu\left[X_{1} \mapsto W(\xi, \mu, 1), \ldots, X_{n} \mapsto W(\xi, \mu, n)\right]\right)\right) \\
& =\mathrm{h}_{\kappa}\left(\left(\xi, \mu\left[X_{1} \mapsto W(\xi, \mu, 1), \ldots, X_{n} \mapsto W(\xi, \mu, n)\right]\right)\right) \\
& =\bigotimes_{w \in \operatorname{pos}(\xi)} \kappa_{k}\left(\left(\xi(w),\left\{X_{j} \mid w \in W(\xi, \mu, j)\right\}\right)\right) \quad(\text { by (2.27) (14.30)) and the fact that B is commutative) })
\end{aligned}
$$

where $k=\operatorname{rk}(\zeta(w))$. Let $w \in \operatorname{pos}(\xi)$. Clearly,

$$
\left.\begin{array}{rl}
\kappa_{k}\left(\left(\xi(w),\left\{X_{j} \mid w \in W(\xi, \mu, j)\right\}\right)\right)= & \bigoplus_{\substack{i \in[n]: \\
X_{i} \in\left\{X_{j} \mid w \in W(\xi, \mu, j)\right\}}} b_{i}=\bigoplus_{\substack{i \in[n]: \\
w \in W(\xi, \mu, i)}} b_{i} \\
=\bigoplus_{\substack{i \in[n]:}} b_{i}=\bigoplus_{\substack{i \in[n]: \\
(\xi, \mu[x \mapsto w]) \in \mathrm{L}_{\mathcal{V}^{\prime} \cup\{x\}}\left(\varphi_{i}\right)}} b_{i} \\
= & \bigoplus_{i \in[n]} b_{i} \otimes \chi\left(L_{i}\right)\left((\xi, \mu[x \mapsto w]) \in L_{i}\right.
\end{array}\right)
$$

Hence,

$$
\llbracket t \rrbracket(\xi, \mu)=\bigotimes_{w \in \operatorname{pos}(\xi)} \llbracket e^{\prime} \rrbracket_{\mathcal{V}^{\prime} \cup\{x\}}(\xi, \mu[x \mapsto w])=\llbracket X_{x} e^{\prime} \rrbracket \mathcal{V}^{\prime}(\xi, \mu)=\llbracket X_{x} e^{\prime} \rrbracket(\xi, \mu)
$$

This proves that $\llbracket t \rrbracket=\llbracket X_{x} e^{\prime} \rrbracket$.
Then, by Observation 14.4.5 we obtain $\llbracket e \rrbracket \mathcal{v}=\llbracket g \rrbracket \mathcal{v}$ where $g$ is obtained from $e$ by replacing the subformula $X_{x} e^{\prime}$ by $t$. Since $t \in \operatorname{MSO}(\Sigma, \mathrm{~B})$, we have $g \in \mathrm{MSO}^{\text {cext }}(\Sigma, \mathrm{B})$. Obviously, $g$ contains one occurrence of a weighted first-order universal quantification less than $e$.

If $g$ does not contain a subformula of the form $X_{x} e^{\prime}$, then we are ready and we can choose $f$ to be $g$. Otherwise, we proceed as before by starting with $g$ and by replacing one occurrence of a weighted first-order universal quantification. Since each replacement reduces the number of weighted first-order universal quantifications, this procedure will terminate. Since each replacement preserves the semantics, we eventually obtain a formula $f$ with the desired properties.

Example 14.4.13. In this example we want to illustrate the construction defined in the proof of Lemma 14.4.12. We consider the string ranked alphabet $\Sigma=\left\{\gamma^{(1)}, \alpha^{(0)}\right\}$ and the semiring Nat $=$ $(\mathbb{N},+, \cdot, 0,1)$. Moreover, we let $e=\left(X_{x} \operatorname{label}_{\alpha}(x) \triangleright\langle 2\rangle\right)$. Obviously, Free $(e)=\emptyset$; we let $\mathcal{V}=\emptyset$.

Then the formula $t$ in $\operatorname{MSO}(\Sigma, N a t)(c f . ~(14.29))$ has the following form (with $n=1$ and $\varphi_{1}=$ $\left.\operatorname{label}_{\alpha}(x)\right)$ :

$$
t=\mathcal{W}_{X_{1}}\left(\left(\forall x .\left(\left(x \in X_{1}\right) \leftrightarrow \operatorname{label}_{\alpha}(x)\right)\right) \triangleright \mathrm{H}(\kappa)\right)
$$

where $\kappa$ is the $\mathbb{N}$-indexed family of mappings $\kappa_{k}: \Sigma_{\left\{X_{1}\right\}}^{(k)} \rightarrow \mathbb{N}$ such that

- $\kappa_{0}((\alpha, \emptyset))=\kappa_{1}((\gamma, \emptyset))=+_{X_{1} \in \emptyset} 2=0$ and
- $\kappa_{0}\left(\left(\alpha,\left\{X_{1}\right\}\right)\right)=\kappa_{1}\left(\left(\gamma,\left\{X_{1}\right\}\right)=+_{X_{1} \in\left\{X_{1}\right\}} 2=2\right.$.

For the tree $\gamma(\alpha) \in \mathrm{T}_{\Sigma_{\emptyset}}^{\mathrm{v}}$, we can calculate as follows:

$$
\begin{aligned}
& \llbracket\left(X_{x} \operatorname{label}_{\alpha}(x) \triangleright\langle 2\rangle\right) \rrbracket_{\emptyset}(\gamma(\alpha)) \\
= & \llbracket \operatorname{label}_{\alpha}(x) \triangleright\langle 2\rangle \rrbracket_{\{x\}}(\gamma(\alpha),[x \mapsto 1]) \cdot \llbracket \operatorname{label}_{\alpha}(x) \triangleright\langle 2\rangle \rrbracket_{\{x\}}(\gamma(\alpha),[x \mapsto \varepsilon]) \\
= & \llbracket\langle 2\rangle \rrbracket_{\{x\}}(\gamma(\alpha),[x \mapsto 1]) \cdot 0=2 \cdot 0=0 .
\end{aligned}
$$

Using the above mentioned instance of $t$, we can calculate as follows:

$$
\begin{aligned}
& \llbracket 十_{X_{1}}\left(\left(\forall x \cdot\left(\left(x \in X_{1}\right) \leftrightarrow \operatorname{label}_{\alpha}(x)\right)\right) \triangleright \mathrm{H}(\kappa)\right) \rrbracket_{\emptyset}(\gamma(\alpha)) \\
&={\underset{W \subseteq\{\varepsilon, 1\}}{ }}_{\mathbb{\llbracket}\left(\forall x \cdot\left(\left(x \in X_{1}\right) \leftrightarrow \operatorname{label}_{\alpha}(x)\right)\right) \triangleright \mathrm{H}(\kappa)}^{\rrbracket_{\left\{X_{1}\right\}}\left(\gamma(\alpha),\left[X_{1} \mapsto W\right]\right)} \\
&=\llbracket g \rrbracket_{\left\{X_{1}\right\}}\left(\gamma(\alpha),\left[X_{1} \mapsto\{\varepsilon, 1\}\right]\right)+\llbracket g \rrbracket_{\left\{X_{1}\right\}}\left(\gamma(\alpha),\left[X_{1} \mapsto\{1\}\right]\right) \\
&+\llbracket g \rrbracket_{\left\{X_{1}\right\}}\left(\gamma(\alpha),\left[X_{1} \mapsto\{\varepsilon\}\right]\right)+\llbracket g \rrbracket_{\left\{X_{1}\right\}}\left(\gamma(\alpha),\left[X_{1} \mapsto \emptyset\right]\right) .
\end{aligned}
$$

Due to (14.30), we have

$$
\begin{align*}
& \left(\gamma(\alpha),\left[X_{1} \mapsto V_{1}^{\prime}\right]\right) \models\left(\forall x .\left(\left(x \in X_{1}\right) \leftrightarrow \operatorname{label}_{\alpha}(x)\right)\right. \text { iff }  \tag{14.32}\\
& V_{1}^{\prime}=\left\{w \in \operatorname{pos}(\xi) \mid(\gamma(\alpha),[x \mapsto w]) \in \mathrm{L}_{\{x\}}\left(\operatorname{label}_{\alpha}(x)\right)\right\}
\end{align*}
$$

Since $V_{1}^{\prime}=\{1\}$, we can continue as follows:

$$
\begin{aligned}
& \llbracket g \rrbracket_{\left\{X_{1}\right\}}\left(\gamma(\alpha),\left[X_{1} \mapsto\{\varepsilon, 1\}\right]\right)+\llbracket g \rrbracket_{\left\{X_{1}\right\}}\left(\gamma(\alpha),\left[X_{1} \mapsto\{1\}\right]\right) \\
& \quad+\llbracket g \rrbracket_{\left\{X_{1}\right\}}\left(\gamma(\alpha),\left[X_{1} \mapsto\{\varepsilon\}\right]\right)+\llbracket g \rrbracket_{\left\{X_{1}\right\}}\left(\gamma(\alpha),\left[X_{1} \mapsto \emptyset\right]\right) \\
= & \llbracket H(\kappa) \rrbracket_{\left\{X_{1}\right\}}\left(\gamma(\alpha),\left[X_{1} \mapsto\{1\}\right]\right) \\
= & \kappa_{0}\left(\left(\alpha,\left\{X_{1}\right\}\right) \cdot \kappa_{1}((\gamma, \emptyset))\right. \\
= & 2 \cdot 0=0 .
\end{aligned}
$$

Now we start to prepare the elimination of weighted conjunction in formulas which do not contain weighted first-order universal quantification. First, for each $e_{1}, e_{2} \in \operatorname{MSO}(\Sigma, \mathrm{~B})$ with Free $\left(e_{1}\right)=\mathcal{U}_{1}$ and Free $\left(e_{2}\right)=\mathcal{U}_{2}$, we define the $\operatorname{MSO}(\Sigma, \mathrm{B})$-formula $\mathrm{hp}\left(e_{1}, e_{2}\right)$ (where hp stands for Hadamard product) with Free $\left(\operatorname{hp}\left(e_{1}, e_{2}\right)\right)=\mathcal{U}_{1} \cup \mathcal{U}_{2}$ (cf. proof of [FSV12, Lm. 5.7]).

For the inductive definition of $\mathrm{hp}\left(e_{1}, e_{2}\right)$ we define the well-founded set

$$
(\operatorname{MSO}(\Sigma, \mathrm{B}) \times \operatorname{MSO}(\Sigma, \mathrm{B}), \prec)
$$

where for every $\left(e_{1}^{\prime}, e_{2}^{\prime}\right),\left(e_{1}, e_{2}\right) \in \operatorname{MSO}(\Sigma, \mathrm{B})$ we let $\left(e_{1}^{\prime}, e_{2}^{\prime}\right) \prec\left(e_{1}, e_{2}\right)$ if

- $e_{1}^{\prime} \prec_{\mathrm{MSO}(\Sigma, \mathrm{B})} e_{1}$ (i.e., $e_{1}^{\prime}$ is a direct subformula of $e_{1}$, cf. Section 14.2)) and $e_{2}^{\prime}=e_{2}$ or
- $e_{1}^{\prime}=e_{1}=\mathrm{H}(\kappa)$ for some $\mathbb{N}$-indexed family $\kappa=\left(\kappa_{k} \mid k \in \mathbb{N}\right)$ and $e_{2}^{\prime} \prec_{\mathrm{MSO}(\Sigma, \mathrm{B})} e_{2}$.


Figure 14.3: Illustration of $\prec$ where moving from a node labeled by $A$ to its direct predecessor labeled by $B$ means that $A \prec B$.

Clearly, $\prec$ is well-founded and $\min _{\prec}(\operatorname{MSO}(\Sigma, \mathrm{B}) \times \operatorname{MSO}(\Sigma, \mathrm{B}))$ is the set of all pairs of the form $\left(\mathrm{H}\left(\kappa_{1}\right), \mathrm{H}\left(\kappa_{2}\right)\right)$ where for each $i \in[2], \kappa_{i}=\left(\left(\kappa_{i}\right)_{k} \mid k \in \mathbb{N}\right)$ is an $\mathbb{N}$-indexed family of mappings $\left(\kappa_{i}\right)_{k}: \Sigma_{\mathcal{V}_{i}}^{(k)} \rightarrow B$ for some finite sets $\mathcal{V}_{i}$. In Figure 14.3 we illustrate the well-founded order $\prec$ by means of an example.

Then we define the $\operatorname{MSO}(\Sigma, \mathrm{B})$-formula $\mathrm{hp}\left(e_{1}, e_{2}\right)$ by induction on $(\operatorname{MSO}(\Sigma, \mathrm{B}) \times \operatorname{MSO}(\Sigma, \mathrm{B}), \prec)$ as follows.
(a) Let $e_{1}=\mathrm{H}\left(\kappa_{1}\right)$ and $e_{2}=\mathrm{H}\left(\kappa_{2}\right)$ where for each $i \in[2]$, the members of the $\mathbb{N}$-indexed family $\kappa_{i}=\left(\left(\kappa_{i}\right)_{k} \mid k \in \mathbb{N}\right)$ have the type $\left(\kappa_{i}\right)_{k}: \Sigma_{\mathcal{U}_{i}}^{(k)} \rightarrow B$. Then we define $\operatorname{hp}\left(e_{1}, e_{2}\right)=\mathrm{H}(\kappa)$, where $\kappa=\left(\kappa_{k} \mid k \in \mathbb{N}\right)$ and for every $k \in \mathbb{N}$ we define the mapping $\kappa_{k}: \Sigma_{\mathcal{U}_{1} \cup \mathcal{U}_{2}}^{(k)} \rightarrow B$ for each $(\sigma, V) \in \Sigma_{\mathcal{U}_{1} \cup \mathcal{U}_{2}}^{(k)}$ by $\kappa_{k}((\sigma, V))=\left(\kappa_{1}\right)_{k}\left(\left(\sigma, V \cap \mathcal{U}_{1}\right)\right) \otimes\left(\kappa_{2}\right)_{k}\left(\left(\sigma, V \cap \mathcal{U}_{2}\right)\right)$.
(b) Let $e_{1}=g_{1}+g_{2}$. We define $\operatorname{hp}\left(\left(g_{1}+g_{2}\right), e_{2}\right)=\operatorname{hp}\left(g_{1}, e_{2}\right)+\operatorname{hp}\left(g_{2}, e_{2}\right)$.
(c) Let $e_{1}=+_{x} g$. Then $x \notin \mathcal{U}_{2}$. We define $\operatorname{hp}\left(+_{x} g, e_{2}\right)=+_{x} \operatorname{hp}\left(g, e_{2}\right)$.
(d) Let $e_{1}=+_{X} g$. Then $X \notin \mathcal{U}_{2}$. We define $\operatorname{hp}\left(+_{X} g, e_{2}\right)=+_{X} \operatorname{hp}\left(g, e_{2}\right)$.
(e) Let $e_{1}=(\varphi \triangleright g)$. We define $\operatorname{hp}\left(\varphi \triangleright g, e_{2}\right)=\left(\varphi \triangleright \mathrm{hp}\left(g, e_{2}\right)\right)$.

In each of the following cases, let $e_{1}=\mathrm{H}\left(\kappa_{1}\right)$ with $\kappa_{1}=\left(\left(\kappa_{1}\right)_{k} \mid k \in \mathbb{N}\right)$ and $\left(\kappa_{1}\right)_{k}: \Sigma_{\mathcal{U}_{1}}^{(k)} \rightarrow B$.
(f) Let $e_{2}=g_{1}+g_{2}$. We define $\left.\operatorname{hp}\left(\mathrm{H}\left(\kappa_{1}\right), g_{1}+g_{2}\right)\right)=\operatorname{hp}\left(\mathrm{H}\left(\kappa_{1}\right), g_{1}\right)+\operatorname{hp}\left(\mathrm{H}\left(\kappa_{1}\right), g_{2}\right)$.
(g) Let $e_{2}=+_{x} g$. Then $x \notin \mathcal{U}_{1}$. We define $\operatorname{hp}\left(\mathrm{H}\left(\kappa_{1}\right),+_{x} g\right)=+_{x} \operatorname{hp}\left(\mathrm{H}\left(\kappa_{1}\right), g\right)$.
(h) Let $e_{2}=+_{X} g$. Then $X \notin \mathcal{U}_{1}$. We define $\operatorname{hp}\left(\mathrm{H}\left(\kappa_{1}\right),+_{X} g\right)=+_{X} \operatorname{hp}\left(\mathrm{H}\left(\kappa_{1}\right), g\right)$.
(i) Let $e_{1}=(\varphi \triangleright g)$. We define $\mathrm{hp}\left(\mathrm{H}\left(\kappa_{1}\right), \varphi \triangleright g\right)=\left(\varphi \triangleright \mathrm{hp}\left(\mathrm{H}\left(\kappa_{1}\right), g\right)\right)$.

Next we prove that, roughly speaking, $e_{1} \times e_{2}$ and $\operatorname{hp}\left(e_{1}, e_{2}\right)$ are equivalent, if B is distributive.
Lemma 14.4.14. (cf. FSV12, Lm. 5.7]) Let B be distributive and let $e_{1}, e_{2} \in \operatorname{MSO}(\Sigma, \mathrm{~B})$ with Free $\left(e_{1}\right)=$ $\mathcal{U}_{1}$ and $\operatorname{Free}\left(e_{2}\right)=\mathcal{U}_{2}$ such that $e_{1} \times e_{2}$ is in normal form. For each finite set $\mathcal{V}$ of variables which contains $\mathcal{U}_{1} \cup \mathcal{U}_{2}$ we have $\llbracket e_{1} \times e_{2} \rrbracket_{\mathcal{V}}=\llbracket \operatorname{hp}\left(e_{1}, e_{2}\right) \rrbracket_{\mathcal{V}}$.

Proof. We prove by induction on $(\operatorname{MSO}(\Sigma, \mathrm{B}) \times \operatorname{MSO}(\Sigma, \mathrm{B}), \prec)$. In the proof we refer to Cases (a)-(i) of the definition of $\operatorname{hp}\left(e_{1}, e_{2}\right)$. Let $\zeta \in \mathrm{T}_{\Sigma_{\mathcal{V}}}^{\mathrm{v}}$.

Let $e_{1}$ and $e_{2}$ be as in Case (a). Then we can calculate as follows:

$$
\begin{aligned}
\llbracket \mathrm{H}\left(\kappa_{1}\right) \times \mathrm{H}\left(\kappa_{2}\right) \rrbracket \mathcal{V}(\zeta) & =\llbracket \mathrm{H}\left(\kappa_{1}\right) \rrbracket \mathcal{V}(\zeta) \otimes \llbracket \mathrm{H}\left(\kappa_{2}\right) \rrbracket \mathcal{V}(\zeta)=\mathrm{h}_{\kappa_{1}\left[\mathcal{U}_{1} \rightsquigarrow \mathcal{V}\right]}(\zeta) \otimes \mathrm{h}_{\kappa_{2}\left[\mathcal{U}_{2} \rightsquigarrow \mathcal{V}\right]}(\zeta) \\
& =\mathrm{h}_{\kappa\left[\mathcal{U}_{1} \cup \mathcal{U}_{2} \rightsquigarrow \mathcal{V}\right]}(\zeta)=\llbracket \mathrm{H}(\kappa) \rrbracket \mathcal{V}(\zeta)=\llbracket \operatorname{hp}\left(\mathrm{H}\left(\kappa_{1}\right), \mathrm{H}\left(\kappa_{2}\right)\right) \rrbracket \mathcal{V}(\zeta)
\end{aligned}
$$

where the third equality holds because

$$
=\bigotimes_{\substack{w \in \operatorname{pos}(\zeta), \zeta(w)=(\sigma, V)}} \kappa_{1}\left(\left(\sigma, \mathcal{U}_{1} \cap V\right) \otimes \bigotimes_{\substack{w \in \operatorname{pos}(\zeta), \zeta(w)=(\sigma, V)}} \kappa_{2}\left(\left(\sigma, \mathcal{U}_{2} \cap V\right)\right)\right.
$$

where the latter equality holds because B is commutative.
Let $e_{1}$ be as in Case (b). Then our statement holds due to right-distributivity of B .
Let $e_{1}$ be as in Case (c). Then

$$
\begin{aligned}
& \left.\llbracket\left(\text { Н }_{x} g\right) \times e_{2}\right) \rrbracket \mathcal{V}(\zeta)=\llbracket \text { W }_{x} g \rrbracket_{\mathcal{V}}(\zeta) \otimes \llbracket e_{2} \rrbracket_{\mathcal{V}}(\zeta) \\
& =\left(\bigoplus_{w \in \operatorname{pos}(\zeta)} \llbracket g \rrbracket \mathcal{V} \cup\{x\}(\zeta[x \mapsto w])\right) \otimes \llbracket e_{2} \rrbracket \mathcal{V}(\zeta) \\
& =\bigoplus_{w \in \operatorname{pos}(\zeta)}\left(\llbracket g \rrbracket_{\mathcal{V} \cup\{x\}}(\zeta[x \mapsto w]) \otimes \llbracket e_{2} \rrbracket_{\mathcal{V}}(\zeta)\right) \quad \text { (by right-distributivity) } \\
& =\bigoplus_{w \in \operatorname{pos}(\zeta)}\left(\llbracket g \rrbracket_{\mathcal{V} \cup\{x\}}(\zeta[x \mapsto w]) \otimes \llbracket e_{2} \rrbracket \rrbracket_{\mathcal{V} \cup\{x\}}(\zeta[x \mapsto w])\right) \\
& \text { (by Lemma 14.4.1, we note that } e_{1} \times e_{2} \text { is in normal form) } \\
& =\bigoplus_{w \in \operatorname{pos}(\zeta)}\left(\llbracket g \rrbracket_{\mathcal{V} \cup\{x\}} \otimes \llbracket e_{2} \rrbracket \mathcal{V} \cup\{x\}\right)(\zeta[x \mapsto w]) \\
& =\bigoplus_{w \in \operatorname{pos}(\zeta)} \llbracket \operatorname{hp}\left(g, e_{2}\right) \rrbracket \mathcal{V \cup \{ x \}}(\zeta[x \mapsto w]) \\
& =\llbracket+_{x} \operatorname{hp}\left(g, e_{2}\right) \rrbracket \mathcal{V}(\zeta) \quad \text { (by definition of semantics) } \\
& =\llbracket \mathrm{hp}\left(\text { Н }_{x} g, e_{2}\right) \rrbracket_{\mathcal{V}}(\zeta) . \quad \text { (by definition of hp) }
\end{aligned}
$$

Let $e_{1}$ be as in Case (d). Then the corresponding equation for $\operatorname{hp}\left(+_{X} g, e_{2}\right)$ can be verified in a similar way as for $\mathrm{hp}\left(+_{x} g, e_{2}\right)$.

Let $e_{1}$ be as in Case (e). Then

$$
\begin{align*}
\llbracket(\varphi \triangleright g) \otimes e_{2} \rrbracket_{\mathcal{V}}(\zeta) & =\llbracket \varphi \triangleright g \rrbracket_{\mathcal{V}}(\zeta) \otimes \llbracket e_{2} \rrbracket_{\mathcal{V}}(\zeta)= \begin{cases}\llbracket g \rrbracket_{\mathcal{V}}(\zeta) \otimes \llbracket e_{2} \rrbracket_{\mathcal{V}}(\zeta) & \text { if } \zeta \in \mathrm{L}_{\mathcal{V}}(\varphi) \\
\mathbb{0} & \text { otherwise }\end{cases} \\
& = \begin{cases}\llbracket \operatorname{hp}\left(g, e_{2}\right) \rrbracket_{\mathcal{V}}(\zeta) & \text { if } \zeta \in \operatorname{L\mathcal {L}}(\varphi) \\
\mathbb{0} & \text { otherwise }\end{cases}  \tag{byI.H.}\\
& =\llbracket \varphi \triangleright \operatorname{hp}\left(g, e_{2}\right) \rrbracket \mathcal{V}(\zeta)=\llbracket \operatorname{hp}\left(\varphi \triangleright g, e_{2}\right) \rrbracket_{\mathcal{V}}(\zeta) .
\end{align*}
$$

Let $e_{1}$ and $e_{2}$ be as in Case (f). Then our statement holds by left-distributivity.

Let $e_{1}$ and $e_{2}$ be as in Case (g). Then the proof is analogous to the proof of Case (c) except that we use left-distributivity instead of right-distributivity.

Let $e_{1}$ and $e_{2}$ be as in Case (h). Then the proof is analogous to the proof of Case (d) except that we use left-distributivity instead of right-distributivity.

Let $e_{1}$ and $e_{2}$ be as in Case (i). Then the proof is analogous to the proof of Case (e).
In Lemma 14.4.14, we assume that $e_{1} \times e_{2}$ is in normal form. We have used this condition in the proof for the case that $e_{1}=+_{x} g$ (cf. Case (c) of the definition of $\left.\operatorname{hp}\left(e_{1}, e_{2}\right)\right)$. More precisely, it is used in the justification of the equality:

$$
\begin{equation*}
\bigoplus_{w \in \operatorname{pos}(\zeta)}\left(\llbracket g \rrbracket_{\mathcal{V} \cup\{x\}}(\zeta[x \mapsto w]) \otimes \llbracket e_{2} \rrbracket_{\mathcal{V}}(\zeta)\right)=\bigoplus_{w \in \operatorname{pos}(\zeta)}\left(\llbracket g \rrbracket_{\mathcal{V} \cup\{x\}}(\zeta[x \mapsto w]) \otimes \llbracket e_{2} \rrbracket_{\mathcal{V} \cup\{x\}}(\zeta[x \mapsto w])\right) \tag{14.33}
\end{equation*}
$$

Here we show that, in general, Equation (14.33) is wrong if we drop the condition that $\left(+_{x} g\right) \times e_{2}$ is in normal form. For this we let $\Sigma=\left\{\gamma^{(1)}, \alpha^{(0)}\right\}, \mathcal{V}=\{x\}, \xi=\gamma(\alpha), \eta(x)=\varepsilon, \zeta=(\xi, \eta)$ in $T_{\Sigma \mathcal{V}}^{v}$ (i.e., $\zeta=(\gamma,\{x\})((\alpha, \emptyset)))$, and

- $\kappa=\left(\kappa_{k} \mid k \in \mathbb{N}\right)$ with $\kappa_{1}((\gamma,\{x\}))=\kappa_{0}((\alpha, \emptyset))=\mathbb{1}$ and $\kappa_{1}((\gamma, \emptyset))=\kappa_{0}((\alpha,\{x\}))=\mathbb{O}$ and
- $\nu=\left(\nu_{k} \mid k \in \mathbb{N}\right)$ with $\nu_{1}((\gamma,\{x\}))=\nu_{0}((\alpha, \emptyset))=\mathbb{0}$ and $\nu_{1}\left((\gamma, \emptyset)=\nu_{0}((\alpha,\{x\}))=\mathbb{1}\right.$.

We let $g=\mathrm{H}(\nu)$ and $e_{2}=\mathrm{H}(\kappa)$. We note that Bound $\left(+_{x} g\right) \cap \operatorname{Free}\left(e_{2}\right)=\{x\}$, hence Free $\left(\left(+_{x} g\right) \times\right.$ $\left.e_{2}\right) \cap \operatorname{Free}\left(\left(+_{x} g\right) \times e_{2}\right) \neq \emptyset$. We can calculate the left-hand side of (14.33) as follows:

$$
\begin{aligned}
& \bigoplus_{w \in \operatorname{pos}(\zeta)}\left(\llbracket g \rrbracket_{\mathcal{V} \cup\{x\}}(\zeta[x \mapsto w]) \otimes \llbracket e_{2} \rrbracket_{\mathcal{V}}(\zeta)\right) \\
= & \bigoplus_{w \in\{\varepsilon, 1\}}\left(\llbracket \mathrm{H}(\nu) \rrbracket_{\{x\}}(\zeta[x \mapsto w]) \otimes \llbracket \mathrm{H}(\kappa) \rrbracket_{\{x\}}(\zeta)\right) \\
= & \bigoplus_{w \in\{\varepsilon, 1\}}\left(\llbracket \mathrm{H}(\nu) \rrbracket_{\{x\}}(\zeta[x \mapsto w]) \otimes \mathbb{1}\right) \\
= & \llbracket \mathrm{H}(\nu) \rrbracket_{\{x\}}(\zeta[x \mapsto \varepsilon]) \oplus \llbracket \mathrm{H}(\nu) \rrbracket_{\{x\}}(\zeta[x \mapsto 1]) \\
= & \llbracket \mathrm{H}(\nu) \rrbracket_{\{x\}}((\gamma,\{x\})((\alpha, \emptyset))) \oplus \llbracket \mathrm{H}(\nu) \rrbracket_{\{x\}}((\gamma, \emptyset)((\alpha,\{x\})))=\mathbb{O} \oplus \mathbb{1}=\mathbb{1} .
\end{aligned}
$$

And we can calculate the right-hand side of (14.33) as follows:

$$
\begin{aligned}
& \bigoplus_{w \in \operatorname{pos}(\zeta)}\left(\llbracket g \rrbracket_{\mathcal{V} \cup\{x\}}(\zeta[x \mapsto w]) \otimes \llbracket \mathrm{H}(\kappa) \rrbracket_{\mathcal{V} \cup\{x\}}(\zeta[x \mapsto w))\right. \\
= & \bigoplus_{w \in\{\varepsilon, 1\}}\left(\llbracket \mathrm{H}(\nu) \rrbracket_{\{x\}}(\zeta[x \mapsto w]) \otimes \llbracket \mathrm{H}(\kappa) \rrbracket_{\{x\}}(\zeta[x \mapsto w))\right. \\
= & \left(\llbracket \mathrm{H}(\nu) \rrbracket_{\{x\}}((\gamma,\{x\})((\alpha, \emptyset))) \otimes \llbracket \mathrm{H}(\kappa) \rrbracket_{\{x\}}((\gamma,\{x\})((\alpha, \emptyset)))\right) \oplus \\
& \left(\llbracket \mathrm{H}(\nu) \rrbracket_{\{x\}}((\gamma, \emptyset)((\alpha,\{x\}))) \otimes \llbracket \mathrm{H}(\kappa) \rrbracket_{\{x\}}((\gamma, \emptyset)((\alpha,\{x\})))\right) \\
= & (\mathbb{O} \otimes \mathbb{1}) \oplus(\mathbb{1} \otimes \mathbb{O})=\mathbb{0} .
\end{aligned}
$$

Thus, without the mentioned property, Equation (14.33) does not hold in general.
In the next lemma we get rid of weighted conjunction in formulas which do not contain weighted firstorder universal quantification. Here we will need commutativity and distributivity of B . We repeatedly apply Lemma 14.4.14.
Lemma 14.4.15. Let B be distributive and let $e \in \operatorname{MSO}^{\text {cext }}(\Sigma, \mathrm{B})$ and $\mathcal{V} \supseteq$ Free $(e)$ be a finite set of variables. If $e$ does not contain a subformula of the form $X_{x} e^{\prime}$, then there exists an $\operatorname{MSO}(\Sigma, \mathrm{B})$-formula $f$ such that Free $(e)=\operatorname{Free}(f)$ and $\llbracket e \rrbracket \mathcal{v}=\llbracket f \rrbracket \mathcal{v}$.

Proof. If $e$ does not contain a subformula of the form $e_{1} \times e_{2}$, then $e \in \operatorname{MSO}(\Sigma, \mathrm{~B})$ and we are done and we can choose $f=e$.

Otherwise, we choose a subformula $e_{1} \times e_{2}$ of $e$ with Free $\left(e_{1}\right)=\mathcal{U}_{1}$ and Free $\left(e_{2}\right)=\mathcal{U}_{2}$ such that $e_{1}, e_{2} \in$ $\operatorname{MSO}(\Sigma, \mathrm{B})$, i.e., neither $e_{1}$ nor $e_{2}$ contains a weighted conjunction. Clearly, such a subformula exists. By Lemma 14.4.6, we can construct $e_{1}^{\prime} \times e_{2}^{\prime} \in \operatorname{MSO}(\Sigma, \mathrm{B})$ in normal form such that Free $\left(e_{1} \times e_{2}\right)=\operatorname{Free}\left(e_{1}^{\prime} \times e_{2}^{\prime}\right)$ and $\llbracket e_{1} \times e_{2} \rrbracket=\llbracket e_{1}^{\prime} \times e_{2}^{\prime} \rrbracket$. By the latter fact and by Lemma 14.4.14, we have Free $\left(e_{1} \times e_{2}\right)=\operatorname{Free}\left(\mathrm{hp}\left(e_{1}^{\prime}, e_{2}^{\prime}\right)\right)$ and $\llbracket e_{1} \times e_{2} \rrbracket=\llbracket \mathrm{hp}\left(e_{1}^{\prime}, e_{2}^{\prime}\right) \rrbracket$.

Then we replace the subformula $e_{1} \times e_{2}$ in $e$ by the $\operatorname{MSO}(\Sigma, \mathrm{B})$-formula $\mathrm{hp}\left(e_{1}^{\prime}, e_{2}^{\prime}\right)$. Let $e^{\prime}$ be the formula obtained in this way. Obviously, Free $(e)=\operatorname{Free}\left(e^{\prime}\right)$ and $e^{\prime}$ contains one occurrence of $\times$ less than $e$. Moreover, by Observation 14.4.5, we have $\llbracket e \rrbracket \mathcal{V}=\llbracket e^{\prime} \rrbracket \mathcal{V}$.

If $e^{\prime}$ does not contain a subformula of the form $e_{1}^{\prime \prime} \times e_{2}^{\prime \prime}$, then we are ready, and $e^{\prime}$ is the desired formula $f \in \operatorname{MSO}(\Sigma, \mathrm{~B})$ with $\operatorname{Free}(e)=\operatorname{Free}(f)$ and $\llbracket e \rrbracket \mathcal{v}=\llbracket f \rrbracket \mathcal{V}$. Otherwise we repeat the transformation described above by replacing some subformula $e_{1}^{\prime \prime} \times e_{2}^{\prime \prime}$. Since, in each step, the number of occurrences of $\times$ becomes smaller, the iteration of the transformation terminates, and after termination, the result is the desired formula $f \in \operatorname{MSO}(\Sigma, \mathrm{~B})$ with $\operatorname{Free}(e)=\operatorname{Free}(f)$ and $\llbracket e \rrbracket_{\mathcal{v}}=\llbracket f \rrbracket \mathcal{v}$.

### 14.4.4 Weighted first-order universal quantification of recognizable step mappings

In Example 14.4 .8 we gave an atomic formula such that its weighted first-order universal quantification is not recognizable. Here we show that if the semantics of a weighted $\mathrm{MSO}^{\text {ext }}$-formula is a recognizable step mapping, then its weighted first-order universal quantification is run recognizable.

Lemma 14.4.16. (cf. DG05, Lm. 4.2], DG07, Lm. 4.4], and DV06, Lm. 5.5]) Let $e \in \operatorname{MSO}^{\mathrm{ext}}(\Sigma, \mathrm{B})$ with $\mathcal{U}=\operatorname{Free}(e)$. Moreover, let $n \in \mathbb{N}_{+}, b_{1}, \ldots, b_{n} \in B$, and $\Sigma_{\mathcal{U} \cup\{x\}}$-fta $A_{1}, \ldots, A_{n}$ such that $\llbracket e \rrbracket_{\mathcal{U} \cup\{x\}}=$ $\bigoplus_{j \in[n]} b_{j} \otimes \chi\left(\mathrm{~L}\left(A_{j}\right)\right)$. Then we can construct a $\left(\Sigma_{\mathcal{V}}, \mathrm{B}\right)$-wta $\mathcal{B}$, where $\mathcal{V}=\operatorname{Free}\left(X_{x} e\right)$, such that $\llbracket \mathcal{B} \rrbracket^{\text {run }}=$ $\llbracket X_{x} e \rrbracket$.

Proof. Since $\mathcal{V}=\mathcal{U} \backslash\{x\}$, we have $\mathcal{V} \cup\{x\}=\mathcal{U} \cup\{x\}$. By Theorem 10.3.1(B) $\Rightarrow(\mathrm{D})$ we can assume that $\left(\mathrm{L}\left(A_{j}\right) \mid j \in[n]\right)$ is a partitioning of $\mathrm{T}_{\Sigma_{\mathcal{V} \cup\{x\}}}$.

First, for each $\xi \in \mathrm{T}_{\Sigma_{\mathcal{V}}}$, we define the relation $\nu_{\xi} \subseteq \operatorname{pos}(\xi) \times[n]$ by

$$
\nu_{\xi}=\left\{(w, j) \in \operatorname{pos}(\xi) \times[n] \mid \xi[x \mapsto w] \in \mathrm{L}\left(A_{j}\right)\right\}
$$

Since $\left(\mathrm{L}\left(A_{j}\right) \mid j \in[n]\right)$ is a partitioning of $\mathrm{T}_{\Sigma_{\mathcal{V} \cup\{x\}}}$, this relation is a mapping $\nu_{\xi}: \operatorname{pos}(\xi) \rightarrow[n]$.
In order to understand the idea for the construction of $\mathcal{B}$, let us perform a few steps in the calculation of $\llbracket X_{x} e \rrbracket(\xi)$ where $\xi \in \mathrm{T}_{\Sigma_{\nu}}$.

$$
\llbracket X_{x} e \rrbracket(\xi)=\bigotimes_{w \in \operatorname{pos}(\xi)} \llbracket e \rrbracket_{\mathcal{V} \cup\{x\}}(\xi[x \mapsto w])=\bigotimes_{w \in \operatorname{pos}(\xi)}\left(\bigoplus_{j \in[n]} b_{j} \otimes \chi\left(\mathrm{~L}\left(A_{j}\right)\right)\right)(\xi[x \mapsto w])=\bigotimes_{w \in \operatorname{pos}(\xi)} b_{\nu_{\xi}(w)}
$$

Before constructing $\mathcal{B}$, we wish to construct a bu deterministic wta $\mathcal{C}$ which computes $\bigotimes_{w \in \operatorname{pos}(\xi)} b_{\nu_{\xi}(w)}$; as input $\mathcal{C}$ receives the information $\xi$ and $\nu_{\xi}$. Since $\mathcal{C}$ can only read trees (and not pairs of the form $(\xi, \nu)$ ), we encode each pair $(\xi, \nu)$ where $\xi \in \mathrm{T}_{\Sigma_{\mathcal{V}}}$ and $\nu: \operatorname{pos}(\xi) \rightarrow[n]$ into one tree. For this purpose, we introduce the new ranked alphabet $n \Sigma_{\mathcal{V}}$ by letting

$$
\left(n \Sigma_{\mathcal{V}}\right)^{(k)}=\Sigma_{\mathcal{V}}^{(k)} \times[n] \text { for each } k \in \mathbb{N}
$$

Instead of $\left(n \Sigma_{\mathcal{V}}\right)^{(k)}$ we write $n \Sigma_{\mathcal{V}}{ }^{(k)}$. Moreover, for $((\sigma, U), j) \in n \Sigma_{\mathcal{V}}$ we also write $(\sigma, U, j)$.
Now it is clear that the two sets $\mathrm{T}_{n \Sigma_{\mathcal{V}}}$ and $\mathrm{T}_{\Sigma_{\nu}}^{+ \text {steps }}$, where $\mathrm{T}_{\Sigma_{\mathcal{\nu}}}^{+ \text {steps }}=\left\{(\xi, \nu) \mid \xi \in \mathrm{T}_{\Sigma_{\mathcal{V}}}, \nu: \operatorname{pos}(\xi) \rightarrow[n]\right\}$, are in a one-to-one correspondence. Indeed,

- each tree $\zeta \in \mathrm{T}_{n \Sigma_{\mathcal{V}}}$ corresponds to the pair $(\xi, \nu)$, where $\operatorname{pos}(\xi)=\operatorname{pos}(\zeta)$ and, for each $w \in \operatorname{pos}(\xi)$, if $\zeta(w)=(\sigma, U, j)$, then $\xi(w)=(\sigma, U)$ and $\nu(w)=j$ and
- each pair $(\xi, \nu) \in \mathrm{T}_{\Sigma_{\nu}}^{+ \text {steps }}$ corresponds to the tree $\zeta \in \mathrm{T}_{n \Sigma \nu}$ where $\operatorname{pos}(\zeta)=\operatorname{pos}(\xi)$ and, for each $w \in \operatorname{pos}(\zeta)$, we let $\zeta(w)=(\xi(w), \nu(w))$.
Due to this one-to-one correspondence, we can assume that elements of $\mathrm{T}_{n \Sigma_{\nu}}$ have the form $(\xi, \nu)$ and, vice versa, that each pair $(\xi, \nu)$ with $\xi \in \mathrm{T}_{\Sigma_{\mathcal{V}}}$ and $\nu: \operatorname{pos}(\xi) \rightarrow[n]$ is an element of $\mathrm{T}_{n \Sigma_{\nu}}$.

Now we can turn back to the expression $\bigotimes_{w \in \operatorname{pos}(\xi)} b_{\nu_{\xi}(w)}$ above and can be more precise with respect to the type and the semantics of $\mathcal{C}$. We will construct a bu deterministic $\left(n \Sigma_{\mathcal{V}}, \mathrm{B}\right)$-wta $\mathcal{C}$ such that, for each $(\xi, \nu) \in \mathrm{T}_{n \Sigma_{\mathcal{V}}}$, we have

$$
\llbracket \mathcal{C} \rrbracket((\xi, \nu))= \begin{cases}\bigotimes_{w \in \operatorname{pos}(\xi)} b_{\nu_{\xi}(w)} & \text { if } \nu=\nu_{\xi}  \tag{14.34}\\ \mathbb{O} & \text { otherwise }\end{cases}
$$

For the time being, let us assume that we have constructed such a $\mathcal{C}$ already. Then, by using the deterministic tree relabeling $\tau: n \Sigma_{\mathcal{V}} \rightarrow \Sigma_{\mathcal{V}}$ defined, for each $(\sigma, U, j)$ by $\tau((\sigma, U, j))=(\sigma, U)$, we can continue the calculation as follows: for each $\xi \in \mathrm{T}_{\Sigma_{\mathcal{V}}}$, we have

$$
\bigotimes_{w \in \operatorname{pos}(\xi)} b_{\nu_{\xi}(w)}=\llbracket \mathcal{C} \rrbracket\left(\left(\xi, \nu_{\xi}\right)\right)=\bigoplus_{(\xi, \nu) \in \tau^{-1}(\xi)} \llbracket \mathcal{C} \rrbracket((\xi, \nu))=\chi(\tau)(\llbracket \mathcal{C} \rrbracket)(\xi)
$$

Lastly, by Theorem 10.10 .1 (closure of $\operatorname{Rec}^{\text {run }}\left(\_, B\right)$ under tree relabelings), we can construct the desired $\left(\Sigma_{\mathcal{V}}, \mathrm{B}\right)$-wta $\mathcal{B}$ such that $\llbracket \mathcal{B} \rrbracket^{\text {run }}=\chi(\tau)(\llbracket \mathcal{C} \rrbracket)$ and we have proved the lemma. (We note that, since $\tau$ is overlapping, in general $\mathcal{B}$ is not bu deterministic.)

Thus it remains to construct the bu deterministic $\left(n \Sigma_{\mathcal{V}}, \mathrm{B}\right)$-wta $\mathcal{C}$ such that (14.34) holds. For this purpose we first construct an (unweighted) total and bu deterministic $n \Sigma_{\mathcal{V}}$ - $\mathrm{fta} C$ such that $\mathrm{L}(C)=L$ where

$$
L=\left\{\left(\xi, \nu_{\xi}\right) \mid \xi \in \mathrm{T}_{\Sigma_{\mathcal{V}}}\right\}
$$

Assume that $C=(\widetilde{Q}, \widetilde{\delta}, \widetilde{F})$ is such an $n \Sigma_{\mathcal{V}}$-fta. Then we construct the bu deterministic $\left(n \Sigma_{\mathcal{V}}, \mathrm{B}\right)$-wta $\mathcal{C}=(\widetilde{Q}, \delta, F)$ such that, for every $k \in \mathbb{N},(\sigma, U, j) \in n \Sigma^{(k)}$, and $q_{1}, \ldots, q_{k}, q \in \widetilde{Q}$, we let

$$
\delta_{k}\left(q_{1} \cdots q_{k},(\sigma, U, j), q\right)= \begin{cases}b_{j} & \text { if } \widetilde{\delta}_{k}\left(q_{1} \cdots q_{k},(\sigma, U, j)\right)=q \\ 0 & \text { otherwise }\end{cases}
$$

and $F_{q}=\mathbb{1}$ if $q \in \widetilde{F}$ and $\mathbb{O}$ otherwise. It is obvious that (14.34) holds.
Thus it remains to construct a total and bu deterministic $n \Sigma_{\mathcal{V}}$ - $\mathrm{fta} C$ such that $\mathrm{L}(C)=L$.
We observe the following:

$$
\begin{aligned}
L & =\left\{\left(\xi, \nu_{\xi}\right) \mid \xi \in \mathrm{T}_{\Sigma_{\nu}}\right\} \\
& =\left\{(\xi, \nu) \in \mathrm{T}_{n \Sigma_{\nu}} \mid(\forall j \in[n], w \in \operatorname{pos}(\xi)): \text { if } \nu(w)=j, \text { then } \xi[x \rightarrow w] \in L\left(A_{j}\right)\right\} \\
& =\bigcap_{j \in[n]}\left\{(\xi, \nu) \in \mathrm{T}_{n \Sigma_{\nu}} \mid(\forall w \in \operatorname{pos}(\xi)): \text { if } \nu(w)=j, \text { then } \xi[x \rightarrow w] \in L\left(A_{j}\right)\right\}
\end{aligned}
$$

Thus, due to Theorems [2.13.3(2) and 2.13.2, it suffices to construct, for each $j \in[n]$, a total and bu deterministic $n \Sigma_{\mathcal{V}}$-fta $C_{j}$ such that $\mathrm{L}\left(C_{j}\right)=L_{j}$ where

$$
L_{j}=\left\{(\xi, \nu) \in \mathrm{T}_{n \Sigma \mathcal{v}} \mid(\forall w \in \operatorname{pos}(\xi)): \text { if } \nu(w)=j, \text { then } \xi[x \rightarrow w] \in L\left(A_{j}\right)\right\}
$$

Let $j \in[n]$. The idea behind the construction of $C_{j}$ follows the one in the proof of [DG07, Lm. 4.4] and it is roughly described as follows. On an input tree $\zeta=(\xi, \nu)$, the $n \Sigma_{\mathcal{V}}$-fta $C_{j}$ simulates the work of
$A_{j}$ on $\xi$ and, whenever a position $w$ of $\zeta$ is encountered for which $\nu(w)=j$ holds, then, additionally, $C_{j}$ splits off a copy of $A_{j}$; this copy behaves as if at $w$ the $x$ would occur (or, in other words: as if $A_{j}$ would have $\xi[x \mapsto w]$ as input tree). Moreover, since we have to guarantee that the placement of $x$ is done at most once in $\xi$, we maintain a bit $d$ in every state of $C_{j}$ which indicates whether the $x$ was placed $(d=1)$ or not $(d=0)$.

In the sequel, we will reuse the standard notations $\delta, F, \widetilde{Q}, \widetilde{\delta}$, and $\widetilde{F}$, which occurred already in the specification of $C$ and $\mathcal{C}$, for other purposes. This will be harmless because below we will not deal with $C$ and $\mathcal{C}$ anymore.

Formally, let $A_{j}=(Q, \delta, F)$ be the $\Sigma_{\mathcal{V} \cup\{x\}}$-fta given in the statement of the lemma. By Theorem 2.13.2, we may assume that $A_{j}$ is total and bu deterministic. We construct the total and bu deterministic $n \Sigma_{\mathcal{V}}$-fta $C_{j}=(\widetilde{Q}, \widetilde{\delta}, \widetilde{F})$, where

- $\widetilde{Q}=\mathcal{P}(Q \times\{0,1\})$,
- $\widetilde{F}=\{P \subseteq Q \times\{0,1\} \mid P \cap(Q \times\{1\}) \subseteq F \times\{1\}\}$, and
- $\widetilde{\delta}$ is defined such that, for every $k \in \mathbb{N},(\sigma, U, l) \in n \Sigma_{\mathcal{V}}^{(k)}$, and $P_{1}, \ldots, P_{k} \in \widetilde{Q}$, we let

$$
\widetilde{\delta}_{k}\left(P_{1} \cdots P_{k},(\sigma, U, l)\right)=P \quad \text { where } P= \begin{cases}P^{\prime} & \text { if } l \neq j \\ P^{\prime} \cup P^{\prime \prime} & \text { otherwise }\end{cases}
$$

and

$$
P^{\prime}=\left\{\left(\delta_{k}\left(p_{1} \cdots p_{k},(\sigma, U)\right), \mathcal{W}_{i \in[k]} d_{i}\right) \mid(\forall i \in[k]):\left(p_{i}, d_{i}\right) \in P_{i} \text { and } \mathcal{H}_{i \in[k]} d_{i} \leq 1\right\}
$$

and

$$
P^{\prime \prime}=\left\{\left(\delta_{k}\left(p_{1} \cdots p_{k},(\sigma, U \cup\{x\})\right), 1\right) \mid(\forall i \in[k]):\left(p_{i}, 0\right) \in P_{i}\right\}
$$

Since $A_{j}$ is total and bu deterministic, the states $\delta_{k}\left(p_{1} \cdots p_{k},(\sigma, U)\right)$ and $\delta_{k}\left(p_{1} \cdots p_{k},(\sigma, U \cup\{x\})\right)$ are defined. Obviously, $P^{\prime}$ formalizes the propagation of the bit value 0 if $+_{i \in[k]} d_{i}=0$, and the propagation of the bit value 1 if $+_{i \in[k]} d_{i}=1$. If $l=j$, then additionally to this propagation, $C_{j}$ can change from the bit vector $(0, \ldots, 0)$ to the bit value 1 , thereby placing the $x$ to this position; the union with $P^{\prime \prime}$ can be described as "splitting off a new copy of $A_{j}$ ". Also it is clear that $C_{j}$ is total and bu deterministic. Now we will show that $C_{j}$ recognizes $L_{j}$.

Since $A_{j}$ and $C_{j}$ are total and bu deterministic, for each $(\xi, \nu) \in \mathrm{T}_{n \Sigma_{\nu}}$ and $w \in \operatorname{pos}(\xi)$, the sets of valid runs $\mathrm{R}_{A_{j}}^{\mathrm{v}}(\xi), \mathrm{R}_{A_{j}}^{\mathrm{v}}(\xi[x \rightarrow w])$, and $\mathrm{R}_{C_{j}}^{\mathrm{v}}((\xi, \nu))$ are singletons. We denote these runs by letting

$$
\mathrm{R}_{A_{j}}^{\mathrm{v}}(\xi)=\left\{\rho_{\xi}\right\}, \mathrm{R}_{A_{j}}^{\mathrm{v}}(\xi[x \rightarrow w])=\left\{\rho_{\xi[x \rightarrow w]}\right\}, \quad \text { and } \quad \mathrm{R}_{C_{j}}^{\mathrm{v}}((\xi, \nu))=\left\{\rho_{(\xi, \nu)}\right\}
$$

By induction on $\mathrm{T}_{n \Sigma_{\mathcal{V}}}$, we prove the following statement.
For each $(\xi, \nu) \in \mathrm{T}_{n \Sigma_{\nu}}$, we have $\rho_{(\xi, \nu)}(\varepsilon)=\left\{\left(\rho_{\xi}(\varepsilon), 0\right)\right\} \cup\left\{\left(\rho_{\xi[x \rightarrow w]}(\varepsilon), 1\right) \mid w \in \operatorname{pos}(\xi), \nu(w)=j\right\}$.
Let $\zeta=(\sigma, U, l)\left(\zeta_{1}, \ldots, \zeta_{k}\right)$ be in $\mathrm{T}_{n \Sigma_{\mathcal{V}}}$ with $\zeta=(\xi, \nu)$ and $\zeta_{i}=\left(\xi_{i}, \nu_{i}\right)$ for every $i \in[k]$. Thus $\xi=(\sigma, U)\left(\xi_{1}, \ldots, \xi_{k}\right)$. By I.H., (14.35) holds for $\zeta_{1}, \ldots, \zeta_{k}$. First we prove that

$$
\begin{align*}
& \left\{\left(\delta_{k}\left(p_{1} \ldots p_{k},(\sigma, U)\right), \mathcal{W}_{i \in[k]} d_{i}\right) \mid(\forall i \in[k]):\left(p_{i}, d_{i}\right) \in \rho_{\left(\xi_{i}, \nu_{i}\right)}(\varepsilon) \text { and } \mathcal{F}_{i \in[k]} d_{i} \leq 1\right\}  \tag{14.36}\\
& =\left\{\left(\rho_{\xi}(\varepsilon), 0\right)\right\} \cup\left\{\left(\rho_{\xi[x \rightarrow u w]}(\varepsilon), 1\right) \mid u \in[k], w \in \operatorname{pos}\left(\xi_{u}\right), \text { and } \nu_{u}(w)=j\right\}
\end{align*}
$$

as follows:

$$
\left\{\left(\delta_{k}\left(p_{1} \cdots p_{k},(\sigma, U)\right), \mathcal{W}_{i \in[k]} d_{i}\right) \mid(\forall i \in[k]):\left(p_{i}, d_{i}\right) \in \rho_{\left(\xi_{i}, \nu_{i}\right)}(\varepsilon) \text { and } \mathcal{W}_{i \in[k]} d_{i} \leq 1\right\}
$$

$$
\begin{aligned}
= & \left\{\left(\delta_{k}\left(p_{1} \cdots p_{k},(\sigma, U)\right), \mathcal{W}_{i \in[k]} d_{i}\right) \mid(\forall i \in[k]):\right. \\
& {\left.\left[\left(p_{i}, d_{i}\right)=\left(\rho_{\xi_{i}}(\varepsilon), 0\right) \text { or }\left(p_{i}, d_{i}\right)=\left(\rho_{\xi_{i}[x \rightarrow w]}(\varepsilon), 1\right), w \in \operatorname{pos}\left(\xi_{i}\right), \nu_{i}(w)=j\right] \text { and } \text { _}_{i \in[k]} d_{i} \leq 1\right\} } \\
= & \left\{\left(\delta_{k}\left(\rho_{\xi_{1}}(\varepsilon) \cdots \rho_{\xi_{k}}(\varepsilon),(\sigma, U)\right), 0\right)\right\} \\
& \cup\left\{\left(\delta_{k}\left(\rho_{\xi_{1}}(\varepsilon) \cdots \rho_{\xi_{u}[x \rightarrow w]}(\varepsilon) \cdots \rho_{\xi_{k}}(\varepsilon),(\sigma, U)\right), 1\right) \mid u \in[k], w \in \operatorname{pos}\left(\xi_{u}\right), \nu_{u}(w)=j\right\} \\
= & \left\{\left(\rho_{\xi}(\varepsilon), 0\right)\right\} \cup\left\{\left(\rho_{\xi[x \rightarrow u w]}(\varepsilon), 1\right) \mid u \in[k], w \in \operatorname{pos}\left(\xi_{u}\right), \nu_{u}(w)=j\right\} .
\end{aligned}
$$

This finishes the proof of（14．36）．Now we can complete the proof of（14．35）．
Case（i）：Let $l \neq j$ ．Then

$$
\begin{aligned}
\rho_{(\xi, \nu)}(\varepsilon) & =\widetilde{\delta}_{k}\left(\rho_{(\xi, \nu)}(1) \ldots \rho_{(\xi, \nu)}(k),(\sigma, U, l)\right)=\widetilde{\delta}_{k}\left(\rho_{\left(\xi_{1}, \nu_{1}\right)}(\varepsilon) \ldots \rho_{\left(\xi_{k}, \nu_{k}\right)}(\varepsilon),(\sigma, U, l)\right) \\
& =\left\{\left(\delta_{k}\left(p_{1} \ldots p_{k},(\sigma, U)\right), \text { 十 }_{i \in[k]} d_{i}\right) \mid(\forall i \in[k]):\left(p_{i}, d_{i}\right) \in \rho_{\left(\xi_{i}, \nu_{i}\right)}(\varepsilon) \text { and } \text { — }_{i \in[k]} d_{i} \leq 1\right\} \\
& =\left\{\left(\rho_{\xi}(\varepsilon), 0\right)\right\} \cup\left\{\left(\rho_{\xi[x \rightarrow u w]}(\varepsilon), 1\right) \mid u \in[k], w \in \operatorname{pos}\left(\xi_{u}\right), \nu_{u}(w)=j\right\} \\
& \left.=\left\{\left(\rho_{\xi}(\varepsilon), 0\right)\right\} \cup\left\{\left(\rho_{\xi[x \rightarrow w]}(\varepsilon), 1\right) \mid w \in \operatorname{pos}(\xi), \nu(w)=j\right\} . \quad \text { (because } \nu(\varepsilon)=l \neq j\right)
\end{aligned}
$$

$\underline{\text { Case（ii）：Let } l=j \text { ．Then }}$

$$
\begin{aligned}
& \rho_{(\xi, \nu)}(\varepsilon)= \widetilde{\delta}_{k}\left(\rho_{\left(\xi_{1}, \nu_{1}\right)}(\varepsilon) \ldots \rho_{\left(\xi_{k}, \nu_{k}\right)}(\varepsilon),(\sigma, U, l)\right) \\
&=\left\{\left(\delta_{k}\left(p_{1} \ldots p_{k},(\sigma, U)\right), \text { 十 }_{i \in[k]} d_{i}\right) \mid(\forall i \in[k]):\left(p_{i}, d_{i}\right) \in \rho_{\left(\xi_{i}, \nu_{i}\right)}(\varepsilon) \text { and } \text { 十 }_{i \in[k]} d_{i} \leq 1\right\} \\
&\left.\cup\left\{\left(\delta_{k}\left(\rho_{\xi_{1}}(\varepsilon) \ldots \rho_{\xi_{k}}(\varepsilon),(\sigma, U \cup\{x\})\right), 1\right)\right\} \quad \text { (by definition of } \widetilde{\delta} \text { and (14.36) for reducing } P^{\prime \prime}\right) \\
&=\left\{\left(\rho_{\xi}(\varepsilon), 0\right)\right\} \cup\left\{\left(\rho_{\xi[x \rightarrow u w]}(\varepsilon), 1\right) \mid u \in[k], w \in \operatorname{pos}\left(\xi_{u}\right), \nu_{u}(w)=j\right\} \cup\left\{\left(\rho_{\xi[x \rightarrow \varepsilon]}(\varepsilon), 1\right)\right\} \\
& \quad \quad \text { (by (14.36) and definition of run) } \\
&=\left\{\left(\rho_{\xi}(\varepsilon), 0\right)\right\} \cup\left\{\left(\rho_{\xi[x \rightarrow w]}(\varepsilon), 1\right) \mid w \in \operatorname{pos}(\xi), \nu(w)=j\right\} .
\end{aligned}
$$

This finishes the proof of（14．35）．
Finally，let $(\xi, \nu) \in \mathrm{T}_{n \Sigma_{\nu}}$ ．Then we can calculate as follows．

$$
\begin{align*}
(\xi, \nu) \in L_{j} & \text { iff }(\forall w \in \operatorname{pos}(\xi)): \nu(w)=j \text { implies } \xi[x \rightarrow w] \in \mathrm{L}\left(A_{j}\right) \\
& \text { iff }(\forall w \in \operatorname{pos}(\xi)): \nu(w)=j \text { implies } \rho_{\xi[x \rightarrow w]}(\varepsilon) \in F \\
& \text { iff }\left\{\left(\rho_{\xi[x \mapsto w]}(\varepsilon), 1\right) \mid w \in \operatorname{pos}(\xi), \nu(w)=j\right\} \subseteq F \times\{1\} \\
& \text { iff } \rho_{(\xi, \nu)}(\varepsilon) \in \widetilde{F}  \tag{14.35}\\
& \text { iff }(\xi, \nu) \in \mathrm{L}\left(C_{j}\right) .
\end{align*}
$$

Hence $\mathrm{L}\left(C_{j}\right)=L_{j}$ ，and this finishes the proof of the lemma．

Finally，we point out the close relationship between Lemma 14.4 .16 and Lemma 14．4．12，both show how to eliminate weighted first－order universal quantification．The difference is that in Lemma 14．4．16 we end up in a wta，whereas in Lemma 14．4．12 we end up in an $\operatorname{MSO}(\Sigma, \mathrm{B})$－formula．

### 14.5 Relationship between a decomposition theorem of wta and B-E-T's theorem

In this final section we want to show a strong relationship between the following two results which we have proved already:
(1) the decomposition of a $(\Sigma, \mathrm{B})$-wta $\mathcal{A}$ (cf. Theorem $11.3 .1(\mathrm{~A}) \Rightarrow(\mathrm{B})$, roughly): we can construct $\tau$, $(G, H), \Theta$, and $\kappa$ such that, for each $\xi \in \mathrm{T}_{\Sigma}: \quad \llbracket \mathcal{A} \rrbracket^{\mathrm{run}}(\xi)=\mathrm{h}_{\mathrm{M}(\Theta, \kappa)}\left(\tau^{-1}(\xi) \cap \mathrm{L}(G, H)\right)$,
(2) the definability of $\llbracket \mathcal{A} \rrbracket^{\text {run }}$ for each $(\Sigma, \mathrm{B})$-wta $\mathcal{A}$ (cf. Theorem 14.3.2): we can construct a sentence $e \in \operatorname{MSO}(\Sigma, \mathrm{~B})$ such that $\llbracket e \rrbracket=\llbracket \mathcal{A} \rrbracket^{\mathrm{run}}$.

We will show that (1) can be used to prove (2) (cf. Theorem 14.5.2) and vice versa (cf. Theorem 14.5.4). Thus Theorem 14.5 .2 can be seen as an alternative proof of "r-recognizable implies definable" (cf. Subsection 14.5.1); also, Theorem 14.5.4 can be seen as an alternative proof of the decomposition result Theorem 11.3.1 (A) $\Rightarrow$ (B) (cf. Subsection 14.5.2).

This section elaborates the ideas of HVD19, Sec. 7.5] for the particular case of the trivial storage type and strong bimonoids as weight algebras.

### 14.5.1 Alternative proof of "r-recognizable implies definable"

We start with an auxiliary lemma which states that the set of definable weighted tree languages is closed under tree relabelings. Actually, we have proved this result already: each definable weighted tree language is r-recognizable (by Theorem 14.3.8), r-recognizable weighted tree languages are closed under tree relabelings (cf. Theorem 10.10.1), and each r-recognizable weighted tree language is definable (by Theorem 14.3.2). But we do not want to use Theorem 14.3 .2 for this closure result. So we show a proof with a direct construction.

Essentially the next lemma has been proved in Her17, Lm. 15] and Her20a, Lm. 5.3.2]. We follow this proof, but we mention that the notion of "definable" in Her17 and Her20a refers to the weighted logic in DG05, DV06, and this is different from the one in the present chapter.

Lemma 14.5.1. Let $\Theta$ be a ranked alphabet, $e \in \operatorname{MSO}(\Theta, \mathrm{~B})$ be a sentence, and $\tau=\left(\tau_{k} \mid k \in \mathbb{N}\right)$ be a $(\Theta, \Sigma)$-tree relabeling. Then we can construct a sentence $f \in \operatorname{MSO}(\Sigma, \mathrm{~B})$ such that $\llbracket f \rrbracket=\chi(\tau)(\llbracket e \rrbracket)$.

Proof. Let $\left(\theta_{1}, \ldots, \theta_{n}\right)$ be an arbitrary, but fixed enumeration of $\Theta$. For each $i \in[n]$, we let $k_{i}=\mathrm{rk}_{\Theta}\left(\theta_{i}\right)$. Moreover, we let $\mathcal{V}=\left\{X_{\theta} \mid \theta \in \Theta\right\}$ be a set of second-order variables. The idea is

- to code, for each $\zeta \in \mathrm{T}_{\Sigma}$, each preimage $\xi \in \tau^{-1}(\zeta)$ as a tree in $\mathrm{T}_{\Sigma_{\mathcal{V}}}$ where the second order variable $X_{\theta}$ represents the symbol $\theta \in \Theta$, and
- to evaluate $e^{\prime}$ on $\xi$ where $e^{\prime}$ is obtained from $e$ by replacing each subformula of the form label ${ }_{\theta}(x)$ by $\left(x \in X_{\theta}\right)$.
Formally, we construct the $\operatorname{MSO}(\Sigma, \mathrm{B})$-formula

$$
f=\left(\boldsymbol{T}_{X_{\theta_{1}}} \cdots \mathcal{F}_{X_{\theta_{n}}}\left(\psi_{\text {part }} \wedge \psi_{\text {relab }}\right) \triangleright e^{\prime}\right)
$$

where

- $\psi_{\text {part }}=\forall x .\left(\bigvee_{i \in[n]}\left(\left(x \in X_{\theta_{i}}\right) \wedge \bigwedge_{\substack{j \in[n]: \\ i \neq j}} \neg\left(x \in X_{\theta_{j}}\right)\right)\right)$,
- $\psi_{\text {relab }}=\forall x .\left(\bigwedge_{i \in[n]}\left(\neg\left(x \in X_{\theta_{i}}\right) \vee \operatorname{label}_{\tau_{k_{i}}\left(\theta_{i}\right)}(x)\right)\right)$, and
- $e^{\prime}$ is obtained from $e$ by replacing each subformula of the form $\operatorname{label}_{\theta}(x)$ by $\left(x \in X_{\theta}\right)$.

We note that $\psi_{\text {part }}$ and $\psi_{\text {relab }}$ are in $\operatorname{MSO}(\Sigma)$ and $e^{\prime}$ is in $\operatorname{MSO}(\Sigma, \mathrm{B})$. Moreover, Free $\left(\psi_{\text {part }}\right)=$ $\operatorname{Free}\left(\psi_{\text {relab }}\right)=\mathcal{V}$ and Free $\left(e^{\prime}\right) \subseteq \mathcal{V}$, hence the formula $f$ is a sentence.

