# ON FINITE GROUPS WITH GIVEN ICO-SUBGROUPS

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ABSTRACT. A subgroup H of a group G is said to be an  $IC\Phi$ -subgroup of G if  $H \cap [H, G] \leq \Phi(H)$ . We analyze the structure of a finite group G under the assumption that some given subgroups of G are  $IC\Phi$ -subgroups of G. A new characterization of finite abelian groups and some new criteria for 2-nilpotence and nilpotence of finite groups will be obtained. Moreover, we will obtain two criteria for a finite group to lie in a given solvably saturated formation containing the class of finite supersolvable groups.

# 1. INTRODUCTION

All groups in this paper are implicitly assumed to be finite. Our notation and terminology are standard. The reader is referred to [6, 10, 14] for unfamiliar definitions on groups and to [7] for unfamiliar definitions on classes of groups.

Given a group G and a subgroup H of G, we say that H is an  $IC\Phi$ -subgroup of G provided that  $H \cap [H, G] \leq \Phi(H)$ . This concept was introduced by Gao and Li in [5] and further investigated by the author in [12]. The papers [5] and [12] contain results that constrain the structure of a group G under the condition that some given subgroups of G are  $IC\Phi$ -subgroups of G. The following theorem is the main result of [12].

**Theorem 1.1.** ([12, Theorem 1.3]) Let p be a prime dividing the order of a group G, and let P be a Sylow p-subgroup of G. Suppose that there is a subgroup D of P with  $1 < |D| \le |P|$  such that any subgroup of P with order |D| is an  $IC\Phi$ -subgroup of G. If |D| = 2 and  $|P| \ge 8$ , assume moreover that any cyclic subgroup of P with order 4 is an  $IC\Phi$ -subgroup of G. Then G is p-nilpotent.

The goal of the present paper is to further study how the structure of a group is influenced by its  $IC\Phi$ -subgroups. Our results show, together with [5] and [12], that one often gets very rich information about a group when some of its subgroups are assumed to be  $IC\Phi$ -subgroups.

Groups some of whose primary subgroups are  $IC\Phi$ -subgroups. Let  $Q_8$  denote the quaternion group with order 8. Recall that a group G is said to be  $Q_8$ -free if no section of G is isomorphic to  $Q_8$ . Our first main result is the following improvement of Theorem 1.1 and [5, Theorem 3.1].

**Theorem 1.2.** Let G be a  $Q_8$ -free group such that any subgroup of G with order 2 is an  $IC\Phi$ -subgroup of G. Then G is 2-nilpotent.

The condition in Theorem 1.2 that G is  $Q_8$ -free is really necessary. For example,  $Z(SL_2(3))$  is the only subgroup of  $SL_2(3)$  with order 2, and  $Z(SL_2(3))$  is an  $IC\Phi$ -subgroup of  $SL_2(3)$ . But  $SL_2(3)$  is not 2-nilpotent.

Our second main result is a generalization of the following result of Gao and Li.

**Theorem 1.3.** ([5, Theorem 3.5]) Let G be a group, and let E be a normal subgroup of G such that G/E is supersolvable. If every maximal subgroup of every Sylow subgroup of E is an  $IC\Phi$ -subgroup of G, then G is supersolvable.

<sup>2010</sup> Mathematics Subject Classification. 20D10, 20D15, 20D20.

Key words and phrases. finite groups,  $IC\Phi$ -subgroups, abelian, 2-nilpotent, nilpotent, solvably saturated formations, maximal subgroups, 2-maximal subgroups, 3-maximal subgroups.

To state our generalization of Theorem 1.3, we recall some definitions. A class of groups  $\mathfrak{F}$  is said to be a *formation* if  $\mathfrak{F}$  is closed under taking homomorphic images and subdirect products. A formation  $\mathfrak{F}$  is said to be *saturated* if whenever G is a group with  $G/\Phi(G) \in \mathfrak{F}$ , we have  $G \in \mathfrak{F}$ . We say that a formation  $\mathfrak{F}$  is *solvably saturated* if whenever G is a group and N is a solvable normal subgroup of G with  $G/\Phi(N) \in \mathfrak{F}$ , we have  $G \in \mathfrak{F}$ . Note that every saturated formation is solvably saturated.

The class of all supersolvable groups is denoted by  $\mathfrak{U}$ . It is well-known that  $\mathfrak{U}$  is a saturated and hence a solvably saturated formation.

With these definitions at hand, we can now state our second main result.

**Theorem 1.4.** Let  $\mathfrak{F}$  be a solvably saturated formation containing  $\mathfrak{U}$ , let G be a group, and let E be a non-trivial normal subgroup of G such that  $G/E \in \mathfrak{F}$ . Let  $t := |\pi(E)|$ , and let  $p_1 < \cdots < p_t$  be the distinct prime divisors of |E|. For each  $1 \leq i \leq t$ , let  $P_i$  be a Sylow  $p_i$ -subgroup of E. Suppose that, for each  $1 \leq i \leq t$ , either  $P_i$  is cyclic or there is a subgroup  $D_i$  of  $P_i$  with  $1 < |D_i| \leq |P_i|$  such that any subgroup of  $P_i$  with order  $|D_i|$  is an  $IC\Phi$ -subgroup of G. If  $p_1 = 2$ ,  $|D_1| = 2$  and  $P_1$  is not  $Q_8$ -free, assume moreover that any cyclic subgroup of  $P_1$  with order 4 is an  $IC\Phi$ -subgroup of G. Then  $G \in \mathfrak{F}$ .

