# A CHARACTERIZATION OF THE GROUPS $P S L_{n}(q)$ AND $P S U_{n}(q)$ BY THEIR 2-FUSION SYSTEMS, $q$ ODD 

JULIAN KASPCZYK


#### Abstract

Let $q$ be a nontrivial odd prime power, and let $n \geq 2$ be a natural number with $(n, q) \neq(2,3)$. We characterize the groups $P S L_{n}(q)$ and $P S U_{n}(q)$ by their 2-fusion systems. This contributes to a programme of Aschbacher aiming at a simplified proof of the classification of finite simple groups.


## 1. Introduction

The classification of finite simple groups (CFSG) is one of the greatest achievements in the history of mathematics. Its proof required around 15,000 pages and spreads out over many hundred articles in various journals. Many mathematicians from all over the world were involved in the proof, whose final steps were published in 2004 by Aschbacher and Smith, after it was prematurely announced as finished already in 1983. Because of its extreme length, a simplified and shortened proof of the CFSG would be very valuable. There are three programmes working towards this goal: the Gorenstein-Lyons-Solomon programme (see [27]), the Meierfrankenfeld-Stellmacher-Stroth programme (see [43]) and Aschbacher's programme.

The goal of Aschbacher's programme is to obtain a new proof of the CFSG by using fusion systems. The standard examples of fusion systems are the fusion categories of finite groups over $p$-subgroups ( $p$ a prime). If $G$ is a finite group and $S$ is a $p$-subgroup of $G$ for some prime $p$, then the fusion category of $G$ over $S$ is defined to be the category $\mathcal{F}_{S}(G)$ given as follows: the objects of $\mathcal{F}_{S}(G)$ are precisely the subgroups of $S$, the morphisms in $\mathcal{F}_{S}(G)$ are precisely the group homomorphisms between subgroups of $S$ induced by conjugation in $G$, and the composition of morphisms in $\mathcal{F}_{S}(G)$ is the usual composition of group homomorphisms. Abstract fusion systems are a generalization of this concept. A fusion system over a finite $p$-group $S$, where $p$ is a prime, is a category whose objects are the subgroups of $S$ and whose morphisms behave as if they are induced by conjugation inside a finite group containing $S$ as a $p$-subgroup. For the precise definition, we refer to [11, Part I, Definition 2.1]. A fusion system is called saturated if it satisfies certain axioms motivated by properties of fusion categories of finite groups over Sylow subgroups (see [11, Part I, Definition 2.2]). If $G$ is a finite group and $S_{1}, S_{2} \in \operatorname{Syl}_{p}(G)$ for some prime $p$, then $\mathcal{F}_{S_{1}}(G)$ and $\mathcal{F}_{S_{2}}(G)$ are easily seen to be isomorphic (in the sense of [12, p. 560]). Given a finite group $G$, a prime $p$ and a Sylow $p$-subgroup $S$ of $G$, we refer to $\mathcal{F}_{S}(G)$ as the $p$-fusion system of $G$.

Originally considered by the representation theorist Puig, fusion systems have become an object of active research in finite group theory, representation theory and algebraic topology. It has always been a problem of great interest in the theory of fusion systems to translate group-theoretic concepts into suitable concepts for fusion systems. For example, there is a notion of normalizers and centralizers of $p$-subgroups in fusion systems, a notion of the center of a fusion system, a notion of factor systems, a notion of normal subsystems of saturated fusion systems and a notion of simple saturated fusion systems (see [11, Parts I and II]). Roughly speaking, Aschbacher's programme consists of the following two steps.

[^0]1. Classify the simple saturated fusion systems on finite 2-groups. Use the original proof of the CFSG as a "template".
2. Use the first step to give a new and simplified proof of the CFSG.

There is the hope that several steps of the original proof of the CFSG become easier when working with fusion systems. For example, in the original proof of the CFSG, the study of centralizers of involutions plays an important role. The $2^{\prime}$-cores of the involution centralizers, i.e. their largest normal odd order subgroups, cause serious difficulties and are obstructions to many arguments. Such difficulties are not present in fusion systems since cores do not exist in fusion systems. This is suggested by the well-known fact that the 2 -fusion system of a finite group $G$ is isomorphic to the 2-fusion system of $G / O(G)$, where $O(G)$ denotes the $2^{\prime}$-core of $G$. For an outline of and recent progress on Aschbacher's programme, we refer to [8].

So far, Aschbacher's programme has focused mainly on Step 1, while not much has been done on Step 2. An important part of Step 2 is to identify finite simple groups from their 2 -fusion systems. The present paper contributes to Step 2 of Aschbacher's programme by characterizing the finite simple groups $P S L_{n}(q)$ and $P S U_{n}(q)$ in terms of their 2-fusion systems, where $n \geq 2$ and where $q$ is a nontrivial odd prime power with $(n, q) \neq(2,3)$.

In order to state our results, we introduce some notation and recall some definitions. Let $G$ be a finite group. A component of $G$ is a quasisimple subnormal subgroup of $G$, and a 2 component of $G$ is a perfect subnormal subgroup $L$ of $G$ such that $L / O(L)$ is quasisimple. The natural homomorphism $G \rightarrow G / O(G)$ induces a one-to-one correspondence between the set of 2components of $G$ and the set of components of $G / O(G)$ (see [28, Proposition 4.7]). We use $Z^{*}(G)$ to denote the full preimage of the center $Z(G / O(G))$ in $G$. In Step 2 of Aschbacher's programme, one may assume that a finite group $G$ is a minimal counterexample to the CFSG. Such a group $G$ has the following property.

$$
\begin{equation*}
\text { Whenever } x \in G \text { is an involution and } J \text { is a 2-component of } C_{G}(x) \text {, } \tag{CK}
\end{equation*}
$$ then $J / Z^{*}(J)$ is a known finite simple group.

By a known finite simple group, we mean a finite simple group appearing in the statement of the CFSG.

For each integer $n \neq 0$, we use $n_{2}$ to denote the 2-part of $n$, i.e. the largest power of 2 dividing $n$. Given odd integers $a, b$ with $|a|,|b|>1$, we write $a \sim b$ provided that $(a-1)_{2}=(b-1)_{2}$ and $(a+1)_{2}=(b+1)_{2}$. If $q$ is a nontrivial prime power and if $n$ is a positive integer, then we write $P S L_{n}^{+}(q)$ for $P S L_{n}(q)$ and $P S L_{n}^{-}(q)$ for $P S U_{n}(q)$. With this notation, we can now state our main results.

Theorem A. Let $q$ be a nontrivial odd prime power, and let $n \geq 2$ be a natural number. Let $G$ be a finite simple group. Suppose that $G$ satisfies (CK) if $n \geq 6$. Then the 2 -fusion system of $G$ is isomorphic to the 2 -fusion system of $P S L_{n}(q)$ if and only if one of the following holds:
(i) $G \cong P S L_{n}^{\varepsilon}\left(q^{*}\right)$ for some nontrivial odd prime power $q^{*}$ and some $\varepsilon \in\{+,-\}$ with $\varepsilon q^{*} \sim q$;
(ii) $n=2,\left|P S L_{2}(q)\right|_{2}=8$, and $G \cong A_{7}$;
(iii) $n=3,(q+1)_{2}=4$, and $G \cong M_{11}$.

Our second main result is an extension of Theorem A. In order to state it, we briefly mention some concepts from the local theory of fusion systems. Let $\mathcal{F}$ be a saturated fusion system on a finite $p$-group $S$ for some prime $p$, and let $\mathcal{E}$ be a normal subsystem of $\mathcal{F}$. In [7, Chapter 6], Aschbacher introduced a subgroup $C_{S}(\mathcal{E})$ of $S$, which plays the role of the centralizer of $\mathcal{E}$ in $S$. In [7, Chapter 9], he defined a normal subsystem $F^{*}(\mathcal{F})$ of $\mathcal{F}$, called the generalized Fitting subsystem of $\mathcal{F}$, and proved that $C_{S}\left(F^{*}(\mathcal{F})\right)=Z\left(F^{*}(\mathcal{F})\right)$, where the latter denotes the center of $F^{*}(\mathcal{F})$.

Theorem B. Let $q$ be a nontrivial odd prime power, and let $n \geq 2$ be a natural number. If $n=2$, suppose that $q \equiv 1$ or $7 \bmod 8$. Let $G$ be a finite simple group, and let $S$ be a Sylow 2 -subgroup of $G$. Suppose that $\mathcal{F}_{S}(G)$ has a normal subsystem $\mathcal{E}$ on a subgroup $T$ of $S$ such that $\mathcal{E}$ is isomorphic to the 2 -fusion system of $\operatorname{PS} L_{n}(q)$ and such that $C_{S}(\mathcal{E})=1$. Then $\mathcal{F}_{S}(G)$ is isomorphic to the 2 -fusion system of $P S L_{n}(q)$. In particular, if $n \leq 5$ or if $G$ satisfies (CK), then one of the properties (i)-(iii) from Theorem A holds.

Corollary C. Let $q$ be a nontrivial odd prime power, and let $n \geq 2$ be a natural number. If $n=2$, suppose that $q \equiv 1$ or $7 \bmod 8$. Let $G$ be a finite simple group, and let $S$ be a Sylow 2 -subgroup of $G$. Suppose that $F^{*}\left(\mathcal{F}_{S}(G)\right)$ is isomorphic to the 2 -fusion system of $P S L_{n}(q)$. Then $\mathcal{F}_{S}(G)$ is isomorphic to the 2 -fusion system of $\operatorname{PS} L_{n}(q)$. In particular, if $n \leq 5$ or if $G$ satisfies (CK), then one of the properties (i)-(iii) from Theorem $A$ holds.

The paper is organized as follows. In Sections 2 and 3, we collect several results needed for the proofs of our main results. Preliminary results on abstract finite groups and abstract fusion systems are proved in Section 2. Section 3 presents some results on linear and unitary groups over finite fields, mainly focussing on 2-local properties and on the automorphisms of these groups.

In Section 4, we will verify Theorem A for the case $n \leq 5$. Our proofs strongly depend on work of Gorenstein and Walter 31] (for $n=2$ ), on work of Alperin, Brauer and Gorenstein [2], 3] (for $n=3$ ) and on work of Mason [40], [41, [42] (for $n=4$ and $n=5$ ).

For $n \geq 6$, we will prove Theorem by induction over $n$. In order to do so, we will consider a finite group $G$ realizing the 2-fusion system of $P S L_{n}(q)$, where $q$ is a nontrivial odd prime power and where $n \geq 6$ is a natural number such that Theorem A is true with $m$ instead of $n$ for any natural number $m$ with $6 \leq m<n$. We will also assume that $O(G)=1$ and that $G$ satisfies (CK). To prove that Theorem $A$ is satisfied for the natural number $n$, we will prove the existence of a normal subgroup $G_{0}$ of $G$ such that $G_{0}$ is isomorphic to a nontrivial quotient of $S L_{n}^{\varepsilon}\left(q^{*}\right)$ for some nontrivial odd prime power $q^{*}$ and some $\varepsilon \in\{+,-\}$ with $\varepsilon q^{*} \sim q$. This will happen in Sections 558

In Section 5, we will introduce some notation and prove some preliminary lemmas. Section 6 describes the 2-components of the centralizers of involutions of $G$. In Section 7, we will use signalizer functor methods to describe the components of the centralizers of certain involutions of $G$. This will be used in Section 8 to construct the subgroup $G_{0}$ of $G$. One of the main tools here will be a version of the Curtis-Tits theorem [30, Chapter 13, Theorem 1.4] and a related theorem of Phan reproved by Bennett and Shpectorov in [14].

Finally, in Section 9, we will give a full proof of Theorem A (basically summarizing Sections 448 , and we will prove Theorem B and Corollary C.

Notation and Terminology. Our notation and terminology are fairly standard. The reader is referred to [24], [28], [37] for unfamiliar definitions on groups and to [11], [19] for unfamiliar definitions on fusion systems.

However, we shall now explain some particularly important notation and definitions (before stating our main results, we already introduced some other important definitions).

Given a map $\alpha: A \rightarrow B$ and an element or a subset $X$ of $A$, we write $X^{\alpha}$ for the image of $X$ under $\alpha$. Also, if $C \subseteq A$ and $D \subseteq B$ such that $C^{\alpha} \subseteq D$, we use $\left.\alpha\right|_{C, D}$ to denote the map $C \rightarrow D, c \mapsto c^{\alpha}$. Given two maps $\alpha: A \rightarrow B$ and $\beta: B \rightarrow C$, we write $\alpha \beta$ for the map $A \rightarrow C, a \mapsto\left(a^{\alpha}\right)^{\beta}$.

Sometimes, we will interprete the symbols + and - as the integers 1 and -1 , respectively. For example, if $n$ is an integer and if $\varepsilon$ is assumed to be an element of $\{+,-\}$, then $n \equiv \varepsilon \bmod 4$ shall express that $n \equiv 1 \bmod 4$ if $\varepsilon=+$ and that $n \equiv-1 \bmod 4$ if $\varepsilon=-$.

Let $G$ be a finite group. We write $G^{\#}$ for the set of non-identity elements of $G$. Given an element $g$ of $G$ and an element or a subset $X$ of $G$, we write $X^{g}$ for $g^{-1} X g$. The inner automorphism
$G \rightarrow G, x \mapsto x^{g}$ is denoted by $c_{g}$. For subgroups $Q$ and $H$ of $G$, we write $\operatorname{Aut}_{H}(Q)$ for the subgroup of $\operatorname{Aut}(Q)$ consisting of all automorphisms of $Q$ of the form $\left.c_{h}\right|_{Q, Q}$, where $h \in N_{H}(Q)$.

We write $L(G)$ for the subgroup of $G$ generated by the components of $G$ and $L_{2^{\prime}}(G)$ for the subgroup of $G$ generated by the 2 -components of $G$. We say that $G$ is core-free if $O(G)=1$. If $G$ is core-free and if $L$ is a subnormal subgroup of $G$, then $L$ is said to be a solvable 2-component of $G$ if $L \cong S L_{2}(3)$ or $P S L_{2}(3)$.

Let $n$ be a natural number. Then we use $E_{2^{n}}$ to denote an elementary abelian 2-group of order $2^{n}$, and we say that $n$ is the rank of $E_{2^{n}}$. The maximal rank of an elementary abelian 2 -subgroup of a finite 2 -group $S$ is said to be the rank of $S$. It is denoted by $m(S)$.

Now let $p$ be a prime, and let $\mathcal{F}$ be a fusion system on a finite $p$-group $S$. Then $S$ is said to be the Sylow group of $\mathcal{F}$, and $\mathcal{F}$ is said to be nilpotent if $\mathcal{F}=\mathcal{F}_{S}(S)$. Given a fusion system $\mathcal{F}_{1}$ on a finite $p$-group $S_{1}$, we say that $\mathcal{F}$ and $\mathcal{F}_{1}$ are isomorphic if there is a group isomorphism $\varphi: S \rightarrow S_{1}$ such that

$$
\operatorname{Hom}_{\mathcal{F}_{1}}\left(Q^{\varphi}, R^{\varphi}\right)=\left\{\left(\left.\varphi^{-1}\right|_{Q^{\varphi}, Q}\right) \psi\left(\left.\varphi\right|_{R, R^{\varphi}}\right) \mid \psi \in \operatorname{Hom}_{\mathcal{F}}(Q, R)\right\}
$$

for all $Q, R \leq S$. In this case, we say that $\varphi$ induces an isomorphism from $\mathcal{F}$ to $\mathcal{F}_{1}$. Let $Q$ be a normal subgroup of $S$. If $P$ and $R$ are subgroups of $S$ containing $Q$ and if $\alpha: P \rightarrow R$ is a morphism in $\mathcal{F}$ such that $Q^{\alpha}=Q$, we write $\alpha / Q$ for the group homomorphism $P / Q \rightarrow R / Q$ induced by $\alpha$. The fusion system $\mathcal{F} / Q$ on $S / Q$ with $\operatorname{Hom}_{\mathcal{F} / Q}(P / Q, R / Q)=\left\{\alpha / Q \mid \alpha \in \operatorname{Hom}_{\mathcal{F}}(P, R), Q^{\alpha}=Q\right\}$ for all $P, R \leq S$ containing $Q$ is said to be the factor system of $\mathcal{F}$ modulo $Q$.

Suppose now that $\mathcal{F}$ is saturated. We write $\mathfrak{f o c}(\mathcal{F})$ for the focal subgroup of $\mathcal{F}$ and $\mathfrak{h n p}(\mathcal{F})$ for the hyperfocal subgroup of $\mathcal{F}$. We say that $\mathcal{F}$ is quasisimple if $\mathcal{F} / Z(\mathcal{F})$ is simple and $\mathfrak{f o c}(\mathcal{F})=S$. A component of $\mathcal{F}$ is a subnormal quasisimple subsystem of $\mathcal{F}$. Given a normal subsystem $\mathcal{E}$ of $S$ and a subgroup $R$ of $S$, we write $\mathcal{E} R$ for the product of $\mathcal{E}$ and $R$, as defined in [7, Chapter 8].

## 2. Preliminaries on finite groups and fusion systems

In this section, we present some general results on finite groups and fusion systems.

### 2.1. Preliminaries on finite groups.

Lemma 2.1. ([37, 3.2.8]) Let $G$ be a finite group, and let $N$ be a normal $p^{\prime}$-subgroup of $G$ for some prime $p$. Set $\bar{G}:=G / N$. If $R$ is a p-subgroup of $G$, then we have $N_{\bar{G}}(\bar{R})=\overline{N_{G}(R)}$ and $C_{\bar{G}}(\bar{R})=\overline{C_{G}(R)}$.

Corollary 2.2. Let $G$ be a finite group, and let $N$ be a normal $p^{\prime}$-subgroup of $G$ for some prime $p$. Set $\bar{G}:=G / N$. If $x \in G$ has order $p$, then we have $C_{\bar{G}}(\bar{x})=\overline{C_{G}(x)}$.

Lemma 2.3. Let $G$ be a finite group, and let $Z$ be a cyclic central subgroup of $G$. Then each $E_{8}$-subgroup of $G / Z$ has an involution which is the image of an involution of $G$.

Proof. Let $Z \leq E \leq G$ such that $E / Z \cong E_{8}$. Let $R$ be a Sylow 2-subgroup of $E$. Then $E=R Z$. It suffices to show that $R$ has an involution not lying in $R \cap Z$. Assume that any involution of $R$ is an element of $R \cap Z$. Then $R$ has a unique involution since $Z$ is cyclic. We have $R /(R \cap Z) \cong R Z / Z=E / Z \cong E_{8}$, and so $R$ is not cyclic. Applying [37, 5.3.7], we conclude that $R$ is generalized quaternion. In particular, $Z(R)$ has order 2 , and so we have $R \cap Z=Z(R)$. Since $R$ is a generalized quaternion group, $R / Z(R)$ is dihedral. In particular, $E / Z \cong R /(R \cap Z)=R / Z(R) \nsubseteq E_{8}$. This contradiction shows that $R$ has an involution not lying in $R \cap Z$, as required.

The following proposition is well-known. We include a proof since we could not find a reference in which it appears in the form given here.

Proposition 2.4. Let $G$ be a finite group, and let $N$ be a normal subgroup of $G$ with odd order. If $L$ is a 2 -component of $G$, then $L N / N$ is a 2 -component of $G / N$. The map from the set of 2components of $G$ to the set of 2-components of $G / N$ sending each 2 -component $L$ of $G$ to $L N / N$ is a bijection. Moreover, if $N \leq K \leq G$ and $K / N$ is a 2 -component of $G / N$, then $O^{2^{\prime}}(K)$ is the associated 2 -component of $G$.

Proof. Let $L$ be a 2-component of $G$. Hence, $L$ is a perfect subnormal subgroup of $G$ such that $L / O(L)$ is quasisimple. Clearly, $L N / N$ is perfect and subnormal in $G / N$. Also, we have $(L N / N) / O(L N / N) \cong L / O(L)$, and so $(L N / N) / O(L N / N)$ is quasisimple. It follows that $L N / N$ is a 2 -component of $G / N$.

Let $N \leq K \leq G$ such that $K / N$ is a 2 -component of $G / N$. In order to prove the second statement of the proposition, it is enough to show that there is precisely one 2-component $L$ of $G$ such that $L N / N=K / N$.

Since $K / N$ is subnormal in $G / N$, we have that $K$ is subnormal in $G$. Therefore, $L:=O^{2^{\prime}}(K)$ is subnormal in $G$. Since $O^{2^{\prime}}(K / N)=K / N$, we have that $K / N=L N / N$. Clearly, $O^{2^{\prime}}(L)=L$. We have $L / O(L) \cong(L N / N) / O(L N / N)=(K / N) / O(K / N)$, and so $L / O(L)$ is quasisimple. Applying [28, Lemma 4.8], we conclude that $L$ is a 2-component of $G$.

Now let $L_{0}$ be a 2-component of $G$ such that $K / N=L_{0} N / N$. Then $K=L_{0} N$. In particular, $L_{0}$ is a subgroup of $K$ with odd index in $K$. Since $L_{0}$ is subnormal in $G$, we have that $L_{0}$ is subnormal in $K$. Applying [13, Lemma 1.1.11], we conclude that $L_{0}=O^{2^{\prime}}\left(L_{0}\right)=O^{2^{\prime}}(K)=L$. The proof of the second statement of the proposition is now complete. The third statement also follows from the above arguments.

Lemma 2.5. Let $G$ be a finite group, and let $n$ be a positive integer. Assume that $L_{1}, \ldots, L_{n}$ are the distinct 2 -components of $G$, and assume that $L_{i} \unlhd G$ for all $1 \leq i \leq n$. Let $x$ be a 2-element of $G$, and let $L$ be a 2-component of $C_{G}(x)$. Then $L$ is a 2 -component of $C_{L_{i}}(x)$ for some $1 \leq i \leq n$.

Proof. By [32, Corollary 3.2], we have $L_{2^{\prime}}\left(C_{G}(x)\right)=L_{2^{\prime}}\left(C_{L_{2^{\prime}}(G)}(x)\right)$, and by [32, Lemma 2.18 (iii)], we have $L_{2^{\prime}}\left(C_{L_{2^{\prime}}(G)}(x)\right)=\prod_{i=1}^{n} L_{2^{\prime}}\left(C_{L_{i}}(x)\right)$. Using basic properties of 2-components, as presented in [28, Proposition 4.7], it is not hard to deduce that $L$ is a 2-component of $C_{L_{i}}(x)$ for some $1 \leq i \leq n$.

The concepts introduced by the following two definitions will play a crucial role in the proof of Theorem A (see [32] for a detailed study of these concepts).