First we analyse the semantics of $\psi_{\text {part }}$ and $\psi_{\text {relab }}$. The following two statements are obvious:

$$
\begin{gather*}
\text { for every } \zeta \in \mathrm{T}_{\Sigma} \text { and } \eta \in \Phi_{\zeta, \mathcal{V}}:(\zeta, \eta) \models \psi_{\text {part }} \quad \text { iff }  \tag{14.37}\\
\left\{\eta\left(X_{\theta_{1}}\right), \ldots, \eta\left(X_{\theta_{n}}\right)\right\} \text { is a partitioning of } \operatorname{pos}(\zeta)
\end{gather*}
$$

and

$$
\begin{align*}
& \text { for every } \zeta \in \mathrm{T}_{\Sigma} \text { and } \eta \in \Phi_{\zeta, \mathcal{V}}:(\zeta, \eta) \models \psi_{\text {relab }} \text { iff } \\
& \quad(\forall w \in \operatorname{pos}(\zeta), i \in[n]):\left(w \in \eta\left(X_{\theta_{i}}\right)\right) \rightarrow\left(\zeta(w) \in \tau_{k_{i}}\left(\theta_{i}\right)\right) \tag{14.38}
\end{align*}
$$

In the following, for every $\zeta \in \mathrm{T}_{\Sigma}$ and $U_{1}, \ldots, U_{n} \subseteq \operatorname{pos}(\zeta)$, we abbreviate the $\mathcal{V}$-assignment $\left[X_{\theta_{1}} \mapsto\right.$ $\left.U_{1}, \ldots, X_{\theta_{n}} \mapsto U_{n}\right]$ for $\zeta$ by $\left[X_{\theta_{i}} \mapsto U_{i} \mid i \in[n]\right]$. Then, putting (14.37) and (14.38) together, we obtain the statement:

$$
\begin{align*}
& \text { for every } \zeta \in \mathrm{T}_{\Sigma} \text { and } \eta \in \Phi_{\zeta, \mathcal{V}}:(\zeta, \eta) \models\left(\psi_{\text {part }} \wedge \psi_{\text {relab }}\right) \quad \text { iff }  \tag{14.39}\\
& \left(\exists \xi \in \mathrm{T}_{\Theta}\right):(\zeta \in \tau(\xi)) \wedge\left(\eta=\left[X_{\theta_{i}} \mapsto \operatorname{pos}_{\theta_{i}}(\xi) \mid i \in[n]\right]\right)
\end{align*}
$$

We note that $\operatorname{pos}(\zeta)=\operatorname{pos}(\xi)$. Moreover, by construction of $e^{\prime}$, we have

$$
\begin{equation*}
\text { for every } \zeta \in \mathrm{T}_{\Sigma} \text { and } \xi \in \tau^{-1}(\zeta): \llbracket e \rrbracket(\xi)=\llbracket e^{\prime} \rrbracket\left(\zeta,\left[X_{\theta_{i}} \mapsto \operatorname{pos}_{\theta_{i}}(\xi) \mid i \in[n]\right]\right) \tag{14.40}
\end{equation*}
$$

Now we prove that $\llbracket f \rrbracket=\chi(\tau)(r)$. Let $\zeta \in \mathrm{T}_{\Sigma}$. Then

$$
\begin{aligned}
& \llbracket f \rrbracket(\zeta) \\
& =\llbracket 十_{X_{\theta_{1}}} \cdots \text { N}_{X_{\theta_{n}}}\left(\psi_{\text {part }} \wedge \psi_{\text {relab }}\right) \triangleright e^{\prime} \rrbracket(\zeta) \\
& =\bigoplus_{W_{1}, \ldots, W_{n} \subseteq \operatorname{pos}(\zeta)} \llbracket\left(\psi_{\text {part }} \wedge \psi_{\text {relab }}\right) \triangleright e^{\prime} \rrbracket\left(\zeta,\left[X_{\theta_{i}} \mapsto W_{i} \mid i \in[n]\right]\right) \\
& = \begin{cases}\bigoplus_{W_{1}, \ldots, W_{n} \subseteq \operatorname{pos}(\zeta)} \llbracket e^{\prime} \rrbracket\left(\zeta,\left[X_{\theta_{i}} \mapsto W_{i} \mid i \in[n]\right]\right) & \text { if }\left(\zeta,\left[X_{\theta_{i}} \mapsto W_{i} \mid i \in[n]\right]\right) \models\left(\psi_{\text {part }} \wedge \psi_{\text {relab }}\right) \\
0 & \text { otherwise }\end{cases}
\end{aligned}
$$

$$
=\bigoplus_{\xi \in \tau^{-1}(\zeta)} \llbracket e^{\prime} \rrbracket\left(\zeta,\left[X_{\theta_{i}} \mapsto \operatorname{pos}_{\theta_{i}}(\xi) \mid i \in[n]\right]\right)
$$

$$
=\bigoplus_{\xi \in \tau^{-1}(\zeta)} \llbracket e \rrbracket(\xi)
$$

$$
=\chi(\tau)(\llbracket e \rrbracket)(\zeta) \quad\left(\text { by }\left(\underline{2.30)} \text { and the fact that } \tau^{-1}(\zeta) \text { is finite }\right)\right.
$$

Now we can show the alternative proof of the fact that r-recognizable implies definable.
Theorem 14.5.2. For each $(\Sigma, \mathrm{B})$-wta $\mathcal{A}$, we can construct a sentence $f \in \operatorname{MSO}(\Sigma, \mathrm{~B})$ such that $\llbracket f \rrbracket=$ $\llbracket \mathcal{A} \rrbracket^{\text {run }}$.

Proof. By Theorem $11.3 .1(\mathrm{~A}) \Rightarrow(\mathrm{B})$, we can construct a ranked alphabet $\Theta$, a deterministic $(\Theta, \Sigma)$-tree relabeling $\tau$, a $\Theta$-local system $(K, H)$, and a family $\kappa=\left(\kappa_{k} \mid k \in \mathbb{N}\right)$ of mappings $\kappa_{k}: \Theta^{(k)} \rightarrow B$ such that, for each $\xi \in \mathrm{T}_{\Sigma}$, we have $\llbracket \mathcal{A} \rrbracket^{\mathrm{run}}(\xi)=\mathrm{h}_{\mathrm{M}(\Theta, \kappa)}\left(\tau^{-1}(\xi) \cap \mathrm{L}(K, H)\right)$. By Observation 2.10.1 we have

$$
\begin{equation*}
\llbracket \mathcal{A} \rrbracket^{\mathrm{run}}(\xi)=\chi(\tau)\left(\chi(\mathrm{L}(K, H)) \otimes \mathrm{h}_{\mathrm{M}(\Theta, \kappa)}\right) \tag{14.41}
\end{equation*}
$$

By Corollary 11.2 .4 we can construct a bu deterministic $\Sigma$-fta $A$ such that $\mathrm{L}(K, H)=\mathrm{L}(A)$. By Lemma 14.1.4, we can construct a sentence $\varphi \in \operatorname{MSO}(\Theta)$ such that $\mathrm{L}(A)=\mathrm{L}(\varphi)$. Then

$$
\begin{array}{rlr}
\chi(\mathrm{L}(K, H)) \otimes \mathrm{h}_{\mathrm{M}(\Theta, \kappa)} & =\chi(\mathrm{L}(A)) \otimes \mathrm{h}_{\mathrm{M}(\Theta, \kappa)}=\chi(\mathrm{L}(\varphi)) \otimes \mathrm{h}_{\mathrm{M}(\Theta, \kappa)} \\
& =\chi(\mathrm{L}(\varphi)) \otimes \llbracket \mathrm{H}(\kappa) \rrbracket & \text { (because } \mathrm{H}(\kappa) \text { is a sentence) } \\
& =\llbracket \varphi \triangleright \mathrm{H}(\kappa) \rrbracket . & \text { (by (14.10)) }
\end{array}
$$

By Lemma 14.5.1, we can construct a sentence $f \in \operatorname{MSO}(\Sigma, \mathrm{~B})$ such that $\llbracket f \rrbracket=\chi(\tau)(\llbracket \varphi \triangleright \mathrm{H}(\kappa) \rrbracket)$. Finally, by (14.41), we obtain $\llbracket f \rrbracket=\llbracket \mathcal{A} \rrbracket^{\text {run }}$.

### 14.5.2 Alternative proof of the decomposition Theorem 11.3.1(A) $\Rightarrow(B)$

The alternative proof uses the fact that, for each $(\Sigma, \mathrm{B})$-wta $\mathcal{A}$, we can construct a sentence $e \in \operatorname{MSO}(\Sigma, \mathrm{~B})$ such that $\llbracket \mathcal{A} \rrbracket^{\text {run }}=\llbracket e \rrbracket$ (cf. Theorem 14.3 .2 ) and two easy technical lemmas. The first lemma claims that the set of models of the $\operatorname{MSO}(\Sigma)$-formula $\varphi$ in (14.19) is a local tree language.

Lemma 14.5.3. Let $\mathcal{A}=(Q, \delta, F)$ be a root weight normalized $(\Sigma, \mathrm{B})$-wta with $\operatorname{supp}(F)=\left\{q_{f}\right\}$ and $\mathcal{U}=\bigcup\left(Q^{k} \times Q \mid k \in \mathbb{N}\right.$ such that $\left.\operatorname{rk}^{-1}(k) \neq \emptyset\right)$. Moreover, let $\varphi$ be the $\operatorname{MSO}(\Sigma)$-formula $\varphi$ in (14.19). Then we can construct a $\Sigma_{\mathcal{U}}$-local system $(G, H)$ such that $\mathrm{L}(G, H)=\mathrm{L}_{\mathcal{U}}(\varphi)$.

Proof. We define the $\Sigma_{\mathcal{U}}$-local system $(G, H)$ such that $G$ is the set of all forks

$$
\left(\left(\sigma_{1},\left\{\left(w_{1}, q_{1}\right)\right\}\right) \cdots\left(\sigma_{k},\left\{\left(w_{k}, q_{k}\right)\right\}\right),(\sigma,\{(w, q)\})\right)
$$

in $\operatorname{Fork}\left(\Sigma_{\mathcal{U}}\right)$ such that

- $\sigma \in \Sigma^{(k)}$ and $\sigma_{1}, \ldots, \sigma_{k} \in \Sigma$ for some $k \in \mathbb{N}$,
- $q, q_{1}, \ldots, q_{k} \in Q$, and
- $w=q_{1} \cdots q_{k}$ and for each $i \in[k]$ we have $w_{i} \in Q^{\mathrm{rk}\left(\sigma_{i}\right)}$.

Moreover, $H=\left\{\left(\sigma,\left\{\left(w, q_{f}\right)\right\}\right) \mid k \in \mathbb{N}, \sigma \in \Sigma^{(k)}, w \in Q^{k}\right\}$. Then it is easy to see that $\mathrm{L}(G, H)=$ $\mathrm{L}_{\mathcal{U}}(\varphi)$.

Now we can give an alternative proof of Theorem $11.3 .1(\mathrm{~A}) \Rightarrow(\mathrm{B})$.
Theorem 14.5.4. Let $\mathcal{A}$ be a ( $\Sigma, \mathrm{B})$-wta. Then we can construct

- a ranked alphabet $\Theta$,
- a deterministic $(\Theta, \Sigma)$-tree relabeling $\tau$,
- a $\Theta$-local system $(G, H)$, and
- a family $\kappa=\left(\kappa_{k} \mid k \in \mathbb{N}\right)$ of mappings $\kappa_{k}: \Theta^{(k)} \rightarrow B$
such that, for each $\xi \in \mathrm{T}_{\Sigma}$, the following holds: $\llbracket \mathcal{A} \rrbracket^{\mathrm{run}}(\xi)=\mathrm{h}_{\mathrm{M}(\Theta, \kappa)}\left(\tau^{-1}(\xi) \cap \mathrm{L}(G, H)\right)$.
Proof. Let $\mathcal{A}=(Q, \delta, F)$. By Theorem 7.3.1 we can assume that $\mathcal{A}$ is root weight normalized, i.e., $\operatorname{supp}(F)$ contains exactly one element $q_{f}$ and $F\left(q_{f}\right)=\mathbb{1}$.

We define $\mathcal{U}=\bigcup\left(Q^{k} \times Q \mid k \in \mathbb{N}\right.$ such that $\left.\operatorname{rk}^{-1}(k) \neq \emptyset\right)$ and $\Theta=\Sigma_{\mathcal{U}}$. We recall the $\operatorname{MSO}(\Sigma, \mathrm{B})$ formula defined in (14.22):

$$
e=\boldsymbol{1}_{X_{1}} \cdots \mathcal{F}_{X_{n}}(\varphi \triangleright \mathrm{H}(\kappa))
$$

and $\kappa=\left(\kappa_{k} \mid k \in \mathbb{N}\right)$ with $\kappa_{k}:\left(\Sigma_{\mathcal{U}}\right)^{(k)} \rightarrow B$. By the proof of Theorem 14.3 .2 , we have $\llbracket \mathcal{A} \rrbracket^{\text {run }}=\llbracket e \rrbracket$.
For each $i \in[n]$, we consider the subformula

$$
e_{i}=\mathcal{F}_{X_{i}} \cdots \mathcal{F}_{X_{n}}(\varphi \triangleright \mathrm{H}(\kappa))
$$

Obviously, Free $\left(e_{i}\right)=\left\{X_{1}, \ldots, X_{i-1}\right\}$ and $X_{i} \notin\left\{X_{1}, \ldots, X_{i-1}\right\}$. Thus we can apply Lemma 14.2.6(2) (with $\left.\mathcal{V}=\operatorname{Free}\left(e_{i}\right)\right)$ and thereby construct the deterministic $\left(\Sigma_{\left\{X_{1}, \ldots, X_{i}\right\}}, \Sigma_{\left\{X_{1}, \ldots, X_{i-1}\right\}}\right)$-tree relabeling $\tau_{i}$ such that

$$
\llbracket \mathcal{F}_{X_{i}} \cdots \mathcal{F}_{X_{n}}(\varphi \triangleright \mathrm{H}(\kappa)) \rrbracket_{\left\{X_{1}, \ldots, X_{i-1}\right\}}=\chi\left(\tau_{i}\right)\left(\llbracket \mathcal{W}_{X_{i+1}} \cdots \mathcal{F}_{X_{n}}(\varphi \triangleright \mathrm{H}(\kappa)) \rrbracket_{\left\{X_{1}, \ldots, X_{i}\right\}}\right) .
$$

Hence, by the $n$-fold application of Lemma 14.2.6(2), we can construct deterministic tree relabelings $\tau_{1}, \ldots, \tau_{n}$ such that

$$
\llbracket e \rrbracket=\llbracket \mathcal{F}_{X_{1}} \cdots \mathcal{F}_{X_{n}}(\varphi \triangleright \mathrm{H}(\kappa)) \rrbracket=\chi\left(\tau_{1}\right)\left(\ldots \chi\left(\tau_{n}\right)(\llbracket \varphi \triangleright \mathrm{H}(\kappa) \rrbracket \mathcal{U}) \ldots\right)
$$

Then we construct the deterministic tree relabeling $\tau$ such that $\tau=\tau_{1} \hat{o} \ldots \hat{o} \tau_{n}$ (cf. Section 2.9) and, by Theorem 2.9.6 we have

$$
\chi\left(\tau_{1}\right)\left(\ldots \chi\left(\tau_{n}\right)\left(\llbracket \varphi \triangleright \mathrm{H}(\kappa) \rrbracket_{\mathcal{U}}\right) \ldots\right)=\chi(\tau)(\llbracket \varphi \triangleright \mathrm{H}(\kappa) \rrbracket \mathcal{U})
$$

Now let $\xi \in \mathrm{T}_{\Sigma}$. We can calculate as follows (where we abbreviate the $\Theta$-algebra homomorphism $\mathrm{h}_{\mathrm{M}(\Theta, \kappa)}$ by $\left.\mathrm{h}_{\kappa}\right)$.

$$
\chi(\tau)(\llbracket \varphi \triangleright \mathrm{H}(\kappa) \rrbracket \mathcal{U})(\xi)=\chi(\tau)\left(\chi\left(\mathrm{L}_{\mathcal{U}}(\varphi)\right) \otimes \llbracket \mathrm{H}(\kappa) \rrbracket_{\mathcal{U}}\right)(\xi)
$$

$$
\left.=\chi(\tau)\left(\chi\left(\mathrm{L}_{\mathcal{U}}(\varphi)\right) \otimes \mathrm{h}_{\kappa[\mathcal{U} \rightsquigarrow \mathcal{U}]} \otimes \chi\left(\mathrm{T}_{\Sigma_{\mathcal{U}}}^{\mathrm{v}}\right)\right)(\xi) \quad \text { (by definition of } \llbracket \mathrm{H}(\kappa) \rrbracket \mathcal{U}\right)
$$

$$
\left.=\chi(\tau)\left(\chi\left(\mathrm{L}_{\mathcal{U}}(\varphi)\right) \otimes \mathrm{h}_{\kappa} \otimes \chi\left(\mathrm{T}_{\Sigma_{\mathcal{U}}}^{\mathrm{v}}\right)\right)(\xi) \quad \text { (because } \mathrm{h}_{\kappa[\mathcal{U} \rightsquigarrow \mathcal{U}]}=\mathrm{h}_{\kappa}\right)
$$

$$
\left.=\chi(\tau)\left(\chi\left(\mathrm{L}_{\mathcal{U}}(\varphi)\right) \otimes \mathrm{h}_{\kappa}\right)(\xi) \quad \quad \text { (because } \mathrm{L}_{\mathcal{U}}(\varphi) \subseteq \mathrm{T}_{\Sigma_{\mathcal{U}}}^{\mathrm{v}}\right)
$$

$$
=\mathrm{h}_{\kappa}\left(\tau^{-1}(\xi) \cap \mathrm{L}_{\mathcal{U}}(\varphi)\right) . \quad \text { (by Observation 2.10.1) }
$$

By Lemma 14.5.3, we can construct a $\Sigma_{\mathcal{U}}$-local system $(G, H)$ such that $\mathrm{L}_{\mathcal{U}}(\varphi)=\mathrm{L}(G, H)$. Hence we obtain $\llbracket \mathcal{A} \rrbracket^{\text {run }}(\xi)=\mathrm{h}_{\kappa}\left(\tau^{-1}(\xi) \cap \mathrm{L}(G, H)\right)$.

## Chapter 15

## Abstract families of weighted tree languages

In the 60 's and 70 's of the previous century, Ginsburg, Greibach, and Hopcroft proposed a unifying concept for the study of closure properties of sets of formal languages GG67, GG69, GGH69, Gin75. This is the concept of abstract family of languages (AFL). Roughly speaking, an AFL is a set of formal languages which is closed under intersection with regular languages, inverse homomorphisms, $\varepsilon$-free homomorphisms, union, concatenation, and Kleene star. An AFL is full if it is closed under arbitrary homomorphisms. Of particular interest are principal AFL; an AFL is principal if it is generated from one formal language by using the mentioned closure properties. For instance, the sets of regular languages, context-free languages, stack languages, nested-stack languages, and recursively enumerable languages are full principal AFL [GG70, Sec. 2].

The importance of principal AFL shows up in one of the main theorems of AFL-theory Gin75, Thm. 5.2.1]: a set $\mathcal{L}$ is a full principal AFL if and only if there exists a finitely encoded abstract family of acceptors (AFA) $\mathcal{D}$ such that $\mathcal{L}$ is set of all formal languages accepted by $\mathcal{D}$. Roughly speaking, an acceptor $\mathcal{D}$ is a one-way nondeterministic finite-state automaton which uses an additional storage (indicated by the type $\mathcal{D}$ ) such as, e.g. counter, pushdown, stack, or nested-stack. Thus, in other words, "each family of languages defined by a set of 'well-behaving' one-way nondeterministic acceptors of a same 'type' is a full principal AFL" (cited from ESvL80, p. 109].

In this chapter we define the concept of abstract family of weighted tree languages. It is inspired by the definitions of abstract family of languages GG67, GG69, GG70, GGH69, Gin75, abstract family of formal power series KS86, abstract family of fuzzy languages Asv03] (also cf. Wec78, Ch. 3]), abstract family of tree series [Kui99a, Thm. 3.5] and ÉK03, Sect. 7], and sheaf of forests [BR94.

In the rest of this chapter, B denotes an arbitrary commutative and $\sigma$-complete semiring.
Thus, by Theorem 9.2.9, we have that $\operatorname{Rec}(\Sigma, \mathrm{B})=\operatorname{Reg}(\Sigma, \mathrm{B})$, and, in particular, the closure results developed in Chapter 10 for $\operatorname{Rec}(\Sigma, \mathrm{B})$ also hold for $\operatorname{Reg}(\Sigma, \mathrm{B})$.

In this chapter we follow the notions and definitions in FV22b. The main theorem is the following (cf. Theorem 15.4.5): for each $n \in \mathbb{N}$, the set $\operatorname{Reg}(n, \mathrm{~B})$ of all $(n, \mathrm{~B})$-weighted tree languages is the smallest principal abstract family of ( $n, \mathrm{~B}$ )-weighted tree languages.

### 15.1 The basic definitions

For each $n \in \mathbb{N}$, an ( $n, \mathrm{~B}$ )-weighted tree language is a $(\Sigma, \mathrm{B})$-weighted tree language for some ranked alphabet $\Sigma$ with $\operatorname{maxrk}(\Sigma) \leq n$.

In this chapter we will often consider a set $\mathcal{L}$ of $(n, \mathrm{~B})$-weighted tree languages and write that " $\mathcal{L}$ is closed under some operation", like scalar multiplication, sum, tree concatenations, or weighted projective bimorphisms (wpb). If we consider a binary operation, then we always assume that the two arguments of the operation are based on the same ranked alphabet. For instance, if we write that a set $\mathcal{L}$ of $(n, \mathrm{~B})$ weighted tree languages is closed under sum, then this means that, for every ranked alphabet $\Sigma$ with $\operatorname{maxrk}(\Sigma) \leq n$, and $r_{1}: \mathrm{T}_{\Sigma} \rightarrow B$ and $r_{2}: \mathrm{T}_{\Sigma} \rightarrow B$ in $\mathcal{L}$, the weighted tree language $r_{1} \oplus r_{2}$ is in $\mathcal{L}$.

As a consequence of the definition of an $(n, \mathrm{~B})$-weighted tree language, we have that a regular $(n, \mathrm{~B})$ weighted tree language is a regular $(\Sigma, \mathrm{B})$-weighted tree language for some ranked alphabet $\Sigma$ with $\operatorname{maxrk}(\Sigma) \leq n$. We denote the set of all regular $(n, \mathrm{~B})$-weighted tree languages by $\operatorname{Reg}(n, \mathrm{~B})$. That is

$$
\operatorname{Reg}(n, \mathrm{~B})=\bigcup(\operatorname{Reg}(\Sigma, \mathrm{B}) \mid \Sigma \text { is a ranked alphabet such that } \operatorname{maxrk}(\Sigma) \leq n)
$$

Similarly to the unweighted string case, we require that a family of weighted tree languages contains at least one element. Here we let this element be a recognizable step mapping which is unit-valued, one-step, and over a very simple ranked alphabet. Formally, let $\mathcal{L}$ be a set of ( $n, \mathrm{~B}$ )-weighted tree languages. We call $\mathcal{L}$ a family of $(n, B)$-weighted tree languages if there exists a ranked alphabet $\Delta$ such that

- $\operatorname{maxrk}(\Delta)=n$,
- $\left|\Delta^{(k)}\right|=1$ for each $k \in[0, n]$, and
- $\chi\left(\mathrm{T}_{\Delta}\right) \in \mathcal{L}$.

If $\Delta$ is such a ranked alphabet, then for each ranked alphabet $\Sigma$ with $\operatorname{maxrk}(\Sigma) \leq n$ and $\xi \in \mathrm{T}_{\Sigma}$, there exists a unique $\zeta \in \mathrm{T}_{\Delta}$ such that $\operatorname{pos}(\zeta)=\operatorname{pos}(\xi)$. We will exploit this obvious fact later.

Let $\mathcal{L}$ be a family of $(n, \mathrm{~B})$-weighted tree languages. We call $\mathcal{L}$ an $(n, \mathrm{~B})$-tree cone if it is closed under wpb. (For the definition of wpb, we refer to cf. Section 10.13. For related concepts, we refer to [Ber79, p. 136] for rational cones, to [Kui99a, p. 8] for recognizable tree cones, and to EÉK03, Sect. 7] for recognizable tree series cones.)

We call $\mathcal{L}$ an abstract family of ( $n, \mathrm{~B}$ )-weighted tree languages (for short: $(n, \mathrm{~B})$ - AFwtL , or just: $\mathrm{AFwtL})$ if $\mathcal{L}$ is an ( $n, \mathrm{~B}$ )-tree cone which is closed under the rational operations, i.e., under sum, tree concatenations, and Kleene-stars.

The above definition of $(n, \mathrm{~B})$-AFwtL is equivalent to the one in FV22b where additionally closure under scalar multiplication was required. However, scalar multiplication of some ( $\Sigma, \mathrm{B}$ )-weighted tree language $r$ with some $b \in B$ can be simulated as the Hadamard product of $r$ and the constant weighted tree language $\widetilde{b}$. In its turn, the Hadamard product of $r$ and $\widetilde{b}$, can be expressed as application $\llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}(r)$ for some $(\Sigma, \Sigma, \mathrm{B})$-wpb $\mathcal{H}$ such that $\llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}(\xi, \zeta)=b$ if $\xi=\zeta$, and $\mathbb{O}$ otherwise (cf. (2.28)). It is obvious how to construct such an $\mathcal{H}$. Thus, scalar multiplication can be simulated by wpb, and hence closure under scalar multiplication can be neglected in the definition of $(n, \mathrm{~B})$-AFwtL. We note that the concept of full abstract family of languages GG67, GG69, GGH69, Gin75 is a particular instance of the concept of AFwtL (cf. FV22b, Sec. 7]).

Let $\mathcal{S}$ be a set of ( $n, \mathrm{~B}$ )-weighted tree languages. We denote the smallest ( $n, \mathrm{~B}$ )-AFwtL which contains $\mathcal{S}$ by $\mathcal{F}(\mathcal{S})$.

Now assume that $\mathcal{L}$ is an $(n, \mathrm{~B})$-AFwtL. Then it is principal if there exists an $(n, \mathrm{~B})$-weighted tree language $g \in \mathcal{L}$ such that $\mathcal{L}=\mathcal{F}(\{g\})$; and if this is the case, then $g$ is called generator (of $\mathcal{L}$ ).

### 15.2 Characterization of tree cones

As defined above, a family of $(n, \mathrm{~B})$-weighted tree languages is an $(n, \mathrm{~B})$-tree cone if it is closed under wpb. Here we characterize the property of being an ( $n, \mathrm{~B}$ )-tree cone by decomposing each wpb into (a) the inverse of a linear and nondeleting tree homormorphism, (b) the Hadamard product with a recognizable weighted tree language, and (c) a linear and nondeleting tree homomorphism. In other words, we make the two homomorphisms and the center language of a wpb explicit (cf. Corollary 15.2.2).

The characterization resembles Nivat's decomposition theorem of gsm Niv68, Arnold's and Dauchet's bimorphism AD76, AD82, Ginsburg's decomposition of a-transducers Gin75, Lm. 3.2.2], and Engelfriet's decomposition of bottom-up tree transducers Eng75a, Thm. 3.5]. Moreover, it is an instance of [FMV11, Thm. 4.2].

Let $\mathcal{H}=(N, S, R, w t)$ be a $(\Sigma, \Psi, \mathrm{B})$-wpb. We recall that $\mathcal{H}$ is a particular $([\Sigma \Psi], \mathrm{B})$-wrtg and hence, we have defined the $(R,[\Sigma \Psi])$-tree homomorphism $\pi: \mathrm{T}_{\mathrm{R}} \rightarrow \mathrm{T}_{[\Sigma \Psi]}$, which extracts from a rule tree the derived $[\Sigma \Psi]$-tree. Moreover, for the wbp $\mathcal{H}$, we have defined the $([\Sigma \Psi], \Sigma)$-tree homomorphism $\pi_{1}$ and the $([\Sigma \Psi], \Psi)$-tree homomorphism $\pi_{2}$, which project a $[\Sigma \Psi]$-tree to its first and second component, respectively. Each of the three tree homomorphisms $\pi, \pi_{1}$, and $\pi_{2}$ is linear and nondeleting. Now we compose (a) $\pi$ and $\pi_{1}$ and (b) $\pi$ and $\pi_{2}$, and thereby obtain two linear and nondeleting tree homomorphisms $h_{1}^{\mathcal{H}}$ and $h_{2}^{\mathcal{H}}$.

Formally, we define $h_{1}^{\mathcal{H}}=\pi_{1} \hat{\circ} \pi$ and $h_{2}^{\mathcal{H}}=\pi_{2} \hat{o} \pi$, where $\hat{o}$ is the syntactic composition of tree homomorphisms defined in Section [2.9, For the convenience of the reader, we recall that $h_{1}^{\mathcal{H}}=\left(\left(h_{1}^{\mathcal{H}}\right)_{k} \mid k \in \mathbb{N}\right)$ is the $(R, \Sigma)$-tree homomorphism and $h_{2}^{\mathcal{H}}=\left(\left(h_{2}^{\mathcal{H}}\right)_{k} \mid k \in \mathbb{N}\right)$ is an $(R, \Psi)$-tree homomorphism such that the following holds.

- For every $k \in \mathbb{N}$ and rule $r=\left(A \rightarrow[\sigma, \psi]\left(A_{1}, \ldots, A_{k}\right)\right)$ with $[\sigma, \psi] \in[\Sigma \Psi]^{(k)}$, we have

$$
\left(h_{1}^{\mathcal{H}}\right)_{k}(r)=\left\{\begin{array}{ll}
\sigma\left(x_{1}, \ldots, x_{k}\right) & \text { if } \sigma \neq \varepsilon \\
x_{1} & \text { otherwise }
\end{array} \quad \text { and } \quad\left(h_{2}^{\mathcal{H}}\right)_{k}(r)= \begin{cases}\psi\left(x_{1}, \ldots, x_{k}\right) & \text { if } \psi \neq \varepsilon \\
x_{1} & \text { otherwise }\end{cases}\right.
$$

- For each chain-rule $r=(A \rightarrow B)$, we have $\left(h_{1}^{\mathcal{H}}\right)_{1}(r)=x_{1}$ and $\left(h_{2}^{\mathcal{H}}\right)_{1}(r)=x_{1}$.

In fact, $h_{1}^{\mathcal{H}}$ and $h_{2}^{\mathcal{H}}$ are simple, i.e., linear, nondeleting, alphabetic, and ordered tree homomorphisms. Moreover, by Theorem 2.9.7, we have $h_{1}^{\mathcal{H}}=\pi_{1} \circ \pi$ and $h_{2}^{\mathcal{H}}=\pi_{2} \circ \pi$.

Now we can prove the characterization of wpb reported above. We mention that the composition of weighted tree transformations involved in the following theorem is associative due to Observation 2.10.2. Moreover, we will use the concepts of characteristic mapping and diagonalization as follows (cf. Subsections 2.10.2 and 2.10.3). Since $h_{1}^{\mathcal{H}}$ and $h_{2}^{\mathcal{H}}$ are particular binary relations, i.e., $h_{1}^{\mathcal{H}} \subseteq \mathrm{T}_{R} \times \mathrm{T}_{\Sigma}$ and $h_{2}^{\mathcal{H}} \subseteq \mathrm{T}_{R} \times \mathrm{T}_{\Psi}$, we can consider the characteristic mappings $\chi\left(\left(h_{1}^{\mathcal{H}}\right)^{-1}\right): \mathrm{T}_{\Sigma} \times \mathrm{T}_{R} \rightarrow B$ and

Theorem 15.2.1. Let $\mathcal{H}$ be a $(\Sigma, \Psi, \mathrm{B})-w p b$. Then $\llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}=\chi\left(\left(h_{1}^{\mathcal{H}}\right)^{-1}\right) ; \overline{\llbracket \mathcal{H} \rrbracket^{\mathrm{wrt}}} ; \chi\left(h_{2}^{\mathcal{H}}\right)$.
Proof. Let $\xi \in \mathrm{T}_{\Sigma}$ and $\zeta \in \mathrm{T}_{\Psi}$. By the definition of $\mathrm{RT}_{\mathcal{H}}\left(\mathrm{T}_{[\Sigma \Psi]}\right)$ in Section 8.1 (notice the fact that each wpb in a particular weighted context-free grammar) and by (10.41), we have

$$
\begin{equation*}
\operatorname{RT}_{\mathcal{H}}(\xi, \zeta)=\left\{d \in \mathrm{~T}_{R} \mid(\xi, \zeta)=\left(h_{1}^{\mathcal{H}}(d), h_{2}^{\mathcal{H}}(d)\right), d \in \mathrm{RT}_{\mathcal{H}}\left(\mathrm{T}_{[\Sigma \Psi]}\right)\right\} \tag{15.1}
\end{equation*}
$$

Then we can calculate as follows.

$$
\begin{array}{rlr}
\llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}(\xi, \zeta)= & \sum_{d \in \mathrm{RT}_{\mathcal{H}}(\xi, \zeta)}^{\oplus} \mathrm{wt}_{\mathcal{H}}(d) \\
= & \sum_{\substack{d \in \mathrm{~T}_{R}: \\
(\xi, \zeta)=\left(h_{1}^{\mathcal{H}}(d), h_{2}^{\mathcal{H}}(d)\right)}} \mathrm{wt}_{\mathcal{H}}(d) \otimes \chi\left(\mathrm{RT}_{\mathcal{H}}\left(\mathrm{T}_{[\Sigma \Psi]}\right)\right)(d) & \quad \text { (by (15.1)) } \\
= & \sum_{\substack{d \in \mathrm{~T}_{R}: \\
\\
\\
\\
(\xi, \zeta)=\left(h_{1}^{\mathcal{H}}(d), h_{2}^{\mathcal{H}}(d)\right) \\
=}} \sum_{d \in \rrbracket^{\mathrm{wrt}}(d)}^{\oplus} \chi\left(h_{1}^{\mathcal{H}}\right)(d, \xi) \otimes \overline{\llbracket \mathcal{H} \rrbracket^{\mathrm{wrt}}}(d, d) \otimes \chi\left(h_{2}^{\mathcal{H}}\right)(d, \zeta) \\
= & \left.\sum_{d, d^{\prime} \in \mathrm{T}_{R}}^{\oplus} \chi\left(h_{1}^{\mathcal{H}}\right)(d, \xi) \otimes \overline{\llbracket \mathcal{H} \rrbracket^{\mathrm{wrt}}}\left(d, d^{\prime}\right) \otimes \chi\left(h_{2}^{\mathcal{H}}\right)\left(d^{\prime}, \zeta\right) \quad \quad \text { (by definition of } \llbracket \mathcal{H} \rrbracket^{\mathrm{wrt}}\right) \\
\end{array}
$$

$$
\begin{aligned}
& =\sum_{d, d^{\prime} \in \mathrm{T}_{R}}^{\oplus} \chi\left(\left(h_{1}^{\mathcal{H}}\right)^{-1}\right)(\xi, d) \otimes \overline{\overline{[\mathcal{H}} \rrbracket^{\mathrm{wrt}}}\left(d, d^{\prime}\right) \otimes \chi\left(h_{2}^{\mathcal{H}}\right)\left(d^{\prime}, \zeta\right) \\
& =\left(\chi\left(\left(h_{1}^{\mathcal{H}}\right)^{-1}\right) ; \overline{\left\lceil\mathcal{H} \rrbracket^{\mathrm{wrt}}\right.} ; \chi\left(h_{2}^{\mathcal{H}}\right)\right)(\xi, \zeta) . \quad \text { (by definition of composition, see Subsection [2.10.3) }
\end{aligned}
$$

An $n$-tree homomorphism is a $(\Sigma, \Psi)$-tree homomorphism where $\Sigma$ and $\Psi$ are ranked alphabets and $\operatorname{maxrk}(\Sigma) \leq n$ and $\operatorname{maxrk}(\Psi) \leq n$. The closure under $n$-tree homomorphisms and closure under inverse $n$-tree homomorphisms are defined in a way analogous to the corresponding closure under tree homomorphisms and inverse tree homomorphisms (cf. Sections 10.11 and 10.12).

A set $\mathcal{L}$ of $(n, \mathrm{~B})$-weighted tree languages is closed under Hadamard product with recognizable ( $n, \mathrm{~B}$ )weighted tree languages if, for every $(\Sigma, \mathrm{B})$-weighted tree language $r$ in $\mathcal{L}$ and for each recognizable ( $\Sigma, \mathrm{B}$ )-weighted tree language $r^{\prime}$, we have $r \otimes r^{\prime}$ is in $\mathcal{L}$.

Corollary 15.2.2. Let B be a commutative and $\sigma$-complete semiring. Let $n \in \mathbb{N}$ and $\mathcal{L}$ be a family of $(n, \mathrm{~B})$-weighted tree languages. Then the following three statements are equivalent.
(A) $\mathcal{L}$ is an $(n, \mathrm{~B})$-tree cone.
(B) $\mathcal{L}$ is closed under (a) simple $n$-tree homomorphisms, (b) inverse of simple $n$-tree homomorphisms, and (c) Hadamard product with recognizable ( $n, \mathrm{~B}$ )-weighted tree languages.
(C) $\mathcal{L}$ is closed under (a) simple $n$-tree homomorphisms, (b) inverse of simple $n$-tree homomorphisms, and (c) Hadamard product with bu deterministically recognizable ( $n, \mathrm{~B}$ )-weighted tree languages.

Proof. Proof of $(\mathrm{A}) \Rightarrow(\mathrm{B})$ : (a) Let $h=\left(h_{k} \mid k \in \mathbb{N}\right)$ be a simple ( $\Sigma, \Psi$ )-tree homomorphism with $\operatorname{maxrk}(\Sigma) \leq n$ and $\operatorname{maxrk}(\Psi) \leq n$. Then we construct the $(\Sigma, \Psi, \mathrm{B})-\mathrm{wpb} \mathcal{H}=(N, S, R, w t)$ as follows. We let $N=S=\{A\}$ and, for every $k \in \mathbb{N}$ and $\sigma \in \Sigma^{(k)}$, we have the following.

If $h_{k}(\sigma)=\psi\left(x_{1}, \ldots, x_{k}\right)$, then $r=(A \rightarrow[\sigma, \psi](A, \ldots, A))$ is in $R$ and $w t(r)=\mathbb{1}$.
If $k=1$ and $h_{k}(\sigma)=x_{1}$, then $r=(A \rightarrow[\sigma, \varepsilon](A))$ is in $R$ and $w t(r)=\mathbb{1}$.
It is obvious that $\llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}=\chi(h)$. Since $\mathcal{L}$ is closed under $(n, B)$-wpb, we obtain that $\mathcal{L}$ is closed under simple $n$-tree homomorphisms.
(b) This can be proved in the same way as (a) except that the order of $\sigma$ and $\psi$ (and $\sigma$ and $\varepsilon$ ) in the rules of $R$ are exchanged.
(c) Let $r: \mathrm{T}_{\Sigma} \rightarrow B$ be a weighted tree language in $\mathcal{L}$ and $\mathcal{A}$ be a $(\Sigma, \mathrm{B})$-wta. By definition, maxrk $(\Sigma) \leq$ n. By the proof of Corollary $\mathbb{1 0 . 1 3 . 1 1}$, we can construct a $(\Sigma, \Sigma, \mathrm{B})$-wpb $\mathcal{H}$ such that $\llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}=\overline{\llbracket \mathcal{A} \rrbracket}$, i.e., $\llbracket \mathcal{H} \rrbracket^{\text {tt }}$ is the diagonalization of $\llbracket \mathcal{A} \rrbracket$ (cf. page 51). Then by Equation (2.33) we have $\llbracket \mathcal{H} \rrbracket^{\text {tt }}(r)=r \otimes \llbracket \mathcal{A} \rrbracket$. Since $\mathcal{L}$ is closed under wpb, we can conclude that $\mathcal{L}$ is closed under Hadamard product with recognizable ( $n, \mathrm{~B}$ )-weighted tree languages.

Proof of $(\mathrm{B}) \Rightarrow(\mathrm{C})$ : It is obvious.
Proof of $(\mathrm{C}) \Rightarrow(\mathrm{A})$ : Let $r: \mathrm{T}_{\Sigma} \rightarrow B$ be in $\mathcal{L}$ and let $\mathcal{H}$ be a $(\Sigma, \Psi, \mathrm{B})-$ wpb with $\operatorname{maxrk}(\Psi) \leq n$. Let $R$ be the set of rules of $\mathcal{H}$. By Theorem 15.2.1 we have

$$
\llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}(r)=\left(\chi\left(\left(h_{1}^{\mathcal{H}}\right)^{-1}\right) ; \overline{\llbracket \mathcal{H} \rrbracket^{\mathrm{wrt}}} ; \chi\left(h_{2}^{\mathcal{H}}\right)\right)(r)
$$

and thus, by Observation 2.10.3 we have

$$
\llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}(r)=\chi\left(h_{2}^{\mathcal{H}}\right)\left(\overline{\llbracket \mathcal{H} \rrbracket^{\mathrm{wrt}}}\left(\chi\left(\left(h_{1}^{\mathcal{H}}\right)^{-1}\right)(r)\right)\right)
$$

Let us denote the weighted tree language $\chi\left(\left(h_{1}^{\mathcal{H}}\right)^{-1}\right)(r)$ by $r^{\prime}$ Since $\mathcal{L}$ is closed under inverse of simple $n$-tree homomorphism, $r^{\prime}$ is in $\mathcal{L}$. Then, by Equation (2.33), we have

$$
\overline{\llbracket \mathcal{H} \rrbracket^{\mathrm{wrt}}}\left(r^{\prime}\right)=r^{\prime} \otimes \llbracket \mathcal{H} \rrbracket^{\mathrm{wrt}}
$$

By Lemma 11.2 .2 (2) and Lemma 11.2 .3 , the weighted tree language $\llbracket \mathcal{H} \rrbracket^{\text {wrt }}$ is bu deterministically recognizable. Since $\mathcal{L}$ is closed under Hadamard product with bu deterministically recognizable $n$-weighted tree languages, we obtain that $\overline{\llbracket \mathcal{H} \rrbracket^{\text {wrt }}}\left(r^{\prime}\right)$ is in $\mathcal{L}$. Let us denote this weighted tree language by $r^{\prime \prime}$. Finally, since $\mathcal{L}$ is closed under simple $n$-tree homomorphism, the weighted tree language $\chi\left(h_{2}^{\mathcal{H}}\right)\left(r^{\prime \prime}\right)$ is in $\mathcal{L}$. Hence $\llbracket \mathcal{H} \rrbracket^{\text {tt }}(r)$ is in $\mathcal{L}$.

## 15.3 $\operatorname{Reg}(n, \mathrm{~B})$ is an AFwtL

Here we prove that the set of regular $(n, \mathrm{~B})$-weighted tree languages is an AFwtL . For each $n \in \mathbb{N}$, we define the ranked alphabet $\Delta_{n}$ by

$$
\Delta_{n}=\bigcup_{k \in[0, n]} \Delta_{n}^{(k)} \text { and } \Delta_{n}^{(k)}=\{[k]\} \text { for each } k \in[0, n]
$$

We note that, for each ranked alphabet $\Sigma$ with $\operatorname{maxrk}(\Sigma)=n$ for some $n \in \mathbb{N}$, the skeleton alphabet of $\Sigma$ is a subset of $\Delta_{n}$, i.e., $[\Sigma] \subseteq \Delta_{n}$. For the definition of the skeleton alphabet we refer to page 228,

Lemma 15.3.1. FV22b, Lm. 6.2,6.9] The $\left(\Delta_{n}, \mathrm{~B}\right)$-weighted tree language $\chi\left(\mathrm{T}_{\Delta_{n}}\right)$ is in $\operatorname{Reg}(n, \mathrm{~B})$. Moreover, $\operatorname{Reg}(n, \mathrm{~B})$ is a family of $(n, \mathrm{~B})$-weighted tree languages.

Proof. We construct the $\left(\Delta_{n}, \mathrm{~B}\right)-$ wrtg $\mathcal{G}=(\{*\},\{*\}, R, \mathrm{wt})$ such that, for each $k \in[0, n], R$ contains the rule $r=(* \rightarrow[k](*, \ldots, *))$ with $k$ occurrences of $*$ in the right-hand side and $\operatorname{wt}(r)=\mathbb{1}$. Since $\llbracket \mathcal{G} \rrbracket=\chi\left(\mathrm{T}_{\Delta_{n}}\right)$, we have $\chi\left(\mathrm{T}_{\Delta_{n}}\right) \in \operatorname{Reg}(n, \mathrm{~B})$.

Since $\operatorname{maxrk}\left(\Delta_{n}\right)=n,\left|\Delta_{n}^{(k)}\right|=1$ for each $k \in[0, n]$, and $\chi\left(\mathrm{T}_{\Delta_{n}}\right) \in \operatorname{Reg}(n, \mathrm{~B})$, we have that $\operatorname{Reg}(n, \mathrm{~B})$ is a family of $(n, \mathrm{~B})$-weighted tree languages.

Theorem 15.3.2. FV22b, Thm. 6.8] $\operatorname{Reg}(n, \mathrm{~B})$ is an $A F w t L$.
Proof. By Lemma 15.3.1, the set $\operatorname{Reg}(n, \mathrm{~B})$ is a family of $(n, \mathrm{~B})$-weighted tree languages.
By Theorem 10.13.9, $\operatorname{Reg}(n, \mathrm{~B})$ is closed under wpb , and hence $\operatorname{Reg}(n, \mathrm{~B})$ is a tree cone. We note that the image of an $(n, \mathrm{~B})$-weighted tree language under a wpb is again an $(n, \mathrm{~B})$-weighted tree language.

Let $r, r_{1}, r_{2}, s \in \operatorname{Reg}(\Sigma, \mathrm{~B})$ for some ranked alphabet $\Sigma$ with $\operatorname{maxrk}(\Sigma) \leq n$. Moreover, let $\alpha \in \Sigma^{(0)}$. We require that $s$ is $\alpha$-proper. By Theorems 10.1.1, 10.6.1, and 10.7.4, the ( $\Sigma, \mathrm{B}$ )-weighted tree languages $r_{1} \oplus r_{2}, r_{1} \circ_{\alpha} r_{2}$, and $s_{\alpha}^{*}$ are in $\operatorname{Reg}(\Sigma, \mathrm{B})$, respectively. (We recall that $\operatorname{Rec}(\Sigma, \mathrm{B})=\operatorname{Reg}(\Sigma, \mathrm{B})$ because we assumed that B is a $\sigma$-complete semiring.) Hence $\operatorname{Reg}(n, \mathrm{~B})$ is closed under the rational operations, i.e., under sum, tree concatenations, and Kleene stars. Hence, $\operatorname{Reg}(n, \mathrm{~B})$ is an AFwtL .

### 15.4 The $\operatorname{AFwtL} \operatorname{Reg}(n, \mathrm{~B})$ is principal

We will prove that the $\left(\Delta_{n}, \mathrm{~B}\right)$-weighted tree language $\chi\left(\mathrm{T}_{\Delta_{n}}\right)$ is the generator of $\operatorname{Reg}(n, \mathrm{~B})$.
Lemma 15.4.1. FV22b, Lm. 6.10] Let $\mathcal{G}$ be a $(\Sigma, \mathrm{B})$-wrtg with maxrk $(\Sigma) \leq n$. Then there exists a $\left(\Delta_{n}, \Sigma\right)$-wpb $\mathcal{H}$ such that $\llbracket \mathcal{G} \rrbracket=\llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}\left(\chi\left(\mathrm{T}_{\Delta_{n}}\right)\right)$.

Proof. Let $\mathcal{G}=(N, S, R, w t)$ with $\operatorname{maxrk}(\Sigma) \leq n$. By Lemma 9.2.3, we can assume that $\mathcal{G}$ is in tree automata form. Thus $[R]=[\Sigma]$ and hence $[R] \subseteq \Delta_{n}$ (also for $n=0$ ).

By Lemma 10.13 .6 , there exists a chain-free $([R], \Sigma, \mathrm{B})$-wpb $\mathcal{H}^{\prime}$ such that $\llbracket \mathcal{G} \rrbracket=\llbracket \mathcal{H}^{\prime} \rrbracket^{\mathrm{tt}}\left(\chi\left(\mathrm{T}_{[R]}\right)\right)$. Since $[R] \subseteq \Delta_{n}$, we can view $\mathcal{H}^{\prime}$ as a $\left(\Delta_{n}, \Sigma, \mathrm{~B}\right)$-wbp $\mathcal{H}$. Obviously, $\llbracket \mathcal{H}^{\prime} \rrbracket=\left.\llbracket \mathcal{H} \rrbracket\right|_{\mathrm{T}_{[R]} \times \mathrm{T}_{\Sigma}}$ and, for each $(b, \xi) \in\left(\mathrm{T}_{\Delta_{n}} \times \mathrm{T}_{\Sigma}\right) \backslash\left(\mathrm{T}_{[R]} \times \mathrm{T}_{\Sigma}\right)$, we have that $\llbracket \mathcal{H} \rrbracket(b, \xi)=\mathbb{0}$. Thus $\llbracket \mathcal{H}^{\prime} \rrbracket\left(\chi\left(\mathrm{T}_{[R]}\right)\right)=\llbracket \mathcal{H} \rrbracket\left(\chi\left(\mathrm{T}_{\Delta_{n}}\right)\right)$. Hence $\llbracket \mathcal{G} \rrbracket=\llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}\left(\chi\left(\mathrm{T}_{\Delta_{n}}\right)\right)$.

Lemma 15.4.2. [FV22b, Lm. 6.11] Let $\operatorname{maxrk}(\Sigma) \leq n$ and $r: \mathrm{T}_{\Sigma} \rightarrow B$. The following statements are equivalent.
(A) $r \in \operatorname{Reg}(n, \mathrm{~B})$.
(B) There exists a $\left(\Delta_{n}, \Sigma\right)$-wpb $\mathcal{H}$ such that $r=\llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}\left(\chi\left(\mathrm{T}_{\Delta_{n}}\right)\right)$.
(C) $r \in \mathcal{F}\left(\left\{\chi\left(\mathrm{~T}_{\Delta_{n}}\right)\right\}\right)$.

Proof. Proof of $(\mathrm{A}) \Rightarrow(\mathrm{B})$ : This follows from Lemma 15.4 .1
Proof of $(\mathrm{B}) \Rightarrow(\mathrm{C})$ : This holds by definition of $\mathcal{F}\left(\left\{\chi\left(\mathrm{T}_{\Delta_{n}}\right)\right\}\right)$.
Proof of $(\mathrm{C}) \Rightarrow(\mathrm{A})$ : By Theorem 15.3.2, the set $\operatorname{Reg}(n, \mathrm{~B})$ is an $\operatorname{AFwtL}$. By Lemma 15.3.1, we have that $\chi\left(\mathrm{T}_{\Delta_{n}}\right)$ is in $\operatorname{Reg}(n, \mathrm{~B})$. Since $\mathcal{F}\left(\left\{\chi\left(\mathrm{T}_{\Delta_{n}}\right)\right\}\right)$ is the smallest AFwtL containing $\chi\left(\mathrm{T}_{\Delta_{n}}\right)$, we obtain that $r \in \operatorname{Reg}(n, \mathrm{~B})$.

The equivalence of $(\mathrm{A})$ and $(\mathrm{C})$ of Lemma 15.4 .2 implies the following theorem.
Theorem 15.4.3. FV22b, Thm. 6.12] The AFwtL $\operatorname{Reg}(n, \mathrm{~B})$ is principal with generator $\chi\left(\mathrm{T}_{\Delta_{n}}\right)$.
Next we prove that the set $\operatorname{Reg}(n, \mathrm{~B})$ is contained in each $(n, \mathrm{~B})$-tree cone.
Corollary 15.4.4. FV22b, Cor. 6.13] For each $(n, \mathrm{~B})$-tree cone $\mathcal{L}$ we have $\operatorname{Reg}(n, \mathrm{~B}) \subseteq \mathcal{L}$.
Proof. Let $\mathcal{G}$ by a $(\Sigma, \mathrm{B})-$ wrtg with $\operatorname{maxrk}(\Sigma) \leq n$. By Lemma 15.4.1 there exists a $\left(\Delta_{n}, \Sigma\right)$-wpb $\mathcal{H}$ such that

$$
\llbracket \mathcal{G} \rrbracket=\llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}\left(\chi\left(\mathrm{~T}_{\Delta_{n}}\right)\right) .
$$

Now let $\mathcal{L}$ be an $(n, B)$-tree cone. Since $\mathcal{L}$ is a family of $(n, B)$-weighted tree languages, there exists a ranked alphabet $\Delta$ such that $\operatorname{maxrk}(\Delta)=n,\left|\Delta^{(k)}\right|=1$ for each $k \in[0, n]$, and $\chi\left(\mathrm{T}_{\Delta}\right)$ is in $\mathcal{L}$. The only difference between $\Delta$ and $\Delta_{n}$ is the fact that the symbols look different. Let $\langle k\rangle$ denote the unique element in $\Delta^{(k)}$.

We construct the $\left(\Delta, \Delta_{n}, \mathrm{~B}\right)-\operatorname{wpb} \mathcal{N}=(\{*\},\{*\}, R, \mathrm{wt})$ such that for each $k \in[0, n]$, the rule

$$
* \rightarrow[\langle k\rangle,[k]](*, \ldots, *)
$$

is in $R$ with $k$ occurrences of $*$. Obviously, $\chi\left(\mathrm{T}_{\Delta_{n}}\right)=\llbracket \mathcal{N} \rrbracket^{\mathrm{tt}}\left(\chi\left(\mathrm{T}_{\Delta}\right)\right)$. Hence,

$$
\llbracket \mathcal{G} \rrbracket=\llbracket \mathcal{H} \rrbracket^{\mathrm{tt}}\left(\llbracket \mathcal{N} \rrbracket^{\mathrm{tt}}\left(\chi\left(\mathrm{~T}_{\Delta}\right)\right)\right)
$$

By Theorem 10.13.7 wpb are closed under composition. Hence there exists a $(\Delta, \Sigma)$-wpb $\mathcal{H}^{\prime}$ such that

$$
\llbracket \mathcal{G} \rrbracket=\llbracket \mathcal{H}^{\prime} \rrbracket^{\mathrm{tt}}\left(\chi\left(\mathrm{~T}_{\Delta}\right)\right)
$$

Since $\chi\left(\mathrm{T}_{\Delta}\right)$ is in $\mathcal{L}$ and $\mathcal{L}$ is an $(n, \mathrm{~B})$-tree cone, we obtain that $\llbracket \mathcal{G} \rrbracket$ is in $\mathcal{L}$.

Theorem 15.4.5. Let B be a commutative and $\sigma$-complete semiring and $n \in \mathbb{N}$. Then $\operatorname{Reg}(n, \mathrm{~B})$ is the smallest principal abstract family of ( $n, \mathrm{~B}$ )-weighted tree languages.

Proof. By Theorem 15.4.3, the set $\operatorname{Reg}(n, \mathrm{~B})$ is a principal $(n, \mathrm{~B})-\mathrm{AFwtL}$. Now let $\mathcal{L}$ be an arbitrary principal ( $n, \mathrm{~B}$ )-AFwtL. Since, in particular, $\mathcal{L}$ is an $(n, \mathrm{~B})$-tree cone, by Corollary 15.4.4 we have $\operatorname{Reg}(n, \mathrm{~B}) \subseteq \mathcal{L}$.

Other examples of principal abstract families of $(n, \mathrm{~B})$-weighted tree languages are the members of the infinite family $\left(\operatorname{Reg}\left(\mathrm{P}^{\ell}, n, \mathrm{~B}\right) \mid \ell \in \mathbb{N}\right)\left(\right.$ cf. [FV22b, Thm. 8.1]). Here $\operatorname{Reg}\left(\mathrm{P}^{\ell}, n, \mathrm{~B}\right)$ is the set of all $(n, \mathrm{~B})$ weighted tree languages which are generated by weighted regular tree grammars with additional storage $\mathrm{P}^{\ell}$, the $\ell$-iterated pushdown storage Gre70, Mas76, Eng86. The particular instance $\left(\operatorname{Reg}\left(\mathrm{P}^{\ell}, n\right.\right.$, Boole) | $\ell \in \mathbb{N}$ ) is known as the OI-hierarchy Wan74, Dam82, ES77, ES78, DG82, Eng91, which starts with the set of regular tree languages $(\ell=0)$ and the set of OI context-free tree languages $(\ell=1)$.

## Chapter 16

## Crisp determinization

We recall that, in a crisp deterministic $(\Sigma, \mathrm{B})$-wta $\mathcal{B}$, the weight of each transition is one of the unit weights $\mathbb{O}$ and $\mathbb{1}$ of the strong bimonoid B ; weights different from these units may only occur as root weights. Also, since crisp determinism implies bu determinism, the two semantics of $\mathcal{B}$ coincide, i.e., $\llbracket \mathcal{B} \rrbracket^{\text {init }}=\llbracket \mathcal{B} \rrbracket^{\text {run }}$ (cf. Theorem 5.3.1). We denoted the semantics of $\mathcal{B}$ by $\llbracket \mathcal{B} \rrbracket$.

In this chapter we will investigate the question under which conditions a given wta can be transformed into an equivalent crisp deterministic wta. More precisely, a $(\Sigma, \mathrm{B})$-wta $\mathcal{A}$ is initial algebra crisp determinizable if there exists an i-equivalent crisp deterministic ( $\Sigma, \mathrm{B}$ )-wta, i.e., there exists a crisp deterministic $(\Sigma, \mathrm{B})$-wta $\mathcal{B}$ such that $\llbracket \mathcal{A} \rrbracket^{\text {init }}=\llbracket \mathcal{B} \rrbracket$. It is run crisp determinizable if there exists an r-equivalent crisp deterministic $(\Sigma, \mathrm{B})$-wta, i.e., there exists a crisp deterministic $(\Sigma, \mathrm{B})$-wta $\mathcal{B}$ such that $\llbracket \mathcal{A} \rrbracket^{\mathrm{run}}=\llbracket \mathcal{B} \rrbracket$. It follows from the previous definitions that, if $B$ is semiring, then $\mathcal{A}$ is initial algebra crisp determinizable if and only if it is run crisp determinizable (because by Corollary $5.3 .3 \llbracket \mathcal{A} \rrbracket^{\text {init }}=\llbracket \mathcal{A} \rrbracket^{\text {run }}$ ).

In the sequel we will identify conditions under which a given $(\Sigma, B)$-wta $\mathcal{A}$ is initial algebra crisp determinizable and under which conditions it is run crisp determinizable. We approach the answers by recalling that each crisp deterministic $(\Sigma, B)$-wta $\mathcal{B}$ recognizes a ( $\Sigma, \mathrm{B}$ )-recognizable step mapping, and vice versa, each ( $\Sigma, B$ )-recognizable step mapping can be obtained in this way (cf. Theorem 10.3.1). Hence, $\mathcal{A}$ is initial algebra crisp determinizable if and only if $\llbracket \mathcal{A} \rrbracket^{\text {init }}$ is a recognizable step mapping, i.e., $\operatorname{im}\left(\llbracket \mathcal{A} \rrbracket^{\text {init }}\right)$ is finite and $\llbracket \mathcal{A} \rrbracket^{\text {init }}$ has the preimage property (and similarly for run crisp determinizable).

Thus, Theorem 10.3.1 provides a blue-print for the answers to the crisp determinization problems. On the negative side, if we consider a $(\Sigma, \mathrm{B})$-wta $\mathcal{A}$ for which $\llbracket \mathcal{A} \rrbracket^{\text {init }}$ is not a recognizable step mapping, then $\mathcal{A}$ is not initial algebra crisp determinizable, and similarly for $\llbracket \mathcal{A} \rrbracket^{\text {run }}$ and run crisp determinizability. For instance, let $\mathcal{A}$ be the $\left(\Sigma, N_{\text {at }}{ }_{\text {max },+}\right)$-wta from Example 3.2 .4 for which $\llbracket \mathcal{A} \rrbracket^{\text {init }}=$ height. Since $\operatorname{im}\left(\right.$ height ) is not finite, $\llbracket \mathcal{A} \rrbracket^{\text {init }}$ is not a recognizable step mapping (by definition). Hence, by Theorem 10.3.1 and the fact that Nat $_{\text {max },+}$ is a semiring, $\mathcal{A}$ is neither initial algebra crisp determinizable nor run crisp determinizable. On the positive side, each $(\Sigma, \mathrm{B})$-wta $\mathcal{A}$ for which $\llbracket \mathcal{A} \rrbracket^{\text {init }}$ is a recognizable step mapping, is initial algebra crisp determinizable, and similarly for the run semantics and run crisp determinizability.

In Section 16.1 we will deal with initial algebra crisp determinization and in Section 16.2 with run crisp determinization. This chapter is based on results and constructions in [FKV21, DFKV20, and DFKV22.

### 16.1 Initial algebra crisp determinization

In this section we will show an equivalence between local finiteness and initial algebra crisp determinizability (cf. Theorem 16.1.6).

If we recall the way in which a $(\Sigma, \mathrm{B})$-wta $\mathcal{A}$ computes $\mathrm{h}_{\mathcal{A}}(\xi)$ for some input tree $\xi \in \mathrm{T}_{\Sigma}$, then we see that the summation and multiplication of $B$ are used in an alternating way. One possibility to guarantee that these computations only produce finitely many values in $B$, is to assume that B is locally finite. Indeed, for each locally finite strong bimonoid $B$, the set $\operatorname{Rec}^{\text {init }}(\Sigma, B)$ contains only recognizable step mappings (cf. BMŠ 06, Thm. 9], [DV06, Lm. 6.1], [FV09, Cor. 3.16], and CDIV10, p. 9]). Since each distributive bounded lattice is a particular locally finite semiring (cf. Subsection 2.6.6), the next lemma can also be compared to [Rah09, Thm. 3.8].

Lemma 16.1.1. (cf. FKV21, Cor. 6.10]) Let (a) B be locally finite or (b) let B be weakly locally finite and $\Sigma$ is monadic. Moreover, let $\mathcal{A}$ be a $(\Sigma, \mathrm{B})$-wta. Then $\operatorname{im}\left(\mathrm{h}_{\mathcal{A}}\right)$ is finite and $\mathcal{A}$ is initial algebra crisp determinizable.

Proof. Let $\mathcal{A}=(Q, \delta, F)$. We recall that $\operatorname{im}(\delta)=\bigcup_{k \in \mathbb{N}} \operatorname{im}\left(\delta_{k}\right)$.
First we show that the set $H=\left\{\mathrm{h}_{\mathcal{A}}(\xi)_{q} \mid \xi \in \mathrm{T}_{\Sigma}, q \in Q\right\}$ is finite.
Case (a): Let B be locally finite. Since $H \subseteq\langle\operatorname{im}(\delta)\rangle_{\{\oplus, \otimes\}}$ and $\langle\operatorname{im}(\delta)\rangle_{\{\oplus, \otimes\}}$ is finite, the set $H$ is finite.
Case (b): Let B be weakly locally finite and $\Sigma$ be monadic. By induction on $\mathrm{T}_{\Sigma}$, we prove that the following statement holds (where $\mathrm{Cl}(\mathrm{im}(\delta))$ is defined in Subsection 2.6.5).

$$
\begin{equation*}
\text { For every } \xi \in \mathrm{T}_{\Sigma} \text { and } q \in Q \text {, we have: } \mathrm{h}_{\mathcal{A}}(\xi)_{q} \in \mathrm{Cl}(\operatorname{im}(\delta)) \tag{16.1}
\end{equation*}
$$

The proof for $\xi \in \Sigma^{(0)}$ is trivial. Let $\xi=\sigma\left(\xi_{1}\right)$. Then $\mathrm{h}_{\mathcal{A}}\left(\xi_{1}\right)_{p} \otimes \delta_{1}(p, \sigma, q) \in \mathrm{Cl}(\mathrm{im}(\delta))$ for each $p \in Q$ because $\mathrm{h}_{\mathcal{A}}\left(\xi_{1}\right)_{p} \in \mathrm{Cl}(\operatorname{im}(\delta))$ by I.H. and $\delta_{1}(p, \sigma, q) \in \operatorname{im}(\delta)$. Moreover, $\mathrm{h}_{\mathcal{A}}(\xi)_{q}=\bigoplus_{p \in Q} \mathrm{~h}_{\mathcal{A}}\left(\xi_{1}\right)_{p} \otimes$ $\delta_{1}(p, \sigma, q)$ is also in $\mathrm{Cl}(\operatorname{im}(\delta))$ because $\mathrm{Cl}(\operatorname{im}(\delta))$ is closed under $\oplus$. This finishes the proof of (16.1).

Due to (16.1), we obtain that $H \subseteq \mathrm{Cl}(\operatorname{im}(\delta))$. Since B is weakly locally finite, we have that $H$ is finite.
Hence, in both Cases (a) and (b), the set $H$ is finite. Then the set $\operatorname{im}\left(\mathrm{h}_{\mathcal{A}}\right)$ is also finite because $\mathrm{h}_{\mathcal{A}}(\xi) \in H^{Q}$ for every $\xi \in \mathrm{T}_{\Sigma}$. Hence the congruence $\operatorname{ker}\left(\mathrm{h}_{\mathcal{A}}\right)$ of $\mathcal{A}$ has finite index. Thus, by Theorem $4.3 .5(\mathrm{C}) \Rightarrow(\mathrm{B})$, the $(\Sigma, \mathrm{B})$-wta $\mathcal{A}$ is initial algebra crisp determinizable.

Lemma 16.1.1 states that each ( $\Sigma, \mathrm{B}$ )-wta is initial algebra crisp determinizable if one of its conditions (a) and (b) holds. In fact, if one traces back the proofs of Lemma 16.1.1, Theorem 4.3.5, and Lemma 4.3.4 then one can construct the corresponding crisp deterministic wta. Since its construction is rather hidden, we want to show this explicitly here (cf. Definition 16.1.2).

In fact, the construction of the corresponding crisp deterministic wta generalizes the well-known subset method which transforms an fta into an equivalent bu deterministic fta (cf. TW68, Thm. 1], Eng75b, Thm. 3.8], and [GS84, Thm. 2.2.6]). The set of states of the bu deterministic fta is the set of all subsets of the state set of the given fta. Here, for a given wta $\mathcal{A}=(Q, \delta, F)$, we represent the states of the equivalent bu deterministic wta by vectors in $B^{Q}$. However, we do not use all vectors in $B^{Q}$, but only those which are images of trees under $\mathrm{h}_{\mathcal{A}}$. This corresponds to using the reachable subsets in the subset method. We generalize the subset method for any wta and strong bimonoid $B$, and we keep the name "subset method" also for the generalized version.

