On c-embedded subgroups of finite groups

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Abstract

Let G be a group and $H \leqslant K \leqslant G$. We say that H is *c-embedded* in G with respect to K if there is a subgroup B of G such that $G = HB$ and $H \cap B \leqslant Z(K)$. Given a finite group G, a prime number p and a Sylow p-subgroup P of G, we investigate the structure of G under the assumption that $N_G(P)$ is p-supersolvable or p-nilpotent and that certain cyclic subgroups of P with order p or 4 are c-embedded in G with respect to P. New characterizations of p-supersolvability and p-nilpotence of finite groups will be obtained.

Keywords: finite groups; p-supersolvable; p-nilpotent; c-embedded.
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1 Introduction

All groups in this paper are implicitly assumed to be finite. We use standard notation and terminology, see for example [10] or [11]. Throughout, p denotes an arbitrary but fixed prime.

Recall that a group G is said to be *p-nilpotent* if G has a normal Hall p'-subgroup. This concept plays an important role in local finite group theory, and many criteria for p-nilpotence of finite groups can be found in the literature. For example, a well-known theorem of Burnside asserts that a group G with Sylow p-subgroup P is p-nilpotent if $P \leqslant Z(N_G(P))$, or equivalently if P is abelian and $N_G(P)$ is p-nilpotent (see [11, Theorem 5.13]). This result has been generalized in many directions, and we shall now consider some of these generalizations.

First let us introduce some notation. Let P be a p-group and i be a positive integer. Then the subgroup of P generated by all elements x of P with $x^{p^i} = 1$ is denoted by $\Omega_i(P)$. We set $\Omega(P) := \Omega_1(P)$ if p is odd and $\Omega(P) := \Omega_2(P)$ if $p = 2$.

In 1974, Laffey [12] published the following generalization of Burnside's p-nilpotency criterion: If G is a group and P is a Sylow p-subgroup of G, then G is p-nilpotent if $\Omega(P) \leqslant Z(P)$ and $N_G(P)$ is p-nilpotent (see [12, p. 136]).

In 2000, Ballester-Bolinches and Guo [3] published the following more general result: If G is a group and P is a Sylow p-subgroup of G, then G is p-nilpotent if $\Omega(P \cap G') \leqslant Z(N_G(P))$ (see [3, Theorem 1]). Moreover, they proved that a group G with Sylow 2-subgroup P is

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2-nilpotent if $\Omega_1(P \cap G') \leqslant Z(P)$, P is quaternion-free and $N_G(P)$ is 2-nilpotent (see [3, Theorem 2]). Here, a group is said to be quaternion-free if it has no section isomorphic to the quaternion group of order 8.

For a non-empty formation $\mathfrak{F}$ and a group G, we use $G^{\mathfrak{F}}$ to denote the $\mathfrak{F}$-*residual* of G, i.e. $G^{\mathfrak{F}}$ is the smallest normal subgroup of G whose quotient lies in $\mathfrak{F}$. As usual, $\mathfrak{N}$ denotes the formation of all nilpotent groups.

In 2004, Asaad [1] published the following result, which extends the above-mentioned results of Ballester-Bolinches and Guo.

Theorem 1. ([1, Theorem 1 (a) $\Leftrightarrow$ (b)]) *Let P be a Sylow p-subgroup of a group G. If $p = 2$, assume that P is quaternion-free. Then the following statements are equivalent:*

(1) *G is p-nilpotent.*

(2) *$N_G(P)$ is p-nilpotent and $\Omega_1(G^{\mathfrak{N}} \cap P \cap P^x) \leqslant Z(P)$ for all $x \in G \setminus N_G(P)$.*

Recall that a group G is said to be *p-solvable* if any chief factor of G is either a p-group or a p'-group. A p-solvable group G is called *p-supersolvable* if every p-chief factor of G has order p. The formation of all p-supersolvable groups is denoted by $\mathfrak{U}_p$. Note that every p-nilpotent group is p-supersolvable. In particular, we have $G^{\mathfrak{U}_p} \leqslant G^{\mathfrak{N}_p}$ for any group G.