Theorem 1.3 is covered by Theorem 1.4. Also, the proof of Theorem 1.4 given here is shorter than the proof of Theorem 1.3 given in [5].

Our third main result shows that Theorem 1.4 remains true when we replace the assumption that the subgroups  $P_1, \ldots, P_t$  are Sylow subgroups of E by the assumption that they are Sylow subgroups of the generalized Fitting subgroup  $F^*(E)$  of E.

**Theorem 1.5.** Let  $\mathfrak{F}$  be a solvably saturated formation containing  $\mathfrak{U}$ , let G be a group, and let E be a non-trivial normal subgroup of G such that  $G/E \in \mathfrak{F}$ . Let  $t := |\pi(F^*(E))|$ , and let  $p_1 < \cdots < p_t$ be the distinct prime divisors of  $|F^*(E)|$ . For each  $1 \leq i \leq t$ , let  $P_i$  be a Sylow  $p_i$ -subgroup of  $F^*(E)$ . Suppose that, for each  $1 \leq i \leq t$ , either  $P_i$  is cyclic or there is a subgroup  $D_i$  of  $P_i$  with  $1 < |D_i| \leq |P_i|$  such that any subgroup of  $P_i$  with order  $|D_i|$  is an  $IC\Phi$ -subgroup of G. If  $p_1 = 2$ ,  $|D_1| = 2$  and  $P_1$  is not  $Q_8$ -free, assume moreover that any cyclic subgroup of  $P_1$  with order 4 is an  $IC\Phi$ -subgroup of G. Then  $G \in \mathfrak{F}$ .

All the above theorems are concerned with groups G such that *some* primary subgroups of G are  $IC\Phi$ -subgroups of G. It is natural to ask what we can say about the structure of a group G when *every* primary subgroup of G is an  $IC\Phi$ -subgroup of G. Clearly, any abelian group has this property. Also, one can check that any subgroup of  $Q_8$  is an  $IC\Phi$ -subgroup of  $Q_8$ . Our fourth main result characterizes the abelian groups as the  $Q_8$ -free groups all of whose primary subgroups are  $IC\Phi$ -subgroups.

**Theorem 1.6.** Let G be a group. Then the following are equivalent:

- (1) G is abelian.
- (2) G is  $Q_8$ -free, and any subgroup of G is an  $IC\Phi$ -subgroup of G.
- (3) G is  $Q_8$ -free, and any primary subgroup of G is an  $IC\Phi$ -subgroup of G.

**Groups all of whose maximal, 2-maximal or 3-maximal subgroups are**  $IC\Phi$ -subgroups. Let n be a positive integer, and let G be a group. A subgroup H of G is said to be n-maximal in G if there is a chain of subgroups  $H = H_0 < H_1 < \cdots < H_n = G$ , where  $H_i$  is maximal in  $H_{i+1}$  for all  $0 \le i \le n-1$ .

There are many results in finite group theory that describe the structure of a group G under the assumption that, for a given positive integer n, all n-maximal subgroups of G satisfy a given property.

Perhaps the most well-known result of this kind is due to Wielandt, who proved that a group G is nilpotent if every maximal subgroup of G is normal in G (see [10, Kapitel III, Hauptsatz 2.3]).

Huppert proved that a group G is supersolvable if every 2-maximal subgroup of G is normal in G (see [9, Satz 23]) or if |G| is divisible by at least three primes and every 3-maximal subgroup of G is normal in G (see [9, Satz 24]). Janko proved that a solvable group G is supersolvable if every 4-maximal subgroup of G is normal in G and |G| is divisible by at least four primes (see [11, Theorem 3]). Huppert's and Janko's results were strengthened by Asaad [1].

Mann [15] proved a number of structural results about groups whose *n*-maximal subgroups, for some positive integer *n*, are subnormal. In the last decade, a number of results have been obtained on groups whose *n*-maximal subgroups, for some positive integer *n*, satisfy certain properties generalizing subnormality, see for example [13, 16, 17, 18] (some of these results only deal with the case n = 2).

Other recent results on n-maximal subgroups and their influence on the structure of groups were obtained for example in [3, 4, 19].

As a development of the research on n-maximal subgroups, we will prove the following three theorems.

**Theorem 1.7.** Let G be a group such that any maximal subgroup of G is an  $IC\Phi$ -subgroup of G. Then G is nilpotent.

**Theorem 1.8.** Let G be a group. Suppose that G has a non-trivial 2-maximal subgroup and that any 2-maximal subgroup of G is an  $IC\Phi$ -subgroup of G. Then G is nilpotent.