A special case of this subset method was given in the proof of Běl02, Thm. 2.1] for the case of string ranked alphabets (i.e., weighted automata) and complete distributive lattices. (We note that distributivity is not assumed in [Běl02, but it is needed because the many-fold composition of matrices in the definition of the acceptance degree, i.e., run semantics, assumes associativity of matrix multiplication, and in its turn, this assumes distributivity [Běl08]). In fact, the notion of deterministic fuzzy automata in Běl02] corresponds to our notion of crisp deterministic wta over some string ranked alphabet and some complete distributive lattice due to Lemma 3.3.3 and Theorem4.3.5(A) $\Leftrightarrow(\mathrm{B})$. In BLB10, Thm. 5 and 8] the subset method was presented for wta over Unitlnt ${ }_{u, i}$ where ( $u, i$ ) are particular pairs of t-conorm $u$ and t-norm $i$. Another special case of the subset method is given in [MZA11, Thm. 3.5] for wta over ( $[0,1], \nabla, \Delta, 0,1$ ) where $\nabla$ and $\Delta$ are some t-conorm and t-norm, respectively, such that $(\nabla, \Delta)$ are finite range. The latter property implies that $([0,1], \nabla, \Delta, 0,1)$ is locally finite (but not vice versa).

Definition 16.1.2. Let $\Sigma$ be a ranked alphabet and $\mathrm{B}=(B, \oplus, \otimes, \mathbb{O}, \mathbb{1})$ be a strong bimonoid. Moreover, let $\mathcal{A}$ be a $(\Sigma, \mathrm{B})$-wta. The subset method transforms $\mathcal{A}$ into the triple $\operatorname{sub}(\mathcal{A})=\left(Q^{\prime}, \delta^{\prime}, F^{\prime}\right)$, where

- $Q^{\prime}=\operatorname{im}\left(\mathrm{h}_{\mathcal{A}}\right)$,
- $\delta^{\prime}=\left(\delta_{k}^{\prime} \mid k \in \mathbb{N}\right)$ is the family of mappings $\delta_{k}^{\prime}:\left(Q^{\prime}\right)^{k} \times \Sigma^{(k)} \times Q^{\prime} \rightarrow B$ defined by

$$
\delta_{k}^{\prime}\left(u_{1} \ldots u_{k}, \sigma, u\right)= \begin{cases}\mathbb{1} & \text { if } \delta_{\mathcal{A}}(\sigma)\left(u_{1}, \ldots, u_{k}\right)=u \\ \mathbb{0} & \text { otherwise }\end{cases}
$$

for every $k \in \mathbb{N}, \sigma \in \Sigma^{(k)}, u_{1}, \ldots, u_{k} \in Q^{\prime}$, and $u \in Q^{\prime}$,

- $F^{\prime}: Q^{\prime} \rightarrow B$ is a mapping defined by $\left(F^{\prime}\right)_{u}=\bigoplus_{q \in Q} u_{q} \otimes F_{q}$ for every $u \in Q^{\prime}$.

Thus, $\operatorname{sub}(\mathcal{A})$ is merely a triple and, in general, not a wta. In particular, $Q^{\prime}$ can be an infinite set. However, $\operatorname{sub}(\mathcal{A})$ is a crisp deterministic wta if and only if $Q^{\prime}$ is finite. But assuming one of the conditions of Lemma 16.1.1, we can prove that $\operatorname{sub}(\mathcal{A})$ is a crisp deterministic wta which is i-equivalent to $\mathcal{A}$.

Theorem 16.1.3. Let $\Sigma$ be a ranked alphabet and $\mathrm{B}=(B, \oplus, \otimes, \mathbb{0}, \mathbb{1})$ be a strong bimonoid such that (a) B is locally finite or (b) B is weakly locally finite and $\Sigma$ is monadic. Then the following two statements hold.
(1) For each $(\Sigma, \mathrm{B})-w t a \mathcal{A}$, the triple $\operatorname{sub}(\mathcal{A})$ can be constructed; moreover, $\operatorname{sub}(\mathcal{A})$ is a crisp deterministic $(\Sigma, B)-w t a$ and $\llbracket \mathcal{A} \rrbracket^{\text {init }}=\llbracket \operatorname{sub}(\mathcal{A}) \rrbracket$.
(2) $\operatorname{Rec}^{\mathrm{init}}(\Sigma, \mathrm{B}) \subseteq \operatorname{Rec}^{\mathrm{run}}(\Sigma, \mathrm{B})$.

Proof. Proof of (1): By Observation [2.9.4, we have $\operatorname{im}\left(\mathrm{h}_{\mathcal{A}}\right)=\langle\emptyset\rangle_{\delta_{\mathcal{A}}(\Sigma)}$ and by Lemma 16.1.1] the set $\operatorname{im}\left(\mathrm{h}_{\mathcal{A}}\right)$ is finite. Hence $\langle\emptyset\rangle_{\delta_{\mathcal{A}}(\Sigma)}$ is finite and by Lemma 2.6.1, it can be constructed. Thus im $\left(\mathrm{h}_{\mathcal{A}}\right)$ can be constructed. It is easy to see that $\operatorname{sub}(\mathcal{A})$ is a crisp deterministic $(\Sigma, B)$-wta.

Moreover, the subset method is the composition of the constructions in the proofs of Theorem 4.3.5(C) $\Rightarrow(\mathrm{A})$ and Lemma 4.3.4. Indeed, in Theorem 4.3.5(C) $\Rightarrow$ (A) we construct the finite algebra $\left(\operatorname{im}\left(\mathrm{h}_{\mathcal{A}}\right), \delta_{\mathrm{aV}(\mathcal{A})}\right)$ and the mapping $F^{\prime}$. Then, using this as input, Lemma 4.3.4 constructs the crisp deterministic $(\Sigma, \mathrm{B})$-wta $\left(\operatorname{im}\left(\mathrm{h}_{\mathcal{A}}\right), \delta^{\prime}, F^{\prime}\right)$, which is nothing else but $\operatorname{sub}(\mathcal{A})$. Hence it follows that $\llbracket \mathcal{A} \rrbracket^{\text {init }}=\llbracket \operatorname{sub}(\mathcal{A}) \rrbracket$.

Proof of (2): Let $\mathcal{A}$ be a $(\Sigma, \mathrm{B})$-wta. By (1) we have $\llbracket \mathcal{A} \rrbracket{ }^{\text {init }}=\llbracket \operatorname{sub}(\mathcal{A}) \rrbracket$, where $\llbracket \operatorname{sub}(\mathcal{A}) \rrbracket$ is a crisp deterministic $(\Sigma, \mathrm{B})$-wta. Since, by Theorem 5.3.1 $\llbracket \operatorname{sub}(\mathcal{A}) \rrbracket=\llbracket \operatorname{sub}(\mathcal{A}) \rrbracket^{\text {run }}$, we have $\operatorname{Rec}^{\text {init }}(\Sigma, \mathrm{B}) \subseteq$ $\operatorname{Rec}^{\text {run }}(\Sigma, \mathrm{B})$.

As a consequence of Theorem 16.1.3, we show a preimage theorem. In general, a preimage theorem states, for some modifier $x \in\{$ run, init $\}$, under which conditions on B , for each $(\Sigma, \mathrm{B})$-wta $\mathcal{A}$, the weighted tree language $\llbracket \mathcal{A} \rrbracket^{x}$ has the preimage property (i.e., for each $b \in B$ we have that $\left(\llbracket \mathcal{A} \rrbracket^{x}\right)^{-1}(b)$ is a recognizable tree language, cf. Section 2.10.2). Clearly, if B is a semiring, then we can disregard the modifier $x$ (cf. Corollary 5.3.3). The next preimage theorem concerns the initial algebra semantics (i.e., $x=$ init).

Corollary 16.1.4. Let (a) B be locally finite or (b) let B be weakly locally finite and $\Sigma$ monadic. Moreover, let $\mathcal{A}$ be a $(\Sigma, \mathrm{B})$-wta. Then $\llbracket \mathcal{A} \rrbracket^{\text {init }}$ has the preimage property. Moreover, for each $b \in B$, we can construct a $\Sigma$-fta $A$ such that $\mathrm{L}(A)=\left(\llbracket \mathcal{A} \rrbracket^{\text {init }}\right)^{-1}(b)$.

Proof. By Theorem 16.1.3 we can construct the crisp deterministic $(\Sigma, \mathrm{B})$-wta $\operatorname{sub}(\mathcal{A})$ such that $\llbracket \mathcal{A} \rrbracket^{\text {init }}=$ $\llbracket \operatorname{sub}(\mathcal{A}) \rrbracket$. Then our statements follow from Theorem $10.3 .1(\mathrm{~A}) \Rightarrow(\mathrm{C})$.

A special case of Corollary 16.1.4(a) was proved in DV06, Lm. 6.1] for locally finite commutative semirings (taking into account that for wta over semirings the run semantics and the initial algebra
semantics coincide cf. Corollary 5.3.3(2)).
Also a kind of reverse of Lemma 16.1.1 holds.
Corollary 16.1.5. Rad10, Lm. 6.1] If for each ranked alphabet $\Sigma$, each ( $\Sigma, \mathrm{B}$ )-wta $\mathcal{A}$ is initial algebra crisp determinizable, then B is locally finite.

Proof. Let $A \subseteq B$ be a finite set. Let $\Sigma$ be a ranked alphabet such that $\left|\Sigma^{(0)}\right| \geq|A \cup\{\mathbb{0}, \mathbb{1}\}|$ and $\left|\Sigma^{(2)}\right| \geq 2$. By Theorem 3.1.5, we can construct a $(\Sigma, \mathrm{B})$-wta $\mathcal{A}$ such that $\operatorname{im}\left(\llbracket \mathcal{A} \rrbracket^{\text {init }}\right)=\langle A\rangle_{\{\oplus, \otimes, 0, \mathbb{1}\}}$. Since, by assumption, $\mathcal{A}$ is initial algebra crisp determinizable, we have that $\llbracket \mathcal{A} \rrbracket^{\text {init }}$ is a recognizable step mapping (by Theorem 10.3.1). Thus the set $\operatorname{im}\left(\llbracket \mathcal{A} \rrbracket^{\text {init }}\right.$ ) is finite (by definition). Hence $\langle A\rangle_{\{\oplus, \otimes, 0, \mathbb{1}\}}$ is finite, which means that $B$ is locally finite.

Hence we obtain the following equivalence between local finiteness and initial algebra crisp determinizability (where the equivalence with Statement (C) is due to Dro22).

Theorem 16.1.6. Let B be a strong bimonoid. Then the following three statements are equivalent.
(A) B is locally finite.
(B) For each ranked alphabet $\Sigma$ and each $(\Sigma, \mathrm{B})$-wta $\mathcal{A}$, we can construct a crisp deterministic ( $\Sigma, \mathrm{B}$ )wta $\mathcal{B}$ such that $\llbracket \mathcal{A} \rrbracket^{\text {init }}=\llbracket \mathcal{B} \rrbracket$.
(C) For each ranked alphabet $\Sigma$ and each $(\Sigma, \mathrm{B})-w t a \mathcal{A}$, the set $\operatorname{im}\left(\llbracket \mathcal{A} \rrbracket^{\mathrm{init}}\right)$ is finite.

Proof. Proof of $(\mathrm{A}) \Rightarrow(\mathrm{B})$ : This follows from Theorem 16.1.3.
Proof of $(\mathrm{B}) \Rightarrow(\mathrm{C})$ : This follows from Theorem 10.3.1
Proof of $(C) \Rightarrow(A)$ : We prove by contraposition. Assume that $B$ is not locally finite. Then there exists a finite subset $A \subseteq B$ such that $\langle A\rangle_{\{\oplus, \otimes, \mathbb{Q}, \mathbb{\mathbb { 1 }},}$ is infinite. Let $\Sigma$ be a ranked alphabet such that $\left|\Sigma^{(0)}\right| \geq|A \cup\{\mathbb{O}, \mathbb{1}\}|$ and $\left|\Sigma^{(2)}\right| \geq 2$. By Theorem 3.1.5, we can construct a ( $\Sigma, \mathrm{B}$ )-wta $\mathcal{A}$ such that $\operatorname{im}\left(\llbracket \mathcal{A} \rrbracket^{\text {init }}\right)=\langle A\rangle_{\{\oplus, \otimes, 0, \mathbb{1}\}}$. Hence (C) does not hold.

A special case of Theorem 16.1.6 was proved in Běl02, Thm. 2.1] for the case of string ranked alphabets (i.e., weighted automata) and complete distributive lattices (cf. the remark on p 328). Moreover, it is easy to see that the Viterbi semiring $([0,1], \max , \cdot, 0,1)$ is not locally finite. Hence, by Theorem 16.1.6, there exists a ranked alphabet $\Sigma$ and a $(\Sigma$, Viterbi)-wta $\mathcal{A}$ such that $\operatorname{im}(\llbracket \mathcal{A} \rrbracket)$ is infinite. The wsa in LP05, Ex. 3.1] can be thought of as such a wta. Also we note that LP05. Thm. 3.4] is a special case of Theorem 16.1.6(A) $\Leftrightarrow(\mathrm{B})$ for lattice-ordered monoids.

### 16.2 Run crisp determinization

In this section we will prove a sufficient condition under which a $(\Sigma, \mathrm{B})$-wta $\mathcal{A}$ is run crisp determinizable (cf. Theorem 16.2.6). We will continue with two characterizations of this property (cf. Theorems 16.2.7 and 16.2.13), where the first one deals with arbitrary strong bimonoids and the second one with past-finite monotonic strong bimonoids.

### 16.2.1 Sufficient condition for run crisp determinizability

Let $\mathcal{A}=(Q, \delta, F)$ be a $(\Sigma, \mathrm{B})$-wta. We recall that, for each $\xi \in \mathrm{T}_{\Sigma}$, we have

$$
\llbracket \mathcal{A} \rrbracket^{\mathrm{run}}(\xi)=\bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}}(\xi)} \mathrm{wt}(\xi, \rho) \otimes F_{\rho(\varepsilon)}
$$

In contrast to the initial algebra semantics, multiplication and summation are not used alternatingly but first multiplication is used (for the computation of $\left.\mathrm{wt}(\xi, \rho) \otimes F_{\rho(\varepsilon)}\right)$ and second summation is used. In the following we will formulate a sufficient condition for run crisp determinizability of $\mathcal{A}$ which is based on the values which occur in the computation of $\llbracket \mathcal{A} \rrbracket^{\text {run }}$ and on structural properties of $\mathcal{A}$.

For the following discussion we recall the definition $\mathrm{H}(\mathcal{A})=\left\{\mathrm{wt}(\xi, \rho) \mid \xi \in \mathrm{T}_{\Sigma}, \rho \in \mathrm{R}_{\mathcal{A}}(\xi)\right\}$ (cf. 7.15 of Section (7.4) and define the set

$$
\begin{equation*}
\mathrm{C}(\mathcal{A})=\left\{\mathrm{wt}(\xi, \rho) \otimes F_{\rho(\varepsilon)} \mid \xi \in \mathrm{T}_{\Sigma}, \rho \in \mathrm{R}_{\mathcal{A}}(\xi)\right\} \tag{16.2}
\end{equation*}
$$

Clearly, if the set $\mathrm{H}(\mathcal{A})$ is finite, then $\mathrm{C}(\mathcal{A})$ is also finite, because $\mathrm{C}(\mathcal{A}) \subseteq \mathrm{H}(\mathcal{A}) \otimes \operatorname{im}(F)$.
Lemma 16.2.1. DFKV22, Lm. 6.5] Let $\mathcal{A}=(Q, \delta, F)$ be a $(\Sigma, \mathrm{B})$-wta. If $\mathrm{H}(\mathcal{A})$ is finite, then we can construct the set $\mathrm{C}(\mathcal{A})$.

Proof. Let $\mathrm{H}(\mathcal{A})$ be finite. By Lemma 7.4.1, we construct the set $\mathrm{H}(\mathcal{A})$. We recall that in the proof of that lemma, for every $n \in \mathbb{N}$ and $q \in Q$, we construct the set

$$
H_{n, q}=\left\{\mathrm{wt}(\xi, \rho) \mid \xi \in \mathrm{T}_{\Sigma}, \operatorname{height}(\xi) \leq n, \rho \in \mathrm{R}_{\mathcal{A}}(q, \xi)\right\}
$$

and denote by $n_{m} \in \mathbb{N}$ the least number such that $H_{n_{m}, q}=H_{n_{m}+1, q}$ for each $q \in Q$.
Now we prove that the set $\mathrm{C}(\mathcal{A})$ can be constructed. Let $n_{m}$ be the number as before and

$$
C=\left\{\operatorname{wt}(\xi, \rho) \otimes F_{\rho(\varepsilon)} \mid \xi \in \mathrm{T}_{\Sigma}, \operatorname{height}(\xi) \leq n_{m}, \rho \in \mathrm{R}_{\mathcal{A}}(\xi)\right\}
$$

It suffices to show that $C=\mathrm{C}(\mathcal{A})$ because we can construct the set $C$. It is obvious that $C \subseteq \mathrm{C}(\mathcal{A})$. For the proof of the other inclusion, let $b \in \mathrm{C}(\mathcal{A})$, i.e., $b=\mathrm{wt}(\xi, \rho) \otimes F_{q}$ for some $\xi \in \mathrm{T}_{\Sigma}, q \in Q$, and $\rho \in \mathrm{R}_{\mathcal{A}}(q, \xi)$. Since $\operatorname{wt}(\xi, \rho) \in \mathrm{H}(\mathcal{A})$, by the proof of constructing the set $\mathrm{H}(\mathcal{A})$ (cf. Lemma 7.4.1), we have $\operatorname{wt}(\xi, \rho) \in H_{n_{m}, q}$, i.e., there exist $\xi^{\prime} \in \mathrm{T}_{\Sigma}$ with height $\left(\xi^{\prime}\right) \leq n_{m}$, and $\rho^{\prime} \in \mathrm{R}_{\mathcal{A}}\left(q, \xi^{\prime}\right)$ such that $\mathrm{wt}(\xi, \rho)=\mathrm{wt}\left(\xi^{\prime}, \rho^{\prime}\right)$. Hence $b \in C$.

We continue with an analysis of $\operatorname{im}\left(\llbracket \mathcal{A} \rrbracket^{\text {run }}\right)$. We can achieve that $\operatorname{im}\left(\llbracket \mathcal{A} \rrbracket^{\text {run }}\right)$ is finite if we guarantee that (a) the set $\mathrm{C}(\mathcal{A})$ is finite and (b) there exists an upper bound $K$ such that each $b \in \mathrm{C}(\mathcal{A})$ is summed up at most $K$ times. To express this more precisely, for each $b \in \mathrm{C}(\mathcal{A})$, we define the mapping $f_{\mathcal{A}, b}: \mathrm{T}_{\Sigma} \rightarrow \mathbb{N}$, called complete run number mapping of $b$, such that for each $\xi \in \mathrm{T}_{\Sigma}$ we let

$$
f_{\mathcal{A}, b}(\xi)=\left|\left\{\rho \in \mathrm{R}_{\mathcal{A}}(\xi) \mid \operatorname{wt}(\xi, \rho) \otimes F_{\rho(\varepsilon)}=b\right\}\right|
$$

We recall from (2.15) that, for each $n \in \mathbb{N}$, the value $n b$ is the sum $b \oplus \ldots \oplus b$ with $n$ summands. Finally, we define the $(\Sigma, B)$-weighted tree language

$$
r_{\mathcal{A}, b}: \mathrm{T}_{\Sigma} \rightarrow B \text { with } r_{\mathcal{A}, b}(\xi)=\left(f_{\mathcal{A}, b}(\xi)\right) b \text { for each } \xi \in \mathrm{T}_{\Sigma}
$$

Thus, if $\mathrm{C}(\mathcal{A})$ is finite, then for each $\xi \in \mathrm{T}_{\Sigma}$, we have

$$
\begin{equation*}
\llbracket \mathcal{A} \rrbracket^{\mathrm{run}}(\xi)=\bigoplus_{b \in \mathrm{C}(\mathcal{A})}\left(f_{\mathcal{A}, b}(\xi)\right) b=\bigoplus_{b \in \mathrm{C}(\mathcal{A})} r_{\mathcal{A}, b}(\xi) \tag{16.3}
\end{equation*}
$$

Overall, if $\mathrm{C}(\mathcal{A})$ is finite and $r_{\mathcal{A}, b}$ is a recognizable step mapping for each $b \in \mathrm{C}(\mathcal{A})$, then $\llbracket \mathcal{A} \rrbracket^{\text {run }}$ is a recognizable step mapping (because recognizable step mappings are closed under sum, cf. Corollary 10.3.2).

In order to find out under which conditions $r_{\mathcal{A}, b}$ is a recognizable step mapping (cf. Lemma 16.2.5), we first prove that $f_{\mathcal{A}, b}$ is an r-recognizable $(\Sigma, N)$-weighted tree language. For this we use the set $\mathrm{H}(\mathcal{A})$. The next lemma generalizes [DSV10, Thm. 11] and [DGMM11, Thm. 6.2(a)] from the string case to the tree case.

Lemma 16.2.2. DFKV22, Thm. 6.6] Let $\mathcal{A}$ be a $(\Sigma, \mathrm{B})$-wta. If $\mathrm{H}(\mathcal{A})$ is finite, then for each $b \in \mathrm{C}(\mathcal{A})$ we can construct a $(\Sigma, \mathbb{N})$-wta $\mathcal{A}_{b}^{\prime}$ such that $\llbracket \mathcal{A}_{b}^{\prime} \rrbracket=f_{\mathcal{A}, b}$.

Proof. Let $\mathrm{H}(\mathcal{A})$ be finite. By Lemma 7.4.1 we construct $\mathrm{H}(\mathcal{A})$. Let $b \in \mathrm{C}(\mathcal{A})$. We define the $(\Sigma, \mathbb{N})$-wta $\mathcal{A}_{b}^{\prime}=\left(Q^{\prime}, \delta^{\prime}, F_{b}^{\prime}\right)$ as follows: $Q^{\prime}=Q \times \mathrm{H}(\mathcal{A})$ and for every $k \in \mathbb{N}, \sigma \in \Sigma^{(k)}$ and $\left(q_{1}, y_{1}\right), \ldots,\left(q_{k}, y_{k}\right),(q, y) \in$ $Q^{\prime}$, let

$$
\delta_{k}^{\prime}\left(\left(q_{1}, y_{1}\right) \cdots\left(q_{k}, y_{k}\right), \sigma,(q, y)\right)= \begin{cases}1 & \text { if }\left(\bigotimes_{i \in[k]} y_{i}\right) \otimes \delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right)=y \\ 0 & \text { otherwise }\end{cases}
$$

and let

$$
\left(F_{b}^{\prime}\right)_{(q, y)}= \begin{cases}1 & \text { if } y \otimes F_{q}=b \\ 0 & \text { otherwise }\end{cases}
$$

Let $\xi \in \mathrm{T}_{\Sigma}$ and $b \in \mathrm{C}(\mathcal{A})$. It is obvious that,

$$
\begin{equation*}
\text { for each } \rho \in \mathrm{R}_{\mathcal{A}_{b}^{\prime}}(\xi), \text { we have } \mathrm{wt}_{\mathcal{A}_{b}^{\prime}}(\xi, \rho) \cdot\left(F_{b}^{\prime}\right)_{\rho^{\prime}(\varepsilon)} \in\{0,1\} \tag{16.4}
\end{equation*}
$$

Next we observe that there exists a bijection between the two sets

$$
\left.\left\{\rho \in \mathrm{R}_{\mathcal{A}}(\xi) \mid \mathrm{wt}_{\mathcal{A}}(\xi, \rho) \otimes F_{\rho(\varepsilon)}=b\right\} \quad \text { and } \quad\left\{\rho^{\prime} \in \mathrm{R}_{\mathcal{A}_{b}^{\prime}}(\xi)\right) \mid \mathrm{wt}_{\mathcal{A}_{b}^{\prime}}\left(\xi, \rho^{\prime}\right) \cdot\left(F_{b}^{\prime}\right)_{\rho^{\prime}(\varepsilon)}=1\right\}
$$

Then it follows that

$$
\begin{align*}
f_{\mathcal{A}, b}(\xi)= & \left|\left\{\rho \in \mathrm{R}_{\mathcal{A}}(\xi) \mid \mathrm{wt}_{\mathcal{A}}(\xi, \rho) \otimes F_{\rho(\varepsilon)}=b\right\}\right| \\
= & \left|\left\{\rho^{\prime} \in \mathrm{R}_{\mathcal{A}_{b}^{\prime}}(\xi) \mid \mathrm{wt}_{\mathcal{A}_{b}^{\prime}}\left(\xi, \rho^{\prime}\right) \cdot\left(F_{b}^{\prime}\right)_{\rho^{\prime}(\varepsilon)}=1\right\}\right| \\
= & \bigoplus_{\substack{\rho^{\prime} \in \mathrm{R}_{\mathcal{A}_{b}^{\prime}}(\xi):}} 1 \\
& \mathrm{wt}_{\mathcal{A}_{b}^{\prime}\left(\xi, \rho^{\prime}\right) \cdot\left(F_{b}^{\prime}\right)_{\rho^{\prime}(\varepsilon)}=1}  \tag{16.4}\\
= & \bigoplus_{\rho^{\prime} \in \mathrm{R}_{\mathcal{A}_{b}^{\prime}}(\xi)} \mathrm{wt}_{\mathcal{A}_{b}^{\prime}}\left(\xi, \rho^{\prime}\right) \cdot\left(F_{b}^{\prime}\right)_{\rho^{\prime}(\varepsilon)} \\
= & \llbracket \mathcal{A}_{b}^{\prime} \rrbracket(\xi)
\end{align*}
$$

Next we analyse the preimage of $f_{\mathcal{A}, b}=\llbracket \mathcal{A}_{b}^{\prime} \rrbracket$, where $\mathcal{A}_{b}^{\prime}$ is the ( $\Sigma$, Nat)-wta of Lemma 16.2.2 For this we prove two preimage theorems for the case that $B$ is the semiring of natural numbers (cf. Lemmas 16.2 .3 and 16.2 .4 ) where the first one is generalization of [BR88, Cor. III.2.5].

Lemma 16.2.3. (cf. DV06, Lm. 6.3(2)] and DFKV22, Lm. 6.3]) Let $\mathcal{A}$ be a ( $\Sigma$, Nat)-wta. Then $\llbracket \mathcal{A} \rrbracket$ has the preimage property and, for each $n \in \mathbb{N}$, we can construct a $\Sigma$-fta $A$ such that $\mathrm{L}(A)=\llbracket \mathcal{A} \rrbracket^{-1}(n)$.

Proof. Let $n \in \mathbb{N}$ and $M=\{k \in \mathbb{N} \mid k>n\}$. Moreover, we let $\sim$ be the equivalence relation on the set $\mathbb{N}$ defined such that its $n+2$ classes are the singleton sets $\{k\}$ for each $k \in[0, n]$ and the set $M$.

As is well known, $\sim$ is a congruence relation on the semiring of natural numbers, which can be seen as follows. Let $n_{1}, n_{2}, n_{1}^{\prime}, n_{2}^{\prime} \in \mathbb{N}$ such that $n_{1} \sim n_{1}^{\prime}$ and $n_{2} \sim n_{2}^{\prime}$. Since, for each $k \in \mathbb{N}$ with $k \leq n$, the equivalence class $\{k\}$ is a singleton and $N$ at is commutative, the only interesting case arises if $n_{1}, n_{1}^{\prime} \in M$. So we assume that $n_{1}, n_{1}^{\prime} \in M$. For the summation, we obviously have $n_{1} \leq n_{1}+n_{2}$ and $n_{1}^{\prime} \leq n_{1}^{\prime}+n_{2}^{\prime}$, and hence $n_{1}+n_{2} \in M$ and $n_{1}^{\prime}+n_{2}^{\prime} \in M$. For the multiplication, if $n_{2} \neq 0 \neq n_{2}^{\prime}$, then similarly we obtain that $n_{1} \cdot n_{2} \in M$ and $n_{1}^{\prime} \cdot n_{2}^{\prime} \in M$. If $n_{2}=0=n_{2}^{\prime}$, then $n_{1} \cdot n_{2} \in\{0\}$ and $n_{1}^{\prime} \cdot n_{2}^{\prime} \in\{0\}$. Hence $\sim$ is a congruence relation on Nat.

Since $\sim$ has finite index, the quotient semiring Nat/ $\sim$ is finite. Let $h: \mathbb{N} \rightarrow \mathbb{N} / \sim$ be the canonical semiring homomorphism, i.e., $h(n)=[n]_{\sim}$ for each $n \in \mathbb{N}$. Then, by Theorem 10.9.3, the ( $\Sigma$, Nat/ $)$ weighted tree language $h \circ \llbracket \mathcal{A} \rrbracket$ is recognizable, and by Corollary 16.1 .4 we have that $(h \circ \llbracket \mathcal{A} \rrbracket)^{-1}(\{n\})$ is
a $\Sigma$-recognizable tree language for each $n \in \mathbb{N}$. Since $(h \circ \llbracket \mathcal{A} \rrbracket)^{-1}(\{n\})=\llbracket \mathcal{A} \rrbracket^{-1}(n)$ for each $n \in \mathbb{N}$, we obtain that $\llbracket \mathcal{A} \rrbracket$ has the preimage property.

Clearly, we can give effectively the congruence classes of $\sim$, i.e., the elements of $\mathbb{N} / \sim$, by choosing only one representative for each congruence class. By Lemma 10.9 .2 and the obvious fact that $h(\mathcal{A})$ is constructable, we can construct the $(\Sigma, \operatorname{Nat} / \sim)-$ wta $h(\mathcal{A})$ such that $\llbracket h(\mathcal{A}) \rrbracket=h \circ \llbracket \mathcal{A} \rrbracket$. Since $\mathbb{N} / \sim$ is finite, by Corollary16.1.4 we can construct a finite-state $\Sigma$-tree automaton $A$ which recognizes $\llbracket h(\mathcal{A}) \rrbracket^{-1}(\{n\})=$ $\llbracket \mathcal{A} \rrbracket^{-1}(n)$.

In the next lemma we deal with the preimage of a set under a recognizable weighted tree language over the semiring of natural numbers. For the sake of simplicity, we call this theorem also preimage theorem. It is a generalization of [BR88, Cor. III.2.4]. For every $m \in \mathbb{N}$ and $n \in \mathbb{N}_{+}$, we define

$$
m+n \cdot \mathbb{N}=\{m+n \cdot j \mid j \in \mathbb{N}\}
$$

Moreover, for each $m \in \mathbb{N}$, we let $\bar{m}=m+n \mathbb{N}$, the equivalence class of $m$ modulo $n$. The semiring of natural numbers modulo $n$ is the semiring Nat $\left./ n \mathrm{Nat}=\left(\{\overline{0}, \ldots, \overline{n-1}\},{ }_{n},{ }_{n}, \overline{0}, \overline{1}\right\}\right)$, where for every $k, \ell \in[0, n-1]$ we define $\bar{k}+{ }_{n} \bar{\ell}=\overline{k+\ell \bmod (n)}$ and $\bar{k} \cdot{ }_{n} \bar{\ell}=\overline{k \cdot \ell \bmod (n)}$.

Lemma 16.2.4. (cf. [DV06, Lm. 6.3(2)] and DFKV22, Lm. 6.4]) Let $\mathcal{A}$ be a ( $\Sigma$, Nat)-wta. Moreover, let $m \in \mathbb{N}$ and $n \in \mathbb{N}_{+}$. Then (a) the $\Sigma$-tree language $\llbracket \mathcal{A} \rrbracket^{-1}(m+n \cdot \mathbb{N})$ is recognizable and (b) we can construct a $\Sigma$-fta $A$ such that $\mathrm{L}(A)=\llbracket \mathcal{A} \rrbracket^{-1}(m+n \cdot \mathbb{N})$.

Proof. Proof of (a): Let $m \in \mathbb{N}$ and $n \in \mathbb{N}_{+}$. If $m<n$, then, by Theorem 10.9.3, $(h \circ \llbracket \mathcal{A} \rrbracket) \in$ $\operatorname{Rec}(\Sigma$, $\mathrm{Nat} / n \mathrm{Nat})$, where $h: \mathbb{N} \rightarrow \mathbb{N} / n \mathbb{N}$ is the canonical semiring homomorphism. Moreover, $\llbracket \mathcal{A} \rrbracket^{-1}(m+n \cdot \mathbb{N})=\llbracket \mathcal{A} \rrbracket^{-1}\left(h^{-1}(\bar{m})\right)=(h \circ \llbracket \mathcal{A} \rrbracket)^{-1}(\bar{m})$. Since Nat/ $n$ Nat is a finite semiring, by Corollary 16.1.4, the $\Sigma$-tree language $(h \circ \llbracket \mathcal{A} \rrbracket)^{-1}(\bar{m})$ is recognizable. Now assume that $m \geq n$. Then there exist $m^{\prime} \in[0, n-1]$ and $k \in \mathbb{N}_{+}$such that $m=m^{\prime}+n \cdot k$. Then

$$
\begin{equation*}
\llbracket \mathcal{A} \rrbracket^{-1}(m+n \cdot \mathbb{N})=\llbracket \mathcal{A} \rrbracket^{-1}\left(m^{\prime}+n \cdot \mathbb{N}\right) \backslash \bigcup_{j=0}^{k-1} \llbracket \mathcal{A} \rrbracket^{-1}\left(m^{\prime}+n \cdot j\right) \tag{16.5}
\end{equation*}
$$

As we saw, the $\Sigma$-tree language $\llbracket \mathcal{A} \rrbracket^{-1}\left(m^{\prime}+n \cdot \mathbb{N}\right)$ is recognizable because $m^{\prime}<n$. Moreover, by Lemma 16.2 .3 , for each $j \in[0, k-1]$, the $\Sigma$-tree language $\llbracket \mathcal{A} \rrbracket^{-1}\left(m^{\prime}+n \cdot j\right)$ is also recognizable. Finally, $\Sigma$-tree languages are closed under union and subtraction (cf. Theorem 2.13.3). Thus, also in this case, the $\Sigma$-tree language $\llbracket \mathcal{A} \rrbracket^{-1}(m+n \cdot \mathbb{N})$ is recognizable.

Proof of (b): We follow the proof of (a). Let $m \in \mathrm{~N}$. Assume that $m<n$. Obviously, we can give effectively the equivalence classes modulo $n$, i.e., the elements of $\mathbb{N} / n \mathbb{N}$, by choosing only one representative for each equivalence class. Then we can construct the ( $\Sigma$, Nat $/ n N a t$ )-wta $h(\mathcal{A})$ (cf. page 213). By Lemma 10.9 .2 we have $\llbracket h(\mathcal{A}) \rrbracket=h \circ \llbracket \mathcal{A} \rrbracket$.

Since Nat $/ n$ Nat is a finite semiring, by Corollary 16.1.4 we can construct a $\Sigma$-fta which recognizes $(h \circ \llbracket \mathcal{A} \rrbracket)^{-1}(\bar{m})=\llbracket \mathcal{A} \rrbracket^{-1}(m+n \cdot \mathbb{N})$.

Now assume that $m \geq n$. Since $m^{\prime}<n$, by the above, we can construct a $\Sigma$-fta which recognizes $\llbracket \mathcal{A} \rrbracket^{-1}\left(m^{\prime}+n \cdot \mathbb{N}\right)$. Moreover, by Lemma 16.2 .3 , for each $j \in[0, k-1]$, we can also construct a $\Sigma$-fta which recognizes $\llbracket \mathcal{A} \rrbracket^{-1}\left(m^{\prime}+n \cdot j\right)$. Thus, for each tree language which occurs on the right-hand side of (16.5) a $\Sigma$-fta can be constructed. Hence, by Theorem 2.13.3, we can construct a $\Sigma$-fta which recognizes $\llbracket \mathcal{A} \rrbracket^{-1}(m+n \cdot \mathbb{N})$.

Now we have collected the necessary preimage theorems to show that, for each ( $\Sigma, \mathrm{B}$ )-wta $\mathcal{A}$ and $b \in \mathrm{C}(\mathcal{A})$, the mapping $r_{\mathcal{A}, b}$ is a recognizable step mapping if $f_{\mathcal{A}, b}$ is bounded or $b$ has finite additive order (cf. Lemma 16.2.5).

We call the mapping $f_{\mathcal{A}, b}$ bounded if there exists $K \in \mathbb{N}$ such that $f_{\mathcal{A}, b}(\xi) \leq K$ for each $\xi \in \mathrm{T}_{\Sigma}$.


Figure 16.1: Illustration of the index $i(b)$ and the period $p(b)$ of $b$ (cf. [FKV21, Fig. 1]).

An element $b \in B$ has finite additive order if $\langle b\rangle_{\{\oplus\}}$ is finite. If this is the case, then there exists a least number $i \in \mathbb{N}_{+}$such that $i b=(i+k) b$ for some $k \in \mathbb{N}_{+}$, and there exists a least number $p \in \mathbb{N}_{+}$ such that $i b=(i+p) b$. We call $i$ and $p$ the index (of b) and the period (of b), respectively, and denote them by $i(b)$ and $p(b)$, respectively. Moreover, we call $i+p-1$, i.e., the number of elements of $\langle b\rangle_{\{\oplus\}}$, the additive order of $b$ (cf. Figure 16.1). In particular, the additive order of $\mathbb{O}$ is 1 because $i(\mathbb{O})=p(\mathbb{O})=1$.

Lemma 16.2.5. DFKV22, Thm. 6.6] Let $\mathcal{A}$ be a $(\Sigma, \mathrm{B})$-wta such that $\mathrm{H}(\mathcal{A})$ is finite and let $b \in \mathrm{C}(\mathcal{A})$. If the mapping $f_{\mathcal{A}, b}$ is bounded or $b$ has finite additive order, then we can construct a crisp deterministic $(\Sigma, \mathrm{B})$-wta $\mathcal{B}_{b}$ such that $\llbracket \mathcal{B}_{b} \rrbracket=r_{\mathcal{A}, b}$.

Proof. By performing Algorithm 3, a crisp deterministic ( $\Sigma, \mathrm{B}$ )-wta $\mathcal{B}_{b}$ is constructed.
We prove that $\llbracket \mathcal{B}_{b} \rrbracket=r_{\mathcal{A}, b}$. If, before execution of line 3 , the variable $i$ has the value $\ell$, then the family $\left(A_{b, j} \mid j \in[0, \ell-1]\right)$ of $\Sigma$-fta has already been constructed by using Lemma 16.2.3; in particular, if $\ell=0$, then this family is empty. Thus, after execution of line 3 , the family $\left(A_{b, j} \mid j \in[0, \ell]\right)$ of $\Sigma$-fta has been constructed (where $A_{b, \ell}$ is also constructed by Lemma 16.2.3).

Then, line 4 asks whether the mapping $f_{\mathcal{A}, b}$ is bounded by $\ell$ (i.e., $\mathrm{T}_{\Sigma} \subseteq \bigcup_{j \in[0, \ell]} \mathrm{L}\left(A_{b, j}\right)$; due to [GS84, Thm. 2.10.3] this is decidable). If so, then the for-loop is exited with $K=\ell$.

If not, then line 6 asks whether $b$ has finite additive order $\ell$ (i.e., $\ell b=j b$ for some $j<\ell$ ). If so, then $\ell=\min \left(\ell^{\prime} \mid \exists j<\ell^{\prime}: \ell^{\prime} b=j b\right)$ and the index $i(b)$ of $b$ and the period $p(b)$ of $b$ are constructed and, due to lines 8-10, the upper part $\left(A_{b, j} \mid j \in[i(b), i(b)+p(b)-1]\right)$ of the family of $\Sigma$-fta (generated in line 3 is reconstructed by using Lemma 16.2 .4 Thereafter the for-loop is exited with $K=\ell$. (We note that $K=i(b)+p(b)-1$, i.e., $K$ is the finite additive order of $b$.)

Since, by our assumption, the mapping $f_{\mathcal{A}, b}$ is bounded or $b$ has finite additive order, the for-loop in lines $2-13$, will eventually terminate.

If the for-loop in lines 2-13 was terminated due to the exit in line 4 , then the family $\left(A_{b, j} \mid j \in[0, K]\right)$ of $\Sigma$-fta was constructed such that $\mathrm{L}\left(A_{b, j}\right)=\llbracket \mathcal{A}_{b}^{\prime} \rrbracket^{-1}(j)$ for each $j \in[0, K]$. Thus, due to line 1 , for each $j \in[0, K]$ and $\xi \in \mathrm{T}_{\Sigma}$, we have: $\xi \in \mathrm{L}\left(A_{b, j}\right)$ iff $f_{\mathcal{A}, b}(\xi)=j$. Hence $\mathrm{L}\left(A_{b, j}\right) \cap \mathrm{L}\left(A_{b, j^{\prime}}\right)=\emptyset$ for every $j, j^{\prime} \in[0, K]$ with $j \neq j^{\prime}$, and for each $\xi \in \mathrm{T}_{\Sigma}$ there exists $j \in[0, K]$ such that $\xi \in \mathrm{L}\left(A_{b, j}\right)$; that means that the family $\left(\mathrm{L}\left(A_{b, j}\right) \mid j \in[0, K]\right)$ is a partitioning of $\mathrm{T}_{\Sigma}$. Since $r_{\mathcal{A}, b}(\xi)=\left(f_{\mathcal{A}, b}(\xi)\right) b$ for each $\xi \in \mathrm{T}_{\Sigma}$, we finally obtain

$$
r_{\mathcal{A}, b}=\bigoplus_{j \in[0, K]}(j b) \otimes \chi\left(\mathrm{L}\left(A_{b, j}\right)\right)
$$

If the for-loop in lines 2-13 was terminated due to the exit in line 11 , then the family $\left(A_{b, j} \mid j \in[0, K]\right)$ of $\Sigma$-fta was constructed such that

- $\mathrm{L}\left(A_{b, j}\right)=\llbracket \mathcal{A}_{b}^{\prime} \rrbracket^{-1}(j)$ for each $j \in[0, i(b)-1]$ and
- $\mathrm{L}\left(A_{b, j}\right)=\llbracket \mathcal{A}_{b}^{\prime} \rrbracket^{-1}(j+p(b) \cdot \mathbb{N})$ for each $j \in[i(b), K]$.

Thus, due to line 1,

- for each $j \in[0, i(b)-1]$ and $\xi \in \mathrm{T}_{\Sigma}$, we have: $\xi \in \mathrm{L}\left(A_{b, j}\right)$ iff $f_{\mathcal{A}, b}(\xi)=j$.

```
Algorithm 3 Construction of the \(\mathrm{cd}(\Sigma, \mathrm{B})\)-wta \(\mathcal{B}_{b}\) such that \(\llbracket \mathcal{B}_{b} \rrbracket=r_{\mathcal{A}, b}(\mathrm{~cd}=\mathrm{crisp}\) deterministic)
Input: (a) ( \(\Sigma, \mathrm{B})\)-wta \(\mathcal{A}\) such that \(\mathrm{H}(\mathcal{A})\) is finite and (b) \(b \in \mathrm{C}(\mathcal{A})\) such that \(f_{\mathcal{A}, b}\) is bounded or \(b\) has
    finite additive order
Output: cd \((\Sigma, \mathrm{B})\)-wta \(\mathcal{B}_{b}\)
    : by applying Lemma 16.2 .2 to \(\mathcal{A}\), we construct the \(\left(\Sigma\right.\), Nat)-wta \(\mathcal{A}_{b}^{\prime}\) s.t. \(\llbracket \mathcal{A}_{b}^{\prime} \rrbracket=f_{\mathcal{A}, b}\);
    for each \(i=0,1,2, \ldots\) do
        by applying Lemma 16.2 .3 to \(\mathcal{A}_{b}^{\prime}\), we construct the \(\Sigma\)-fta \(A_{b, i}\) s.t. \(\mathrm{L}\left(A_{b, i}\right)=\llbracket \mathcal{A}_{b}^{\prime} \rrbracket^{-1}(i)\);
        if \(\mathrm{T}_{\Sigma} \subseteq \bigcup_{j \in[0, i]} \mathrm{L}\left(A_{b, j}\right)\) then let \(K \leftarrow i\) and exit the for-loop; \(\quad \triangleright f_{b, \mathcal{A}}\) is bounded by \(K\)
        end if
        if \(i b=j b\) for some \(j<i\) then
            \(i(b) \leftarrow \min \left(j \in \mathbb{N}_{+} \mid j<i\right.\) and \(\left.j b=i b\right)\) and \(p(b) \leftarrow i-i(b) ; \triangleright\) construct the index and period of
    \(b\)
        for each \(j \in[i(b), i(b)+p(b)-1]\) do
            by applying Lemma 16.2 .4 to \(\mathcal{A}_{b}^{\prime}\), we construct the \(\Sigma\)-fta \(A_{b, j}\) s.t. \(\mathrm{L}\left(A_{b, j}\right)=\llbracket \mathcal{A}_{b}^{\prime} \rrbracket^{-1}(j+p(b) \cdot \mathbb{N})\)
            end for
            let \(K \leftarrow i(b)+p(b)-1\) and exit the for-loop \(\quad \triangleright b\) has additive order \(K\)
        end if
    end for
    for each \(j \in[0, K]\) do
        by applying Theorem 4.3.6 \((\mathrm{B}) \Rightarrow(\mathrm{C})\) to \(A_{b, j}\), we construct the \(\mathrm{cd}(\Sigma, \mathrm{B})\)-wta \(\mathcal{C}_{b, j}\) s.t.
```

$$
\llbracket \mathcal{C}_{b, j} \rrbracket=\chi\left(\mathrm{L}\left(A_{b, j}\right)\right) ;
$$

by applying Theorem 10.2 .1 (4) to $j b$ and $\mathcal{C}_{b, j}$, we construct the $\mathrm{cd}(\Sigma, \mathrm{B})$-wta $\mathcal{D}_{b, j}$ s.t.

$$
\llbracket \mathcal{D}_{b, j} \rrbracket=(j b) \otimes \llbracket \mathcal{C}_{b, j} \rrbracket ;
$$

end for
by applying iteratively Theorem $10.1 .1(2)$ to the members of the finite family ( $\left.\mathcal{D}_{b, j} \mid j \in[0, K]\right)$, we construct the $\mathrm{cd}(\Sigma, \mathrm{B})$-wta $\mathcal{B}_{b}$ s.t.

$$
\llbracket \mathcal{B}_{b} \rrbracket=\bigoplus_{j \in[0, K]} \llbracket \mathcal{D}_{b, j} \rrbracket
$$

return $\mathcal{B}_{b}$

- for each $j \in[i(b), K]$ and $\xi \in \mathrm{T}_{\Sigma}$, we have: $\xi \in \mathrm{L}\left(A_{b, j}\right)$ iff $f_{\mathcal{A}, b}(\xi) \in(j+p(b) \cdot \mathbb{N})$.

Hence also in this case the family $\left(\mathrm{L}\left(A_{b, j}\right) \mid j \in[0, K]\right)$ is a partitioning of $\mathrm{T}_{\Sigma}$ and as above we obtain

$$
r_{\mathcal{A}, b}=\bigoplus_{j \in[0, K]}(j b) \otimes \chi\left(\mathrm{L}\left(A_{b, j}\right)\right)
$$

In lines $14-17$, the family $\left(\mathcal{D}_{b, j} \mid j \in[0, K]\right)$ of crisp deterministic $(\Sigma, \mathrm{B})$-wta is constructed such that

$$
\llbracket \mathcal{D}_{b, j} \rrbracket=(j b) \otimes \chi\left(\mathrm{L}\left(A_{b, j}\right)\right)
$$

Finally, in line 18 , the crisp deterministic $(\Sigma, \mathrm{B})$-wta $\mathcal{B}_{b}$ is constructed such that $\llbracket \mathcal{B}_{b} \rrbracket=\bigoplus_{j \in[0, K]} \llbracket \mathcal{D}_{b, j} \rrbracket$. Hence

$$
\llbracket \mathcal{B}_{b} \rrbracket=\bigoplus_{j \in[0, K]}(j b) \otimes \chi\left(\mathrm{L}\left(A_{b, j}\right)\right)=r_{\mathcal{A}, b}
$$

Next we can show the main theorem of this section. It originated from [DSV10, Thm. 11].

Theorem 16.2.6. DFKV22, Thm. 6.6] Let $\Sigma$ be a ranked alphabet, B be a strong bimonoid, and $\mathcal{A}$ be $a(\Sigma, \mathrm{~B})$-wta such that $\mathrm{H}(\mathcal{A})$ is finite. If, for each $b \in \mathrm{C}(\mathcal{A})$, the mapping $f_{\mathcal{A}, b}$ is bounded or $b$ has finite additive order, then we can construct a crisp deterministic $(\Sigma, B)$-wta $\mathcal{B}$ such that $\llbracket \mathcal{A} \rrbracket^{\text {run }}=\llbracket \mathcal{B} \rrbracket$.

Proof. Since $\mathrm{H}(\mathcal{A})$ is finite, also $\mathrm{C}(\mathcal{A})$ is finite. By Lemma 16.2.1, we construct $\mathrm{C}(\mathcal{A})$. Then, by Lemma 16.2.5, for each $b \in \mathrm{C}(\mathcal{A})$, we construct a crisp deterministic $(\Sigma, \mathrm{B})$-wta $\mathcal{B}_{b}$ such that $\llbracket \mathcal{B}_{b} \rrbracket=r_{\mathcal{A}, b}$. Then by (16.3), we have $\llbracket \mathcal{A} \rrbracket^{\text {run }}=\bigoplus_{b \in \mathrm{C}(\mathcal{A})} \llbracket \mathcal{B}_{b} \rrbracket$. Hence, by Theorem 10.1.1(2), we can construct a crisp deterministic $(\Sigma, \mathrm{B})$-wta $\mathcal{B}$ such that $\llbracket \mathcal{A} \rrbracket^{\text {run }}=\llbracket \mathcal{B} \rrbracket$.

There is a slight alternative to the construction in the proof of Theorem 16.2.6 if B is bi-locally finite. Instead of computing the sets $\mathrm{H}(\mathcal{A})$ and $\mathrm{C}(\mathcal{A})$ (by Lemmas 7.4.1 and 16.2.1), we can construct the set $\langle\operatorname{wts}(\mathcal{A})\rangle_{\{\otimes\}}$ (by Lemma 2.6.1). Then we could replace, in Lemmas 16.2 .2 and 16.2.5, the sets $\mathrm{H}(\mathcal{A})$ and $\mathrm{C}(\mathcal{A})$ by the set $\langle\operatorname{wts}(\mathcal{A})\rangle_{\{\otimes\}}$, and we are also able to construct a crisp deterministic wta which is run-equivalent to $\mathcal{A}$. However, this modified construction has the disadvantage that, in the construction of $\mathcal{A}_{b}^{\prime}$ in Lemma 16.2 .2 , its state space contains, in general, too many useless states, because many of the elements of $\langle\operatorname{wts}(\mathcal{A})\rangle_{\{\otimes\}}$ cannot be generated by $\mathcal{A}$. Note, however, that bi-local finiteness is a very strong restriction. In contrast, Theorem 16.2 .6 holds for all strong bimonoids and gives a structural condition for the wta $\mathcal{A}$ which ensures its crisp determinizability.

### 16.2.2 Equivalence of bi-local finiteness and run crisp determinizability

First we recall, for a $(\Sigma, \mathrm{B})$-wta $\mathcal{A}=(Q, \delta, F)$, the definitions

$$
\mathrm{H}(\mathcal{A})=\left\{\mathrm{wt}(\xi, \rho) \mid \xi \in \mathrm{T}_{\Sigma}, \rho \in \mathrm{R}_{\mathcal{A}}(\xi)\right\} \text { and } \mathrm{C}(\mathcal{A})=\left\{\mathrm{wt}(\xi, \rho) \otimes F_{\rho(\varepsilon)} \mid \xi \in \mathrm{T}_{\Sigma}, \rho \in \mathrm{R}_{\mathcal{A}}(\xi)\right\}
$$

Then we can prove the following characterization of bi-locally finiteness in terms of crisp determinization, where the equivalence with Statement (C) is due to Dro22]. It can be compared directly to Theorem 16.1.6. For the comparison we recall that each locally finite strong bimonoid is bi-locally finite, but not vice versa (cf. the Euler diagram in Figure 2.5). We note that in Lemma 3.1.2 we have already proved that $(\mathrm{A}) \Rightarrow(\mathrm{C})$.

Theorem 16.2.7. (cf. FKV21, Cor. 7.7]) Let B be a strong bimonoid. Then the following three statements are equivalent.
(A) B is bi-locally finite.
(B) For every ranked alphabet $\Sigma$ and $(\Sigma, \mathrm{B})-w t a \mathcal{A}$, we can construct a crisp deterministic $(\Sigma, \mathrm{B})$-wta $\mathcal{B}$ such that $\llbracket \mathcal{A} \rrbracket^{\mathrm{run}}=\llbracket \mathcal{B} \rrbracket$.
(C) For every ranked alphabet $\Sigma$ and $(\Sigma, \mathrm{B})-w t a \mathcal{A}$, the set $\operatorname{im}\left(\llbracket \mathcal{A} \rrbracket^{\mathrm{run}}\right)$ is finite.

Thus, in particular, if B is bi-locally finite, then $\operatorname{Rec}^{\text {run }}(\Sigma, \mathrm{B}) \subseteq \operatorname{Rec}^{\text {init }}(\Sigma, \mathrm{B})$. Moreover, if B is locally finite, then $\operatorname{Rec}^{\mathrm{run}}(\Sigma, \mathrm{B})=\operatorname{Rec}^{\mathrm{init}}(\Sigma, \mathrm{B})$.

Proof. Proof of $(\mathrm{A}) \Rightarrow(\mathrm{B})$ : Since B is bi-locally finite, the set $\mathrm{H}(\mathcal{A})$ is finite and each $b \in \mathrm{C}(\mathcal{A})$ has finite additive order. Hence the statement follows from Theorem 16.2.6.

Proof of $(B) \Rightarrow(C)$ : It follows from Theorem $10.3 .1(A) \Rightarrow(C)$.
Proof of $(\mathrm{C}) \Rightarrow(\mathrm{A})$ : This can be obtained by an easy adaptation of [DSV10, Lm. 12] from weighted string automata to wta over string ranked alphabet as follows.

First we show that the additive monoid $(B, \oplus, 0)$ is locally finite. Since $\oplus$ is commutative and associative, it suffices to show that, for each $b \in B$, the monoid $\langle b\rangle_{\{\oplus\}}$ is finite. Thus, let $b \in B$ and


Figure 16.2: $\quad$ The $(\Sigma, B)$-wta $\mathcal{A}$ with $\llbracket \mathcal{A} \rrbracket^{\text {run }}\left(\gamma^{n}(\alpha)\right)=n b$.


Figure 16.3: The $(\Sigma, B)$-wta $\mathcal{A}^{\prime}$ with $\llbracket \mathcal{A}^{\prime} \rrbracket^{\text {run }}\left(\gamma_{2} \gamma_{1}^{l_{m}} \gamma_{2} \gamma_{1}^{l_{m-1}} \gamma_{2} \ldots \gamma_{1}^{l_{1}} \beta\right)=b_{l_{1}} \otimes \ldots \otimes b_{l_{m}}$.
$\Sigma=\left\{\gamma^{(1)}, \alpha^{(0)}\right\}$. We construct the $(\Sigma, \mathrm{B})$-wta $\mathcal{A}=(Q, \delta, F)$, where $Q=\{p, q\}, F_{p}=\mathbb{O}, F_{q}=\mathbb{1}$, and $\delta$ is defined as follows (cf. Figure 16.2):

- $\delta_{0}(\varepsilon, \alpha, p)=\delta_{1}(p, \gamma, p)=\delta_{1}(q, \gamma, q)=\mathbb{1}$,
- $\delta_{1}(p, \gamma, q)=b$, and
- $\delta_{0}(\varepsilon, \alpha, q)=\delta_{1}(q, \gamma, p)=\mathbb{0}$.

Then, for each $n \in \mathbb{N}$, we have $\llbracket \mathcal{A} \rrbracket^{\text {run }}\left(\gamma^{n}(\alpha)\right)=n b$ (cf. (2.15)). Hence, $\langle b\rangle_{\{\oplus\}} \subseteq \operatorname{im}\left(\llbracket \mathcal{A} \rrbracket^{\text {run }}\right.$ ). By (C), the set $\operatorname{im}\left(\llbracket \mathcal{A} \rrbracket^{\text {run }}\right)$ is finite. Hence $\langle b\rangle_{\{\oplus\}}$ is also finite.

Next we prove that the multiplicative monoid $(B, \otimes, \mathbb{1})$ is locally finite. Let $n \in \mathbb{N}$ and $b_{1}, \ldots, b_{n} \in B$. We show that the set

$$
B^{\prime}=\left\{b_{l_{1}} \otimes \ldots \otimes b_{l_{m}} \mid m \in \mathbb{N}, l_{1}, \ldots, l_{m} \in[n]\right\}
$$

is finite. Let $\Sigma=\left\{\gamma_{1}^{(1)}, \gamma_{2}^{(1)}, \beta^{(0)}\right\}$. We construct the $(\Sigma, \mathrm{B})$-wta $\mathcal{A}^{\prime}=\left(Q^{\prime}, \delta^{\prime}, F^{\prime}\right)$ with $Q^{\prime}=\left\{q_{0}, \ldots, q_{n}\right\}$, $F_{q_{0}}^{\prime}=\mathbb{1}$ and $F_{q}^{\prime}=\mathbb{0}$ for every $q \in Q \backslash\left\{q_{0}\right\}$, and $\delta^{\prime}$ is defined as follows (cf. Figure 16.3):

- $\delta_{0}^{\prime}\left(\varepsilon, \beta, q_{0}\right)=\mathbb{1}$,
- $\delta_{1}^{\prime}\left(q_{i-1}, \gamma_{1}, q_{i}\right)=\mathbb{1}$ for each $i \in[n]$,
- $\delta_{1}^{\prime}\left(q_{i}, \gamma_{2}, q_{0}\right)=b_{i}$ for each $i \in[n]$, and
- $\delta_{k}^{\prime}\left(q_{1}^{\prime} \cdots q_{k}^{\prime}, \sigma, q^{\prime}\right)=\mathbb{O}$ for each other combination of $k \in \mathbb{N}, \sigma \in \Sigma^{(k)}$ and $q_{1}^{\prime} \cdots q_{k}^{\prime} \in Q^{k}$ and $q^{\prime} \in Q^{\prime}$. Hence, $\llbracket \mathcal{A}^{\prime} \rrbracket^{\text {run }}\left(\gamma_{2} \gamma_{1}^{l_{m}} \gamma_{2} \gamma_{1}^{l_{m-1}} \gamma_{2} \ldots \gamma_{1}^{l_{1}} \beta\right)=b_{l_{1}} \otimes \ldots \otimes b_{l_{m}}$ for every $m \in \mathbb{N}$ and $l_{1}, \ldots, l_{m} \in[n]$. Thus $B^{\prime} \subseteq \operatorname{im}\left(\llbracket \mathcal{A}^{\prime} \rrbracket^{\text {run }}\right)$. By $(\mathrm{C})$, the set $\operatorname{im}\left(\llbracket \mathcal{A}^{\prime} \rrbracket^{\text {run }}\right)$ is finite, and therefore $B^{\prime}$ is finite. Hence $(C) \Rightarrow(A)$.

For the proof of the next statement, let $\mathcal{A}$ be a $(\Sigma, B)$-wta. If B is bi-locally finite, then by $(\mathrm{A}) \Rightarrow(\mathrm{B})$ we can construct a crisp deterministic $(\Sigma, B)$-wta $\mathcal{B}$ such that $\llbracket \mathcal{A} \rrbracket^{\text {run }}=\llbracket \mathcal{B} \rrbracket$. Since, in particular, $\llbracket \mathcal{B} \rrbracket=\llbracket \mathcal{B} \rrbracket$ init, we have $\operatorname{Rec}^{\text {run }}(\Sigma, \mathrm{B}) \subseteq \operatorname{Rec}^{\text {init }}(\Sigma, \mathrm{B})$.

If $B$ is locally finite, then this inclusion together with Theorem16.1.3(2) imply that $\operatorname{Rec}^{\text {run }}(\Sigma, B)=$ $\operatorname{Rec}^{\mathrm{init}}(\Sigma, \mathrm{B})$.

As a consequence of Theorem $16.2 .7(\mathrm{~A}) \Rightarrow(\mathrm{B})$ and Theorem $10.3 .1(\mathrm{~A}) \Rightarrow(\mathrm{C})$, we show one more preimage theorem.

Corollary 16.2.8. DFKV22, Cor. 6.9] Let B be bi-locally finite and $\mathcal{A}$ a $(\Sigma, \mathrm{B})$-wta. Then, for each $b \in B$, we can construct a $\Sigma$ - $\mathrm{fta} A$ such that $\mathrm{L}(A)=\left(\llbracket \mathcal{A} \rrbracket^{\text {run }}\right)^{-1}(b)$.

### 16.2.3 Characterization of run crisp determinizability for past-finite monotonic strong bimonoids

In this subsection we show a characterization of run crisp determinizability for wta over past-finite monotonic strong bimonoids (cf. Theorem 16.2.13). Each past-finite monotonic strong bimonoid shares many properties with the semiring of natural numbers, like (a) having a partial order on its carrier set (which is not necessarily total), (b) being zero-sum free and zero-divisor free, (c) monotonicity of its operations with respect to that partial order, and (d) a strong kind of well-foundedness (called pastfiniteness). However, in general, distributivity is not required and also does not follow from the axioms. The idea of past-finite strong bimonoids and the characterization stems from DFKV22.

First we define these strong bimonoids. Let $(B, \preceq)$ be a partially ordered set. We say that $\preceq$ is past-finite if for each $b \in B$ the set $\operatorname{past}(b)=\{a \in B \mid a \preceq b\}$ is finite. In the sequel, we write $a \prec b$ if $a \preceq b$ and $a \neq b$.

Thus, for each past-finite partial order $\preceq$, the relation $\prec$ is well-founded. However, there exists a well-founded relation $\prec$ such that the relation $\preceq$ (defined by $a \preceq b$ if $a \prec b$ or $a=b$ ) is a partial order and it is not past-finite. For instance, let $B=\left\{b, a_{1}, a_{2}, \ldots\right\}$ and $\prec=\left\{\left(a_{1}, b\right),\left(a_{2}, b\right), \ldots\right\}$.

Let us consider a strong bimonoid B and let $\preceq$ be a partial order on $B$. We say that B is monotonic with respect to $\preceq$ (cf. [BFGM05, Def. 5] and [DFKV22]) if

$$
\begin{equation*}
\text { for every } a, b \in B \text {, we have } a \preceq a \oplus b \text { and } \tag{16.6}
\end{equation*}
$$

$$
\begin{equation*}
\text { for every } a, b, c \in B \backslash\{\mathbb{D}\} \text { with } b \neq \mathbb{1} \text { we have: } a \otimes c \prec a \otimes b \otimes c \tag{16.7}
\end{equation*}
$$

From (16.7) we easily obtain that

$$
\begin{equation*}
\text { for every } a, b \in B \backslash\{\mathbb{O}\} \text { with } b \neq \mathbb{1} \text { we have: } a \prec a \otimes b \text { and } a \prec b \otimes a . \tag{16.8}
\end{equation*}
$$

If B is monotonic with respect to $\preceq$, then we also write that $(\mathrm{B}, \preceq)=(B, \oplus, \otimes, \mathbb{0}, \mathbb{1}, \preceq)$ is a monotonic strong bimonoid. A monotonic strong bimonoid ( $\mathrm{B}, \preceq$ ) has several properties BFGM05, Lm. 14].

Lemma 16.2.9. If ( $B, \preceq$ ) is a monotonic strong bimonoid, then
(1) $\mathbb{O} \preceq b$ for each $b \in B$, and $\mathbb{1} \preceq b$ for each $b \in B \backslash\{\mathbb{O}\}$,
(2) $B$ is positive, i.e., zero-sum free and zero-divisor free,
(3) B is one-summand free, i.e., for every $a, b \in B$ if $a \oplus b=\mathbb{1}$, then $a, b \in\{\mathbb{0}, \mathbb{1}\}$, and
(4) B is one-product free, i.e., for every $a, b \in B$ if $a \otimes b=\mathbb{1}$, then $a=b=\mathbb{1}$.

Proof. Let $a, b \in B$.
Proof of (1): By (16.6), we obtain $\mathbb{O} \preceq \mathbb{O} \oplus b=b$. If $b \neq \mathbb{D}$, then by (16.8), $\mathbb{1} \preceq \mathbb{1} \otimes b=b$.
Proof of (2): First we prove zero-sum freeness. Let $b \in B \backslash\{\mathbb{D}\}$. Then, by Item (1) and (16.6), we have $\mathbb{0} \prec b \preceq a \oplus b$. Hence B is zero-sum free.

Next we show zero-divisor freeness. Let $a, b \in B \backslash\{\mathbb{O}\}$. Then, by Item (1) and (16.8), © $\prec a \preceq a \otimes b$. Thus B is zero-divisor free.

Proof of (3): Assume that $a \notin\{\mathbb{0}, \mathbb{1}\}$. Then, by (1) and (16.6), we obtain $\mathbb{1} \prec a \preceq a \oplus b$. Hence B is one-summand free.

Proof of (4): We show by contradiction that $a \otimes b=\mathbb{1}$ implies $a=\mathbb{1}$ and $b=\mathbb{1}$. Assume that $a \neq \mathbb{1}$ or $b \neq \mathbb{1}$ and $a \otimes b=\mathbb{1}$. Apparently, $a, b \in B \backslash\{\mathbb{O}\}$. Hence by (1) and (16.8) we obtain $\mathbb{1} \preceq b \prec a \otimes b$ or $\mathbb{1} \preceq a \prec a \otimes b$. This contradicts to $a \otimes b=\mathbb{1}$. Consequently, B is one-product free.

Lemma 16.2 .9 ( 1 ) and (16.8) imply that, for each $b \in B \backslash\{\mathbb{0}, \mathbb{1}\}$ and $n \in \mathbb{N}$, we have $\mathbb{0} \prec \mathbb{1} \prec b^{n} \prec b^{n+1}$, i.e.,

$$
\mathbb{O} \prec \mathbb{1} \prec b \prec b \otimes b \prec b \otimes b \otimes b \cdots .
$$

Hence, if $(\mathrm{B}, \preceq)$ is a finite monotonic strong bimonoid, then $B$ has only two elements $\mathbb{O}$ and $\mathbb{1}$, we have $\mathbb{O} \prec \mathbb{1}$ and $\mathbb{1} \oplus \mathbb{1}=\mathbb{1}$. Thus $B$ is isomorphic to the Boolean semiring Boole.