By [13, Theorem 2.2], if P is a Sylow p-subgroup of a group G, then we have $P \cap G^{\mathfrak{N}} = P \cap G^{\mathfrak{N}_p}$, where $\mathfrak{N}_p$ denotes the formation of all p-nilpotent groups. Therefore, the nilpotent residual $G^{\mathfrak{N}}$ can be replaced by the p-nilpotent residual $G^{\mathfrak{N}_p}$ in Theorem 1. In view of this observation, it is natural to ask whether Theorem 1 still remains true when the nilpotent residual $G^{\mathfrak{N}}$ is replaced by the p-supersolvable residual $G^{\mathfrak{U}_p}$. We will show that $G^{\mathfrak{N}}$ cannot only be replaced by $G^{\mathfrak{U}_p}$, but that it is enough to require in (2) that the normalizer $N_G(P)$ is p-nilpotent and that every minimal subgroup of $G^{\mathfrak{U}_p} \cap P \cap P^x$ is central in P or complemented in G for all $x \in G \setminus N_G(P)$. We will also show that Theorem 1 remains true if we additionally replace "p-nilpotent" by "p-supersolvable" in both statements of the theorem and furthermore assume G to be p-solvable.

In fact, our results are slightly more general than just stated, and they also deal with the case that $P \in \mathrm{Syl}_p(G)$ is not quaternion-free when $p = 2$. In order to state our results in full generality, we introduce the following definition.

Definition 2. Let G be a group and $H \leqslant K \leqslant G$. Then H is said to be *c-embedded* in G with respect to K if there is a subgroup B of G such that $G = HB$ and $H \cap B \leqslant Z(K)$.

Let G be a group and $H \leqslant K \leqslant G$ such that H is c-embedded in G with respect to K, so that there exists a subgroup B of G with $G = HB$ and $H \cap B \leqslant Z(K)$. If $H \cap B = 1$ then H is complemented in G, and if $H \cap B = H$ then H is central in K. If $1 < H \cap B < H$, then H can be described as being between complemented in G and central in K. Note that if H is a minimal subgroup of G, then H is c-embedded in G with respect to K if and only if H is complemented in G or central in K.

Having introduced the concept of c-embedded subgroups, we can now state our main results.

Theorem A. *Let P be a Sylow p-subgroup of a p-solvable group G. Then G is p-supersolvable if and only if the following two conditions are satisfied:*

(1) $N_G(P)$ *is p-supersolvable.*

(2) *For all* $x \in G \setminus N_G(P)$, *the following hold: Any subgroup of* $G^{\mathfrak{U}_p} \cap P \cap P^x$ *with order* p *is c-embedded in* G *with respect to* P. *Moreover, if* $p = 2$ *and* $G^{\mathfrak{U}_p} \cap P \cap P^x$ *is not quaternion-free, any cyclic subgroup of* $G^{\mathfrak{U}_p} \cap P \cap P^x$ *with order* 4 *is c-embedded in* G *with respect to* P.

The p-solvability condition on G cannot be dropped in Theorem A. For example, the conditions (1) and (2) from Theorem A are satisfied for $G = A_5$ and $p = 5$ since a Sylow 5-subgroup of A_5 is cyclic of order 5 and has a normalizer of order 10, but A_5 is not 5-supersolvable.

Theorem B. *Let* P *be a Sylow* p*-subgroup of a group* G. *Then* G *is* p*-nilpotent if and only if the following two conditions are satisfied:*

(1) $N_G(P)$ *is p-nilpotent.*

(2) *For all* $x \in G \setminus N_G(P)$, *the following hold: Any subgroup of* $G^{\mathfrak{U}_p} \cap P \cap P^x$ *with order* p *is c-embedded in* G *with respect to* P. *Moreover, if* $p = 2$ *and* $G^{\mathfrak{U}_p} \cap P \cap P^x$ *is not quaternion-free, any cyclic subgroup of* $G^{\mathfrak{U}_p} \cap P \cap P^x$ *with order* 4 *is c-embedded in* G *with respect to* P.

2 Preliminaries

In this section, we collect some lemmas needed for the proofs of our main results. The following well-known result can be deduced from [2, Theorem 1 and Proposition 1].