Definition 2.6. Let $G$ be a finite group, $k$ be a positive integer and $A$ be an elementary abelian 2-subgroup of $G$.
(i) For each nontrivial elementary abelian 2-subgroup $E$ of $G$, we define

$$
\Delta_{G}(E):=\bigcap_{a \in E^{\#}} O\left(C_{G}(a)\right) .
$$

(ii) We say that $G$ is $k$-balanced with respect to $A$ if whenever $E$ is a subgroup of $A$ of $\operatorname{rank} k$ and $a$ is a non-trivial element of $A$, we have

$$
\Delta_{G}(E) \cap C_{G}(a) \leq O\left(C_{G}(a)\right)
$$

(iii) We say that $G$ is $k$-balanced if whenever $E$ is an elementary abelian 2-subgroup of $G$ of rank $k$ and $a$ is an involution of $G$ centralizing $E$, we have

$$
\Delta_{G}(E) \cap C_{G}(a) \leq O\left(C_{G}(a)\right) .
$$

(iv) By saying that $G$ is balanced (respectively, balanced with respect to $A$ ), we mean that $G$ is 1-balanced (respectively, 1-balanced with respect to $A$ ).

Definition 2.7. Let $G$ be a finite quasisimple group, and let $k$ be a positive integer. Then $G$ is said to be locally $k$-balanced if whenever $H$ is a subgroup of $\operatorname{Aut}(G)$ containing $\operatorname{Inn}(G)$, we have

$$
\Delta_{H}(E)=1
$$

for any elementary abelian 2-subgroup $E$ of $H$ of rank $k$. We say that $G$ is locally balanced if $G$ is locally 1 -balanced.

We need the following proposition for the proof of Theorem A, It includes [32, Theorem 6.10] and some additional statements, which should be also known. We include a proof for the convenience of the reader.
Proposition 2.8. Let $k$ be a positive integer, and let $G$ be a finite group. For each elementary abelian 2-subgroup $A$ of $G$ of rank at least $k+1$, let

$$
W_{A}:=\left\langle\Delta_{G}(E) \mid E \leq A, m(E)=k\right\rangle .
$$

Then, for any elementary abelian 2-subgroup $A$ of $G$ of rank at least $k+1$, the following hold:
(i) $\left(W_{A}\right)^{g}=W_{A^{g}}$ for all $g \in G$.
(ii) Suppose that $A$ has rank at least $k+2$ and that $G$ is $k$-balanced with respect to $A$. Then $W_{A}$ has odd order. Moreover, if $A_{0}$ is a subgroup of $A$ of rank at least $k+1$, then we have $W_{A}=W_{A_{0}}$ and $N_{G}\left(A_{0}\right) \leq N_{G}\left(W_{A}\right)$.
In order to prove Proposition 2.8, we need the following theorem.
Theorem 2.9. ([32, Theorem 6.9]) Let $k$ be a positive integer, $G$ be a finite group and $A$ be an elementary abelian 2-subgroup of $G$ of rank at least $k+2$. Suppose that $G$ is $k$-balanced with respect to $A$. Then we obtain an A-signalizer functor on $G$ (in the sense of [25, Definition 4.37]) by defining

$$
\theta\left(C_{G}(a)\right):=\left\langle\Delta_{G}(E) \cap C_{G}(a): E \leq A, m(E)=k\right\rangle
$$

for each $a \in A^{\#}$.
We also need the following lemma.
Lemma 2.10. Let the notation be as in Theorem 2.9. Suppose that $A_{0}$ is subgroup of $A$ of rank $k+1$. Then we have

$$
\theta(G, A):=\left\langle\theta\left(C_{G}(a)\right) \mid a \in A^{\#}\right\rangle=\left\langle\Delta_{G}(E) \mid E \leq A_{0}, m(E)=k\right\rangle=: W_{A_{0}}
$$

Proof. To prove this, we follow arguments found on pp. 40-41 of [40].
Since $\theta$ is an $A$-signalizer functor on $G, \theta\left(C_{G}(a)\right)$ is $A$-invariant and in particular $A_{0}$-invariant for each $a \in A^{\#}$. Consequently, $\theta(G, A)$ is $A_{0}$-invariant. By the Solvable Signalizer Functor Theorem [37, 11.3.2], $\theta$ is complete (in the sense of [25, Definition 4.37]). In particular, $\theta(G, A)$ has odd order. Applying [28, Proposition 11.23], we conclude that

$$
\theta(G, A)=\left\langle C_{\theta(G, A)}(E) \mid E \leq A_{0}, m(E)=k\right\rangle .
$$

Since $\theta$ is complete, we have $C_{\theta(G, A)}(a)=\theta\left(C_{G}(a)\right)$ for each $a \in A^{\#}$. By definition of $\theta$ and since $G$ is $k$-balanced with respect to $A$, we have $\theta\left(C_{G}(a)\right) \leq O\left(C_{G}(a)\right)$ for each $a \in A^{\#}$. So, if $E$ is a subgroup of $A_{0}$ of rank $k$, then

$$
C_{\theta(G, A)}(E)=\bigcap_{a \in E^{\#}} C_{\theta(G, A)}(a)=\bigcap_{a \in E^{\#}} \theta\left(C_{G}(a)\right) \leq \bigcap_{a \in E^{\#}} O\left(C_{G}(a)\right)=\Delta_{G}(E) .
$$

It follows that $\theta(G, A) \leq W_{A_{0}}$.
Let $E \leq A_{0}$ with $m(E)=k$. Clearly, $\Delta_{G}(E)$ is $A$-invariant. As a consequence of [28], Proposition 11.23], we have

$$
\Delta_{G}(E)=\left\langle\Delta_{G}(E) \cap C_{G}(a) \mid a \in A^{\#}\right\rangle .
$$

By definition of $\theta$, we have $\Delta_{G}(E) \cap C_{G}(a) \leq \theta\left(C_{G}(a)\right)$ for each $a \in A^{\#}$. It follows that $\Delta_{G}(E) \leq$ $\theta(G, A)$. Consequently, $W_{A_{0}} \leq \theta(G, A)$.

Proof of Proposition 2.8. It is straightforward to verify (i).
To verify (ii), let $A$ be an elementary abelian 2 -subgroup of $G$ of rank at least $k+2$ such that $G$ is $k$-balanced with respect to $A$. Let $\theta$ be the $A$-signalizer functor on $G$ given by Theorem 2.9 , and let $\theta(G, A):=\left\langle\theta\left(C_{G}(a)\right) \mid a \in A^{\#}\right\rangle$. As a consequence of Lemma 2.10, we have $\theta(G, A)=W_{A}$. By the proof of Lemma 2.10, $W_{A}=\theta(G, A)$ has odd order.

Now let $A_{0}$ be a subgroup of $A$ of rank at least $k+1$. By Lemma 2.10, $W_{A}=\theta(G, A) \leq W_{A_{0}} \leq$ $W_{A}$, and so $W_{A}=W_{A_{0}}$. Finally, if $g \in N_{G}\left(A_{0}\right)$, then $\left(W_{A}\right)^{g}=\left(W_{A_{0}}\right)^{g}=W_{\left(A_{0}\right)^{g}}=W_{A_{0}}=W_{A}$, and hence $N_{G}\left(A_{0}\right) \leq N_{G}\left(W_{A}\right)$.

### 2.2. Preliminaries on fusion systems.

Lemma 2.11. Let $p$ be a prime, $G$ be a finite group, $N$ be a normal subgroup of $G$ and $S \in$ $\operatorname{Syl}_{p}(G)$. Then the canonical group isomorphism $S /(S \cap N) \rightarrow S N / N$ induces an isomorphism from $\mathcal{F}_{S}(G) /(S \cap N)$ to $\mathcal{F}_{S N / N}(G / N)$.

Proof. Let $\varphi$ denote the canonical group isomorphism $S /(S \cap N) \rightarrow S N / N$. Let $P$ and $Q$ be two subgroups of $S$ such that $S \cap N$ is contained in both $P$ and $Q$. Set $\widetilde{P}:=P /(S \cap N), \widetilde{Q}:=Q /(S \cap N)$, $\bar{P}:=P N / N$ and $\bar{Q}:=Q N / N$. Moreover, define $\widetilde{\mathcal{F}}:=\mathcal{F}_{S}(G) /(S \cap N)$ and $\overline{\mathcal{F}}:=\mathcal{F}_{S N / N}(G / N)$. It is enough to show that

$$
\operatorname{Hom}_{\overline{\mathcal{F}}}(\bar{P}, \bar{Q})=\left\{\left(\left.\varphi^{-1}\right|_{\bar{P}, \widetilde{P}}\right) \alpha\left(\left.\varphi\right|_{\widetilde{Q}, \bar{Q}}\right) \mid \alpha \in \operatorname{Hom}_{\widetilde{\mathcal{F}}}(\widetilde{P}, \widetilde{Q})\right\}
$$

Let $\alpha \in \operatorname{Hom}_{\widetilde{\mathcal{F}}}(\widetilde{P}, \widetilde{Q})$. Then there exists $g \in G$ with $P^{g} \leq Q$ and $\alpha=\left(c_{g} \mid P, Q\right) /(S \cap N)$. By a direct calculation, $\left(\left.\varphi^{-1}\right|_{\bar{P}, \widetilde{P}}\right) \alpha\left(\left.\varphi\right|_{\widetilde{Q}, \bar{Q}}\right)=\left.c_{g N}\right|_{\bar{P}, \bar{Q}} \in \operatorname{Hom}_{\overline{\mathcal{F}}}(\bar{P}, \bar{Q})$.

Now let $\bar{\alpha} \in \operatorname{Hom}_{\overline{\mathcal{F}}}(\bar{P}, \bar{Q})$. Then there exists $g \in G$ with $\bar{P}^{g N} \leq \bar{Q}$ and $\bar{\alpha}=\left.c_{g N}\right|_{\bar{P}, \bar{Q}}$. Clearly, $P^{g} \leq Q N$. Since $S \cap N \leq Q$, we have that $Q$ is a Sylow $p$-subgroup of $Q N$. Since $P^{g}$ is a $p$-subgroup of $Q N$, it follows that there exists an element $n \in N$ with $P^{g n} \leq Q$. Set $\alpha:=\left(\left.c_{g n}\right|_{P, Q}\right) /(S \cap N)$. Then a direct calculation shows that $\bar{\alpha}=\left(\left.\varphi^{-1}\right|_{\bar{P}, \widetilde{P}}\right) \alpha\left(\left.\varphi\right|_{\widetilde{Q}, \bar{Q}}\right)$.

Corollary 2.12. ([11, Part II, Exercise 2.1]) Let p be a prime, $G$ be a finite group and $S \in \operatorname{Syl}_{p}(G)$. Then the canonical group isomorphism $S \rightarrow \bar{S}:=S O_{p^{\prime}}(G) / O_{p^{\prime}}(G)$ induces an isomorphism from $\mathcal{F}_{S}(G)$ to $\mathcal{F}_{\bar{S}}\left(G / O_{p^{\prime}}(G)\right)$.
Lemma 2.13. Let $K_{1}$ and $K_{2}$ be two quasisimple finite groups. If the 2-fusion systems of $K_{1}$ and $K_{2}$ are isomorphic, then the 2-fusion systems of $K_{1} / Z\left(K_{1}\right)$ and $K_{2} / Z\left(K_{2}\right)$ are isomorphic.

Proof. Suppose that the 2 -fusion systems of $K_{1}$ and $K_{2}$ are isomorphic. Let $S_{i}$ be a Sylow 2subgroup of $K_{i}$ and $\mathcal{F}_{i}:=\mathcal{F}_{S_{i}}\left(K_{i}\right)$ for $i \in\{1,2\}$. As a consequence of [23, Corollary 1], we have $Z\left(\mathcal{F}_{i}\right)=S_{i} \cap Z^{*}\left(K_{i}\right)$ for $i \in\{1,2\}$. Since $K_{1}$ and $K_{2}$ are quasisimple, we have $Z^{*}\left(K_{i}\right)=Z\left(K_{i}\right)$ and hence $Z\left(\mathcal{F}_{i}\right)=S_{i} \cap Z\left(K_{i}\right)$ for $i \in\{1,2\}$. Since $\mathcal{F}_{1} \cong \mathcal{F}_{2}$, it follows that

$$
\mathcal{F}_{1} /\left(S_{1} \cap Z\left(K_{1}\right)\right)=\mathcal{F}_{1} / Z\left(\mathcal{F}_{1}\right) \cong \mathcal{F}_{2} / Z\left(\mathcal{F}_{2}\right)=\mathcal{F}_{2} /\left(S_{2} \cap Z\left(K_{2}\right)\right)
$$

Applying Lemma 2.11, we may conclude that the 2-fusion system of $K_{1} / Z\left(K_{1}\right)$ is isomorphic to the 2 -fusion system of $K_{2} / Z\left(K_{2}\right)$.

Lemma 2.14. Let $S$ be a finite 2-group, and let $A$ and $B$ be normal subgroups of $S$ such that $S$ is the internal direct product of $A$ and $B$. Suppose that $A \cong Q_{8}$. Let $\mathcal{F}$ be a (not necessarily saturated) fusion system on $S$. Assume that $A$ and $B$ are strongly $\mathcal{F}$-closed and that there is an automorphism $\alpha \in \operatorname{Aut}_{\mathcal{F}}(S)$ such that $\left.\alpha\right|_{A, A}$ has order 3 , while $\left.\alpha\right|_{B, B}=\mathrm{id}_{B}$. Then each strongly $\mathcal{F}$-closed subgroup of $S$ contains or centralizes $A$.
Proof. Let $C$ be a strongly $\mathcal{F}$-closed subgroup of $S$ not containing $A$. Our task is to show that $C$ centralizes $A$.

Since $A$ and $C$ are strongly $\mathcal{F}$-closed, we have that $A \cap C$ is strongly $\mathcal{F}$-closed. In particular, $\alpha$ normalizes $A \cap C$. It is easy to see that an automorphism of $Q_{8}$ with order 3 does not normalize
any maximal subgroup of $Q_{8}$. So, as $\left.\alpha\right|_{A, A}$ has order 3 and normalizes $A \cap C$, we have that $A \cap C$ has order 1 or 2 .

By [37, 8.2.7], we have

$$
[C,\langle\alpha\rangle]=[[C,\langle\alpha\rangle],\langle\alpha\rangle] .
$$

We claim that $[C,\langle\alpha\rangle] \leq A \cap C$. Let $c \in C$ and $\beta \in\langle\alpha\rangle$. Let $a \in A$ and $b \in B$ such that $c=a b$. Since $A$ and $B$ commute and since $\beta$ normalizes $A$ and centralizes $B$, we have

$$
[c, \beta]=c^{-1} c^{\beta}=b^{-1} a^{-1} a^{\beta} b^{\beta}=a^{-1} a^{\beta} \in A \cap C
$$

Thus $[C,\langle\alpha\rangle] \leq A \cap C$, as asserted.
Since $A \cap C$ has order 1 or 2 , we have $[A \cap C,\langle\alpha\rangle]=1$. So it follows that

$$
[C,\langle\alpha\rangle]=[[C,\langle\alpha\rangle],\langle\alpha\rangle] \leq[A \cap C,\langle\alpha\rangle]=1 .
$$

Now we prove that $C$ centralizes $A$. Let $c \in C$ and $a \in A, b \in B$ with $c=a b$. We have $c^{-1} c^{\alpha} \in[C,\langle\alpha\rangle]=1$, whence $c^{\alpha}=c$. Thus $a b=(a b)^{\alpha}=a^{\alpha} b$ and hence $a=a^{\alpha}$. As remarked above, $\alpha$ does not normalize any maximal subgroup of $A$. So $a$ cannot have order 4. By the structure of $A \cong Q_{8}$, it follows that $a \in Z(A)$. This implies that $c=a b$ centralizes $A$.

We need the following definition in order to state the next proposition.
Definition 2.15. A nonabelian finite simple group $G$ is said to be a Goldschmidt group provided that one of the following holds:
(1) $G$ has an abelian Sylow 2-subgroup.
(2) $G$ is isomorphic to a finite simple group of Lie type in characteristic 2 of Lie rank 1.

Proposition 2.16. Let $G$ be a finite group, and let $S$ be a Sylow 2-subgroup of $G$. Assume that for each 2 -component $L$ of $G$, the factor group $L / Z^{*}(L)$ is a known finite simple group. Let $\mathfrak{L}_{2^{\prime}}$ denote the set of 2 -components $L$ of $G$ such that $L / Z^{*}(L)$ is not a Goldschmidt group. Then the following hold:
(i) Let $L$ be a 2-component of $G$. Then $\mathcal{F}_{S \cap L}(L)$ is a component of $\mathcal{F}_{S}(G)$ if and only if $L \in \mathfrak{L}_{2^{\prime}}$.
(ii) The map from $\mathfrak{L}_{2^{\prime}}$ to the set of components of $\mathcal{F}_{S}(G)$ sending each element $L$ of $\mathfrak{L}_{2^{\prime}}$ to $\mathcal{F}_{S \cap L}(L)$ is a bijection.
Proof. Let $L$ be a 2-component of $G$. Set $\mathcal{G}:=\mathcal{F}_{S \cap L}(L)$. Since $L$ is subnormal in $G$, we have that $\mathcal{G}$ is subnormal in $\mathcal{F}_{S}(G)$ (see [11, Part I, Proposition 6.2]). Therefore, $\mathcal{G}$ is a component of $\mathcal{F}_{S}(G)$ if and only if $\mathcal{G}$ is quasisimple. We have $\mathfrak{f o c}(\mathcal{G})=S \cap L^{\prime}=S \cap L$ by the focal subgroup theorem [24, Chapter 7, Theorem 3.4], and so $\mathcal{G}$ is quasisimple if and only if $\mathcal{G} / Z(\mathcal{G})$ is simple. As a consequence of [23, Corollary 1], we have $Z(\mathcal{G})=S \cap Z^{*}(L)$. Lemma 2.11 implies that $\mathcal{G} / Z(\mathcal{G})$ is isomorphic to the 2 -fusion system of $L / Z^{*}(L)$. By [10, Theorem 5.6.18], the 2 -fusion system of $L / Z^{*}(L)$ is simple if and only if $L \in \mathfrak{L}_{2^{\prime}}$. So $\mathcal{G}$ is a component of $\mathcal{F}_{S}(G)$ if and only if $L \in \mathfrak{L}_{2^{\prime}}$, and (i) holds.
(ii) follows from [9, (1.8)].

Lemma 2.17. Let $G$ be a finite group with $O(G)=1$, and let $S$ be a Sylow 2-subgroup of $G$. Let $n \geq 1$ be a natural number, and let $L_{1}, \ldots, L_{n}$ be pairwise distinct subgroups of $G$ such that $L_{i}$ is either a component or a solvable 2-component of $G$ for each $1 \leq i \leq n$. Set $Q:=$ $\left(S \cap L_{1}\right) \cdots\left(S \cap L_{n}\right)$. Assume that $Q \unlhd S$ and that $\mathcal{F}_{S}(G) / Q$ is nilpotent. Then, if $L_{0}$ is a component or a solvable 2 -component of $G$, we have $L_{0}=L_{i}$ for some $1 \leq i \leq n$.
Proof. Let $L^{s}(G)$ denote the subgroup of $G$ generated by the components and the solvable 2components of $G$. By [37, 6.5.2] and [28, Proposition 13.5], $L^{s}(G)$ is the central product of the subgroups of $G$ which are components or solvable 2-components. Set $L:=L_{1} \cdots L_{n} \unlhd L^{s}(G)$.

Let $\mathcal{G}:=\mathcal{F}_{S \cap L^{s}(G)}\left(L^{s}(G)\right)$. Clearly, $S \cap L=\left(S \cap L_{1}\right) \cdots\left(S \cap L_{n}\right)=Q$. Lemma 2.11 implies that the 2-fusion system of $L^{s}(G) / L$ is isomorphic to $\mathcal{G} / Q$. By hypothesis, $\mathcal{F}_{S}(G) / Q$ is nilpotent,
and so $\mathcal{G} / Q$ is nilpotent. So the 2-fusion system of $L^{s}(G) / L$ is nilpotent. Applying [39, Theorem 1.4], we conclude that $L^{s}(G) / L$ is 2-nilpotent.

Now let $L_{0}$ be a component or a solvable 2-component of $G$. If $L_{0} \leq L$, then we have $L_{0}=L_{i}$ for some $1 \leq i \leq n$ since otherwise $L_{0} \leq Z(L)$, which is impossible. So it suffices to show that $L_{0} \leq L$.

If $L_{0}$ is a component of $G$, then $L_{0} /\left(L_{0} \cap L\right)$ is both perfect and 2-nilpotent, which implies that $L_{0} \leq L$, as needed.

Suppose now that $L_{0}$ is a solvable 2-component of $G$. Assume that $L_{0} \not \leq L$. Then $L_{0} \cap L \leq$ $Z\left(L_{0}\right)$. Since $L_{0}$ is a solvable 2-component of $G$, it follows that $L_{0} /\left(L_{0} \cap L\right)$ is isomorphic to $S L_{2}(3)$ or $P S L_{2}(3)$. On the other hand, $L_{0} /\left(L_{0} \cap L\right)$ is 2-nilpotent. This contradiction shows that $L_{0} \leq L$, as required.

Corollary 2.18. Let $G$ be a finite group, and let $S$ be a Sylow 2-subgroup of $G$. Let $n \geq 1$ be a natural number, and let $L_{1}, \ldots, L_{n}$ be pairwise distinct 2 -components of $G$. Assume that $Q:=\left(S \cap L_{1}\right) \cdots\left(S \cap L_{n}\right)$ is a normal subgroup of $S$ and that $\mathcal{F}_{S}(G) / Q$ is nilpotent. Then, if $L_{0}$ is a 2-component of $G$, we have $L_{0}=L_{i}$ for some $1 \leq i \leq n$.

Proposition 2.19. Let $p$ be a prime, and let $\mathcal{E}$ be a simple saturated fusion system on a finite p-group $T$. Suppose that $\mathcal{E}$ is tamely realized (in the sense of [4, Section 2.2]) by a nonabelian known finite simple group $K$ such that $\operatorname{Out}(K)$ is p-nilpotent. Assume moreover that $G$ is a nonabelian finite simple group containing a Sylow p-subgroup $S$ with $T \leq S$ such that $\mathcal{E} \unlhd \mathcal{F}_{S}(G)$ and $C_{S}(\mathcal{E})=1$. Then $\mathcal{F}_{S}(G)$ is tamely realized by a subgroup $L$ of $\operatorname{Aut}(K)$ containing $\operatorname{Inn}(K)$ such that the index of $\operatorname{Inn}(K)$ in $L$ is coprime to $p$.