We call a monotonic strong bimonoid $(B, \oplus, \otimes, \mathbb{0}, \mathbb{1}, \preceq)$ past-finite if $(B, \preceq)$ is past-finite. In DFKV22, Ex. 2.2-2.4] a number of past-finite monotonic strong bimonoids are shown. Here we only mention the following examples:
(a) the semiring $($ Boole,$\leq)$, where $0 \leq 1$,
(b) the semiring (Nat, $\leq$ ) of natural numbers, where $\leq$ is the common relation "less than or equal to" on $\mathbb{N}$,
(c) the semiring $\left(\operatorname{Nat}_{\text {max }},+, \leq\right)$, where $\leq$ is the natural extension of $\leq$ to $\mathbb{N}_{-\infty}$,
(d) the semiring $\left(\mathcal{P}_{\mathrm{f}}\left(\Gamma^{*}\right), \cup, \cdot, \emptyset,\{\varepsilon\}, \preceq\right)$ where $\mathcal{P}_{\mathrm{f}}\left(\Gamma^{*}\right)$ is the set of finite subsets of $\Gamma^{*}$ and $L_{1} \preceq L_{2}$ if there exists an injective mapping $f: L_{1} \rightarrow L_{2}$ such that $w$ is a substring of $f(w)$ for each $w \in L_{1}$,
(e) the semiring $\left(\mathcal{P}_{\mathrm{f}}(\mathbb{N}), \cup,+, \emptyset,\{0\}, \preceq\right)$ where $\mathcal{P}_{\mathrm{f}}(\mathbb{N})$ is the set of finite subsets of $\mathbb{N}$, the operation + is extended in the usual way to sets, and $N_{1} \preceq N_{2}$ if there exists an injective mapping $f: N_{1} \rightarrow N_{2}$ such that $n \leq f(n)$ for each $n \in N_{1}$, and
(f) the plus-plus strong bimonoid $\left(\mathrm{PP}_{\mathbb{N}}, \leq\right)=\left(\mathbb{N}_{\mathbb{O}}, \oplus,+, \mathbb{0}, 0, \leq\right)$ of natural numbers (cf. Example 2.6.10), where $\leq$ is the usual order on $\mathbb{N}$ together with $\mathbb{O} \leq x$ for each $x \in \mathbb{N}$. Then $\left(\operatorname{PP}_{\mathbb{N}}, \leq\right)$ is a past-finite monotonic strong bimonoid which is not a semiring.
The set of past-finite monotonic strong bimonoids shares another property with the semiring of natural numbers (cf. Lemma 16.2.3).

Theorem 16.2.10. [DFKV22, Thm. 6.10] Let $(\mathrm{B}, \preceq)$ be past-finite monotonic strong bimonoid and $\mathcal{A} a$ ( $\Sigma, \mathrm{B})-w t a$ and $b \in B$.
(1) The tree language $\left(\llbracket \mathcal{A} \rrbracket^{\text {run }}\right)^{-1}(b)$ is recognizable. Hence $\llbracket \mathcal{A} \rrbracket^{\text {run }}$ has the preimage property.
(2) If the set $\operatorname{past}(b)$ can be constructed, then we can construct a finite-state $\Sigma$-tree automaton which recognizes $\llbracket \mathcal{A} \rrbracket^{-1}(b)$.

## Proof. Let $b \in B$.

Proof of (1): We define the set $C=B \backslash \operatorname{past}(b)=\{a \in B \mid a \npreceq b\}$. Moreover, we define the equivalence relation $\sim$ on the set $B$ such that its classes are the singleton sets $\{a\}$ for each $a \in \operatorname{past}(b)$ and the set $C$.

We claim that $\sim$ is a congruence. To show this let $c, c^{\prime} \in C$ and $d \in B$. Since ( $\mathrm{B}, \preceq$ ) is monotonic, we have $c \preceq c \oplus d$ and $c^{\prime} \preceq c^{\prime} \oplus d$. Hence $c \oplus d \in C$ and $c^{\prime} \oplus d \in C$. By commutativity of $\oplus$ we also have $d \oplus c \in C$ and $d \oplus c^{\prime} \in C$. Thus $\oplus$ obeys the structure of the equivalence classes. Now let $d \neq \mathbb{0}$. Then we obtain $c \preceq c \otimes d$ and $c \preceq d \otimes c$, showing that $c \otimes d \in C$ and $d \otimes c \in C$, and similarly $c^{\prime} \otimes d \in C$ and $d \otimes c^{\prime} \in C$. Hence $\sim$ is a congruence on the strong bimonoid B .

By definition, the quotient strong bimonoid of B modulo $\sim$ is finite. Let $h: B \rightarrow B / \sim$ be the canonical strong bimonoid homomorphism, i.e., for each $c \in B$ we let $h(c)=[c]_{\sim}$. By Lemma 10.9.3, we obtain $\left(h \circ \llbracket \mathcal{A} \rrbracket^{\text {run }}\right) \in \operatorname{Rec}^{\text {run }}(\Sigma, \mathrm{B} / \sim)$. Moreover $\left(\llbracket \mathcal{A} \rrbracket^{\text {run }}\right)^{-1}(b)=\left(h \circ \llbracket \mathcal{A} \rrbracket^{\text {run }}\right)^{-1}(\{b\})$. Since $B / \sim$ is finite, by Corollary 16.2.8, the $\Sigma$-tree language $\left(h \circ \llbracket \mathcal{A} \rrbracket^{\text {run }}\right)^{-1}(\{b\})$ is recognizable. Thus $\llbracket \mathcal{A} \rrbracket^{\text {run }}$ has the preimage property.

Proof of (2): Assume that past $(b)$ is constructed. Then we can construct the congruence $\sim$ as defined in the proof of Statement 1 and the canonical strong bimonoid homomorphism $h: B \rightarrow B / \sim$. Since $\mathrm{B} / \sim$ is finite, by Observation 10.9 .1 , we can construct the $(\Sigma, \mathrm{B} / \sim)$-wta $h(\mathcal{A})$ such that $\llbracket h(\mathcal{A}) \rrbracket^{\text {run }}=h \circ \llbracket \mathcal{A} \rrbracket^{\text {run }}$. Since $B / \sim$ is finite, by Corollary 16.2 .8 , we can construct a finite-state $\Sigma$-tree automaton which recognizes $\left(\llbracket h(\mathcal{A}) \rrbracket^{\text {run }}\right)^{-1}(\{b\})$.

By Theorems 10.3 .1 and $16.2 .10(1), \mathcal{A}$ is run crisp determinizable if and only if $\operatorname{im}\left(\llbracket \mathcal{A} \rrbracket^{\text {run }}\right)$ is finite. As first step towards our characterization theorem we analyse the impact of this finiteness property on
the structure of a $(\Sigma, \mathrm{B})$-wta $\mathcal{A}$ where $(\mathrm{B}, \preceq)$ is a past-finite monotonic strong bimonoid. In particular, we consider local-non-zero runs and small loops of $\mathcal{A}$. (We recall that the definition of a run on a context and the weight of such a run can be found in Section 6.1)

For each $\xi \in \mathrm{T}_{\Sigma}(\{z\})$, we call a run $\rho \in \mathrm{R}_{\mathcal{A}}(\xi)$ local-non-zero if for each $w \in \operatorname{pos}_{\Sigma}(\xi)$ we have $\delta_{k}(\rho(w 1) \cdots \rho(w k), \sigma, \rho(w)) \neq \mathbb{O}$ where $k=\operatorname{rk}(\xi(w))$ and $\sigma=\xi(w)$. We note that, for every $\xi \in \mathrm{T}_{\Sigma}$ and $\rho \in \mathrm{R}_{\mathcal{A}}(\xi)$, the run $\rho$ is local-successful (cf. Subsection 7.1.1) if it is local-non-zero and $F_{\rho(\varepsilon)} \neq \mathbb{0}$.

We say that small loops of $\mathcal{A}$ have weight $\mathbb{1}$ if, for every $q \in Q, c \in \mathrm{C}_{\Sigma}$, and local-non-zero $\rho \in$ $\mathrm{R}_{\mathcal{A}}(q, c, q)$, if height $(c)<|Q|$, then $\mathrm{wt}(c, \rho)=\mathbb{1}$.

Theorem 16.2.11. DFKV22, Thm. 7.1] Let $(\mathrm{B}, \preceq)$ be a past-finite monotonic strong bimonoid and $\mathcal{A}$ be a local-trim $(\Sigma, \mathrm{B})$-wta. If $\operatorname{im}\left(\llbracket \mathcal{A} \rrbracket^{\mathrm{run}}\right)$ is finite, then small loops of $\mathcal{A}$ have weight $\mathbb{1}$.

Proof. We prove by contraposition. Suppose there exist $q \in Q, c \in \mathrm{C}_{\Sigma}$, and local-non-zero $\rho \in \mathrm{R}_{\mathcal{A}}(q, c, q)$ such that height $(c)<|Q|$ and $\mathrm{wt}(c, \rho) \neq \mathbb{1}$. Since $\rho$ is local-non-zero and $(\mathrm{B}, \preceq)$ is monotonic, we have that $\mathbb{1} \prec \mathrm{wt}(c, \rho)$. Since $\mathcal{A}$ is local-trim, the state $q$ is local-useful and thus there exist $\xi \in \mathrm{T}_{\Sigma}, \theta \in \mathrm{R}_{\mathcal{A}}(q, \xi)$, $c^{\prime} \in \mathrm{C}_{\Sigma}, q^{\prime} \in Q$ with $F_{q^{\prime}} \neq \mathbb{O}$, and $\rho^{\prime} \in \mathrm{R}_{\mathcal{A}}\left(q^{\prime}, c^{\prime}, q\right)$, and $\theta$ and $\rho^{\prime}$ are local-non-zero. By Theorem 6.1.3, for each $n \in \mathbb{N}$, we have

$$
\mathrm{wt}\left(c^{\prime}\left[c^{n}[\xi]\right], \rho^{\prime}\left[\rho^{n}[\theta]\right]\right)=l_{c^{\prime}, \rho^{\prime}} \otimes\left(l_{c, \rho}\right)^{n} \otimes \mathrm{wt}(\xi, \theta) \otimes\left(r_{c, \rho}\right)^{n} \otimes r_{c^{\prime}, \rho^{\prime}}
$$

Since $\mathbb{1} \prec \operatorname{wt}(c, \rho)=l_{c, \rho} \otimes r_{c, \rho}$, we have $\mathbb{1} \prec l_{c, \rho}$ or $\mathbb{1} \prec r_{c, \rho}$, because B is one-product free. Thus, by monotonicity of ( $\mathrm{B}, \preceq$ ), we obtain

$$
\begin{equation*}
\operatorname{wt}\left(c^{\prime}\left[c^{0}[\xi]\right], \rho^{\prime}\left[\rho^{0}[\theta]\right]\right) \prec \operatorname{wt}\left(c^{\prime}\left[c^{1}[\xi]\right], \rho^{\prime}\left[\rho^{1}[\theta]\right]\right) \prec \ldots . \tag{16.9}
\end{equation*}
$$

Next we define a sequence $\xi_{1}, \xi_{2}, \xi_{3}, \ldots$ of trees in $\mathrm{T}_{\Sigma}$ such that the elements $\llbracket \mathcal{A} \rrbracket^{\text {run }}\left(\xi_{1}\right), \llbracket \mathcal{A} \rrbracket^{\text {run }}\left(\xi_{2}\right)$, $\llbracket \mathcal{A} \rrbracket^{\text {run }}\left(\xi_{3}\right), \ldots$ are pairwise different as follows. We let $\xi_{1}=c^{\prime}[c[\xi]]$. Since $(\mathrm{B}, \preceq)$ is past-finite, the set $P_{1}=\operatorname{past}\left(\llbracket \mathcal{A} \rrbracket^{\mathrm{run}}\left(\xi_{1}\right)\right)$ is finite. By (16.9) we choose $n_{2}$ such that $\operatorname{wt}\left(c^{\prime}\left[c^{n_{2}}[\xi]\right], \rho^{\prime}\left[\rho^{n_{2}}[\theta]\right]\right) \notin P_{1}$ and let $\xi_{2}=c^{\prime}\left[c^{n_{2}}[\xi]\right]$. Since $\rho^{\prime}\left[\rho^{n_{2}}[\theta]\right] \in \mathrm{R}_{\mathcal{A}}\left(q^{\prime}, \xi_{2}\right)$ and $(\mathrm{B}, \preceq)$ is monotonic, we have

$$
\mathrm{wt}\left(\xi_{2}, \rho^{\prime}\left[\rho^{n_{2}}[\theta]\right]\right) \preceq \mathrm{wt}\left(\xi_{2}, \rho^{\prime}\left[\rho^{n_{2}}[\theta]\right]\right) \otimes F_{q^{\prime}} \preceq \bigoplus_{\kappa \in \mathrm{R}_{\mathcal{A}}\left(\xi_{2}\right)} \mathrm{wt}\left(\xi_{2}, \kappa\right) \otimes F_{\kappa(\varepsilon)}=\llbracket \mathcal{A} \rrbracket^{\mathrm{run}}\left(\xi_{2}\right) .
$$

(Note that $F_{q^{\prime}}$ may be $\mathbb{1}$.) Hence $\llbracket \mathcal{A} \rrbracket^{\text {run }}\left(\xi_{2}\right) \notin P_{1}$. Put $P_{2}=\operatorname{past}\left(\llbracket \mathcal{A} \rrbracket^{\text {run }}\left(\xi_{2}\right)\right)$. Then we choose $n_{3} \in \mathbb{N}$ such that $\operatorname{wt}\left(c^{\prime}\left[c^{n_{3}}[\xi]\right], \rho^{\prime}\left[\rho^{n_{3}}[\theta]\right]\right) \notin P_{1} \cup P_{2}$ and let $\xi_{3}=c^{\prime}\left[c^{n_{3}}[\xi]\right]$. As before, we have $\llbracket \mathcal{A} \rrbracket^{\text {run }}\left(\xi_{3}\right) \notin$ $P_{1} \cup P_{2}$. Continuing this process, we obtain the desired sequence of trees. It means that $\operatorname{im}\left(\llbracket \mathcal{A} \rrbracket^{\text {run }}\right)$ is infinite.

We recall, for a $(\Sigma, \mathrm{B})$-wta $\mathcal{A}=(Q, \delta, F)$, the definitions

$$
\mathrm{H}(\mathcal{A})=\left\{\mathrm{wt}(\xi, \rho) \mid \xi \in \mathrm{T}_{\Sigma}, \rho \in \mathrm{R}_{\mathcal{A}}(\xi)\right\} \text { and } \mathrm{C}(\mathcal{A})=\left\{\mathrm{wt}(\xi, \rho) \otimes F_{\rho(\varepsilon)} \mid \xi \in \mathrm{T}_{\Sigma}, \rho \in \mathrm{R}_{\mathcal{A}}(\xi)\right\}
$$

Also a kind of inverse of Theorem 16.2.11 holds.
Lemma 16.2.12. DFKV22, Lm. 5.5, 6.7] Let B be a one-product free strong bimonoid and $\mathcal{A}$ be a ( $\Sigma, \mathrm{B}$ )-wta. If small loops of $\mathcal{A}$ have weight $\mathbb{1}$, then
(1) for every $\xi \in \mathrm{T}_{\Sigma}, q^{\prime} \in Q$, and local-non-zero $\kappa \in \mathrm{R}_{\mathcal{A}}\left(q^{\prime}, \xi\right)$, there exist $\xi^{\prime} \in \mathrm{T}_{\Sigma}$ and $\kappa^{\prime} \in \mathrm{R}_{\mathcal{A}}\left(q^{\prime}, \xi^{\prime}\right)$ such that height $\left(\xi^{\prime}\right)<|Q|$ and $\mathrm{wt}(\xi, \kappa)=\mathrm{wt}\left(\xi^{\prime}, \kappa^{\prime}\right)$ and
(2) the set $\mathrm{H}(\mathcal{A})$ is finite.

Proof. Proof of (1): The idea is to cut out iteratively local-non-zero, small loops with weight $\mathbb{1}$. Formally, let $\xi \in \mathrm{T}_{\Sigma}, q^{\prime} \in Q$, and $\kappa \in \mathrm{R}_{\mathcal{A}}\left(q^{\prime}, \xi\right)$ be a local-non-zero run. We may assume that height $(\xi) \geq|Q|$. Applying Theorem 6.1.4 (for $n=1$ and $n=0$ ), there exist $c, c^{\prime} \in \mathrm{C}_{\Sigma}, \zeta \in \mathrm{T}_{\Sigma}, q \in Q, \rho^{\prime} \in \mathrm{R}_{\mathcal{A}}\left(q^{\prime}, c^{\prime}, q\right)$,
$\rho \in \mathrm{R}_{\mathcal{A}}(q, c, q)$, and $\theta \in \mathrm{R}_{\mathcal{A}}(q, \zeta)$ such that $\xi=c^{\prime}[c[\zeta]], \kappa=\rho^{\prime}[\rho[\theta]], \operatorname{height}(c)>0, \operatorname{height}(c[\zeta])<|Q|$, and

$$
\begin{aligned}
\mathrm{wt}(\xi, \kappa)= & \mathrm{wt}\left(c^{\prime}[c[\zeta]], \rho^{\prime}[\rho[\theta]]\right)=l_{c^{\prime}, \rho^{\prime}} \otimes l_{c, \rho} \otimes \mathrm{wt}(\zeta, \theta) \otimes r_{c, \rho} \otimes r_{c^{\prime}, \rho^{\prime}} \quad(\text { for } n=1) \\
& \operatorname{wt}\left(c^{\prime}[\zeta], \rho^{\prime}[\theta]\right)=l_{c^{\prime}, \rho^{\prime}} \otimes \operatorname{wt}(\zeta, \theta) \otimes r_{c^{\prime}, \rho^{\prime}}(\text { for } n=0)
\end{aligned}
$$

Since $\kappa$ is local-non-zero, also the runs $\rho^{\prime}, \rho$, and $\theta$ are local-non-zero. Thus wt $(c, \rho)=\mathbb{1}$ by our assumption.

Moreover, by Observation 6.1.1, we have $\mathrm{wt}(c, \rho)=l_{c, \rho} \otimes r_{c, \rho}$. Since B is one-product free, we have $l_{c, \rho}=r_{c, \rho}=\mathbb{1}$. Hence we have $\mathrm{wt}(\xi, \kappa)=\mathrm{wt}\left(c^{\prime}[\zeta], \rho^{\prime}[\theta]\right)$.

Obviously, $\rho^{\prime}[\theta] \in \mathrm{R}_{\mathcal{A}}\left(q^{\prime}, c^{\prime}[\zeta]\right)$ and $\operatorname{size}\left(c^{\prime}[\zeta]\right)<\operatorname{size}(\xi)$. If height $\left(c^{\prime}[\zeta]\right)<|Q|$, then we are ready. Otherwise we continue with $c^{\prime}[\zeta], q^{\prime}$, and $\rho^{\prime}[\theta]$ as before. After finitely many steps, we obtain $\xi^{\prime} \in T_{\Sigma}$ and $\kappa^{\prime} \in \mathrm{R}_{\mathcal{A}}\left(q^{\prime}, \xi^{\prime}\right)$ with height $\left(\xi^{\prime}\right)<|Q|$ as required.

Proof of (2): If small loops of $\mathcal{A}$ have weight $\mathbb{1}$, then by Statement 1 of this lemma we have

$$
\mathrm{H}(\mathcal{A})=\left\{\mathrm{wt}(\xi, \rho)\left|\xi \in \mathrm{T}_{\Sigma}, \operatorname{height}(\xi)<|Q|, \text { and } \rho \in \mathrm{R}_{\mathcal{A}}(\xi)\right\}\right.
$$

Hence $\mathrm{H}(\mathcal{A})$ is finite.
Now we can prove the main result of this subsection.

Theorem 16.2.13. DFKV22, Thm. 6.10, 7.1] Let $\Sigma$ be a ranked alphabet, $(\mathrm{B}, \preceq)=(B, \oplus, \otimes, \mathbb{O}, \mathbb{1}, \preceq)$ be a past-finite monotonic strong bimonoid, and $\mathcal{A}$ be a local-trim $(\Sigma, \mathrm{B})-w t a$. Then the following two statements are equivalent.
(A) $\mathcal{A}$ is run crisp determinizable.
(B) Small loops of $\mathcal{A}$ have weight $\mathbb{1}$ and, for each $b \in \mathrm{C}(\mathcal{A})$, the mapping $f_{\mathcal{A}, b}$ is bounded or $b$ has finite additive order.

Proof. Proof of $(\mathrm{A}) \Rightarrow(\mathrm{B})$ : Since $\mathcal{A}$ is run crisp determinizable, the set $\operatorname{im}\left(\llbracket \mathcal{A} \rrbracket^{\text {run }}\right)$ is finite by Theorem 10.3.1. Thus, by Theorem 16.2.11 small loops of $\mathcal{A}$ have weight $\mathbb{1}$.

Now let $b \in \mathrm{C}(\mathcal{A})$. If the mapping $f_{\mathcal{A}, b}$ is not bounded, then there exists an infinite sequence $\xi_{1}, \xi_{2}, \ldots$ of trees in $\mathrm{T}_{\Sigma}$ such that $f_{\mathcal{A}, b}\left(\xi_{1}\right)<f_{\mathcal{A}, b}\left(\xi_{2}\right)<\ldots$ By Equality (16.3), we have $\left(f_{\mathcal{A}, b}\left(\xi_{i}\right)\right) b \preceq \llbracket \mathcal{A} \rrbracket^{\text {run }}\left(\xi_{i}\right)$ for each $i \in \mathbb{N}$. Thus $\left(f_{\mathcal{A}, b}\left(\xi_{i}\right)\right) b \in P$, where $P=\bigcup_{a \in \operatorname{im}(\llbracket \mathcal{A} \rrbracket \text { run })} \operatorname{past}(a)$. Since $\operatorname{im}\left(\llbracket \mathcal{A} \rrbracket^{\text {run }}\right)$ is finite and $(\mathrm{B}, \preceq)$ is past-finite, the set $P$ is also finite. Hence $\left(f_{\mathcal{A}, b}\left(\xi_{i}\right)\right) b=\left(f_{\mathcal{A}, b}\left(\xi_{j}\right)\right) b$ for some $i, j \in \mathbb{N}$ with $i<j$, which implies that $b$ has finite additive order.

Proof of $(\mathrm{B}) \Rightarrow(\mathrm{A})$ : This implication follows from Lemma 16.2.12 (we recall that each monotonic strong bimonoid is one-product free) and Theorem 16.2.6.

Finally we show an application of Theorem 16.2.13,
Theorem 16.2.14. DFKV20, Thm. 10] Let $(\mathrm{B}, \preceq)$ be an additively locally finite and past-finite monotonic strong bimonoid. Moreover, let $\mathcal{A}$ be a ( $\Sigma, \mathrm{B})$-wta which contains at least one local-useful state. It is decidable whether $\mathcal{A}$ is run crisp determinizable.

Proof. Let $\mathcal{A}=(Q, \Sigma, \delta)$. By Theorem 7.1.4, we can construct a $(\Sigma, \mathrm{B})$-wta $\mathcal{A}^{\prime}$ such that $\mathcal{A}^{\prime}$ is local-trim and $\llbracket \mathcal{A}^{\prime} \rrbracket^{\text {run }}=\llbracket \mathcal{A} \rrbracket^{\text {run }}$. Hence we may assume that $\mathcal{A}$ is local-trim. Since B is additively locally finite, each $b \in \mathrm{C}(\mathcal{A})$ has finite additive order. Thus, by Theorem 16.2.13, $\mathcal{A}$ is run crisp determinizable if and only if small loops of $\mathcal{A}$ have weight $\mathbb{1}$. The latter property is decidable because (a) there exist only finitely many $c \in \mathrm{C}_{\Sigma}$ such that height $(c)<|Q|$, and (b) since ( $\mathrm{B}, \preceq$ ) is monotonic, for every $c \in \mathrm{C}_{\Sigma}, q \in Q$, and $\rho \in \mathrm{R}_{\mathcal{A}}(q, c, q)$ we have $\operatorname{wt}(c, \rho)=\mathbb{1}$ if and only if for each $v \in \operatorname{pos}(c)$ we have $\delta_{k}(\rho(v 1) \cdots \rho(v k), \sigma, \rho(v))=\mathbb{1}$ where $\sigma=c(v)$ and $k=\operatorname{rk}(\sigma)$, and (c) this is decidable because B has an effective test for $\mathbb{1}$.

The decidability problem addressed in Theorem 16.2 .14 is meaningful, because there exists an additively locally finite and past-finite monotonic semiring and a wta over that semiring which is not run crisp determinizable. Such a semiring and wta is the arctic semiring $\mathrm{Nat}_{\mathrm{max},+}$ and the ( $\Sigma, \mathrm{Nat}_{\mathrm{max},+}$ )-wta $\mathcal{A}$ in Example 3.2.4, respectively. We recall that $\llbracket \mathcal{A} \rrbracket^{\text {init }}=\llbracket \mathcal{A} \rrbracket^{\text {run }}=$ height (cf. Theorem [5.3.1). As we mentioned $\mathcal{A}$ is not run crisp determinizable because im(height) is not finite.

We refer the reader to DFKV20, DFKV22] for further decidability and undecidability results for wta and wsa over past-finite monotonic strong bimonoids.

## Chapter 17

## Determinization of wta over semirings

In this chapter, we consider as weight algebras only semirings. $\mathrm{A}(\Sigma, \mathrm{B})$-wta $\mathcal{A}$ over some semiring B is bu determinizable (or just: determinizable) if there exists an equivalent bu deterministic ( $\Sigma, \mathrm{B}$ )-wta $\mathcal{B}$, i.e., $\llbracket \mathcal{A} \rrbracket=\llbracket \mathcal{B} \rrbracket$. ${ }^{1}$

If $B$ is locally finite (e.g., finite), then we can apply the subset method of Definition 16.1.2 to $\mathcal{A}$ and obtain an i-equivalent crisp deterministic $(\Sigma, B)$-wta $\mathcal{B}$. In Section 17.1 , it will turn out that, in general, the subset method will not yield a bu deterministic wta, i.e., it is not able to determinize each wta. One might think that there exist other, successful approaches. However, there are recognizable weighted tree languages which are not bu deterministically recognizable (as already in the string case, cf. [KLMP04, pp.354]). Hence, if $r$ is such a weighted tree language, then each wta $\mathcal{A}$ with $\llbracket \mathcal{A} \rrbracket=r$ is not bu determinizable. For example, the mapping height : $\mathrm{T}_{\Sigma} \rightarrow \mathbb{N}$ is recognizable by a ( $\Sigma, \mathrm{Nat}_{\mathrm{max},+}$ )-wta (cf. Example 3.2.4) but it is not bu deterministically recognizable (cf. Corollary 6.1.5). In Section 17.2 we show two further recognizable weighted tree languages which are not bu deterministically recognizable.

In Section 17.3 we show an advanced version of the subset method which is based on factorization and it is called determinization by factorization (cf. Definition 17.3.1). We elaborate sufficient conditions under which determinization by factorization transforms a given wta into a bu deterministic wta which is equivalent to the given wta. More precisely, if $B$ is an extremal and commutative semiring for which there exists a maximal factorization, and the given $(\Sigma, B)$-wta $\mathcal{A}$ has the twinning property, then $\mathcal{A}$ is bu determinizable (cf. Theorem 17.3.2) and a bu deterministic wta equivalent to $\mathcal{A}$ can be obtained by applying determinization by factorization to $\mathcal{A}$. The approach of bu determinization of wta by factorization was published in BVM10 (also cf. Büc14, Sec. 5]), and it is based on the determinization by factorization of wsa KM05]. In its turn, the latter is a generalization of determinization of wsa over the tropical semiring Moh97.

We mention that in Pau20, Sect. 4.5] the decidability of the determinization of ( $\Sigma$, Real $\mathrm{Rax}_{\text {max }}$ )-wta was considered, where Real $\max +$ is the semifield $\left(\mathbb{R}_{-\infty}, \max ,+,-\infty, 0\right)$. One of the main results is Pau20, Thm. 4.33] which states the following. For a finitely ambiguous ( $\Sigma, \operatorname{Real}_{\max ,+}$ )-wta $\mathcal{A}$ it is decidable whether there exists a deterministic $\left(\Sigma\right.$, Real $\left._{\text {max, }+}\right)$-wta $\mathcal{A}^{\prime}$ such that $\llbracket \mathcal{A} \rrbracket=\llbracket \mathcal{A}^{\prime} \rrbracket$. If such an automaton $\llbracket \mathcal{A}^{\prime} \rrbracket$ exists, then it can be constructed. Here finitely ambiguous means that there exists an integer $M \geq 1$ such that, for each tree $\xi \in \mathrm{T}_{\Sigma}$, there are at most $M$ accepting runs of $\mathcal{A}$ on $\xi$. We also mention that the base of the construction of $\mathcal{A}^{\prime}$ is not the factorization as in this chapter.

Also, we mention that in DSF21 wta over particular semirings were considered which are based on groups. Let $G=(G, \otimes, \mathbb{1})$ be a group. Then $\operatorname{Sem}(G)=\left(\mathcal{P}_{\text {fin }}(G), \cup, \otimes, \emptyset,\{\mathbb{1}\}\right)$ is a semiring where $\otimes$ is

[^16]lifted to finite subsets of $G$ in the straightforward way (cf. Example 2.6.9(17)). A group-weighted tree automaton over $\Sigma$ and $G$ (for short: ( $\Sigma, \mathrm{G})$-gwta) is a ( $\Sigma$, Sem $(\mathrm{G})$ )-wta. In [DSF21, Thm. 1] it is proved that each $(\Sigma, G)$-gwta is sequentializable if it has the twinning property. $A(\Sigma, G)$-gwta is sequential if it is bu deterministic and for each transition $\left(q_{1} \cdots q_{k}, \sigma, q\right)$ we have that $\left|\delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right)\right| \leq 1$. It is easy to see that each crisp deterministic $(\Sigma, G)$-gwta is sequential, and each sequential ( $\Sigma, G)$-gwta is bu deterministic. Moreover, both inclusions are strict.

Finally, we mention that in Dör22 a general framework for the determinization was established, which is based on a theory of factorizations in (multiplicative) monoids Dör22, Sec. 3.3]). Roughly speaking, the main determinization result [Dör22, Thm. 3.78] says the following. Let $\mathrm{B}=(B, \oplus, \otimes, \mathbb{0}, \mathbb{1})$ be a semiring; moreover, let $\mathrm{M}=(M, \otimes, \mathbb{1})$ be a finitely generated submonoid of $(B, \otimes, \mathbb{1})$ which satisfies a certain monotonicity property and admits centering factorizations; then $\mathrm{B}^{\prime}=\left(\langle M\rangle_{\oplus}, \oplus, \otimes, \mathbb{0}, \mathbb{1}\right)$ is a semiring. Moreover, let $\mathcal{A}$ be a $\left(\Sigma, \mathrm{B}^{\prime}\right)$-wta which satisfies a certain twinning property. Then [Dör22, Thm. 3.78] says that, if $\mathcal{A}$ is finitely $M$-ambiguous or $\mathrm{B}^{\prime}$ is additively idempotent, then $\mathcal{A}$ is M -sequentializable, which means sequential and the transition weights are in $M$. As illustrated in Dör22, Ex. 3.95], the twinning properties of [BVM10] and of Dör22, Thm. 3.78] are incomparable.

In this chapter B is a semiring and $\mathcal{A}=(Q, \delta, F)$ denotes an arbitrary $(\Sigma, \mathrm{B})$-wta unless specified otherwise.

### 17.1 Applying the subset method to wta over arbitrary semirings

One can be tempted to apply the subset method (cf. Definition 16.1.2) to a $(\Sigma, \mathrm{B})$-wta $\mathcal{A}$ where B is a not necessarily locally finite semiring. Then one obtains the triple $\operatorname{sub}(\mathcal{A})=\left(Q^{\prime}, \delta^{\prime}, F^{\prime}\right)$. We note that $Q^{\prime}$ is finite if and only if $\operatorname{sub}(\mathcal{A})$ is crisp deterministic wta which is equivalent to $\mathcal{A}$.

Lemma 17.1.1. If $\operatorname{sub}(\mathcal{A})$ is a crisp deterministic $(\Sigma, B)$-wta, then $\llbracket \mathcal{A} \rrbracket$ is a recognizable step mapping.
Proof. Let $\operatorname{sub}(\mathcal{A})$ be a crisp deterministic $(\Sigma, \mathrm{B})$-wta. Then by Theorem $16.1 .3(1)$ we have $\llbracket \operatorname{sub}(\mathcal{A}) \rrbracket=$ $\llbracket \mathcal{A} \rrbracket$. Hence the statement follows from Theorem 10.3 .1 .

This lemma shows the limit of the subset method: if $\llbracket \mathcal{A} \rrbracket$ is not a recognizable step mapping, then $\operatorname{sub}(\mathcal{A})$ is not a crisp determinisitic wta, i.e., the subset method does not yield a wta. In the following we give an example of a (not bu deterministic) ( $\Sigma, \mathrm{B}$ )-wta $\mathcal{A}$ which cannot be determinized by the subset method because $\llbracket \mathcal{A} \rrbracket$ is not a recognizable step mapping (also cf. [BVM10, Ex. 4.1]).

Example 17.1.2. We consider the tropical semifield $\operatorname{Rat}_{\min ,+}=\left(\mathbb{Q}_{\infty}, \min ,+, \infty, 0\right)$ and the ranked alphabet $\Sigma=\left\{\sigma^{(2)}, \alpha^{(0)}\right\}$. Moreover, we let $\mathcal{A}=(Q, \delta, F)$ be the ( $\Sigma{\text {, } \text { Rat }_{\min ,+} \text { )-wta given by }}^{\text {b }}$

- $Q=\left\{p, p^{\prime}\right\}$ and $F_{p}=0$ and $F_{p^{\prime}}=\infty$ and
- $\delta_{0}(\varepsilon, \alpha, p)=1, \delta_{2}(p p, \sigma, p)=1, \delta_{2}\left(p p, \sigma, p^{\prime}\right)=0.5$, and $\delta_{2}\left(p^{\prime} p, \sigma, p\right)=1.5$, and all other values of $\delta_{0}$ and $\delta_{2}$ are $\infty$.
Figure 17.1 shows the fta-hypergraph of $\mathcal{A}$. Obviously, $\mathcal{A}$ is not bu deterministic. Moreover, for each $\xi \in \mathrm{T}_{\Sigma}$, we have

$$
\mathrm{h}_{\mathcal{A}}(\xi)_{p^{\prime}}=\left\{\begin{array}{ll}
\infty & \text { if } \xi=\alpha \\
\operatorname{size}(\xi)-0.5 & \text { otherwise }
\end{array} \quad \text { and } \quad \mathrm{h}_{\mathcal{A}}(\xi)_{p}=\operatorname{size}(\xi) .\right.
$$

Thus, for each $\xi \in \mathrm{T}_{\Sigma}$, we have $\llbracket \mathcal{A} \rrbracket(\xi)=\llbracket \mathcal{A} \rrbracket(\xi)=\operatorname{size}(\xi) \|^{2}$ Obviously, size is not a recognizable step mapping, because im(size) is an infinite set. Thus, by the contraposition of Lemma 17.1.1 $\operatorname{sub}(\mathcal{A})=$

[^17]

Figure 17.1: The ( $\left.\Sigma, \operatorname{Rat}_{\text {min },+}\right)$-wta $\mathcal{A}$.
$\left(Q^{\prime}, \delta^{\prime}, F^{\prime}\right)$ is not a crisp deterministic wta. In particular, the set $Q^{\prime}$ is not finite. We obtain that, in general, the subset method is not appropriate to determinize wta.

Finally, we note that the reverse of Lemma 17.1.1 does not hold. For instance, consider the $\left(\Sigma, \mathrm{Nat}_{\text {max },+}\right)$-wta $\mathcal{A}$ of Example 3.2 .4 which recognizes the weighted tree language height : $\mathrm{T}_{\Sigma} \rightarrow \mathbb{N}$. In the definition of $\mathcal{A}$, let us change the mapping $F$ by defining $F_{h}=-\infty$, and let us denote the obtained wta by $\mathcal{A}^{\prime}$. Obviously, $\llbracket \mathcal{A}^{\prime} \rrbracket=\widetilde{-\infty}$, i.e., a recognizable step mapping. On the other hand, $Q^{\prime}=\left\{\left.\binom{n}{0} \right\rvert\, n \in \mathbb{N}\right\}$, i.e., an infinite set. Hence $\operatorname{sub}(\mathcal{A})$ is not a crisp deterministic ( $\Sigma$, $\operatorname{Nat}_{\text {max },+}$ )-wta.

### 17.2 Negative results for determinization

We have seen that the subset method does not yield the desired determinization. Actually, there is no determinization which works for each $(\Sigma, \mathrm{B})$-wta because there exist recognizable weighted tree languages which are not bu deterministically recognizable. For instance, the weighted tree language height is not bu deterministically recognizable (cf. Corollary 6.1.5). In this section we show two further examples of such weighted tree languages.

Theorem 17.2.1. Den17 The weighted tree language zigzag : $\mathrm{T}_{\Sigma} \rightarrow \mathbb{N}$ defined in Example 3.2.15 is in $\operatorname{Rec}\left(\Sigma, \operatorname{Nat}_{\max ,+}\right)$, but not in $\operatorname{bud}-\operatorname{Rec}\left(\Sigma, \operatorname{Nat}_{\max ,+}\right)$. Hence, each $\left(\Sigma, \operatorname{Nat}_{\mathrm{max},+}\right)$-wta which recognizes zigzag is not bu determinizable.

Proof. We recall that $\Sigma=\left\{\sigma^{(2)}, \alpha^{(0)}\right\}$ and $\operatorname{Nat}_{\text {max },+}=\left(\mathbb{N}_{-\infty}, \max ,+,-\infty, 0\right)$. We prove the claim by contradiction. For this we assume that there exists a bu deterministic ( $\Sigma$, Nat $_{\text {max },+}$ )-wta $\mathcal{A}=(Q, \delta, F)$ such that $\llbracket \mathcal{A} \rrbracket=$ zigzag.

By Lemma 4.2.1 (1), for every $\xi \in \mathrm{T}_{\Sigma}$ we have $\left|\mathrm{Q}_{\neq-\infty}^{\mathrm{h}_{\mathcal{A}}}(\xi)\right| \leq 1$. If there exists a tree $\xi \in \mathrm{T}_{\Sigma}$ such that $\left|\mathrm{Q}_{\neq-\infty}^{\mathrm{h}_{\mathcal{A}}}(\xi)\right|=0$, then $\llbracket \mathcal{A} \rrbracket(\xi)=-\infty \neq \operatorname{zigzag}(\xi)$. Thus for every $\xi \in \mathrm{T}_{\Sigma}$ we have $\left|\mathrm{Q}_{\neq-\infty}^{\mathrm{h}_{\mathcal{A}}}(\xi)\right|=1$. Let us denote, for each $\xi \in \mathrm{T}_{\Sigma}$, the unique element in $\mathrm{Q}_{\neq-\infty}^{\mathrm{h}_{\mathcal{A}}}(\xi)$ by $q_{\xi}$.

If there exists a $\xi \in \mathrm{T}_{\Sigma}$ such that $F_{q_{\xi}}=-\infty$, then $\llbracket \mathcal{A} \rrbracket(\xi)=-\infty \neq \operatorname{zigzag}(\xi)$. Thus $F_{q_{\xi}} \neq-\infty$ for every $\xi \in \mathrm{T}_{\Sigma}$. In conclusion we know:

$$
\begin{equation*}
\left(\forall \xi \in \mathrm{T}_{\Sigma}\right):\left(\llbracket \mathcal{A} \rrbracket(\xi)=\mathrm{h}_{\mathcal{A}}(\xi)_{q_{\xi}}+F_{q_{\xi}}=\operatorname{zigzag}(\xi)\right) \wedge\left((\forall q \in Q): \text { if } q \neq q_{\xi}, \text { then } \mathrm{h}_{\mathcal{A}}(\xi)_{q}=-\infty\right) \tag{17.1}
\end{equation*}
$$

Since $Q$ is finite and $\operatorname{zigzag}\left(\mathrm{T}_{\Sigma}\right)$ is not, there exist trees $\xi_{1}, \xi_{2} \in \mathrm{~T}_{\Sigma}$ such that $q_{\xi_{1}}=q_{\xi_{2}}$ and $\operatorname{zigzag}\left(\xi_{1}\right) \neq \operatorname{zigzag}\left(\xi_{2}\right)$. Since $\mathcal{A}$ is bu deterministic, we have $q_{\sigma\left(\alpha, \xi_{1}\right)}=q_{\sigma\left(\alpha, \xi_{2}\right)}$. Let us abbreviate
$q_{\xi_{1}}$ by $q$ and $q_{\sigma\left(\alpha, \xi_{1}\right)}$ by $q^{\prime}$, respectively. Due to the definition of zigzag we have that $\operatorname{zigzag}\left(\sigma\left(\alpha, \xi_{1}\right)\right)=$ $\operatorname{zigzag}\left(\sigma\left(\alpha, \xi_{2}\right)\right)=1$.

Then we can derive the following sequence of implications:

$$
\begin{array}{rlrl}
\operatorname{zigzag}\left(\sigma\left(\alpha, \xi_{1}\right)\right) & = & \operatorname{zigzag}\left(\sigma\left(\alpha, \xi_{2}\right)\right) \\
\Rightarrow \quad \mathrm{h}_{\mathcal{A}}\left(\sigma\left(\alpha, \xi_{1}\right)\right)_{q^{\prime}}+F_{q^{\prime}}= & \mathrm{h}_{\mathcal{A}}\left(\sigma\left(\alpha, \xi_{2}\right)\right)_{q^{\prime}}+F_{q^{\prime}} \\
& \left(\text { by (17.1) and because } q_{\sigma\left(\alpha, \xi_{1}\right)}=q_{\sigma\left(\alpha, \xi_{2}\right)}\right) \\
\Rightarrow \quad \mathrm{h}_{\mathcal{A}}\left(\sigma\left(\alpha, \xi_{1}\right)\right)_{q^{\prime}}= & \left.\mathrm{h}_{\mathcal{A}}\left(\sigma\left(\alpha, \xi_{2}\right)\right)_{q^{\prime}} \quad \text { (because } F_{q^{\prime}} \neq-\infty\right) \\
\Rightarrow \quad \mathrm{h}_{\mathcal{A}}(\alpha)_{q_{\alpha}}+\mathrm{h}_{\mathcal{A}}\left(\xi_{1}\right)_{q}+\delta_{2}\left(q_{\alpha} q, \sigma, q^{\prime}\right)= & \mathrm{h}_{\mathcal{A}}(\alpha)_{q_{\alpha}}+\mathrm{h}_{\mathcal{A}}\left(\xi_{2}\right)_{q}+\delta_{2}\left(q_{\alpha} q, \sigma, q^{\prime}\right) \\
& & \left(\text { by definition of } \mathrm{h}_{\mathcal{A}} \text { and because } q=q_{\xi_{1}}=q_{\xi_{2}}\right) \\
\Rightarrow \quad \mathrm{h}_{\mathcal{A}}\left(\xi_{1}\right)_{q}= & \mathrm{h}_{\mathcal{A}}\left(\xi_{2}\right)_{q} \\
& \left(\operatorname{because} \mathrm{~h}_{\mathcal{A}}(\alpha)_{q_{\alpha}} \neq-\infty \neq \delta_{2}\left(q_{\alpha} q, \sigma, q^{\prime}\right)\right) \\
\Rightarrow \quad & \mathrm{h}_{\mathcal{A}}\left(\xi_{1}\right)_{q}+F_{q}= & \mathrm{h}_{\mathcal{A}}\left(\xi_{2}\right)_{q}+F_{q} \quad\left(\text { because } F_{q} \neq-\infty\right) \\
\Rightarrow \quad \operatorname{zigzag}\left(\xi_{1}\right) & = & \operatorname{zigzag}\left(\xi_{2}\right) \quad(\text { by (17.1) })
\end{array}
$$

This is a contradiction. Thus zigzag $\notin \operatorname{bud}-\operatorname{Rec}\left(\Sigma, \operatorname{Nat}_{\max ,+}\right)$. As a consequence, each $\left(\Sigma, N a t_{\text {max },+}\right)$-wta which recognizes zigzag (including the one in Example 3.2.15) is not bu determinizable.

Theorem 17.2.2. (cf. BV03, Thm. 6.3]) The weighted tree language $(\exp +1): \mathrm{T}_{\Sigma} \rightarrow \mathbb{N}$ defined in Example 3.2.9 is in $\operatorname{Rec}(\Sigma$, Rat), but not in bud-Rec $(\Sigma$, Rat), where Rat $=(\mathbb{Q},+, \cdot, 0,1)$ is the field over the rational numbers. Hence, each ( $\Sigma$, Rat)-wta which recognizes $(\exp +1)$ is not bu determinizable.

Proof. We recall that $\Sigma=\left\{\gamma^{(1)}, \alpha^{(0)}\right\}$. In Example 3.2.9, we gave a ( $\left.\Sigma, \mathrm{Nat}\right)$-wta over the semiring Nat $=(\mathbb{N},+, \cdot, 0,1)$ of natural numbers which recognizes $(\exp +1)$. We can view this wta as a ( $\Sigma$, Rat)wta (cf. Section 3.7), which we denote by $\mathcal{A}_{\mathbb{Q}}$ in this proof. Hence $(\exp +1) \in \operatorname{Rec}(\Sigma, \operatorname{Rat})$.

Next we prove that $(\exp +1) \notin \operatorname{bud}-\operatorname{Rec}(\Sigma, \operatorname{Rat})$, and we prove this by contradiction. We assume that $(\exp +1) \in \operatorname{bud}-\operatorname{Rec}(\Sigma, R a t)$. Then, by Theorem 7.3 .3 there exists a bu deterministic ( $\Sigma$, Rat)-wta $\mathcal{A}=(Q, \delta, F)$ with identity root weights such that $\llbracket \mathcal{A} \rrbracket=(\exp +1)$.

By Lemma $4.2 .1(1)$, for every $n \in \mathbb{N}$ we have $\left|\mathrm{Q}_{\neq 0}^{\mathrm{h} \mathcal{A}}\left(\gamma^{n}(\alpha)\right)\right| \leq 1$. If $\left|\mathrm{Q}_{\neq 0}^{\mathrm{h} \mathcal{A}}\left(\gamma^{n}(\alpha)\right)\right|=0$ or $\mathrm{Q}_{\neq 0}^{\mathrm{h} \mathcal{A}}\left(\gamma^{n}(\alpha)\right) \cap$ $\operatorname{supp}(F)=\emptyset$, then $\llbracket \mathcal{A} \rrbracket\left(\gamma^{n}(\alpha)\right)=0$ which is a contradiction. Thus $\left|\mathrm{Q}_{\neq 0}^{\mathrm{h}_{\mathcal{A}}}\left(\gamma^{n}(\alpha)\right) \cap \operatorname{supp}(F)\right|=1$. We denote the state in the set $\mathrm{Q}_{\neq 0}^{\mathrm{h} \mathcal{A}}\left(\gamma^{n}(\alpha)\right) \cap \operatorname{supp}(F)$ by $q_{n}$. Thus

$$
\begin{equation*}
\llbracket \mathcal{A} \rrbracket\left(\gamma^{n}(\alpha)\right)=\mathrm{h}_{\mathcal{A}}\left(\gamma^{n}(\alpha)\right)_{q_{n}}=2^{n}+1 . \tag{17.2}
\end{equation*}
$$

It is an easy exercise to show that, for every $q \in Q$ and $n \geq 1$ :

$$
\delta_{0}(\varepsilon, \alpha, q)= \begin{cases}2 & \text { if } q=q_{0} \\ 0 & \text { otherwise }\end{cases}
$$

and

$$
\delta_{1}\left(q_{n-1}, \gamma, q\right)= \begin{cases}\left(2^{n-1}+1\right)^{-1} \cdot\left(2^{n}+1\right) & \text { if } q=q_{n} \\ 0 & \text { otherwise }\end{cases}
$$

Since $Q$ is finite, there exist $n, i \in \mathbb{N}$ such that $0 \leq i<n$ and $q_{i}=q_{n}$. Then we can calculate as follows:

$$
\begin{aligned}
\llbracket \mathcal{A} \rrbracket\left(\gamma^{n+1}(\alpha)\right) & =\mathrm{h}_{\mathcal{A}}\left(\gamma^{n}(\alpha)\right)_{q_{n}} \cdot \delta_{1}\left(q_{n}, \gamma, q_{n+1}\right)=\left(2^{n}+1\right) \cdot \delta_{1}\left(q_{n}, \gamma, q_{n+1}\right) \\
& =\left(2^{n}+1\right) \cdot \delta_{1}\left(q_{i}, \gamma, q_{i+1}\right)=\left(2^{n}+1\right) \cdot\left(2^{i}+1\right)^{-1} \cdot\left(2^{i+1}+1\right) \neq 2^{n+1}+1 .
\end{aligned}
$$

where the inequality follows from a straightforward calculation. However, $\llbracket \mathcal{A} \rrbracket\left(\gamma^{n+1}(\alpha)\right) \neq 2^{n+1}+1$ which contradicts Equation (17.2). Hence $(\exp +1) \notin \operatorname{bud}-\operatorname{Rec}(\Sigma, \operatorname{Rat})$. As a consequence, each ( $\Sigma$, Rat)-wta which recognizes $(\exp +1)$ (including $\mathcal{A}_{\mathbb{Q}}$ from the beginning of the proof) is not bu determinizable.

### 17.3 Positive result for determinization

In this section we recall the approach of BVM10 (based on KM05) and adapt it to our notations.
We recall that $\mathrm{V}(\mathcal{A})=\left(B^{Q}, \delta_{\mathcal{A}}\right)$ is the vector algebra of $\mathcal{A}$, and that $\mathbb{O}_{Q} \in B^{Q}$ is the $Q$-vector over B which contains $\mathbb{0}$ in each component (cf. Section 2.7). We assume that there exists $\alpha \in \Sigma^{(0)}$ such that $\delta_{\mathcal{A}}(\alpha)() \neq \mathbb{O}_{Q}$.

In this section we use the addition of $B^{Q}$-vectors and the multiplication of a $Q$-vector with a scalar both from the left and the right (cf. Section 2.7). Also we recall that, for every $\xi \in \mathrm{T}_{\Sigma}$ and $q \in Q$ we have $\mathrm{h}_{\mathcal{A}}(\xi)_{q}=\bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}}(q, \xi)} \mathrm{wt}(\xi, \rho)$ (cf. (5.2)).

In Subsection 17.3.1 we start by defining the concepts of factorization and twinning property, we show the construction of the triple $\operatorname{det}_{(f, g)}(\mathcal{A})$ for some wta $\mathcal{A}$ and some factorization $(f, g)$, and we show the main determinization theorem of this section (cf. Theorem 17.3.2) and an application to a particular wta. In Subsections 17.3 .2 and 17.3.3, we elaborate properties and examples of factorizations and of the twinning property, respectively. Finally, in Subsection 17.3.4, we show the proof of Theorem 17.3.2,

### 17.3.1 Determinization by factorization

The idea of factorization is to represent each $Q$-vector $u \in B^{Q}$ as a scalar product $b \cdot u^{\prime}$ where $b \in B$ and $u^{\prime} \in B^{Q}$; moreover, the way in which $Q$-vectors are split into elements of $B$ and remaining $Q$-vectors in $B^{Q}$ should be uniform. Formally, a factorization over $B^{Q}$ (or: factorization) is a pair $(f, g)$ of mappings, where

$$
f: B^{Q} \backslash\left\{\mathbb{O}_{Q}\right\} \rightarrow B^{Q} \text { and } g: B^{Q} \backslash\left\{\mathbb{O}_{Q}\right\} \rightarrow B
$$

such that $u=g(u) \cdot f(u)$ for every $u \in B^{Q} \backslash\left\{\mathbb{O}_{Q}\right\}$. Thus $\mathbb{O}_{Q} \notin \operatorname{im}(f)$ and $\mathbb{O} \notin \operatorname{im}(g)$. We call $g(u)$ the common factor of $u$ and $f(u)$ the remainder of $u$. The factorization $(f, g)$ is called

- maximal if, for every $u \in B^{Q} \backslash\left\{\mathbb{O}_{Q}\right\}$ and $b \in B$ with $b \cdot u \neq \mathbb{O}_{Q}$, we have $f(u)=f(b \cdot u)$, and
- trivial if $f(u)=u$ and $g(u)=\mathbb{1}$ for every $u \in B^{Q} \backslash\left\{\mathbb{O}_{Q}\right\}$.

We recall from Section 6.1 that, for every $c \in \mathrm{C}_{\Sigma}$ and $p, q \in Q$, we denote the set of all runs of $\mathcal{A}$ on $c$ and the set of all $(q, p)$-runs of $\mathcal{A}$ on $c$ by $\mathrm{R}_{\mathcal{A}}(c)$ and $\mathrm{R}_{\mathcal{A}}(q, c, p)$, respectively. Also the weight of a run on a context is defined in Section 6.1 Moreover, for every $\xi \in \mathrm{T}_{\Sigma}, \rho \in \mathrm{R}_{\mathcal{A}}(\xi)$, and $w \in \operatorname{pos}(\xi)$, we defined the run $\left.\rho\right|^{w}$ on the context $\left.\xi\right|^{w}$.

Additionally, for each $\xi \in \mathrm{T}_{\Sigma} \cup \mathrm{C}_{\Sigma}$ and $R \subseteq \mathrm{R}_{\mathcal{A}}(\xi)$, we define

$$
\mathrm{wt}(R)=\bigoplus_{\rho \in R} \mathrm{wt}(\xi, \rho)
$$

By Observation 2.6.7 if B is extremal, then there exists $\rho \in R$ such that $\mathrm{wt}(\xi, \rho)=\mathrm{wt}(R)$. We call such a run victorious in $R$.

The wta $\mathcal{A}$ has the twinning property if, for every $p, q \in Q, \xi \in \mathrm{~T}_{\Sigma}$, and $c \in \mathrm{C}_{\Sigma}$, the following holds:

$$
\begin{aligned}
& \text { If } \operatorname{wt}\left(\mathrm{R}_{\mathcal{A}}(p, c, p)\right) \neq \mathbb{0} \text { and } \operatorname{wt}\left(\mathrm{R}_{\mathcal{A}}(p, \xi)\right) \neq \mathbb{0} \text { and } \\
& \operatorname{wt}\left(\mathrm{R}_{\mathcal{A}}(q, c, q)\right) \neq \mathbb{0} \text { and } \operatorname{wt}\left(\mathrm{R}_{\mathcal{A}}(q, \xi)\right) \neq \mathbb{0} \\
& \text { then } \operatorname{wt}\left(\mathrm{R}_{\mathcal{A}}(p, c, p)\right)=\operatorname{wt}\left(\mathrm{R}_{\mathcal{A}}(q, c, q)\right)
\end{aligned}
$$

The following determinization by factorization is based on KM05, Sect. 3.3] and [BVM10, p. 11]. Roughly speaking, it differs from the subset method as follows. In the subset method, for every $u_{1}, \ldots, u_{k} \in B^{Q}$ and $\sigma \in \Sigma^{(k)}$, the transition

$$
\left(u_{1} \cdots u_{k}, \sigma, \delta_{\mathcal{A}}(\sigma)\left(u_{1}, \ldots, u_{k}\right)\right) \text { with weight } \mathbb{1}
$$

is constructed. In the determinization by factorization, the $Q$-vector $\delta_{\mathcal{A}}(\sigma)\left(u_{1}, \ldots, u_{k}\right)$ is factorized as

$$
\delta_{\mathcal{A}}(\sigma)\left(u_{1}, \ldots, u_{k}\right)=\underbrace{g\left(\delta_{\mathcal{A}}(\sigma)\left(u_{1}, \ldots, u_{k}\right)\right)}_{\text {common factor in } B} \cdot \underbrace{f\left(\delta_{\mathcal{A}}(\sigma)\left(u_{1}, \ldots, u_{k}\right)\right)}_{\text {remainder in } B^{Q}}
$$

and the transition

$$
\left(u_{1} \cdots u_{k}, \sigma, f\left(\delta_{\mathcal{A}}(\sigma)\left(u_{1}, \ldots, u_{k}\right)\right)\right) \text { with weight } g\left(\delta_{\mathcal{A}}(\sigma)\left(u_{1}, \ldots, u_{k}\right)\right)
$$

is constructed. Thus, the remainder $f\left(\delta_{\mathcal{A}}(\sigma)\left(u_{1}, \ldots, u_{k}\right)\right)$ is the target state of the transition and the common factor $g\left(\delta_{\mathcal{A}}(\sigma)\left(u_{1}, \ldots, u_{k}\right)\right)$ is its weight. We do this because, as we will see, in certain cases the set of remainders will be a finite set, while the set of vectors produced by the subset method is infinite.

In the following we give the definition of determinization by factorization. Like the subset method, it is not a construction in the sense described in the Introduction, because the set $Q^{\prime}$ might be infinite and in this case the triple $\left(Q^{\prime}, \delta^{\prime}, F^{\prime}\right)$ is not a bu deterministic wta. We will need further conditions to assure that $\left(Q^{\prime}, \delta^{\prime}, F^{\prime}\right)$ is a bu determinstic wta (cf. Theorem 17.3.2).

Definition 17.3.1. BVM10, p. 11] Let $\Sigma$ be a ranked alphabet and $\mathrm{B}=(B, \oplus, \otimes, 0, \mathbb{1})$ be a semiring. Moreover, let $\mathcal{A}=(Q, \delta, F)$ be a $(\Sigma, \mathrm{B})$-wta and $(f, g)$ be a factorization over $B^{Q}$. The determinization by factorization transforms $\mathcal{A}$ and $(f, g)$ into the triple $\operatorname{det}_{(f, g)}(\mathcal{A})=\left(Q^{\prime}, \delta^{\prime}, F^{\prime}\right)$ where

- $Q^{\prime}$ is the smallest set $P \subseteq B^{Q}$ such that, for every $k \in \mathbb{N}, \sigma \in \Sigma^{(k)}, u_{1}, \ldots, u_{k} \in P$ : if $\delta_{\mathcal{A}}(\sigma)\left(u_{1}, \ldots, u_{k}\right) \neq \mathbb{O}_{Q}$, then $f\left(\delta_{\mathcal{A}}(\sigma)\left(u_{1}, \ldots, u_{k}\right)\right) \in P$,
- $\delta^{\prime}=\left(\delta_{k}^{\prime} \mid k \in \mathbb{N}\right)$ is the family of mappings $\delta_{k}^{\prime}:\left(Q^{\prime}\right)^{k} \times \Sigma^{(k)} \times Q^{\prime} \rightarrow B$ defined for every $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}, u_{1}, \ldots, u_{k} \in Q^{\prime}$, and $u \in Q^{\prime}$ by

$$
\delta_{k}^{\prime}\left(u_{1} \ldots u_{k}, \sigma, u\right)=\left\{\begin{array}{lc}
g\left(\delta_{\mathcal{A}}(\sigma)\left(u_{1}, \ldots, u_{k}\right)\right) & \text { if } \delta_{\mathcal{A}}(\sigma)\left(u_{1}, \ldots, u_{k}\right) \neq \mathbb{O}_{Q} \text { and } \\
& u=f\left(\delta_{\mathcal{A}}(\sigma)\left(u_{1}, \ldots, u_{k}\right)\right) \\
0 & \text { otherwise }
\end{array}\right.
$$

- $F_{u}^{\prime}=\bigoplus_{q \in Q} u_{q} \otimes F_{q}$ for every $u \in Q^{\prime}$.

We call the triple $\operatorname{det}_{(f, g)}(\mathcal{A})$ the determinization of $\mathcal{A}$ by $(f, g)$.
In the following, we list some properties of determinization by factorization.
(1) Since $\mathbb{O}_{Q} \notin \operatorname{im}(f)$, we have $\mathbb{O}_{Q} \notin Q^{\prime}$.
(2) If $Q^{\prime}$ is finite, then $\operatorname{det}_{(f, g)}(\mathcal{A})$ is a bu deterministic $(\Sigma, \mathrm{B})$-wta. However, even in this case $\operatorname{det}_{(f, g)}(\mathcal{A})$ is not necessarily crisp deterministic.
(3) If $(f, g)$ is the trivial factorization, then the determinization by factorization is the same as the subset method except that the latter may generate the state $\mathbb{D}_{Q}$.
(4) If $(f, g)$ is the trivial factorization and B is locally finite, then the determinization by factorization is the same as the subset method except that the latter may generate the state $\mathbb{0}_{Q}$; in particular, the state set $Q^{\prime}$ is finite. Hence $\operatorname{det}_{(f, g)}(\mathcal{A})$ is a bu deterministic $(\Sigma, \mathrm{B})$-wta. Moreover, $\operatorname{sub}(\mathcal{A})$ and $\operatorname{det}_{(f, g)}(\mathcal{A})$ are equivalent, because the state $\mathbb{D}_{Q}$ is useless for $\operatorname{sub}(\mathcal{A})$. More precisely, (a) the operation $\delta_{\mathcal{A}}(\sigma)$ yields $\mathbb{O}_{Q}$ if there exists an argument which is $\mathbb{O}_{Q}(c f .(3.2))$ and (b) $F^{\prime}\left(\mathbb{O}_{Q}\right)=\mathbb{0}$. Thus, if $\mathbb{O}_{Q} \in \operatorname{im}\left(\mathrm{~h}_{\mathcal{A}}\right)$, then we can also drop $\mathbb{O}_{Q}$ from the state set without changing the semantics of $\operatorname{sub}(\mathcal{A})$. Then, by applying the construction of Theorem 7.2.1 to $\operatorname{det}_{(f, g)}(\mathcal{A})$ (which transforms a given wta into an equivalent total one such that bu determinism is preserved), we obtain a crisp deterministic wta which is equivalent to $\operatorname{det}_{(f, g)}(\mathcal{A})$. This construction can be compared to the construction of the Nerode wsa in CDIV10, Sec. 6].
(5) If B is the semiring Boole, then there exists exactly one factorization $(f, g)$, viz. the trivial and maximal factorization defined by $g(u)=1$ and $f(u)=u$. Since $\mathbb{B}$ is finite, determinization by factorization is a particular case of the one described in (4).
The main result of this chapter will be the following theorem.

Theorem 17.3.2. BVM10, Thm. 5.2] Let $\Sigma$ be a ranked alphabet, B be an extremal and commutative semiring, $\mathcal{A}$ be a $(\Sigma, \mathrm{B})$-wta with the twinning property, and $(f, g)$ be a maximal factorization over $B^{Q}$. Then
(1) $\operatorname{det}_{(f, g)}(\mathcal{A})$ is a bu deterministic $(\Sigma, \mathrm{B})-w t a$ (finiteness),
(2) $\llbracket \operatorname{det}_{(f, g)}(\mathcal{A}) \rrbracket=\llbracket \mathcal{A} \rrbracket$ (correctness), and
(3) $\operatorname{det}_{(f, g)}(\mathcal{A})$ is minimal with respect to the number of states among all bu deterministic $(\Sigma, \mathrm{B})-$ wta which are obtained from $\mathcal{A}$ by determinization by factorization (minimality).

Proof. It follows from Theorems $17.3 .17,17.3 .18$ and 17.3 .19 which are proved in the next subsections.

We show six examples of extremal and commutative semirings:

- the semiring Boole $=(\mathbb{B}, \vee, \wedge, 0,1)$,
- the arctic semiring $\mathrm{Nat}_{\text {max },+}=\left(\mathbb{N}_{-\infty}, \max ,+,-\infty, 0\right)$,
- the semiring $\operatorname{Nat}_{\max ,+, n}=\left([0, n]_{-\infty}, \max , \hat{+}_{n},-\infty, 0\right)$ where $[0, n]_{-\infty}$ is an abbreviation of $[0, n] \cup$ $\{-\infty\}$ (defined in Example 2.6.9(9)),
- the tropical semifield $\operatorname{Rat}_{\min ,+}=\left(\mathbb{Q}_{\infty}, \min ,+, \infty, 0\right)$,
- the semifield $\left(\mathbb{R}_{\geq 0}, \max , \cdot, 0,1\right)$, and
- the semiring Viterbi $=([0,1], \max , \cdot, 0,1)$.

We note that, due to Lemma 17.3 .6 (below), Theorem 17.3 .2 cannot be used to determinize a $(\Sigma, \mathrm{B})$ wta $(Q, \delta, F)$ with at least two states if B is not zero-divisor free, because in this case there does not exist a maximal factorization of $B^{Q}$.

Example 17.3.3. Again, we consider the $\left(\Sigma\right.$, Rat $\left._{\min ,+}\right)$-wta $\mathcal{A}=(Q, \delta, F)$ given in Example 17.1.2, For convenience, we recall that

- $Q=\left\{p, p^{\prime}\right\}$ and $F_{p}=0$ and $F_{p^{\prime}}=\infty$ and
- $\delta_{0}(\varepsilon, \alpha, p)=1, \delta_{2}(p p, \sigma, p)=1, \delta_{2}\left(p p, \sigma, p^{\prime}\right)=0.5$, and $\delta_{2}\left(p^{\prime} p, \sigma, p\right)=1.5$, and all other values of $\delta_{0}$ and $\delta_{2}$ are $\infty$.
As it turned out in Example 17.1.2, the subset method is not applicable to determinize $\mathcal{A}$. Now we determinize $\mathcal{A}$ by factorization according to Definition 17.3.1,

For this, we define the mappings $f:\left(\mathbb{Q}_{\infty}\right)^{Q} \backslash\left\{\infty_{Q}\right\} \rightarrow\left(\mathbb{Q}_{\infty}\right)^{Q}$ and $g:\left(\mathbb{Q}_{\infty}\right)^{Q} \backslash\left\{\infty_{Q}\right\} \rightarrow \mathbb{Q}_{\infty}$ defined for each $u=\binom{u_{p}}{u_{p^{\prime}}}$ in $\left(\mathbb{Q}_{\infty}\right)^{Q} \backslash\left\{\infty_{Q}\right\}$ by

$$
f(u)_{q}=u_{q}-g(u) \text { for each } q \in Q \quad \text { and } \quad g(u)=\min \left(u_{p}, u_{p^{\prime}}\right)
$$

Then $(f, g)$ is a maximal factorization.
Then $\operatorname{det}_{(f, g)}(\mathcal{A})=\left(Q^{\prime}, \delta^{\prime}, F^{\prime}\right)$, which we will abbreviate by $\mathcal{A}^{\prime}$ in the sequel, is defined by (cf. Figure (17.2)

- $Q^{\prime}=\left\{\binom{0}{\infty},\binom{0.5}{0}\right\}$ and $F^{\prime}\left(\binom{0}{\infty}\right)=0$ and $F^{\prime}\left(\binom{0.5}{0}\right)=0.5$
- $\delta_{0}^{\prime}\left(\varepsilon, \alpha,\binom{0}{\infty}\right)=1$ and

$$
\begin{array}{cc}
\delta_{2}^{\prime}\left(\binom{0}{\infty}\binom{0}{\infty}, \sigma,\binom{0.5}{0}\right)=0.5, & \delta_{2}^{\prime}\left(\binom{0}{\infty}\binom{0.5}{0}, \sigma,\binom{0.5}{0}\right)=1, \\
\delta_{2}^{\prime}\left(\binom{0.5}{0}\binom{0}{\infty}, \sigma,\binom{0.5}{0}\right)=1, & \delta_{2}^{\prime}\left(\binom{0.5}{0}\binom{0.5}{0}, \sigma,\binom{0.5}{0}\right)=1.5
\end{array}
$$

and $\delta_{0}^{\prime}(\varepsilon, \alpha, p)=\delta_{2}^{\prime}(p q, \sigma, r)=\infty$ for every other combination of $p, q, r \in Q^{\prime}$.
Since $Q^{\prime}$ is finite, $\operatorname{det}_{(f, g)}(\mathcal{A})$ is a bu deterministic $\left(\Sigma, \operatorname{Rat}_{\min ,+}\right)$-wta. For two transitions of $\operatorname{det}_{(f, g)}(\mathcal{A})$, we illustrate the way in which they are constructed. We abbreviate vectors of the form $\binom{a-c}{b-c}$ by $\binom{a}{b}-c$ for every $a, b, c \in \mathbb{Q}_{\geq 0}$.