Lemma 3. *Let* G *be a* p*-solvable minimal non-*p*-supersolvable group. Then the following hold:*

(1) $G^{\mathfrak{U}_p}$ *is a* p*-group,* $G^{\mathfrak{U}_p}$ *has exponent* p *if* p *is odd and exponent at most* 4 *if* $p = 2$.

(2) $G^{\mathfrak{U}_p}/\Phi(G^{\mathfrak{U}_p})$ *is a chief factor of* G.

Lemma 4. *Let* G *be a* p*-solvable minimal non-*p*-supersolvable group,* H *be a proper subgroup of* $G^{\mathfrak{U}_p}$ *and* B *be a subgroup of* G *such that* $G = HB$. *Then* $B = G$.

Proof. Assume for the sake of contradiction that B is a proper subgroup of G. Let M be a maximal subgroup of G such that $B \leqslant M$. Then $G^{\mathfrak{U}_p} \nleqslant M$ because otherwise $G = HB \leqslant M$. Thus $G = MG^{\mathfrak{U}_p}$.

We have $\Phi(G^{\mathfrak{U}_p}) \leqslant G^{\mathfrak{U}_p} \cap \Phi(G) \leqslant G^{\mathfrak{U}_p} \cap M$. By Lemma 3(1), $G^{\mathfrak{U}_p}$ is a p-group. Since $G^{\mathfrak{U}_p}/\Phi(G^{\mathfrak{U}_p})$ is elementary abelian, $(G^{\mathfrak{U}_p} \cap M)/\Phi(G^{\mathfrak{U}_p})$ is normal in $G^{\mathfrak{U}_p}/\Phi(G^{\mathfrak{U}_p})$. So $G^{\mathfrak{U}_p} \cap M$ is normal in $G^{\mathfrak{U}_p}$. Since $G^{\mathfrak{U}_p} \cap M$ is also normal in M and since $G = MG^{\mathfrak{U}_p}$, it follows that $G^{\mathfrak{U}_p} \cap M \trianglelefteq G$.

As $G^{\mathfrak{U}_p}/\Phi(G^{\mathfrak{U}_p})$ is a chief factor of G by Lemma 3(2), it follows that either $G^{\mathfrak{U}_p} \cap M = \Phi(G^{\mathfrak{U}_p})$ or $G^{\mathfrak{U}_p} \cap M = G^{\mathfrak{U}_p}$. In the latter case, we have $H \leqslant M$ and so $G = HB \leqslant M$, a contradiction. Thus $G^{\mathfrak{U}_p} \cap M = \Phi(G^{\mathfrak{U}_p})$.

Now we have $G^{\mathfrak{U}_p} = G^{\mathfrak{U}_p} \cap HB = H(G^{\mathfrak{U}_p} \cap B) \leqslant H(G^{\mathfrak{U}_p} \cap M) = H\Phi(G^{\mathfrak{U}_p})$. Thus $G^{\mathfrak{U}_p} = H\Phi(G^{\mathfrak{U}_p})$ and so $G^{\mathfrak{U}_p} = H$. This is a contradiction since H is assumed to be a proper subgroup of $G^{\mathfrak{U}_p}$. So we have $B = G$. □

Lemma 5. *Let G be a group and L be a subgroup of G. Then $L^{\mathfrak{U}_p} \leqslant G^{\mathfrak{U}_p}$.*

Proof. Set $L_0 := L \cap G^{\mathfrak{U}_p}$. Then $L/L_0 \cong LG^{\mathfrak{U}_p}/G^{\mathfrak{U}_p} \leqslant G/G^{\mathfrak{U}_p}$. As subgroups of p-supersolvable groups are p-supersolvable, it follows that L/L_0 is p-supersolvable. Thus $L^{\mathfrak{U}_p} \leqslant L_0 \leqslant G^{\mathfrak{U}_p}$. □

Lemma 6. *Let G be a p-supersolvable group and P be a Sylow p-subgroup of G. Suppose that $N_G(P)$ is p-nilpotent. Then G is p-nilpotent.*