Proof. Set $\mathcal{F}:=\mathcal{F}_{S}(G)$. By a result of Bob Oliver, namely by [44, Corollary 2.4], $\mathcal{F}$ is tamely realized by a subgroup $L$ of $\operatorname{Aut}(K)$ containing $\operatorname{Inn}(K)$. We are going to show that the index of $\operatorname{Inn}(K)$ in $L$ is coprime to $p$.

Let $S_{0}$ be a Sylow $p$-subgroup of $L$. Then $\mathcal{F} \cong \mathcal{F}_{S_{0}}(L)$. Clearly, $O^{p}(G)=G$, and so $\mathfrak{h n p}(\mathcal{F})=S$ by the hyperfocal subgroup theorem [19, Theorem 1.33]. It follows that $\mathfrak{h n p}\left(\mathcal{F}_{S_{0}}(L)\right)=S_{0}$.

By the hyperfocal subgroup theorem [19, Theorem 1.33], $S_{0}=\mathfrak{h n p}\left(\mathcal{F}_{S_{0}}(L)\right)=O^{p}(L) \cap S_{0}$. Consequently, $O^{p}(L)$ has $p^{\prime}$-index in $L$, whence $O^{p}(L)=L$. So we have $O^{p}(L / \operatorname{Inn}(K))=L / \operatorname{Inn}(K)$. On the other hand, $L / \operatorname{Inn}(K)$ is $p$-nilpotent since $\operatorname{Out}(K)$ is $p$-nilpotent. It follows that $L / \operatorname{Inn}(K)$ is a $p^{\prime}$-group, as claimed.

## 3. Auxiliary results on linear and unitary groups

In this section, we collect several results on linear and unitary groups needed for the proofs of our main results. Some of the results stated here are known, while others seem to be new. For the convenience of the reader, we also include proofs of known results when we could not find a reference in which they appear in the form stated here.
3.1. Basic definitions. We begin with some basic definitions. Let $q$ be a nontrivial prime power, and let $n$ be a positive integer. The general linear group $G L_{n}(q)$ is the group of all invertible $n \times n$ matrices over $\mathbb{F}_{q}$ under matrix multiplication. The special linear group $S L_{n}(q)$ is the subgroup of $G L_{n}(q)$ consisting of all $n \times n$ matrices over $\mathbb{F}_{q}$ with determinant 1 . The center of $G L_{n}(q)$ consists of all scalar matrices $\lambda I_{n}$ with $\lambda \in\left(\mathbb{F}_{q}\right)^{*}$. We have $Z\left(S L_{n}(q)\right)=S L_{n}(q) \cap Z\left(G L_{n}(q)\right)$. Set $P G L_{n}(q):=G L_{n}(q) / Z\left(G L_{n}(q)\right)$ and $P S L_{n}(q):=S L_{n}(q) / Z\left(S L_{n}(q)\right)$. By [35, Kapitel II, Satz 6.10] and [35, Kapitel II, Hauptsatz 6.13], $S L_{n}(q)$ is quasisimple if $n \geq 2$ and $(n, q) \neq(2,2),(2,3)$.

As in [35, Kapitel II, Bemerkung $10.5(\mathrm{~b})$ ], we consider the general unitary group $G U_{n}(q)$ as the subgroup of $G L_{n}\left(q^{2}\right)$ consisting of all $\left(a_{i j}\right) \in G L_{n}\left(q^{2}\right)$ satisfying the condition $\left(\left(a_{i j}\right)^{q}\right)\left(a_{i j}\right)^{t}=I_{n}$. The special unitary group $S U_{n}(q)$ is the subgroup of $G U_{n}(q)$ consisting of all elements of $G U_{n}(q)$ with determinant 1. By [35, Kapitel II, Hilfssatz 8.8], we have $S L_{2}(q) \cong S U_{2}(q)$. The center of $G U_{n}(q)$ consists of all scalar matrices $\lambda I_{n}$, where $\lambda \in\left(\mathbb{F}_{q^{2}}\right)^{*}$ and $\lambda^{q+1}=1$. We have $Z\left(S U_{n}(q)\right)=$
$S U_{n}(q) \cap Z\left(G U_{n}(q)\right)$. Set $P G U_{n}(q):=G U_{n}(q) / Z\left(G U_{n}(q)\right)$ and $P S U_{n}(q):=S U_{n}(q) / Z\left(S U_{n}(q)\right)$. By [33, Theorems 11.22 and 11.26], $S U_{n}(q)$ is quasisimple if $n \geq 2$ and $(n, q) \neq(2,2),(2,3),(3,2)$.

We write $(P) G L_{n}^{+}(q)$ and $(P) S L_{n}^{+}(q)$ for $(P) G L_{n}(q)$ and $(P) S L_{n}(q)$, respectively. Also, we write $(P) G L_{n}^{-}(q)$ for $(P) G U_{n}(q)$ and $(P) S L_{n}^{-}(q)$ for $P S U_{n}(q)$.
3.2. Central extensions of $P S L_{n}(q)$ and $P S U_{n}(q)$. In the proofs of the following two lemmas, we use the terminology of [5, Section 33].
Lemma 3.1. Let $n \geq 3$ be a natural number, and let $q$ be a nontrivial odd prime power. Let $H$ be a perfect central extension of $P S L_{n}(q)$. Then there is a subgroup $Z \leq Z\left(S L_{n}(q)\right)$ such that $H \cong S L_{n}(q) / Z$.

Proof. By [29, pp. 312-313], the Schur multiplier of $\operatorname{PS} L_{n}(q)$ is isomorphic to $C_{(n, q-1)} \cong Z\left(S L_{n}(q)\right)$. From [5, 33.6], we see that this is just another way to say that $S L_{n}(q)$ is the universal covering group of $\operatorname{PS} L_{n}(q)$. Applying [5, 33.6] again, we conclude that $H \cong S L_{n}(q) / Z$ for some $Z \leq Z\left(S L_{n}(q)\right)$.
Lemma 3.2. Let $n \geq 3$ be a natural number, and let $q$ be a nontrivial odd prime power. Let $H$ be a perfect central extension of $\operatorname{PSU}_{n}(q)$. Assume that $(n, q) \neq(4,3)$ or that $Z(H)$ is a 2-group. Then there is a subgroup $Z \leq Z\left(S U_{n}(q)\right)$ such that $H \cong S U_{n}(q) / Z$.
Proof. Suppose that $(n, q) \neq(4,3)$. By [29, pp. 312-313], the Schur multiplier of $P S U_{n}(q)$ is isomorphic to $C_{(n, q+1)} \cong Z\left(S U_{n}(q)\right)$. As in the proof of Lemma 3.1. we conclude that $H \cong$ $S U_{n}(q) / Z$ for some $Z \leq Z\left(S U_{n}(q)\right)$.

Suppose now that $(n, q)=(4,3)$ and that $Z(H)$ is a 2-group. Let $G:=P S U_{4}(3)$, and let $\widetilde{G}$ be the universal covering group of $G$. Clearly, the Schur multiplier of $G$ is isomorphic to $Z(\widetilde{G})$. By [29, pp. 312-313], the Schur multiplier of $G$ is isomorphic to $C_{4} \times C_{3} \times C_{3}$. Thus $Z(\widetilde{G}) \cong C_{4} \times C_{3} \times C_{3}$. Clearly, if $Z \leq Z(\widetilde{G})$, then $Z(\widetilde{G} / Z)=Z(\widetilde{G}) / Z$. Let $Q$ be the unique Sylow 3 -subgroup of $Z(\widetilde{G})$. By [5, 33.6], $\widetilde{G}$ is a central extension of $S U_{4}(3)$ and of $H$. Since $S U_{4}(3)$ has a center of order 4, we have $S U_{4}(3) \cong \widetilde{G} / Q$. Let $Z \leq Z(\widetilde{G})$ with $H \cong \widetilde{G} / Z$. As $Z(H)$ is a 2 -group, we have $Q \leq Z$, whence $H \cong \widetilde{G} / Z \cong(\widetilde{G} / Q) /(Z / Q)$ is isomorphic to a quotient of $S U_{4}(3)$ by a central subgroup.
3.3. Involutions. In this subsection, we collect several results on the involutions of the groups $(P) G L_{n}^{\varepsilon}(q)$ and $(P) S L_{n}^{\varepsilon}(q)$, where $q$ is a nontrivial odd prime power, $n \geq 2$ and $\varepsilon \in\{+,-\}$.
Lemma 3.3. Let $q$ be a nontrivial odd prime power, and let $n \geq 2$. Let $T$ be an element of $G L_{n}(q)$ such that $T^{2}=\lambda I_{n}$ for some $\lambda \in \mathbb{F}_{q}^{*}$. Then one of the following holds:
(i) There is some $\mu \in \mathbb{F}_{q}^{*}$ such that $\lambda=\mu^{2}$, and $T$ is $G L_{n}(q)$-conjugate to a diagonal matrix with diagonal entries in $\{\mu,-\mu\}$.
(ii) $n$ is even, $\lambda$ is a non-square element of $\mathbb{F}_{q}^{*}$, and $T$ is $G L_{n}(q)$-conjugate to the matrix

$$
\left(\begin{array}{ll} 
& I_{n / 2} \\
\lambda I_{n / 2} &
\end{array}\right)
$$

Moreover, we have $C_{G L_{n}(q)}(T) \cong G L_{\frac{n}{2}}\left(q^{2}\right)$.
Proof. We identify the field $\mathbb{F}_{q}$ with the subfield of $\mathbb{F}_{q^{2}}$ consisting of all $x \in \mathbb{F}_{q^{2}}$ satisfying $x^{q}=x$. It is easy to note that any element of $\mathbb{F}_{q}^{*}$ is the square of an element of $\mathbb{F}_{q^{2}}^{*}$. Let $\mu \in \mathbb{F}_{q^{2}}^{*}$ with $\lambda=\mu^{2}$.

If $\mu \in \mathbb{F}_{q}$, then basic linear algebra shows that $T$ is diagonalizable over $\mathbb{F}_{q}$, and it follows that (i) holds.

Assume now that $\mu \notin \mathbb{F}_{q}$. Then $\lambda$ is a non-square element of $\mathbb{F}_{q}^{*}$. Let $V$ be an $n$-dimensional vector space over $\mathbb{F}_{q}$, and let $B$ be an ordered basis of $V$. Let $\varphi$ be the element of $G L(V)$ such
that $\varphi$ is represented by $T$ with respect to $B$. Clearly, $(1, \mu)$ is an $\mathbb{F}_{q^{-}}$-basis of $\mathbb{F}_{q^{2}}$. Using that $\varphi^{2}=\lambda \mathrm{id}_{V}$, one can check that $V$ becomes a vector space over $\mathbb{F}_{q^{2}}$ by defining

$$
(x+y \mu) v:=x v+y v^{\varphi}
$$

for all $x, y \in \mathbb{F}_{q}$ and $v \in V$. Let $m$ be the dimension of $V$ over $\mathbb{F}_{q^{2}}$, and let $\left(v_{1}, \ldots, v_{m}\right)$ be an $\mathbb{F}_{q^{2}}$-basis of $V$. Then $B_{0}:=\left(v_{1}, \ldots, v_{m}, \mu v_{1}, \ldots, \mu v_{m}\right)$ is an $\mathbb{F}_{q^{-}}$basis of $V$. In particular, $n=2 m$ is even. For $1 \leq i \leq m$, we have $v_{i}^{\varphi}=\mu v_{i}$ and $\left(\mu v_{i}\right)^{\varphi}=\left(v_{i}\right)^{\varphi^{2}}=\lambda v_{i}$. So, with respect to $B_{0}, \varphi$ is represented by the matrix

$$
M:=\left(\begin{array}{ll} 
& I_{n / 2} \\
\lambda I_{n / 2} &
\end{array}\right)
$$

It follows that $T$ and $M$ are $G L_{n}(q)$-conjugate.
Let $\psi$ be an automorphism of $V$ as an $\mathbb{F}_{q}$-vector space centralizing $\varphi$. For $x, y \in \mathbb{F}_{q}$ and $v \in V$, we have

$$
((x+y \mu) v)^{\psi}=\left(x v+y v^{\varphi}\right)^{\psi}=x v^{\psi}+y v^{\psi \varphi}=(x+y \mu) v^{\psi}
$$

whence $\psi$ is $\mathbb{F}_{q^{2}}$-linear. Conversely, if $\psi$ is $\mathbb{F}_{q^{2}}$-linear, then

$$
v_{i}^{\psi \varphi}=\mu v_{i}^{\psi}=\left(\mu v_{i}\right)^{\psi}=v_{i}^{\varphi \psi}
$$

and hence $\psi \varphi=\varphi \psi$. It follows that the centralizer of $\varphi$ in the general linear group of $V$ as an $\mathbb{F}_{q^{-}}$ vector space is equal to the general linear group of $V$ as an $\mathbb{F}_{q^{2} \text {-vector space. Thus } C_{G L_{n}(q)}(T) \cong, ~}^{\text {. }}$. $G L_{\frac{n}{2}}\left(q^{2}\right)$. So (ii) holds.
Lemma 3.4. Let $q$ be a nontrivial odd prime power, and let $n \geq 2$ be a natural number. Let $T \in G U_{n}(q)$.
(i) If $T^{2}=\lambda I_{n}$ for some $\lambda \in \mathbb{F}_{q^{2}}^{*}$, then $\lambda$ is a square in $\mathbb{F}_{q^{2}}^{*}$.
(ii) If $T^{2}=\rho^{2} I_{n}$ for some $\rho \in \mathbb{F}_{q^{2}}^{*}$ with $\rho^{q+1}=1$, then $T$ is $G U_{n}(q)$-conjugate to a diagonal matrix with diagonal entries in $\{\rho,-\rho\}$.
(iii) If $T^{2}=\rho^{2} I_{n}$ for some $\rho \in \mathbb{F}_{q^{2}}^{*}$ with $\rho^{q+1} \neq 1$, then $n$ is even, and we have $C_{G U_{n}(q)}(T) \cong$ $G L_{\frac{n}{2}}\left(q^{2}\right)$.

Proof. Suppose that $T^{2}=\lambda I_{n}$ for some $\lambda \in \mathbb{F}_{q^{2}}^{*}$. Since $T^{2} \in G U_{n}(q)$, we have that $\lambda^{q+1}=1$. It is easy to see that any element $x$ of $\mathbb{F}_{q^{2}}^{*}$ with $x^{q+1}=1$ is a square in $\mathbb{F}_{q^{2}}^{*}$. So (i) holds.

A proof of (ii) and (iii) can be extracted from [47, pp. 314-315].
Proposition 3.5. Let $q$ be a nontrivial odd prime power, and let $n \geq 2$ be a natural number. Let $\rho$ be an element of $\mathbb{F}_{q}^{*}$ of order $(n, q-1)$. For each even natural number $i$ with $2 \leq i<n$, let

$$
\tilde{t}_{i}:=\left(\begin{array}{ll}
I_{n-i} & \\
& -I_{i}
\end{array}\right) \in S L_{n}(q)
$$

and let $t_{i}$ be the image of $\widetilde{t_{i}}$ in $P S L_{n}(q)$.
(i) Assume that $n$ is odd. Then each involution of $P S L_{n}(q)$ is $P S L_{n}(q)$-conjugate to $t_{i}$ for some even $2 \leq i<n$.
(ii) Assume that $n$ is even and that there is some $\mu \in \mathbb{F}_{q}^{*}$ with $\rho=\mu^{2}$. For each odd natural number $i$ with $1 \leq i<n$, the matrix

$$
\widetilde{t_{i}}:=\left(\begin{array}{ll}
\mu I_{n-i} & \\
& -\mu I_{i}
\end{array}\right)
$$

lies in $S L_{n}(q)$. Let $t_{i}$ denote the image of $\widetilde{t_{i}}$ in $P S L_{n}(q)$ for each odd $1 \leq i<n$. Then each involution of $P S L_{n}(q)$ is $P S L_{n}(q)$-conjugate to $t_{i}$ for some (even or odd) $1 \leq i \leq \frac{n}{2}$.
(iii) Assume that $n$ is even and that $\rho$ is a non-square element of $\mathbb{F}_{q}$. Let

$$
\widetilde{w}:=\left(\begin{array}{ll} 
& I_{n / 2} \\
\rho I_{n / 2} &
\end{array}\right)
$$

If $\widetilde{w} \in S L_{n}(q)$, then each involution of $P S L_{n}(q)$ is $P S L_{n}(q)$-conjugate to to $t_{i}$ for some even $2 \leq i \leq \frac{n}{2}$ or to $w:=\widetilde{w} Z\left(S L_{n}(q)\right) \in P S L_{n}(q)$. If $\widetilde{w} \notin S L_{n}(q)$, then each involution of $P S L_{n}(q)$ is $P S L_{n}(q)$-conjugate to $t_{i}$ for some even $2 \leq i \leq \frac{n}{2}$.
Proof. We follow arguments found in the proof of [46, Lemma 1.1].
Assume that $n$ is odd. Then $Z\left(S L_{n}(q)\right)$ has odd order, and therefore, any involution of $P S L_{n}(q)$ is the image of an involution of $S L_{n}(q)$. As a consequence of Lemma 3.3, each involution of $S L_{n}(q)$ is $S L_{n}(q)$-conjugate to $\widetilde{t_{i}}$ for some even $2 \leq i<n$. So (i) follows.

Assume now that $n$ is even and that $\rho=\mu^{2}$ for some $\mu \in \mathbb{F}_{q}^{*}$. Note that $Z\left(S L_{n}(q)\right)$ equals $\left\langle\rho I_{n}\right\rangle$. We claim that $\mu^{n}=-1$. Since $\mu^{2 n}=\rho^{n}=1$, we have that $\mu^{n}=1$ or -1 . If $\mu^{n}=1$, then $\mu \in\langle\rho\rangle$, and so $\rho$ is a square in $\langle\rho\rangle$, which is impossible. So we have $\mu^{n}=-1$. It follows that $\widetilde{t_{i}} \in S L_{n}(q)$ for each odd $1 \leq i<n$. Now let $T \in S L_{n}(q)$ such that $T Z\left(S L_{n}(q)\right) \in P S L_{n}(q)$ is an involution. Then we have $T^{2}=\rho^{\ell} I_{n}=\mu^{2 \ell} I_{n}$ for some $1 \leq \ell \leq(n, q-1)$. Using Lemma 3.3, we conclude that $T$ is $S L_{n}(q)$-conjugate to a diagonal matrix $D \in S L_{n}(q)$ with diagonal entries in $\left\{\mu^{\ell},-\mu^{\ell}\right\}$. Let $1 \leq i<n$ such that $-\mu^{\ell}$ occurs precisely $i$ times as a diagonal entry of $D$. If $i$ is odd, we may assume that $D=\mu^{\ell-1} \widetilde{t_{i}}$, and if $i$ is even, we may assume that $D=\mu^{\ell} \widetilde{t_{i}}$. In either case, the image of $D$ in $P S L_{n}(q)$ is $t_{i}$. Hence, $T Z\left(S L_{n}(q)\right)$ is $P S L_{n}(q)$-conjugate to $t_{i}$. Noticing that $t_{i}$ is $P S L_{n}(q)$-conjugate to $t_{n-i}$, we conclude that (ii) holds.

Now assume that $n$ is even and that $\rho$ is a non-square element of $\mathbb{F}_{q}$. Again let $T$ be an element of $S L_{n}(q)$ such that $T Z\left(S L_{n}(q)\right) \in P S L_{n}(q)$ is an involution. We have $T^{2}=\rho^{\ell} I_{n}$ for some $1 \leq \ell \leq(n, q-1)$. Assume that $\ell$ is even. Then Lemma 3.3 implies that $T$ or $-T$ is $S L_{n}(q)$ conjugate to $\rho^{\frac{\ell}{2}} \tilde{t}_{i}$ for some even $2 \leq i \leq \frac{n}{2}$. It follows that $T Z\left(S L_{n}(q)\right)$ is $P S L_{n}(q)$-conjugate to $t_{i}$ for some even $2 \leq i \leq \frac{n}{2}$. Assume now that $\ell$ is odd. As $\rho$ is not a square in $\mathbb{F}_{q}$, but $\rho^{\ell-1}$ is a square in $\mathbb{F}_{q}, \rho^{\ell}$ cannot be a square in $\mathbb{F}_{q}$. Using Lemma 3.3. we may conclude that $T$ is $G L_{n}(q)$-conjugate to the matrix

$$
M:=\left(\begin{array}{ccccc}
0 & \rho^{\ell} & & & \\
1 & 0 & & & \\
& & \ddots & & \\
& & & 0 & \rho^{\ell} \\
& & & 1 & 0
\end{array}\right) \in S L_{n}(q)
$$

It is rather easy to see that $T$ and $M$ are even conjugate in $S L_{n}(q)$. Let $k:=\frac{\ell-1}{2}$. It is not hard to show that the matrices

$$
\left(\begin{array}{cc}
0 & \rho^{\ell} \\
1 & 0
\end{array}\right) \text { and }\left(\begin{array}{cc}
0 & \rho^{k+1} \\
\rho^{k} & 0
\end{array}\right)
$$

are $S L_{2}(q)$-conjugate. So it follows that $M$ and hence $T$ is $S L_{n}(q)$-conjugate to $\rho^{k} M_{2}$, where

$$
M_{2}:=\left(\begin{array}{ccccc}
0 & \rho & & & \\
1 & 0 & & & \\
& & \ddots & & \\
& & & 0 & \rho \\
& & & 1 & 0
\end{array}\right) \in S L_{n}(q)
$$

Consequently, the images of $T$ and $M_{2}$ in $P S L_{n}(q)$ are conjugate. Furthermore, as $\operatorname{det}\left(M_{2}\right)=$ $\operatorname{det}(\widetilde{w})$, we see that $\widetilde{w} \in S L_{n}(q)$. Also, $\widetilde{w}$ is $S L_{n}(q)$-conjugate to $M_{2}$, and so $T Z\left(S L_{n}(q)\right)$ is $P S L_{n}(q)$-conjugate to $w$.