Figure 17.2: $\operatorname{The}\left(\Sigma, \operatorname{Rat}_{\text {min },+}\right)$-wta $\operatorname{det}_{(f, g)}(\mathcal{A})=\left(Q^{\prime}, \delta^{\prime}, F^{\prime}\right)$.
$\underline{\delta_{0}^{\prime}\left(\varepsilon, \alpha,\binom{0}{\infty}\right)}$ : We have $\delta_{\mathcal{A}}(\alpha)()=\binom{1}{\infty} \neq\binom{\infty}{\infty}$ and $g\left(\binom{1}{\infty}\right)=1$. Thus

$$
f\left(\delta_{\mathcal{A}}(\alpha)()\right)=f\left(\binom{1}{\infty}\right)=\binom{1}{\infty}-g\left(\binom{1}{\infty}\right)=\binom{1}{\infty}-1=\binom{0}{\infty} .
$$

Thus

$$
\delta_{0}^{\prime}\left(\varepsilon, \alpha,\binom{0}{\infty}\right)=g\left(\delta_{\mathcal{A}}(\alpha)()\right)=g\left(\binom{1}{\infty}\right)=1
$$

$\underline{\delta_{2}^{\prime}\left(\binom{0}{\infty}\binom{0.5}{0}, \sigma,\binom{0.5}{0}\right): \text { We have }}$

$$
\delta_{\mathcal{A}}(\sigma)\left(\binom{0}{\infty}\binom{0.5}{0}\right)=\binom{\min (0+0.5+1, \infty+0.5+1.5)}{0+0.5+0.5}=\binom{1.5}{1} .
$$

Since $g\left(\left({ }_{1}^{1.5}\right)\right)=1$, we have

$$
f\left(\delta_{\mathcal{A}}(\sigma)\left(\binom{0}{\infty},\binom{0.5}{0}\right)\right)=f\left(\binom{1.5}{1}\right)=\binom{1.5}{1}-g\left(\binom{1.5}{1}\right)=\binom{1.5}{1}-1=\binom{0.5}{0} .
$$

Thus we obtain

$$
\delta_{2}^{\prime}\left(\binom{0}{\infty}\binom{0.5}{0}, \sigma,\binom{0.5}{0}\right)=g\left(\delta_{\mathcal{A}}(\sigma)\left(\binom{0}{\infty}\binom{0.5}{0}\right)\right)=g\left(\binom{1.5}{1}\right)=1 .
$$

Finally we show how the value of $F^{\prime}\left(\binom{0.5}{0}\right)$ is obtained.

$$
F^{\prime}\left(\binom{0.5}{0}\right)=\left(\begin{array}{ll}
0.5 & 0
\end{array}\right) \cdot\binom{0}{\infty}=\min (0.5+0,0+\infty)=0.5 .
$$

Indeed, $\llbracket \operatorname{det}_{(f, g)}(\mathcal{A}) \rrbracket(\xi)=\operatorname{size}(\xi)$ for each $\xi \in \mathrm{T}_{\Sigma}$ which can be seen as follows. Let us abbreviate $\mathrm{h}_{\operatorname{det}_{(f, g)}(\mathcal{A})}$ by h, and the states $\binom{0}{\infty}$ and $\binom{0.5}{0}$ by $q_{1}$ and $q_{2}$, respectively. Then for each $\xi \in \mathrm{T}_{\Sigma}$ we have

$$
\mathrm{h}(\xi)_{q_{1}}=\left\{\begin{array}{ll}
1 & \text { if } \xi=\alpha \\
\infty & \text { otherwise }
\end{array} \text { and } \mathrm{h}(\xi)_{q_{2}}= \begin{cases}\infty & \text { if } \xi=\alpha \\
\operatorname{size}(\xi)-0.5 & \text { otherwise }\end{cases}\right.
$$

This can be proved easily by induction on $\mathrm{T}_{\Sigma}$. Then

$$
\llbracket \operatorname{det}_{(f, g)}(\mathcal{A}) \rrbracket(\xi)=\min \left(\mathrm{h}(\xi)_{q_{1}}+F^{\prime}\left(q_{1}\right), \mathrm{h}(\xi)_{q_{2}}+F^{\prime}\left(q_{2}\right)\right)=\operatorname{size}(\xi)
$$

where the last equality follows from case analysis $\xi=\alpha$ or $\xi \neq \alpha$.
In BVM10, Ex. 3.1] another example of a wta $\mathcal{A}$ is shown for which $\operatorname{sub}(\mathcal{A})$ is not finite, $\operatorname{but}^{\operatorname{det}}{ }_{(f, g)}(\mathcal{A})$ is a bu deterministic wta which is equivalent to $\mathcal{A}$.

### 17.3.2 Examples and properties of factorizations

We start with some examples of factorizations for extremal and commutative semirings.
Example 17.3.4. In the following list we show pairs where each pair consists of an extremal and commutative semiring B and a maximal factorization $(f, g)$ over $B^{Q}$. We assume that $u \in B^{Q} \backslash\left\{\mathbb{O}_{Q}\right\}$.

1. the semiring Boole $=(\mathbb{B}, \vee, \wedge, 0,1)$ and the trivial factorization
2. $\operatorname{Nat}_{\max ,+}=\left(\mathbb{N}_{-\infty}, \max ,+,-\infty, 0\right)$ and $g(u)=\min \left(u_{q} \mid q \in Q, u_{q} \neq-\infty\right)$ and $f(u)_{q}=u_{q}-g(u)$ for each $q \in Q$
3. $\operatorname{Nat}_{\max ,+, n}=\left([0, n]_{-\infty}, \max , \hat{+}_{n},-\infty, 0\right)$ where $n \in \mathbb{N}_{+}$and $g(u)=\min \left(u_{q} \mid q \in Q, u_{q} \neq-\infty\right)$ and $f(u)_{q}=u_{q}-g(u)$ for every $q \in Q$
4. $\operatorname{Rat}_{\min ,+}=\left(\mathbb{Q}_{\infty}, \min ,+, \infty, 0\right)$ and $g(u)=\min \left(u_{q} \mid q \in Q\right)$ and $f(u)_{q}=u_{q}-g(u)$ for each $q \in Q$
5. $\left(\mathbb{R}_{\geq 0}, \max , \cdot, 0,1\right)$ and $g(u)=\max \left(u_{q} \mid q \in Q\right)$ and $f(u)=\frac{1}{g(u)} \cdot u$
6. the semiring Viterbi $=([0,1], \max , \cdot, 0,1)$ and $g(u)=\max \left(u_{q} \mid q \in Q\right)$ and $f(u)=\frac{1}{g(u)} \cdot u$

Next we will show some properties of factorizations.
The last two examples in Example 17.3 .4 show particular semifields with maximal factorizations. In fact, for each zero-sum free semifield, we can show a general construction of a maximal factorization.

Lemma 17.3.5. BVM10, Lm. 4.2] Let B be a zero-sum free semifield. Then $(f, g)$ is a maximal factorization where $g(u)=\bigoplus_{q \in Q} u_{q}$ and $f(u)=g(u)^{-1} \cdot u$ for each $u \in B^{Q} \backslash\left\{\mathbb{O}_{Q}\right\}$.

Proof. First we show that $(f, g)$ is a factorization. Let $u \in B^{Q} \backslash\left\{\mathbb{D}_{Q}\right\}$. Since B is zero-sum free, $g(u) \neq \mathbb{0}$ and hence $g(u) \cdot f(u)=g(u) \cdot\left(g(u)^{-1} \cdot u\right)=\left(g(u) \otimes g(u)^{-1}\right) \cdot u=u$.

Second we show that $(f, g)$ is maximal. Let $b \in B$ such that $b \cdot u \neq \mathbb{O}_{Q}$. Moreover let $q \in Q$. Then

$$
\begin{aligned}
(f(b \cdot u))_{q} & =\left(g(b \cdot u)^{-1} \cdot(b \cdot u)\right)_{q}=\left(\bigoplus_{q^{\prime} \in Q} b \otimes u_{q^{\prime}}\right)^{-1} \otimes b \otimes u_{q} \\
& =\left(b \otimes \bigoplus_{q^{\prime} \in Q} u_{q^{\prime}}\right)^{-1} \otimes b \otimes u_{q}=\left(\bigoplus_{q^{\prime} \in Q} u_{q^{\prime}}\right)^{-1} \otimes b^{-1} \otimes b \otimes u_{q} \\
& =g(u)^{-1} \otimes u_{q}=(f(u))_{q} .
\end{aligned}
$$

The next lemma shows that, for a commutative semiring with zero-divisors and $|Q| \geq 2$, there does not exist a maximal factorization.

Lemma 17.3.6. BVM10, Lm. 4.4] If B is commutative and $(f, g)$ is a maximal factorization, then $|Q|=1$ or B is zero-divisor free.

Proof. If $|Q|=1$, then the statement holds. So assume that $|Q| \geq 2$ and let $a_{1} \in B \backslash\{\mathbb{O}\}$ and $a_{2} \in B$ be such that $a_{1} \otimes a_{2}=\mathbb{O}$. We choose a pair $q_{1}, q_{2} \in Q$ such that $q_{1} \neq q_{2}$. For each $i \in\{1,2\}$, we define the $Q$-vector $u_{i} \in B^{Q}$ as the $Q$-vector whose $q_{i}$-component is $\mathbb{1}$ while the other components are $\mathbb{O}$. Since $(f, g)$ is maximal and $a_{1} \otimes a_{2}=\mathbb{0}$, we have that

$$
\begin{aligned}
f\left(u_{1}\right) & =f\left(a_{1} \cdot u_{1}\right)=f\left(a_{1} \cdot u_{1}+\left(a_{1} \otimes a_{2}\right) \cdot u_{2}\right)=f\left(a_{1} \cdot\left(u_{1}+a_{2} \cdot u_{2}\right)\right) \\
& =f\left(u_{1}+a_{2} \cdot u_{2}\right)
\end{aligned}
$$

Let $u=u_{1}+a_{2} \cdot u_{2}$. Since $f\left(u_{1}\right)=f(u)$ and $(f, g)$ is a factorization, we obtain the equalities

$$
\begin{align*}
g\left(u_{1}\right) \otimes f\left(u_{1}\right)_{q_{1}} & =\left(u_{1}\right)_{q_{1}}=\mathbb{1}  \tag{I}\\
g\left(u_{1}\right) \otimes f\left(u_{1}\right)_{q_{2}} & =\left(u_{1}\right)_{q_{2}}=\mathbb{0}  \tag{II}\\
g(u) \otimes f\left(u_{1}\right)_{q_{1}} & =u_{q_{1}}=\mathbb{1}  \tag{III}\\
g(u) \otimes f\left(u_{1}\right)_{q_{2}} & =u_{q_{2}}=a_{2} \tag{IV}
\end{align*}
$$

By (I) and (III), and using commutativity, we derive

$$
g\left(u_{1}\right)=g\left(u_{1}\right) \otimes\left(g(u) \otimes f\left(u_{1}\right)_{q_{1}}\right)=\left(g\left(u_{1}\right) \otimes f\left(u_{1}\right)_{q_{1}}\right) \otimes g(u)=g(u)
$$

Then by (II) and (IV) we obtain that $a_{2}=\mathbb{0}$. Hence B is zero-divisor free.
If $|Q|=1$, then $\mathcal{A}$ is already bu deterministic and there is no need to determinize it. Thus, as a consequence of Lemma 17.3.6, if we want to determinize by means of maximal factorization a ( $\Sigma, \mathrm{B}$ )-wta which is not bu deterministic, then B has to be zero-divisor free.

The next lemma shows that, e.g., in the representation

$$
f\left(\delta_{\mathcal{A}}(\sigma)\left(f\left(\delta_{\mathcal{A}}(\alpha)()\right), f\left(\delta_{\mathcal{A}}(\alpha)()\right)\right)\right)
$$

of a state of $Q^{\prime}$, the inner two occurrences of $f$ can be dropped without changing the value of the expression, i.e.,

$$
f\left(\delta_{\mathcal{A}}(\sigma)\left(f\left(\delta_{\mathcal{A}}(\alpha)()\right), f\left(\delta_{\mathcal{A}}(\alpha)()\right)\right)\right)=f\left(\delta_{\mathcal{A}}(\sigma)\left(\delta_{\mathcal{A}}(\alpha)(), \delta_{\mathcal{A}}(\alpha)()\right)\right)=f\left(\mathrm{~h}_{\mathcal{A}}(\sigma(\alpha, \alpha))\right)
$$

The next lemma also says that dropping the $f$ from $\delta_{\mathcal{A}}(\sigma)\left(\ldots, f\left(u_{i}\right), \ldots\right) \neq \mathbb{O}_{Q}$ preserves the non-zero property if $B$ is zero-divisor free.

Lemma 17.3.7. BVM10, Lm. 5.5] Let B be commutative and $(f, g)$ be a maximal factorization. Furthermore, let $k \in \mathbb{N}, \sigma \in \Sigma^{(k)}$, and $u_{1}, \ldots, u_{k} \in B^{Q} \backslash\left\{\mathbb{Q}_{Q}\right\}$. Then the following two statements hold.
(1) If $\delta_{\mathcal{A}}(\sigma)\left(u_{1}, \ldots, u_{k}\right) \neq \mathbb{O}_{Q}$, then $\delta_{\mathcal{A}}(\sigma)\left(f\left(u_{1}\right), \ldots, f\left(u_{k}\right)\right) \neq \mathbb{O}_{Q}$ and
$f\left(\delta_{\mathcal{A}}(\sigma)\left(u_{1}, \ldots, u_{k}\right)\right)=f\left(\delta_{\mathcal{A}}(\sigma)\left(f\left(u_{1}\right), \ldots, f\left(u_{k}\right)\right)\right)$.
(2) If B is zero-divisor free and $\delta_{\mathcal{A}}(\sigma)\left(f\left(u_{1}\right), \ldots, f\left(u_{k}\right)\right) \neq \mathbb{O}_{Q}$, then $\delta_{\mathcal{A}}(\sigma)\left(u_{1}, \ldots, u_{k}\right) \neq \mathbb{O}_{Q}$.

Proof. Clearly we have ( $\star$ )

$$
\begin{array}{rlr}
\delta_{\mathcal{A}}(\sigma)\left(u_{1}, \ldots, u_{k}\right) & =\delta_{\mathcal{A}}(\sigma)\left(g\left(u_{1}\right) \cdot f\left(u_{1}\right), \ldots, g\left(u_{k}\right) \cdot f\left(u_{k}\right)\right) \\
& =\left(g\left(u_{1}\right) \otimes \cdots \otimes g\left(u_{k}\right)\right) \cdot \delta_{\mathcal{A}}(\sigma)\left(f\left(u_{1}\right), \ldots, f\left(u_{k}\right)\right) . & ((f, g) \text { factorization) }
\end{array}
$$

Proof of (1). Let $\delta_{\mathcal{A}}(\sigma)\left(u_{1}, \ldots, u_{k}\right) \neq \mathbb{O}_{Q}$. By $(\star)$ also $\delta_{\mathcal{A}}(\sigma)\left(f\left(u_{1}\right), \ldots, f\left(u_{k}\right)\right) \neq \mathbb{O}_{Q}$. Applying $f$ to $(\star)$ and using that $(f, g)$ is maximal, we obtain that $f\left(\delta_{\mathcal{A}}(\sigma)\left(u_{1}, \ldots, u_{k}\right)\right)=f\left(\delta_{\mathcal{A}}(\sigma)\left(f\left(u_{1}\right), \ldots, f\left(u_{k}\right)\right)\right)$.

Proof of (2). We assume that $\delta_{\mathcal{A}}(\sigma)\left(f\left(u_{1}\right), \ldots, f\left(u_{k}\right)\right) \neq \mathbb{O}_{Q}$. Since $g\left(u_{i}\right) \neq \mathbb{0}$ for every $i \in[k]$, and since B is zero-divisor free, $(\star)$ yields that $\delta_{\mathcal{A}}(\sigma)\left(u_{1}, \ldots, u_{k}\right) \neq \mathbb{O}_{Q}$.

Let B be commutative and $(f, g)$ a maximal factorization. Then the set $Q^{\prime}$ of states of $\operatorname{det}_{(f, g)}(\mathcal{A})$ can be enumerated by enumerating $\mathrm{T}_{\Sigma}$ in the sense that $Q^{\prime} \subseteq f\left(\mathrm{~h}_{\mathcal{A}}\left(\mathrm{T}_{\Sigma}\right) \backslash\left\{\mathbb{O}_{Q}\right\}\right.$ ) (cf. Lemma 17.3.9). As preparation we have the following easy observation.
Lemma 17.3.8. BVM10, Obs. 4.5] Let $\operatorname{det}_{(f, g)}(\mathcal{A})=\left(Q^{\prime}, \delta^{\prime}, F^{\prime}\right)$. Moreover, let $\left(Q_{n}^{\prime} \mid n \in \mathbb{N}\right)$ be the family defined by $Q_{0}^{\prime}=\emptyset$, and for every $n \in \mathbb{N}$ we let

$$
Q_{n+1}^{\prime}=Q_{n}^{\prime} \cup\left\{f\left(\delta_{\mathcal{A}}(\sigma)\left(u_{1}, \ldots, u_{k}\right)\right) \mid k \in \mathbb{N}, \sigma \in \Sigma^{(k)}, u_{1}, \ldots, u_{k} \in Q_{n}^{\prime}, \delta_{\mathcal{A}}(\sigma)\left(u_{1}, \ldots, u_{k}\right) \neq \mathbb{O}_{Q}\right\}
$$

Then $Q^{\prime}=\bigcup\left(Q_{n}^{\prime} \mid n \in \mathbb{N}\right)$. Moreover, $Q^{\prime}$ is finite iff there exists an $N \in \mathbb{N}$ with $Q_{N+1}^{\prime}=Q_{N}^{\prime}$.

Proof. We define the mapping $h: \mathcal{P}\left(B^{Q}\right) \rightarrow \mathcal{P}\left(B^{Q}\right)$ for each $U \in \mathcal{P}\left(B^{Q}\right)$ by

$$
h(U)=U \cup\left\{f\left(\delta_{\mathcal{A}}(\sigma)\left(u_{1}, \ldots, u_{k}\right)\right) \mid k \in \mathbb{N}, \sigma \in \Sigma^{(k)}, u_{1}, \ldots, u_{k} \in U, \delta_{\mathcal{A}}(\sigma)\left(u_{1}, \ldots, u_{k}\right) \neq \mathbb{O}_{Q}\right\}
$$

It is easy to see that $h$ is continuous. Moreover, we have

$$
\begin{aligned}
Q^{\prime} & =\bigcap\left(U \mid U \in \mathcal{P}\left(B^{Q}\right), h(U) \subseteq U\right) \\
& =\bigcup\left(h^{n}(\emptyset) \mid n \in \mathbb{N}\right) \\
& =\bigcup\left(Q_{n}^{\prime} \mid n \in \mathbb{N}\right)
\end{aligned}
$$

$$
\text { (by definition of } Q^{\prime} \text { and } h \text { ) }
$$

(by Theorem 2.6.17)
(by definition of $Q_{n}^{\prime}$ and $h$ )
The second statement of the lemma is obvious.
Then we can show the following enumeration lemma (which is based on KM05, Lm. 2]). Intuitively, it is obtained by iterating Lemma 17.3.7, thereby pulling out $f$ from $Q_{1}^{\prime}$, then from $Q_{2}^{\prime}$, then from $Q_{3}^{\prime}$, and so on.

Lemma 17.3.9. BVM10, Lm. 5.8] Let B be commutative, $(f, g)$ a maximal factorization, and $\operatorname{det}_{(f, g)}(\mathcal{A})=\left(Q^{\prime}, \delta^{\prime}, F^{\prime}\right)$. Then $Q^{\prime} \subseteq f\left(\mathrm{~h}_{\mathcal{A}}\left(\mathrm{T}_{\Sigma}\right) \backslash\left\{\mathbb{0}_{Q}\right\}\right)$.

Proof. Again, we prove by case analysis on the cardinality of $Q$.


$$
\begin{equation*}
\text { for each } u \in B \backslash\{\mathbb{O}\}, \text { we have } f(u)=f(\mathbb{1}) \text {. } \tag{17.3}
\end{equation*}
$$

To see this we compute:

$$
\begin{array}{rlr}
f(u) & =f(u \cdot \mathbb{1}) \quad\left(\text { where on the left-hand side } u \in B^{Q} \text { and on the right-hand side } u \in B\right) \\
& =f(\mathbb{1}) . & \text { (because }(f, g) \text { is maximal) }
\end{array}
$$

By our assumption on $\mathcal{A}$, there exists an $\alpha \in \Sigma^{(0)}$ such that $\delta_{\mathcal{A}}(\alpha)() \neq \mathbb{0}$. Hence

$$
\mathrm{h}_{\mathcal{A}}(\alpha)=\delta_{\mathcal{A}}(\alpha)() \neq \mathbb{0}
$$

and thus, by (17.3), we have $f\left(\mathrm{~h}_{\mathcal{A}}(\alpha)\right)=f(\mathbb{1})$ and $f\left(\mathrm{~h}_{\mathcal{A}}\left(\mathrm{T}_{\Sigma}\right) \backslash\{\mathbb{D}\}\right)=\{f(\mathbb{1})\}$.
Next, by induction on $\mathbb{N}_{+}$, we show that $Q_{n}^{\prime}=\{f(\mathbb{1})\}$ for each $n \in \mathbb{N}_{+}$. The I.B. follows from $f\left(\delta_{\mathcal{A}}(\alpha)()\right)=f(\mathbb{1})$ and (17.3); then the I.S. can be proved by using (17.3).

Thus $Q^{\prime}=\{f(\mathbb{1})\}$ and we obtain $Q^{\prime}=f\left(\mathrm{~h}_{\mathcal{A}}\left(\mathrm{T}_{\Sigma}\right) \backslash\{\mathbb{D}\}\right)$.
Case (b): Let $|Q|>1$. By Lemma 17.3.6, B is zero-divisor free. Using Lemma 17.3.8, it suffices to prove the following statement by induction on $\mathbb{N}$ :

$$
\begin{equation*}
\text { for each } n \in \mathbb{N} \text {, we have } Q_{n}^{\prime} \subseteq f\left(\mathrm{~h}_{\mathcal{A}}\left(\mathrm{T}_{\Sigma}\right) \backslash\left\{\mathbb{O}_{Q}\right\}\right) \tag{17.4}
\end{equation*}
$$

I.B.: Let $n=0$. Then (17.4) is trivially true.
I.S.: Let $n=n^{\prime}+1$ for some $n^{\prime} \in \mathbb{N}$. We assume that (17.4) holds for $n^{\prime}$. Let $u \in Q_{n}^{\prime}$. If $u \in Q_{n^{\prime}}^{\prime}$, then we are done. Otherwise, there exist $k \in \mathbb{N}, \sigma \in \Sigma^{(k)}$, and $u_{1}, \ldots, u_{k} \in Q_{n^{\prime}}^{\prime}$ such that $\delta_{\mathcal{A}}(\sigma)\left(u_{1}, \ldots, u_{k}\right) \neq \mathbb{O}_{Q}$ and $u=f\left(\delta_{\mathcal{A}}(\sigma)\left(u_{1}, \ldots, u_{k}\right)\right)$. By I.H., for each $i \in[k]$ there exists $\xi_{i} \in \mathrm{~T}_{\Sigma}$ such that $u_{i}=f\left(\mathrm{~h}_{\mathcal{A}}\left(\xi_{i}\right)\right)$. Hence

$$
u=f\left(\delta_{\mathcal{A}}(\sigma)\left(u_{1}, \ldots, u_{k}\right)\right)=f\left(\delta_{\mathcal{A}}(\sigma)\left(f\left(\mathrm{~h}_{\mathcal{A}}\left(\xi_{1}\right)\right), \ldots, f\left(\mathrm{~h}_{\mathcal{A}}\left(\xi_{k}\right)\right)\right)\right)
$$

By assumption $\delta_{\mathcal{A}}(\sigma)\left(u_{1}, \ldots, u_{k}\right) \neq \mathbb{O}_{Q}$, i.e., $\delta_{\mathcal{A}}(\sigma)\left(f\left(\mathrm{~h}_{\mathcal{A}}\left(\xi_{1}\right)\right), \ldots, f\left(\mathrm{~h}_{\mathcal{A}}\left(\xi_{k}\right)\right)\right) \neq \mathbb{O}_{Q}$. By Lemma 17.3.7(2) (using zero-divisor freeness), $\delta_{\mathcal{A}}(\sigma)\left(\mathrm{h}_{\mathcal{A}}\left(\xi_{1}\right), \ldots, \mathrm{h}_{\mathcal{A}}\left(\xi_{k}\right)\right) \neq \mathbb{O}_{Q}$. Thus, we can continue with:

$$
\begin{aligned}
f\left(\delta_{\mathcal{A}}(\sigma)\left(f\left(\mathrm{~h}_{\mathcal{A}}\left(\xi_{1}\right)\right), \ldots, f\left(\mathrm{~h}_{\mathcal{A}}\left(\xi_{k}\right)\right)\right)\right) & =f\left(\delta_{\mathcal{A}}(\sigma)\left(\mathrm{h}_{\mathcal{A}}\left(\xi_{1}\right), \ldots, \mathrm{h}_{\mathcal{A}}\left(\xi_{k}\right)\right)\right) \\
& =f\left(\mathrm{~h}_{\mathcal{A}}\left(\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)\right)\right) \in f\left(\mathrm{~h}_{\mathcal{A}}\left(\mathrm{T}_{\Sigma}\right) \backslash\left\{\mathbb{D}_{Q}\right\}\right)
\end{aligned}
$$

### 17.3.3 On the twinning property

Obviously, there exist $(\Sigma, B)$-wta which do not have the twinning property. For example, consider the $\left(\Sigma\right.$, Nat $\left._{\text {max },+}\right)$-wta $\mathcal{A}=(Q, \delta, F)$ with $\Sigma=\left\{\gamma^{(1)}, \alpha^{(0)}\right\}, Q=\{p, q\}$, and

$$
\begin{array}{ll}
\delta_{0}(\varepsilon, \alpha, p)=2 & \delta_{1}(p, \gamma, p)=5 \\
\delta_{0}(\varepsilon, \alpha, q)=3 & \delta_{1}(q, \gamma, q)=7
\end{array}
$$

and the weight of each other transition is $-\infty$. Then, for $\xi=\alpha$ and context $c=\gamma(z)$, we have

$$
\begin{aligned}
\mathrm{wt}\left(\mathrm{R}_{\mathcal{A}}(p, \xi)\right)=2 \neq-\infty & \operatorname{wt}\left(\mathrm{R}_{\mathcal{A}}(p, c, p)\right)=5 \neq-\infty \\
\operatorname{wt}\left(\mathrm{R}_{\mathcal{A}}(q, \xi)\right)=3 \neq-\infty & \operatorname{wt}\left(\mathrm{R}_{\mathcal{A}}(q, c, q)\right)=7 \neq-\infty
\end{aligned}
$$

and $\operatorname{wt}\left(\mathrm{R}_{\mathcal{A}}(p, c, p)\right) \neq \operatorname{wt}\left(\mathrm{R}_{\mathcal{A}}(q, c, q)\right)$.
But, in particular, each bu deterministic wta has the twinning property.
Observation 17.3.10. The conditions $\mathrm{wt}\left(\mathrm{R}_{\mathcal{A}}(p, \xi)\right) \neq \mathbb{O}$ and $\mathrm{wt}\left(\mathrm{R}_{\mathcal{A}}(q, \xi)\right) \neq \mathbb{0}$ imply that $p, q \in \mathrm{Q}_{\neq \mathbb{\mathcal { A }}}^{\mathrm{R}_{\mathcal{A}}}(\xi)$ (cf. Section 4.1). If $\mathcal{A}$ is bu deterministic, then by Lemma4.2.1(2) it follows that $p=q$. Hence, each bu deterministic wta has the twinning property.

In the next example we illustrate that proving the twinning property can be a tedious task even for a small wta.

Example 17.3.11. We prove that the wta $\mathcal{A}$ given in Example 17.1.2 has the twinning property. For the sake of convenience, we repeat the definition of that ( $\Sigma$, Rat $\operatorname{Rin}_{\min ,+}$ )-wta.

We recall that $\operatorname{Rat}_{\min ,+}=\left(\mathbb{Q}_{\infty}, \min ,+, \infty, 0\right)$ is the tropical semifield. We consider the ranked alphabet $\Sigma=\left\{\sigma^{(2)}, \alpha^{(0)}\right\}$ and the $\left(\Sigma\right.$, Rat $\left._{\min ,+}\right)$-wta $\mathcal{A}=(Q, \delta, F)$ given by

- $Q=\left\{p, p^{\prime}\right\}$ and $F_{p}=0$ and $F_{p^{\prime}}=\infty$ and
- $\delta_{0}(\varepsilon, \alpha, p)=1, \delta_{2}(p p, \sigma, p)=1, \delta_{2}\left(p p, \sigma, p^{\prime}\right)=0.5$, and $\delta_{2}\left(p^{\prime} p, \sigma, p\right)=1.5$, and all other values of $\delta_{0}$ and $\delta_{2}$ are $\infty$.

Next we show that $\mathcal{A}$ has the twinning property. First, by induction on $\mathrm{T}_{\Sigma}$, we prove that the following statement holds:

For every $\xi \in \mathrm{T}_{\Sigma}, \rho \in \mathrm{R}_{\mathcal{A}}(p, \xi)$, and $\rho^{\prime} \in \mathrm{R}_{\mathcal{A}}\left(p^{\prime}, \xi\right)$ we have
(1) $\mathrm{wt}(\xi, \rho)=\infty$ or $\mathrm{wt}(\xi, \rho)=\operatorname{size}(\xi)$ and
(2) $\operatorname{wt}\left(\xi, \rho^{\prime}\right)=\infty$ or $\operatorname{wt}\left(\xi, \rho^{\prime}\right)=\operatorname{size}(\xi)-0.5$.
I.B.: Let $\xi=\alpha$. Then (1) holds because $\mathrm{R}_{\mathcal{A}}(p, \xi)=\{\rho\}$, where $\rho(\varepsilon)=p$. Hence wt $(\xi, \rho)=$ $\delta_{0}(\varepsilon, \alpha, p)=1=\operatorname{size}(\xi)$. Also, (2) holds because $\mathrm{R}_{\mathcal{A}}\left(p^{\prime}, \xi\right)=\{\rho\}$, where $\rho(\varepsilon)=p^{\prime}$. Hence wt $(\xi, \rho)=$ $\delta_{0}\left(\varepsilon, \alpha, p^{\prime}\right)=\infty$.
I.S.: Let $\xi=\sigma\left(\xi_{1}, \xi_{2}\right)$. For the proof, let $\rho \in \mathrm{R}_{\mathcal{A}}(p, \xi)$. Then

$$
\begin{equation*}
\mathrm{wt}(\xi, \rho)=\mathrm{wt}\left(\xi_{1},\left.\rho\right|_{1}\right)+\mathrm{wt}\left(\xi_{2},\left.\rho\right|_{2}\right)+\delta_{2}(\rho(1) \rho(2), \sigma, p) \tag{17.6}
\end{equation*}
$$

where $\left.\rho\right|_{i}$ is the run induced by $\rho$ at position $i$ for each $i \in[2]$; in particular, $\left.\rho\right|_{i} \in \mathrm{R}_{\mathcal{A}}\left(\rho(i), \xi_{i}\right)$.
For the proof of (17.5)(1), we distinguish three cases as follows.
Case (a): Let $\rho(1) \rho(2)=p p$. By I.H., part (1), $\operatorname{wt}\left(\xi_{i},\left.\rho\right|_{i}\right)=\infty$ or $\operatorname{wt}\left(\left.\xi_{i} \rho\right|_{i}\right)=\operatorname{size}\left(\xi_{i}\right)$ for each $i \in[2]$. If $\overline{\mathrm{wt}\left(\xi_{i},\left.\rho\right|_{i}\right)}=\infty$ for some $i \in[2]$, then by (17.6) we have $\mathrm{wt}(\xi, \rho)=\infty$. Otherwise, again by (17.6), we have $\operatorname{wt}(\xi, \rho)=\operatorname{size}\left(\xi_{1}\right)+\operatorname{size}\left(\xi_{2}\right)+\delta_{2}(p p, \sigma, p)=\operatorname{size}\left(\xi_{1}\right)+\operatorname{size}\left(\xi_{2}\right)+1=\operatorname{size}(\xi)$.

Case (b): Let $\rho(1) \rho(2)=p^{\prime} p$. By I.H., part (2), $\operatorname{wt}\left(\xi_{1},\left.\rho\right|_{1}\right)=\infty$ or wt $\left(\xi_{1},\left.\rho\right|_{1}\right)=\operatorname{size}\left(\xi_{1}\right)-0.5$, and by I.H., part (1), $\operatorname{wt}\left(\xi_{2},\left.\rho\right|_{2}\right)=\infty$ or $\operatorname{wt}\left(\xi_{2},\left.\rho\right|_{2}\right)=\operatorname{size}\left(\xi_{2}\right)$. If $\operatorname{wt}\left(\left.\xi_{i} \rho\right|_{i}\right)=\infty$ for some $i \in[2]$, then by (17.6)
we have wt $(\xi, \rho)=\infty$. Otherwise, also by (17.6), we have wt $(\xi, \rho)=\operatorname{size}\left(\xi_{1}\right)-0.5+\operatorname{size}\left(\xi_{2}\right)+\delta_{2}\left(p^{\prime} p, \sigma, p\right)=$ $\operatorname{size}\left(\xi_{1}\right)-0.5+\operatorname{size}\left(\xi_{2}\right)+1.5=\operatorname{size}(\xi)$.

Case (c): Let $\rho(1) \rho(2) \notin\left\{p p, p^{\prime} p\right\}$. Then $\delta_{2}(\rho(1) \rho(2), \sigma, p)=\infty$ and thus by (17.6) we have wt $(\xi, \rho)=$ $\infty$.

For the proof of $(\overline{17.5})(2)$, we distinguish two cases.
Case (a): Let $\rho(1) \rho(2)=p p$. By I.H., part (1), $\mathrm{wt}\left(\xi_{i},\left.\rho\right|_{i}\right)=\infty$ or $\mathrm{wt}\left(\xi_{i},\left.\rho\right|_{i}\right)=\operatorname{size}\left(\xi_{i}\right)$ for each $i \in[2]$. If $\overline{\mathrm{wt}\left(\xi_{i},\left.\rho\right|_{i}\right)}=\infty$ for some $i \in[2]$, then by (17.6) we have $\mathrm{wt}(\xi, \rho)=\infty$. Otherwise, by (17.6), we have $\mathrm{wt}(\xi, \rho)=\operatorname{size}\left(\xi_{1}\right)+\operatorname{size}\left(\xi_{2}\right)+\delta_{2}\left(p p, \sigma, p^{\prime}\right)=\operatorname{size}\left(\xi_{1}\right)+\operatorname{size}\left(\xi_{2}\right)+0.5=\operatorname{size}(\xi)-0.5$.

Case (b): Let $\rho(1) \rho(2) \neq p p$. Then $\delta_{2}\left(\rho(1) \rho(2), \sigma, p^{\prime}\right)=\infty$ and thus by (17.6) we have $\mathrm{wt}(\xi, \rho)=\infty$.
This finishes the proof of (17.5).

Next we prove the following statement for contexts. For every $c \in \mathrm{C}_{\Sigma} \backslash\{z\}$ :
(1) for every $\rho \in \mathrm{R}_{\mathcal{A}}(p, c, p): \operatorname{wt}(c, \rho)=\infty$ or $\mathrm{wt}(c, \rho)=\operatorname{size}(c)-1$ and
(2) for every $\rho \in \mathrm{R}_{\mathcal{A}}\left(p, c, p^{\prime}\right)$ : $\operatorname{wt}(c, \rho)=\infty$ or $\mathrm{wt}(c, \rho)=\operatorname{size}(c)-0.5$ and
(3) for every $\rho \in \mathrm{R}_{\mathcal{A}}\left(p^{\prime}, c, p\right): \mathrm{wt}(c, \rho)=\infty$ or $\mathrm{wt}(c, \rho)=\operatorname{size}(c)-1.5$, and
(4) for every $\rho \in \mathrm{R}_{\mathcal{A}}\left(p^{\prime}, c, p^{\prime}\right): \operatorname{wt}(c, \rho)=\infty \operatorname{or} \operatorname{wt}(c, \rho)=\operatorname{size}(c)-1$.

For the proof, we recall the relation $\prec_{\mathrm{C}_{\Sigma}}$ on $\mathrm{C}_{\Sigma}$ defined in Chapter 6; for every $c_{1}, c_{2} \in \mathrm{C}_{\Sigma}$ we have $c_{1} \prec_{\mathrm{C}_{\Sigma}} c_{2}$ if there exists an elementary context $c \in \mathrm{eC}_{\Sigma}$ such that $c_{2}=c\left[c_{1}\right]$. We denote the restriction of $\prec_{\mathrm{C}_{\Sigma}}$ to $\mathrm{C}_{\Sigma} \backslash\{z\}$ also by $\prec_{\mathrm{C}_{\Sigma}}$ and we note that $\prec_{\mathrm{C}_{\Sigma}}$ is well-founded and $\min _{\prec_{\mathrm{C}_{\Sigma}}}\left(\mathrm{C}_{\Sigma} \backslash\{z\}\right)=\mathrm{e} \mathrm{C}_{\Sigma}$, i.e., the set of elementary contexts.

We prove (17.7) by induction on $\left(\mathrm{C}_{\Sigma} \backslash\{z\}, \prec_{\mathrm{C}_{\Sigma}}\right)$
I.B.: We distinguish the two cases that (a) $c=\sigma(z, \xi)$ and (b) $c=\sigma(\xi, z)$ for some $\xi \in \mathrm{T}_{\Sigma}$. Then we proceed as follows.
(1) Let $\rho \in \mathrm{R}_{\mathcal{A}}(p, c, p)$. If $\mathrm{wt}(c, \rho) \neq \infty$, then the following two cases are possible:

- $c$ has the form (a), $\rho(1)=\rho(2)=p$, and $\left.\rho\right|_{2} \in \mathrm{R}_{\mathcal{A}}(p, \xi)$. By (17.5)(1), we have wt $\left(\xi,\left.\rho\right|_{2}\right)=$ $\operatorname{size}(\xi)$. Hence $\operatorname{wt}(c, \rho)=0+\operatorname{wt}\left(\left.\rho\right|_{2}\right)+\delta_{2}(p p, \sigma, p)=\operatorname{size}(\xi)+1=\operatorname{size}(c)-1$.
- $c$ has the form $(\mathrm{b}),\left(\rho(1)=\rho(2)=p\right.$ and $\left.\left.\rho\right|_{1} \in \mathrm{R}_{\mathcal{A}}(p, \xi)\right)$ or $\left(\rho(1)=p^{\prime}, \rho(2)=p\right.$, and $\left.\left.\rho\right|_{1} \in \mathrm{R}_{\mathcal{A}}\left(p^{\prime}, \xi\right)\right)$. In the first case, by symmetry, we get as before $\mathrm{wt}(c, \rho)=\operatorname{size}(c)-1$. In the second case, by (17.5) $(2)$, we have $\mathrm{wt}\left(\xi,\left.\rho\right|_{1}\right)=\operatorname{size}(\xi)-0.5$. Hence $\mathrm{wt}(c, \rho)=\mathrm{wt}\left(\left.\rho\right|_{1}\right)+0+$ $\delta_{2}\left(p^{\prime} p, \sigma, p\right)=\operatorname{size}(\xi)-0.5+1.5=\operatorname{size}(\xi)+1=\operatorname{size}(c)-1$.
(2) Let $\rho \in \mathrm{R}_{\mathcal{A}}\left(p, c, p^{\prime}\right)$. If $\mathrm{wt}(c, \rho) \neq \infty$, then
- $c$ has the form (a), $\rho(1)=p^{\prime}, \rho(2)=p$ and $\left.\rho\right|_{2} \in \mathrm{R}_{\mathcal{A}}(p, \xi)$. By (17.5)(1), we have wt $\left(\xi,\left.\rho\right|_{2}\right)=$ $\operatorname{size}(\xi)$. Hence wt $(c, \rho)=0+\mathrm{wt}\left(\left.\rho\right|_{2}\right)+\delta_{2}\left(p^{\prime} p, \sigma, p\right)=\operatorname{size}(\xi)+1.5=\operatorname{size}(c)-0.5$.
(3) Let $\rho \in \mathrm{R}_{\mathcal{A}}\left(p^{\prime}, c, p\right)$. If $\mathrm{wt}(c, \rho) \neq \infty$, then the following two cases are possible:
- $c$ has the form (a), $\rho(1)=\rho(2)=p$, and $\left.\rho\right|_{2} \in \mathrm{R}_{\mathcal{A}}(p, \xi)$. By (17.5)(1), we have wt $\left(\xi,\left.\rho\right|_{2}\right)=$ $\operatorname{size}(\xi)$. Hence $\mathrm{wt}(c, \rho)=0+\mathrm{wt}\left(\left.\rho\right|_{2}\right)+\delta_{2}\left(p p, \sigma, p^{\prime}\right)=\operatorname{size}(\xi)+0.5=\operatorname{size}(c)-1.5$.
- $c$ has the form (b), $\rho(1)=\rho(2)=p$, and $\left.\rho\right|_{1} \in \mathrm{R}_{\mathcal{A}}(p, \xi)$. By symmetry with the previous case, we obtain $\mathrm{wt}(c, \rho)=\operatorname{size}(c)-1.5$.
(4) Let $\rho \in \mathrm{R}_{\mathcal{A}}\left(p^{\prime}, c, p^{\prime}\right)$. We have $\mathrm{wt}(c, \rho)=\infty$ because $\delta_{2}\left(p^{\prime} q, \sigma, p\right)=\delta_{2}\left(q p^{\prime}, \sigma, p\right)=\infty$ for each $q \in Q$.
I.S.: We distinguish the two cases that (a) $c=\sigma\left(c^{\prime}, \xi\right)$ and (b) $c=\sigma\left(\xi, c^{\prime}\right)$ for some $c^{\prime} \in \mathrm{C}_{\Sigma}$ and $\xi \in \mathrm{T}_{\Sigma}$. Then we proceed as follows.
(1) Let $\rho \in \mathrm{R}_{\mathcal{A}}(p, c, p)$. If $\operatorname{wt}(c, \rho) \neq \infty$, then the following two cases are possible:
- $c$ has the form (a), $\rho(1)=\rho(2)=p,\left.\rho\right|_{1} \in \mathrm{R}_{\mathcal{A}}\left(p, c^{\prime}, p\right)$, and $\left.\rho\right|_{2} \in \mathrm{R}_{\mathcal{A}}(p, \xi)$. By I.H., part (1), we have $\mathrm{wt}\left(c^{\prime},\left.\rho\right|_{1}\right)=\operatorname{size}\left(c^{\prime}\right)-1$ and by (17.5) (1), we have $\mathrm{wt}\left(\xi,\left.\rho\right|_{2}\right)=\operatorname{size}(\xi)$. Hence $\mathrm{wt}(c, \rho)=\mathrm{wt}\left(c^{\prime},\left.\rho\right|_{1}\right)+\mathrm{wt}\left(\xi,\left.\rho\right|_{2}\right)+\delta_{2}(p p, \sigma, p)=\operatorname{size}\left(c^{\prime}\right)-1+\operatorname{size}(\xi)+1=\operatorname{size}(c)-1$.
- $c$ has the form (a), $\rho(1)=p^{\prime}, \rho(2)=p,\left.\rho\right|_{1} \in \mathrm{R}_{\mathcal{A}}\left(p^{\prime}, c^{\prime}, p\right)$, and $\left.\rho\right|_{2} \in \mathrm{R}_{\mathcal{A}}(p, \xi)$. By I.H., part (3) we have $\mathrm{wt}\left(c^{\prime},\left.\rho\right|_{1}\right)=\operatorname{size}\left(c^{\prime}\right)-1.5$ and by (17.5) (1), we have $\mathrm{wt}\left(\xi,\left.\rho\right|_{2}\right)=\operatorname{size}(\xi)$. Hence $\mathrm{wt}(c, \rho)=\mathrm{wt}\left(c^{\prime},\left.\rho\right|_{1}\right)+\mathrm{wt}\left(\xi,\left.\rho\right|_{2}\right)+\delta_{2}\left(p^{\prime} p, \sigma, p\right)=\operatorname{size}\left(c^{\prime}\right)-1.5+\operatorname{size}(\xi)+1.5=\operatorname{size}(c)-1$.
- $c$ has the form (b), $\rho(1)=\rho(2)=p,\left.\rho\right|_{1} \in \mathrm{R}_{\mathcal{A}}(p, \xi)$, and $\left.\rho\right|_{2} \in \mathrm{R}_{\mathcal{A}}\left(p, c^{\prime}, p\right)$. By symmetry with the previous case, we obtain $\mathrm{wt}(c, \rho)=\operatorname{size}(c)-1$.
- $c$ has the form (b), $\rho(1)=p^{\prime}, \rho(2)=p,\left.\rho\right|_{1} \in \mathrm{R}_{\mathcal{A}}\left(p^{\prime}, \xi\right)$ and $\left.\rho\right|_{2} \in \mathrm{R}_{\mathcal{A}}\left(p, c^{\prime}, p\right)$. By (17.5) (2), we have $\mathrm{wt}\left(\xi,\left.\rho\right|_{1}\right)=\operatorname{size}(\xi)-0.5$ and by I.H., part (1), we have $\mathrm{wt}\left(c^{\prime},\left.\rho\right|_{2}\right)=\operatorname{size}\left(c^{\prime}\right)-1$. Hence $\mathrm{wt}(c, \rho)=\mathrm{wt}\left(\xi,\left.\rho\right|_{1}\right)+\mathrm{wt}\left(c^{\prime},\left.\rho\right|_{2}\right)+\delta_{2}\left(p^{\prime} p, \sigma, p\right)=\operatorname{size}(\xi)-0.5+\operatorname{size}\left(c^{\prime}\right)-1+1.5=\operatorname{size}(c)-1$.
(2) Let $\rho \in \mathrm{R}_{\mathcal{A}}\left(p, c, p^{\prime}\right)$. If $\mathrm{wt}(c, \rho) \neq \infty$, then we have the following cases.
- $c$ has the form (a), $\rho(1)=p^{\prime}, \rho(2)=p,\left.\rho\right|_{1} \in \mathrm{R}_{\mathcal{A}}\left(p^{\prime}, c^{\prime}, p^{\prime}\right)$, and $\left.\rho\right|_{2} \in \mathrm{R}_{\mathcal{A}}(p, \xi)$. By I.H., part (4) and by (17.5) (1), we have $\mathrm{wt}\left(c^{\prime},\left.\rho\right|_{1}\right)=\operatorname{size}\left(c^{\prime}\right)-1$ and $\mathrm{wt}\left(\xi,\left.\rho\right|_{2}\right)=\operatorname{size}(\xi)$, respectively. Hence $\mathrm{wt}(c, \rho)=\mathrm{wt}\left(c^{\prime},\left.\rho\right|_{1}\right)+\mathrm{wt}\left(\xi,\left.\rho\right|_{2}\right)+\delta_{2}\left(p^{\prime} p, \sigma, p\right)=\operatorname{size}\left(c^{\prime}\right)-1+\operatorname{size}(\xi)+1.5=\operatorname{size}(c)-0.5$.
- $c$ has the form (a), $\rho(1)=\rho(2)=p,\left.\rho\right|_{1} \in \mathrm{R}_{\mathcal{A}}\left(p, c^{\prime}, p^{\prime}\right)$, and $\left.\rho\right|_{2} \in \mathrm{R}_{\mathcal{A}}(p, \xi)$. By I.H., part (2) and by (17.5) (1), we have $\mathrm{wt}\left(c^{\prime},\left.\rho\right|_{1}\right)=\operatorname{size}\left(c^{\prime}\right)-0.5$ and $\mathrm{wt}\left(\xi,\left.\rho\right|_{2}\right)=\operatorname{size}(\xi)$, respectively. Hence $\mathrm{wt}(c, \rho)=\mathrm{wt}\left(c^{\prime},\left.\rho\right|_{1}\right)+\mathrm{wt}\left(\xi,\left.\rho\right|_{2}\right)+\delta_{2}(p p, \sigma, p)=\operatorname{size}\left(c^{\prime}\right)-0.5+\operatorname{size}(\xi)+1=\operatorname{size}(c)-0.5$.
- $c$ has the form (b), $\rho(1)=p^{\prime}, \rho(2)=p,\left.\rho\right|_{1} \in \mathrm{R}_{\mathcal{A}}\left(p^{\prime}, \xi\right)$, and $\left.\rho\right|_{2} \in \mathrm{R}_{\mathcal{A}}\left(p, c^{\prime}, p^{\prime}\right)$. By (17.5) (2) and by I.H., part $(2) \mathrm{wt}\left(\xi,\left.\rho\right|_{1}\right)=\operatorname{size}(\xi)-0.5$ and $\mathrm{wt}\left(c^{\prime},\left.\rho\right|_{2}\right)=\operatorname{size}\left(c^{\prime}\right)-0.5$, respectively. Hence $\mathrm{wt}(c, \rho)=\mathrm{wt}\left(\xi,\left.\rho\right|_{1}\right)+\mathrm{wt}\left(c^{\prime},\left.\rho\right|_{2}\right)+\delta_{2}\left(p^{\prime} p, \sigma, p\right)=\operatorname{size}(\xi)-0.5+\operatorname{size}\left(c^{\prime}\right)-0.5+1.5=\operatorname{size}(c)-0.5$.
- $c$ has the form (b), $\rho(1)=\rho(2)=p,\left.\rho\right|_{1} \in \mathrm{R}_{\mathcal{A}}(p, \xi)$, and $\left.\rho\right|_{2} \in \mathrm{R}_{\mathcal{A}}\left(p, c^{\prime}, p^{\prime}\right)$. By symmetry, we have $\mathrm{wt}(c, \rho)=\operatorname{size}(c)-0.5$.
(3) Let $\rho \in \mathrm{R}_{\mathcal{A}}\left(p^{\prime}, c, p\right)$. If $\mathrm{wt}(c, \rho) \neq \infty$, then the following two cases are possible:
- $c$ has the form (a), $\rho(1)=\rho(2)=p,\left.\rho\right|_{1} \in \mathrm{R}_{\mathcal{A}}\left(p, c^{\prime}, p\right)$, and $\left.\rho\right|_{2} \in \mathrm{R}_{\mathcal{A}}(p, \xi)$. By I.H., part (1) and by (17.5) (1), we have $\mathrm{wt}\left(c^{\prime},\left.\rho\right|_{1}\right)=\operatorname{size}\left(c^{\prime}\right)-1$ and $\mathrm{wt}\left(\xi,\left.\rho\right|_{2}\right)=\operatorname{size}(\xi)$, respectively. Hence $\mathrm{wt}(c, \rho)=\mathrm{wt}\left(c^{\prime},\left.\rho\right|_{1}\right)+\mathrm{wt}\left(\xi,\left.\rho\right|_{2}\right)+\delta_{2}\left(p p, \sigma, p^{\prime}\right)=\operatorname{size}\left(c^{\prime}\right)-1+\operatorname{size}(\xi)+0.5=\operatorname{size}(c)-1.5$.
- $c$ has the form (b), $\rho(1)=\rho(2)=p,\left.\rho\right|_{1} \in \mathrm{R}_{\mathcal{A}}(p, \xi)$, and $\left.\rho\right|_{2} \in \mathrm{R}_{\mathcal{A}}\left(p, c^{\prime}, p\right)$. By symmetry with the previous case, we obtain $\mathrm{wt}(c, \rho)=\operatorname{size}(c)-1.5$.
(4) Let $\rho \in \mathrm{R}_{\mathcal{A}}\left(p^{\prime}, c, p^{\prime}\right)$. If $\mathrm{wt}(c, \rho) \neq \infty$, then the following two cases are possible:
- $c$ has the form (a), $\rho(1)=\rho(2)=p,\left.\rho\right|_{1} \in \mathrm{R}_{\mathcal{A}}\left(p, c^{\prime}, p^{\prime}\right)$, and $\left.\rho\right|_{2} \in \mathrm{R}_{\mathcal{A}}(p, \xi)$. By I.H., part (2) and by (17.5) (1), we have $\mathrm{wt}\left(c^{\prime},\left.\rho\right|_{1}\right)=\operatorname{size}\left(c^{\prime}\right)-0.5$ and $\mathrm{wt}\left(\xi,\left.\rho\right|_{2}\right)=\operatorname{size}(\xi)$, respectively. Hence $\mathrm{wt}(c, \rho)=\mathrm{wt}\left(c^{\prime},\left.\rho\right|_{1}\right)+\mathrm{wt}\left(\xi,\left.\rho\right|_{2}\right)+\delta_{2}\left(p p, \sigma, p^{\prime}\right)=\operatorname{size}\left(c^{\prime}\right)-0.5+\operatorname{size}(\xi)+0.5=\operatorname{size}(c)-1$.
- $c$ has the form (b), $\rho(1)=\rho(2)=p,\left.\rho\right|_{1} \in \mathrm{R}_{\mathcal{A}}(p, \xi)$, and $\left.\rho\right|_{2} \in \mathrm{R}_{\mathcal{A}}\left(p, c^{\prime}, p^{\prime}\right)$. By symmetry with the previous case, we obtain $\mathrm{wt}(c, \rho)=\operatorname{size}(c)-1$.

This finishes the proof of the statement 17.7 .
Now we prove that $\mathcal{A}$ has the twinning property. In fact, we show that even the following stronger property holds: for every context $c \in \mathrm{C}_{\Sigma}$, if $\operatorname{wt}\left(\mathrm{R}_{\mathcal{A}}(p, c, p)\right) \neq \infty$ and $\operatorname{wt}\left(\mathrm{R}_{\mathcal{A}}\left(p^{\prime}, c, p^{\prime}\right)\right) \neq \infty$, then $\operatorname{wt}\left(\mathrm{R}_{\mathcal{A}}(p, c, p)\right)=\operatorname{wt}\left(\mathrm{R}_{\mathcal{A}}\left(p^{\prime}, c, p^{\prime}\right)\right)$. The proof is as follows. If $c=z$, then $\mathrm{R}_{\mathcal{A}}(p, c, p)=\{\rho\}$ with $\rho(\varepsilon)=p$ and $\mathrm{R}_{\mathcal{A}}\left(p^{\prime}, c, p^{\prime}\right)=\left\{\rho^{\prime}\right\}$ with $\rho^{\prime}(\varepsilon)=p^{\prime}$. Hence $\operatorname{wt}\left(\mathrm{R}_{\mathcal{A}}(p, c, p)\right)=0=\operatorname{wt}\left(\mathrm{R}_{\mathcal{A}}\left(p^{\prime}, c, p^{\prime}\right)\right)$. If $c \neq z$, then the condition $\operatorname{wt}\left(\mathrm{R}_{\mathcal{A}}(p, c, p)\right) \neq \infty$ implies that there exists a run $\rho \in \mathrm{R}_{\mathcal{A}}(p, c, p)$ with $\mathrm{wt}(c, \rho) \neq \infty$. By (17.7) (1) we know that $\mathrm{wt}(c, \rho)=\operatorname{size}(c)-1$ for every such $\rho$, hence by the fact that summation of Rat $\mathrm{min}_{\mathrm{m},+}$
is min, we obtain that $\operatorname{wt}\left(\mathrm{R}_{\mathcal{A}}(p, c, p)\right)=\operatorname{size}(c)-1$. Similarly, the conditions $\mathrm{wt}\left(\mathrm{R}_{\mathcal{A}}\left(p^{\prime}, c, p^{\prime}\right)\right) \neq \infty$ and (17.7) (4) imply that $\operatorname{wt}\left(\mathrm{R}_{\mathcal{A}}\left(p^{\prime}, c, p^{\prime}\right)\right)=\operatorname{size}(c)-1$.

This ends the proof of the fact that $\mathcal{A}$ has the twinning property.
Thus, the $\left(\Sigma\right.$, Rat $\left._{\text {min },+}\right)$-wta $\mathcal{A}$ given in Example 17.1 .2 has the twinning property. Also the other requirements of Theorem 17.3 .2 are satisfied, because $\operatorname{Rat}_{\min ,+}=\left(\mathbb{Q}_{\infty}, \min ,+, \infty, 0\right)$ is an extremal semiring and the pair $(f, g)$ is a maximal factorization with $f:\left(\mathbb{Q}_{\infty}\right)^{Q} \backslash\left\{\infty_{Q}\right\} \rightarrow\left(\mathbb{Q}_{\infty}\right)^{Q}$ and $g$ : $\left(\mathbb{Q}_{\infty}\right)^{Q} \backslash\left\{\infty_{Q}\right\} \rightarrow \mathbb{Q}_{\infty}$ defined for each $u=\binom{u_{p}}{u_{p^{\prime}}}$ in $\mathbb{Q}_{\infty}^{Q} \backslash\left\{\infty_{Q}\right\}$ by

$$
f(u)_{q}=u_{q}-g(u) \text { for each } q \in Q \quad \text { and } \quad g(u)=\min \left(u_{p}, u_{p^{\prime}}\right)
$$

Hence, for the bu deterministic $\left(\Sigma, \operatorname{Rat}_{\text {min },+}\right)-\operatorname{wta}^{\operatorname{det}}{ }_{(f, g)}(\mathcal{A})$ (cf. Figure 17.2 ), Theorem 17.3 .2 guarantees that

- $\llbracket \operatorname{det}_{(f, g)}(\mathcal{A}) \rrbracket=\llbracket \mathcal{A} \rrbracket=$ size and
- $\operatorname{det}_{(f, g)}(\mathcal{A})$ is minimal with respect to the number of states among all bu deterministic $\left(\Sigma, \operatorname{Rat}_{\text {min },+}\right)$ wta which are equivalent to $\mathcal{A}$ and obtained by determinization by factorization.
We note that the bu deterministic ( $\Sigma, \mathrm{Nat}_{\mathrm{min},+}$ )-wta of Example 3.2.3, which also recognizes size, has only one state, and hence it is smaller than $\operatorname{det}_{(f, g)}(\mathcal{A})$ (where, in this comparison, we disregard the difference between $\mathrm{Nat}_{\mathrm{min},+}$ and $\mathrm{Rat}_{\min ,+}$ ). However, by means of two transformations, viz. weight pushing and forward bisimulation, we can transform $\operatorname{det}_{(f, g)}(\mathcal{A})$ into the ( $\Sigma, \operatorname{Rat}_{\text {min },+}$ )-wta of Example 3.2.3 (cf. Figure 17.3). Let us briefly explain these transformations.

Since Rat min,+ is a commutative semifield, we can apply weight pushing to $\operatorname{det}_{(f, g)}(\mathcal{A})$ (cf. Section 7.3). More precisely, we define the mapping $\lambda\left(\binom{0}{\infty}\right)=0$ and $\lambda\left(\binom{0.5}{0}\right)=0.5$. Then Figure 17.3 shows the $\left(\Sigma, \operatorname{Rat}_{\text {min },+}\right)$-wta $\operatorname{push}_{\lambda}\left(\operatorname{det}_{(f, g)}(\mathcal{A})\right)$. By Lemma 7.3.2 we have that $\llbracket \operatorname{det}_{(f, g)}(\mathcal{A}) \rrbracket=\llbracket \operatorname{push}_{\lambda}\left(\operatorname{det}_{(f, g)}(\mathcal{A})\right) \rrbracket$.

For the second transformation (viz., forward bisimulation, cf. HMM07), we define the equivalence relation $R=Q \times Q$. Thus, the factor set $Q / R$ contains one equivalence class, namely $Q$. One can easily check that $R$ is a forward bisimulation on $\operatorname{push}_{\lambda}\left(\operatorname{det}_{(f, g)}(\mathcal{A})\right)$ in the sense of [HMM07, Def. 1]. Thus we can construct the forward aggregate $\left(\Sigma\right.$, $\left.\operatorname{Rat}_{\text {min },+}\right)$-wta $\left(\operatorname{push}_{\lambda}\left(\operatorname{det}_{(f, g)}(\mathcal{A})\right)\right) / R$ (cf. [HMM07, Def. 3]). By [HMM07, Thm. 6] we have $\llbracket \operatorname{push}_{\lambda}\left(\operatorname{det}_{(f, g)}(\mathcal{A})\right) \rrbracket=\llbracket\left(\operatorname{push}_{\lambda}\left(\operatorname{det}_{(f, g)}(\mathcal{A})\right)\right) / R \rrbracket$. (We note that the relation $R$ is not a forward bisimulation on $\operatorname{det}_{(f, g)}(\mathcal{A})$ because $F\left(\binom{0}{\infty}\right)=0 \neq 0.5=F\left(\binom{0.5}{0}\right)$.) It is obvious that the forward aggregate $\left(\Sigma, \operatorname{Rat}_{\text {min },+}\right)$-wta $\left(\operatorname{push}_{\lambda}\left(\operatorname{det}_{(f, g)}(\mathcal{A})\right)\right) / R$ is exactly the $\left(\Sigma, \operatorname{Rat}_{\text {min },+}\right)$-wta $\mathcal{A}$ of Example 3.2.3 (modulo state renaming and the change of the weight algebra from Nat ${ }_{\text {min },+}$ to Rat $_{\text {min },+}$ ).

Finally, we mention that in BVM10, Sect. 5.5], BF12, and Büc14, Sect. 5.4] the question of deciding the twinning property is investigated.

### 17.3.4 Proof of Theorem 17.3 .2

We organize the proof of Theorem 17.3 .2 as follows:
(1) finiteness of $\operatorname{det}_{(f, g)}(\mathcal{A})$ (cf. Theorem 17.3.17),
(2) correctness of $\operatorname{det}_{(f, g)}(\mathcal{A})$ (cf. Theorem 17.3.18), and
(3) minimality of $\operatorname{det}_{(f, g)}(\mathcal{A})$ (cf. Theorem 17.3.19).