Proof. Set $\overline{G} := G/O_{p'}(G)$. Then $\overline{G}$ is p-supersolvable and $O_{p'}(\overline{G}) = 1$. So $\overline{G}$ is p-closed by [5, Lemma 2.1.6]. Consequently $\overline{P} \trianglelefteq \overline{G}$, and so $\overline{G} = N_{\overline{G}}(\overline{P}) = \overline{N_G(P)}$. Now, since $N_G(P)$ is p-nilpotent, we have that $\overline{G} = \overline{N_G(P)}$ is p-nilpotent. This implies that G is p-nilpotent. □

3 Proofs of Theorems A and B

Proof of Theorem A. Let G and P be as in the statement of Theorem A. Assume that G is p-supersolvable. Then, since subgroups of p-supersolvable groups are p-supersolvable, we have that $N_G(P)$ is p-supersolvable, whence condition (1) from Theorem A is satisfied. Also $G^{\mathfrak{U}_p} = 1$, so that condition (2) from Theorem A is trivially satisfied.

Suppose now that conditions (1) and (2) from Theorem A are satisfied. We have to show that G is p-supersolvable. To prove this, we assume that G is not p-supersolvable, and we are going to derive a contradiction from this assumption. Since G is not p-supersolvable, G has a minimal non-p-supersolvable subgroup, say L. Without loss of generality, we assume that $P \cap L \in \mathrm{Syl}_p(L)$. As G is p-solvable, we have that L is p-solvable, and so $L^{\mathfrak{U}_p}$ is a p-group by Lemma 3(1). In particular $L^{\mathfrak{U}_p} \leqslant P \cap L$. In order to obtain the desired contradiction, we proceed in a number of steps.

1) *Any subgroup of $L^{\mathfrak{U}_p}$ with order p is c-embedded in G with respect to P. Moreover, if $p = 2$ and $L^{\mathfrak{U}_p}$ is not quaternion-free, any cyclic subgroup of $L^{\mathfrak{U}_p}$ with order 4 is c-embedded in G with respect to P.*

Since $N_G(P)$ is p-supersolvable and L is not p-supersolvable, we have $L \not\leqslant N_G(P)$. Let $x \in L \setminus N_G(P)$. Using Lemma 5 and the fact that $L^{\mathfrak{U}_p} \trianglelefteq L$, we see that $L^{\mathfrak{U}_p} = L^{\mathfrak{U}_p} \cap (L^{\mathfrak{U}_p})^x \cap G^{\mathfrak{U}_p} \leqslant P \cap P^x \cap G^{\mathfrak{U}_p}$. Since condition (2) from Theorem A is satisfied by assumption, it follows that any subgroup of $L^{\mathfrak{U}_p}$ with order p is c-embedded in G with respect to P.

Assume that $p = 2$ and that $L^{\mathfrak{U}_p}$ is not quaternion-free. Then $P \cap P^x \cap G^{\mathfrak{U}_p}$ is not quaternion-free either, and so the validity of condition (2) from Theorem A implies that any cyclic subgroup of $L^{\mathfrak{U}_p}$ with order 4 is c-embedded in G with respect to P.

2) *If H is a proper subgroup of $L^{\mathfrak{U}_p}$ which is c-embedded in G with respect to P, then $H \leqslant Z(P)$.*

Let H be a proper subgroup of $L^{\mathfrak{U}_p}$ such that H is c-embedded in G with respect to P. Hence there is a subgroup B of G such that $G = HB$ and $H \cap B \leqslant Z(P)$. Set $B_0 := L \cap B$. Then $L = L \cap HB = HB_0$. Lemma 4 implies that $L = B_0 \leqslant B$. So it follows that $H = H \cap B \leqslant Z(P)$.

3) $L^{\mathfrak{U}_p} \leqslant Z(P)$.

Suppose that $L^{\mathfrak{U}_p}$ has exponent p. Let $x \in L^{\mathfrak{U}_p}$. We show that $x \in Z(P)$. Clearly, we only need to consider the case $x \neq 1$. Then $|\langle x \rangle| = p$. We have $|L^{\mathfrak{U}_p}| > p$ since L would be p-supersolvable otherwise. Hence $\langle x \rangle$ is a proper subgroup of $L^{\mathfrak{U}_p}$. So 1) and 2) imply that $x \in Z(P)$. As x was arbitrarily chosen, it follows that $L^{\mathfrak{U}_p} \leqslant Z(P)$.