Lemma 3.6. Let $q$ be a nontrivial odd prime power and let $n \geq 4$ be an even natural number. Let $\rho$ be an element of $\mathbb{F}_{q}^{*}$ of order $(n, q-1)$. Suppose that $\rho$ is a non-square element of $\mathbb{F}_{q}$ and that

$$
\widetilde{w}:=\left(\begin{array}{ll} 
& I_{n / 2} \\
\rho I_{n / 2} &
\end{array}\right.
$$

lies in $S L_{n}(q)$. Denote the image of $\widetilde{w}$ in $P S L_{n}(q)$ by $w$. Set $C:=C_{P S L_{n}(q)}(w)$. Let $P$ be a Sylow 2-subgroup of $C$. Then the following hold:
(i) $C$ has a unique 2-component $J$, and $J$ is isomorphic to a nontrivial quotient of $S L_{\frac{n}{2}}\left(q^{2}\right)$.
(ii) We have $P \cap J \unlhd P$, and the factor system $\mathcal{F}_{P}(C) /(P \cap J)$ is nilpotent.
(iii) If $n \geq 6$, then $P$ has rank at least 4 .

Proof. Set $C_{0}:=C_{S L_{n}(q)}(\widetilde{w}) / Z\left(S L_{n}(q)\right) \leq C$. By a direct argument, $C_{0}$ has index 2 in $C$. So the 2-components of $C$ are precisely the 2-components of $C_{0}$. One may deduce from Lemma 3.3 that $C_{S L_{n}(q)}(\widetilde{w})$ has a normal subgroup $\widetilde{J}$ isomorphic to $S L_{\frac{n}{2}}\left(q^{2}\right)$ such that the corresponding factor group is cyclic. Let $J$ be the image of $\widetilde{J}$ in $P S L_{n}(q)$. Then $J$ is isomorphic to a nontrivial quotient of $S L_{\frac{n}{2}}\left(q^{2}\right)$. Moreover, $J \unlhd C_{0}$ and $C_{0} / J$ is cyclic. Therefore, $J$ is the only 2 -component of $C_{0}$ and hence the only 2-component of $C$. Thus (i) holds.

We have $P \cap J \unlhd P$ because $J \unlhd C$. By Lemma 2.11, the factor system $\mathcal{F}_{P}(C) /(P \cap J)$ is isomorphic to the 2 -fusion system of $C / J$. Since $C_{0}$ has index 2 in $C$ and $C_{0} / J$ is abelian, we have that $C / J$ is 2-nilpotent. So $C / J$ has a nilpotent 2 -fusion system, and (ii) follows.

We now prove (iii). Assume that $n \geq 6$. Let $u$ denote the image of

$$
\left(\begin{array}{ccccc}
0 & \rho & & & \\
1 & 0 & & & \\
& & \ddots & & \\
& & & 0 & \rho \\
& & & 1 & 0
\end{array}\right) \in S L_{n}(q)
$$

in $P S L_{n}(q)$. It is easy to see that there exist $a, b \in \mathbb{F}_{q}$ with $a^{2} \rho-b^{2} \rho^{2}=1$. Let $s$ be the image of

$$
\left(\begin{array}{ccccc}
-b \rho & a \rho & & & \\
-a & b \rho & & & \\
& & \ddots & & \\
& & & -b \rho & a \rho \\
& & & -a & b \rho
\end{array}\right) \in S L_{n}(q)
$$

in $P S L_{n}(q)$. By a direct calculation, $s \in C_{P S L_{n}(q)}(u)$. Another direct calculation shows that $s$ is an involution. Let $z_{1}$ denote the image of

$$
\left(\begin{array}{cc}
-I_{2} & \\
& I_{n-2}
\end{array}\right) \in S L_{n}(q)
$$

in $P S L_{n}(q)$, and let $z_{2}$ denote the image of

$$
\left(\begin{array}{ccc}
I_{2} & & \\
& -I_{2} & \\
& & I_{n-4}
\end{array}\right) \in S L_{n}(q)
$$

in $P S L_{n}(q)$. Then one can easily verify that $\left\langle s, u, z_{1}, z_{2}\right\rangle \leq C_{P S L_{n}(q)}(u)$ is isomorphic to $E_{16}$. So a Sylow 2-subgroup of $C_{P S L_{n}(q)}(u)$ has rank at least 4. This is also true for $P$ as $w$ and $u$ are conjugate (see Proposition 3.5).
Lemma 3.7. Let $n \geq 2$ be a natural number and let $\varepsilon \in\{+,-\}$. Also, let $T \in G L_{n}^{\varepsilon}(3) \backslash Z\left(G L_{n}^{\varepsilon}(3)\right)$ such that $T^{2} \in Z\left(G L_{n}^{\varepsilon}(3)\right)$. Then $C_{G L_{n}^{\varepsilon}(3)}(T)$ is core-free.

Proof. By Lemmas 3.3 and 3.4 , we either have $C_{G L_{n}^{\varepsilon}(3)}(T) \cong G L_{i}^{\varepsilon}(3) \times G L_{n-i}^{\varepsilon}(3)$ for some $1 \leq i<$ $n$, or $n$ is even and $C_{G L_{n}^{\varepsilon}(3)}(T) \cong G L_{n / 2}(9)$. So we have that $C_{G L_{n}^{\varepsilon}(3)}(T)$ is core-free.

It is easy to deduce the following two corollaries from Lemma 3.7.
Corollary 3.8. Let $n \geq 2$ be a natural number and let $\varepsilon \in\{+,-\}$. Then any involution centralizer in $S L_{n}^{\varepsilon}(3)$ is core-free.
Corollary 3.9. Let $n \geq 2$ be a natural number and let $\varepsilon \in\{+,-\}$. Then any involution centralizer in $P G L_{n}^{\varepsilon}(3)$ is core-free.
3.4. Sylow 2-subgroups and 2-fusion systems. In this subsection, we consider several properties of Sylow 2-subgroups and 2 -fusion systems of linear and unitary groups.
Lemma 3.10. ([18, p. 142]) Let $q$ be a nontrivial odd prime power. Let $k, s \in \mathbb{N}$ such that $2^{k}$ is the 2 -part of $q-1$ and that $2^{s}$ is the 2 -part of $q+1$. Then:
(i) Assume that $q \equiv 1 \bmod 4$. Then

$$
\left\{\left(\begin{array}{ll}
\lambda & \\
& \mu
\end{array}\right): \lambda, \mu \text { are 2-elements of } \mathbb{F}_{q}^{*}\right\} \cdot\left\langle\left(\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right)\right\rangle
$$

is a Sylow 2-subgroup of $G L_{2}(q)$. In particular, the Sylow 2-subgroups of $G L_{2}(q)$ are isomorphic to the wreath product $C_{2^{k}}$ l $C_{2}$.
(ii) If $q \equiv 3 \bmod 4$, then the Sylow 2-subgroups of $G L_{2}(q)$ are semidihedral of order $2^{s+2}$.

Lemma 3.11. ([18, p. 143]) Let $q$ be a nontrivial odd prime power. Let $k, s \in \mathbb{N}$ such that $2^{k}$ is the 2 -part of $q-1$ and that $2^{s}$ is the 2 -part of $q+1$. Then:
(i) If $q \equiv 1 \bmod 4$, then the Sylow 2 -subgroups of $G U_{2}(q)$ are semidihedral of order $2^{k+2}$.
(ii) If $q \equiv 3 \bmod 4$, then the Sylow 2-subgroups of $G U_{2}(q)$ are isomorphic to the wreath product $C_{2^{s}} \backslash C_{2}$. If $\varepsilon \in \mathbb{F}_{q^{2}}^{*}$ has order $2^{s}$, then a Sylow 2-subgroup of $G U_{2}(q)$ is concretely given by

$$
W:=\left\{\left(\begin{array}{ll}
\lambda & \\
& \mu
\end{array}\right): \lambda, \mu \in\langle\varepsilon\rangle\right\} \cdot\left\langle\left(\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right)\right\rangle .
$$

Lemma 3.12. (35, Kapitel II, Satz 8.10 a)]) If $q$ is a nontrivial odd prime power, then a Sylow 2 -subgroup of $S L_{2}(q)$ is generalized quaternion of order $\left(q^{2}-1\right)_{2}$.
Lemma 3.13. (35, Kapitel II, Satz 8.10 b)]) If $q$ is a nontrivial odd prime power, then $P S L_{2}(q)$ has dihedral Sylow 2 -subgroups of order $\frac{1}{2}\left(q^{2}-1\right)_{2}$.
Lemma 3.14. ([18, Lemma 1]) Let $q$ be a nontrivial odd prime power and let $\varepsilon \in\{+,-\}$. Let $r$ be a positive integer. Let $W_{r}$ be a Sylow 2-subgroup of $G L_{2^{r}}^{\varepsilon}(q)$. Then $W_{r}$ 〕 $C_{2}$ is isomorphic to a Sylow 2-subgroup of $G L_{2^{r+1}}^{\varepsilon}(q)$. A Sylow 2-subgroup of $G L_{2^{r+1}}^{\varepsilon}(q)$ is concretely given by

$$
\left\{\left(\begin{array}{ll}
A & \\
& B
\end{array}\right): A, B \in W_{r}\right\} \cdot\left\langle\left(\begin{array}{cc} 
& I_{2^{r}} \\
I_{2^{r}} &
\end{array}\right)\right\rangle .
$$

Lemma 3.15. ([18, Theorem 1]) Let $q$ be a nontrivial odd prime power and let $n$ be a positive integer. Let $\varepsilon \in\{+,-\}$. Let $0 \leq r_{1}<\cdots<r_{t}$ such that $n=2^{r_{1}}+\cdots+2^{r_{t}}$. Let $W_{i} \in$ $\operatorname{Syl}_{2}\left(G L_{2^{r_{i}}}^{\varepsilon}(q)\right)$ for all $1 \leq i \leq t$. Then $W_{1} \times \cdots \times W_{t}$ is isomorphic to a Sylow 2-subgroup of $G L_{n}^{\varepsilon}(q)$. A Sylow 2-subgroup of $G L_{n}^{\varepsilon}(q)$ is concretely given by

$$
\left\{\left(\begin{array}{ccc}
A_{1} & & \\
& \ddots & \\
& & A_{t}
\end{array}\right): A_{i} \in W_{i}\right\} .
$$

Lemma 3.16. Let $q$ be a prime power with $q \equiv 3 \bmod 4$. Let $W$ be a Sylow 2-subgroup of $G L_{2}(q)$, and let $m \in \mathbb{N}$ such that $|W|=2^{m}$. Then:
(i) $W$ is semidihedral. In particular, there are elements $a, b \in W$ with $\operatorname{ord}(a)=2^{m-1}$ and $\operatorname{ord}(b)=2$ such that $a^{b}=a^{2^{m-2}-1}$.
(ii) We have $W \cap S L_{2}(q)=\left\langle a^{2}\right\rangle\langle a b\rangle$.
(iii) Let $1 \leq \ell \leq 2^{m-1}$. If $\ell$ is odd, then $a^{\ell}$ has determinant -1 , and $a^{\ell} b$ has determinant 1 . If $\ell$ is even, then $a^{\ell}$ has determinant 1, and $a^{\ell} b$ has determinant -1 .
(iv) The involutions of $W$ are precisely the elements $a^{2^{m-2}}$ and $a^{\ell} b$, where $2 \leq \ell \leq 2^{m-1}$ is even.

Proof. By Lemma 3.10 (ii), we have (i).
Let $W_{0}:=W \cap S L_{2}(q)$. By Lemma 3.12 , $W_{0}$ is generalized quaternion. Also, $W_{0}$ is a maximal subgroup of $W$ since $S L_{2}(q)$ has index $q-1$ in $G L_{2}(q)$ and $q \equiv 3 \bmod 4$. By [24, Chapter 5 , Theorem 4.3 (ii) (b)], we have $\Phi(W)=\left\langle a^{2}\right\rangle$. So the maximal subgroups of $W$ are precisely the groups $M_{1}:=\langle a\rangle, M_{2}:=\left\langle a^{2}\right\rangle\langle b\rangle$ and $M_{3}:=\left\langle a^{2}\right\rangle\langle a b\rangle$. One can check that $M_{1} \cong C_{2^{n-1}}$, $M_{2} \cong D_{2^{n-1}}$ and $M_{3} \cong Q_{2^{n-1}}$. Consequently, $W_{0}=\left\langle a^{2}\right\rangle\langle a b\rangle$, and (ii) holds.
(iii) follows from (ii) since any element of $W \backslash W_{0}$ has determinant -1 .

The proof of (iv) is an easy exercise.
Lemma 3.17. Let $q$ be a nontrivial odd prime power, $n$ a positive integer and $\varepsilon \in\{+,-\}$. Let $0 \leq r_{1}<\cdots<r_{t}$ such that $n=2^{r_{1}}+\cdots+2^{r_{t}}$. Then there is a Sylow 2 -subgroup $W$ of $G:=G L_{n}^{\varepsilon}(q)$ containing all diagonal matrices in $G$ with 2 -power order such that $C_{W}\left(W \cap S L_{n}^{\varepsilon}(q)\right)$ consists precisely of the matrices

$$
\left(\begin{array}{ccc}
\lambda_{1} I_{2^{r_{1}}} & & \\
& \ddots & \\
& & \lambda_{t} I_{2^{r_{t}}}
\end{array}\right)
$$

where $\lambda_{1}, \ldots, \lambda_{t}$ are 2-elements of $\mathbb{F}_{q}^{*}$ if $G=G L_{n}(q)$ and 2-elements of $\mathbb{F}_{q^{2}}^{*}$ with $\lambda_{i}^{q+1}=1$ (for each $1 \leq i \leq t$ ) if $G=G U_{n}(q)$.
Proof. Using Lemmas 3.10 and 3.11 , one can check that the centralizer of a Sylow 2-subgroup of $S L_{2}^{\varepsilon}(q)$ inside a Sylow 2-subgroup of $G L_{2}^{\varepsilon}(q)$ is the Sylow 2-subgroup of $Z\left(G L_{2}^{\varepsilon}(q)\right)$. Applying Lemma 3.14 and arguing by induction, one can see that a similar statement holds for the centralizer of a Sylow 2-subgroup of $S L_{2^{r}}^{\varepsilon}(q)$ inside a Sylow 2-subgroup of $G L_{2^{r}}^{\varepsilon}(q)$ for all $r \geq 0$. Now we may apply Lemma 3.15 to obtain a Sylow 2-subgroup of $G$ with the desired properties.
Lemma 3.18. Let $q$ be a nontrivial odd prime power, $n$ a positive integer and $\varepsilon \in\{+,-\}$. Let $G:=S L_{n}^{\varepsilon}(q)$, and let $S$ be a Sylow 2-subgroup of $G$. Then we have $Z\left(\mathcal{F}_{S}(G)\right)=S \cap Z(G)$.
Proof. Let $0 \leq r_{1}<\cdots<r_{t}$ such that $n=2^{r_{1}}+\cdots+2^{r_{t}}$. By Lemma 3.17, we may assume that $Z(S)$ consists precisely of the matrices

$$
\left(\begin{array}{ccc}
\lambda_{1} I_{2^{r_{1}}} & & \\
& \ddots & \\
& & \lambda_{t} I_{2^{r_{t}}}
\end{array}\right)
$$

where $\lambda_{1}, \ldots, \lambda_{t}$ are 2-elements of $\mathbb{F}_{q}^{*}$ with $\lambda_{1}^{2^{r_{1}}} \cdots \lambda_{t}^{2^{r_{t}}}=1$ if $G=S L_{n}(q)$ and 2-elements of $\mathbb{F}_{q^{2}}^{*}$ with $\lambda_{i}^{q+1}=1$ (for each $1 \leq i \leq t$ ) and $\lambda_{1}^{2^{r_{1}}} \cdots \lambda_{t}^{2^{r} t}=1$ if $G=S U_{n}(q)$. Moreover, by Lemma 3.17, we may assume that $S$ contains each diagonal matrix in $G$ of 2-power order.

Let $x$ be an element of $Z(S)$ with diagonal blocks $\lambda_{1} I_{2^{r}}, \ldots, \lambda_{t} I_{2^{r} t}$. One can easily see that $x$ is $G$-conjugate to any diagonal matrix in $G$ that is obtained from $x$ by permuting its diagonal entries. It follows that, if $\lambda_{i} \neq \lambda_{j}$ for some $1 \leq i \neq j \leq t$, then $x \notin Z\left(\mathcal{F}_{S}(G)\right)$. This implies $Z\left(\mathcal{F}_{S}(G)\right)=S \cap Z(G)$.
Proposition 3.19. Let $n$ be a positive integer. Let $q, q^{*}$ be nontrivial odd prime powers, and let $\varepsilon, \varepsilon^{*} \in\{+,-\}$. If $\varepsilon q \sim \varepsilon^{*} q^{*}$, then the 2 -fusion systems of $S L_{n}^{\varepsilon}(q)$ and $S L_{n}^{\varepsilon^{*}}\left(q^{*}\right)$ are isomorphic.

Proof. Assume that $\varepsilon \neq \varepsilon^{*}$. From $\varepsilon q \sim \varepsilon^{*} q^{*}$, it is easy to deduce that $\varepsilon q \equiv \varepsilon^{*} q^{*} \bmod 8$ and $\left(q^{2}-1\right)_{2}=\left(\left(q^{*}\right)^{2}-1\right)_{2}$. So, in view of the remarks at the bottom of p. 11 of [15], we may apply [15, Proposition 3.3 (a)] to conclude that the 2 -fusion system of $S L_{n}^{\varepsilon}(q)$ is isomorphic to the 2-fusion system of $S L_{n}^{\varepsilon^{*}}\left(q^{*}\right)$.

Assume now that $\varepsilon=\varepsilon^{*}$. Using Dirichlet's theorem [21, Theorem 3.3.1], one can easily see that there is an odd prime $q_{0}$ with $\varepsilon q \sim \varepsilon q^{*} \sim-\varepsilon q_{0}$. By the preceding paragraph, both the 2 -fusion system of $S L_{n}^{\varepsilon}(q)$ and the 2 -fusion system of $S L_{n}^{\varepsilon}\left(q^{*}\right)$ are isomorphic to the 2 -fusion system of $S L_{n}^{-\varepsilon}\left(q_{0}\right)$. Consequently, the 2-fusion systems of $S L_{n}^{\varepsilon}(q)$ and $S L_{n}^{\varepsilon^{*}}\left(q^{*}\right)$ are isomorphic.
Proposition 3.20. Let $n$ be a positive integer. Let $q, q^{*}$ be nontrivial odd prime powers, and let $\varepsilon, \varepsilon^{*} \in\{+,-\}$. If $\varepsilon q \sim \varepsilon^{*} q^{*}$, then the 2 -fusion systems of $P S L_{n}^{\varepsilon}(q)$ and $P S L_{n}^{\varepsilon^{*}}\left(q^{*}\right)$ are isomorphic.
Proof. Let $S$ and $S^{*}$ be Sylow 2-subgroups of $G:=S L_{n}^{\varepsilon}(q)$ and $G^{*}:=S L_{n}^{\varepsilon^{*}}\left(q^{*}\right)$, respectively. By Proposition 3.19, $\mathcal{F}:=\mathcal{F}_{S}(G)$ and $\mathcal{F}^{*}:=\mathcal{F}_{S^{*}}\left(G^{*}\right)$ are isomorphic. Therefore, $\mathcal{F} / Z(\mathcal{F})$ and $\mathcal{F}^{*} / Z\left(\mathcal{F}^{*}\right)$ are isomorphic. Lemma 3.18 implies that $\mathcal{F} /(S \cap Z(G))$ and $\mathcal{F}^{*} /\left(S^{*} \cap Z\left(G^{*}\right)\right)$ are isomorphic. Now the proposition follows from Lemma 2.11.

The following lemma shows together with [10, Theorem 5.6.18] that the 2-fusion system of $P S L_{n}(q)$ is simple whenever $q$ is odd and $n \geq 3$.

Lemma 3.21. Let $q$ be a nontrivial odd prime power and $n \geq 2$ a natural number such that $(n, q) \neq(2,3)$. Moreover, let $\varepsilon$ be an element of $\{+,-\}$. Then $P S L_{n}^{\varepsilon}(q)$ is a Goldschmidt group if and only if $n=2$ and $q \equiv 3$ or $5 \bmod 8$.
Proof. Set $G:=P S L_{n}^{\varepsilon}(q)$.
Assume that $n=2$. Then $G \cong P S L_{2}(q)$. By Lemma 3.13, $G$ has dihedral Sylow 2-subgroups of order $\frac{1}{2}(q-1)_{2}(q+1)_{2}$. So, if $q \equiv 3$ or $5 \bmod 8$, then $G$ has abelian Sylow 2-subgroups and is thus a Goldschmidt group. If $q \equiv 1$ or $7 \bmod 8$, then the Sylow 2 -subgroups of $G$ are dihedral of order at least 8 and hence nonabelian. Moreover, if $q \equiv 1$ or $7 \bmod 8$, then [48, Theorem 37] shows that $G$ is not isomorphic to a finite simple group of Lie type in characteristic 2 of Lie rank 1. So $G$ is not a Goldschmidt group if $q \equiv 1$ or $7 \bmod 8$.

Assume now that $n \geq 3$. Again, we see from [48, Theorem 37] that there is no finite simple group of Lie type in characteristic 2 of Lie rank 1 which is isomorphic to $G$. Also, $G$ has a subgroup isomorphic to $S L_{2}^{\varepsilon}(q) \cong S L_{2}(q)$, and therefore, the Sylow 2-subgroups of $G$ are nonabelian. Consequently, $G$ is not a Goldschmidt group.
Lemma 3.22. Let $n$ be a positive integer, $q$ a nontrivial odd prime power and $\varepsilon \in\{+,-\}$. Let $E$ be the subgroup of $S L_{n}^{\varepsilon}(q)$ consisting of the diagonal matrices in $S L_{n}^{\varepsilon}(q)$ with diagonal entries in $\{1,-1\}$. Then $|E|=2^{n-1}$. Moreover, any elementary abelian 2-subgroup of $S L_{n}^{\varepsilon}(q)$ is conjugate to a subgroup of $E$.
Proof. It is straightforward to check that $|E|=2^{n-1}$.
Let $E_{0}$ be an elementary abelian 2-subgroup of $S L_{n}^{\varepsilon}(q)$. We show that $E_{0}$ is conjugate to a subgroup of $E$. Using Dirichlet's theorem [21, Theorem 3.3.1], one can see that there is an odd prime number $q^{*}$ with $-q \sim q^{*}$, and Proposition 3.19 shows that the 2-fusion systems of $S U_{n}(q)$ and $S L_{n}\left(q^{*}\right)$ are isomorphic. Therefore, it is enough to consider the case $\varepsilon=+$.

Since $E_{0}$ is an elementary abelian 2-group, any two elements of $E_{0}$ commute, and any element of $E_{0}$ is diagonalizable (see Lemma 3.3). It follows that $E_{0}$ is simultaneously diagonalizable, and this implies that $E_{0}$ is conjugate to a subgroup of $E$.