## Finiteness of $\operatorname{det}_{(f, g)}(\mathcal{A})$ (proof of Theorem 17.3.2(1))

Here we show sufficient conditions under which $\operatorname{det}_{(f, g)}(\mathcal{A})$ is a $(\Sigma, \mathrm{B})$-wta (cf. Theorem 17.3.17).
Let $\xi \in \mathrm{T}_{\Sigma}, w \in \operatorname{pos}(\xi)$, and $\rho \in \mathrm{R}_{\mathcal{A}}(\xi)$ such that $\rho(w)=\rho(\varepsilon)$ and assume that B is extremal and commutative. Then the following kind of Bellman optimality property holds (where "optimality" has to be replaced by "victory"): If $\rho$ is victorious in $\mathrm{R}_{\mathcal{A}}(\rho(\varepsilon), \xi)$ ), then $\left.\rho\right|^{w}$ is victorious for $\mathrm{R}_{\mathcal{A}}\left(\rho(\varepsilon),\left.\xi\right|^{w}, \rho(w)\right)$


Figure 17.3: The following $\left(\Sigma\right.$, Rat $\left._{\text {min },+}\right)$-wta are shown clockwise, starting up-left: (a) $\mathcal{A}$ of Figure 17.1 (b) $\operatorname{det}_{(f, g)}(\mathcal{A}),(\mathrm{c}) \operatorname{push}_{\lambda}\left(\operatorname{det}_{(f, g)}(\mathcal{A})\right)$, and (d) $\left(\operatorname{push}_{\lambda}\left(\operatorname{det}_{(f, g)}(\mathcal{A})\right)\right) / R$. The wta in (b), (c), and (d) are bu deterministic.
(modulo right multiplication with $\mathrm{wt}\left(\left.\xi\right|_{w},\left.\rho\right|_{w}\right)$ ). The property is based on the following trivial fact.
Let B be extremal, $b \in B$, and $B_{1}, B_{2} \subseteq B$ be finite subsets.
If $b \in B_{1}, B_{1} \subseteq B_{2}$, and $b=\bigoplus_{b^{\prime} \in B_{2}} b^{\prime}$, then $b=\bigoplus_{b^{\prime} \in B_{1}} b^{\prime}$.
Observation 17.3.12. BVM10, Obs. 5.12] Let B be extremal and commutative. Moreover, let $\xi \in \mathrm{T}_{\Sigma}$, $w \in \operatorname{pos}(\xi)$, and $\rho \in \mathrm{R}_{\mathcal{A}}(\xi)$ such that $\rho(w)=\rho(\varepsilon)$ and $\operatorname{wt}(\xi, \rho)=\operatorname{wt}^{( }\left(\mathrm{R}_{\mathcal{A}}(\rho(\varepsilon), \xi)\right)$. Then

$$
\mathrm{wt}(\xi, \rho)=\mathrm{wt}\left(\left.\xi\right|^{w},\left.\rho\right|^{w}\right) \otimes \mathrm{wt}\left(\left.\xi\right|_{w},\left.\rho\right|_{w}\right)=\mathrm{wt}\left(\mathrm{R}_{\mathcal{A}}\left(\rho(\varepsilon),\left.\xi\right|^{w}, \rho(\varepsilon)\right)\right) \otimes \mathrm{wt}\left(\left.\xi\right|_{w},\left.\rho\right|_{w}\right)
$$

Proof. We have

$$
\begin{align*}
& \mathrm{wt}\left(\mathrm{R}_{\mathcal{A}}\left(\rho(\varepsilon),\left.\xi\right|^{w}, \rho(\varepsilon)\right)\right) \otimes \mathrm{wt}\left(\left.\xi\right|_{w},\left.\rho\right|_{w}\right) \\
& =\left(\bigoplus_{\nu \in \mathrm{R}_{\mathcal{A}}\left(\rho(\varepsilon),\left.\xi\right|^{w}, \rho(\varepsilon)\right)} \operatorname{wt}\left(\left.\xi\right|^{w}, \nu\right)\right) \otimes \mathrm{wt}\left(\left.\xi\right|_{w},\left.\rho\right|_{w}\right) \\
& =\bigoplus_{\nu \in \mathrm{R}_{\mathcal{A}}\left(\rho(\varepsilon),\left.\xi\right|^{w}, \rho(\varepsilon)\right)} \operatorname{wt}\left(\left.\xi\right|^{w}, \nu\right) \otimes \operatorname{wt}\left(\left.\xi\right|_{w},\left.\rho\right|_{w}\right) \\
& =\operatorname{wt}(\xi, \rho) \\
& =\operatorname{wt}\left(\left.\xi\right|^{w},\left.\rho\right|^{w}\right) \otimes \operatorname{wt}\left(\left.\xi\right|_{w},\left.\rho\right|_{w}\right)
\end{align*}
$$

$$
=\bigoplus_{\nu \in \mathrm{R}_{\mathcal{A}}\left(\rho(\varepsilon),\left.\xi\right|^{w}, \rho(\varepsilon)\right)} \mathrm{wt}\left(\left.\xi\right|^{w}, \nu\right) \otimes \mathrm{wt}\left(\left.\xi\right|_{w},\left.\rho\right|_{w}\right) \quad \quad \text { (by distributivity) }
$$

(by commutativity)
At ( $\star$ ) we have used (17.8) with

- $b=\mathrm{wt}(\xi, \rho)$,
- $B_{1}=\left\{\operatorname{wt}\left(\left.\xi\right|^{w}, \nu\right) \otimes \operatorname{wt}\left(\left.\xi\right|_{w},\left.\rho\right|_{w}\right) \mid \nu \in \mathrm{R}_{\mathcal{A}}\left(\rho(\varepsilon),\left.\xi\right|^{w}, \rho(\varepsilon)\right)\right\}$, and
- $B_{2}=\left\{\mathrm{wt}(\xi, \nu) \mid \nu \in \mathrm{R}_{\mathcal{A}}(\rho(\varepsilon), \xi)\right\}$.

Obviously, (a) $b \in B_{1}$, (b) by commutativity we have $B_{1} \subseteq B_{2}$, and (c) by assumption (viz. wt $(\xi, \rho)=$ $\left.\operatorname{wt}\left(\mathrm{R}_{\mathcal{A}}(\rho(\varepsilon), \xi)\right)\right)$ we have $b=\bigoplus_{b^{\prime} \in B_{2}} b^{\prime}$.

The next definitions are taken from [BVM10, Def. 5.13]. Let $S \subseteq Q$. Then we define

$$
\mathcal{C}^{\prime}(S)=\left\{(\xi, \kappa) \mid \xi \in \mathrm{T}_{\Sigma}, \kappa: S \rightarrow \mathrm{R}_{\mathcal{A}}(\xi),(\forall q \in S): \kappa_{q} \in \mathrm{R}_{\mathcal{A}}(q, \xi), \mathrm{wt}\left(\xi, \kappa_{q}\right) \neq \mathbb{O}\right\}
$$

We set $\mathcal{C}^{\prime}=\bigcup_{S \subseteq Q} \mathcal{C}^{\prime}(S)$.
In the following, we define (a) the mapping $\overline{\mathrm{wt}}: \mathcal{C}^{\prime} \rightarrow B^{Q}$ and (b) for each $S \subseteq Q$, the family $\left(U(\xi, \kappa) \mid(\xi, \kappa) \in \mathcal{C}^{\prime}(S)\right)$ and (c) the sets $\mathcal{C}(S) \subseteq \mathcal{C}^{\prime}(S)$ and $\mathcal{C}$. To this end, let $S \subseteq Q$ and $(\xi, \kappa) \in \mathcal{C}^{\prime}(S)$. Then

- for each $q \in Q$ we set $\overline{\mathrm{wt}}(\xi, \kappa)_{q}=\mathrm{wt}\left(\xi, \kappa_{q}\right)$ if $q \in S$, otherwise we set $\overline{\mathrm{wt}}(\xi, \kappa)_{q}=\mathbb{0}$,
- we define $U(\xi, \kappa)$ to be the set of all pairs $\left(w_{1}, w_{2}\right) \in \operatorname{pos}(\xi) \times \operatorname{pos}(\xi)$ such that $w_{1}<_{\text {pref }} w_{2}$ and for every $q \in S$ we have $\kappa_{q}\left(w_{1}\right)=\kappa_{q}\left(w_{2}\right)$, and
- we have $(\xi, \kappa) \in \mathcal{C}(S)$ iff for every $\left(w_{1}, w_{2}\right) \in U(\xi, \kappa)$ and $q \in S$ we have

$$
\operatorname{wt}\left(\left.\xi\right|_{w_{1}},\left.\kappa_{q}\right|_{w_{1}}\right)=\operatorname{wt}\left(\mathrm{R}_{\mathcal{A}}\left(\kappa_{q}\left(w_{1}\right),\left.\xi\right|_{w_{1}}\right)\right)
$$

i.e., $\left.\kappa_{q}\right|_{w_{1}}$ is victorious in $\left.\mathrm{R}_{\mathcal{A}}\left(\kappa_{q}\left(w_{1}\right),\left.\xi\right|_{w_{1}}\right)\right)$. We set $\mathcal{C}=\bigcup_{S \subseteq Q} \mathcal{C}(S)$.

Lemma 17.3.13. BVM10, Lm. 5.14] Let B be commutative and extremal. For each $\xi \in \mathrm{T}_{\Sigma}$ there exists $(\xi, \kappa) \in \mathcal{C}$ such that, for each $q \in Q$, we have $\overline{\mathrm{wt}}(\xi, \kappa)_{q}=\mathrm{wt}\left(\mathrm{R}_{\mathcal{A}}(q, \xi)\right)$.

Proof. We begin by showing the following statement:

$$
\begin{equation*}
\text { for each } \xi \in \mathrm{T}_{\Sigma} \text { there exists a } \kappa: Q \rightarrow \mathrm{R}_{\mathcal{A}}(\xi) \text { such that } \tag{17.9}
\end{equation*}
$$

for each $q \in Q$ we have $\kappa_{q} \in \mathrm{R}_{\mathcal{A}}(q, \xi)$ and $P(\xi, \kappa, q)$ holds,
where $P(\xi, \kappa, q)$ is the abbreviation of the statement

$$
\text { for every } w \in \operatorname{pos}(\xi): \operatorname{wt}\left(\left.\xi\right|_{w},\left.\kappa_{q}\right|_{w}\right)=\operatorname{wt}\left(\mathrm{R}_{\mathcal{A}}\left(\kappa_{q}(w),\left.\xi\right|_{w}\right)\right) .
$$

We prove (17.9) by induction on $\mathrm{T}_{\Sigma}$. Let $\xi=\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)$. By I.H., for each $i \in[k]$, there exists a $\kappa_{i}: Q \rightarrow \mathrm{R}_{\mathcal{A}}\left(\xi_{i}\right)$ such that, for each $q \in Q$, we have $\left(\kappa_{i}\right)_{q} \in \mathrm{R}_{\mathcal{A}}\left(q, \xi_{i}\right)$ and the statement $P\left(\xi_{i}, \kappa_{i}, q\right)$ holds. We define $\kappa$ as follows: for every $q \in Q$, we let $\kappa_{q} \in \mathrm{R}_{\mathcal{A}}(\xi)$ be such that

- $\kappa_{q}(\varepsilon)=q$,
- for every $i \in[k]$ and $w \in \operatorname{pos}\left(\xi_{i}\right)$, we let $\kappa_{q}(i w)=\left(\kappa_{i}\right)_{p_{i}}(w)$, where the states $p_{1}, \ldots, p_{k}$ are defined such that

$$
\begin{equation*}
\left(\bigotimes_{i \in[k]} \mathrm{wt}\left(\xi_{i},\left(\kappa_{i}\right)_{p_{i}}\right)\right) \otimes \delta_{k}\left(p_{1} \cdots p_{k}, \sigma, q\right)=\bigoplus_{q_{1}, \ldots q_{k} \in Q}\left(\bigotimes_{i \in[k]} \mathrm{wt}\left(\xi_{i},\left(\kappa_{i}\right)_{q_{i}}\right)\right) \otimes \delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right), \tag{17.10}
\end{equation*}
$$

where, for each $i \in[k]$, the state $p_{i}$ is defined such that

$$
\begin{equation*}
\bigoplus_{q_{i} \in Q} \operatorname{wt}\left(\xi_{i},\left(\kappa_{i}\right)_{q_{i}}\right)=\operatorname{wt}\left(\xi_{i},\left(\kappa_{i}\right)_{p_{i}}\right) . \tag{17.11}
\end{equation*}
$$

We recall that such a choice of $p_{i}$ exists by Observation [2.6.7
Let $q \in Q$. We show that $P(\xi, \kappa, q)$ holds. For this, let $w \in \operatorname{pos}(\xi)$. We proceed by case analysis. Case (a): Let $w=\varepsilon$. Then

$$
\begin{aligned}
& \mathrm{wt}\left(\xi, \kappa_{q}\right)=\left(\bigotimes_{i \in[k]} \mathrm{wt}\left(\left.\xi\right|_{i},\left.\kappa_{q}\right|_{i}\right)\right) \otimes \delta_{k}\left(\kappa_{q}(1) \cdots \kappa_{q}(k), \sigma, \kappa_{q}(\varepsilon)\right) \\
& =\left(\bigotimes_{i \in[k]} \mathrm{wt}\left(\xi_{i},\left(\kappa_{i}\right)_{p_{i}}\right)\right) \otimes \delta_{k}\left(p_{1} \cdots p_{k}, \sigma, q\right) \quad \quad \text { (by definition of } \kappa \text { ) } \\
& =\left(\bigotimes_{i \in[k] q_{i} \in Q} \bigoplus_{i} \mathrm{wt}\left(\xi_{i},\left(\kappa_{i}\right)_{q_{i}}\right)\right) \otimes \delta_{k}\left(p_{1} \cdots p_{k}, \sigma, q\right) \quad \text { (by (17.11)) } \\
& =\bigoplus_{q_{1}, \ldots, q_{k} \in Q}\left(\bigotimes_{i \in[k]} \operatorname{wt}\left(\xi_{i},\left(\kappa_{i}\right)_{q_{i}}\right)\right) \otimes \delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right) \quad \quad \text { (by distributivity) } \\
& \left.=\bigoplus_{q_{1}, \ldots, q_{k} \in Q}\left(\bigotimes_{i \in[k]} \operatorname{wt}\left(\mathrm{R}_{\mathcal{A}}\left(q_{i}, \xi_{i}\right)\right)\right) \otimes \delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right) \quad \quad \text { (by } P\left(\xi_{i}, \kappa_{i}, q_{i}\right) \text { for } w=\varepsilon\right) \\
& =\bigoplus_{q_{1}, \ldots, q_{k} \in Q} \bigoplus_{\rho_{1} \in \mathrm{R}_{\mathcal{A}}\left(q_{1}, \xi_{1}\right)} \ldots \bigoplus_{\rho_{k} \in \mathrm{R}_{\mathcal{A}}\left(q_{k}, \xi_{k}\right)}\left(\bigotimes_{i \in[k]} \mathrm{wt}\left(\xi_{i}, \rho_{i}\right)\right) \otimes \delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right) \quad \text { (by distributivity) } \\
& =\bigoplus_{\rho \in \mathbb{R}_{\mathcal{A}}(q, \xi)} \operatorname{wt}(\xi, \rho)=\operatorname{wt}\left(\mathrm{R}_{\mathcal{A}}(q, \xi)\right)=\operatorname{wt}\left(\mathrm{R}_{\mathcal{A}}\left(\kappa_{q}(\varepsilon), \xi\right)\right) .
\end{aligned}
$$

Case (b): Let $w=i v$. Then $\left.\operatorname{wt}\left(\left.\xi\right|_{i v},\left.\kappa_{q}\right|_{i v}\right)=\operatorname{wt}\left(\left.\xi_{i}\right|_{v},\left.\left(\kappa_{i}\right)_{p_{i}}\right|_{v}\right)\right)=\operatorname{wt}\left(\mathrm{R}_{\mathcal{A}}\left(\left(\kappa_{i}\right)_{p_{i}}(v),\left.\xi_{i}\right|_{v}\right)\right)=$ $\left.\operatorname{wt}\left(\overline{\mathrm{R}_{\mathcal{A}}\left(\kappa_{q}(i v)\right.},\left.\xi\right|_{i v}\right)\right)$, where the second equality holds by $P\left(\xi_{i}, \kappa_{i}, p_{i}\right)$ for $v$.

This finishes the proof of (17.9).
Now we prove the statement of our lemma. Let $\xi \in \mathrm{T}_{\Sigma}$. By (17.9) there exists a $\kappa: Q \rightarrow \mathrm{R}_{\mathcal{A}}(\xi)$ such that for every $q \in Q$ we have $\kappa_{q} \in \mathrm{R}_{\mathcal{A}}(q, \xi)$ and $P(\xi, \kappa, q)$. Let $S=\left\{q \in Q \mid \operatorname{wt}\left(\mathrm{R}_{\mathcal{A}}(q, \xi)\right) \neq \mathbb{0}\right\}$ and $\kappa^{\prime}=\left.\kappa\right|_{S}$. We note that $\left(\xi, \kappa^{\prime}\right) \in \mathcal{C}(S)$ (and hence, $\left(\xi, \kappa^{\prime}\right) \in \mathcal{C}$ ), because

- for every $q \in S: \kappa_{q}^{\prime} \in \mathrm{R}_{\mathcal{A}}(q, \xi)$ and $\operatorname{wt}\left(\xi, \kappa_{q}^{\prime}\right)=\operatorname{wt}\left(\mathrm{R}_{\mathcal{A}}(q, \xi)\right) \neq \mathbb{O}$ (hence $\left(\xi, \kappa^{\prime}\right) \in \mathcal{C}^{\prime}(S)$ ) and
- for every $w \in \operatorname{pos}(\xi): \operatorname{wt}\left(\left.\xi\right|_{w}, \kappa_{q}^{\prime} \mid w\right)=\operatorname{wt}\left(\mathrm{R}_{\mathcal{A}}\left(\kappa_{q}^{\prime}(w),\left.\xi\right|_{w}\right)\right)$ by (17.9) (hence $\left.\left(\xi, \kappa^{\prime}\right) \in \mathcal{C}(S)\right)$.

Then, for each $q \in S$, we have $\kappa_{q}^{\prime} \in \mathrm{R}_{\mathcal{A}}(q, \xi)$ and $P\left(\xi, \kappa^{\prime}, q\right)$. Moreover, for each $q \in S$ :

$$
\overline{\mathrm{wt}}\left(\xi, \kappa^{\prime}\right)_{q}=\mathrm{wt}\left(\xi, \kappa_{q}^{\prime}\right)=\mathrm{wt}\left(\mathrm{R}_{\mathcal{A}}(q, \xi)\right)
$$

If $q \in Q \backslash S$, then also $\overline{\mathrm{wt}}\left(\xi, \kappa^{\prime}\right)_{q}=\mathbb{0}=\operatorname{wt}\left(\mathrm{R}_{\mathcal{A}}(q, \xi)\right)$ by definitions of $\overline{\mathrm{wt}}$ and of $S$.

The next lemma shows how one slice can be cut out.
Lemma 17.3.14. BVM10, Lm. 5.15] Let B be extremal and commutative and $\mathcal{A}$ have the twinning property. Moreover, let $S \subseteq Q$ and $(\xi, \kappa) \in \mathcal{C}(S)$ such that $U(\xi, \kappa) \neq \emptyset$. Then there exist $\left(\xi^{\prime}, \kappa^{\prime}\right) \in \mathcal{C}(S)$ and $b \in B$ such that $\overline{\mathrm{wt}}(\xi, \bar{\kappa})=b \cdot \overline{\mathrm{wt}}\left(\xi^{\prime}, \kappa^{\prime}\right)$ and $|U(\xi, \kappa)|>\left|U\left(\xi^{\prime}, \kappa^{\prime}\right)\right|$.

Proof. Since $U(\xi, \kappa) \neq \emptyset$, there exists a pair $\left(w_{1}, w_{2}\right) \in U(\xi, \kappa)$ such that for every $\left(w_{1}^{\prime}, w_{2}^{\prime}\right) \in U(\xi, \kappa)$, if $w_{1}^{\prime} \leq_{\text {pref }} w_{1}$, then $w_{1}^{\prime}=w_{1}$. (Thus, intuitively, there does not exist a repetition of states above $w_{1}$.) We construct $\xi^{\prime}=\xi\left[\left.\xi\right|_{w_{2}}\right]_{w_{1}}$ and, for every $q \in S$ and $w \in \operatorname{pos}\left(\xi^{\prime}\right)$, we set $\kappa_{q}^{\prime}(w)=\kappa_{q}\left(w_{2} v\right)$ if $w=w_{1} v$ and $\kappa_{q}^{\prime}(w)=\kappa_{q}(w)$ otherwise. Before defining $b \in B$, we show that $\kappa$ has the following property:

$$
\begin{equation*}
\left(\forall q^{\prime}, \bar{q} \in S\right): \operatorname{wt}\left(\mathrm{R}_{\mathcal{A}}\left(\kappa_{q^{\prime}}\left(w_{1}\right),\left.\left(\left.\xi\right|^{w_{2}}\right)\right|_{w_{1}}, \kappa_{q^{\prime}}\left(w_{2}\right)\right)\right)=\operatorname{wt}\left(\mathrm{R}_{\mathcal{A}}\left(\kappa_{\bar{q}}\left(w_{1}\right),\left.\left(\left.\xi\right|^{w_{2}}\right)\right|_{w_{1}}, \kappa_{\bar{q}}\left(w_{2}\right)\right)\right) \tag{17.12}
\end{equation*}
$$

We prove (17.12) by using the twinning property of $\mathcal{A}$. Roughly speaking, the twinning property is an implication where the premise has two conditions, viz., (i) certain states repeat and (ii) certain weights are not $\mathbb{O}$. For (i) we remark that $\left(w_{1}, w_{2}\right) \in U(\xi, \kappa)$ and thus $\kappa_{q^{\prime}}\left(w_{1}\right)=\kappa_{q^{\prime}}\left(w_{2}\right)$ and $\kappa_{\bar{q}}\left(w_{1}\right)=\kappa_{\bar{q}}\left(w_{2}\right)$. For (ii), by definition, we have $\mathrm{wt}\left(\xi, \kappa_{q^{\prime}}\right) \neq \mathbb{O}$ and $\mathrm{wt}\left(\xi, \kappa_{\bar{q}}\right) \neq \mathbb{O}$. Thus, using commutativity,

- $\mathbb{O} \neq \operatorname{wt}\left(\xi, \kappa_{q^{\prime}}\right)=\operatorname{wt}\left(\xi_{w_{1}},\left.\kappa_{q^{\prime}}\right|_{w_{1}}\right) \otimes \operatorname{wt}\left(\left.\left(\left.\xi\right|^{w_{2}}\right)\right|_{w_{1}},\left.\left(\left.\kappa_{q^{\prime}}\right|^{w_{2}}\right)\right|_{w_{1}}\right) \otimes \operatorname{wt}\left(\left.\xi\right|_{w_{2}},\left.\kappa_{q^{\prime}}\right|_{w_{2}}\right)$ and
- $\mathbb{O} \neq \operatorname{wt}\left(\xi, \kappa_{\bar{q}}\right)=\mathrm{wt}\left(\xi_{w_{1}},\left.\kappa_{\bar{q}}\right|_{w_{1}}\right) \otimes \operatorname{wt}\left(\left.\left(\left.\xi\right|^{w_{2}}\right)\right|_{w_{1}},\left.\left(\left.\kappa_{\bar{q}}\right|^{w_{2}}\right)\right|_{w_{1}}\right) \otimes \mathrm{wt}\left(\left.\xi\right|_{w_{2}},\left.\kappa_{\bar{q}}\right|_{w_{2}}\right)$
and hence
- $\operatorname{wt}\left(\left.\left(\left.\xi\right|^{w_{2}}\right)\right|_{w_{1}},\left.\left(\left.\kappa_{q^{\prime}}\right|^{w_{2}}\right)\right|_{w_{1}}\right) \otimes \operatorname{wt}\left(\left.\xi\right|_{w_{2}},\left.\kappa_{q^{\prime}}\right|_{w_{2}}\right) \neq \mathbb{O}$ and
$\cdot \operatorname{wt}\left(\left.\left(\left.\xi\right|^{w_{2}}\right)\right|_{w_{1}},\left.\left(\left.\kappa_{\bar{q}}\right|^{w_{2}}\right)\right|_{w_{1}}\right) \otimes \operatorname{wt}\left(\left.\xi\right|_{w_{2}},\left.\kappa_{\bar{q}}\right|_{w_{2}}\right) \neq \mathbb{0}$.
Since $\mathbb{C}$ is annihilating, each of the four values

$$
\operatorname{wt}\left(\left.\left(\left.\xi\right|^{w_{2}}\right)\right|_{w_{1}},\left.\left(\left.\kappa_{q^{\prime}}\right|^{w_{2}}\right)\right|_{w_{1}}\right), \quad \operatorname{wt}\left(\left.\xi\right|_{w_{2}}, \kappa_{q^{\prime}} \mid w_{w_{2}}\right), \quad \operatorname{wt}\left(\left.\left(\left.\xi\right|^{w_{2}}\right)\right|_{w_{1}},\left.\left(\left.\kappa_{\bar{q}}\right|^{w_{2}}\right)\right|_{w_{1}}\right), \quad \text { and } \quad \operatorname{wt}\left(\left.\xi\right|_{w_{2}},\left.\kappa_{\bar{q}}\right|_{w_{2}}\right)
$$

is different from $\mathbb{0}$. Since $B$ is extremal, and hence by Observation 2.6.11 $(3,4)$ it is zero-sum free, we obtain that

- $\operatorname{wt}\left(\mathrm{R}_{\mathcal{A}}\left(\kappa_{q^{\prime}}\left(w_{1}\right),\left.\left(\left.\xi\right|^{w_{2}}\right)\right|_{w_{1}}, \kappa_{q^{\prime}}\left(w_{2}\right)\right)\right) \neq \mathbb{O}$ and $\operatorname{wt}\left(\mathrm{R}_{\mathcal{A}}\left(\kappa_{q^{\prime}}\left(w_{2}\right),\left.\xi\right|_{w_{2}}\right)\right) \neq \mathbb{O}$ and
$\bullet \operatorname{wt}\left(\mathrm{R}_{\mathcal{A}}\left(\kappa_{\bar{q}}\left(w_{1}\right),\left.\left(\left.\xi\right|^{w_{2}}\right)\right|_{w_{1}}, \kappa_{\bar{q}}\left(w_{2}\right)\right)\right) \neq \mathbb{O}$ and $\operatorname{wt}\left(\mathrm{R}_{\mathcal{A}}\left(\kappa_{\bar{q}}\left(w_{2}\right),\left.\xi\right|_{w_{2}}\right)\right) \neq \mathbb{0}$.
Hence (17.12) follows by the twinning property of $\mathcal{A}$.
Next we define $b$ by case analysis as follows:

$$
b= \begin{cases}0 & \text { if } S=\emptyset \\ \operatorname{wt}\left(\mathrm{R}_{\mathcal{A}}\left(\kappa_{q}\left(w_{1}\right),\left.\left(\left.\xi\right|^{w_{2}}\right)\right|_{w_{1}}, \kappa_{q}\left(w_{2}\right)\right)\right) \text { for some } q \in S & \text { otherwise }\end{cases}
$$

We note that, by (17.12), $b$ does not depend on the choice of $q$. Now we show that the following three stataments hold:
(1) $\left(\xi^{\prime}, \kappa^{\prime}\right) \in \mathcal{C}(S)$,
(2) $\overline{\mathrm{wt}}(\xi, \kappa)=b \cdot \overline{\mathrm{wt}}\left(\xi^{\prime}, \kappa^{\prime}\right)$, and
(3) $|U(\xi, \kappa)|>\left|U\left(\xi^{\prime}, \kappa^{\prime}\right)\right|$.

We begin with the proof of Statement (1). It is easy to see that $\left(\xi^{\prime}, \kappa^{\prime}\right) \in \mathcal{C}^{\prime}(S)$. Now let $\left(w_{1}^{\prime}, w_{2}^{\prime}\right) \in$ $U\left(\xi^{\prime}, \kappa^{\prime}\right)$ and $q \in S$. We show that $\operatorname{wt}\left(\left.\xi^{\prime}\right|_{w_{1}^{\prime}},\left.\kappa_{q}^{\prime}\right|_{w_{1}^{\prime}}\right)=\operatorname{wt}\left(\mathrm{R}_{\mathcal{A}}\left(\kappa_{q}^{\prime}\left(w_{1}^{\prime}\right),\left.\xi^{\prime}\right|_{w_{1}^{\prime}}\right)\right)$. Note that $w_{1}^{\prime} \nless \mathrm{pref} w_{1}$. We distinguish two cases.

Case (a): There exist $v_{1}, v_{2} \in \mathbb{N}^{*}$ such that $w_{1}^{\prime}=w_{1} v_{1}$ and $w_{2}^{\prime}=w_{1} v_{2}$ (cf. Fig. 17.4(a)). Since $\kappa_{q}^{\prime}\left|\overline{w_{1}}=\kappa_{q}\right|_{w_{2}}$ for every $q \in S$, we obtain that $\left(w_{2} v_{1}, w_{2} v_{2}\right) \in U(\xi, \kappa)$. Hence

$$
\mathrm{wt}\left(\left.\xi^{\prime}\right|_{w_{1}^{\prime}},\left.\kappa_{q}^{\prime}\right|_{w_{1}^{\prime}}\right)=\mathrm{wt}\left(\left.\xi\right|_{w_{2} v_{1}},\left.\kappa_{q}\right|_{w_{2} v_{1}}\right)=\mathrm{wt}\left(\mathrm{R}_{\mathcal{A}}\left(\kappa_{q}\left(w_{2} v_{1}\right),\left.\xi\right|_{w_{2} v_{1}}\right)\right)=\mathrm{wt}\left(\mathrm{R}_{\mathcal{A}}\left(\kappa_{q}^{\prime}\left(w_{1}^{\prime}\right),\left.\xi^{\prime}\right|_{w_{1}^{\prime}}\right)\right) .
$$

Case (b): Otherwise, $\left.\kappa_{q}^{\prime}\right|_{w_{1}^{\prime}}=\left.\kappa_{q}\right|_{w_{1}^{\prime}}$ (cf. Fig. 17.4(b)). Thus $\left(w_{1}^{\prime}, w_{2}^{\prime}\right) \in U(\xi, \kappa)$ and

$$
\mathrm{wt}\left(\left.\xi^{\prime}\right|_{w_{1}^{\prime}},\left.\kappa_{q}^{\prime}\right|_{w_{1}^{\prime}}\right)=\mathrm{wt}\left(\left.\xi\right|_{w_{1}^{\prime}},\left.\kappa_{q}\right|_{w_{1}^{\prime}}\right)=\mathrm{wt}\left(\mathrm{R}_{\mathcal{A}}\left(\kappa_{q}\left(w_{1}^{\prime}\right),\left.\xi\right|_{w_{1}^{\prime}}\right)\right)=\mathrm{wt}\left(\mathrm{R}_{\mathcal{A}}\left(\kappa_{q}^{\prime}\left(w_{1}^{\prime}\right),\left.\xi^{\prime}\right|_{w_{1}^{\prime}}\right)\right)
$$



Figure 17.4: The Cases (a) and (b) for $\left(w_{1}^{\prime}, w_{2}^{\prime}\right) \in U\left(\kappa^{\prime}\right)$ (cf. [BVM10, Fig. 7]).

Now we prove Statement (2) for the non-trivial case of $b$, i.e., $S \neq \emptyset$. Let $q \in S$. Then

$$
\begin{aligned}
& \operatorname{wt}\left(\xi, \kappa_{q}\right) \\
& =\operatorname{wt}\left(\left.\xi\right|^{w_{1}},\left.\kappa_{q}\right|^{w_{1}}\right) \otimes \operatorname{wt}\left(\left.\left(\left.\xi\right|^{w_{2}}\right)\right|_{w_{1}},\left.\left(\left.\kappa_{q}\right|^{w_{2}}\right)\right|_{w_{1}}\right) \otimes \operatorname{wt}\left(\left.\xi\right|_{w_{2}},\left.\kappa_{q}\right|_{w_{2}}\right) \\
& =\operatorname{wt}\left(\left.\xi\right|^{w_{1}},\left.\kappa_{q}\right|^{w_{1}}\right) \otimes \operatorname{wt}\left(\mathrm{R}_{\mathcal{A}}\left(\kappa_{q}\left(w_{1}\right),\left.\left(\left.\xi\right|^{w_{2}}\right)\right|_{w_{1}}, \kappa_{q}\left(w_{2}\right)\right)\right) \otimes \operatorname{wt}\left(\left.\xi\right|_{w_{2}},\left.\kappa_{q}\right|_{w_{2}}\right) \\
& =b \otimes \operatorname{wt}\left(\left.\xi\right|^{w_{1}},\left.\kappa_{q}\right|^{w_{1}}\right) \otimes \operatorname{wt}\left(\left.\xi\right|_{w_{2}},\left.\kappa_{q}\right|_{w_{2}}\right) \\
& =b \otimes \operatorname{wt}\left(\xi^{\prime}, \kappa_{q}^{\prime}\right) .
\end{aligned}
$$

Finally, for the proof of Statement (3), we remark that $\kappa^{\prime}$ is obtained from $\kappa$ by removing the cycle $\left(w_{1}, w_{2}\right)$, and that this process does not introduce new cycles. Hence $|U(\xi, \kappa)|>\left|U\left(\xi^{\prime}, \kappa^{\prime}\right)\right|$.

The following lemma is used for the proof that our cutting process can only end in a finite set of trees. For this we define the finite set

$$
\mathcal{F}=\left\{(\xi, \kappa) \in \mathcal{C}^{\prime}\left|\operatorname{height}(\xi)<|Q|^{|Q|}\right\}\right.
$$

Next we will use $U^{-1}(\emptyset)$ as shorthand for the set $\{(\xi, \kappa) \mid U(\xi, \kappa)=\emptyset\}$.
Lemma 17.3.15. BVM10, Lm. 5.16] $U^{-1}(\emptyset) \subseteq \mathcal{F}$.
Proof. Let $(\xi, \kappa) \in \mathcal{C}^{\prime} \backslash \mathcal{F}$. We show that $U(\xi, \kappa) \neq \emptyset$ follows. First of all, there exists an $S \subseteq Q$ such that $(\xi, \kappa) \in \mathcal{C}^{\prime}(S)$. Since $(\xi, \kappa) \notin \mathcal{F}$, there exists a $w \in \operatorname{pos}(\xi)$ such that $|w| \geq|Q|^{|S|}$. Hence, there exist $k \in \mathbb{N}, w_{1}, \ldots, w_{k} \in \mathbb{N}^{*}$, and $u_{1}, \ldots, u_{k} \in Q^{S}$ such that $k>|Q|^{|S|}$, $w_{i} \in \operatorname{pos}(\xi)$ for every $i \in[k]$, $w_{1}<_{\text {pref }} w_{2}<_{\text {pref }} \ldots<_{\text {pref }} w_{k}$, and $\left.\kappa_{q}\right|_{w_{i}} \in \mathrm{R}_{\mathcal{A}}\left(\left(u_{i}\right)_{q}, \xi\right)$ for every $i \in[k]$ and $q \in S$. By the pigeon-hole principle, there exist $i, j \in[k]$ such that $i<j$ and $u_{i}=u_{j}$. Hence $\kappa_{q}\left(w_{i}\right)=\left(u_{i}\right)_{q}=\left(u_{j}\right)_{q}=\kappa_{q}\left(w_{j}\right)$ for every $q \in S$, which means that $\left(w_{i}, w_{j}\right) \in U(\xi, \kappa)$.

We recall that $\operatorname{det}_{(f, g)}(\mathcal{A})=\left(Q^{\prime}, \delta^{\prime}, F^{\prime}\right)$, and by Lemma 17.3 .9 we have that $Q^{\prime} \subseteq f\left(\mathrm{~h}_{\mathcal{A}}\left(\mathrm{T}_{\Sigma}\right) \backslash\left\{\mathbb{0}_{Q}\right\}\right)$ if B is commutative and $(f, g)$ is a maximal factorization. If we could show that $\mathrm{h}_{\mathcal{A}}\left(\mathrm{T}_{\Sigma}\right)$ is finite, then we have shown that $Q^{\prime}$ is finite. In the next lemma we prove that $\mathrm{h}_{\mathcal{A}}\left(\mathrm{T}_{\Sigma}\right)$ is finite under the conditions that B is commutative and extremal and $\mathcal{A}$ has the twinning property. The idea of the proof is the following. Let us consider a large tree $\xi \in \mathrm{T}_{\Sigma}$ for which we want to compute $\mathrm{h}_{\mathcal{A}}(\xi)$, which is nothing else but the $Q$-vector $\left.\left(\operatorname{wt}^{( } \mathrm{R}_{\mathcal{A}}(q, \xi)\right) \mid q \in Q\right)$. Then, by Lemma 17.3.13, there exists $\kappa: Q \rightarrow \mathrm{R}_{\mathcal{A}}(\xi)$ such that, for each $q \in Q$, we have $\operatorname{wt}\left(\mathrm{R}_{\mathcal{A}}(q, \xi)\right)=\mathrm{wt}\left(\xi, \kappa_{q}\right)=\overline{\mathrm{wt}}(\xi, \kappa)_{q}$ (which needs extremality), i.e., $\kappa_{q}$ is victorious for $\mathrm{R}_{\mathcal{A}}(q, \xi)$. If $\xi$ is large enough, then there exists a pair $\left(w_{1}, w_{2}\right) \in \operatorname{pos}(\xi) \times \operatorname{pos}(\xi)$ such that $\kappa_{q}\left(w_{1}\right)=\kappa_{q}\left(w_{2}\right)$. Now we can cut out the slice of $\xi$ which is determined by $w_{1}$ and $w_{2}$. By Lemma 17.3.14 we obtain a pair $\left(\xi^{\prime}, \kappa^{\prime}\right)$ such that $\overline{\mathrm{wt}}(\xi, \kappa)=b \cdot \overline{\mathrm{wt}}\left(\xi^{\prime}, \kappa^{\prime}\right)$ where $b$ is the weight of the victorious run on the slice which we cut out, and $\xi^{\prime}$ contains one less pair of positions with repeating
states. After repeating this cutting process, we end up in a pair $\left(\xi^{\prime \prime}, \kappa^{\prime \prime}\right)$ such that $\xi^{\prime \prime}$ does not contain a pair of positions with repeating states, i.e., $\left(\xi^{\prime \prime}, \kappa^{\prime \prime}\right) \in U^{-1}(\emptyset)$. By Lemma 17.3.15, the set $U^{-1}(\emptyset)$ is finite.

The following lemma is based on [KM05, Thm. 5].
Lemma 17.3.16. BVM10, Lm. 5.9] If B is commutative and extremal and $\mathcal{A}$ has the twinning property, then there exists a finite set $P \subseteq B^{Q}$ with $\mathrm{h}_{\mathcal{A}}\left(\mathrm{T}_{\Sigma}\right) \subseteq\{b \cdot u \mid b \in B, u \in P\}$.

Proof. Let $\xi \in \mathrm{T}_{\Sigma}$. We set $P=\{\overline{\mathrm{wt}}(\xi, \kappa) \mid(\xi, \kappa) \in \mathcal{F})$.
Let $n \in \mathbb{N}$ be maximal such that there exist $\left(\xi_{1}, \kappa_{1}\right), \ldots,\left(\xi_{n}, \kappa_{n}\right) \in \mathcal{C}$ and $b_{1}, \ldots, b_{n} \in B$ such that

- $\xi=\xi_{1}$,
- $\mathrm{h}_{\mathcal{A}}(\xi)=\overline{\mathrm{wt}}\left(\xi_{1}, \kappa_{1}\right)$ and $\overline{\mathrm{wt}}\left(\xi_{1}, \kappa_{1}\right)=b_{i} \cdot \overline{\mathrm{wt}}\left(\xi_{i}, \kappa_{i}\right)$ for every $i \in[n]$, and
- $\left|U\left(\xi_{i+1}, \kappa_{i+1}\right)\right|<\left|U\left(\xi_{i}, \kappa_{i}\right)\right|$ for every $i \in[n-1]$.

We claim that

$$
\begin{equation*}
n>0 \tag{17.13}
\end{equation*}
$$

and

$$
\begin{equation*}
\left(\xi_{n}, \kappa_{n}\right) \in \mathcal{F} \tag{17.14}
\end{equation*}
$$

which allows us to derive

$$
\mathrm{h}_{\mathcal{A}}(\xi)=\overline{\mathrm{wt}}\left(\xi_{1}, \kappa_{1}\right)=b_{n} \cdot \overline{\mathrm{wt}}\left(\xi_{n}, \kappa_{n}\right) \in\{b \cdot u \mid b \in B, u \in P\}
$$

Claim (17.13) follows from Lemma 17.3 .13 if we set $b_{1}=1$. Finally, we prove (17.14). Assume that $U\left(\xi_{n}, \kappa_{n}\right) \neq \emptyset$. By Lemma 17.3 .14 there exist $\left(\xi^{\prime}, \kappa^{\prime}\right)$ and $b^{\prime}$ such that $\overline{\mathrm{wt}}\left(\xi_{n}, \kappa_{n}\right)=b^{\prime} \cdot \overline{\mathrm{wt}}\left(\xi^{\prime}, \kappa^{\prime}\right)$ and $\left|U\left(\xi^{\prime}, \kappa^{\prime}\right)\right|<\left|U\left(\xi_{n}, \kappa_{n}\right)\right|$. Using $\kappa_{n+1}=\kappa^{\prime}$ and $b_{n+1}=b^{\prime} \cdot b_{n}$, we see that $n$ was not maximal. Hence, $U\left(\xi_{n}, \kappa_{n}\right)=\emptyset$, and by Lemma 17.3.15, $\left(\xi_{n}, \kappa_{n}\right) \in \mathcal{F}$.

Theorem 17.3.17. BVM10, Thm. 5.10] Let B be commutative and extremal and let $(f, g)$ be a maximal factorization of $B^{Q}$. Moreover, let $\mathcal{A}$ have the twinning property. Then $\operatorname{det}_{(f, g)}(\mathcal{A})$ is a $(\Sigma, \mathrm{B})$-wta.

Proof. Let $\operatorname{det}_{(f, g)}(\mathcal{A})=\left(Q^{\prime}, \delta^{\prime}, F^{\prime}\right)$. By Lemma 17.3.9, we have $Q^{\prime} \subseteq f\left(\mathrm{~h}_{\mathcal{A}}\left(\mathrm{T}_{\Sigma}\right) \backslash\left\{\mathbb{0}_{Q}\right\}\right)$. We show that $Q^{\prime}$ is finite. Lemma 17.3 .16 yields that there exists a finite set $P \subseteq B^{Q}$ such that $\mathrm{h}_{\mathcal{A}}\left(\mathrm{T}_{\Sigma}\right) \subseteq\{b \cdot u \mid b \in$ $B, u \in P\}$. We calculate

$$
Q^{\prime} \subseteq f\left(\mathrm{~h}_{\mathcal{A}}\left(\mathrm{T}_{\Sigma}\right) \backslash\left\{\mathbb{D}_{Q}\right\}\right) \subseteq f\left(\{b \cdot u \mid b \in B, u \in P\} \backslash\left\{\mathbb{O}_{Q}\right\}\right) \subseteq f\left(P \backslash\left\{\mathbb{D}_{Q}\right\}\right)
$$

because $(f, g)$ is maximal. Hence, $Q^{\prime}$ is finite.

## Correctness of $\operatorname{det}_{(f, g)}(\mathcal{A})$ (proof of Theorem 17.3.2(2))

Theorem 17.3.18. BVM10, Thm. 5.4] Let $(f, g)$ be a factorization and B be ${\operatorname{commutative} . \text { If } \operatorname{det}_{(f, g)}(\mathcal{A})}^{(\mathcal{A}}$ is a wta, then $\llbracket \mathcal{A} \rrbracket=\llbracket \operatorname{det}_{(f, g)}(\mathcal{A}) \rrbracket$.

Proof. Let $\operatorname{det}_{(f, g)}(\mathcal{A})=\left(Q^{\prime}, \delta^{\prime}, F^{\prime}\right)$ be a wta, i.e., $Q^{\prime}$ be finite. We abbreviate $\operatorname{det}_{(f, g)}(\mathcal{A})$ by $\mathcal{A}^{\prime}$. By induction on $\mathrm{T}_{\Sigma}$, we prove that the following statement holds:

$$
\begin{equation*}
\text { For every } \xi \in \mathrm{T}_{\Sigma} \text {, we have: } \mathrm{h}_{\mathcal{A}}(\xi)={\underset{u \in Q^{\prime}}{ } \mathrm{h}_{\mathcal{A}^{\prime}}(\xi)_{u} \cdot u . . . . . . . .} \tag{17.15}
\end{equation*}
$$

Let $\xi=\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)$. Since $\mathcal{A}^{\prime}$ is bu deterministic, by Lemma 4.2.1(1) there are $u_{1}^{\prime}, \ldots, u_{k}^{\prime} \in Q^{\prime}$ such that, for every $i \in[k]$ and $u \in Q^{\prime} \backslash\left\{u_{i}^{\prime}\right\}$, we have $\mathrm{h}_{\mathcal{A}^{\prime}}\left(\xi_{i}\right)_{u}=\mathbb{0}$. We derive $(\star)$ :

$$
\begin{align*}
\mathrm{h}_{\mathcal{A}}(\xi) & =\delta_{\mathcal{A}}(\sigma)\left(\mathrm{h}_{\mathcal{A}}\left(\xi_{1}\right), \ldots, \mathrm{h}_{\mathcal{A}}\left(\xi_{k}\right)\right) \\
& =\delta_{\mathcal{A}}(\sigma)\left(\underset{u_{1} \in Q^{\prime}}{ } \mathrm{h}_{\mathcal{A}^{\prime}}\left(\xi_{1}\right)_{u_{1}} \cdot u_{1}, \ldots, \underset{u_{k} \in Q^{\prime}}{ } \mathrm{h}_{\mathcal{A}^{\prime}}\left(\xi_{k}\right)_{u_{k}} \cdot u_{k}\right)  \tag{byI.H.}\\
& =\delta_{\mathcal{A}}(\sigma)\left(\mathrm{h}_{\mathcal{A}^{\prime}}\left(\xi_{1}\right)_{u_{1}^{\prime}} \cdot u_{1}^{\prime}, \ldots, \mathrm{h}_{\mathcal{A}^{\prime}}\left(\xi_{k}\right)_{u_{k}^{\prime}} \cdot u_{k}^{\prime}\right) \\
& =\left(\mathrm{h}_{\mathcal{A}^{\prime}}\left(\xi_{1}\right)_{u_{1}^{\prime}} \otimes \ldots \otimes \mathrm{h}_{\mathcal{A}^{\prime}}\left(\xi_{k}\right)_{u_{k}^{\prime}}\right) \cdot \delta_{\mathcal{A}}(\sigma)\left(u_{1}^{\prime}, \ldots, u_{k}^{\prime}\right)
\end{align*}
$$

(Lemma 3.6.4)
where Lemma 3.6.4 uses the assumption that B is commutative.
Moreover, we derive ( $\dagger$ ): for every $u \in Q^{\prime}$

$$
\begin{aligned}
\mathrm{h}_{\mathcal{A}^{\prime}}(\xi)_{u} & =\delta_{\mathcal{A}^{\prime}}(\sigma)\left(\mathrm{h}_{\mathcal{A}^{\prime}}\left(\xi_{1}\right), \ldots \mathrm{h}_{\mathcal{A}^{\prime}}\left(\xi_{k}\right)\right)_{u} \\
& =\bigoplus_{u_{1}, \ldots, u_{k} \in Q^{\prime}} \mathrm{h}_{\mathcal{A}^{\prime}}\left(\xi_{1}\right)_{u_{1}} \otimes \ldots \otimes \mathrm{~h}_{\mathcal{A}^{\prime}}\left(\xi_{k}\right)_{u_{k}} \otimes \delta_{k}^{\prime}\left(u_{1} \ldots u_{k}, \sigma, u\right) \\
& =\mathrm{h}_{\mathcal{A}^{\prime}}\left(\xi_{1}\right)_{u_{1}^{\prime}} \otimes \ldots \otimes \mathrm{h}_{\mathcal{A}^{\prime}}\left(\xi_{k}\right)_{u_{k}^{\prime}} \otimes \delta_{k}^{\prime}\left(u_{1}^{\prime} \ldots u_{k}^{\prime}, \sigma, u\right)
\end{aligned}
$$

Now we distinguish two cases.
Case (a): Let $\delta_{\mathcal{A}}(\sigma)\left(u_{1}^{\prime}, \ldots, u_{k}^{\prime}\right)=\mathbb{O}_{Q}$. Then $(\star)$ implies $\mathrm{h}_{\mathcal{A}}(\xi)=\mathbb{O}_{Q}$. By definition of $\delta^{\prime}$, we have $\delta_{k}^{\prime}\left(\overline{u_{1}^{\prime} \ldots u_{k}^{\prime}}, \sigma, u\right)=\mathbb{O}$ for every $u \in Q^{\prime}$. Hence, $(\dagger)$ implies $\mathrm{h}_{\mathcal{A}^{\prime}}(\xi)=\mathbb{O}_{Q}$.


$$
\begin{aligned}
\mathrm{h}_{\mathcal{A}}(\xi) & =\left(\mathrm{h}_{\mathcal{A}^{\prime}}\left(\xi_{1}\right)_{u_{1}^{\prime}} \otimes \ldots \otimes \mathrm{h}_{\mathcal{A}^{\prime}}\left(\xi_{k}\right)_{u_{k}^{\prime}}\right) \cdot \delta_{\mathcal{A}}(\sigma)\left(u_{1}^{\prime}, \ldots, u_{k}^{\prime}\right) \\
& =\left(\mathrm{h}_{\mathcal{A}^{\prime}}\left(\xi_{1}\right)_{u_{1}^{\prime}} \otimes \ldots \otimes \mathrm{h}_{\mathcal{A}^{\prime}}\left(\xi_{k}\right)_{u_{k}^{\prime}} \otimes g\left(\delta_{\mathcal{A}}(\sigma)\left(u_{1}^{\prime}, \ldots, u_{k}^{\prime}\right)\right)\right) \cdot u^{\prime} \\
& =\mathrm{h}_{\mathcal{A}^{\prime}}(\xi)_{u^{\prime}} \cdot u^{\prime} \quad((f, g) \text { is a factorization) }) \\
& ={\left.\mathcal{W y ~ d e f i n i t i o n ~ o f ~} \delta_{k}^{\prime}\left(u_{1}^{\prime} \cdots u_{k}^{\prime}, \sigma, u^{\prime}\right) \text { and by }(\dagger)\right)} \mathrm{h}_{\mathcal{A}^{\prime}}(\xi)_{u} \cdot u .
\end{aligned}
$$

Now we show that $\llbracket \mathcal{A} \rrbracket(\xi)=\llbracket \mathcal{A}^{\prime} \rrbracket(\xi)$ for every $\xi \in \mathrm{T}_{\Sigma}$.

$$
\begin{align*}
& \llbracket \mathcal{A}^{\prime} \rrbracket(\xi)=\bigoplus_{u \in Q^{\prime}} \mathrm{h}_{\mathcal{A}^{\prime}}(\xi)_{u} \otimes F_{u}^{\prime} \\
& \left.=\bigoplus_{u \in Q^{\prime}} \mathrm{h}_{\mathcal{A}^{\prime}}(\xi)_{u} \otimes\left(\bigoplus_{q \in Q} u_{q} \otimes F_{q}\right) \quad \quad \quad \text { (by construction of } \operatorname{det}_{(f, g)}(\mathcal{A})\right) \\
& =\bigoplus_{u \in Q^{\prime}} \bigoplus_{q \in Q}\left(\mathrm{~h}_{\mathcal{A}^{\prime}}(\xi)_{u} \otimes u_{q} \otimes F_{q}\right) \quad \quad \text { (by left-distributivity) } \\
& =\bigoplus_{q \in Q} \bigoplus_{u \in Q^{\prime}}\left(\mathrm{h}_{\mathcal{A}^{\prime}}(\xi)_{u} \otimes u_{q} \otimes F_{q}\right) \quad \quad \quad \text { (by commutativity of } \oplus \text { ) } \\
& =\bigoplus_{q \in Q}\left[\left(\bigoplus_{u \in Q^{\prime}} \mathrm{h}_{\mathcal{A}^{\prime}}(\xi)_{u} \otimes u_{q}\right) \otimes F_{q}\right] \quad \quad \text { (by right-distributivity) } \\
& =\bigoplus_{q \in Q}\left[\left(\underset{u \in Q^{\prime}}{ } \mathrm{h}_{\mathcal{A}^{\prime}}(\xi)_{u} \cdot u\right)_{q} \otimes F_{q}\right] \\
& =\bigoplus_{q \in Q} \mathrm{~h}_{\mathcal{A}}(\xi)_{q} \otimes F_{q}  \tag{17.15}\\
& =\llbracket \mathcal{A} \rrbracket(\xi) .
\end{align*}
$$

Minimality of $\operatorname{det}_{(f, g)}(\mathcal{A})$ (proof of Theorem 17.3.2(3))
The following theorem corresponds to KM05, Thm. 3].

Theorem 17.3.19. BVM10, Thm. 5.6] Let B be commutative and let $(f, g)$ and $(\tilde{f}, \tilde{g})$ be factorizations such that $(f, g)$ is maximal. Moreover, let $\operatorname{det}_{(f, g)}(\mathcal{A})=\left(Q^{\prime}, \delta^{\prime}, F^{\prime}\right)$ and $\operatorname{det}_{(\tilde{f}, \tilde{g})}(\mathcal{A})=(\tilde{Q}, \tilde{\delta}, \tilde{F})$. Then $Q^{\prime}=f(\tilde{Q})$; hence $\left|Q^{\prime}\right| \leq|\tilde{Q}|$, and if $\operatorname{det}_{(\tilde{f}, \tilde{g})}(\mathcal{A})$ is a wta, then so is $\operatorname{det}_{(f, g)}(\mathcal{A})$.

Proof. We prove by case analysis on the cardinality of $Q$.
Case (a): Let $|Q|=1$. Then we can identify $B^{Q}$ with $B$. Since $(f, g)$ is maximal, we have that $f\left(\overline{B \backslash\{\mathbb{O}\})}=\{f(\mathbb{1})\}\right.$. Since $Q^{\prime} \neq \emptyset, \tilde{Q} \neq \emptyset$, and $\tilde{Q} \subseteq B \backslash\{\mathbb{O}\}$, we obtain that $Q^{\prime}=\{f(\mathbb{1})\}=f(\tilde{Q})$.

Case (b): Let $|Q|>1$. By Lemma 17.3.6, B is zero-divisor free. Note that ( $\star$ ) for every $u \in B^{Q} \backslash\left\{\mathbb{O}_{Q}\right\}$, we have $\tilde{g}(u) \cdot \tilde{f}(u)=u=g(u) \cdot f(u)$, and by applying $f$ we obtain $f(\tilde{f}(u))=f(u)=f(f(u))$ because $(f, g)$ is maximal.

We begin with the proof of $f(\tilde{Q}) \subseteq Q^{\prime}$. Using Lemma 17.3.8, it suffices to prove the following statement by induction on $\mathbb{N}$ :

$$
\text { for every } n \in \mathbb{N} \text {, we have } f\left(\tilde{Q}_{n}\right) \subseteq Q^{\prime}
$$

I.B.: Since $\tilde{Q}_{0}=\emptyset$, the statement holds trivially.
I.S.: Let $n \in \mathbb{N}$ and $\tilde{u} \in f\left(\tilde{Q}_{n+1}\right)$. If $\tilde{u} \in f\left(\tilde{Q}_{n}\right)$, then we are ready. Otherwise, there exist $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $\tilde{u}_{1}, \ldots, \tilde{u}_{k} \in \tilde{Q}_{n}$ such that $\tilde{u}=f\left(\tilde{f}\left(\delta_{\mathcal{A}}(\sigma)\left(\tilde{u}_{1}, \ldots, \tilde{u}_{k}\right)\right)\right)$. Hence

$$
\begin{array}{rlr}
\tilde{u} & =f\left(\tilde{f}\left(\delta_{\mathcal{A}}(\sigma)\left(\tilde{u}_{1}, \ldots, \tilde{u}_{k}\right)\right)\right) \\
& =f\left(\delta_{\mathcal{A}}(\sigma)\left(\tilde{u}_{1}, \ldots, \tilde{u}_{k}\right)\right) \\
& =f\left(\delta_{\mathcal{A}}(\sigma)\left(f\left(\tilde{u}_{1}\right), \ldots, f\left(\tilde{u}_{k}\right)\right)\right) \\
& \in Q^{\prime}
\end{array} \quad \text { (Lemma 17.3.7(2) and Lemma 17.3.7(1)) }
$$

Now we prove $Q^{\prime} \subseteq f(\tilde{Q})$. Using Lemma 17.3 .8 again, it suffices to prove the following statement by induction on $\mathbb{N}$ :

$$
\text { for every } n \in \mathbb{N} \text {, we have } Q_{n}^{\prime} \subseteq f(\tilde{Q})
$$

I.B.: Since $Q_{0}^{\prime}=\emptyset$, the statement holds trivially.
I.S.: Let $n \in \mathbb{N}$ and $u^{\prime} \in Q_{n+1}^{\prime}$. Then there are $k \in \mathbb{N}, \sigma \in \Sigma^{(k)}$, and $u_{1}^{\prime}, \ldots, u_{k}^{\prime} \in Q_{n}^{\prime}$ such that $u^{\prime}=f\left(\delta_{\mathcal{A}}(\sigma)\left(u_{1}^{\prime}, \ldots, u_{k}^{\prime}\right)\right)$. By I.H., there exist $\tilde{u}_{1}, \ldots, \tilde{u}_{k} \in \tilde{Q}$ such that $u_{i}^{\prime}=f\left(\tilde{u}_{i}\right)$ for every $i \in[k]$. Hence

$$
\begin{align*}
u^{\prime} & =f\left(\delta_{\mathcal{A}}(\sigma)\left(f\left(\tilde{u}_{1}\right), \ldots, f\left(\tilde{u}_{k}\right)\right)\right) \\
& =f\left(\delta_{\mathcal{A}}(\sigma)\left(\tilde{u}_{1}, \ldots, \tilde{u}_{k}\right)\right) \\
& =f\left(\tilde{f}\left(\delta_{\mathcal{A}}(\sigma)\left(\tilde{u}_{1}, \ldots, \tilde{u}_{k}\right)\right)\right) \\
& \in f(\tilde{Q})
\end{align*}
$$

$$
=f\left(\delta_{\mathcal{A}}(\sigma)\left(\tilde{u}_{1}, \ldots, \tilde{u}_{k}\right)\right) \quad(\text { Lemma 17.3.7(2) and Lemma 17.3.7(1) }
$$

### 17.3.5 Applying determinization by factorization to bu deterministic wta

In this subsection let $\mathcal{A}=(Q, \delta, F)$ be a bu deterministic ( $\Sigma, \mathrm{B})$-wta. Then of course there is no need to determinize $\mathcal{A}$. However, in order to show the robustness of determinization by factorization, we apply it to $\mathcal{A}$ and a maximal factorization $(f, g)$. We prove that the resulting triple $\operatorname{det}_{(f, g)}(\mathcal{A})=\left(Q^{\prime}, \delta^{\prime}, F^{\prime}\right)$ is indeed a bu deterministic wta (i.e., $Q^{\prime}$ is finite) by showing that $\left|Q^{\prime}\right| \leq|Q|$. This is due to the fact that $\mathrm{h}_{\mathcal{A}}\left(\mathrm{T}_{\Sigma}\right)$ consists of single-valued elements of $B^{Q}$ and that $f$ takes single-valued $Q$-vectors to $q$-unit vectors. We note that we do not require that B is extremal (which we do in Theorem 17.3.17).

Formally, for every $b \in B$ and $q \in Q$, we let $b_{q}$ be the element of $B^{Q}$ such that $\left(b_{q}\right)_{q}=b$ and $\left(b_{q}\right)_{p}=\mathbb{0}$ for each $p \in Q \backslash\{q\}$. Then we define the set of single-valued $Q$-vectors over $B$, denoted by $B_{\mathrm{sv}}^{Q}$, as the set

$$
B_{\mathrm{sv}}^{Q}=\left\{b_{q} \in B^{Q} \mid q \in Q, b \in B\right\}
$$

By Lemma 4.2.1(1), the following is obvious.
Observation 17.3.20. $\mathrm{h}_{\mathcal{A}}\left(\mathrm{T}_{\Sigma}\right) \subseteq B_{\mathrm{sv}}^{Q}$.
Then we can prove the following result.
Corollary 17.3.21. Büc14, Lm. 5.3.11] Let B be commutative and $\mathcal{A}=(Q, \delta, F)$ be a bu deterministic $(\Sigma, \mathrm{B})$-wta. Moreover, let $(f, g)$ be a maximal factorization and $\operatorname{det}_{(f, g)}(\mathcal{A})=\left(Q^{\prime}, \delta^{\prime}, F^{\prime}\right)$. Then we have $\left|Q^{\prime}\right| \leq|Q|$.

Proof. By Lemma 17.3 .9 and Observation 17.3 .20 , we have

$$
Q^{\prime} \subseteq f\left(\mathrm{~h}_{\mathcal{A}}\left(\mathrm{T}_{\Sigma}\right) \backslash\left\{\mathbb{O}_{Q}\right\}\right) \subseteq f\left(B_{\mathrm{sv}}^{Q} \backslash\left\{\mathbb{O}_{Q}\right\}\right)
$$

Since $(f, g)$ is maximal, for each $q \in Q$ and $b_{q} \in B_{\mathrm{sv}}^{Q}$, we have $f\left(b_{q}\right)=f\left(b \cdot \mathbb{1}_{q}\right)=f\left(\mathbb{1}_{q}\right)$. Hence $\left|f\left(B_{\mathrm{sv}}^{Q} \backslash\left\{\mathbb{O}_{Q}\right\}\right)\right| \leq\left|f\left(\left\{\mathbb{1}_{q} \mid q \in Q\right\}\right)\right| \leq|Q|$, which proves our lemma.

## Chapter 18

## Support of recognizable weighted tree languages

In this chapter we will consider the question whether the support of a recognizable ( $\Sigma, \mathrm{B}$ )-weighted tree language is a recognizable $\Sigma$-tree language. We will start with a negative result by showing a string ranked alphabet and a ( $\Sigma, \operatorname{lnt}$ )-wta $\mathcal{A}$ (where Int is the ring of integers) such that $\operatorname{supp}(\llbracket \mathcal{A} \rrbracket)$ is not recognizable. We will continue with positive results and show support theorems; a support theorem states conditions under which the support of a recognizable weighted tree language is a recognizable tree language. The support theorems concern the run semantics or the initial algebra semantics; moreover, they are based on additional requirements on the wta or on the strong bimonoid.

### 18.1 Negative result for support

Lemma 18.1.1. BR88, Ex. III. 3.1] For the $(\Sigma, \operatorname{lnt})-$ wta $\mathcal{A}$ in Example 3.2.14 we have supp $(\llbracket \mathcal{A} \rrbracket) \notin$ $\operatorname{Rec}(\Sigma)$.

Proof. We recall that $\Sigma=\left\{\sigma^{(1)}, \gamma^{(1)}, \alpha^{(0)}\right\}$ and that $\llbracket \mathcal{A} \rrbracket(\xi)=\left|\operatorname{pos}_{\gamma}(\xi)\right|-\left|\operatorname{pos}_{\sigma}(\xi)\right|$ for each $\xi \in \mathrm{T}_{\Sigma}$. Hence

$$
\operatorname{supp}(\llbracket \mathcal{A} \rrbracket)=\left\{\xi \in \mathrm{T}_{\Sigma}| | \operatorname{pos}_{\gamma}(\xi)\left|\neq\left|\operatorname{pos}_{\sigma}(\xi)\right|\right\}\right.
$$

By contradiction, we can easily prove that $\operatorname{supp}(\llbracket \mathcal{A} \rrbracket)$ is not recognizable. For this, we assume that $\operatorname{supp}(\llbracket \mathcal{A} \rrbracket) \in \operatorname{Rec}(\Sigma)$. The complement of $\operatorname{supp}(\llbracket \mathcal{A} \rrbracket))$ is the tree language

$$
L=\left\{\xi \in \mathrm{T}_{\Sigma}| | \operatorname{pos}_{\gamma}(\xi)\left|=\left|\operatorname{pos}_{\sigma}(\xi)\right|\right\} .\right.
$$

Since, by Theorem 2.13.3 recognizable $\Sigma$-tree languages are closed under complement, $L$ is also recognizable. However, this contradicts to Example 6.1 .7 in which we proved that $L$ is not a recognizable $\Sigma$-tree language. Hence $\operatorname{supp}(\llbracket \mathcal{A} \rrbracket)$ is not recognizable.

The following result immediately follows from Lemma 18.1.1
Theorem 18.1.2. There exists a string ranked alphabet $\Sigma$ such that $\operatorname{supp}(\operatorname{Rec}(\Sigma, \operatorname{lnt})) \backslash \operatorname{Rec}(\Sigma) \neq \emptyset$.

### 18.2 Positive results for support

In this section we elaborate some support theorems. By a support theorem we mean a result which, for a certain set of recognizable weighted tree languages, guarantees that the support of each element of
that set is a recognizable tree language. We start with a support theorem which can be derived from the preimage theorems. Then we consider (arbitrary) wta and classify the support theorem according to two criteria: the type of semantics of the wta (initial algebra semantics or run semantics) and the set of strong bimonoids. Finally, we compare some of the sets of strong bimonoids for which we have a support theorem.

### 18.2.1 Consequences of preimage theorems

In Chapter 16 we have proved some preimage theorems. Roughly speaking, each of them states conditions such that the following implication holds: if a recognizable weighted tree language $r: \mathrm{T}_{\Sigma} \rightarrow B$ satisfies the conditions, then for each $b \in B$, the $\Sigma$-tree language $r^{-1}(b)$ is recognizable. In a straightforward way, each of these preimage theorems implies a support theorem. In the next corollary we merely collect these support theorems (without dealing with their relationship).

Corollary 18.2.1. Let $\Sigma$ be a ranked alphabet, B be a strong bimonoid, and $r: \mathrm{T}_{\Sigma} \rightarrow B$. If one of the following conditions is satisfied:
(1) $r$ is a recognizable step mapping,
(2) $r \in \operatorname{Rec}^{\text {init }}(\Sigma, \mathrm{B})$ and (a) B is locally finite or (b) B is weakly locally finite and $\Sigma$ is monadic,
(3) $r \in \operatorname{Rec}^{\text {run }}(\Sigma, \mathrm{B})$ and B is bi-locally finite,
(4) $r \in \operatorname{Rec}^{\mathrm{run}}(\Sigma, \mathrm{B})$ and $(\mathrm{B}, \preceq)$ is a past-finite monotonic strong bimonoid,
then $\operatorname{supp}(r) \in \operatorname{Rec}(\Sigma)$. Moreover, if $r$ is given effectively, then in each case we can construct $a \Sigma$-fta $A$ such that $\operatorname{supp}(r)=\mathrm{L}(A)$.

Proof. Since $\operatorname{supp}(r)=\mathrm{T}_{\Sigma} \backslash r^{-1}(\mathbb{O})$ and the set of $\Sigma$-recognizable tree languages is closed under complement (cf. Theorem 2.13.3), the following equivalence holds:

$$
\begin{equation*}
r^{-1}(\mathbb{O}) \text { is recognizable if and only if } \operatorname{supp}(r) \text { is recognizable. } \tag{18.1}
\end{equation*}
$$

Then, each of the four implications is justified by equivalence (18.1) and the corresponding preimage theorem: (1) Theorem $10.3 .1(B) \Rightarrow(C)$, (2) Corollary 16.1.4 (3) Corollary 16.2.8 and (4) Theorem 16.2.10(1).

Now let $r$ be given effectively. Then, in each of the Cases (1)-(3), by the corresponding preimage theorem and Theorem 2.13.3 we can construct a $\Sigma$ - $\mathrm{fta} A$ such that $\operatorname{supp}(r)=\mathrm{L}(A)$. In Case $(4)$ we use Theorem 16.2.10 (2) and Theorem 2.13.3 to construct the $\Sigma$-fta $A$; note that past(0) $=\{\mathbb{D}\}$.

### 18.2.2 Both semantics and positive strong bimonoids

Corollary 3.4 .2 is a support theorem for the Boolean semiring. If we analyse the proof of the underlying Theorem 3.4.1, then we realize that we have used the fact that the semiring Boole is positive, i.e., zerosum free and zero-divisor free. In this subsection we generalize Corollary 3.4.2 from Boole to an arbitrary positive strong bimonoid.

We define the particular mapping sgn : $B \rightarrow \mathbb{B}$ for each $b \in B$ by

$$
\operatorname{sgn}(b)= \begin{cases}1 & \text { if } b \neq \mathbb{0} \\ 0 & \text { otherwise }\end{cases}
$$

Lemma 18.2.2. Let B be positive. Then $\operatorname{sgn}: B \rightarrow \mathbb{B}$ is a strong bimonoid homomorphism from B to Boole.