Suppose now that $L^{\mathfrak{U}_p}$ does not have exponent p. Then, by Lemma 3(1), $p = 2$ and $L^{\mathfrak{U}_2}$ has exponent 4. Therefore L is a group appearing in [4, Theorem 9] as a group of Type 3. In particular, $L^{\mathfrak{U}_2} = P \cap L$ is a non-abelian special 2-group, $\Phi(L^{\mathfrak{U}_2}) \leqslant Z(L)$ and $|L^{\mathfrak{U}_2}/\Phi(L^{\mathfrak{U}_2})| = 2^{2m}$, $|\Phi(L^{\mathfrak{U}_2})| \leqslant 2^m$ for some positive integer m.

We claim that $L^{\mathfrak{U}_2}$ is not quaternion-free. Assume that $m = 1$. Then $|L^{\mathfrak{U}_2}| = 8$, and $L^{\mathfrak{U}_2}$ cannot be dihedral because then $L^{\mathfrak{U}_2}/\Phi(L^{\mathfrak{U}_2})$ would not be a chief factor of L. Consequently $L^{\mathfrak{U}_2}$ is isomorphic to the quaternion group of order 8 and in particular not quaternion-free. Assume now that $m > 1$. Let R be a maximal subgroup of $\Phi(L^{\mathfrak{U}_2})$. Then $L^{\mathfrak{U}_2}/R$ is extraspecial of order $2^{2m+1} \geqslant 2^5$, and so $L^{\mathfrak{U}_2}/R$ has a section isomorphic to the quaternion group of order 8 (see [8, Chapter 5, Theorem 5.2]). Hence $L^{\mathfrak{U}_2}$ is not quaternion-free.

Now let $x \in L^{\mathfrak{U}_2}$. We show that $x \in Z(P)$. Clearly, we only need to consider the case $x \neq 1$. Then $|\langle x \rangle| = 2$ or 4. Also $\langle x \rangle$ is a proper subgroup of $L^{\mathfrak{U}_2}$. So 1) and 2) imply that $x \in Z(P)$. As x was arbitrarily chosen, it follows that $L^{\mathfrak{U}_2} \leqslant Z(P)$.

4) *The final contradiction.*

Set $N := N_G(L^{\mathfrak{U}_p})$. By 3), we have $P \leqslant C_G(L^{\mathfrak{U}_p})$. Since $C_G(L^{\mathfrak{U}_p}) \trianglelefteq N$, the Frattini argument implies that $N = N_N(P)C_G(L^{\mathfrak{U}_p})$.

Clearly $L^{\mathfrak{U}_p} \trianglelefteq N_N(P)$. Let $1 = L_0 \leqslant L_1 \leqslant \cdots \leqslant L_t = L^{\mathfrak{U}_p}$ be a part of a chief series of $N_N(P)$ below $L^{\mathfrak{U}_p}$, i.e. $L_i \trianglelefteq N_N(P)$ for all $0 \leqslant i \leqslant t$ and L_{i+1}/L_i is a chief factor of $N_N(P)$ for all $0 \leqslant i < t$. Since $N_G(P)$ is p-supersolvable and $N_N(P) \leqslant N_G(P)$, we have that $N_N(P)$ is p-supersolvable. Consequently $|L_{i+1}/L_i| = p$ for all $0 \leqslant i < t$.

For each $0 \leqslant i \leqslant t$, we have $C_G(L^{\mathfrak{U}_p}) \leqslant C_G(L_i) \leqslant N_G(L_i)$ and hence $N = N_N(P)C_G(L^{\mathfrak{U}_p}) \leqslant N_G(L_i)$. In particular, we have $L_i \trianglelefteq L$ for all $0 \leqslant i \leqslant t$. Consequently, each chief factor of L below $L^{\mathfrak{U}_p}$ has order p. So it follows that L is p-supersolvable. This contradiction completes the proof. □

Proof of Theorem B. Let P be a Sylow p-subgroup of a group G. Assume that G is p-nilpotent. Then, since subgroups of p-nilpotent groups are p-nilpotent, we have that $N_G(P)$ is p-nilpotent, whence condition (1) from Theorem B is satisfied. Also $G^{\mathfrak{U}_p} \leqslant G^{\mathfrak{N}_p} = 1$, so that condition (2) from Theorem B is trivially satisfied.