**Theorem 1.9.** Let G be a group. Suppose that G has a non-trivial 3-maximal subgroup and that any 3-maximal subgroup of G is an  $IC\Phi$ -subgroup of G. Then either G is nilpotent or  $G \cong SL_2(3)$ .

# 2. Preliminaries

In this section, we collect some results needed for the proofs of our main results.

**Lemma 2.1.** ([5, Lemma 2.1]) Let G be a group, H be an  $IC\Phi$ -subgroup of G, and N be a normal subgroup of G. Then the following hold:

- (1) If  $H \leq K \leq G$ , then H is an  $IC\Phi$ -subgroup of K.
- (2) If  $N \leq H$ , then H/N is an  $IC\Phi$ -subgroup of G/N.
- (3) If H is a p-group for some prime divisor p of |G| and N is a p'-group, then HN/N is an  $IC\Phi$ -subgroup of G/N.

**Lemma 2.2.** Let G be a group possessing a proper non-trivial  $IC\Phi$ -subgroup H. Then G is not simple.

*Proof.* Since H is an  $IC\Phi$ -subgroup of G, we have  $H \cap [H,G] \leq \Phi(H)$ . If G = [H,G], then it follows that  $H \leq \Phi(H)$ , which is impossible. Therefore, [H,G] is a proper subgroup of G. Also, [H,G] is normal in G. If  $[H,G] \neq 1$ , it follows that G is not simple. If [H,G] = 1, then  $H \leq Z(G)$ , and again it follows that G is not simple.  $\Box$ 

**Lemma 2.3.** Let G be a group, and let H be an  $IC\Phi$ -subgroup of G. Suppose that  $G' \leq H$ . Then G is nilpotent.

*Proof.* We have  $[G', G] \leq H \cap [H, G] \leq \Phi(H)$ . Applying [10, Kapitel III, Hilfssatz 3.3], we conclude that  $[G', G] \leq \Phi(G)$ . It follows that

$$[(G/\Phi(G))', G/\Phi(G)] = [G'\Phi(G)/\Phi(G), G/\Phi(G)] = [G', G]\Phi(G)/\Phi(G) = 1.$$

So the lower central series of  $G/\Phi(G)$  terminates at 1. Consequently,  $G/\Phi(G)$  is nilpotent, and [10, Kapitel III, Satz 3.7] implies that G is nilpotent.

**Lemma 2.4.** ([7, Theorem 3.4.11], [10, Kapitel III, Satz 5.2]) Let G be a minimal non-nilpotent group. Then:

- (1)  $|G| = p^a q^b$  with distinct prime numbers p, q and positive integers a, b, where G has a normal Sylow p-subgroup P and cyclic Sylow q-subgroups.
- (2)  $P/\Phi(P)$  is a chief factor of G.
- (3) If P is abelian, then P is elementary abelian.

**Lemma 2.5.** Let G be a  $Q_8$ -free minimal non-2-nilpotent group. Then G has an elementary abelian Sylow 2-subgroup.

*Proof.* By [10, Kapitel IV, Satz 5.4], G is minimal non-nilpotent. Lemma 2.4 (1) implies that G has a normal Sylow 2-subgroup P.

Assume that P is not elementary abelian. Then P is non-abelian by Lemma 2.4 (3). Since G is  $Q_8$ -free, we have that P is  $Q_8$ -free. By a result of Ward, namely by [2, Theorem 56.1], any non-abelian  $Q_8$ -free 2-group has a characteristic maximal subgroup. Since P is normal in G, it follows that there is a maximal subgroup  $P_1$  of P which is normal in G. We have  $\Phi(P) \leq P_1$ , and  $P/\Phi(P)$  is a chief factor of G by Lemma 2.4 (2). It follows that  $\Phi(P) = P_1$ . This implies that P is cyclic and hence abelian. On the other hand, we have observed above that P is non-abelian. This is a contradiction.

So we have that P is elementary abelian, and the lemma follows.

**Lemma 2.6.** ([6, Chapter 7, Theorem 4.5]) Let p be a prime number, and let G be a group. If  $N_G(H)/C_G(H)$  is a p-group for any non-trivial p-subgroup H of G, then G is p-nilpotent.

**Lemma 2.7.** ([14, 7.2.2]) Let G be a non-trivial group, and let p be the smallest prime divisor of |G|. Suppose that the Sylow p-subgroups of G are cyclic. Then G is p-nilpotent.

**Lemma 2.8.** ([21, Appendix C, Theorem 6.3]) Let p be a prime number, and let P be a normal p-subgroup of a group G such that  $G/C_G(P)$  is a p-group. Then  $P \leq Z_{\infty}(G)$ .

To state the next two lemmas, we recall some definitions. Let  $\mathfrak{F}$  be a formation, let G be a group, and let H/K be a chief factor of G. Then H/K is said to be  $\mathfrak{F}$ -central if  $H/K \rtimes G/C_G(H/K) \in \mathfrak{F}$ . Note that H/K is  $\mathfrak{U}$ -central if and only if H/K is cyclic. A normal subgroup N of G is said to be  $\mathfrak{F}$ -central if any chief factor of G below N is  $\mathfrak{F}$ -central. The product of all  $\mathfrak{F}$ -central normal subgroups of G is denoted by  $Z_{\mathfrak{F}}(G)$ .