Lemma 3.23. Let $q$ be a nontrivial odd prime power, $n \geq 3$ a natural number and $S$ a Sylow 2 -subgroup of $P S L_{n}(q)$. Then $\operatorname{Aut}_{P S L_{n}(q)}(S)=\operatorname{Inn}(S)$.
Proof. Let $R \in \operatorname{Syl}_{2}\left(S L_{n}(q)\right)$ such that $S$ is the image of $R$ in $P S L_{n}(q)$. Let $T$ be a Sylow 2-subgroup of $G L_{n}(q)$ with $R \leq T$. By [36, Theorem 1], we have $N_{G L_{n}(q)}(R)=T C_{G L_{n}(q)}(T)$.

So we have that $\operatorname{Aut}_{S L_{n}(q)}(R)$ is a 2-group. Since the image of $N_{S L_{n}(q)}(R)$ in $P S L_{n}(q)$ equals $N_{P S L_{n}(q)}(S)$ (see [35, Kapitel I, Hilfssatz 7.7 c )]), it follows that Aut ${ }_{P S L_{n}(q)}(S)$ is a 2-group, and this implies Aut ${ }_{P S L_{n}(q)}(S)=\operatorname{Inn}(S)$.
3.5. $k$-connectivity. In this subsection, we prove some connectivity properties of the Sylow 2subgroups of $S L_{n}(q)$ and $P S L_{n}(q)$, where $q$ is a nontrivial odd prime power and $n \geq 6$. We will work with the following definition (see [32, Section 8]):
Definition 3.24. Let $S$ be a finite 2-group, and let $k$ be a positive integer. If $A$ and $B$ are elementary abelian subgroups of $S$ of rank at least $k$, then $A$ and $B$ are said to be $k$-connected if there is a sequence

$$
A=A_{1}, A_{2}, \ldots, A_{n}=B \quad(n \geq 1)
$$

of elementary abelian subgroups $A_{i}, 1 \leq i \leq n$, of $S$ with rank at least $k$ such that

$$
A_{i} \subseteq A_{i+1} \text { or } A_{i+1} \subseteq A_{i}
$$

for all $1 \leq i \leq n-1$. The group $S$ is said to be $k$-connected if any two elementary abelian subgroups of $S$ of rank at least $k$ are $k$-connected.

Lemma 3.25. (32, Lemma 8.4]) Let $S$ be a finite 2-group, and let $k$ be a positive integer. If $S$ has a normal elementary abelian subgroup of rank at least $2^{k-1}+1$, then $S$ is $k$-connected.

Lemma 3.26. Let $q$ be a nontrivial odd prime power with $q \equiv 1 \bmod 4$, and let $n \geq 6$ be a natural number. Then the Sylow 2-subgroups of $P S L_{n}(q)$ and those of $S L_{n}(q)$ are 3-connected.

Proof. Let $W_{0}$ be the unique Sylow 2-subgroup of $G L_{1}(q)$, and let $W_{1}$ be the Sylow 2-subgroup of $G L_{2}(q)$ given in Lemma 3.10 (i). For each $r \geq 2$, let $W_{r}$ be the Sylow 2-subgroup of $G L_{2^{r}}(q)$ obtained from $W_{r-1}$ by the construction given in the last statement of Lemma 3.14. Let $0 \leq r_{1}<$ $\cdots<r_{t}$ such that $n=2^{r_{1}}+\cdots+2^{r_{t}}$, and let $W$ be the Sylow 2-subgroup of $G L_{n}(q)$ obtained from $W_{r_{1}}, \ldots, W_{r_{t}}$ by using the last statement of Lemma 3.15.

Let $R$ denote the subgroup of $G L_{n}(q)$ consisting of all diagonal matrices $D \in G L_{n}(q)$, where $D^{2} \in Z\left(G L_{n}(q)\right)$ and any diagonal element of $D$ is a 2 -element of $\mathbb{F}_{q}^{*}$. It is easy to note that $R \unlhd W$.

Set $R_{0}:=R \cap S L_{n}(q)$. Then $\Omega_{1}\left(R_{0}\right)$, the subgroup of $R_{0}$ generated by all involutions of $R_{0}$, is elementary abelian of order $2^{n-1} \geq 2^{5}$, and $\Omega_{1}\left(R_{0}\right) \unlhd W \cap S L_{n}(q)$. Also, $R_{0} Z\left(S L_{n}(q)\right) / Z\left(S L_{n}(q)\right)$ is a normal elementary abelian subgroup of $\left(W \cap S L_{n}(q)\right) Z\left(S L_{n}(q)\right) / Z\left(S L_{n}(q)\right)$, and one can easily check that the order of $R_{0} Z\left(S L_{n}(q)\right) / Z\left(S L_{n}(q)\right)$ is at least $2^{5}$. Lemma 3.25 implies that $W \cap S L_{n}(q)$ and its image in $P S L_{n}(q)$ are 3-connected.

Lemma 3.25 and the proof of Lemma 3.26 show that we also have the following:
Lemma 3.27. Let $q$ be a nontrivial odd prime power with $q \equiv 1 \bmod 4$, and let $n \geq 6$ be a natural number. Then the Sylow 2-subgroups of $P S L_{n}(q)$ and those of $S L_{n}(q)$ are 2-connected.

We now study the case $q \equiv 3 \bmod 4$.
Lemma 3.28. Let $q$ be a nontrivial odd prime power with $q \equiv 3 \bmod 4$, and let $n \geq 6$ be a natural number. Then the Sylow 2-subgroups of $P S L_{n}(q)$ and those of $S L_{n}(q)$ are 2 -connected. If $n \geq 10$, then we even have that the Sylow 2-subgroups of $P S L_{n}(q)$ and those of $S L_{n}(q)$ are 3-connected.

Proof. Let $W_{0}$ denote the unique Sylow 2-subgroup of $G L_{1}(q)$, and let $W_{1}$ be a Sylow 2-subgroup of $G L_{2}(q)$. By Lemma 3.10 (ii), $W_{1}$ is semidihedral. Let $m \in \mathbb{N}$ with $\left|W_{1}\right|=2^{m}$. Also, let $h, a \in W_{1}$ such that $\operatorname{ord}(h)=2^{m-1}, \operatorname{ord}(a)=2$ and $h^{a}=h^{2^{m-2}-1}$. Set $z:=-I_{2}=h^{2^{m-2}}$. For each $r \geq 2$, let $W_{r}$ be the Sylow 2-subgroup of $G L_{2^{r}}(q)$ obtained from $W_{r-1}$ by the construction given in the last statement of Lemma 3.14. Let $0 \leq r_{1}<\cdots<r_{t}$ such that $n=2^{r_{1}}+\cdots+2^{r_{t}}$, and let $W$ be the Sylow 2-subgroup of $G L_{n}(q)$ obtained from $W_{r_{1}}, \ldots, W_{r_{t}}$ by using the last statement of Lemma 3.15.

Given a natural number $\ell \geq 1$ and elements $x_{1}, \ldots, x_{\ell} \in G L_{2}(q)$, we write $\operatorname{diag}\left(x_{1}, \ldots, x_{\ell}\right)$ for the block diagonal matrix

$$
\left(\begin{array}{ccc}
x_{1} & & \\
& \ddots & \\
& & x_{\ell}
\end{array}\right)
$$

For each natural number $r \geq 1$, let $A_{r}$ denote the subgroup of $G L_{2^{r}}(q)$ consisting of the matrices $\operatorname{diag}\left(x_{1}, \ldots, x_{2^{r-1}}\right)$, where either $x_{i} \in\langle z\rangle$ for all $1 \leq i \leq 2^{r-1}$ or $x_{i}$ is an element of $\langle h\rangle$ with order 4 for all $1 \leq i \leq 2^{r-1}$. By induction over $r$, one can see that $A_{r} \unlhd W_{r}$ for all $r \geq 1$. Also, let $\widetilde{A_{r}}:=\Omega_{1}\left(A_{r}\right)$ for all $r \geq 1$. Clearly, $\widetilde{A_{r}} \unlhd W_{r}$ for all $r \geq 1$.

We now consider two cases.
Case 1: $n$ is even.
Let $E$ be the subgroup of $G L_{n}(q)$ consisting of the matrices $\operatorname{diag}\left(x_{1}, \ldots, x_{\frac{n}{2}}\right)$, where either $x_{i} \in\langle z\rangle$ for all $1 \leq i \leq \frac{n}{2}$ or $x_{i}$ is an element of $\langle h\rangle$ with order 4 for all $1 \leq i \leq \frac{n}{2}$. Let $\widetilde{E}:=\Omega_{1}(E)$. Since $A_{r_{i}} \unlhd W_{r_{i}}$ for all $1 \leq i \leq t$, we have that $E$ and $\widetilde{E}$ are normal subgroups of $W$. Lemma 3.16 (iii) shows that $E \leq W \cap S L_{n}(q)$.

As $\widetilde{E}$ is elementary abelian of order $2^{\frac{n}{2}}$, Lemma 3.25 implies that $W \cap S L_{n}(q)$ is 2-connected, and even 3 -connected if $n \geq 10$. Since $E Z\left(S L_{n}(q)\right) / Z\left(S L_{n}(q)\right)$ is a normal elementary abelian subgroup of $\left(W \cap S L_{n}(q)\right) Z\left(S L_{n}(q)\right) / Z\left(S L_{n}(q)\right)$ with order $2^{\frac{n}{2}}$, Lemma 3.25 also shows that a Sylow 2-subgroup is 2-connected, and even 3-connected if $n \geq 10$.

Case 2: $n$ is odd.
Now let $E$ denote the subgroup of $G L_{n}(q)$ consisting of the matrices

$$
\left(\begin{array}{c|ccc}
1 & & & \\
\hline & x_{1} & & \\
& & \ddots & \\
& & & x_{\frac{n-1}{2}}
\end{array}\right),
$$

where $x_{i} \in\langle z\rangle$ for all $1 \leq i \leq \frac{n-1}{2}$. Since $\widetilde{A_{r_{i}}} \unlhd W_{r_{i}}$ for all $2 \leq i \leq t$, we have that $E$ is a normal subgroup of $W \cap S L_{n}(q)$. Moreover, $E$ is elementary abelian of order $2^{\frac{n-1}{2}}$. Lemma 3.25 implies that $W \cap S L_{n}(q)$ is 2-connected, and even 3 -connected if $n \geq 11$. There is nothing else to show since the Sylow 2-subgroups of $P S L_{n}(q)$ are isomorphic to those of $S L_{n}(q)$ (as $n$ is odd).

We show next that the groups $S L_{n}(q)$, where $6 \leq n \leq 9$ and $q \equiv 3 \bmod 4$, and the groups $P S L_{n}(q)$, where $7 \leq n \leq 9$ and $q \equiv 3 \bmod 4$, also have 3 -connected Sylow 2-subgroups.

Lemma 3.29. Let $q$ be a nontrivial odd prime power with $q \equiv 3 \bmod 4$. Then the Sylow 2subgroups of $S L_{6}(q)$ and those of $S L_{7}(q)$ are 3 -connected.
Proof. Let $W_{1}$ be a Sylow 2-subgroup of $G L_{2}(q)$, let $W_{2}$ be the Sylow 2-subgroup of $G L_{4}(q)$ obtained from $W_{1}$ by the construction given in the last statement of Lemma 3.14, and let $W$ be the Sylow 2-subgroup of $G L_{6}(q)$ obtained from $W_{1}$ and $W_{2}$ by using the last statement of Lemma 3.15 .

From Lemma 3.15, we see that the Sylow 2-subgroups of $S L_{7}(q)$ are isomorphic to those of $G L_{6}(q)$. So it is enough to show that $W$ and $W \cap S L_{6}(q)$ are 3-connected. Given elements $x_{1}, x_{2}, x_{3} \in G L_{2}(q)$, we write $\operatorname{diag}\left(x_{1}, x_{2}, x_{3}\right)$ for the block diagonal matrix

$$
\left(\begin{array}{lll}
x_{1} & & \\
& x_{2} & \\
& & x_{3}
\end{array}\right)
$$

Let $A$ be the subgroup of $W \cap S L_{6}(q)$ consisting of the matrices $\operatorname{diag}\left(x_{1}, x_{2}, x_{3}\right)$, where $x_{i} \in\left\langle-I_{2}\right\rangle$ for $1 \leq i \leq 3$. Clearly, $A \cong E_{8}$. We prove the following:
(1) If $E$ is an elementary abelian subgroup of $W$ of rank at least 3 , then $E$ is 3-connected to an elementary abelian subgroup of $W \cap S L_{6}(q)$ of rank at least 3 .
(2) If $E$ is an elementary abelian subgroup of $W \cap S L_{6}(q)$ of rank at least 3, then $E$ is 3 -connected to $A$ in $W \cap S L_{6}(q)$.
By (1) and (2), any elementary abelian subgroup of $W$ of rank at least 3 is 3 -connected to $A$, and so $W$ is 3 -connected. Similarly, (2) implies that $W \cap S L_{6}(q)$ is 3 -connected.

Let $Z:=\left\langle\operatorname{diag}\left(-I_{2}, I_{2}, I_{2}\right), \operatorname{diag}\left(I_{2},-I_{2},-I_{2}\right)\right\rangle$. Since $Z \leq Z(W)$, we have that any elementary abelian subgroup of $W$ of rank at least 3 is 3 -connected to an $E_{8}$-subgroup of $W$ containing $Z$. Also, any elementary abelian subgroup of $W \cap S L_{6}(q)$ of rank at least 3 is 3 -connected (in $\left.W \cap S L_{6}(q)\right)$ to an $E_{8}$-subgroup of $W \cap S L_{6}(q)$ containing $Z$. Therefore, we only need to consider $E_{8}$-subgroups containing $Z$ in order to prove (1) and (2).

So let $E$ be an $E_{8}$-subgroup of $W$ with $Z \leq E$, and let $s \in E \backslash Z$. Suppose that $s=$ $\operatorname{diag}\left(s_{1}, s_{2}, s_{3}\right)$, where $s_{1}, s_{2}, s_{3} \in W_{1}$. Then $[E, A]=1$, and it is easy to deduce that $E$ is 3connected to $A$, so that $E$ satisfies (1). Also, if $E \leq W \cap S L_{6}(q)$, it is easy to deduce that $E$ satisfies (2).

Suppose now that

$$
s=\left(\begin{array}{lll}
s_{1} & & \\
& & s_{2} \\
& s_{3} &
\end{array}\right)
$$

for some $s_{1}, s_{2}, s_{3} \in W_{1}$. Since $s^{2}=I_{6}$, we have $s_{2}=s_{3}^{-1}$. Let $a$ be an involution of $W_{1}$ with $a \neq-I_{2}$. Set $s^{*}:=\operatorname{diag}\left(I_{2}, a, a^{s_{2}}\right)$ and $E^{*}:=\left\langle Z, s^{*}\right\rangle \cong E_{8}$. Clearly, $E^{*} \leq W \cap S L_{6}(q)$. It is easy to check that $\left[E, E^{*}\right]=1$, which implies that $E$ is 3-connected to $E^{*}$. So $E$ satisfies (1). If $E \leq W \cap S L_{6}(q)$, then $E$ is 3-connected to $E^{*}$ in $W \cap S L_{6}(q)$, and $E^{*}$ is 3-connected to $A$ in $W \cap S L_{6}(q)$ since $\left[E^{*}, A\right]=1$. Therefore, $E$ satisfies (2) when $E \leq W \cap S L_{6}(q)$.

Let $q$ be a nontrivial odd prime power with $q \equiv 3 \bmod 4$. A Sylow 2-subgroup of $P S L_{7}(q)$ is isomorphic to a Sylow 2-subgroup of $S L_{7}(q)$. So, by Lemma 3.29, the Sylow 2-subgroups of $P S L_{7}(q)$ are 3-connected.

We need the following lemma in order to prove that the Sylow 2-subgroups of $S L_{n}(q)$ and $P S L_{n}(q)$ are 3-connected when $n \in\{8,9\}$.
Lemma 3.30. Let $q$ be a nontrivial odd prime power with $q \equiv 3 \bmod 4$, and let $V$ be a Sylow 2 -subgroup of $G L_{4}(q)$. Let $u \in V$ with $u^{2}=I_{4}$ or $u^{2}=-I_{4}$. Then there is an involution $v \in V \backslash\left\langle u,-I_{4}\right\rangle$ which commutes with $u$.

Proof. Fix a Sylow 2-subgroup $W_{1}$ of $G L_{2}(q)$, and let $W_{2}$ be the Sylow 2-subgroup of $G L_{4}(q)$ obtained from $W_{1}$ by the construction given in the last statement of Lemma 3.14. By Sylow's Theorem, we may assume that $V=W_{2}$. Let $a$ be an involution of $W_{1}$ with $a \neq-I_{2}$.

First, we consider the case that

$$
u=\left(\begin{array}{ll}
x & \\
& y
\end{array}\right)
$$

with elements $x, y \in W_{1}$. If $x \notin\left\langle-I_{2}\right\rangle$ or $y \notin\left\langle-I_{2}\right\rangle$, then

$$
\left(\begin{array}{cc}
-I_{2} & \\
& I_{2}
\end{array}\right) \in W_{2}
$$

is an involution commuting with $u$ and not lying in $\left\langle u,-I_{4}\right\rangle$. If $x, y \in\left\langle-I_{2}\right\rangle$, then we may choose

$$
v:=\left(\begin{array}{ll}
a & \\
& a
\end{array}\right) .
$$

Assume now that

$$
u=\left(\begin{array}{ll} 
& x \\
y &
\end{array}\right)
$$

with elements $x, y \in W_{1}$. Let

$$
v:=\left(\begin{array}{ll}
a & \\
& a^{x}
\end{array}\right) .
$$

As $a$ is an involution of $W_{1}$, we have that $v$ is an involution of $W_{2}$. By a direct calculation (using that $\left.x y \in\left\langle-I_{2}\right\rangle\right)$, $v$ has the desired properties.

Lemma 3.31. Let $q$ be a nontrivial odd prime power with $q \equiv 3 \bmod 4$. Then the Sylow 2subgroups of $S L_{8}(q)$ and those of $S L_{9}(q)$ are 3 -connected.

Proof. Fix a Sylow 2-subgroup $W_{1}$ of $G L_{2}(q)$, let $W_{2}$ be the Sylow 2-subgroup of $G L_{4}(q)$ obtained from $W_{1}$ by the construction given in the last statement of Lemma 3.14, and let $W$ be the Sylow 2-subgroup of $G L_{8}(q)$ obtained from $W_{2}$ by the construction given in the last statement of Lemma 3.14. Set $S:=W \cap S L_{8}(q)$.

From Lemma 3.15, we see that the Sylow 2-subgroups of $S L_{9}(q)$ are isomorphic to those of $G L_{8}(q)$. So it is enough to show that $W$ and $S$ are 3-connected.

Given a natural number $\ell \geq 1$ and $x_{1}, \ldots, x_{\ell}$ of $G L_{2}(q) \cup G L_{4}(q)$, we write $\operatorname{diag}\left(x_{1}, \ldots, x_{\ell}\right)$ for the block diagonal matrix

$$
\left(\begin{array}{lll}
x_{1} & & \\
& \ddots & \\
& & x_{\ell}
\end{array}\right)
$$

Set

$$
A:=\left\{\operatorname{diag}\left(x_{1}, x_{2}, x_{3}, x_{4}\right) \mid x_{i} \in\left\langle-I_{2}\right\rangle \forall 1 \leq i \leq 4\right\} \leq S
$$

and

$$
Z:=\left\langle-I_{8}\right\rangle \leq S
$$

Clearly, $A \cong E_{16}$. Since $Z \leq Z(W)$, we have that any elementary abelian subgroup of $W$ of rank at least 3 is 3 -connected to an $E_{8}$-subgroup of $W$ containing $Z$. Similarly, any elementary abelian subgroup of $S$ of rank at least 3 is 3 -connected to an $E_{8}$-subgroup of $S$ containing $Z$. So it suffices to prove that any $E_{8}$-subgroup $E$ of $W$ with $Z \leq E$ is 3 -connected to $A$, where $E$ is even 3 -connected in $S$ to $A$ if $E \leq S$. Thus let $E$ be an $E_{8}$-subgroup of $W$ containing $Z$, and let $x, y \in E$ with $E=\langle Z, x, y\rangle$.