Proof. Let $b_{1}, b_{2} \in B$. Then

$$
\begin{aligned}
\operatorname{sgn}\left(b_{1} \oplus b_{2}\right)=1 & \text { iff } b_{1} \oplus b_{2} \neq \mathbb{O} \text { iff* } b_{1} \neq \mathbb{O} \vee b_{2} \neq \mathbb{0} \\
& \text { iff }\left(\operatorname{sgn}\left(b_{1}\right)=1\right) \vee\left(\operatorname{sgn}\left(b_{2}\right)=1\right) \text { iff }\left(\operatorname{sgn}\left(b_{1}\right) \vee \operatorname{sgn}\left(b_{2}\right)\right)=1
\end{aligned}
$$

where at equivalence iff* from right to left we have used the fact that $\mathbf{B}$ is zero-sum free. Thus $\operatorname{sgn}\left(b_{1} \oplus\right.$ $\left.b_{2}\right)=\operatorname{sgn}\left(b_{1}\right) \vee \operatorname{sgn}\left(b_{2}\right)$. Also

$$
\begin{aligned}
\operatorname{sgn}\left(b_{1} \otimes b_{2}\right)=1 & \text { iff } b_{1} \otimes b_{2} \neq \mathbb{O} \text { iff }^{*} b_{1} \neq \mathbb{O} \wedge b_{2} \neq \mathbb{0} \\
& \text { iff }\left(\operatorname{sgn}\left(b_{1}\right)=1\right) \wedge\left(\operatorname{sgn}\left(b_{2}\right)=1\right) \text { iff }\left(\operatorname{sgn}\left(b_{1}\right) \wedge \operatorname{sgn}\left(b_{2}\right)\right)=1
\end{aligned}
$$

where at equivalence iff* from right to left we have used the fact that $B$ is zero-divisor free. Hence $\operatorname{sgn}\left(b_{1} \otimes b_{2}\right)=\operatorname{sgn}\left(b_{1}\right) \wedge \operatorname{sgn}\left(b_{2}\right)$. Moreover, $\operatorname{sgn}(\mathbb{O})=0$ and $\operatorname{sgn}(\mathbb{1})=1$. Thus sgn is a strong bimonoid homomorphism.

Our next support theorem can be compared to FV09, Thm. 3.12]. In Lemma 18.2.3 we prove that the support of the run semantics of a wta is the same as the support of its initial algebra semantics if the strong bimonoid is positive. Moreover, an fta can be constructed which recognizes the support. Since each monotonic strong bimonoid is positive (cf. Subsection 16.2.3), it contains Corollary 18.2.1(4). In GV22, Thm. 4.1] a direct proof of the Lemma 18.2.3(1) is given.

Lemma 18.2.3. Let B be positive and $\mathcal{A}$ be a $(\Sigma, \mathrm{B})$-wta. Then the following two statements hold.
(1) $\operatorname{supp}\left(\llbracket \mathcal{A} \rrbracket^{\text {init }}\right)=\operatorname{supp}\left(\llbracket \mathcal{A} \rrbracket^{\text {run }}\right)$.
(2) We can construct a $\Sigma$-fta $A$ such that $\mathrm{L}(A)=\operatorname{supp}\left(\llbracket \mathcal{A} \rrbracket^{\text {init }}\right)=\operatorname{supp}\left(\llbracket \mathcal{A} \rrbracket^{\text {run }}\right)$.

Hence both $\operatorname{supp}\left(\llbracket \mathcal{A} \rrbracket^{\text {run }}\right)$ and $\operatorname{supp}\left(\llbracket \mathcal{A} \rrbracket^{\text {init }}\right)$ are recognizable $\Sigma$-tree languages.
Proof. Proof of (1): We consider the strong bimonoid homomorphism sgn : B $\rightarrow \mathbb{B}$ (cf. Lemma 18.2.2). Then we have

$$
\begin{aligned}
\operatorname{supp}\left(\llbracket \mathcal{A} \rrbracket^{\text {init }}\right) & =\operatorname{supp}\left(\operatorname{sgn} \circ \llbracket \mathcal{A} \rrbracket^{\text {init }}\right) \\
& =\operatorname{supp}\left(\llbracket \operatorname{sgn}(\mathcal{A}) \rrbracket^{\text {init }}\right) \\
& =\operatorname{supp}\left(\llbracket \operatorname{sgn}(\mathcal{A}) \rrbracket^{\text {run }}\right) \\
& =\operatorname{supp}\left(\operatorname{sgn} \circ \llbracket \mathcal{A} \rrbracket^{\mathrm{run}}\right) \\
& =\operatorname{supp}\left(\llbracket \mathcal{A} \rrbracket^{\text {run }}\right) .
\end{aligned}
$$

$$
=\operatorname{supp}\left(\llbracket \operatorname{sgn}(\mathcal{A}) \rrbracket^{\text {init }}\right) \quad(\text { by Lemma 10.9.2 })
$$

$$
=\operatorname{supp}\left(\llbracket \operatorname{sgn}(\mathcal{A}) \rrbracket^{\text {run }}\right) \quad(\text { by Theorem 3.4.1 })
$$

$$
=\operatorname{supp}\left(\operatorname{sgn} \circ \llbracket \mathcal{A} \rrbracket^{\text {run }}\right) \quad \text { (by Lemma 10.9.2) }
$$

Proof of $(2)$ : By the above, we have $\operatorname{supp}\left(\llbracket \mathcal{A} \rrbracket^{\text {init }}\right)=\operatorname{supp}\left(\llbracket \operatorname{sgn}(\mathcal{A}) \rrbracket^{\text {init }}\right)$. Now we construct the $(\Sigma$, Boole)-wta $\operatorname{sgn}(\mathcal{A})(\mathrm{cf}$. Observation 10.9.1). Then we apply Corollary 3.4.2 (B) $\Rightarrow(\mathrm{A})$ to $\operatorname{sgn}(\mathcal{A})$ and construct a $\Sigma$-fta $A$ such that $\mathrm{L}(A)=\operatorname{supp}\left(\llbracket \operatorname{sgn}(\mathcal{A}) \rrbracket^{\text {init }}\right)$. By Statement (1), we also have $\mathrm{L}(A)=$ $\operatorname{supp}\left(\llbracket \operatorname{sgn}(\mathcal{A}) \rrbracket^{\text {run }}\right)$.

It is clear that Lemma 18.2 .3 generalizes also Corollary 3.4.2 because Boole is positive. However, we have used Corollary 3.4 .2 for the proof of Lemma 18.2 .3 .

We note that, in spite of Lemma 18.2.3(1), there exists a positive strong bimonoid B and ( $\Sigma, \mathrm{B}$ )-wta $\mathcal{A}$ such that $\llbracket \mathcal{A} \rrbracket^{\text {init }} \neq \llbracket \mathcal{A} \rrbracket^{\text {run }}$; witnesses of this fact are shown in Examples 5.2.1 and 5.2.2, as well as in the proof of Theorem 5.2.5.

The following theorem is a generalization of [FV09, Thm. 3.12] from positive semirings to positive strong bimonoids.

Theorem 18.2.4. Let $\Sigma$ be a ranked alphabet and B be positive. Then

$$
\operatorname{Rec}(\Sigma)=\operatorname{supp}\left(\operatorname{Rec}^{\mathrm{init}}(\Sigma, \mathrm{~B})\right)=\operatorname{supp}\left(\operatorname{Rec}^{\mathrm{run}}(\Sigma, \mathrm{~B})\right)
$$

Proof. The inclusion $\operatorname{Rec}(\Sigma) \subseteq \operatorname{supp}\left(\operatorname{Rec}^{\text {init }}(\Sigma, \mathrm{B})\right)$ follows from $(\mathrm{B}) \Rightarrow(\mathrm{C})$ of Theorem 4.3.6 (without using that B is positive). The inclusion $\operatorname{supp}\left(\operatorname{Rec}^{\mathrm{init}}(\Sigma, \mathrm{B})\right) \subseteq \operatorname{Rec}(\Sigma)$ follows from Lemma 18.2.3(2). Lastly, the second equality follows from Lemma 18.2.3(1).

At this point we can prove that Int is not a Fatou extension of Nat (cf. Section 3.7).
Lemma 18.2.5. KS86, Ex. 8.1 on p. 129] The ring Int is not a Fatou extension of the semiring Nat.
Proof. Let $r=\llbracket \mathcal{A} \rrbracket$, where $\mathcal{A}$ is the $\left(\Sigma\right.$, Int)-wta in Example 3.2.14 By Theorem 10.4.1, $r^{\prime}=r \otimes r$ is also ( $\Sigma$, Int)-recognizable. Moreover, $\operatorname{im}\left(r^{\prime}\right) \subseteq \mathbb{N}$, hence $r^{\prime} \in \operatorname{Rec}(\Sigma, \operatorname{lnt}) \cap \mathbb{N}^{T_{\Sigma}}$. Next we observe that $\operatorname{supp}\left(r^{\prime}\right)=\operatorname{supp}(r)$ and hence, by Lemma 18.1.1, the $\Sigma$-tree language $\operatorname{supp}\left(r^{\prime}\right)$ is not recognizable. Since the semiring Nat is positive, by Theorem 18.2.4, we obtain that $r^{\prime} \notin \operatorname{Rec}(\Sigma$, Nat $)$.

### 18.2.3 Both semantics and commutative semirings which are not rings

Here we show a support theorem which deals with commutative semirings which are not rings. It is based on the following result.

Theorem 18.2.6. Wan97, Thm. 2.1] (also cf. DK21, Lm. 9.3]) Let B be a commutative semiring which is not a ring. Then there exists a semiring homomorphism from B onto the Boolean semiring.

Theorem 18.2.7. Let $\Sigma$ be a ranked alphabet and B be a commutative semiring which is not a ring. Moreover, let $L \subseteq \mathrm{~T}_{\Sigma}$ be a tree language. If $\chi_{\mathrm{B}}(L) \in \operatorname{Rec}(\Sigma, \mathrm{B})$, then $L$ is in $\operatorname{Rec}(\Sigma)$.

Proof. Let $\chi_{\mathrm{B}}(L) \in \operatorname{Rec}(\Sigma, \mathrm{B})$. By Theorem 18.2.6, there exists a semiring homomorphism $h: B \rightarrow \mathbb{B}$. Clearly, $h\left(\chi_{\mathrm{B}}(L)\right)=\chi_{\text {Boole }}(L)$. Then, by Theorem 10.9.3, we have $\chi_{\text {Boole }}(L) \in \operatorname{Rec}(\Sigma$, Boole $)$. Thus, since $L=\operatorname{supp}\left(\chi_{\text {Boole }}(L)\right)$, we have $L \in \operatorname{supp}(\operatorname{Rec}(\Sigma$, Boole $))$. By Corollary 3.4.2 we obtain that $L \in$ $\operatorname{Rec}(\Sigma)$.

### 18.2.4 Run semantics and commutative, zero-sum free strong bimonoids

In Kir11, Thm. 3.1] (also cf. Kir09, Thm. 1]) it was proved that the support of every recognizable $(\Gamma, B)$-weighted string language is a recognizable string language provided that the weight algebra $B$ is a commutative zero-sum free semiring. This generalizes Wan98, Cor. 5.2] where this result was proved for commutative quasi-positive semirings, because each commutative quasi-positive semiring is a commutative zero-sum free semiring by definition. Here we generalize [Kir11, Thm. 3.1] to the tree case by adapting the proof technique from Kir11] (also cf. [DH15, Göt17, Thm. 4.6], and [FHV18, Thm. 4.3(1)]). Distributivity is not needed.

## In this subsection we assume that B is commutative.

Let us consider a $(\Sigma, \mathrm{B})$-wta $\mathcal{A}$, a tree $\xi \in \mathrm{T}_{\Sigma}$, and a run $\rho \in \mathrm{R}_{\mathcal{A}}(\xi)$. By definition, $\mathrm{wt}_{\mathcal{A}}(\xi, \rho)$ is the product of some occurrences of elements of $\operatorname{im}(\delta)$ (cf. Observation 3.1.1). Since $\operatorname{im}(\delta)$ is finite, we can sort its elements in a vector of $n$ components, where $n=|\operatorname{im}(\delta)|$. Then, since $\otimes$ is commutative, we can represent $\mathrm{wt}_{\mathcal{A}}(\xi, \rho)$ as a vector $\bar{z} \in \mathbb{N}^{n}$ of the multiplicities of these occurrences. Next we will formalize this representation.

Let $\vec{b}=\left(b_{1}, \ldots, b_{n}\right)$ be an arbitrary, but fixed enumeration of $\operatorname{im}(\delta)$. Then $b_{i} \neq b_{j}$ for $i \neq j$. Let $\left(z_{1}, \ldots, z_{n}\right),\left(z_{1}^{\prime}, \ldots, z_{n}^{\prime}\right) \in \mathbb{N}^{n}$. Then we define

$$
\left(z_{1}, \ldots, z_{n}\right)+\left(z_{1}^{\prime}, \ldots, z_{n}^{\prime}\right)=\left(z_{1}+z_{1}^{\prime}, \ldots, z_{n}+z_{n}^{\prime}\right)
$$

It is clear that the binary operation + is associative. By identifying each $b_{j} \in \operatorname{im}(\delta)$ with $(\underbrace{0, \ldots, \mathbb{0}}_{j-1}, b_{j}, \underbrace{\mathbb{0}, \ldots, \mathbb{0}}_{n-j})$, we have

$$
\left(z_{1}, \ldots, z_{n}\right)+b_{j}=\left(z_{1}, \ldots, z_{j-1}, z_{j}+1, z_{j+1}, \ldots, z_{n}\right)
$$

For the inductive definition of the representation, we use the well-founded set (TR, $\prec)$ defined in Section 3.1 on p. 63 For the sake of convenience we recall that $\mathrm{TR}=\left\{(\xi, \rho) \mid \xi \in \mathrm{T}_{\Sigma}, \rho \in \mathrm{R}_{\mathcal{A}}(\xi)\right\}$ and that the well-founded relation $\prec$ on TR is the binary relation

$$
\prec=\left\{\left(\left(\left.\xi\right|_{i},\left.\rho\right|_{i}\right),(\xi, \rho)\right) \mid(\xi, \rho) \in \mathrm{TR}, i \in[\operatorname{rk}(\xi(\varepsilon))]\right\} .
$$

Clearly $\min _{\prec}(\mathrm{TR})=\left\{(\alpha, \rho) \mid \alpha \in \Sigma^{(0)}, \rho:\{\varepsilon\} \rightarrow Q\right\}$. Then we define the mapping $\overline{(.)}: \mathrm{TR} \rightarrow \mathbb{N}^{n}$ by induction on (TR, $\prec)$ for every $\xi \in \mathrm{T}_{\Sigma}$ and $\rho \in \mathrm{R}_{\mathcal{A}}(\xi)$ by

$$
\overline{(\xi, \rho)}=(0, \ldots, 0)+\overline{\left(\left.\xi\right|_{1},\left.\rho\right|_{1}\right)}+\ldots+\overline{\left(\left.\xi\right|_{k},\left.\rho\right|_{k}\right)}+\delta_{k}(\rho(1) \cdots \rho(k), \xi(\varepsilon), \rho(\varepsilon))
$$

where $k=\operatorname{rk}(\xi(\varepsilon))$. We have added the vector $(0, \ldots, 0)$ in order to have $\overline{(\xi, \rho)}$ well defined for the case $k=0$.

Moreover, each vector $\vec{z}=\left(z_{1}, \ldots, z_{n}\right) \in \mathbb{N}^{n}$ represents an element of $B$. Formally, we define the mapping $\llbracket . \rrbracket: \mathbb{N}^{n} \rightarrow B$ for each $\vec{z}=\left(z_{1}, \ldots, z_{n}\right) \in \mathbb{N}^{n}$ by

$$
\llbracket \vec{z} \rrbracket=b_{1}^{z_{1}} \otimes \ldots \otimes b_{n}^{z_{n}}
$$

Since B is commutative,

$$
\begin{equation*}
\text { for every } \xi \in \mathrm{T}_{\Sigma} \text { and } \rho \in \mathrm{R}_{\mathcal{A}}(\xi), \text { we have } \mathrm{wt}_{\mathcal{A}}(\xi, \rho)=\llbracket \overline{(\xi, \rho)} \rrbracket \tag{18.2}
\end{equation*}
$$

In other words, $\mathrm{wt}_{\mathcal{A}}=\llbracket . \rrbracket \circ \overline{(.)}$. In this sense, the vector $\overline{(\xi, \rho)}$ represents $\mathrm{wt}_{\mathcal{A}}(\xi, \rho)$.

Next we will analyse the set $\llbracket . \rrbracket^{-1}(\mathbb{O})$ of vectors $\vec{z} \in \mathbb{N}^{n}$ which are mapped to $\mathbb{O}$. (We note that $(0, \ldots, 0) \notin \llbracket \cdot \rrbracket^{-1}(\mathbb{O})$, because $b_{1}^{0} \otimes \ldots \otimes b_{n}^{0}=\mathbb{1}$.)

For this, we consider the partially ordered set $\left(\mathbb{N}^{n}, \leq\right)$, where the binary relation $\leq$ is defined as follows: for all vectors $\vec{z}=\left(z_{1}, \ldots, z_{n}\right) \in \mathbb{N}^{n}$ and $\vec{y}=\left(y_{1}, \ldots, y_{n}\right) \in \mathbb{N}^{n}$, we define $\vec{z} \leq \vec{y}$ if $z_{i} \leq y_{i}$ for each $i \in\left[n \rrbracket\right.$. We note that, for every $\vec{z}, \vec{y} \in \mathbb{N}^{n}$, if $\llbracket \vec{z} \rrbracket=\mathbb{O}$ and $\vec{z} \leq \vec{y}$, then $\llbracket \vec{y} \rrbracket=\mathbb{O}$ (because $\mathbb{O}$ is annihilating). Let $M \subseteq \mathbb{N}^{n}$. An element $\vec{z} \in M$ is minimal in $M$ if, for each $\vec{y} \in M$, the assumption $\vec{y} \leq \vec{z}$ implies $\vec{y}=\vec{z}$. We denote by $\min (M)$ the set of all minimal elements in $M$. Thus, in particular, $\min (\emptyset)=\emptyset$. The following result is called Dickson's Lemma.

Lemma 18.2.8. Dic13] (cf. also Kir11, Lm. 2.1] and KR08]) For each $M \subseteq \mathbb{N}^{n}$, the set min $(M)$ is finite.

By Lemma 18.2 .8 the set $\min \left(\llbracket . \rrbracket^{-1}(\mathbb{O})\right)$ is finite. Hence there exists a smallest number $m \in \mathbb{N}$ satisfying $\min \left(\llbracket \cdot \rrbracket^{-1}(\mathbb{O})\right) \subseteq\{0, \ldots, m\}^{n}$, and we call this number $m$ the degree of $\vec{b}$ and denote it by $\operatorname{dg}(\vec{b})$. (We note that if B is zero-divisor free and $\mathbb{O} \notin \operatorname{im}(\delta)$, then $\llbracket \cdot \rrbracket^{-1}(\mathbb{O})=\emptyset$, hence $\operatorname{dg}(\vec{b})=0$. We also note that, for any other enumeration $\overrightarrow{b^{\prime}}$ of $\operatorname{im}(\delta)$, we have $\operatorname{dg}(\vec{b})=\operatorname{dg}\left(\overrightarrow{b^{\prime}}\right)$.)

The next lemma states that, if $\llbracket \vec{z} \rrbracket=\mathbb{C}$ for some $\vec{z}$, then also the evaluation of $\vec{z}^{\prime}$ results in $\mathbb{O}$ where $\vec{z}^{\prime}$ is obtained from $\vec{z}$ by restricting the components to $\operatorname{dg}(\vec{b})$. Formally, for every $\vec{z} \in \mathbb{N}^{n}$, we define the cut of $\vec{z}$, denoted by $\lfloor\vec{z}\rfloor_{\operatorname{dg}(\vec{b})}$, to be the vector $\lfloor\vec{z}\rfloor_{\operatorname{dg}(\vec{b})} \in \mathbb{N}^{n}$ with

$$
\left(\lfloor\vec{z}\rfloor_{\operatorname{dg}(\vec{b})}\right)_{i}=\min \left\{z_{i}, \operatorname{dg}(\vec{b})\right\}
$$

as $i$-th component for each $i \in[n]$.


Figure 18.1: The Hasse diagram of tuples $\mathbb{N}^{3}$ with $\min \left(\llbracket \cdot \rrbracket^{-1}(\mathbb{O})\right)=\left\{\overrightarrow{z_{1}}, \overrightarrow{z_{2}}, \overrightarrow{z_{3}}\right\}$.

Lemma 18.2.9. ([Kir11 Lm. 4.1]) For every $\vec{z} \in \mathbb{N}^{n}$, we have $\llbracket \vec{z} \rrbracket=\mathbb{O}$ iff $\llbracket\lfloor\vec{z}\rfloor_{\operatorname{dg}(\vec{b})} \rrbracket=\mathbb{O}$.

Proof. Let $\llbracket\lfloor\vec{z}\rfloor_{\operatorname{dg}(\vec{b})} \rrbracket=\mathbb{O}$. Since $\lfloor\vec{z}\rfloor_{\mathrm{dg}(\vec{b})} \leq \vec{z}$ and $\mathbb{O}$ is an annihilator for $\otimes$, we have $\llbracket \vec{z} \rrbracket=\mathbb{0}$.
Now let $\llbracket \vec{z} \rrbracket=\mathbb{0}$. Then there exists a $\vec{y} \in \min \left(\llbracket \cdot \rrbracket^{-1}(\mathbb{O})\right)$ such that $\vec{y} \leq \vec{z}$. We prove that $\bar{y} \leq\lfloor\vec{z}\rfloor_{\operatorname{dg}(\vec{b})}$. For this let $\bar{y}=\left(y_{1}, \ldots, y_{n}\right), \vec{z}=\left(z_{1}, \ldots, z_{n}\right)$, and $i \in[n]$. If $z_{i} \leq \operatorname{dg}(\vec{b})$, then $y_{i} \leq z_{i}=\left(\lfloor\vec{z}\rfloor_{\operatorname{dg}(\vec{b})}\right)_{i}$. If $z_{i}>\operatorname{dg}(\vec{b})$, then $y_{i} \leq \operatorname{dg}(\vec{b})=\left(\lfloor\vec{z}\rfloor_{\operatorname{dg}(\vec{b})}\right)_{i}$. Hence $\vec{y} \leq\lfloor\vec{z}\rfloor_{\operatorname{dg}(\vec{b})}$ and thus $\llbracket\lfloor\vec{z}\rfloor_{\operatorname{dg}(\vec{b})} \rrbracket=\mathbb{0}$.

In Figure 18.1 we indicate the partial order on $\mathbb{N}^{3}$. We assume that $\min \left(\llbracket . \rrbracket^{-1}(\mathbb{0})\right)=\left\{\overrightarrow{z_{1}}, \overrightarrow{z_{2}}, \overrightarrow{z_{3}}\right\}$. For each $i \in\{1,2,3\}$ we indicate the set $\left\{\vec{y} \mid \overrightarrow{z_{i}} \leq \vec{y}\right\}$ by a shaded filter. Clearly, for each $\vec{y}$ in one of these filters we have $\llbracket \vec{y} \rrbracket=\mathbb{0}$. We also indicate the set $\{0, \ldots, j\}^{n}$ for $j=0, j=1$, and $j=\operatorname{dg}(\vec{b})$.

Now we can prove the main support theorem of this subsection. We follow the proof of the corresponding result [Kir11, Thm. 3.1] for weighted string automata over semirings. This result has been generalized to (a) weighted unranked tree automaton over zero-sum free, commutative strong bimonoids DH15 and (b) weighted unranked tree automata over zero-sum free, commutative, zero-preserving tvmonoids Göt17, Thm. 4.7]. We also refer to [FHV18, Thm. 4.4] where the support theorem was proved for weighted tree automata over $\sigma$-complete and commutative strong bimonoids $\sqrt{1}$.

[^18]Theorem 18.2.10. Let $\Sigma$ be a ranked alphabet, B be a commutative strong bimonoid, and $\mathcal{A}$ be a $(\Sigma, \mathrm{B})-$ wta. If (a) B is zero-sum free or (b) $\mathcal{A}$ is bu deterministic and root weight normalized, then $\operatorname{supp}\left(\llbracket \mathcal{A} \rrbracket^{\text {run }}\right) \in$ $\operatorname{Rec}(\Sigma)$.

Proof. Let $\mathcal{A}=(Q, \delta, F)$ be a $(\Sigma, \mathrm{B})$-wta. If $\mathcal{A}$ is not root weight normalized, then by Theorem [7.3.1 we can construct a run equivalent and root weight normalized ( $\Sigma, \mathrm{B}$ )-wta. Hence, in both cases (a) and (b) we may assume that $\mathcal{A}$ is root weight normalized. Thus, there exists $q_{f} \in Q$ such that $F_{q_{f}}=\mathbb{1}$ and $F_{q}=0$ for each $q \in Q \backslash\left\{q_{f}\right\}$. Let $\vec{b}=\left(b_{1}, \ldots, b_{n}\right) \in B^{n}$ be an arbitrary but fixed enumeration of $\operatorname{im}(\delta)$.

Then we define the $\Sigma$ - $\mathrm{fta} A=\left(Q^{\prime}, \delta^{\prime}, F^{\prime}\right)$ which counts, up to the threshold $\operatorname{dg}(\vec{b})$, the number of occurrences of the $b_{j}$ 's in the runs of $\mathcal{A}$. Formally, we let $T=\{0, \ldots, \operatorname{dg}(\vec{b})\}$ and we define

- $Q^{\prime}=Q \times T^{n}$,
- $F^{\prime}=\left\{\left(q_{f}, \vec{z}\right) \in Q^{\prime} \mid \llbracket \vec{z} \rrbracket \neq \mathbb{0}\right\}$,
- we define for each $k \in \mathbb{N}$

$$
\begin{aligned}
\delta_{k}^{\prime}=\left\{\left(\left(q_{1}, \vec{z}_{1}\right) \cdots\left(q_{k}, \vec{z}_{k}\right), \sigma,(q, \vec{z})\right) \mid\right. & \sigma \in \Sigma^{(k)},\left(\left(q_{1}, \vec{z}_{1}\right) \cdots\left(q_{k}, \vec{z}_{k}\right)\right) \in\left(Q^{\prime}\right)^{k},(q, \vec{z}) \in Q^{\prime}, \\
& \left.\vec{z}=\left\lfloor(0, \ldots, 0)+\vec{z}_{1}+\ldots+\vec{z}_{k}+\delta_{k}\left(q_{1} \ldots q_{k}, \sigma, q\right)\right\rfloor_{\mathrm{dg}(\vec{b})}\right\} .
\end{aligned}
$$

Now we prove that $\operatorname{supp}\left(\llbracket \mathcal{A} \rrbracket^{\text {run }}\right)=\mathrm{L}(A)$. First, for each $\xi \in \mathrm{T}_{\Sigma}$, we define the mapping

$$
\varphi: \mathrm{R}_{\mathcal{A}}(\xi) \rightarrow \mathrm{R}_{A}^{\mathrm{v}}(\xi)
$$

as follows. For every $\rho \in \mathrm{R}_{\mathcal{A}}(\xi)$, we define the family $(\varphi(\rho)(w) \mid w \in \operatorname{pos}(\xi))$ of states $\varphi(\rho)(w) \in Q^{\prime}$.
For this definition, we recall the binary relation $\prec$ on $\operatorname{pos}(\xi)$ defined by $w_{1} \prec w_{2}$ if there exists $i \in \mathbb{N}$ such that $w_{1}=w_{2} i$, for every $w_{1}, w_{2} \in \operatorname{pos}(\xi)$ (cf. the proof of Lemma 10.8.1). Then $\prec$ is well-founded and $\min _{\prec}(\operatorname{pos}(\xi))=\operatorname{pos}_{\Sigma^{(0)}}(\xi)$, i.e., it is the set of leaves of $\xi$.

Now we define $\varphi(\rho)$ by induction on $(\operatorname{pos}(\xi), \prec)$ as follows. Let $w \in \operatorname{pos}(\xi)$ with $\xi(w)=\sigma$ for some $\sigma \in \Sigma^{(k)}$ with $k \in \mathbb{N}$. We assume that $\varphi(\rho)(w i)=\left(q_{i}, \overrightarrow{z_{i}}\right)$ for each $i \in[k]$. Then we define

$$
\varphi(\rho)(w)=\left(\rho(w),\left\lfloor(0, \ldots, 0)+\vec{z}_{1}+\ldots+\vec{z}_{k}+\delta_{k}\left(q_{1} \ldots q_{k}, \sigma, \rho(w)\right)\right\rfloor_{\operatorname{dg}(\vec{b})}\right) .
$$

It is clear that $\varphi$ is bijective. Moreover, by induction on $\mathrm{T}_{\Sigma}$, we can prove the following:

$$
\begin{equation*}
\text { for every } \xi \in \mathrm{T}_{\Sigma},(q, \vec{z}) \in Q^{\prime}, \text { and } \rho^{\prime} \in \mathrm{R}_{A}^{\mathrm{v}}((q, \vec{z}), \xi): \quad \vec{z}=\left[\overline{\left(\xi, \varphi^{-1}\left(\rho^{\prime}\right)\right)}\right\rfloor_{\mathrm{dg}(\vec{b})} \tag{18.3}
\end{equation*}
$$

Let $\xi=\sigma\left(\xi_{1}, \ldots, \xi_{k}\right),(q, \vec{z}) \in Q^{\prime}$, and $\rho^{\prime} \in \mathrm{R}_{A}^{\mathrm{v}}((q, \vec{z}), \xi)$. Then, for each $i \in[k]$, there exist $\left(q_{i}, \overrightarrow{z_{i}}\right) \in Q^{\prime}$ such that $\left.\rho^{\prime}\right|_{i} \in \mathrm{R}_{A}^{\mathrm{v}}\left(\left(q_{i}, \overrightarrow{z_{i}}\right), \xi_{i}\right)$ and

$$
\begin{equation*}
\vec{z}=\left\lfloor(0, \ldots, 0)+\overrightarrow{z_{1}}+\ldots+\overrightarrow{z_{k}}+\delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right)\right\rfloor_{\operatorname{dg}(\vec{b})} . \tag{18.4}
\end{equation*}
$$

Then we can calculate as follows (where we abbreviate $\lfloor\cdot\rfloor_{\mathrm{dg}(\vec{b})}$ by $\lfloor\cdot\rfloor$ ):

$$
\begin{array}{rlr} 
& \left\lfloor\overline{\left(\sigma\left(\xi_{1}, \ldots, \xi_{k}\right), \varphi^{-1}\left(\rho^{\prime}\right)\right)}\right\rfloor \\
= & \left\lfloor(0, \ldots, 0)+\overline{\left(\xi_{1}, \varphi^{-1}\left(\left.\rho^{\prime}\right|_{1}\right)\right)}+\ldots+\overline{\left(\xi_{k}, \varphi^{-1}\left(\rho^{\prime} \mid{ }_{k}\right)\right.}+\delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right)\right\rfloor \\
= & \left\lfloor(0, \ldots, 0)+\left\lfloor\overline{\left(\xi_{1}, \varphi^{-1}\left(\rho^{\prime}| |^{\prime}\right)\right.}\right\rfloor+\ldots+\left\lfloor\overline{\left(\xi_{k}, \varphi^{-1}\left(\rho^{\prime} \mid{ }_{k}\right)\right.}\right\rfloor+\delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right)\right\rfloor \\
& \left.\quad \text { (because }\left\lfloor\overrightarrow{y_{1}}+\overrightarrow{y_{2}}\right\rfloor=\left\lfloor\overrightarrow{y_{1}}+\left\lfloor\overrightarrow{y_{2}}\right\rfloor\right\rfloor=\left\lfloor\left\langle\overrightarrow{y_{1}}\right\rfloor+\left\lfloor\overrightarrow{y_{2}}\right\rfloor\right\rfloor \text { for every } \overrightarrow{y_{1}}, \overrightarrow{y_{2}} \in T^{n}\right) \\
= & \left\lfloor(0, \ldots, 0)+\overrightarrow{z_{1}}+\ldots+\overrightarrow{z_{k}}+\delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right)\right\rfloor  \tag{byI.H.}\\
= & \text { (by I.H.) } \\
\text { (by (18.4) }
\end{array}
$$

This proves (18.3).

Now let $\xi \in \mathrm{T}_{\Sigma}$. We note that, if B is not zero-sum free, then $\mathcal{A}$ is bu deterministic. Then

$$
\begin{aligned}
& \xi \in \operatorname{supp}\left(\llbracket \mathcal{A} \rrbracket^{\text {run }}\right) \text { iff } \llbracket \mathcal{A} \rrbracket^{\text {run }}(\xi) \neq \mathbb{O} \\
& \text { iff } \left.\bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}}\left(q_{f}, \xi\right)} \mathrm{wt}_{\mathcal{A}}(\xi, \rho) \neq \mathbb{0} \quad \quad \text { (because } \operatorname{supp}(F)=\left\{q_{f}\right\} \text { and } F_{q_{f}}=\mathbb{1}\right) \\
& \rho \in \mathrm{R}_{\mathcal{A}}\left(q_{f}, \xi\right) \\
& \operatorname{iff}\left(\exists \rho \in \mathrm{R}_{\mathcal{A}}\left(q_{f}, \xi\right)\right): \operatorname{wt}_{\mathcal{A}}(\xi, \rho) \neq 0 \quad \text { (because } \mathrm{B} \text { is zero-sum free or } \mathcal{A} \text { is bu deterministic) } \\
& \text { iff }\left(\exists \vec{z} \in T^{n}\right)\left(\exists \rho^{\prime} \in \mathrm{R}_{A}^{\mathrm{v}}\left(\left(q_{f}, \vec{z}\right), \xi\right)\right): \mathrm{wt}_{\mathcal{A}}\left(\xi, \varphi^{-1}\left(\rho^{\prime}\right)\right) \neq \mathbb{0} \\
& \text { (because } \varphi \text { is bijective) } \\
& \text { iff }\left(\exists \vec{z} \in T^{n}\right)\left(\exists \rho^{\prime} \in \mathrm{R}_{A}^{\mathrm{v}}\left(\left(q_{f}, \vec{z}\right), \xi\right)\right): \llbracket \overline{\left(\xi, \varphi^{-1}\left(\rho^{\prime}\right)\right)} \rrbracket \neq \mathbb{0} \\
& \text { (by 18.2) } \\
& \text { iff }\left(\exists \vec{z} \in T^{n}\right)\left(\exists \rho^{\prime} \in \mathrm{R}_{A}^{\mathrm{v}}\left(\left(q_{f}, \vec{z}\right), \xi\right)\right): \llbracket\left[\overline{\left(\xi, \varphi^{-1}\left(\rho^{\prime}\right)\right)}\right\rfloor_{\mathrm{dg}(\vec{b})} \rrbracket \neq \mathbb{O} \quad \text { (Lemma 18.2.9) } \\
& \text { iff }\left(\exists \vec{z} \in T^{n}\right)\left(\exists \rho^{\prime} \in \mathrm{R}_{A}^{\mathrm{v}}\left(\left(q_{f}, \vec{z}\right), \xi\right)\right): \llbracket \vec{z} \rrbracket \neq 0 \\
& \text { iff }\left(\exists q^{\prime} \in F^{\prime}\right): \quad \mathrm{R}_{A}^{\mathrm{v}}\left(q^{\prime}, \xi\right) \neq \emptyset \quad \text { (by construction of } F^{\prime} \text { ) } \\
& \text { iff } \xi \in \mathrm{L}(A) \text {. }
\end{aligned}
$$

As a direct consequence of Theorem 18.2.10 we obtain a support theorem for wta over each strong bimonoid in which the summation is a t-conorm (cf. Example 2.6.10(4) and thus, in particular, for wta over Unitlnt ${ }_{u, i}$ where $u$ is a t-conorm and $i$ is a t-norm. The latter consequence generalizes the case $\lambda=0$ of [San68, Thm. 7] from UnitInt ${ }_{\text {max,min }}$ to UnitInt ${ }_{u, i}$ for each tuple $(u, i)$ of t-conorm $u$ and t-norm $i$.

Corollary 18.2.11. Let $\mathrm{B}=([0,1], u, \otimes, 0,1)$ be a commutative strong bimonoid such that $u$ is a tconorm. Moreover, let $\mathcal{A}$ be a $(\Sigma, \mathrm{B})$-wta. Then $\operatorname{supp}\left(\llbracket \mathcal{A} \rrbracket^{\text {run }}\right) \in \operatorname{Rec}(\Sigma)$.

Proof. We show that $\mathrm{B}=([0,1], u, \otimes, 0,1)$ is zero-sum free. Then the statement follows from Theorem 18.2.10.

Let $a, b \in[0,1]$ and $u(a, b)=0$. Using the boundary condition and monotonicity condition of $u$ (cf. Example 2.6.10(4)), we have

$$
\begin{aligned}
a & =u(a, 0) \\
& \leq u(a, b) \\
& =0
\end{aligned}
$$

(by the boundary condition for $u$ )
(by monotonicity condition for $u$ )
(by assumption)
This implies that $a=0$. In a similar way we can derive that $b=0$. Hence $\mathrm{B}=([0,1], u, \otimes, 0,1)$ is zero-sum free.

In Theorem 18.2.10(a), the strong bimonoid B is assumed to be commutative and zero-sum free. The theorem is not constructive because, in order to construct an fta which accepts the support of a run recognizable weighted tree language, we should construct $\operatorname{dg}(\vec{b})$. However, it is not clear how to construct $\operatorname{dg}(\vec{b})$. We mention that in Kir11] the zero generation problem (ZGP) is considered, and the decidability of the ZGP implies that $\operatorname{dg}(\vec{b})$ can be constructed and hence the fta in the support theorem can be constructed.

In the following theorem, we require additionally that $B$ is multiplicatively idempotent. Then we do not need to compute $\operatorname{dg}(\vec{b})$ and we can construct the desired fta. We note that our assumption on B implies that the ZGP is decidable. However, we do not wish to include the ZGP in our proof. For more details on ZGP we refer to, e.g., Kir11, DH15, Göt17, FHV18.

Theorem 18.2.12. Let $\Sigma$ be a ranked alphabet and B be a zero-sum free, commutative, and multiplicatively idempotent strong bimonoid. Then, for each $(\Sigma, \mathrm{B})$-wta $\mathcal{A}$, we can construct a $\Sigma$-fta $A$ such that $\mathrm{L}(A)=\operatorname{supp}\left(\llbracket \mathcal{A} \rrbracket^{\text {run }}\right)$.


Figure 18.2: Some support theorems proved in this chapter.

Proof. Let $\mathcal{A}=(Q, \delta, F)$ be a $(\Sigma, \mathrm{B})$-wta. By Theorem 7.3.1 we can assume that $\mathcal{A}$ is root weight normalized and $F_{q_{f}}=\mathbb{1}$ and $F_{q}=\mathbb{O}$ for each $q \in Q \backslash\left\{q_{f}\right\}$. Let $\vec{b}=\left(b_{1}, \ldots, b_{n}\right) \in B^{n}$ be an arbitrary but fixed enumeration of $\operatorname{im}(\delta)$.

Since $B$ is commutative and multiplicatively idempotent, it is clear that $\min \left(\llbracket \cdot \rrbracket^{-1}(\mathbb{O})\right) \subseteq\{0,1\}^{n}$. Hence, instead of $\operatorname{dg}(\vec{b})$, we can use 1 as threshold for the frequency with which an element of im $(\delta)$ can occur in the weight of a run. More precisely, we construct the $\Sigma$-fta $A$ in the same way as in the proof of Theorem 18.2.10 except that we define the set $Q^{\prime}$ of states of $A$ by $Q^{\prime}=Q \times\{0,1\}^{n}$. We obtain $\mathrm{L}(A)=\operatorname{supp}\left(\llbracket \mathcal{A} \rrbracket^{\text {run }}\right)$.

Finally we mention that in [Kir14, Thm. 3.5] the following characterization result was shown: for each semiring B, the supports of all recognizable B-weighted string languages are recognizable if and only if in every finitely generated subsemiring of $B$, there exists a congruence of finite index such that $\{0\}$ is a singleton congruence class. We refer the reader for further support theorems for wsa to [BR88, DK21].

### 18.2.5 Comparison of support theorems

Finally, we will make two comparisons of the support theorems shown in Figure 18.2, First we compare some support theorems concerning run semantics, then we do the same for initial algebra semantics. Since each such support theorem requires a particular subset of strong bimonoids, we compare these support theorems by comparing the corresponding sets of strong bimonoids with respect to set inclusion. We denote the involved sets of strong bimonoids as follows:
$\mathcal{C}$ : set of all strong bimonoids,
$\mathcal{C}_{1}$ : set of all positive strong bimonoids,
$\mathcal{C}_{2}$ : set of all commutative semirings which are not rings,
$\mathcal{C}_{3}$ : set of all commutative, zero-sum free strong bimonoids,
$\mathcal{C}_{4}$ : set of all locally finite strong bimonoids,
$\mathcal{C}_{5}$ : set of all bi-locally finite strong bimonoids.
Hence, first we compare the sets $\mathcal{C}, \mathcal{C}_{1}, \mathcal{C}_{2}, \mathcal{C}_{3}$, and $\mathcal{C}_{5}$ (cf. Theorem 18.2.14), and second the sets $\mathcal{C}, \mathcal{C}_{1}$, $\mathcal{C}_{2}$, and $\mathcal{C}_{4}$ (cf. Theorem 18.2.15).

The following is observation is trivial, but useful.
Observation 18.2.13. $\left(\mathcal{C}_{1} \cap \mathcal{C}_{2}\right) \subseteq \mathcal{C}_{3}$.
Theorem 18.2.14. DM20, Dro22 Figure 18.3 shows the Euler diagram of the sets $\mathcal{C}_{1}, \mathcal{C}_{2}$, and $\mathcal{C}_{3}$. Moreover, for each of the shown seven nonempty regions $X$, we show two strong bimonoids $\mathrm{B}_{i}$ and $\mathrm{B}_{j}$ in the form of the fraction $\frac{\mathrm{B}_{i}}{\mathrm{~B}_{j}}$ such that $\mathrm{B}_{i} \in X \cap \mathcal{C}_{5}$ and $\mathrm{B}_{j} \in X \backslash \mathcal{C}_{5}$.


Figure 18.3: The Euler diagram of the sets $\mathcal{C}_{1}, \mathcal{C}_{2}$, and $\mathcal{C}_{3}$.

Proof. For the sake of a more complete picture, we also show the set $\mathcal{C}$ of all strong bimonoids in Figure 18.3. In the following, we prove that each of the seven regions $X$ which are shown in the figure is not empty. Together with Observation 18.2 .13 this proves that Figure 18.3 is the Euler diagram of $\mathcal{C}_{1}$, $\mathcal{C}_{2}$, and $\mathcal{C}_{3}$. Interleaved, we prove the existence of $\mathrm{B}_{i}$ and $\mathrm{B}_{j}$ with the desired properties.

We organize the proof as follows. Since there are four sets involved (viz. $\mathcal{C}_{1}, \mathcal{C}_{2}, \mathcal{C}_{3}$, and $\mathcal{C}_{5}$ ) there exist 16 Boolean combinations of these sets. Each Boolean combination is identified by an expression of the form $D \backslash E$ where $D$ is the intersection of some of the sets in $\left\{\mathcal{C}_{1}, \mathcal{C}_{2}, \mathcal{C}_{3}, \mathcal{C}_{5}\right\}$ and $E$ is the union of those sets in $\left\{\mathcal{C}_{1}, \mathcal{C}_{2}, \mathcal{C}_{3}, \mathcal{C}_{5}\right\}$ which do not occur in $D$. We order the 16 Boolean combinations by increasing number of sets occurring in $E$, starting with $E=\emptyset$.
$\mathcal{C}_{1} \cap \mathcal{C}_{2} \cap \mathcal{C}_{3} \cap \mathcal{C}_{5} \neq \emptyset:$ By Observation 18.2.13, $\mathcal{C}_{1} \cap \mathcal{C}_{2} \cap \mathcal{C}_{3} \cap \mathcal{C}_{5}=\mathcal{C}_{1} \cap \mathcal{C}_{2} \cap \mathcal{C}_{5}$. The Boolean semiring $B_{1}=$ Boole $=(\mathbb{B}, \vee, \wedge, 0,1)$ is in $\mathcal{C}_{1} \cap \mathcal{C}_{2} \cap \mathcal{C}_{5}$.
$\underline{\left(\mathcal{C}_{2} \cap \mathcal{C}_{3} \cap \mathcal{C}_{5}\right) \backslash \mathcal{C}_{1} \neq \emptyset}$ : Let $A=\{a, b\}$ be a set. The commutative semiring

$$
\mathrm{B}_{2}=\mathrm{PS}_{A}=(\mathcal{P}(A), \cup, \cap, \emptyset, A)
$$

is not zero-divisor free, because, e.g., $\{a\} \cap\{b\}=\emptyset$ and $\{a\} \neq \emptyset$ and $\{b\} \neq \emptyset$. Hence $\mathrm{B}_{2} \notin \mathcal{C}_{1}$. Since $\mathrm{B}_{2}$ is zero-sum free, finite, and not a ring, we have $B_{2} \in \mathcal{C}_{2} \cap \mathcal{C}_{3} \cap \mathcal{C}_{5}$.
$\underline{\left(\mathcal{C}_{1} \cap \mathcal{C}_{3} \cap \mathcal{C}_{5}\right) \backslash \mathcal{C}_{2} \neq \emptyset: \text { We consider the algebra }}$

$$
\mathrm{B}_{3}=\left(\{0,1,2,3,4\},+^{\prime}, \min , 0,4\right)
$$

where $a+^{\prime} b=\min (a+b, 4)$. Obviously $\mathrm{B}_{3}$ is a commutative strong bimonoid. Since $\mathrm{B}_{3}$ is also finite and positive, it is in $\mathcal{C}_{1} \cap \mathcal{C}_{3} \cap \mathcal{C}_{5}$. $\mathrm{B}_{3}$ is not distributive, because $\min \left(2,2+{ }^{\prime} 2\right)=2$ and $\min (2,2)+{ }^{\prime} \min (2,2)=$ $2+^{\prime} 2=4$. Hence $\mathrm{B}_{3} \notin \mathcal{C}_{2}$.
$\left(\mathcal{C}_{1} \cap \mathcal{C}_{2} \cap \mathcal{C}_{5}\right) \backslash \mathcal{C}_{3}=\emptyset:$ This follows from Observation 18.2.13,
$\underline{\left(\mathcal{C}_{1} \cap \mathcal{C}_{2} \cap \mathcal{C}_{3}\right) \backslash \mathcal{C}_{5} \neq \emptyset}$ : By Observation 18.2.13, $\left(\mathcal{C}_{1} \cap \mathcal{C}_{2} \cap \mathcal{C}_{3}\right) \backslash \mathcal{C}_{5}=\left(\mathcal{C}_{1} \cap \mathcal{C}_{2}\right) \backslash \mathcal{C}_{5}$. The semiring

$$
\mathrm{B}_{4}=\mathrm{Nat}=(\mathbb{N},+, \cdot, 0,1)
$$

of natural numbers is positive and commutative, it is not a ring and not bi-locally finite.
$\underline{\left(\mathcal{C}_{3} \cap \mathcal{C}_{5}\right) \backslash\left(\mathcal{C}_{1} \cup \mathcal{C}_{2}\right) \neq \emptyset:}$ We consider the algebra

$$
\mathrm{B}_{5}=\left(\{0,1,2,3\},+^{\prime}, \cdot 4,0,1\right)
$$

where $a+{ }^{\prime} b=\min (a+b, 3)$ and $\cdot_{4}$ is multiplication modulo 4. Obviously, $\mathrm{B}_{5}$ is a commutative strong bimonoid. Since $B_{5}$ is commutative, finite, and zero-sum free, we have $B_{5} \in \mathcal{C}_{3} \cap \mathcal{C}_{5}$. Since $2 \cdot{ }_{4} 2=0$,
$\mathrm{B}_{5} \notin \mathcal{C}_{1}$. The strong bimonoid $\mathrm{B}_{5}$ is not distributive, because $2 \cdot{ }_{4}\left(3+^{\prime} 1\right)=2$ and $2 \cdot{ }_{4} 3+^{\prime} 2 \cdot{ }_{4} 1=3$. Thus $B_{5} \notin \mathcal{C}_{2}$.

$$
\begin{aligned}
& \underline{\left(\mathcal{C}_{2} \cap \mathcal{C}_{5}\right) \backslash\left(\mathcal{C}_{1} \cup \mathcal{C}_{3}\right) \neq \emptyset:} \text { The semiring } \\
& \qquad \mathrm{B}_{6}=\operatorname{PS}_{\{a, b\}}=(\mathcal{P}(\{a, b\}), \cap, \cup\{a, b\}, \emptyset)
\end{aligned}
$$

is commutative, not a ring, and finite; hence $\mathrm{B}_{6} \in \mathcal{C}_{2} \cap \mathcal{C}_{5}$. Moreover, $\mathrm{B}_{6}$ is not zero-sum free; hence $\mathrm{B}_{6} \notin \mathcal{C}_{1} \cup \mathcal{C}_{3}$.
$\underline{\left(\mathcal{C}_{2} \cap \mathcal{C}_{3}\right) \backslash\left(\mathcal{C}_{1} \cup \mathcal{C}_{5}\right) \neq \emptyset}$ : We consider the commutative semiring

$$
\mathrm{B}_{7}=(\mathbb{N} \times \mathbb{N},+, \times,(0,0),(1,1))
$$

with componentwise addition and multiplication. This is not a ring (hence $\mathrm{B}_{7} \in \mathcal{C}_{2}$ ), and it is zero-sum free (hence $\left.\mathrm{B}_{7} \in \mathcal{C}_{3}\right)$. Moreover, $\mathrm{B}_{7}$ is not zero-divisor free because, e.g., $(1,0) \times(0,1)=(0,0)$ (hence $\mathrm{B}_{7} \notin \mathcal{C}_{1}$ ) and not bi-locally finite (hence $\mathrm{B}_{7} \notin \mathcal{C}_{5}$ ).
$\left(\mathcal{C}_{1} \cap \mathcal{C}_{5}\right) \backslash\left(\mathcal{C}_{2} \cup \mathcal{C}_{3}\right) \neq \emptyset:$ By Observation 18.2.13, $\left(\mathcal{C}_{1} \cap \mathcal{C}_{5}\right) \backslash\left(\mathcal{C}_{2} \cup \mathcal{C}_{3}\right)=\left(\mathcal{C}_{1} \cap \mathcal{C}_{5}\right) \backslash \mathcal{C}_{3}$. We consider the algebra

$$
\mathrm{B}_{8}=(\{0, a, b, 1\},+, \cdot, 0,1)
$$

where + is defined as supremum with respect to the partial order $0<a<b<1$; moreover, the operation $\cdot$ is determined by $x \cdot y=x$ for every $x, y \in\{a, b\}$. Then $(\{0, a, b, 1\},+, 0)$ and $(\{0, a, b, 1\}, \cdot, 1)$ are commutative monoids and $x \cdot 0=0$. Hence $\mathrm{B}_{8}$ is a strong bimonoid.

Moreover, $B_{8}$ is zero-sum free and zero-divisor free (hence $B_{8} \in \mathcal{C}_{1}$ ) and $B_{8}$ is finite (hence $B_{8} \in \mathcal{C}_{5}$ ). Since $\cdot$ is not commutative $(a \cdot b \neq b \cdot a)$, we have $\mathrm{B}_{8} \notin \mathcal{C}_{3}$.
$\underline{\left(\mathcal{C}_{1} \cap \mathcal{C}_{3}\right) \backslash\left(\mathcal{C}_{2} \cup \mathcal{C}_{5}\right) \neq \emptyset}$ : The tropical bimonoid

$$
\mathrm{B}_{9}=\operatorname{TropBM}=\left(\mathbb{N}_{\infty},+, \min , 0, \infty\right)
$$

is positive and commutative, hence $\mathrm{B}_{9} \in \mathcal{C}_{1} \cap \mathcal{C}_{3}$. But $\mathrm{B}_{9}$ is neither distributive nor bi-locally finite.
$\underline{\left(\mathcal{C}_{1} \cap \mathcal{C}_{2}\right) \backslash\left(\mathcal{C}_{3} \cup \mathcal{C}_{5}\right)=\emptyset: \text { This follows from Observation 18.2.13, }}$
$\overline{\mathcal{C}_{1} \backslash\left(\mathcal{C}_{2} \cup \mathcal{C}_{3} \cup \mathcal{C}_{5}\right) \neq \emptyset: ~ B y ~ O b s e r v a t i o n ~ 18.2 .13, ~} \mathcal{C}_{1} \backslash\left(\mathcal{C}_{2} \cup \mathcal{C}_{3} \cup \mathcal{C}_{5}\right)=\mathcal{C}_{1} \backslash\left(\mathcal{C}_{3} \cup \mathcal{C}_{5}\right)$. Let $\Sigma$ be an alphabet. We consider the formal language semiring

$$
\mathrm{B}_{10}=\operatorname{Lang}_{\Sigma}=\left(\mathcal{P}\left(\Sigma^{*}\right), \cup, \circ, \emptyset,\{\varepsilon\}\right)
$$

This is a positive, non-commutative, and not bi-locally finite semiring.
$\underline{\mathcal{C}_{2} \backslash\left(\mathcal{C}_{1} \cup \mathcal{C}_{3} \cup \mathcal{C}_{5}\right) \neq \emptyset: ~ B y ~ O b s e r v a t i o n ~ 18.2 .13, ~} \mathcal{C}_{2} \backslash\left(\mathcal{C}_{1} \cup \mathcal{C}_{3} \cup \mathcal{C}_{5}\right)=\mathcal{C}_{2} \backslash\left(\mathcal{C}_{3} \cup \mathcal{C}_{5}\right)$. We consider the commutative semiring

$$
\mathrm{B}_{11}=(\mathbb{Z} \times \mathbb{N},+, \cdot,(0,0),(1,1))
$$

with pointwise addition and pointwise multiplication. $\mathrm{B}_{11}$ is not a ring, not zero-sum free, and not bi-locally finite.
$\underline{\mathcal{C}_{3} \backslash\left(\mathcal{C}_{1} \cup \mathcal{C}_{2} \cup \mathcal{C}_{5}\right) \neq \emptyset}$ : We consider the strong bimonoid

$$
\mathrm{B}_{12}=\left(\mathbb{N},+,{ }_{4}, 0,1\right)
$$

with $a \cdot{ }_{4} b=(a \cdot b) \bmod 4$. Then $\mathrm{B}_{12}$ is commutative and zero-sum free, i.e., $\mathrm{B}_{12} \in \mathcal{C}_{3}$. Since $2 \cdot{ }_{4} 2=0$, $\mathrm{B}_{12}$ contains zero-divisors, and hence $\mathrm{B}_{12} \notin \mathcal{C}_{1}$. $\mathrm{B}_{12}$ is not a semiring, because, e.g., $2 \cdot 4(3+1)=0$ and $2 \cdot{ }_{4} 3+2 \cdot{ }_{4} 1=2+2=4$ ). Hence $\mathrm{B}_{12} \notin \mathcal{C}_{2}$. Also $\mathrm{B}_{12}$ is not bi-locally finite (because $\langle 1\rangle_{+}$is not finite). Thus $\mathrm{B}_{12} \in C_{3} \backslash\left(\mathcal{C}_{1} \cup \mathcal{C}_{2} \cup \mathcal{C}_{5}\right)$.
$\underline{\mathcal{C}_{5} \backslash\left(\mathcal{C}_{1} \cup \mathcal{C}_{2} \cup \mathcal{C}_{3}\right) \neq \emptyset: ~ W e ~ c o n s i d e r ~ t h e ~ r i n g ~}$

$$
\mathrm{B}_{13}=\operatorname{Intmod} 4=\left(\{0,1,2,3\},{ }_{4}, \cdot 4,0,1\right)
$$

as defined in Example 2.6.9 (5). Since $B_{13}$ is finite, $B_{13} \in \mathcal{C}_{5}$. Since $B_{13}$ is a ring and is not zero-sum free, $\mathrm{B}_{13} \notin \mathcal{C}_{1} \cup \mathcal{C}_{2} \cup \mathcal{C}_{3}$.
$\underline{\mathcal{C} \backslash\left(\mathcal{C}_{1} \cup \mathcal{C}_{2} \cup \mathcal{C}_{3} \cup \mathcal{C}_{5}\right): ~ T h e ~ r i n g ~}$

$$
\mathrm{B}_{14}=\operatorname{lnt}=(\mathbb{Z},+, \cdot, 0,1)
$$

is neither zero-sum free nor bi-locally finite.
In particular, it follows from the proof of Theorem 18.2 .14 that each of the support theorems Theorem 18.2.4. Theorem 18.2 .7 Theorem 18.2 .10 and Corollary $18.2 .1(3)$, for $\mathcal{C}_{1}, \mathcal{C}_{2}, \mathcal{C}_{3}$, and $\mathcal{C}_{5}$ respectively, has its own benefit with respect to run semantics. More precisely, let $i \in\{1,2,3,5\}$. Then there exists $\mathrm{B} \in \mathcal{C}_{i} \backslash\left(\mathcal{C}_{j} \cup \mathcal{C}_{k} \cup \mathcal{C}_{\ell}\right)$ with pairwise different $j, k, \ell \in\{1,2,3,5\} \backslash\{i\}$ such that the support theorem for $\mathcal{C}_{i}$ implies the property $\operatorname{supp}\left(\llbracket \mathcal{A} \rrbracket^{\text {run }}\right) \in \operatorname{Rec}(\Sigma)$ for each $(\Sigma, B)$-wta $\mathcal{A}$, and this property does not follow from the support theorems for $\mathcal{C}_{j}, \mathcal{C}_{k}$, and $\mathcal{C}_{\ell}$.

Next we compare the sets $\mathcal{C}_{1}, \mathcal{C}_{2}$, and $\mathcal{C}_{4}$.
Theorem 18.2.15. DM20, Dro22 Figure 18.4 shows the Euler diagram of the sets $\mathcal{C}_{1}, \mathcal{C}_{2}$, and $\mathcal{C}_{4}$. Thus, for every $\mathcal{D}_{1} \in\left\{\mathcal{C}_{1}, \mathcal{C} \backslash \mathcal{C}_{1}\right\}, \mathcal{D}_{2} \in\left\{\mathcal{C}_{2}, \mathcal{C} \backslash \mathcal{C}_{2}\right\}$, and $\mathcal{D}_{4} \in\left\{\mathcal{C}_{4}, \mathcal{C} \backslash \mathcal{C}_{4}\right\}$, we have $\mathcal{D}_{1} \cap \mathcal{D}_{2} \cap \mathcal{D}_{4} \neq \emptyset$.

Proof. $\underline{\mathcal{C}_{1} \cap \mathcal{C}_{2} \cap \mathcal{C}_{4} \neq \emptyset}$ : The Boolean semiring

$$
\mathrm{B}_{1}=\text { Boole }=(\mathbb{B}, \vee, \wedge, 0,1)
$$

is in $\mathcal{C}_{1} \cap \mathcal{C}_{2} \cap \mathcal{C}_{4}$.
$\underline{\left(\mathcal{C}_{1} \cap \mathcal{C}_{2}\right) \backslash \mathcal{C}_{4} \neq \emptyset}$ : The commutative semiring

$$
\mathrm{B}_{2}=\mathrm{Nat}=(\mathbb{N},+, \cdot, 0,1)
$$

is positive and not a ring and not locally finite.
$\underline{\left(\mathcal{C}_{1} \cap \mathcal{C}_{4}\right) \backslash \mathcal{C}_{2} \neq \emptyset}$ : We consider the strong bimonoid

$$
\mathrm{B}_{3}=(\{0, a, b, 1\},+, \cdot, 0,1)
$$

which is the same as $\mathrm{B}_{8}$ on page 377 , i.e., + is defined as supremum with respect to the partial order $0<a<b<1$; moreover, the operation $\cdot$ is determined by $x \cdot y=x$ for every $x, y \in\{a, b\}$.

This $B_{3}$ is zero-sum free and zero-divisor free (hence $B_{3} \in \mathcal{C}_{1}$ ) and $B_{3}$ is finite (hence $B_{3} \in \mathcal{C}_{4}$ ). Since - is not commutative $(a \cdot b \neq b \cdot a)$, we have $\mathrm{B}_{3} \notin \mathcal{C}_{2}$.
$\underline{\left(\mathcal{C}_{2} \cap \mathcal{C}_{4}\right) \backslash \mathcal{C}_{1} \neq \emptyset:}$ We consider the algebra

$$
\mathrm{B}_{4}=\left(\{-1,0,1\} \times\{0,1\},+^{\prime}, \cdot,(0,0),(1,1)\right)
$$

where $\left(a_{1}, a_{2}\right)+^{\prime}\left(b_{1}, b_{2}\right)=\left(\max \left(-1, \min \left(a_{1}+b_{1}, 1\right)\right), \min \left(a_{2}+b_{2}, 1\right)\right)$ (i.e., $+^{\prime}$ is defined componentwise by using the usual addition except that the first component is truncated by 1 from above and by -1 from below, and the second component is truncated by 1 from above). The multiplication is defined componentwise. Obviously, $\mathrm{B}_{4}$ is a finite commutative semiring.

Since there does not exist an element $\left(a_{1}, a_{2}\right)$ such that $(0,1)+^{\prime}\left(a_{1}, a_{2}\right)=(0,0), \mathrm{B}_{4}$ is not a ring. Hence $\mathrm{B}_{4} \in \mathcal{C}_{2}$. $\mathrm{B}_{4}$ is not zero-sum free, because $(-1,0)+^{\prime}(1,0)=(0,0)$. Hence $\mathrm{B}_{4} \notin \mathcal{C}_{1}$.
$\underline{\mathcal{C}_{1} \backslash\left(\mathcal{C}_{2} \cup \mathcal{C}_{4}\right) \neq \emptyset}$ : Let $\Sigma$ be an alphabet. We consider the formal language semiring

$$
\mathrm{B}_{5}=\operatorname{Lang}_{\Sigma}=\left(\mathcal{P}\left(\Sigma^{*}\right), \cup, \circ, \emptyset,\{\varepsilon\}\right)
$$

This is a positive, non-commutative and not locally finite semiring.
$\underline{\mathcal{C}_{2} \backslash\left(\mathcal{C}_{1} \cup \mathcal{C}_{4}\right) \neq \emptyset:}$ We consider the commutative semiring

$$
\mathrm{B}_{6}=(\mathbb{Z} \times \mathbb{N},+, \cdot,(0,0),(1,1))
$$



Figure 18.4: The Euler diagram of the sets $\mathcal{C}_{1}, \mathcal{C}_{2}$, and $\mathcal{C}_{4}$.
with pointwise addition and pointwise multiplication. $\mathrm{B}_{6}$ is not a ring, not zero-sum free, and not locally finite.
$\underline{\mathcal{C}_{4} \backslash\left(\mathcal{C}_{1} \cup \mathcal{C}_{2}\right) \neq \emptyset: ~ T h e ~ r i n g ~}$

$$
\mathrm{B}_{7}=\operatorname{Intmod} 4=\left(\{0,1,2,3\},+_{4}, \cdot 4,0,1\right)
$$

as defined in Example 2.6.9(5), is in $\mathcal{C}_{4} \backslash\left(\mathcal{C}_{1} \cup \mathcal{C}_{2}\right)$.
$\underline{\mathcal{C} \backslash\left(\mathcal{C}_{1} \cup \mathcal{C}_{2} \cup \mathcal{C}_{4}\right): ~ T h e ~ r i n g ~}$

$$
\mathrm{B}_{8}=\operatorname{lnt}=(\mathbb{Z},+, \cdot, 0,1)
$$

is neither zero-sum free nor locally finite.
In particular, it follows from the proof of Theorem 18.2.15 that each of the support theorems Theorem 18.2.4. Theorem 18.2.7 and Corollary 18.2.1(2), for $\mathcal{C}_{1}, \mathcal{C}_{2}$, and $\mathcal{C}_{4}$ respectively, has its own benefit with respect to initial algebra semantics. More precisely, let $i \in\{1,2,4\}$. Then there exists $\mathrm{B} \in \mathcal{C}_{i} \backslash\left(\mathcal{C}_{j} \cup \mathcal{C}_{k}\right)$ with different $j, k \in\{1,2,4\} \backslash\{i\}$ such that the support theorem for $\mathcal{C}_{i}$ implies the property $\operatorname{supp}\left(\llbracket \mathcal{A} \rrbracket^{\text {init }}\right) \in$ $\operatorname{Rec}(\Sigma)$ for each $(\Sigma, \mathrm{B})$-wta $\mathcal{A}$, and this property does not follow from the support theorems for $\mathcal{C}_{j}$ and $\mathcal{C}_{k}$.

## Chapter 19

## Corollaries and theorems for wta over bounded lattices

In the literature, $(\Sigma, \mathrm{L})$-wta have been investigated where L is a bounded lattice (or a bounded lattice with some further restrictions, like distributivity), e.g., [IF75, MM02, ÉL07, BLB10, MZA11, GZ12, GZA12, GZ16, GZ17, Gho18, Gho22 and for the string case DV10, DV12; for a survey we refer to Rah09. We also refer to Asv96 for a bibliography on fuzzy automata, grammars, and languages. In this chapter, we will present some part of the theory of wta over bounded lattices which follows from the results of the previous chapters. This attempt is reasonable because, due to Observation 2.6.13(2) and (3), i.e.,

- each bounded lattice is a particular bi-locally finite and commutative strong bimonoid, and
- each distributive bounded lattice is a particular locally finite and commutative semiring,
respectively. We will use the above facts without reference in the proofs of the corollaries and theorems in this chapter. For each lattice-oriented result that we present here, we will only refer to the covering strong bimonoid-oriented result of some previous chapter; there the reader can find references to the literature.

In this chapter, $\mathrm{L}=(L, \vee, \wedge, \mathbb{O}, \mathbb{1})$ denotes a bounded lattice.

### 19.1 Definition of wta over bounded lattices

Since a bounded lattice is a particular strong bimonoid, Section 3.1 provides the definition of a ( $\Sigma, \mathrm{L}$ )-wta. This definition coincides with the definition of weighted tree automata over bounded lattices as it is given in the literature, e.g. ÉL07 (apart from notational variations). In the same way, the notions of crisp deterministic $(\Sigma, \mathrm{L})$-wta and bu deterministic $(\Sigma, \mathrm{L})$-wta are defined. The definitions of the run semantics and initial algebra semantics of a $(\Sigma, \mathrm{L})$-wta are also given in Section 3.1 For the sake of convenience, we repeat their main parts here.