**Lemma 2.9.** ([8, Lemma 3.3]) Let  $\mathfrak{F}$  be a solvably saturated formation containing  $\mathfrak{U}$ . Let G be a group and E be a normal subgroup of G such that  $G/E \in \mathfrak{F}$  and  $E \leq Z_{\mathfrak{U}}(G)$ . Then  $G \in \mathfrak{F}$ .

**Lemma 2.10.** ([20, Theorem B]) Let  $\mathfrak{F}$  be a formation. Let G be a group, and let E be a normal subgroup of G such that  $F^*(E) \leq Z_{\mathfrak{F}}(G)$ . Then  $E \leq Z_{\mathfrak{F}}(G)$ .

**Lemma 2.11.** ([14, 5.3.7]) Let p be a prime number, and let P be a p-group such that P has a unique subgroup with order p. Then either P is cyclic, or p = 2 and P is a generalized quaternion group.

**Lemma 2.12.** ([1, Lemma 2.1]) Let G be a group such that the trivial subgroup 1 is a 2-maximal subgroup of G and such that G has no non-trivial 2-maximal subgroups. Then |G| = pq, where p and q are prime numbers (not necessarily distinct).

**Lemma 2.13.** ([1, Lemma 2.3]) Let G be a group such that the trivial subgroup 1 is a 3-maximal subgroup of G and such that G has no non-trivial 3-maximal subgroups. Then |G| = pqr, where p, q and r are prime numbers (not necessarily distinct).

**Lemma 2.14.** ([6, Chapter 6, Theorem 1.5]) Let G be a solvable group, and let M be a maximal subgroup of G. Then M has prime power index in G.

**Lemma 2.15.** ([14, 5.3.3])  $\operatorname{Aut}(Q_8)$  is isomorphic to the symmetric group  $S_4$ .

**Lemma 2.16.** ([14, 8.6.10]) Let G be a 2-closed group with order 24 such that the Sylow 2-subgroup of G is isomorphic to  $Q_8$ . Then either  $G \cong Q_8 \times C_3$  or  $G \cong SL_2(3)$ .

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### 3. Proof of Theorem 1.2

Proof of Theorem 1.2. Suppose that the theorem is false, and let G be a minimal counterexample.

Let L be a proper subgroup of G. Then L is  $Q_8$ -free since G is  $Q_8$ -free. Also, by hypothesis, any subgroup of L with order 2 is an  $IC\Phi$ -subgroup of G. Lemma 2.1 (1) implies that any subgroup of L with order 2 is an  $IC\Phi$ -subgroup of L. Consequently, L satisfies the hypotheses of the theorem, and so L is 2-nilpotent by the minimality of G. It follows that G is a minimal non-2-nilpotent group.

Let  $P \in \text{Syl}_2(G)$ . Then  $P \neq 1$ . By hypothesis, any subgroup of P with order 2 is an  $IC\Phi$ subgroup of G. By Lemma 2.5, P is elementary abelian. In particular, P has no cyclic subgroup with order 4. Applying Theorem 1.1 or [5, Theorem 3.1], we conclude that G is 2-nilpotent. This contradiction completes the proof.

### 4. Proofs of Theorems 1.4 and 1.5

**Lemma 4.1.** Let p be a prime number, let G be a group, and let P be a non-trivial normal psubgroup of G. Suppose that there is a subgroup D of P with  $1 < |D| \le |P|$  such that any subgroup of P with order |D| is an  $IC\Phi$ -subgroup of G. If p = 2, |D| = 2 and P is not  $Q_8$ -free, assume moreover that any cyclic subgroup of P with order 4 is an  $IC\Phi$ -subgroup of G. Then  $P \le Z_{\infty}(G)$ .

Proof. Let q be a prime divisor of |G| with  $q \neq p$ , and let Q be a Sylow q-subgroup of G. Set H := PQ. We have  $H \leq G$  since P is normal in G. Note that P is a Sylow p-subgroup of H. By Lemma 2.1 (1), any subgroup of P with order |D| is an  $IC\Phi$ -subgroup of H. Also, if p = 2, |D| = 2 and P is not  $Q_8$ -free, we have that any cyclic subgroup of P with order 4 is an  $IC\Phi$ -subgroup of H. Theorems 1.1 and 1.2 imply that H is p-nilpotent. This implies that  $H = P \times Q$ , and so we have  $Q \leq C_G(P)$ . Since q was arbitrarily chosen, it follows that  $O^p(G) \leq C_G(P)$ . Hence,  $G/C_G(P)$  is a p-group. Lemma 2.8 implies that  $P \leq Z_{\infty}(G)$ .

Proof of Theorem 1.4. Suppose that the theorem is false, and let (G, E) be a counterexample such that |G| + |E| is minimal.