We consider a number of cases. Below, $a$ will always denote an involution of $W_{1}$ with $a \neq-I_{2}$.
Case 1: $x=\operatorname{diag}\left(-I_{4}, I_{4}\right)$ and $y=\operatorname{diag}\left(b_{1}, b_{2}\right)$ for some $b_{1}, b_{2} \in W_{2}$.
We determine an involution $y_{1} \in C_{W}(E) \backslash\langle Z, x\rangle$ such that $\left\langle Z, x, y_{1}\right\rangle \cong E_{8}$ is 3-connected to $A$. In the case that $E \leq S$, we determine $y_{1}$ such that $y_{1} \in S$ and such that $\left\langle Z, x, y_{1}\right\rangle$ is 3 -connected to $A$ in $S$. The existence of such an involution $y_{1}$ easily implies that $E$ is 3 -connected to $A$, and even 3 -connected to $A$ in $S$ if $E \leq S$. The involution $y_{1}$ is given by the following table in dependence of $y$. In each row, $r_{1}, r_{2}, r_{3}, r_{4}$ are assumed to be elements of $W_{1}$ such that $y$ is equal to the matrix given in the column " $y$ " and such that the conditions in the column "Conditions" (if any) are satisfied. The column " $y_{1}$ " gives the involution $y_{1}$ with the desired properties. For each row, one can verify the stated properties of $y_{1}$ by a direct calculation or by using the previous rows.

| Case | $y$ | Conditions | $y_{1}$ |
| :---: | :---: | :---: | :---: |
| 1.1 | $\left(\begin{array}{llll}r_{1} & & & \\ & r_{2} & & \\ & & r_{3} & \\ & & & r_{4}\end{array}\right)$ |  | $y$ |
| 1.2 | $\left(\begin{array}{llll}r_{1} & & & \\ & r_{2} & & \\ & & & r_{3} \\ & & r_{4} & \end{array}\right)$ | $\left\langle r_{1}, r_{2}\right\rangle \not \subset\left\langle-I_{2}\right\rangle$ | $\left(\begin{array}{lll}r_{1} & & \\ & r_{2} & \\ & & I_{4}\end{array}\right)$ |
| 1.3 | $\left(\begin{array}{llll}r_{1} & & & \\ & r_{2} & & \\ & & & r_{3} \\ & & r_{4} & \end{array}\right)$ | $r_{1}, r_{2} \leq\left\langle-I_{2}\right\rangle$ | $\left(\begin{array}{lll}a & & \\ & a & \\ & & I_{4}\end{array}\right)$ |
| 1.4 | $\left(\begin{array}{llll}r_{2} & r_{1} & & \\ & & r_{3} & \\ & & & r_{4}\end{array}\right)$ | $\left\langle r_{3}, r_{4}\right\rangle \notin\left\langle-I_{2}\right\rangle$ | $\left(\begin{array}{lll}I_{4} & & \\ & r_{3} & \\ & & r_{4}\end{array}\right)$ |
| 1.5 | $\left(\begin{array}{llll}r_{2} & r_{1} & & \\ & & r_{3} & \\ & & & r_{4}\end{array}\right)$ | $r_{3}, r_{4} \leq\left\langle-I_{2}\right\rangle$ | $\left(\begin{array}{lll}I_{4} & & \\ & a & \\ & & a\end{array}\right)$ |
| 1.6 | $\left(\begin{array}{llll}r_{2} & r_{1} & & \\ & & & \\ & & & r_{3} \\ & & & \end{array}\right)$ |  | $\left(\begin{array}{lll} & r_{1} & \\ r_{2} & \\ & & I_{4}\end{array}\right)$ |

Case 2: $x=\operatorname{diag}\left(a_{1}, a_{2}\right)$ and $y=\operatorname{diag}\left(b_{1}, b_{2}\right)$ for some $a_{1}, a_{2}, b_{1}, b_{2} \in W_{2}$.
Set $x_{1}:=\operatorname{diag}\left(-I_{4}, I_{4}\right)$. Since $E=\langle Z, x, y\rangle \cong E_{8}$, the elements $x$ and $y$ cannot be both contained in $\left\langle Z, x_{1}\right\rangle$. Without loss of generality, we may assume that $y \notin\left\langle Z, x_{1}\right\rangle$. Then $E_{1}:=$ $\left\langle Z, x_{1}, y\right\rangle \cong E_{8}$. The group $E_{1}$ is 3 -connected to $A$ by Case 1 , and it is 3 -connected to $E$ since $E$ and $E_{1}$ commute. Hence, $E$ is 3 -connected to $A$. Clearly, if $E \leq S$, then $E$ is even 3-connected in $S$ to $A$.

Case 3: There are $a_{1}, a_{2}, b_{1}, b_{2} \in W_{2}$ with

$$
\{x, y\}=\left\{\left(\begin{array}{ll}
a_{1} & \\
& a_{2}
\end{array}\right),\left(\begin{array}{ll} 
& b_{1} \\
b_{2} &
\end{array}\right)\right\} .
$$

Without loss of generality, we assume that

$$
x=\left(\begin{array}{ll}
a_{1} & \\
& a_{2}
\end{array}\right) \text { and } y=\left(\begin{array}{ll} 
& b_{1} \\
b_{2} &
\end{array}\right) .
$$

Since $x$ and $y$ are commuting involutions, we have $b_{1}=b_{2}^{-1}$ and $a_{2}=a_{1}{ }^{b_{1}}$. By Lemma 3.30, there is an involution $\widetilde{a_{1}} \in W_{2} \backslash\left\langle a_{1},-I_{4}\right\rangle$ which commutes with $a_{1}$. Set

$$
y_{1}:=\left(\begin{array}{cc}
\widetilde{a_{1}} & \\
& \widetilde{a_{1}}
\end{array}\right) .
$$

It is easy to see that $y_{1} \in S$, and $y_{1}$ is an involution since $\widetilde{a_{1}}$ is an involution of $W_{2}$. We have $\left[x, y_{1}\right]=1$ since $\widetilde{a_{1}}$ commutes with $a_{1}$ and ${\widetilde{a_{1}}}^{b_{1}}$ commutes with $a_{1}{ }^{b_{1}}=a_{2}$. A direct calculation using that $b_{1}=b_{2}^{-1}$ shows that we also have $\left[y, y_{1}\right]=1$. Thus $E=\langle Z, x, y\rangle$ commutes with $E_{1}:=\left\langle Z, x, y_{1}\right\rangle$. Since $\widetilde{a_{1}} \notin\left\langle a_{1},-I_{4}\right\rangle$, we have $y_{1} \notin\langle Z, x\rangle$ and hence $E_{1} \cong E_{8}$. Applying Case 2, it follows that $E$ is 3 -connected to $A$ (and even 3 -connected in $S$ to $A$ when $E \leq S$ ).

Case 4: There are $a_{1}, a_{2}, b_{1}, b_{2} \in W_{2}$ with

$$
x=\left(\begin{array}{ll} 
& a_{1} \\
a_{2} &
\end{array}\right) \text { and } y=\left(\begin{array}{ll} 
& b_{1} \\
b_{2} &
\end{array}\right) .
$$

This case can be reduced to Case 3 since $E=\langle Z, x, y\rangle=\langle Z, x, x y\rangle$.

Let $q$ be a nontrivial odd prime power with $q \equiv 3 \bmod 4$. A Sylow 2-subgroup of $\operatorname{PSL}_{9}(q)$ is isomorphic to a Sylow 2-subgroup of $S L_{9}(q)$. So, by Lemma 3.31, the Sylow 2-subgroups of $P S L_{9}(q)$ are 3-connected.

Lemma 3.32. Let $q$ be a nontrivial odd prime power with $q \equiv 3 \bmod 4$. Then the Sylow 2subgroups of $P S L_{8}(q)$ are 3 -connected.

Proof. Let $W_{1}$ be a Sylow 2-subgroup of $G L_{2}(q)$. Let $W_{2}$ be the Sylow 2-subgroup of $G L_{4}(q)$ obtained from $W_{1}$ by the construction given in the last statement of Lemma 3.14 and let $W_{3}$ be the Sylow 2-subgroup of $G L_{8}(q)$ obtained from $W_{2}$ by the construction given in the last statement of Lemma 3.14. Set $S:=W_{3} \cap S L_{8}(q)$. For each subgroup or element $X$ of $S L_{8}(q)$, let $\bar{X}$ denote the image of $X$ in $P S L_{8}(q)$. We prove that $\bar{S}$ is 3-connected.

Given a natural number $\ell \geq 1$ and $x_{1}, \ldots, x_{\ell}$ of $G L_{2}(q) \cup G L_{4}(q)$, we write $\operatorname{diag}\left(x_{1}, \ldots, x_{\ell}\right)$ for the block diagonal matrix

$$
\left(\begin{array}{ccc}
x_{1} & & \\
& \ddots & \\
& & x_{\ell}
\end{array}\right)
$$

Set

$$
A:=\left\{\operatorname{diag}\left(x_{1}, x_{2}, x_{3}, x_{4}\right) \mid x_{i} \in\left\langle-I_{2}\right\rangle \forall 1 \leq i \leq 4\right\} \leq S .
$$

We have $\bar{A} \cong E_{8}$.
Set

$$
Z:=\left\langle\operatorname{diag}\left(-I_{4}, I_{4}\right)\right\rangle .
$$

We have $\bar{Z} \leq Z(\bar{S})$. Using this, it is easy to note that any elementary abelian subgroup of $\bar{S}$ of rank at least 3 is 3 -connected to an $E_{8}$-subgroup of $\bar{S}$ containing $\bar{Z}$. Hence, it suffices to prove that any $E_{8}$-subgroup of $\bar{S}$ containing $\bar{Z}$ is 3 -connected to $\bar{A}$.

Let $x, y \in S$ and $B:=\langle\bar{Z}, \bar{x}, \bar{y}\rangle$. Suppose that $B \cong E_{8}$. Considering a number of cases, we will prove that $B$ is 3 -connected to $\bar{A}$. Below, $a$ will always denote an involution of $W_{1}$ with $a \neq-I_{2}$.

Case 1: $x=\operatorname{diag}\left(r_{1}, r_{2}, r_{3}, r_{4}\right)$ and $y=\operatorname{diag}\left(m_{1}, m_{2}\right)$ for some $r_{1}, r_{2}, r_{3}, r_{4} \in W_{1}$ and $m_{1}, m_{2} \in$ $W_{2}$.

We consider a number of subcases. These subcases are given by the rows of the table below. In each row, we assume that $s_{1}, s_{2}, s_{3}, s_{4}$ are elements of $W_{1}$ such that $y$ is equal to the matrix given in the column " $y$ ". We also assume that the conditions in the column "Conditions" (if any) are satisfied. The column " $y_{1}$ " gives an element of $S$ such that $\overline{y_{1}}$ is an involution in $C_{\bar{S}}(\bar{E}) \backslash\langle\bar{Z}, \bar{x}\rangle$ and such that $\left\langle\bar{Z}, \bar{x}, \overline{y_{1}}\right\rangle$ is 3 -connected to $\bar{A}$. The existence of such an element $y_{1}$ easily implies that $B$ is 3 -connected to $\bar{A}$.

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| Case | $y$ | Conditions | $y_{1}$ |
| :---: | :---: | :---: | :---: |
| 1.1 | $\left(\begin{array}{llll}s_{1} & & & \\ & s_{2} & & \\ & & s_{3} & \\ & & & s_{4}\end{array}\right)$ |  | $y$ |
| 1.2 | $\left(\begin{array}{llll}s_{2} & s_{1} & & \\ & & & \\ & & s_{3} & \\ & & & s_{4}\end{array}\right)$ | $x \notin A$ | $\left(\begin{array}{lll}I_{4} & & \\ & -I_{2} & \\ & & I_{2}\end{array}\right)$ |
| 1.3 | $\left(\begin{array}{llll}s_{2} & s_{1} & & \\ & & & \\ & & s_{3} & \\ & & & s_{4}\end{array}\right)$ | $x \in A$ | $\left(\begin{array}{lll}a & & \\ & a^{s_{2}^{-1}} & \\ & & I_{4}\end{array}\right)$ |
| 1.4 | $\left(\begin{array}{llll}s_{2} & s_{1} & & \\ & & & \\ & & & s_{3} \\ & & s_{4} & \end{array}\right)$ | $x \notin A$ | $\left(\begin{array}{llll}I_{2} & & & \\ & -I_{2} & & \\ & & I_{2} & \\ & & & -I_{2}\end{array}\right)$ |
| 1.5 | $\left(\begin{array}{llll} & s_{1} & & \\ s_{2} & & & \\ & & & s_{3} \\ & & s_{4} & \end{array}\right)$ | $x \in A$ | $\left(\begin{array}{llll}a & & & \\ & a^{s_{2}^{-1}} & & \\ & & a & \\ & & & a^{s_{4}^{-1}}\end{array}\right)$ |

The subcase that $y$ has the form

$$
\left(\begin{array}{llll}
s_{1} & & & \\
& s_{2} & & \\
& & & s_{3} \\
& & s_{4} &
\end{array}\right)
$$

can be easily reduced to Cases 1.2 and 1.3 .
Case 2: There are $r_{1}, r_{2}, r_{3}, r_{4} \in W_{1}$ and $m_{1}, m_{2} \in W_{2}$ with

$$
x=\left(\begin{array}{llll} 
& r_{1} & & \\
r_{2} & & & \\
& & r_{3} & \\
& & & r_{4}
\end{array}\right) \text { and } y=\left(\begin{array}{ll}
m_{1} & \\
& m_{2}
\end{array}\right)
$$

Case 2.1: There are $s_{1}, s_{2}, s_{3}, s_{4} \in W_{1}$ with

$$
y=\left(\begin{array}{llll}
s_{1} & & & \\
& s_{2} & & \\
& & s_{3} & \\
& & & s_{4}
\end{array}\right) \text { or } y=\left(\begin{array}{cccc} 
& s_{1} & & \\
s_{2} & & & \\
& & s_{3} & \\
& & & s_{4}
\end{array}\right)
$$

Noticing that $\langle\bar{Z}, \bar{x}, \bar{y}\rangle=\langle\bar{Z}, \bar{x}, \bar{x} \bar{y}\rangle$, this case can be reduced to Case 1 .
Case 2.2: There are $s_{1}, s_{2}, s_{3}, s_{4} \in W_{1}$ with

$$
y=\left(\begin{array}{llll}
s_{1} & & & \\
& s_{2} & & \\
& & & s_{3} \\
& & s_{4} &
\end{array}\right)
$$

Since $B \cong E_{8}$, we have $\varepsilon x^{y}=x$, where $\varepsilon \in\{+,-\}$. By a direct calculation, we have

$$
x^{y}=\left(\begin{array}{cccc} 
& s_{1}^{-1} r_{1} s_{2} & & \\
s_{2}^{-1} r_{2} s_{1} & & & \\
& & r_{4}^{s_{4}} & \\
& & & r_{3}^{s_{3}}
\end{array}\right)
$$

As $x=\varepsilon x^{y}$, we have $r_{1}=\varepsilon s_{1}^{-1} r_{1} s_{2}, r_{2}=\varepsilon s_{2}^{-1} r_{2} s_{1}, r_{3}=\varepsilon r_{4}^{s_{4}}$ and $r_{4}=\varepsilon r_{3}^{s_{3}}$. Note that $\varepsilon s_{1}^{r_{1}}=s_{2}$ and $\varepsilon s_{2}^{r_{2}}=s_{1}$.

We now consider a number of subsubcases. These subsubcases are given by the rows of the table below. The columns "Condition 1" and "Condition 2" describe the subsubcase under consideration. The column " $y_{1}$ " gives an element $y_{1} \in S$ such that $\overline{y_{1}}$ is an involution in $C_{\bar{S}}(\bar{E}) \backslash\langle\bar{Z}, \bar{x}\rangle$ and such that $\left\langle\bar{Z}, \bar{x}, \overline{y_{1}}\right\rangle$ is 3 -connected to $\bar{A}$. In each subsubcase, one can see from the above calculations and from the previous cases that $y_{1}$ indeed has the stated properties. The existence of such an element $y_{1}$ easily implies that $B$ is 3 -connected to $\bar{A}$ in all subsubcases.

| Case | Condition 1 | Condition 2 | $y_{1}$ |
| :---: | :---: | :---: | :---: |
| 2.2.1 | $x^{2}=I_{8}=y^{2}$ | $\left\langle r_{3}, r_{4}\right\rangle \not \leq\left\langle-I_{2}\right\rangle$ | $\left(\begin{array}{llll}\varepsilon s_{1} & & & \\ & s_{2} & & \\ & & \varepsilon r_{3} & \\ & & & r_{4}\end{array}\right)$ |
| 2.2.2 | $x^{2}=I_{8}=y^{2}$ | $\left\langle r_{3}, r_{4}\right\rangle \leq\left\langle-I_{2}\right\rangle$ | $\left(\begin{array}{llll} & r_{1} & & \\ r_{2} & & \\ & & \varepsilon a & \\ & & & a^{s_{3}}\end{array}\right)$ |
| 2.2.3 | $x^{2}=-I_{8}=y^{2}$ |  | $\left(\begin{array}{llll}\varepsilon s_{1} & & & \\ & s_{2} & & \\ & & \varepsilon r_{3} & \\ & & & r_{4}\end{array}\right)$ |
| 2.2.4 | $x^{2}=I_{8}, y^{2}=-I_{8}$ | $\left\langle r_{3}, r_{4}\right\rangle \not \leq\left\langle-I_{2}\right\rangle$ | $\left(\begin{array}{lll}I_{4} & & \\ & \varepsilon r_{3} & \\ & & r_{4}\end{array}\right)$ |
| 2.2.5 | $x^{2}=I_{8}, y^{2}=-I_{8}$ | $\left\langle r_{3}, r_{4}\right\rangle \leq\left\langle-I_{2}\right\rangle$ | $\left(\begin{array}{lll}I_{4} & & \\ & \varepsilon a & \\ & & \varepsilon a^{s_{3}}\end{array}\right)$ |

The case that $x^{2}=-I_{8}$ and $y^{2}=I_{8}$ can be easily reduced to Cases 2.2.4 and 2.2.5.
Case 2.3: There are $s_{1}, s_{2}, s_{3}, s_{4} \in W_{1}$ with

$$
y=\left(\begin{array}{llll} 
& s_{1} & & \\
s_{2} & & & \\
& & & s_{3} \\
& & s_{4} &
\end{array}\right)
$$

Since $\langle\bar{Z}, \bar{x}, \bar{y}\rangle=\langle\bar{Z}, \bar{x}, \bar{x} \bar{y}\rangle$, this case can be reduced to Case 2.2.
Case 3: There are $r_{1}, r_{2}, r_{3}, r_{4} \in W_{1}$ and $m_{1}, m_{2} \in W_{2}$ with

$$
x=\left(\begin{array}{llll}
r_{1} & & & \\
& r_{2} & & \\
& & & r_{3}
\end{array}\right) \text { and } y=\left(\begin{array}{lll}
m_{1} & \\
& & m_{2}
\end{array}\right) \text {. }
$$

This case can be reduced to Case 2.

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Case 4: There are $r_{1}, r_{2}, r_{3}, r_{4} \in W_{1}$ and $m_{1}, m_{2} \in W_{2}$ with

$$
x=\left(\begin{array}{cccc} 
& r_{1} & & \\
r_{2} & & \\
& & & r_{3} \\
& & r_{4} &
\end{array}\right) \text { and } y=\left(\begin{array}{lll}
m_{1} & \\
& m_{2}
\end{array}\right) \text {. }
$$

In view of Cases 1-3, we may assume that

$$
y=\left(\begin{array}{llll} 
& s_{1} & & \\
s_{2} & & & \\
& & & s_{3} \\
& & s_{4} &
\end{array}\right)
$$

for some $s_{1}, s_{2}, s_{3}, s_{4} \in W_{1}$. Since $\langle\bar{Z}, \bar{x}, \bar{y}\rangle=\langle\bar{Z}, \bar{x}, \bar{x} \bar{y}\rangle$, we can now reduce the given case to Case 1.

Case 5: There are $a_{1}, a_{2}, b_{1}, b_{2} \in W_{2}$ with

$$
\{x, y\}=\left\{\left(\begin{array}{ll}
a_{1} & \\
& a_{2}
\end{array}\right),\left(\begin{array}{cc} 
& b_{1} \\
b_{2} &
\end{array}\right)\right\} .
$$

Without loss of generality, we assume that

$$
x=\left(\begin{array}{ll}
a_{1} & \\
& a_{2}
\end{array}\right) \text { and } y=\left(\begin{array}{ll} 
& b_{1} \\
b_{2} &
\end{array}\right) .
$$

We have $x^{2} \in\left\langle-I_{8}\right\rangle$ since $B=\langle\bar{Z}, \bar{x}, \bar{y}\rangle \cong E_{8}$, and hence $a_{1}{ }^{2} \in\left\langle-I_{4}\right\rangle$. So, by Lemma 3.30, there is an involution $\widetilde{a_{1}} \in W_{2} \backslash\left\langle a_{1},-I_{4}\right\rangle$ which commutes with $a_{1}$. Set

$$
y_{1}:=\left(\begin{array}{cc}
\widetilde{a_{1}} & \\
& \widetilde{a_{1}}
\end{array}\right) .
$$

Clearly, $\overline{y_{1}}$ is an involution of $\bar{S}$. As $[x, y] \in\left\langle-I_{8}\right\rangle$, we have $a_{1}{ }^{b_{1}} \in\left\{a_{2},-a_{2}\right\}$. Since $a_{1}$ and $\widetilde{a_{1}}$ commute, it follows that $\widetilde{a_{1}}{ }^{b_{1}}$ and $a_{2}$ commute. So we have $\left[x, y_{1}\right]=1$ and hence $\left[\bar{x}, \overline{y_{1}}\right]=1$. Using that $y^{2} \in\left\langle-I_{8}\right\rangle$, one can easily verify that $\left[y, y_{1}\right]=1$ and hence $\left[\bar{y}, \overline{y_{1}}\right]=1$. As $\widetilde{a_{1}} \notin\left\langle a_{1},-I_{4}\right\rangle$, we have $\overline{y_{1}} \notin\langle\bar{Z}, \bar{x}\rangle$.

Now $\left\langle\bar{Z}, \bar{x}, \overline{y_{1}}\right\rangle$ is an $E_{8}$-subgroup of $\bar{S}$ which commutes with $B$ and which is 3-connected to $\bar{A}$ by Cases $1-4$. Thus $B$ is 3 -connected to $\bar{A}$.

Case 6: There are $a_{1}, a_{2}, b_{1}, b_{2} \in W_{2}$ with

$$
x=\left(\begin{array}{ll} 
& a_{1} \\
a_{2} &
\end{array}\right) \text { and } y=\left(\begin{array}{ll} 
& b_{1} \\
b_{2} &
\end{array}\right) .
$$

Noticing that $\langle\bar{Z}, \bar{x}, \bar{y}\rangle=\langle\bar{Z}, \bar{x}, \bar{x} \bar{y}\rangle$, we can reduce this case to Case 5 .
We summarize the above lemmas in the following corollary.
Corollary 3.33. Let $q$ be a nontrivial odd prime power and $n \geq 6$. Then the following hold:
(i) The Sylow 2-subgroups of $S L_{n}(q)$ and those of $P S L_{n}(q)$ are 2-connected.
(ii) The Sylow 2-subgroups of $S L_{n}(q)$ are 3-connected.
(iii) If $q \equiv 1 \bmod 4$ or $n \geq 7$, then the Sylow 2-subgroups of $P S L_{n}(q)$ are 3 -connected.

Unfortunately, the Sylow 2-subgroups of $P S L_{6}(q)$ are not 3 -connected when $q \equiv 3 \bmod 4$ (this is not terribly difficult to observe).

Corollary 3.34. Let $q$ be a nontrivial odd prime power and $n \geq 6$. Let $G=S L_{n}(q)$, or $G=$ $P S L_{n}(q)$ and $n \geq 7$ if $q \equiv 3 \bmod 4$. For any Sylow 2 -subgroup $S$ of $G$ and any elementary abelian subgroup $A$ of $S$ with $m(A) \leq 3$, there is some elementary abelian subgroup $B$ of $S$ with $A<B$ and $m(B)=4$.
Proof. By Corollary 3.33, $S$ is 2-connected and 3-connected. Applying [32, Lemma 8.7], the claim follows.
3.6. Generation. Next we discuss some generational properties of $(P) S L_{n}(q)$ and $(P) S U_{n}(q)$, where $n \geq 3$ and $q$ is a nontrivial odd prime power. We need the following definition (see 32, Section 8]).