Let $\mathfrak{Y}$ denote the class of all groups G such that condition (2) from Theorem B is satisfied for any Sylow p-subgroup P of G. Note that if condition (2) from Theorem B is satisfied for one Sylow p-subgroup of a group G, then it is satisfied for all Sylow p-subgroups of G, so that $G \in \mathfrak{Y}$. Let $\mathscr{Z}_p$ denote the class of all $\mathfrak{N}$-groups with p-nilpotent normalizers of Sylow p-subgroups. To complete the proof of Theorem B, we show that the class $\mathscr{Z}_p$ is contained in the class $\mathfrak{N}_p$ of all p-nilpotent groups. Suppose that this is not true, and choose a non-p-nilpotent $\mathscr{Z}_p$-group G of minimal order.

Arguing similarly as in [6, Example 2], we see that $\mathfrak{Y}$ is subgroup-closed and that $X/N \in \mathfrak{Y}$ whenever $X \in \mathfrak{Y}$ and N is a normal p'-subgroup of X. Applying [6, Theorem A], we conclude that G is p-solvable.

Let $P \in \mathrm{Syl}_p(G)$. Then $N_G(P)$ is p-nilpotent and hence p-supersolvable as $G \in \mathscr{Z}_p$, whence condition (1) from Theorem A is satisfied. Also, condition (2) from Theorem A is satisfied since it is identical to condition (2) from Theorem B, which holds as $G \in \mathfrak{Y}$. So Theorem A implies that G is p-supersolvable. Applying Lemma 6, we conclude that G is p-nilpotent. This contradiction shows that $\mathscr{Z}_p$ is contained in $\mathfrak{N}_p$, as wanted. □

4 Remarks and open questions

One might wonder whether Theorems A and B remain true when, in condition (2) of the theorems, "c-embedded in G with respect to P" is replaced by "c-embedded in P with respect to P". For the case $p = 2$, the answer is negative, as the following example shows.

Example 7. Let $G := S_4$, the symmetric group of degree 4, and let P be a Sylow 2-subgroup of G. Then P is dihedral of order 8, and we have $N_G(P) = P$. Any subgroup of P with order 2 is either complemented or central in P and thus c-embedded in P with respect to P. However, G is not 2-nilpotent (or, equivalently, not 2-supersolvable).

For the case $p = 3$, the following example shows that Theorem A does not remain true when "c-embedded in G with respect to P" is replaced by "c-embedded in P with respect to P".

Example 8. Let G be the group indexed in GAP [7] as SmallGroup(216,153). Then G is solvable and hence 3-solvable. Let P be a Sylow 3-subgroup of G. Then $N_G(P)$ is 3-supersolvable, and any subgroup of P with order 3 is complemented or central in P and thus c-embedded in P with respect to P. However, G is not 3-supersolvable.

We were not able to answer the following question.

Question 9. Suppose that p is odd. Does Theorem B remain true when, in condition (2) from Theorem B, "c-embedded in G with respect to P" is replaced by "c-embedded in P with respect to P"?

Using GAP [7], we have checked that there are no counterexamples of order up to 2000.

Wei, Wang and Liu [13] obtained the following characterization of p-nilpotent groups: A group G with Sylow p-subgroup P is p-nilpotent if and only if every minimal subgroup of $P \cap G^{\mathfrak{N}_p}$ is complemented in P and $N_G(P)$ is p-nilpotent (see [13, Corollary 2.3]). In a less general form, this had been proved before by Guo and Shum [9, Theorem 2.1]. Note that if the correct answer to Question 9 is positive, then this would generalize both Theorem B and the mentioned result of Wei, Wang and Liu.

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