Set  $p := p_1$ . We claim that E is p-nilpotent. This follows from Lemma 2.7 when  $P_1$  is cyclic. Assume now that  $P_1$  is not cyclic. Then  $P_1$  has a subgroup  $D_1$  with  $1 < |D_1| \le |P_1|$  such that any subgroup of  $P_1$  with order  $|D_1|$  is an  $IC\Phi$ -subgroup of G. Also, if p = 2,  $|D_1| = 2$  and  $P_1$ is not  $Q_8$ -free, then any cyclic subgroup of  $P_1$  with order 4 is an  $IC\Phi$ -subgroup of G. Applying Lemma 2.1 (1), Theorem 1.1 and Theorem 1.2, we conclude that E is p-nilpotent, as claimed.

Assume that  $O_{p'}(E) \neq 1$ . From Lemma 2.1 (3), we see that  $(G/O_{p'}(E), E/O_{p'}(E))$  satisfies the hypotheses of the theorem. So we have  $G/O_{p'}(E) \in \mathfrak{F}$  by the minimality of (G, E). Therefore,  $(G, O_{p'}(E))$  also satisfies the hypotheses of the theorem. The minimality of (G, E) implies that  $G \in \mathfrak{F}$ . This is a contradiction, and so we have  $O_{p'}(E) = 1$ .

We show now that  $E \leq Z_{\mathfrak{U}}(G)$ . Since E is p-nilpotent and since  $O_{p'}(E) = 1$ , we have that  $E = P_1$ . If  $P_1$  is cyclic, then it follows that  $E = P_1 \leq Z_{\mathfrak{U}}(G)$ . If  $P_1$  is not cyclic, then the hypotheses of the theorem and Lemma 4.1 imply that  $E = P_1 \leq Z_{\infty}(G)$  and thus  $E \leq Z_{\mathfrak{U}}(G)$ .

Now Lemma 2.9 implies that  $G \in \mathfrak{F}$ . This contradiction completes the proof.

Proof of Theorem 1.5. Arguing as at the beginning of the proof of Theorem 1.4, we see that  $F^*(E)$  is  $p_1$ -nilpotent. With  $N_1 := O_{(p_1)'}(F^*(E))$ , we thus have  $F^*(E)/N_1 \cong P_1$ .

Assume that t > 1. Then  $P_2 \in \text{Syl}_{p_2}(N_1)$ , and again we can argue as at the beginning of the proof of Theorem 1.4 to see that  $N_1$  is  $p_2$ -nilpotent. With  $N_2 := O_{(p_2)'}(N_1)$ , we thus have  $N_1/N_2 \cong P_2$ .

Repeating this argumentation, we see that G has a Sylow tower of supersolvable type, i.e. there is a chain  $F^*(E) = N_0 > N_1 > \cdots > N_t = 1$  of normal subgroups of  $F^*(E)$  such that  $N_{i-1}/N_i \cong P_i$  for all  $1 \le i \le t$ . It follows that  $F^*(E)$  is solvable.

As is well-known,  $F^*(E)$  is generated by F(E) together with the components of E. Every component of E is a non-solvable subgroup of E. Since  $F^*(E)$  is solvable, it follows that E does not possess any components. Consequently, we have  $F^*(E) = F(E)$ .

Let  $1 \leq i \leq t$ . Since F(E) is nilpotent and  $P_i \in Syl_{p_i}(F(E))$ , we have that  $P_i$  is characteristic in F(E). As  $F(E) \leq G$ , it follows that  $P_i \leq G$ . If  $P_i$  is cyclic, then  $P_i \leq Z_{\mathfrak{U}}(G)$ . If  $P_i$  is not cyclic, then the hypotheses of the theorem and Lemma 4.1 imply that  $P_i \leq Z_{\infty}(G)$  and hence  $P_i \leq Z_{\mathfrak{U}}(G)$ . Since *i* was arbitrarily chosen, it follows that  $F(E) \leq Z_{\mathfrak{U}}(G)$ . 

Applying Lemmas 2.10 and 2.9, we conclude that  $G \in \mathfrak{F}$ .

### **PROOF OF THEOREM 1.6**

**Lemma 4.2.** Let p be a prime number, and let P be a p-group such that any subgroup of P is an  $IC\Phi$ -subgroup of P. If p = 2, assume moreover that P is  $Q_8$ -free. Then P is abelian.

*Proof.* Suppose that the lemma is false, and let P be a minimal counterexample. We will derive a contradiction in three steps.

(1) P' is minimal normal in P, and there are no minimal normal subgroups of P other than P'. Clearly  $P \neq 1$ , and so P has a minimal normal subgroup, say N. We show that N = P'.

Let  $N \leq H \leq P$ . By hypothesis, H is an  $IC\Phi$ -subgroup of P. Lemma 2.1 (2) shows that H/Nis an  $IC\Phi$ -subgroup of P/N. Since H was arbitrarily chosen, it follows that any subgroup of P/Nis an  $IC\Phi$ -subgroup of P/N. Also, if p = 2, then P/N is  $Q_8$ -free since P is  $Q_8$ -free. Therefore, P/N satisfies the hypotheses of the lemma, and so P/N is abelian by the minimality of P.

It follows that  $P' \leq N$ . Noticing that  $P' \neq 1$  since P is not abelian, we conclude that N = P', as required.

(2) We have |P'| = p, and there is no subgroup of P with order p other than P'.