Definition 3.35. Let $G$ be a finite group, let $S$ be a Sylow 2 -subgroup of $G$, and let $k$ be a positive integer. We say that $G$ is $k$-generated if

$$
G=\Gamma_{S, k}(G):=\left\langle N_{G}(T) \mid T \leq S, m(T) \geq k\right\rangle .
$$

The following two lemmas will later prove to be useful.
Lemma 3.36. (see [6]) Let $q$ be a nontrivial odd prime power. Then the groups $S L_{3}(q), P S L_{3}(q)$, $S U_{3}(q)$ and $\mathrm{PSU}_{3}(q)$ are 2-generated.
Lemma 3.37. Let $q$ be a nontrivial odd prime power, and let $n \geq 4$ be a natural number. Moreover, let $\varepsilon \in\{+,-\}$ and $Z \leq Z\left(S L_{n}^{\varepsilon}(q)\right)$. Assume that one of the following holds:
(i) $n \geq 5$,
(ii) $q \equiv \varepsilon \bmod 8$,
(iii) $Z=1$.

Then $S L_{n}^{\varepsilon}(q) / Z$ is 3-generated.
We need the following lemma in order to prove Lemma 3.37 .
Lemma 3.38. (see [45], [14]) Let $q>2$ be a prime power, and let $n \geq 3$ be a natural number. Let $\varepsilon \in\{+,-\}$. Define

$$
U_{1}:=\left\{\left(\begin{array}{ll}
A & \\
& I_{n-2}
\end{array}\right): A \in S L_{2}^{\varepsilon}(q)\right\}
$$

and

$$
U_{n-1}:=\left\{\left(\begin{array}{ll}
I_{n-2} & \\
& A
\end{array}\right): A \in S L_{2}^{\varepsilon}(q)\right\} .
$$

Moreover, for each $2 \leq i \leq n-2$, let

$$
U_{i}:=\left\{\left(\begin{array}{ccc}
I_{i-1} & & \\
& A & \\
& & I_{n-i-1}
\end{array}\right): A \in S L_{2}^{\varepsilon}(q)\right\} .
$$

Then the following hold:
(i) We have $S L_{n}^{\varepsilon}(q)=\left\langle U_{i}: 1 \leq i \leq n-1\right\rangle$.
(ii) For each $1 \leq i \leq n-2$, there is a monomial matrix $m_{i}$ in $S L_{n}^{\varepsilon}(q)$ with $U_{i}^{m_{i}}=U_{i+1}$.

Proof of Lemma 3.37. Let $q$ be a nontrivial odd prime power, and let $n \geq 4$ be a natural number. Moreover, let $\varepsilon \in\{+,-\}$ and $Z \leq Z\left(S L_{n}^{\varepsilon}(q)\right)$. Suppose that one of the conditions $n \geq 5$, $q \equiv \varepsilon \bmod 8$ or $Z=1$ is satisfied. We have to show that $S L_{n}^{\varepsilon}(q) / Z$ is 3 -generated.

Let $U_{1}, \ldots, U_{n-1}$ denote the $S L_{2}^{\varepsilon}(q)$-subgroups of $S L_{n}^{\varepsilon}(q)$ corresponding to the $2 \times 2$ blocks along the main diagonal (as in Lemma 3.38). Let $E$ be the subgroup of $S L_{n}^{\varepsilon}(q)$ consisting of the diagonal matrices in $S L_{n}^{\varepsilon}(q)$ with diagonal entries in $\{-1,1\}$.

Assume that $n \geq 5$. Then one can easily see that, for each $i \in\{1, \ldots, n-1\}$, there is an $E_{8}$-subgroup $E_{i}$ of $E$ with $E_{i} \cap Z\left(S L_{n}^{\varepsilon}(q)\right)=1$ and $\left[E_{i}, U_{i}\right]=1$. Hence, $U_{i} Z / Z$ centralizes
$E_{i} Z / Z \cong E_{8}$ for each $i \in\{1, \ldots, n-1\}$. Now, if $S$ is a Sylow 2-subgroup of $S L_{n}^{\varepsilon}(q) / Z$ containing $E Z / Z$, we have $U_{i} Z / Z \leq \Gamma_{S, 3}\left(S L_{n}^{\varepsilon}(q) / Z\right)$ for each $i \in\{1, \ldots, n-1\}$, and Lemma 3.38 (i) implies that $S L_{n}^{\varepsilon}(q) / Z$ is 3 -generated.

We now consider the case $n=4$. By hypothesis, $Z=1$ or $q \equiv \varepsilon \bmod 8$. Let

If $Z=1$, set $y:=-I_{4}$. If $q \equiv \varepsilon \bmod 8$, let $\lambda$ be an element of $\mathbb{F}_{q^{2}}^{*}$ of order 8 such that $\lambda^{q-\varepsilon}=1$. Note that $\lambda \in \mathbb{F}_{q}^{*}$ if $\varepsilon=+$. Also, if $q \equiv \varepsilon \bmod 8$ and $|Z|=2$, let $y:=\lambda^{2} I_{4} \in S L_{4}^{\varepsilon}(q)$, and if $q \equiv \varepsilon \bmod 8$ and $|Z|=4$, let $y:=\operatorname{diag}(\lambda, \lambda, \lambda,-\lambda) \in S L_{4}^{\varepsilon}(q)$.

Let $S_{0}$ be a Sylow 2-subgroup of $U$ containing $E \cap U$. Let $\widetilde{S}$ be a Sylow 2-subgroup of $S L_{4}^{\varepsilon}(q)$ containing $S_{0}$ and $y$. Denote the image of $\widetilde{S}$ in $S L_{4}^{\varepsilon}(q) / Z$ by $S$. We have $S \cap U Z / Z=S_{0} Z / Z \in$ $\operatorname{Syl}_{2}(U Z / Z)$. By Lemma $3.36, U Z / Z \cong U \cong S L_{3}^{\varepsilon}(q)$ is 2 -generated. So we have

$$
U Z / Z=\Gamma_{S_{0} Z / Z, 2}(U Z / Z)=\left\langle N_{U Z / Z}(T) \mid T \leq S_{0} Z / Z, m(T) \geq 2\right\rangle
$$

Let $T \leq S_{0} Z / Z$ with $m(T) \geq 2$ and $\widehat{T}:=\langle T, y Z\rangle$. Clearly, $y Z$ is an involution of $S$ not contained in $U Z / Z$ and centralizing $U Z / Z$. Therefore, we have that $m(\widehat{T}) \geq 3$ and $N_{U Z / Z}(T) \leq$ $N_{S L_{n}^{\varepsilon}(q) / Z}(\widehat{T})$. It follows that $U Z / Z \leq \Gamma_{S, 3}\left(S L_{n}^{\varepsilon}(q) / Z\right)$. In particular, $U_{i} Z / Z \leq \Gamma_{S, 3}\left(S L_{n}^{\varepsilon}(q) / Z\right)$ for $i \in\{1,2\}$.

From Lemma 3.38 (ii), we see that there is some $m \in S L_{4}^{\varepsilon}(q)$ such that $U_{2}{ }^{m}=U_{3}$ and such that $m$ normalizes $\langle E, y\rangle$. So $m Z$ normalizes $\langle E Z / Z, y Z\rangle$. It is easy to note that $\langle E Z / Z, y Z\rangle \cong E_{8}$, and so we have $m Z \in \Gamma_{S, 3}\left(S L_{n}^{\varepsilon}(q) / Z\right)$. It follows that $U_{3} Z / Z=\left(U_{2} Z / Z\right)^{m Z} \leq \Gamma_{S, 3}\left(S L_{n}^{\varepsilon}(q) / Z\right)$.

So we have $U_{i} Z / Z \leq \Gamma_{S, 3}\left(S L_{n}^{\varepsilon}(q) / Z\right)$ for $i \in\{1,2,3\}$, and Lemma 3.38 (i) implies that $S L_{n}^{\varepsilon}(q) / Z$ is 3-generated.
3.7. Automorphisms of $(P) S L_{n}(q)$. Fix a prime number $p$, a positive integer $f$ and a natural number $n \geq 2$. Set $q:=p^{f}$ and $T:=S L_{n}(q)$. We now briefly describe the structure of $\operatorname{Aut}(T / Z)$, where $Z \leq Z(T)$, referring to [20] and [17, Section 2.1] for further details.

Let $\operatorname{Inndiag}(T):=\operatorname{Aut}_{G L_{n}(q)}(T)$. Note that

$$
\operatorname{Inndiag}(T) / \operatorname{Inn}(T) \cong C_{(n, q-1)}
$$

The map

$$
\phi: T \rightarrow T,\left(a_{i j}\right) \mapsto\left(a_{i j}{ }^{p}\right)
$$

is an automorphism of $T$ with order $f$. One can check that $\phi$ normalizes $\operatorname{Inndiag}(T)$. Set

$$
P \Gamma L_{n}(q):=\operatorname{Inndiag}(T)\langle\phi\rangle .
$$

It is easy to note that $\langle\phi\rangle \cap \operatorname{Inndiag}(T)=1$, so that $P \Gamma L_{n}(q)$ is the inner semidirect product of $\operatorname{Inndiag}(T)$ and $\langle\phi\rangle$.

The map

$$
\iota: T \rightarrow T, a \mapsto\left(a^{t}\right)^{-1}
$$

is an automorphism of $T$ with order 2 . If $n=2$, then $\iota$ turns out to be an inner automorphism of $T$, while we have $\iota \notin P \Gamma L_{n}(q)$ when $n \geq 3$. By a direct calculation, $\iota$ normalizes $\operatorname{Inndiag}(T)$ and commutes with $\phi$. In particular, $A:=P \Gamma L_{n}(q)\langle\iota\rangle$ is a subgroup of $\operatorname{Aut}(T)$, and we have

$$
A / \operatorname{Inndiag}(T) \cong C_{f} \times C_{a},
$$

where $a=1$ if $n=2$ and $a=2$ if $n \geq 3$.
Now let $Z$ be a central subgroup of $T$. It can be easily checked that the natural homomorphism $\operatorname{Aut}(T) \rightarrow \operatorname{Aut}(T / Z)$ is injective. The image of $\operatorname{Inndiag}(T)$ under this homomorphism will be denoted by $\operatorname{Inndiag}(T / Z)$. By abuse of notation, we denote the image of $P \Gamma L_{n}(q)$ in $\operatorname{Aut}(T / Z)$ again by $P \Gamma L_{n}(q)$ and the images of $\iota$ and $\phi$ again by $\iota$ and $\phi$, respectively.

With this notation, we have

$$
\operatorname{Aut}(T / Z)=P \Gamma L_{n}(q)\langle\iota\rangle
$$

Note that the natural homomorphism $\operatorname{Aut}(T) \rightarrow \operatorname{Aut}(T / Z)$ is an isomorphism and that it induces an isomorphism $\operatorname{Out}(T) \rightarrow \operatorname{Out}(T / Z)$.

The elements of $\operatorname{Inndiag}(T / Z) \backslash \operatorname{Inn}(T / Z)$ are said to be the (non-trivial) diagonal automorphisms of $T / Z$. An automorphism of $T / Z$ is called a field automorphism if it is conjugate to $\phi^{i}$ for some $1 \leq i<f$. The automorphisms of the form $\alpha \iota$, where $\alpha \in \operatorname{Inndiag}(T / Z)$, are said to be the graph automorphisms of $T / Z$. An automorphism of $T / Z$ is said to be a graph-field automorphism if it is conjugate to an automorphism of the form $\phi^{i} \iota$ for some $1 \leq i<f$. We remark that these definitions are particular cases of more general definitions, see [48, Chapter 10].

Proposition 3.39. Let $q$ be a nontrivial prime power, and let $n \geq 2$. Then $\operatorname{Out}\left(P S L_{n}(q)\right)$ is 2-nilpotent.

Proof. From the above remarks, it is easy to see that $\operatorname{Out}\left(P S L_{n}(q)\right)$ is supersolvable. By [38, Lemma 2.4 (4)], any supersolvable finite group is 2-nilpotent, and so the proposition follows.

The following proposition also follows from the above remarks.
Proposition 3.40. Let $n \geq 2$ be a natural number. Then $\operatorname{Out}\left(S L_{n}(3)\right)$ is a 2-group.
3.8. Automorphisms of $(P) S U_{n}(q)$. Let $p$ be a prime number, $f$ be a positive integer and $n \geq 3$ be a natural number. Set $q:=p^{f}$ and $T:=S U_{n}(q)$. We now briefly describe the structure of $\operatorname{Aut}(T / Z)$, where $Z \leq Z(T)$, referring to [20] and [17, Section 2.3] for further details.

Let $\operatorname{Inndiag}(T):=\operatorname{Aut}_{G U_{n}(q)}\left(S U_{n}(q)\right)$. It is rather easy to note that

$$
\operatorname{Inndiag}(T) / \operatorname{Inn}(T) \cong C_{(n, q+1)}
$$

The map

$$
\phi: T \rightarrow T,\left(a_{i j}\right) \mapsto\left(a_{i j}{ }^{p}\right)
$$

is an automorphism of $T$ with order $2 f$. One can check that $\phi$ normalizes Inndiag $(T)$. Set

$$
P \Gamma U_{n}(q):=\operatorname{Inndiag}(T)\langle\phi\rangle .
$$

It is rather easy to note that $\langle\phi\rangle \cap \operatorname{Inndiag}(T)=1$, so that $P \Gamma U_{n}(q)$ is the inner semidirect product of Inndiag $(T)$ and $\langle\phi\rangle$. Note that

$$
P \Gamma U_{n}(q) / \operatorname{Inndiag}(T) \cong C_{2 f} .
$$

Now let $Z$ be a central subgroup of $T$. It can be easily checked that the natural homomorphism $\operatorname{Aut}(T) \rightarrow \operatorname{Aut}(T / Z)$ is injective. The image of $\operatorname{Inndiag}(T)$ under this homomorphism will be denoted by Inndiag $(T / Z)$. By abuse of notation, we denote the image of $P \Gamma U_{n}(q)$ in $\operatorname{Aut}(T / Z)$ again by $P \Gamma U_{n}(q)$ and the image of $\phi$ again by $\phi$.

With this notation, we have

$$
\operatorname{Aut}(T / Z)=P \Gamma U_{n}(q)
$$

Note that the natural homomorphism $\operatorname{Aut}(T) \rightarrow \operatorname{Aut}(T / Z)$ is an isomorphism and that it induces an isomorphism $\operatorname{Out}(T) \rightarrow \operatorname{Out}(T / Z)$.

The elements of $\operatorname{Inndiag}(T / Z) \backslash \operatorname{Inn}(T / Z)$ are said to be the (non-trivial) diagonal automorphisms of $T / Z$. An automorphism of $T / Z$ is called a field automorphism if it is conjugate to $\phi^{i}$ for some $1 \leq i<2 f$ such that $\phi^{i}$ has odd order. The automorphisms of the form $\alpha \phi^{i}$, where $\phi^{i}$ has even order and $\alpha \in \operatorname{Inndiag}(T / Z)$, are said to be the graph automorphisms of $T / Z$. There are no graph-field automorphisms of $T / Z$.
Proposition 3.41. Let $q$ be a nontrivial prime power, and let $n \geq 3$. Then $\operatorname{Out}\left(P S U_{n}(q)\right)$ is 2-nilpotent.

Proof. We see from the above remarks that $\operatorname{Out}\left(P S U_{n}(q)\right)$ is supersolvable. So $\operatorname{Out}\left(P S U_{n}(q)\right)$ is 2-nilpotent by [38, Lemma 2.4 (4)].

The following proposition also follows from the above remarks.
Proposition 3.42. Let $n \geq 3$ be a natural number. Then $\operatorname{Out}\left(S U_{n}(3)\right)$ is a 2-group.
3.9. Some lemmas. We now prove several results on the automorphism groups of $(P) S L_{n}(q)$ and $(P) S U_{n}(q)$, where $n \geq 2$ and $q$ is a nontrivial odd prime power.

Lemma 3.43. Let $q$ be a nontrivial odd prime power. Also, let $T:=S L_{2}(q)$ and $S \in \operatorname{Syl}_{2}(T)$. Suppose that $\alpha$ and $\beta$ are 2-elements of $\operatorname{Aut}(T)$ such that $S^{\alpha}=S=S^{\beta}$ and $\left.\alpha\right|_{S, S}=\left.\beta\right|_{S, S}$. Then $\alpha=\beta$.

Proof. Let $\gamma:=\alpha \beta^{-1} \in C_{\operatorname{Aut}(T)}(S)$. We have $C_{\operatorname{Inndiag}(T)}(S)=1$ by [29, Lemma 4.10.10]. Therefore, it suffices to show that $\gamma \in \operatorname{Inndiag}(T)$. Clearly, the images of $\alpha$ and $\beta^{-1}$ in $\operatorname{Aut}(T) / \operatorname{Inndiag}(T)$ are 2-elements of $\operatorname{Aut}(T) / \operatorname{Inndiag}(T)$. Since $\operatorname{Aut}(T) / \operatorname{Inndiag}(T)$ is abelian,

$$
\gamma \cdot \operatorname{Inndiag}(T)=(\alpha \cdot \operatorname{Inndiag}(T)) \cdot\left(\beta^{-1} \cdot \operatorname{Inndiag}(T)\right)
$$

is still a 2-element of $\operatorname{Aut}(T) / \operatorname{Inndiag}(T)$. By [29, Lemma 4.10.10], $C_{\operatorname{Aut}(T)}(S)$ is a $2^{\prime}$-group, and so $\gamma$ has odd order. Therefore, $\gamma \cdot \operatorname{Inndiag}(T)$ has odd order. It follows that $\gamma \in \operatorname{Inndiag}(T)$, as required.
Lemma 3.44. Let $q=p^{f}$, where $p$ is an odd prime and $f$ is a positive integer. Let $T:=P S L_{2}(q)$, and let $\alpha$ be an involution of $\operatorname{Aut}(T)$. Suppose that $C_{T}(\alpha)$ has a 2-component $K$. Then we have $2 \mid f,(f, p) \neq(2,3)$ and $K \cong P S L_{2}\left(p^{\frac{f}{2}}\right)$. In particular, $K$ is a component of $C_{T}(\alpha)$.
Proof. Note that $C_{T}(\alpha) \cong C_{\operatorname{Inn}(T)}(\alpha)$.
Assume that $\alpha \in \operatorname{Inndiag}(T)$. Noticing that $\operatorname{Inndiag}(T) \cong P G L_{2}(q)$, we see from Lemma 3.3 that $C_{\operatorname{Inndiag}(T)}(\alpha)$ is solvable. Thus $C_{T}(\alpha) \cong C_{\operatorname{Inn}(T)}(\alpha)$ is solvable, and $C_{T}(\alpha)$ has no 2components, a contradiction to the choice of $\alpha$.

So we have $\alpha \notin \operatorname{Inndiag}(T)$. By the structure of $\operatorname{Aut}\left(P S L_{2}(q)\right)$ and since $\alpha$ has order 2, we can write $\alpha$ as a product of an inner-diagonal automorphism and a field automorphism of order 2. In particular, $f$ must be even. Consulting [29, Proposition 4.9.1 (d)], we see that $\alpha$ itself is a field automorphism. So we can apply [29, Proposition 4.9 .1 (b)] to conclude that $C_{\operatorname{Inndiag}(T)}(\alpha) \cong$ $\operatorname{Inndiag}\left(P S L_{2}\left(p^{\frac{f}{2}}\right)\right) \cong P G L_{2}\left(p^{\frac{f}{2}}\right)$. Consequently, $K$ is isomorphic to a 2-component of $P G L_{2}\left(p^{\frac{f}{2}}\right)$. It follows that $(f, p) \neq(2,3)$ and $K \cong P S L_{2}\left(p^{\frac{f}{2}}\right)$.

Before we state the next lemma, we introduce some notational conventions for adjoint Chevalley groups. Given a nontrivial prime power $q$, we denote $A_{1}(q)$ also by $B_{1}(q)$ and by $C_{1}(q)$. Moreover, $B_{2}(q)$ will be also denoted by $C_{2}(q)$, and $A_{3}(q)$ will be also denoted by $D_{3}(q)$. We also set $D_{2}(q):=A_{1}(q) \times A_{1}(q)$ and ${ }^{2} D_{2}(q):=A_{1}\left(q^{2}\right)$.
Lemma 3.45. Let $q=p^{f}$, where $p$ is an odd prime and $f$ is a positive integer. Let $n \geq 3$ be a natural number and $\varepsilon \in\{+,-\}$. Let $T:=P S L_{n}^{\varepsilon}(q)$, and let $\alpha$ be an involution of $\operatorname{Aut}(T)$. Suppose that $C_{T}(\alpha)$ has a 2-component $K$. Then $K$ is in fact a component, and one of the following holds:
(i) $K \cong S L_{i}^{\varepsilon}(q)$ for some $2 \leq i<n$, where $i>2$ if $q=3$;
(ii) $n$ is even, and $K$ is isomorphic to a nontrivial quotient of $S L_{\frac{n}{2}}\left(q^{2}\right)$;
(iii) $\varepsilon=+, f$ is even, $K \cong P S L_{n}\left(p^{\frac{f}{2}}\right)$ or $K \cong P S U_{n}\left(p^{\frac{f}{2}}\right)$;
(iv) $q \neq 3, n=3$ or 4 , and $K \cong P S L_{2}(q)$;
(v) $n$ is odd, $n \geq 5$ and $K \cong B_{\frac{n-1}{2}}(q)$;
(vi) $n$ is even and $K \cong C_{\frac{n}{2}}(q)$;
(vii) $n$ is even, $n \geq 6$ and ${ }^{2} K \cong D_{\frac{n}{2}}(q)$;
(viii) $n$ is even, $n \geq 6$ and $K \cong{ }^{2} D_{\frac{n}{2}}(q)$.

Here, the (twisted) Chevalley groups appearing in (v)-(viii) are adjoint.
Proof. It can be shown that any involution of $\operatorname{Aut}(T)$ is an inner-diagonal automorphism, a field automorphism, a graph automorphism, or a graph-field automorphism (see [17, Section 3.1.3] or [29, Section 4.9]).