Let $\mathcal{A}=(Q, \delta, F)$ be a $(\Sigma, \mathrm{L})$-wta. We note that, due to (3.1), for each $\xi=\sigma\left(\xi_{1}, \ldots, \xi_{k}\right)$ and $\rho \in \mathrm{R}_{\mathcal{A}}(\xi)$, we have

$$
\begin{equation*}
\operatorname{wt}_{\mathcal{A}}(\xi, \rho)=\left(\bigwedge_{i \in[k]} \operatorname{wt}_{\mathcal{A}}\left(\xi_{i},\left.\rho\right|_{i}\right)\right) \wedge \delta_{k}(\rho(1) \cdots \rho(k), \sigma, \rho(\varepsilon)) \tag{19.1}
\end{equation*}
$$

and the run semantics of $\mathcal{A}$ is defined, for each $\xi \in \mathrm{T}_{\Sigma}$, by

$$
\llbracket \mathcal{A} \rrbracket^{\mathrm{run}}(\xi)=\bigvee_{\rho \in \mathrm{R}_{\mathcal{A}}(\xi)} \mathrm{wt}(\xi, \rho) \wedge F_{\rho(\varepsilon)} .
$$

Moreover, due to (3.2), the interpretation function $\delta_{\mathcal{A}}$ of the vector algebra $\mathrm{V}(\mathcal{A})=\left(L^{Q}, \delta_{\mathcal{A}}\right)$ is defined, for every $k \in \mathbb{N}, \sigma \in \Sigma^{(k)}, v_{1}, \ldots, v_{k} \in L^{Q}$, and $q \in Q$, by

$$
\begin{equation*}
\delta_{\mathcal{A}}(\sigma)\left(v_{1}, \ldots, v_{k}\right)_{q}=\bigvee_{q_{1} \cdots q_{k} \in Q^{k}}\left(\bigwedge_{i \in[k]}\left(v_{i}\right)_{q_{i}}\right) \wedge \delta_{k}\left(q_{1} \cdots q_{k}, \sigma, q\right) \tag{19.2}
\end{equation*}
$$

We recall that the unique $\Sigma$-algebra homomorphism from $\mathrm{T}_{\Sigma}$ to $\mathrm{V}(\mathcal{A})$ is denoted by $\mathrm{h}_{\mathcal{A}}$. Then the initial algebra semantics of $\mathcal{A}$ is defined, for each $\xi \in \mathrm{T}_{\Sigma}$, by

$$
\llbracket \mathcal{A} \rrbracket^{\mathrm{init}}(\xi)=\bigvee_{q \in Q} \mathrm{~h}_{\mathcal{A}}(\xi)_{q} \wedge F_{q}
$$

For instance, in Asv03, ÉL07 and in Gho22 the initial algebra semantics was used (where L is a $\sigma$-complete distributive lattice and a $\sigma$-complete orthomodular lattice, respectively). In Běl02] the run semantics was used (for wsa where $L$ is a complete distributive lattice, cf. the remark on page 328).

As we have seen in Example $\left\lceil 5.2 .4\right.$, there exists a $\left(\Sigma, N_{5}\right)$-wta $\mathcal{A}$ such that $\llbracket \mathcal{A} \rrbracket^{\text {run }} \neq \llbracket \mathcal{A} \rrbracket^{\text {init }}$. We recall that $\mathcal{A}$ is not bu deterministic and that $\mathrm{N}_{5}$ (cf. Figure 2.3) is not distributive. However, in the following cases run semantics and initial algebra semantics coincide.

Corollary 19.1.1. Let $\Sigma$ be a ranked alphabet and $L$ be a bounded lattice. For each $(\Sigma, \mathrm{L})$-wta $\mathcal{A}$ the following two statements hold.
(1) If $\mathcal{A}$ is bu deterministic, then $\llbracket \mathcal{A} \rrbracket^{\text {run }}=\llbracket \mathcal{A} \rrbracket^{\text {init }}$.
(2) If $L$ is distributive, then $\llbracket \mathcal{A} \rrbracket^{\text {run }}=\llbracket \mathcal{A} \rrbracket^{\text {init }}$.

Proof. Proof of (1) and (2) follows from Theorem 5.3 .1 and Corollary 5.3.3 (2), respectively.

Due to Corollary 19.1.1, for each $(\Sigma, \mathrm{L})-w \operatorname{ta} \mathcal{A}$, if $\mathcal{A}$ is bu deterministic or L is distributive, then we write $\llbracket \mathcal{A} \rrbracket$ instead of $\llbracket \mathcal{A} \rrbracket^{\text {run }}$ and $\llbracket \mathcal{A} \rrbracket^{\text {init }}$. Moreover, if L is distributive, then for an $i$-recognizable or $r$-recognizable weighted tree language $r$, we say that $r$ is recognizable and we denote the set $\operatorname{Rec}^{\text {run }}(\Sigma, \mathrm{L})$ (and hence, $\operatorname{Rec}^{\mathrm{init}}(\Sigma, \mathrm{L})$ ) by $\operatorname{Rec}(\Sigma, \mathrm{L})$.

### 19.2 Crisp determinization

Corollary 19.2.1. Let $\Sigma$ be a ranked alphabet and L be a bounded lattice. For each $(\Sigma, \mathrm{L})$-wta $\mathcal{A}$ the following statements hold.
(1) We can construct a crisp deterministic $(\Sigma, \mathrm{L})$-wta $\mathcal{B}$ such that $\llbracket \mathcal{A} \rrbracket^{\text {run }}=\llbracket \mathcal{B} \rrbracket$.
(2) If $L$ is locally finite, then we can construct a crisp deterministic $(\Sigma, L)$-wta $\mathcal{B}$ such that $\llbracket \mathcal{A} \rrbracket^{\text {init }}=\llbracket \mathcal{B} \rrbracket$. Thus, in particular, $\operatorname{Rec}^{\text {run }}(\Sigma, \mathrm{L})=\operatorname{cd}-\operatorname{Rec}(\Sigma, \mathrm{L})$ and, if L is locally finite, then $\operatorname{Rec}^{\text {init }}(\Sigma, \mathrm{L})=$ $\operatorname{cd}-\operatorname{Rec}(\Sigma, \mathrm{L})$.

Proof. Proof of (1): Since L is a particular bi-locally finite strong bimonoid, the set $\mathrm{H}(\mathcal{A})$ (cf. (7.15)) is finite and each $b \in L$ has finite additive order. Hence we can apply Theorem 16.2 .6 and obtain the desired crisp deterministic $(\Sigma, \mathrm{L})$-wta $\mathcal{B}$ with $\llbracket \mathcal{A} \rrbracket^{\text {run }}=\llbracket \mathcal{B} \rrbracket$.

Proof of (2): It follows from Theorem 16.1.3,
Finally, the equality of the sets follow from Statements (1) and (2) and the trivial fact that $\operatorname{cd}-\operatorname{Rec}(\Sigma, \mathrm{L}) \subseteq \operatorname{Rec}^{\mathrm{run}}(\Sigma, \mathrm{L}) \cap \operatorname{Rec}^{\mathrm{init}}(\Sigma, \mathrm{L})$.

Corollary 19.2 .1 (2) can be applied, e.g., to the locally finite bounded lattice $\left(\mathbb{N}_{\infty}, \max , \min , 0, \infty\right)$. Moreover, it can be applied to each finite lattice and to each distributive bounded lattice.

A direct consequence of Corollary 19.2 .1 and Corollary 4.3.3 is the stability of the set of run recognizable weighted tree languages under changing the bounded lattice $L_{1}$ into the bounded lattice $L_{2}$ if $L_{1}$ and $L_{2}$ have the same carrier set (as, e.g., $N_{5}$ and $M_{3}$ ) and similarly for initial algebra recognizable weighted tree languages over locally finite bounded lattices. Moreover, the expressive power of wta with run semantics over bounded lattices is the same as that of wta with initial algebra semantics over locally finite bounded lattices Dro22].
Corollary 19.2.2. Let $L_{1}$ and $L_{2}$ be bounded lattices with the same carrier set. Then the following three statements hold.
(1) $\operatorname{Rec}^{\text {run }}\left(\Sigma, \mathrm{L}_{1}\right)=\operatorname{cd}-\operatorname{Rec}\left(\Sigma, \mathrm{L}_{1}\right)=\operatorname{cd}-\operatorname{Rec}\left(\Sigma, \mathrm{L}_{2}\right)=\operatorname{Rec}^{\mathrm{run}}\left(\Sigma, \mathrm{L}_{2}\right)$.
(2) If $L_{1}$ and $L_{2}$ are locally finite, then $\operatorname{Rec}^{\text {init }}\left(\Sigma, L_{1}\right)=\operatorname{cd}-\operatorname{Rec}\left(\Sigma, L_{1}\right)=\operatorname{cd}-\operatorname{Rec}\left(\Sigma, L_{2}\right)=\operatorname{Rec}^{\text {init }}\left(\Sigma, L_{2}\right)$.
(3) If $\mathrm{L}_{2}$ is locally finite, then $\operatorname{Rec}^{\text {run }}\left(\Sigma, \mathrm{L}_{1}\right)=\operatorname{Rec}^{\text {init }}\left(\Sigma, \mathrm{L}_{2}\right)$.

Proof. Proof of (1): The equalities $\operatorname{Rec}^{\text {run }}\left(\Sigma, \mathrm{L}_{1}\right)=\operatorname{cd}-\operatorname{Rec}\left(\Sigma, \mathrm{L}_{1}\right)$ and $\operatorname{cd}-\operatorname{Rec}\left(\Sigma, \mathrm{L}_{2}\right)=\operatorname{Rec}^{\text {run }}\left(\Sigma, \mathrm{L}_{2}\right)$ follow from Corollary 19.2.1 The equality $\operatorname{cd}-\operatorname{Rec}\left(\Sigma, L_{1}\right)=\operatorname{cd}-\operatorname{Rec}\left(\Sigma, L_{2}\right)$ follows from Corollary 4.3 .3 (and Theorem 5.3.1 and the convention on page 117).

Proof of (2): This proof is analogous to the proof of (1).
 $\operatorname{Rec}^{\text {init }}\left(\Sigma, \mathrm{L}_{2}\right)$.

We recall from Section 2.14 that a $(\Sigma, \mathrm{L})$-weighted tree language $r$ is a recognizable step mapping if there exist $n \in \mathbb{N}_{+}, b_{1}, \ldots, b_{n} \in L$, and recognizable $\Sigma$-tree languages $L_{1}, \ldots, L_{n}$ such that

$$
r=\bigvee_{i \in[n]} b_{i} \wedge \chi\left(L_{i}\right)
$$

Corollary 19.2.3. Let $\Sigma$ be a ranked alphabet and $\mathrm{L}=(L, \vee, \wedge, \mathbb{D}, \mathbb{1})$ be a bounded lattice. For each ( $\Sigma, \mathrm{L}$ )-wta $\mathcal{A}$, the following statements hold.
(1) $\llbracket \mathcal{A} \rrbracket^{\text {run }}$ is a recognizable step mapping.
(2) If $L$ is locally finite, then $\llbracket \mathcal{A} \rrbracket^{\text {init }}$ is a recognizable step mapping.

Moreover, in (1) we can construct $n \in \mathbb{N}_{+}, b_{1}, \ldots, b_{n} \in L$, and $\Sigma$-fta $A_{1}, \ldots, A_{n}$ such that $\llbracket \mathcal{A} \rrbracket^{\text {run }}=$ $\bigvee_{i \in[n]} b_{i} \wedge \chi\left(\mathrm{~L}\left(A_{i}\right)\right)$. The same holds for $\llbracket \mathcal{A} \rrbracket^{\text {init }}$ in (2).

Proof. The proof follows from Corollary 19.2 .1 and Theorem $10.3 .1(\mathrm{~A}) \Leftrightarrow(\mathrm{B})$. We note that (1) also was already proved in Corollary 7.4.4.

Corollary 19.2.4. Let $\Sigma$ be a ranked alphabet, $\mathrm{L}=(L, \vee, \wedge, \mathbb{O}, \mathbb{1})$ be a bounded lattice, and $r: \mathrm{T}_{\Sigma} \rightarrow L$. Then the following two statements are equivalent.
(A) We can construct a $(\Sigma, \mathrm{L})$-wta $\mathcal{A}$ such that $r=\llbracket \mathcal{A} \rrbracket^{\text {run }}$.
(B) $r$ is a recognizable step mapping and we can construct $n \in \mathbb{N}_{+}, b_{1}, \ldots, b_{n} \in L$ and $\Sigma$-fta $A_{1}, \ldots, A_{n}$ such that $r=\bigvee_{i=1}^{n} b_{i} \wedge \chi\left(\mathrm{~L}\left(A_{i}\right)\right)$.
Moreover, if L is distributive, then $\llbracket \mathcal{A} \rrbracket^{\text {run }}$ in condition (A) can be replaced by $\llbracket \mathcal{A} \rrbracket$.

Proof. Proof of $(\mathrm{A}) \Rightarrow(\mathrm{B})$ : This follows from Corollary 19.2 .3 (1) and Theorem $10.3 .1(\mathrm{~A}) \Rightarrow(\mathrm{B})$.
Proof of $(\mathrm{B}) \Rightarrow(\mathrm{A})$ : By Theorem $10.3 .1(\mathrm{~B}) \Rightarrow(\mathrm{A})$, we can construct a crisp deterministic $(\Sigma, \mathrm{L})$-wta $\mathcal{A}$ such that $r=\llbracket \mathcal{A} \rrbracket$. By the convention on page $382, \llbracket \mathcal{A} \rrbracket$ is an abbreviation of $\llbracket \mathcal{A} \rrbracket^{\text {run }}$.

Let L be distributive. Then the statement follows from Corollary 19.1.1(2).

### 19.3 Relationship between several sets of recognizable weighted tree languages

Here we study the relationship between several sets of r-recognizable, i-recognizable, or crisp deterministically recognizable weighted tree languages where the weight algebra is a bounded lattice, locally finite bounded lattice, distributive bounded lattice, or finite chain. By combining these modes of recognizability and weight algebras, we obtain twelve sets of recognizable weighted tree languages.

It will turn out that eleven of the twelve sets are equal; let us denote this set by $C$. The exception is the set of i-recognizable weighted tree languages over bounded lattices. Moreover, we show that $C$ is a subset of the latter set and, if the ranked alphabet is large enough, then the inclusion is strict. Thus, roughly speaking, using initial algebra semantics over bounded lattices, one can specify strictly more weighted tree languages than using another combination. The results of this section are taken from [FV22a].

More precisely, we consider the set
$\mathcal{Q}=\left\{\operatorname{Rec}^{y}(\Sigma, Z) \mid y \in\{\right.$ run, init $\left.\}, Z \in\{\mathrm{BL}, \mathrm{lfBL}, \mathrm{dBL}, \mathrm{FC}\}\right\} \cup\{\operatorname{cd}-\operatorname{Rec}(\Sigma, Z) \mid Z \in\{\mathrm{BL}, \mathrm{lfBL}, \mathrm{dBL}, \mathrm{FC}\}\}$, where

- BL, lfBL, dBL, FC denote the sets of bounded lattices, locally finite bounded lattices, distributive bounded lattices, and finite chains,
- $\operatorname{Rec}^{y}(\Sigma, Z)=\bigcup_{\mathrm{L} \in Z} \operatorname{Rec}^{y}(\Sigma, \mathrm{~L})$, i.e., the set of all mappings $r: \mathrm{T}_{\Sigma} \rightarrow L$ where $L$ is the carrier set of some $\mathrm{L} \in Z$ and $r$ is a $y$-recognizable $(\Sigma, \mathrm{L})$-weighted tree language, and
- $\operatorname{cd}-\operatorname{Rec}(\Sigma, Z)=\bigcup_{\mathrm{L} \in Z} \operatorname{cd}-\operatorname{Rec}(\Sigma, \mathrm{L})$, i.e., the set of all mappings $r: \mathrm{T}_{\Sigma} \rightarrow L$ where $L$ is the carrier set of some $\mathrm{L} \in Z$ and $r$ is a crisp deterministically recognizable $(\Sigma, \mathrm{L})$-weighted tree language.
Our goal is to present the inclusion diagram of $\mathcal{Q}$ (cf. Theorem 19.3.5), i.e., the Hasse diagram of $\mathcal{Q}$ with set inclusion as partial order.

By definition we have $\mathrm{FC} \subset \mathrm{dBL}$ and by Observation 2.6.16, we have $\mathrm{dBL} \subset \mathrm{lfBL} \subset \mathrm{BL}$. Thus
for each $y \in\{$ run, init $\}$, we have $\operatorname{Rec}^{y}(\Sigma, \mathrm{FC}) \subseteq \operatorname{Rec}^{y}(\Sigma, \mathrm{dBL}) \subseteq \operatorname{Rec}^{y}(\Sigma, \mathrm{lfBL}) \subseteq \operatorname{Rec}^{y}(\Sigma, \mathrm{BL})$
and we have

$$
\begin{equation*}
\operatorname{cd}-\operatorname{Rec}(\Sigma, \mathrm{FC}) \subseteq \operatorname{cd}-\operatorname{Rec}(\Sigma, \mathrm{dBL}) \subseteq \operatorname{cd}-\operatorname{Rec}(\Sigma, \mathrm{lfBL}) \subseteq \operatorname{cd}-\operatorname{Rec}(\Sigma, \mathrm{BL}) \tag{19.4}
\end{equation*}
$$

In Figure 19.1 (ignoring the ovals for the time being) we show all the twelve elements of $\mathcal{Q}$, organized according to their known subset relationships; these relationships follow from (a) Equations (19.3) and (19.4), (b) the trivial fact that each crisp deterministic wta is a wta, and (c) Theorem 5.3.1. We note that this is not a Hasse diagram; in particular, the edges do not show strict inclusions but merely inclusions.

From the trivial fact that $\operatorname{cd}-\operatorname{Rec}(\Sigma, \mathrm{BL}) \subseteq \operatorname{Rec}^{\text {run }}(\Sigma, \mathrm{BL}) \cap \operatorname{Rec}^{\text {init }}(\Sigma, \operatorname{lfBL})$ and from Corollary 19.2.1, we directly obtain four equalities (corresponding to the four ovals in Figure 19.1)
Corollary 19.3.1. FV22a The following four statements hold.
(1) $\operatorname{Rec}^{\text {run }}(\Sigma, \mathrm{BL})=\operatorname{cd}-\operatorname{Rec}(\Sigma, \mathrm{BL})$.
(2) $\operatorname{Rec}^{\text {run }}(\Sigma, \mathrm{lfBL})=\operatorname{cd}-\operatorname{Rec}(\Sigma, \mathrm{lfBL})=\operatorname{Rec}^{\text {init }}(\Sigma, \mathrm{lfBL})$.
(3) $\operatorname{Rec}^{\text {run }}(\Sigma, \mathrm{dBL})=\operatorname{cd}-\operatorname{Rec}(\Sigma, \mathrm{dBL})=\operatorname{Rec}^{\mathrm{init}}(\Sigma, \mathrm{dBL})$.
(4) $\operatorname{Rec}^{\text {run }}(\Sigma, \mathrm{FC})=\operatorname{cd}-\operatorname{Rec}(\Sigma, \mathrm{FC})=\operatorname{Rec}^{\mathrm{init}}(\Sigma, \mathrm{FC})$.

Next we prove that $\operatorname{cd}-\operatorname{Rec}(\Sigma, \mathrm{BL}) \subseteq \operatorname{cd}-\operatorname{Rec}(\Sigma, \mathrm{FC})$.
Lemma 19.3.2. Dro22 FV22a For every crisp deterministic ( $\Sigma, \mathrm{L}$ )-wta $\mathcal{A}$ we can construct a finite chain $\mathrm{L}^{\prime}$ with carrier set $\operatorname{wts}(\mathcal{A}) \cup\{\mathbb{O}\}$ and a crisp deterministic $\left(\Sigma, \mathrm{L}^{\prime}\right)$-wta $\mathcal{B}$ such that $\llbracket \mathcal{A} \rrbracket=\llbracket \mathcal{B} \rrbracket$. Thus, in particular, $\operatorname{cd}-\operatorname{Rec}(\Sigma, \mathrm{BL}) \subseteq \operatorname{cd}-\operatorname{Rec}(\Sigma, \mathrm{FC})$.

Proof. Let $\mathcal{A}=(Q, \delta, F)$. We note that $\mathbb{1} \in \operatorname{wts}(\mathcal{A})$ because $\Sigma^{(0)} \neq \emptyset \neq Q$ and $\mathcal{A}$ is crisp deterministic. We construct the finite chain $\mathrm{L}^{\prime}=(\operatorname{wts}(\mathcal{A}) \cup\{\mathbb{O}\}, \leq, \mathbb{O}, \mathbb{1})$ where $\leq$ is an arbitrary linear order on $\mathrm{wts}(\mathcal{A}) \cup\{\mathbb{O}\}$.


Figure 19.1: Illustration of some of the set inclusions (vertical and diagonal lines) and some of the equalities (ovals) of the elements of $\mathcal{Q}$.

It is obvious that, for the crisp deterministic $\left(\Sigma, \mathrm{L}^{\prime}\right)$-wta $\mathcal{B}=(Q, \delta, F)$, we have $\llbracket \mathcal{A} \rrbracket=\llbracket \mathcal{B} \rrbracket$. This also proves that $\operatorname{cd}-\operatorname{Rec}(\Sigma, \mathrm{BL}) \subseteq \operatorname{cd}-\operatorname{Rec}(\Sigma, \mathrm{FC})$.

From the previous results, we obtain an intermediate answer to our goal.
Corollary 19.3.3. FV22a Let $\mathcal{Q}^{\prime}=\mathcal{Q} \backslash\left\{\operatorname{Rec}^{\text {init }}(\Sigma, \mathrm{BL})\right\}$, i.e., the set of the following eleven elements of $\mathcal{Q}$ :

- $\operatorname{Rec}^{\text {run }}(\Sigma, \mathrm{BL}), \operatorname{Rec}^{\mathrm{run}}(\Sigma, \operatorname{lfBL}), \operatorname{Rec}^{\text {run }}(\Sigma, \mathrm{dBL}), \operatorname{Rec}^{\text {run }}(\Sigma, \mathrm{FC})$,
- $\operatorname{Rec}^{\text {init }}(\Sigma$, lfBL $), \operatorname{Rec}^{\text {init }}(\Sigma, \mathrm{dBL}), \operatorname{Rec}^{\text {init }}(\Sigma, \mathrm{FC})$, and
- $\operatorname{cd}-\operatorname{Rec}(\Sigma, \mathrm{BL}), \operatorname{cd}-\operatorname{Rec}(\Sigma, \mathrm{lfBL}), \operatorname{cd}-\operatorname{Rec}(\Sigma, \mathrm{dBL})$, and $\operatorname{cd}-\operatorname{Rec}(\Sigma, \mathrm{FC})$.

Then all elements of $\mathcal{Q}^{\prime}$ are equal.
Proof. By Lemma 19.3.2, $\operatorname{cd}-\operatorname{Rec}(\Sigma, \mathrm{BL}) \subseteq \operatorname{cd}-\operatorname{Rec}(\Sigma, \mathrm{FC})$. Hence, by the inclusions shown in Figure 19.1 and Corollary 19.3.1 all elements of $\mathcal{Q}^{\prime}$ are equal.

Next we prove that $\operatorname{Rec}^{\text {init }}(\Sigma, \mathrm{BL}) \backslash \operatorname{Rec}^{\text {init }}(\Sigma, \mathrm{lfBL}) \neq \emptyset$ under some mild conditions on $\Sigma$. For this, we consider the bounded lattice $\mathrm{FL}(2+2)$ in Example 2.6.15(9).

Lemma 19.3.4. FV22a] If $\left|\Sigma^{(0)}\right| \geq 6$ and $\left|\Sigma^{(2)}\right| \geq 2$, then we can construct a $(\Sigma, \mathrm{FL}(2+2))$-wta $\mathcal{A}$ such that $\llbracket \mathcal{A} \rrbracket^{\text {init }} \notin \operatorname{Rec}^{\text {init }}(\Sigma$, lfBL $)$. In particular, for such a $\Sigma$ we have $\operatorname{Rec}^{\text {init }}(\Sigma, \mathrm{BL}) \backslash \operatorname{Rec}^{\text {init }}(\Sigma, \mathrm{lfBL}) \neq \emptyset$.

Proof. We consider the finite set $A=\{a, b, c, d\}$ of generators of $\mathrm{FL}(2+2)$ (cf. Figure 2.4). Since $b \vee d=\mathbb{1}$ and $a \wedge c=\mathbb{O}$ and $\mathrm{FL}(2+2)$ is generated by $A$, we have $\langle A\rangle_{\{\vee, \wedge, 0, \mathbb{1}\}}=\langle A\rangle_{\{\vee, \wedge\}}=\mathrm{FL}(2+2)$, which is an infinite set. By Theorem 3.1.5, we can construct a $(\Sigma, \operatorname{FL}(2+2))$-wta $\mathcal{A}$ such that $\operatorname{im}\left(\llbracket \mathcal{A} \rrbracket^{\text {init }}\right)=$ $\langle A\rangle_{\{\vee, \wedge, 0, \mathbb{1}\}}$, i.e., $\operatorname{im}\left(\llbracket \mathcal{A} \rrbracket^{\text {init }}\right)$ is an infinite set.

On the other hand, by Corollary 19.2.3, each element $r \in \operatorname{Rec}^{\text {init }}(\Sigma, \operatorname{lfBL})$ is a recognizable step mapping, and hence $\operatorname{im}(r)$ is finite. Thus $\llbracket \mathcal{A} \rrbracket^{\text {init }} \notin \operatorname{Rec}^{\text {init }}(\Sigma$, lfBL $)$.

From Corollary 19.3.3, the fact that $\operatorname{Rec}^{\text {init }}(\Sigma, \operatorname{lfBL}) \subseteq \operatorname{Rec}^{\text {init }}(\Sigma, \mathrm{BL})$, and Lemma 19.3.4 we obtain the final picture of the Hasse diagram for $\mathcal{Q}$.

Theorem 19.3.5. FV22a Let $\Sigma$ be a ranked alphabet. Let $\mathcal{Q}=\left\{\operatorname{Rec}^{y}(\Sigma, Z) \mid y \in\{\right.$ run, init $\}, Z \in$ $\{\mathrm{BL}, \mathrm{lfBL}, \mathrm{dBL}, \mathrm{FC}\}\} \cup\{\operatorname{cd}-\operatorname{Rec}(\Sigma, Z) \mid Z \in\{\mathrm{BL}, \mathrm{lfBL}, \mathrm{dBL}, \mathrm{FC}\}\}$. Then the following three statements hold.
(1) $\operatorname{Rec}^{\text {run }}(\Sigma, \mathrm{BL})=\operatorname{Rec}^{\text {run }}(\Sigma, \mathrm{lfBL})=\operatorname{Rec}^{\text {run }}(\Sigma, \mathrm{dBL})=\operatorname{Rec}^{\text {run }}(\Sigma, \mathrm{FC})=\operatorname{Rec}^{\mathrm{init}}(\Sigma, \mathrm{lfBL})=$ $\operatorname{Rec}^{\mathrm{init}}(\Sigma, \mathrm{dBL})=\operatorname{Rec}^{\text {init }}(\Sigma, \mathrm{FC})=\operatorname{cd}-\operatorname{Rec}(\Sigma, \mathrm{BL})=\operatorname{cd}-\operatorname{Rec}(\Sigma, \operatorname{lfBL})=\operatorname{cd}-\operatorname{Rec}(\Sigma, \mathrm{dBL})=$ $\operatorname{cd}-\operatorname{Rec}(\Sigma, \mathrm{FC})$. We denote this set by $C$.
(2) $C \subseteq \operatorname{Rec}^{\text {init }}(\Sigma, \mathrm{BL})$.
(3) If $\left|\Sigma^{(0)}\right| \geq 6$ and $\left|\Sigma^{(2)}\right| \geq 2$, then we have $C \subset \operatorname{Rec}^{\text {init }}(\Sigma, \mathrm{BL})$. In this case, the Hasse diagram for $\mathcal{Q}$ with set inclusion as partial order consists of the two nodes $C$ and $\operatorname{Rec}^{\text {init }}(\Sigma, \mathrm{BL})$.

We note that since $b \vee d=\mathbb{1}$ and $a \wedge c=\mathbb{O}$ in $\mathrm{FL}(2+2)$, the condition $\left|\Sigma^{(0)}\right| \geq 6$ in Lemma 19.3.4 and hence in Theorem 19.3.5 can be weakened to $\left|\Sigma^{(0)}\right| \geq 4$ (cf. the proof of Theorem 3.1.5).

### 19.4 Support

Corollary 19.4.1. Let $\Sigma$ be a ranked alphabet and L be a bounded lattice. For each $(\Sigma, \mathrm{L})$-wta $\mathcal{A}$ the following statements hold.
(1) The $\Sigma$-tree language $\operatorname{supp}\left(\llbracket \mathcal{A} \rrbracket^{\text {run }}\right)$ is recognizable.
(2) If $L$ is locally finite, then the $\Sigma$-tree language $\operatorname{supp}\left(\llbracket \mathcal{A} \rrbracket^{\text {init }}\right)$ is recognizable.

Moreover, for the tree language $\operatorname{supp}\left(\llbracket \mathcal{A} \rrbracket^{\text {run }}\right)$ in (1), we can construct a $\Sigma$-fta which recognizes that tree language. The same holds for the tree languages supp $\left(\llbracket \mathcal{A} \rrbracket^{\text {init }}\right)$ in (2).

Proof. Proof of (1) and (2): They follow from Corollary 18.2 .1 (3) and (2), respectively. We note that (1) also follows from Theorem 18.2.10(a) because $L$ is zero-sum free.

Now consider the tree language $\operatorname{supp}\left(\llbracket \mathcal{A} \rrbracket^{\text {run }}\right)$ in (1). By Corollary 19.2.1 (1), we can construct a crisp deterministic $(\Sigma, \mathrm{L})$-wta $\mathcal{B}$ such that $\llbracket \mathcal{A} \rrbracket^{\text {run }}=\llbracket \mathcal{B} \rrbracket$. Then $\operatorname{supp}\left(\llbracket \mathcal{A} \rrbracket^{\text {run }}\right)=\mathrm{T}_{\Sigma} \backslash \llbracket \mathcal{B} \rrbracket^{-1}(\mathbb{O})$. By Theorem 10.3.1(C), we can construct a $\Sigma$-fta which recognizes $\llbracket \mathcal{B} \rrbracket^{-1}(\mathbb{O})$. Finally, it is easy to construct a $\Sigma$-fta which recognizes the complement tree language $\left.\mathrm{T}_{\Sigma} \backslash \llbracket \mathcal{B}\right]^{-1}(\mathbb{O})$ (cf. e.g [GS84, Thm. 2.4.2]).

By using Theorem 18.2.12, we can give an alternative proof for the constructability of a $\Sigma$-fta which recognizes $\operatorname{supp}\left(\llbracket \mathcal{A} \rrbracket^{\text {run }}\right)$ in (1). Moreover, by using Corollary 19.2.1(2) and Theorem 10.3.1(C), we can give an alternative proof for the constructability a $\Sigma$-fta which recognizes $\operatorname{supp}\left(\llbracket \mathcal{A} \rrbracket^{\text {init }}\right)$ in (2).

### 19.5 Closure results

Here we instantiate the closure results of Chapter 10 to the particular case where L is a (distributive) bounded lattice (cf. Figure 10.18 for the more general cases).

Corollary 19.5.1. Let $\Sigma$ be a ranked alphabet and $L$ be a bounded lattice. The following statements hold.
(1) The sets $\operatorname{Rec}^{\text {run }}(\Sigma, \mathrm{L})$ and $\operatorname{Rec}^{\text {init }}(\Sigma, \mathrm{L})$ are closed under sum (cf. Theorem 10.1.1).
(2) If $L$ is distributive, then the set $\operatorname{Rec}(\Sigma, L)$ is closed under

- scalar multiplications (cf. Theorem 10.2.1),
- Hadamard product (cf. Theorem 10.4.1),
- top-concatenations (cf. Corollary 10.5.2),
- tree concatenations (cf. Corollary 10.6.2),
- Kleene stars (cf. Corollary 10.7.6), and
- yield-intersection (cf. Theorem 10.8.2).
(3) The sets $\operatorname{Rec}^{\text {run }}\left(\Sigma,,_{-}\right)$and $\operatorname{Rec}^{\text {init }}\left(\Sigma,,_{-}\right)$are closed under homomorphisms between two bounded lattices (cf. Theorem 10.9.3).
(4) The set $\operatorname{Rec}^{\text {run }}(-, \mathrm{L})$ is closed under
- tree relabelings (cf. Theorem 10.10.1) and
- linear, nondeleting, and productive tree homomorphisms (cf. Corollary 10.11.2).
(5) If $L$ is distributive, then the set $\operatorname{Rec}(-, L)$ is closed under
- inverse of linear tree homomorphisms (cf. Theorem 10.12.2),
- weighted projective bimorphisms (cf. Corollary 10.13.10).


### 19.6 Characterization by rational operations

We recall that an operation on the set of $(\Sigma, \mathrm{L})$-weighted tree languages is a rational operation if it is the sum, a tree concatenation, or a Kleene-star. Moreover, the set of rational ( $\Sigma, \mathrm{L}$ )-weighted tree languages, denoted by $\operatorname{Rat}(\Sigma, \mathrm{L})$, is the smallest set of $(\Sigma, \mathrm{L})$-weighted tree languages which contains each polynomial $(\Sigma, \mathrm{L})$-weighted tree language and is closed under the rational operations.

Due to the need for extra symbols at which tree concatenation can take place (cf. [TW68, Sect. 3]), we have introduced in Section 12.1 the concept of 0 -extension and defined the sets of extended rational ( $\Sigma, \mathrm{L}$ )weighted tree languages, denoted by $\operatorname{Rat}(\Sigma, \mathrm{L})^{\text {ext }}$, and the set of extended recognizable $(\Sigma, \mathrm{L})$-weighted tree languages, denoted by $\operatorname{Rec}(\Sigma, L)^{\text {ext }}$, which formalize these extensions.

Then Theorem 12.1 .2 implies the following Kleene theorem for distributive bounded lattices.
Corollary 19.6.1. Let $\Sigma$ be a ranked alphabet and $L$ a distributive bounded lattice. Then

$$
\operatorname{Rec}(\Sigma, \mathrm{L})^{\mathrm{ext}}=\operatorname{Rat}(\Sigma, \mathrm{L})^{\mathrm{ext}}
$$

### 19.7 Characterization by elementary operations

Since each run recognizable weighted tree language is a recognizable step mapping (cf. Corollary 19.2.1), we can show that the notions of representable, restricted representable, and $\times$-restricted representable (as defined in Chapter 13) coincide. This results in Médvédév's theorem for wta over bounded lattices.

Theorem 19.7.1. Let $\Sigma$ be a ranked alphabet, L be a bounded lattice, and $r: \mathrm{T}_{\Sigma} \rightarrow L$. Then the following four statements are equivalent.
(A) We can construct a $(\Sigma, \mathrm{L})-w t a \mathcal{A}$ such that $r=\llbracket \mathcal{A} \rrbracket^{\text {run }}$.
(B) We can construct an $e \in \operatorname{Rep} \operatorname{Ex}(\Sigma, \mathrm{~L})$ such that $r=\llbracket e \rrbracket$.
(C) We can construct an $e \in \operatorname{RepEx}^{r}(\Sigma, \mathrm{~L})$ such that $r=\llbracket e \rrbracket$.
(D) We can construct an $e \in \operatorname{RepEx}^{\times r}(\Sigma, \mathrm{~L})$ such that $r=\llbracket e \rrbracket$.

Proof. The implication $(\mathrm{A}) \Rightarrow(\mathrm{D})$ holds by Lemma 13.2.2. The implications $(\mathrm{D}) \Rightarrow(\mathrm{C})$ and $(\mathrm{C}) \Rightarrow(\mathrm{B})$ hold by definition.

Proof of $(\mathrm{B}) \Rightarrow(\mathrm{A})$ : By induction on $(\operatorname{RepEx}(\mathrm{L}), \prec)$ we prove the following statement:

$$
\text { For each } e \in \operatorname{Rep} \operatorname{Ex}(\mathrm{~L}) \text {, we can construct a }(\Sigma, \mathrm{L}) \text {-wta } \mathcal{A} \text { such that } \llbracket e \rrbracket=\llbracket \mathcal{A} \rrbracket^{\mathrm{run}} \text {. }
$$

I.B.: For the case that $e=\mathrm{RT}_{\Sigma, \sigma, b}$ or $e=\mathrm{NXT}_{\Sigma, \widetilde{\gamma}, b}$, the statement follows from Corollary 13.1.3 and Theorem 10.3.1 $(\mathrm{B}) \Rightarrow(\mathrm{A})$.
I.S.: We distinguish four cases.

Case (a): Let $e=e_{1}+e_{2}$. Then the statement follows from the I.H. and Theorem 10.1.1(1).
Case (b): Let $e=e_{1} \times e_{2}$. By I.H. we can construct $(\Sigma, \mathrm{L})$-wta $\mathcal{A}_{1}$ and $\mathcal{A}_{2}$ such that $\llbracket e_{1} \rrbracket=\llbracket \mathcal{A}_{1} \rrbracket^{\text {run }}$ and $\llbracket e_{2} \rrbracket=\llbracket \mathcal{A}_{2} \rrbracket^{\text {run }}$. By Corollary $19.2 .1(1)$, we can construct crisp deterministic $(\Sigma, \mathrm{L})$-wta $\mathcal{A}_{1}^{\prime}$ and $\mathcal{A}_{2}^{\prime}$ such that $\llbracket \mathcal{A}_{1} \rrbracket^{\text {run }}=\llbracket \mathcal{A}_{1}^{\prime} \rrbracket$ and $\llbracket \mathcal{A}_{2} \rrbracket^{\text {run }}=\llbracket \mathcal{A}_{2}^{\prime} \rrbracket$. Then the statement follows from Theorem $10.4 .1(3)$.

Case (c): Let $e=\tau\left(e^{\prime}\right)$ where $\tau$ is a $(\Delta, \Sigma)$-tree relabeling. Then the statement follows from the I.H. and Theorem 10.10.1

Case (d): Let $e=\operatorname{REST}\left(e^{\prime}\right)$. By I.H. we can construct a $(\Sigma, \mathrm{L})$-wta $\mathcal{A}^{\prime}$ such that $\llbracket \mathcal{A}^{\prime} \rrbracket$ run $=\llbracket e^{\prime} \rrbracket$. By Corollary 19.2 .1 , we can construct a crisp deterministic $(\Sigma, \mathrm{L})$-wta $\mathcal{B}$ such that $\llbracket \mathcal{B} \rrbracket=\llbracket \mathcal{A}^{\prime} \rrbracket^{\text {run }}$. By Lemma 13.3.1, we can construct a $(\Sigma, \mathrm{L})$-wta $\mathcal{A}$ such that $\llbracket \mathcal{A} \rrbracket^{\text {run }}=\llbracket \operatorname{REST}\left(e^{\prime}\right) \rrbracket$.

### 19.8 Characterization by weighted MSO-logic

In this situation where $L$ is a bounded lattice, we can prove that run recognizability of a $(\Sigma, L)$-weighted tree language is equivalent to its definability in $\mathrm{MSO}^{\text {ext }}(\Sigma, \mathrm{L})$, and the transformations from wta to formula and vice versa are constructive (cf. Theorem 19.8.4). We recall the syntax of the formulas in $\mathrm{MSO}^{\text {ext }}(\Sigma, \mathrm{L})$ from Section 14.4 It was given by the following EBNF with nonterminal $e$ :

$$
\begin{equation*}
e::=\mathrm{H}(\kappa)|(\varphi \triangleright e)|(e+e)|(e \times e)|+_{x} e\left|+_{X} e\right| X_{x} e \mid X_{X} e \tag{19.5}
\end{equation*}
$$

In particular, no restriction is needed on the use of weighted conjunction, weighted first-order universal quantification, and weighted second-order universal quantification.

For the proof of Theorem 19.8.4 in particular, we have to prove that weighted first-order universal quantification and weighted second-order universal quantification preserve run recognizability.

Lemma 19.8.1. Let $e \in \operatorname{MSO}^{\text {ext }}(\Sigma, \mathrm{L})$ with $\mathcal{U}=\operatorname{Free}(e)$ and $x$ any first-order variable. If there exists a $\left(\Sigma_{\mathcal{U}}, \mathrm{L}\right)$-wta $\mathcal{A}$ with $\llbracket \mathcal{A} \rrbracket^{\text {run }}=\llbracket e \rrbracket$, then we can construct a $\left(\Sigma_{\mathcal{V}}, \mathrm{L}\right)$-wta $\mathcal{B}$ such that $\mathcal{V}=\operatorname{Free}\left(X_{x} e\right)$ and $\llbracket \mathcal{B} \rrbracket^{\text {run }}=\llbracket X_{x} e \rrbracket$.

Proof. Let $\mathcal{A}$ be a $\left(\Sigma_{\mathcal{U}}, \mathrm{L}\right)$-wta such that $\llbracket \mathcal{A} \rrbracket^{\text {run }}=\llbracket e \rrbracket$. By Lemma 14.3.4 we can construct a $\left(\Sigma_{\mathcal{U} \cup\{x\}}, \mathrm{L}\right)$ wta $\mathcal{A}^{\prime}$ such that $\llbracket \mathcal{A}^{\prime} \rrbracket^{\text {run }}=\llbracket e \rrbracket_{\mathcal{U} \cup\{x\}}$. Since $\mathcal{U} \cup\{x\}=\mathcal{V} \cup\{x\}$, we have that $\llbracket \mathcal{A}^{\prime} \rrbracket^{\text {run }}=\llbracket e \rrbracket_{\mathcal{V} \cup\{x\}}$.

Since L is a bounded lattice, $\llbracket \mathcal{A}^{\prime} \rrbracket^{\text {run }}$ is a recognizable step mapping. More precisely, by Corollary 19.2.3. we can construct $n \in \mathbb{N}, b_{1}, \ldots, b_{n} \in L$, and $\Sigma_{\mathcal{V} \cup\{x\}}$ - $\mathrm{fta} A_{1}, \ldots, A_{n}$ such that

$$
\llbracket \mathcal{A}^{\prime} \rrbracket^{\mathrm{run}}=\bigvee_{j \in[n]} b_{j} \wedge \chi\left(\mathrm{~L}\left(A_{j}\right)\right)
$$

Then by Lemma 14.4 .16 we can construct a $\left(\Sigma_{\mathcal{V}}, \mathrm{L}\right)$-wta $\mathcal{B}$, where $\mathcal{V}=\operatorname{Free}\left(X_{x} e\right)$, such that $\llbracket \mathcal{B} \rrbracket^{\text {run }}=$ $\llbracket X_{x} e \rrbracket$.

As preparation for the second preservation result, we state that the consistency lemma for wta (cf. Lemma 14.3.4) also holds if we replace $\operatorname{MSO}(\Sigma, \mathrm{L})$ by $\operatorname{MSO}^{\text {ext }}(\Sigma, \mathrm{L})$.

Lemma 19.8.2. Let $\mathcal{V}$ be a finite set of variables, $e \in \operatorname{MSO}^{\text {ext }}(\Sigma, \mathrm{L})$ with $\operatorname{Free}(e)=\mathcal{V}$, and $\mathcal{A}$ be a $\left(\Sigma_{\mathcal{V}}, \mathrm{L}\right)$-wta such that $\llbracket \mathcal{A} \rrbracket^{\text {run }}=\llbracket e \rrbracket$. Moreover, let $V$ be a first-order variable or a second-order variable. Then we can construct a $\left(\Sigma_{\mathcal{V} \cup\{V\}}, \mathrm{L}\right)$-wta $\mathcal{A}^{\prime}$ such that $\llbracket \mathcal{A}^{\prime} \rrbracket^{\text {run }}=\llbracket e \rrbracket \mathcal{V} \cup\{V\}$.

Proof. The proof is the same as the one for Lemma 14.3.4 except that in Case (b)(ii) we have to use the consistency lemma for $\operatorname{MSO}^{\text {ext }}(\Sigma, \mathrm{L})$ (i.e., Lemma 14.4.1) instead of Lemma 14.2.1.

Lemma 19.8.3. Let $e \in \operatorname{MSO}^{\text {ext }}(\Sigma, \mathrm{L})$ with $\mathcal{U}=\operatorname{Free}(e)$ and $X$ any second-order variable. If there exists a $\left(\Sigma_{\mathcal{U}}, \mathrm{L}\right)$-wta $\mathcal{A}$ with $\llbracket \mathcal{A} \rrbracket^{\text {run }}=\llbracket e \rrbracket$, then we can construct a crisp deterministic $\left(\Sigma_{\mathcal{V}}, \mathrm{L}\right)$-wta $\mathcal{B}$ such that $\mathcal{V}=\operatorname{Free}\left(X_{X} e\right)$ and $\llbracket \mathcal{B} \rrbracket=\llbracket X_{X} e \rrbracket$.

Proof. Let $\zeta \in \mathrm{T}_{\Sigma_{\mathcal{V}}}$. By definition, we have that

$$
\llbracket X_{X} e \rrbracket(\zeta)= \begin{cases}\bigwedge_{W \subseteq \operatorname{pos}(\zeta)} \llbracket e \rrbracket \mathcal{V} \cup\{X\} \\ 0 & \text { if } \zeta\left[X \mapsto \mathrm{~T}_{\Sigma_{\mathcal{V}}}^{\mathrm{v}}\right. \\ & \text { otherwise }\end{cases}
$$

which is equivalent to

$$
\llbracket X_{X} e \rrbracket(\zeta)=\left(\bigwedge_{W \subseteq \operatorname{pos}(\zeta)} \llbracket e \rrbracket_{\mathcal{V} \cup\{X\}}(\zeta[X \mapsto W])\right) \wedge \chi\left(\mathrm{T}_{\Sigma_{\mathcal{V}}}^{\mathrm{v}}\right)(\zeta)
$$

As auxiliary tool, we define the deterministic $\left(\Sigma_{\mathcal{V} \cup\{X\}}, \Sigma_{\mathcal{V}}\right)$-tree relabeling $\tau=\left(\tau_{k} \mid k \in \mathbb{N}\right)$ with $\tau_{k}((\sigma, \mathcal{W}))=(\sigma, \mathcal{W} \backslash\{X\})$ for each $(\sigma, \mathcal{W}) \in \Sigma_{\mathcal{V} \cup\{X\}}$. Then we have

$$
\llbracket X_{X} e \rrbracket(\zeta)=\left(\bigwedge_{\xi \in \tau^{-1}(\zeta)} \llbracket e \rrbracket \mathcal{V} \cup\{X\}(\xi)\right) \wedge \chi\left(\mathrm{T}_{\Sigma_{\mathcal{V}}}^{\mathrm{v}}\right)(\zeta)
$$

Let $\mathcal{A}$ be a $\left(\Sigma_{\mathcal{U}}, \mathrm{L}\right)$-wta with $\llbracket \mathcal{A} \rrbracket^{\text {run }}=\llbracket e \rrbracket$. By Lemma 19.8.2, we can construct a $\left(\Sigma_{\mathcal{U} \cup\{X\}}, \mathrm{L}\right)$-wta $\mathcal{A}^{\prime}$ such that $\llbracket \mathcal{A}^{\prime} \rrbracket^{\text {run }}=\llbracket e \rrbracket_{\mathcal{U} \cup\{X\}}$. Since $\mathcal{U} \cup\{X\}=\mathcal{V} \cup\{X\}$, we have

$$
\llbracket X_{X} e \rrbracket(\zeta)=\left(\bigwedge_{\xi \in \tau^{-1}(\zeta)} \llbracket \mathcal{A}^{\prime} \rrbracket^{\mathrm{run}}(\xi)\right) \wedge \chi\left(\mathrm{T}_{\Sigma_{\mathcal{V}}}^{\mathrm{v}}\right)(\zeta)
$$

Since L is a bounded lattice, $\llbracket \mathcal{A}^{\prime} \rrbracket^{\text {run }}$ is a recognizable step mapping. More precisely, by Corollary 19.2.3, we can construct $n \in \mathbb{N}, b_{1}, \ldots, b_{n} \in L$, and $\Sigma_{\mathcal{V} \cup\{X\}}$-fta $A_{1}, \ldots, A_{n}$ such that

$$
\llbracket \mathcal{A}^{\prime} \rrbracket^{\mathrm{run}}=\bigvee_{i \in[n]} b_{i} \wedge \chi\left(\mathrm{~L}\left(A_{i}\right)\right)
$$

Moreover, by Observation 2.14.1, we can assume that the family ( $\mathrm{L}\left(A_{i}\right) \mid i \in[n]$ ) is a partitioning of $\mathrm{T}_{\Sigma_{\mathcal{V} \cup\{X\}}}$. Thus

$$
\llbracket X_{X} e \rrbracket(\zeta)=\left(\bigwedge_{\xi \in \tau^{-1}(\zeta)} \bigvee_{i \in[n]} b_{i} \wedge \chi\left(\mathrm{~L}\left(A_{i}\right)\right)(\xi)\right) \wedge \chi\left(\mathrm{T}_{\Sigma_{\mathcal{V}}}^{\mathrm{v}}\right)(\zeta)
$$

Since the tree languages $\mathrm{L}\left(A_{1}\right), \ldots, \mathrm{L}\left(A_{n}\right)$ are pairwise disjoint and $\wedge$ is idempotent, we obtain

$$
\llbracket X_{X} e \rrbracket(\zeta)=\left(\bigwedge_{i \in[n]} m_{i}(\zeta)\right) \wedge \chi\left(\mathrm{T}_{\Sigma_{\mathcal{V}}}^{\mathrm{v}}\right)(\zeta)
$$

where, for each $i \in[n]$, the mapping $m_{i}: \mathrm{T}_{\Sigma_{\mathcal{V}}} \rightarrow L$ is defined, for each $\theta \in \mathrm{T}_{\Sigma_{\mathcal{V}}}$, by

$$
m_{i}(\theta)= \begin{cases}b_{i} & \text { if } \tau^{-1}(\theta) \cap \mathrm{L}\left(A_{i}\right) \neq \emptyset \\ \mathbb{1} & \text { otherwise }\end{cases}
$$

Since $\tau\left(\mathrm{L}\left(A_{i}\right)\right)=\left\{\theta \in \mathrm{T}_{\Sigma_{\mathcal{V}}} \mid \tau^{-1}(\theta) \cap \mathrm{L}\left(A_{i}\right) \neq \emptyset\right\}$, the mapping $m_{i}$ can be written as the polynomial weighted tree language

$$
m_{i}=\left(b_{i} \wedge \chi_{\mathrm{L}}\left(\tau\left(\mathrm{~L}\left(A_{i}\right)\right)\right)\right) \vee\left(\mathbb{1} \wedge \chi_{\mathrm{L}}\left(\mathrm{~T}_{\Sigma_{\mathcal{V}}} \backslash \tau\left(\mathrm{L}\left(A_{i}\right)\right)\right)\right)
$$

By Corollary 10.10 .2 (by instantiating $\Sigma$ and $\Delta$ by $\Sigma_{\mathcal{V} \cup\{X\}}$ and $\Sigma_{\mathcal{V}}$, respectively), for each $i \in[n]$, we can construct a $\Sigma_{\mathcal{V}}$-fta $B_{i}$ such that $\mathrm{L}\left(B_{i}\right)=\tau\left(\mathrm{L}\left(A_{i}\right)\right)$. Thus we have

$$
m_{i}=\left(b_{i} \wedge \chi_{\mathrm{L}}\left(\mathrm{~L}\left(B_{i}\right)\right)\right) \vee\left(\mathbb{1} \wedge \chi_{\mathrm{L}}\left(\mathrm{~T}_{\Sigma_{\mathcal{V}}} \backslash \mathrm{L}\left(B_{i}\right)\right)\right)
$$

By Theorem 2.13.3, we can construct a $\Sigma_{\mathcal{V}}$-fta $\overline{B_{i}}$ such that $\mathrm{L}\left(\overline{B_{i}}\right)=\mathrm{T}_{\Sigma_{\mathcal{V}}} \backslash \mathrm{L}\left(B_{i}\right)$. Thus

$$
m_{i}=\left(b_{i} \wedge \chi_{\mathrm{L}}\left(\mathrm{~L}\left(B_{i}\right)\right)\right) \vee\left(\mathbb{1} \wedge \chi_{\mathrm{L}}\left(\mathrm{~L}\left(\overline{B_{i}}\right)\right)\right)
$$

and hence $m_{i}$ is a recognizable step mapping.
By Corollary 10.4.2, $\bigwedge_{i \in[n]} m_{i}$ is a recognizable step mapping. Hence by Theorem $10.3 .1(\mathrm{~B}) \Rightarrow(\mathrm{A})$ we can construct a crisp deterministic $\left(\Sigma_{\mathcal{V}}, \mathrm{L}\right)$-wta $\mathcal{C}$ such that $\llbracket \mathcal{C} \rrbracket=\bigwedge_{i \in[n]} m_{i}$. Thus we obtain

$$
\llbracket X_{X} e \rrbracket(\zeta)=\llbracket \mathcal{C} \rrbracket(\zeta) \wedge \chi\left(T_{\Sigma_{\mathcal{V}}}^{\mathrm{v}}\right)(\zeta)
$$

By Lemma 14.1.5, we can construct a $\Sigma_{\mathcal{V}}$-fta $D$ such that $\mathrm{L}(D)=\mathrm{T}_{\Sigma_{\mathcal{V}}}^{v}$. Then by Theorem 10.4 .3 (2), we can construct a crisp deterministic $\left(\Sigma_{\mathcal{V}}, \mathrm{L}\right)$-wta $\mathcal{B}$ such that $\llbracket \mathcal{B} \rrbracket=\llbracket \mathcal{C} \rrbracket \wedge \chi(\mathrm{L}(D))$. Hence $\llbracket \mathcal{B} \rrbracket=\llbracket \mathrm{X}_{X} e \rrbracket$.

Now we can prove B-E-T's theorem for wta over arbitrary bounded lattices.
Theorem 19.8.4. Let $\Sigma$ be a ranked alphabet, L a bounded lattice, and $r: \mathrm{T}_{\Sigma} \rightarrow L$. Then the following four statements are equivalent.
(A) We can construct $a(\Sigma, \mathrm{~L})$-wta $\mathcal{B}$ such that $r=\llbracket \mathcal{B} \rrbracket^{\text {run }}$.
(B) We can construct a $(\Sigma, \mathrm{L})$-recognizable step formula e such that $\operatorname{Free}(e)=\emptyset$ and $r=\llbracket e \rrbracket$.
(C) We can construct a sentence $e \in \operatorname{MSO}(\Sigma, \mathrm{~L})$ such that $r=\llbracket e \rrbracket$
(D) We can construct a sentence $e \in \operatorname{MSO}^{\operatorname{ext}}(\Sigma, \mathrm{L})$ such that $r=\llbracket e \rrbracket$.

Proof. Proof of $(\mathrm{A}) \Rightarrow(\mathrm{B})$ : Let $\mathcal{B}$ be a $(\Sigma, \mathrm{L})$-wta such that $r=\llbracket \mathcal{B} \rrbracket^{\text {run }}$. By Corollary 19.2.3, $r$ is a recognizable step mapping and we can construct $n \in \mathbb{N}_{+}, b_{1}, \ldots, b_{n} \in L$, and $\Sigma$-fta $A_{1}, \ldots, A_{n}$ such that $r=\bigvee_{i \in[n]} b_{i} \wedge \chi\left(\mathrm{~L}\left(A_{i}\right)\right)$. By Lemma 14.2.3, we can construct sentences $\varphi_{1}, \ldots, \varphi_{n}$ in $\operatorname{MSO}(\Sigma)$ such that $r=\llbracket\left(\varphi_{1} \triangleright\left\langle b_{1}\right\rangle\right)+\ldots+\left(\varphi_{n} \triangleright\left\langle b_{n}\right\rangle\right) \rrbracket$. Then we let $e=\left(\varphi_{1} \triangleright\left\langle b_{1}\right\rangle\right)+\ldots+\left(\varphi_{n} \triangleright\left\langle b_{n}\right\rangle\right)$.

Proof of $(\mathrm{B}) \Rightarrow(\mathrm{C})$ : Since each $(\Sigma, \mathrm{L})$-recognizable step formula is in $\operatorname{MSO}(\Sigma, \mathrm{L})$, this is trivial.
Proof of $(C) \Rightarrow(D)$ : Since $\operatorname{MSO}(\Sigma, L) \subset \operatorname{MSO}^{\text {ext }}(\Sigma, L)$, this is trivial.
Proof of $(\mathrm{D}) \Rightarrow(\mathrm{A})$ : We follow the proof of Theorem 14.3 .8 and extend it appropriately. Formally, by induction on $\left(\mathrm{MSO}^{\mathrm{ext}}(\Sigma, \mathrm{L}), \prec_{\mathrm{MSO}^{\mathrm{ext}}}(\Sigma, \mathrm{L})\right.$ ), we prove the following statement.

$$
\begin{align*}
& \text { For every } e \in \operatorname{MSO}^{\text {ext }}(\Sigma, \mathrm{L}) \text { and } \mathcal{V}=\operatorname{Free}(e) \\
& \text { we can construct a }\left(\Sigma_{\mathcal{V}}, \mathrm{L}\right) \text {-wta } \mathcal{B} \text { such that } \llbracket \mathcal{B} \rrbracket^{\text {run }}=\llbracket e \rrbracket . \tag{19.6}
\end{align*}
$$

The proofs for the cases that $e=\mathrm{H}(\kappa), e=\left(\varphi \triangleright e^{\prime}\right), e=e_{1}+e_{2}, e=+_{x} e^{\prime}$, or $e=+_{X} e^{\prime}$ are the same as in the proof of Theorem 14.3.8 (except that for $e_{1}+e_{2}$ we use Lemma 19.8.2 instead of Lemma 14.3 .4 .

Let $e=e_{1} \times e_{2}$. Let $\mathcal{V}_{1}=\operatorname{Free}\left(e_{1}\right)$ and $\mathcal{V}_{2}=\operatorname{Free}\left(e_{2}\right)$. By I.H. we can construct a $\left(\Sigma_{\mathcal{V}_{1}}, \mathrm{~L}\right)$-wta $\mathcal{A}_{1}$ such that $\llbracket \mathcal{A}_{1} \rrbracket^{\text {run }}=\llbracket e_{1} \rrbracket$. By iterated application of Lemma 19.8.2, we can construct a $\left(\Sigma_{\mathcal{V}}, \mathrm{L}\right)$-wta $\mathcal{A}_{1}^{\prime}$ such that $\llbracket \mathcal{A}_{1}^{\prime} \rrbracket^{\text {run }}=\llbracket e_{1} \rrbracket \mathcal{V}$. By Theorem 16.2 .7 we can construct a crisp deterministic $\left(\Sigma_{\mathcal{V}}, \mathrm{L}\right)$-wta $\mathcal{A}_{1}^{\prime \prime}$ such that $\llbracket \mathcal{A}_{1}^{\prime \prime} \rrbracket=\llbracket e_{1} \rrbracket \mathcal{V}$. In the same way we can construct a crisp deterministic $\left(\Sigma_{\mathcal{V}}, \mathrm{L}\right)$-wta $\mathcal{A}_{2}^{\prime \prime}$ such that $\llbracket \mathcal{A}_{2}^{\prime \prime} \rrbracket=\llbracket e_{2} \rrbracket \mathcal{V}$. By Theorem 10.4.1 $(3)$, we can construct a crisp deterministic $\left(\Sigma_{\mathcal{V}}, \mathrm{L}\right)$-wta $\mathcal{B}$ such that $\llbracket \mathcal{B} \rrbracket=\llbracket \mathcal{A}_{1}^{\prime \prime} \rrbracket \otimes \llbracket \mathcal{A}_{2}^{\prime \prime} \rrbracket$. Thus $\llbracket \mathcal{B} \rrbracket=\llbracket e_{1} \times e_{2} \rrbracket$.

Let $e=X_{x} e^{\prime}$. By I.H. and Lemma 19.8.1 we can construct a $\left(\Sigma_{\mathcal{V}}, \mathrm{L}\right)$-wta $\mathcal{B}$ such that $\llbracket \mathcal{B} \rrbracket^{\text {run }}=\llbracket e \rrbracket$.
Let $e=X_{X} e^{\prime}$. By I.H. and Lemma 19.8 .3 , we can construct a $\left(\Sigma_{\mathcal{V}}, \mathrm{L}\right)$-wta $\mathcal{B}$ such that $\llbracket \mathcal{B} \rrbracket^{\text {run }}=$ $\llbracket e \rrbracket$.

We note that in [DV12, Thm. 5.3] a characterization of weighted string languages recognizable over bi-locally finite commutative strong bimonoids in terms of the unrestricted weighted MSO-logic of DG05, DG07, DG09 was proved. When comparing the version of Theorem 19.8.4 in which $\Sigma$ is a string ranked alphabet with [DV12, Thm. 5.3], one has to handle the differences between $\operatorname{MSO}^{\text {ext }}(\Sigma, \mathrm{B})$ and the weighted MSO-logic of DG05, DG07, DG09, DV12; in FSV12, Sec. 5] syntactic transformations for both directions are shown.

### 19.9 Abstract families of weighted tree languages

In Chapter 15 we have defined and investigated abstract families of weighted tree languages over commutative and $\sigma$-complete semirings. Since each $\sigma$-complete lattice is a commutative and $\sigma$-complete semiring, we can immediately apply the main result of that chapter (cf. Theorem 15.4.5) to the case of $\sigma$-complete lattices.

For this, we recall that, for each $n \in \mathbb{N}$, an $(n, \mathrm{~L})$-weighted tree language is a $(\Sigma, \mathrm{L})$-weighted tree language for some ranked alphabet $\Sigma$ with $\operatorname{maxrk}(\Sigma) \leq n$, and $\operatorname{Reg}(n, \mathrm{~L})$ denotes the set of all regular $(n, \mathrm{~L})$-weighted tree languages. Moreover, a family $\mathcal{L}$ of $(n, \mathrm{~L})$-weighted tree languages is an abstract family of ( $n, \mathrm{~L}$ )-weighted tree languages if $\mathcal{L}$ is an $(n, \mathrm{~L})$-tree cone which is closed under the rational operations, i.e., under sum, tree concatenations, and Kleene-stars.

Corollary 19.9.1. Let L be a $\sigma$-complete lattice and $n \in \mathbb{N}$. Then $\operatorname{Reg}(n, \mathrm{~L})$ is the smallest principal abstract family of $(n, \mathrm{~L})$-weighted tree languages.

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[^0]:    ${ }^{1}$ Since we will not deal with convergence questions, we prefer the latter notion.

[^1]:    ${ }^{2}$ We draw a tree with its root up and the leaves down, hence bottom-up means: from the leaves towards the root.

[^2]:    ${ }^{1}$ not to be confused with the set $[0,1]=\{0,1\}$ of natural numbers.

[^3]:    ${ }^{2}$ Stb refers to one of the authors

[^4]:    ${ }^{3}$ We note that the $\mathbb{O}$ of the semiring $B$ can be different from the 0 of the monoid $V$.

[^5]:    ${ }^{4}\left(\Gamma^{*}, \widehat{\Gamma_{e}}\right)$ should not be confused with the free monoid $\left(\Gamma^{*}, \cdot, \varepsilon\right)$.
    ${ }^{5}$ This is exactly the binary relation which we have used to prove that the $\Sigma$-term algebra over $H$ is freely generated by $H$ over the set of all $\Sigma$-algebras; since we will use it often, we give it a specific denotation.

[^6]:    ${ }^{6}$ For each $k \in \mathbb{N}$ with $k>\operatorname{maxrk}(\Sigma)$ we have $\delta_{k}=\emptyset$, because $\Sigma^{(k)}=\emptyset$.

[^7]:    ${ }^{7}$ When citing results of GS84 we use the numbers in the arXiv-version of that book.
    ${ }^{8}$ When citing results of Eng75b we use the numbers in the arXiv-version of those lecture notes.

[^8]:    ${ }^{1}$ For each $k \in \mathbb{N}$ with $k>\operatorname{maxrk}(\Sigma)$ we have $\delta_{k}: \emptyset \rightarrow B$, because $\Sigma^{(k)}=\emptyset$.

[^9]:    ${ }^{2}$ For instance, Equations 11.4 and 11.5 show that the run alone does not determine the value of wt.

[^10]:    ${ }^{3}$ For the case of wsa, in CDIV10, Prop. 6.2] the relation $\operatorname{ker}\left(\mathrm{h}_{\mathcal{A}}\right)$ was defined and called "Nerode right congruence". We refrain from also calling our relation $\operatorname{ker}\left(\mathrm{h}_{\mathcal{A}}\right)$ "Nerode congruence relation", because the Nerode congruence relation from classical formal language theory means something else: it is based on a language $L \subseteq \Gamma^{*}$ and defines two strings $w_{1}$ and $w_{2}$ to be equivalent if for each $u \in \Gamma^{*}: w_{1} u \in L$ iff $w_{2} u \in L$ (also cf. Koz92 for tree languages). Our $\operatorname{ker}\left(\mathrm{h}_{\mathcal{A}}\right)$ is of different nature.

[^11]:    ${ }^{4}$ called 'multilinear representation of $\mathrm{T}_{\Sigma}$ ' in BR82

[^12]:    ${ }^{1}$ This does not mean that $\mathcal{A}$ and $\mathcal{B}$ are i-equivalent because the latter concept is defined for wta over the same strong bimonoid.

[^13]:    ${ }^{1}$ We need this condition because we wish to define the semantics of a wcfg in terms of rule trees (cf. Section 2.12)

[^14]:    ${ }^{2}$ The multiplication • of matrices is associative only if the multiplication $\otimes$ of $B$ distributes over $\oplus$.

[^15]:    ${ }^{1}$ where "'s" distributes over "-"

[^16]:    ${ }^{1}$ We recall that $\llbracket \mathcal{A} \rrbracket=\llbracket \mathcal{A} \rrbracket^{\text {init }}=\llbracket \mathcal{A} \rrbracket^{\text {run }}$ because B is a semiring.

[^17]:    ${ }^{2}$ Indeed, we obtained $\mathcal{A}$ by starting from the bu deterministic wta shown in Example 3.2 .3 and by "unfolding" the transition on $\sigma$.

[^18]:    ${ }^{1}$ (We note that, by Observation 2.6.11(7), $\sigma$-completeness implies zero-sum freeness.)