We have |P'| = p since P' is minimal normal in P. Assume that there is a subgroup Q of P with |Q| = p and  $Q \neq P'$ . Set  $H := P'Q \leq P$ . Note that  $\Phi(H) = 1$ .

By (1), Q is not normal in P. So we have  $Q \not\leq Z(P)$  and hence  $H \not\leq Z(P)$ . Thus  $[H, P] \neq 1$ . Clearly  $[H, P] \leq P$  and  $[H, P] \leq P'$ . So we have [H, P] = P' by (1).

By hypothesis, H is an  $IC\Phi$ -subgroup of P. It follows that  $P' = H \cap P' = H \cap [H, P] \leq$  $\Phi(H) = 1$ . This is a contradiction, and so there is no subgroup of P with order p other than P'.

(3) The final contradiction.

By (2), P has precisely one subgroup with order p. Moreover, P cannot be a generalized quaternion group since P is  $Q_8$ -free by hypothesis. Lemma 2.11 implies that P is cyclic and hence abelian. This final contradiction completes the proof. 

*Proof of Theorem 1.6.* (1)  $\Rightarrow$  (2): Suppose that G is abelian. Then any section of G is abelian. In particular, G is  $Q_8$ -free. Also, if H is a subgroup of G, then  $H \cap [H,G] = H \cap 1 = 1 \leq \Phi(H)$ , so that H is an  $IC\Phi$ -subgroup of G. Thus (2) holds.

 $(2) \Rightarrow (3)$ : Clear.

 $(3) \Rightarrow (1)$ : Suppose that G is  $Q_8$ -free and that any primary subgroup of G is an  $IC\Phi$ -subgroup of G. If G = 1, then there is nothing to show. Thus we assume that  $G \neq 1$ . Set  $t := |\pi(G)|$ , and let  $p_1, \ldots, p_t$  be the distinct prime divisors of |G|. For each  $1 \leq i \leq t$ , let  $P_i$  be a Sylow  $p_i$ -subgroup of G.

Let  $1 \le i \le t$ . Since any primary subgroup of G is an  $IC\Phi$ -subgroup of G, we have that  $P_i$  is an  $IC\Phi$ -subgroup of G. Theorem 1.1 implies that G is  $p_i$ -nilpotent. Since i was arbitrarily chosen, we have that G is p-nilpotent for any prime divisor p of |G|. Consequently, G is nilpotent, and so we have  $G = P_1 \times \cdots \times P_t$ .

Let  $1 \leq i \leq t$ . Then any subgroup of  $P_i$  is an  $IC\Phi$ -subgroup of G. Lemma 2.1 (1) implies that any subgroup of  $P_i$  is an  $IC\Phi$ -subgroup of  $P_i$ . Also,  $P_i$  is  $Q_8$ -free since G is  $Q_8$ -free. Lemma 4.2 implies that  $P_i$  is abelian.

Consequently, G is a direct product of abelian groups, and so G is abelian as well.

PROOFS OF THEOREMS 1.7, 1.8 AND 1.9

Proof of Theorem 1.7. Suppose that the theorem is false, and let G be a minimal counterexample.

Clearly, G has a non-trivial maximal subgroup M. By hypothesis, M is an  $IC\Phi$ -subgroup of G. Lemma 2.2 implies that G is not simple.

Let N be a proper non-trivial normal subgroup of G, and let  $N \leq M \leq G$  such that M/N is a maximal subgroup of G/N. Then M is a maximal subgroup of G. So M is an  $IC\Phi$ -subgroup of G. Lemma 2.1 (2) implies that M/N is an  $IC\Phi$ -subgroup of G/N. Since M was arbitrarily chosen, it follows that any maximal subgroup of G/N is an  $IC\Phi$ -subgroup of G/N. The minimality of G implies that G/N is nilpotent.

It follows that (G/N)' = G'N/N is a proper subgroup of G/N. This implies that G' is a proper subgroup of G. So there is a maximal subgroup M of G with  $G' \leq M$ . By hypothesis, M is an  $IC\Phi$ -subgroup of G. Lemma 2.3 implies that G is nilpotent. This is a contradiction to the choice of G, and so the proof is complete.

Proof of Theorem 1.8. Suppose that the theorem is false, and let G be a (not necessarily minimal) counterexample.

Let M be a maximal subgroup of G. By hypothesis, every maximal subgroup of M is an  $IC\Phi$ subgroup of G. Lemma 2.1 (1) implies that any maximal subgroup of M is an  $IC\Phi$ -subgroup of M. So M is nilpotent by Theorem 1.7. Since M was arbitrarily chosen, it follows that any maximal subgroup of G is nilpotent. Consequently, G is minimal non-nilpotent.

By Lemma 2.4 (1), we have  $|G| = p^a q^b$  with distinct prime numbers p, q and positive integers a, b, where G has a normal Sylow p-subgroup P and cyclic Sylow q-subgroups.

We claim that b = 1. Assume, for sake of contradiction, that  $b \ge 2$ . Clearly, G/P is cyclic with order  $q^b$ . Let  $P \le H < G$  such that H/P is the unique 2-maximal subgroup of G/P. Then H is a 2-maximal subgroup of G. By hypothesis, H is an  $IC\Phi$ -subgroup of G, and we have  $G' \le P \le H$ . Lemma 2.3 implies that G is nilpotent. This contradiction shows that b = 1, as claimed.

It follows that any maximal subgroup of P is a 2-maximal subgroup of G. Consequently, any maximal subgroup of P is an  $IC\Phi$ -subgroup of G. Also, we have |P| > p, since otherwise G would not possess a non-trivial 2-maximal subgroup. Applying Theorem 1.1, we conclude that G is p-nilpotent. Therefore, G has a normal Sylow q-subgroup. Consequently, any Sylow subgroup of G is normal in G. It follows that G is nilpotent. This contradiction completes the proof.  $\Box$ 

We need the following lemma to prove Theorem 1.9.

**Lemma 4.3.** Let G be a group such that any 3-maximal subgroup of G is an  $IC\Phi$ -subgroup of G. Then G is solvable.

*Proof.* Suppose that the lemma is false, and let G be a minimal counterexample. We will derive a contradiction in several steps.

(1) G has a non-trivial 2-maximal subgroup.

Clearly, G has a non-trivial maximal subgroup M. Any maximal subgroup of M is a 2-maximal subgroup of G. Consequently, the set of 2-maximal subgroups of G is not empty. If 1 is the only 2-maximal subgroup of G, then G is solvable as a consequence of Lemma 2.12, a contradiction. So G has a non-trivial 2-maximal subgroup.

(2) G has a non-trivial 3-maximal subgroup.

By (1), G has a non-trivial 2-maximal subgroup, say H. Any maximal subgroup of H is a 3-maximal subgroup of G. Consequently, the set of 3-maximal subgroups of G is not empty. If 1 is the only 3-maximal subgroup of G, then G is solvable as a consequence of Lemma 2.13, a contradiction. So G has a non-trivial 3-maximal subgroup.

(3) G is not simple.

By (2), G has a non-trivial 3-maximal subgroup. By hypothesis, any 3-maximal subgroup of G is an  $IC\Phi$ -subgroup of G. Consequently, G has a proper non-trivial  $IC\Phi$ -subgroup. Lemma 2.2 implies that G is not simple.

(4) Any proper subgroup of G is solvable.

Let M be a maximal subgroup of G. It suffices to show that M is solvable. If M has no non-trivial maximal subgroup, then M has prime order, and so M is solvable.

Suppose now that M has a non-trivial maximal subgroup. Then the set of 2-maximal subgroups of M is not empty. If 1 is the only 2-maximal subgroup of M, then M is solvable as a consequence of Lemma 2.12.

By hypothesis, any 2-maximal subgroup of M is an  $IC\Phi$ -subgroup of G. Lemma 2.1 (1) implies that any 2-maximal subgroup of M is an  $IC\Phi$ -subgroup of M. So, if M has a non-trivial 2-maximal subgroup, then Theorem 1.8 implies that M is nilpotent and hence solvable.

(5) The final contradiction.

By (3), G has a proper non-trivial normal subgroup, say N. Let  $N \leq H \leq G$  such that H/N is a 3-maximal subgroup of G/N. Then H is a 3-maximal subgroup of G. So, by hypothesis, H is an  $IC\Phi$ -subgroup of G. Lemma 2.1 (2) implies that H/N is an  $IC\Phi$ -subgroup of G/N. Since H was arbitrarily chosen, it follows that any 3-maximal subgroup of G/N is an  $IC\Phi$ -subgroup of G/N. The minimality of G implies that G/N is solvable. Also, N is solvable by (4). It follows that G is solvable. This final contradiction completes the proof.

Proof of Theorem 1.9. Let G be a non-nilpotent group such that G has a non-trivial 3-maximal subgroup and such that any 3-maximal subgroup of G is an  $IC\Phi$ -subgroup of G. Our task is to show that G is isomorphic to  $SL_2(3)$ . We accomplish the proof step by step.

(1) If M is a non-nilpotent maximal subgroup of G, then there exist distinct prime numbers p and q such that |M| = pq and such that |G:M| is a power of p.

Let M be a non-nilpotent maximal subgroup of G. Then M has a non-trivial maximal subgroup, and so the set of 2-maximal subgroups of M is not empty. By hypothesis, any 2-maximal subgroup of M is an  $IC\Phi$ -subgroup of G. Lemma 2.1 (1) implies that any 2-maximal subgroup of M is an  $IC\Phi$ -subgroup of M. Since M is not nilpotent, it follows from Theorem 1.8 that the trivial subgroup 1 is the only 2-maximal subgroup of M. Applying Lemma 2.12, we conclude that |M| = pq, where p and q are prime numbers. We have  $p \neq q$  since M is not nilpotent.

By Lemma 4.3, G is solvable. So, by Lemma 2.14, the index |G : M| is a power of a prime number r. Assume that  $r \notin \{p,q\}$ . As a solvable group, G has a Sylow system. Hence, there exist  $P \in \operatorname{Syl}_p(G)$ ,  $Q \in \operatorname{Syl}_q(G)$  and  $R \in \operatorname{Syl}_r(G)$  such that P, Q and R are pairwise permutable. Since |P| = p and |Q| = q, we have that R is maximal in RQ and that RQ is maximal in G. Consequently, any maximal subgroup of R is a 3-maximal subgroup of G. Therefore, any maximal subgroup of R is an  $IC\Phi$ -subgroup of G. We have |R| > r, since otherwise G would not possess a non-trivial 3-maximal subgroup. Applying Theorem 1.1, we conclude that G is r-nilpotent. This implies that  $M = O_{r'}(G) \leq G$ . Since |G : M| = |R| > r, it follows that M cannot be maximal in G. This contradiction shows that  $r \in \{p,q\}$ , and without loss of generality, we may assume that r = p.

(2) G is minimal non-nilpotent.

Assume that G is not minimal non-nilpotent. Then G has a non-nilpotent maximal subgroup M. By (1), there exist distinct prime numbers p and q such that |M| = pq and such that |G:M| is a power of p.

Let P be a Sylow p-subgroup of G, and let Q be a Sylow q-subgroup of G. Then |G:P| = q, and so we have  $|P| \ge p^3$ , since otherwise G would not possess a non-trivial 3-maximal subgroup. Assume that  $Q \leq G$ . Then M/Q is a minimal subgroup of G/Q, and we have  $|G/Q| = |P| \geq p^3$ . This is a contradiction to the maximality of M in G. Consequently, Q is not normal in G.

Any 2-maximal subgroup of P is a 3-maximal subgroup of G. So, by hypothesis, any 2-maximal subgroup of P is an  $IC\Phi$ -subgroup of G. If p is odd or if p = 2 and  $P \not\cong Q_8$ , then Theorems 1.1 and 1.2 imply that G is p-nilpotent, which is impossible since Q is not normal in G. So we have p = 2 and  $P \cong Q_8$ .

Now let N be a minimal normal subgroup of G. Since G is solvable, N is a primary subgroup of G. Also  $N \not\leq Q$  since |Q| = q and Q is not normal in G. So we have  $N \leq P$ .

Since  $P \cong Q_8$ , we have that Z(P) is the only subgroup of P with order 2. Hence  $Z(P) \leq N$ , and the minimal normality of N in G implies that N = Z(P). It follows that N is the only subgroup of G with order 2. Consequently,  $N \in Syl_2(M)$ , which easily implies that M is nilpotent. This is a contradiction, and so (2) holds.

(3) G has a normal Sylow 2-subgroup P. We have  $P \cong Q_8$  and |G:P| = 3.

In order to prove this, we use similar arguments as in the proof of Theorem 1.8.

By (2), G is minimal non-nilpotent. So, by Lemma 2.4 (1), we have  $|G| = p^a q^b$  with distinct prime numbers p, q and positive integers a, b, where G has a normal Sylow p-subgroup P and cyclic Sylow q-subgroups.

Assume that  $b \ge 3$ . Clearly, G/P is cyclic with order  $q^b$ . Let  $P \le H < G$  such that H/P is the unique 3-maximal subgroup of G/P. Then H is a 3-maximal subgroup of G. By hypothesis, H is an  $IC\Phi$ -subgroup of G, and we have  $G' \le P \le H$ . Lemma 2.3 implies that G is nilpotent. This contradiction shows that  $b \le 2$ .

Assume that b = 2. Then any maximal subgroup of P is 3-maximal in G, and so we have that any maximal subgroup of P is an  $IC\Phi$ -subgroup of G. We have |P| > p, since otherwise G would not possess a non-trivial 3-maximal subgroup. Theorem 1.1 implies that G is p-nilpotent. But Gis also q-nilpotent, and so G is nilpotent. This contradiction shows that b = 1.

It follows that any 2-maximal subgroup of P is 3-maximal in G. Therefore, any 2-maximal subgroup of P is an  $IC\Phi$ -subgroup of G. We have  $|P| \ge p^3$ , since otherwise G would not possess a non-trivial 3-maximal subgroup. If p is odd or if p = 2 and  $P \not\cong Q_8$ , then Theorems 1.1 and 1.2 imply that G is p-nilpotent, which leads to a contradiction as above. So we have p = 2 and  $P \cong Q_8$ .

If U is a non-trivial proper subgroup of P, then U is a cyclic 2-group, and this implies that  $N_G(U)/C_G(U)$  is a 2-group. Since G is not 2-nilpotent, Lemma 2.6 implies that  $G/C_G(P)$  is not a 2-group. Since  $G/C_G(P)$  is isomorphic to a subgroup of  $\operatorname{Aut}(P)$ , and since  $\operatorname{Aut}(P)$  has order 24 by Lemma 2.15, we conclude that  $|G/C_G(P)|$  is divisible 3. Therefore, |G| is divisible by 3. Since P has prime index in G, it follows that |G:P| = 3.

(4) Conclusion.

Applying Lemma 2.16, we deduce from (3) that  $G \cong SL_2(3)$ . So we have reached the desired conclusion.

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