Case 1: $\alpha \in \operatorname{Inndiag}(T)$, or $\alpha$ is a graph automorphism.
Set $C^{*}:=C_{\operatorname{Inndiag}(T)}(\alpha)$ and $L^{*}:=O^{p^{\prime}}\left(C^{*}\right)$. One can see from [29, Theorem 4.2.2 and Table 4.5.1] that $C^{*} / L^{*}$ is solvable and that one of the following holds:
(1) $L^{*}$ is the central product of two subgroups isomorphic to $S L_{i}^{\varepsilon}(q)$ and $S L_{n-i}^{\varepsilon}(q)$ for some natural number $i$ with $1 \leq i \leq \frac{n}{2}$,
(2) $n$ is even and $L^{*}$ is isomorphic to a nontrivial quotient of $S L_{\frac{n}{2}}\left(q^{2}\right)$,
(3) $n$ is odd and $L^{*} \cong B_{\frac{n-1}{2}}(q)$,
(4) $n$ is even and $L^{*} \cong C_{\frac{n}{2}}^{2}(q)$,
(5) $n$ is even and $L^{*} \cong D_{\frac{n}{2}}^{2}(q)$,
(6) $n$ is even and $L^{*} \cong{ }^{2} D_{\frac{n}{2}}(q)$,
where the (twisted) Chevalley groups appearing in the last four cases are adjoint. Since $C_{T}(\alpha)$ is isomorphic to $C_{\operatorname{Inn}(T)}(\alpha) \unlhd C^{*}$, we have that $K$ is isomorphic to a 2-component of $C^{*}$ and thus isomorphic to a 2 -component of $L^{*}$. Therefore, one of the conditions (i)-(viii) is satisfied.

Case 2: $\alpha$ is a field automorphism or a graph-field automorphism.
Again, let $C^{*}:=C_{\operatorname{Inndiag}(T)}(\alpha)$. Since the field automorphisms of $P S U_{n}(q)$ have odd order and $P S U_{n}(q)$ has no graph-field automorphisms, we have $\varepsilon=+$. Also, $f$ is even since $\alpha$ is a field automorphism or a graph-field automorphism of order 2. From [29, Proposition 4.9.1 (a), (b)], we see that $C^{*} \cong P G L_{n}\left(p^{\frac{f}{2}}\right)$ if $\alpha$ is a field automorphism and that $C^{*} \cong P G U_{n}\left(p^{\frac{f}{2}}\right)$ if $\alpha$ is a graph-field automorphism. Since $K$ is isomorphic to a 2-component of $C^{*}$, it follows that (iii) is satisfied.
Corollary 3.46. Let $q=p^{f}$, where $p$ is an odd prime and $f$ is a positive integer. Let $n \geq 2$ be a natural number and $\varepsilon \in\{+,-\}$. Let $Z$ be a central subgroup of $S L_{n}^{\varepsilon}(q)$ and let $T:=S L_{n}^{\varepsilon}(q) / Z$. Let $\alpha$ be an involution of $\operatorname{Aut}(T)$, and let $K$ be a 2-component of $C_{T}(\alpha)$. Then the following hold:
(i) $K$ is a component of $C_{T}(\alpha)$, and $K / Z(K)$ is a known finite simple group.
(ii) $K / Z(K) \not \approx M_{11}$.
(iii) Assume that $K / Z(K) \cong P S L_{k}^{\varepsilon^{*}}\left(q^{*}\right)$ for some positive integer $2 \leq k \leq n$, some nontrivial odd prime power $q^{*}$ and some $\varepsilon^{*} \in\{+,-\}$. Then one of the following holds:
(a) $q^{*}=q$;
(b) $q^{*}=q^{2}, n \geq 4$ is even, $k=\frac{n}{2}$, and $\varepsilon^{*}=+$ if $n \geq 6$;
(c) $f$ is even, $k=n, q^{*}=p^{\frac{f}{2}}$.

Proof. Set $\bar{T}:=T / Z(T) \cong P S L_{n}^{\varepsilon}(q)$. Let $\bar{\alpha}$ be the automorphism of $\bar{T}$ induced by $\alpha$.
Clearly, $\bar{K}$ is a 2-component of $\overline{C_{T}(\alpha)}$. It is easy to note that $\overline{C_{T}(\alpha)}$ is a normal subgroup of $C_{\bar{T}}(\bar{\alpha})$. So $\bar{K}$ is a 2 -component of $C_{\bar{T}}(\bar{\alpha})$. Lemmas 3.44 and 3.45 imply that $\bar{K}$ is a component of $C_{\bar{T}}(\bar{\alpha})$ and that $\bar{K} / Z(\bar{K})$ is a known finite simple group. Applying [37, 6.5.1], we conclude that $K^{\prime}$ is a component of $C_{T}(\alpha)$. We have $K=K^{\prime}$ since $K$ is a 2 -component of $C_{T}(\alpha)$, and so it follows that $K$ is a component of $C_{T}(\alpha)$. Also, $K / Z(K) \cong \bar{K} / Z(\bar{K})$, so that $K / Z(K)$ is a known finite simple group. Hence (i) holds.

If $K / Z(K) \cong M_{11}$, then $\bar{K} / Z(\bar{K}) \cong M_{11}$, which is not possible by Lemmas 3.44 and 3.45 . So (ii) holds.

Suppose that $K / Z(K) \cong P S L_{k}^{\varepsilon^{*}}\left(q^{*}\right)$ for some positive integer $2 \leq k \leq n$, some nontrivial odd prime power $q^{*}$ and some $\varepsilon^{*} \in\{+,-\}$. By Lemmas 3.44 and 3.45, one of the following holds:
(1) $\bar{K} / Z(\bar{K}) \cong P S L_{i}^{\varepsilon}(q)$ for some $2 \leq i<n$;
(2) $n$ is even and $\bar{K} / Z(\bar{K})$ is isomorphic to $P S L_{\frac{n}{2}}\left(q^{2}\right)$;
(3) $f$ is even, $\bar{K} \cong P S L_{n}\left(p^{\frac{f}{2}}\right)$ or $P S U_{n}\left(p^{\frac{f}{2}}\right)$;
(4) $q \neq 3, n=3$ or $4, \bar{K} \cong P S L_{2}(q)$;
(5) $n$ is odd, $n \geq 5, \bar{K} \cong B_{\frac{n-1}{2}}(q)$;
(6) $n$ is even, $n \geq 4, \bar{K} \cong C_{\frac{n}{2}}(q)$;
(7) $n$ is even, $n \geq 6, \bar{K} \cong D_{\frac{n}{2}}(q)$;
(8) $n$ is even, $n \geq 6, \bar{K} \cong{ }^{2} D_{\frac{n}{2}}(q)$.

Here, the (twisted) Chevalley groups appearing in (5)-(8) are adjoint. On the other hand, we have $\bar{K} / Z(\bar{K}) \cong P S L_{k}^{\varepsilon^{*}}\left(q^{*}\right)$. Now, if (1) holds, then $P S L_{k}^{\varepsilon^{*}}\left(q^{*}\right) \cong P S L_{i}^{\varepsilon}(q)$ for some $2 \leq i<n$, and [48, Theorem 37] shows that this is only possible when $q^{*}=q$, so that (a) holds. Similarly, if (2) holds, then we have (b). Moreover, (3) implies (c) and (4) implies (a). As Theorem [48, Theorem 37] shows, the cases (5) and (6) cannot occur, while (7) and (8) can only occur when $n=6$. As above, one can see that if $n=6$ and (7) or (8) holds, then we have (a).

Lemma 3.47. Let $n \geq 3$ and $\varepsilon \in\{+,-\}$. Then $S L_{n}^{\varepsilon}(3)$ is locally balanced (in the sense of Definition 2.7).
Proof. Set $T:=S L_{n}^{\varepsilon}(3)$. Let $H$ be a subgroup of $\operatorname{Aut}(T)$ containing $\operatorname{Inn}(T)$, and let $x$ be an involution of $H$. It is enough to show that $O\left(C_{H}(x)\right)=1$.

Assume that $O\left(C_{H}(x)\right) \neq 1$. Then $x \in \operatorname{Inndiag}(T)$ by [29, Theorem 7.7.1]. By Propositions 3.40 and 3.42, Out $(T)$ is a 2-group. This implies $O\left(C_{H}(x)\right)=O\left(C_{\operatorname{Inn}(T)}(x)\right)=O\left(C_{\operatorname{Inndiag}(T)}(x)\right)$. Since $x$ is an involution of $\operatorname{Inndiag}(T) \cong P G L_{n}^{\varepsilon}(3)$, we have $O\left(C_{\operatorname{Inndiag}(T)}(x)\right)=1$ by Corollary 3.9. Thus $O\left(C_{H}(x)\right)=1$. This contradiction completes the proof.

Lemma 3.48. Let $n \geq 3$ be a natural number, let $q$ be a nontrivial odd power, and let $\varepsilon \in\{+,-\}$. Then any non-trivial quotient of $S L_{n}^{\varepsilon}(q)$ is locally 2-balanced (in the sense of Definition 2.7).
Proof. By [25, Theorem 4.61] or [29, Theorem 7.7.4], $P S L_{n}^{\varepsilon}(q)$ is locally 2-balanced. Let $K$ be a non-trivial quotient of $S L_{n}^{\varepsilon}(q)$. As we have seen, there is an isomorphism $\operatorname{Aut}(K) \rightarrow$ $\operatorname{Aut}\left(P S L_{n}^{\varepsilon}(q)\right)$ mapping $\operatorname{Inn}(K)$ to $\operatorname{Inn}\left(P S L_{n}^{\varepsilon}(q)\right)$. So the local 2-balance of $K$ follows from the local 2-balance of $P S L_{n}^{\varepsilon}(q)$.

Lemma 3.49. Let $q$ be a nontrivial odd prime power and $n \geq 4$ be a natural number. Let $Z \leq Z\left(S L_{n}(q)\right)$ and $T:=S L_{n}(q) / Z$. Let $K_{1}$ be the image of

$$
\left\{\left(\begin{array}{ll}
A & \\
& I_{n-2}
\end{array}\right): A \in S L_{2}(q)\right\}
$$

in $T$, and let $K_{2}$ be the image of

$$
\left\{\left(\begin{array}{ll}
I_{2} & \\
& B
\end{array}\right): B \in S L_{n-2}(q)\right\}
$$

in $T$. Let $\alpha$ be an automorphism of $T$ with odd order such that $\alpha$ normalizes $K_{1}$ and centralizes $K_{2}$. Then $\left.\alpha\right|_{K_{1}, K_{1}}$ is an inner automorphism.
Proof. By hypothesis, $q=p^{f}$ for some odd prime number $p$ and some positive integer $f$. We have $\alpha \in P \Gamma L_{n}(q)$ since $\alpha$ has odd order and $\left|\operatorname{Aut}(T) / P \Gamma L_{n}(q)\right|=2$. So there are some $m \in G L_{n}(q)$ and some $1 \leq r \leq f$ such that, for each element $\left(a_{i j}\right)$ of $S L_{n}(q), \alpha$ maps $\left(a_{i j}\right) Z$ to $\left(\left(a_{i j}\right)^{p^{r}}\right)^{m} Z$.

Let $x$ be the image of $\operatorname{diag}(-1,-1,1, \ldots, 1) \in S L_{n}(q)$ in $T$. Then $x$ is the unique involution of $K_{1}$, and so we have $x^{\alpha}=x$. This easily implies that

$$
m=\left(\begin{array}{ll}
m_{1} & \\
& m_{2}
\end{array}\right)
$$

for some $m_{1} \in G L_{2}(q)$ and some $m_{2} \in G L_{n-2}(q)$.
Since $\alpha$ centralizes $K_{2}$, we have $\left(\left(a_{i j}\right)^{p^{r}}\right)^{m_{2}}=\left(a_{i j}\right)$ for all $\left(a_{i j}\right) \in S L_{n-2}(q)$. Therefore, the automorphism $S L_{n-2}(q) \rightarrow S L_{n-2}(q),\left(a_{i j}\right) \mapsto\left(a_{i j}\right)^{p^{p}}$ is an element of $\operatorname{Inndiag}\left(S L_{n-2}(q)\right)$. This implies $r=f$.

Thus, under the isomorphism $\operatorname{Aut}\left(S L_{2}(q)\right) \rightarrow \operatorname{Aut}\left(K_{1}\right)$ induced by the canonical isomorphism $S L_{2}(q) \rightarrow K_{1}$, the automorphism $\left.\alpha\right|_{K_{1}, K_{1}}$ of $K_{1}$ corresponds to the inner-diagonal automorphism $\widetilde{\alpha}: S L_{2}(q) \rightarrow S L_{2}(q), a \mapsto a^{m_{1}}$, and this automorphism has odd order since $\alpha$ has odd order. The index of $\operatorname{Inn}\left(S L_{2}(q)\right)$ in $\operatorname{Inndiag}\left(S L_{2}(q)\right)$ is 2 , and so it follows that $\widetilde{\alpha} \in \operatorname{Inn}\left(S L_{2}(q)\right)$. Consequently, $\left.\alpha\right|_{K_{1}, K_{1}} \in \operatorname{Inn}\left(K_{1}\right)$.

By using similar arguments as in the proof of Lemma 3.49, one can prove the following lemma.
Lemma 3.50. Let $q$ be a nontrivial odd prime power and $n \geq 4$ be a natural number. Let $Z \leq Z\left(S U_{n}(q)\right)$ and $T:=S U_{n}(q) / Z$. Let $K_{1}$ be the image of

$$
\left\{\left(\begin{array}{ll}
A & \\
& I_{n-2}
\end{array}\right): A \in S U_{2}(q)\right\}
$$

in $T$, and let $K_{2}$ be the image of

$$
\left\{\left(\begin{array}{ll}
I_{2} & \\
& B
\end{array}\right): B \in S U_{n-2}(q)\right\}
$$

in $T$. Let $\alpha$ be an automorphism of $T$ with odd order such that $\alpha$ normalizes $K_{1}$ and centralizes $K_{2}$. Then $\left.\alpha\right|_{K_{1}, K_{1}}$ is an inner automorphism.

Our next goal is to prove the following lemma.
Lemma 3.51. Let $q$ and $q^{*}$ be nontrivial odd prime powers. Let $L$ be a group isomorphic to $S L_{2}\left(q^{*}\right)$. Let $Q$ be a Sylow 2-subgroup of L. Moreover, let $V$ be a Sylow 2-subgroup of $G L_{2}(q)$ and $V_{0}:=V \cap S L_{2}(q)$. Suppose that there is a group isomorphism $\psi: V_{0} \rightarrow Q$. Let $v_{1}, v_{2}, v_{3}$ be elements of $V$ such that $v_{3}=v_{1} v_{2}$ and such that the square of any element of $\left\{v_{1}, v_{2}, v_{3}\right\}$ lies in $Z\left(G L_{2}(q)\right)$. For each $i \in\{1,2,3\}$, let $\alpha_{i}$ be a 2-element of $\operatorname{Aut}(L)$ normalizing $Q$ such that

$$
\left.\alpha_{i}\right|_{Q, Q}=\psi^{-1}\left(\left.c_{v_{i}}\right|_{V_{0}, V_{0}}\right) \psi
$$

Then we have

$$
\bigcap_{i=1}^{3} O\left(C_{L}\left(\alpha_{i}\right)\right)=1 .
$$

To prove Lemma 3.51, we need to prove some other lemmas.
Lemma 3.52. Let $q$ be a nontrivial odd prime power with $q \equiv 1 \bmod 4$, and let $k \in \mathbb{N}$ with $(q-1)_{2}=2^{k}$. Let $G$ be a group isomorphic to $S L_{2}(q)$ and $Q \in \operatorname{Syl}_{2}(G)$. Then the following hold:
(i) There are elements $a, b \in Q$ such that $\operatorname{ord}(a)=2^{k}$, $\operatorname{ord}(b)=4, a^{b}=a^{-1}$ and $b^{2}=a^{2^{k-1}}$.
(ii) Let $a$ and $b$ be as in (i). Then there is a group isomorphism $\varphi: G \rightarrow S L_{2}(q)$ such that

$$
a^{\varphi}=\left(\begin{array}{cc}
\lambda & 0 \\
0 & \lambda^{-1}
\end{array}\right)
$$

for some $\lambda \in \mathbb{F}_{q}^{*}$ with order $2^{k}$ and

$$
b^{\varphi}=\left(\begin{array}{cc}
0 & 1 \\
-1 & 0
\end{array}\right)
$$

Proof. (i) follows from Lemma 3.12 .
We now prove (ii). Assume that $k \geq 3$. By Lemma 3.10 (i),

$$
\left\{\left(\begin{array}{cc}
\mu & 0 \\
0 & \mu^{-1}
\end{array}\right): \mu \text { is a } 2 \text {-element of } \mathbb{F}_{q}^{*}\right\}\left\langle\left(\begin{array}{cc}
0 & 1 \\
-1 & 0
\end{array}\right)\right\rangle
$$

is a Sylow 2-subgroup of $S L_{2}(q)$. Choose a group isomorphism $\psi: G \rightarrow S L_{2}(q)$ such that $Q^{\psi}=R$. Clearly, since $k \geq 3, Q$ has only one cyclic subgroup of order $2^{k}$. This implies that

$$
a^{\psi}=\left(\begin{array}{cc}
\lambda & 0 \\
0 & \lambda^{-1}
\end{array}\right)
$$

for some $\lambda \in \mathbb{F}_{q}^{*}$ with order $2^{k}$. Since $b \notin\langle a\rangle$, we have

$$
b^{\psi}=\left(\begin{array}{cc}
0 & \mu \\
-\mu^{-1} & 0
\end{array}\right)
$$

for some 2-element $\mu$ of $\mathbb{F}_{q}^{*}$. Composing $\psi$ with the automorphism

$$
S L_{2}(q) \rightarrow S L_{2}(q), A \mapsto\left(\begin{array}{cc}
\mu^{-1} & 0 \\
0 & 1
\end{array}\right) A\left(\begin{array}{cc}
\mu & 0 \\
0 & 1
\end{array}\right)
$$

we get a group isomorphism $\varphi: G \rightarrow S L_{2}(q)$ with the desired properties. This completes the proof of (ii) for the case $k \geq 3$.

Assume now that $k=2$. Let $\psi: G \rightarrow S L_{2}(q)$ be a group isomorphism. We have $\left(a^{\psi}\right)^{2}=-I_{2}$ since $-I_{2}$ is the only involution of $S L_{2}(q)$ and $\operatorname{ord}\left(a^{2}\right)=2$. So, by Lemma 3.3, we may assume that

$$
a^{\psi}=\left(\begin{array}{cc}
\lambda & 0 \\
0 & \lambda^{-1}
\end{array}\right)
$$

for some $\lambda \in \mathbb{F}_{q}^{*}$ with order 4 . Since $a^{b}=a^{-1}$, we have

$$
\left(\begin{array}{cc}
\lambda & 0 \\
0 & \lambda^{-1}
\end{array}\right)^{b^{\psi}}=\left(\begin{array}{cc}
\lambda^{-1} & 0 \\
0 & \lambda
\end{array}\right) .
$$

This implies that

$$
b^{\psi}=\left(\begin{array}{cc}
0 & \mu \\
-\mu^{-1} & 0
\end{array}\right)
$$

for some $\mu \in \mathbb{F}_{q}^{*}$. Again we may compose $\psi$ with a suitable diagonal automorphism of $S L_{2}(q)$ to obtain a group isomorphism $\varphi: G \rightarrow S L_{2}(q)$ with the desired properties.

By using similar arguments as in the proof of Lemma 3.52, one can prove the following lemma.
Lemma 3.53. Let $q$ be a nontrivial odd prime power with $q \equiv 3 \bmod 4$, and let $s \in \mathbb{N}$ with $(q+1)_{2}=2^{s}$. Let $G$ be a group isomorphic to $S U_{2}(q)$ and $Q \in \operatorname{Syl}_{2}(G)$. Then the following hold:
(i) There are elements $a, b \in Q$ such that $\operatorname{ord}(a)=2^{s}$, $\operatorname{ord}(b)=4, a^{b}=a^{-1}$ and $b^{2}=a^{2^{s-1}}$.
(ii) Let $a$ and $b$ be as in (i). Then there is a group isomorphism $\varphi: G \rightarrow S U_{2}(q)$ such that

$$
a^{\varphi}=\left(\begin{array}{cc}
\lambda & 0 \\
0 & \lambda^{-1}
\end{array}\right)
$$

for some $\lambda \in \mathbb{F}_{q^{2}}^{*}$ with order $2^{s}$ and

$$
b^{\varphi}=\left(\begin{array}{cc}
0 & 1 \\
-1 & 0
\end{array}\right) .
$$

Lemma 3.54. Let $q$ be a nontrivial odd prime power with $q \equiv 1 \bmod 4$. Let $\rho$ be a generating element of the Sylow 2-subgroup of $\mathbb{F}_{q}^{*}$, and let

$$
a:=\left(\begin{array}{ll}
\rho & \\
& \rho^{-1}
\end{array}\right), \quad b:=\left(\begin{array}{cc}
0 & 1 \\
-1 & 0
\end{array}\right) .
$$

Let $V$ be the Sylow 2-subgroup of $G L_{2}(q)$ given by Lemma 3.10 (i), and let $v, w \in V$ such that $v^{2}, w^{2},(v w)^{2} \in Z\left(G L_{2}(q)\right)$. Then one of the following holds:
(i) $\{v, w, v w\} \cap Z\left(G L_{2}(q)\right) \neq \emptyset$.
(ii) There exist $r, s \in\{v, w, v w\}$ with $a^{r}=a, b^{r}=b^{3}$ and $a^{s}=a^{-1}$.

Proof. It is easy to note that (i) holds if $v$ and $w$ are diagonal matrices.
Suppose now that $v$ or $w$ is not a diagonal matrix. If neither $v$ nor $w$ is a diagonal matrix, then $v w$ is a diagonal matrix. So there exist $r, s \in\{v, w, v w\}$ such that

$$
r=\left(\begin{array}{ll}
\lambda_{1} & \\
& \lambda_{2}
\end{array}\right), \quad s=\left(\begin{array}{ll} 
& \mu_{1} \\
\mu_{2} &
\end{array}\right)
$$

where $\lambda_{1}, \lambda_{2}, \mu_{1}$ and $\mu_{2}$ are 2-elements of $\mathbb{F}_{q}^{*}$.
If $\lambda_{1}=\lambda_{2}$, then (i) holds. Assume now that $\lambda_{1} \neq \lambda_{2}$. Then $\lambda_{2}=-\lambda_{1}$ since $r^{2} \in Z\left(G L_{2}(q)\right)$, and a direct calculation shows that $a^{r}=a, b^{r}=b^{3}$ and $a^{s}=a^{-1}$.

Lemma 3.55. Let $q$ be a nontrivial odd prime power with $q \equiv 3 \bmod 4$, and let $k \in \mathbb{N}$ with $(q+1)_{2}=2^{k}$. Let $V$ be a Sylow 2-subgroup of $G L_{2}(q)$.
(i) There exist $x, y \in V$ with $\operatorname{ord}(x)=2^{k+1}$, ord $(y)=2$ and $x^{y}=x^{-1+2^{k}}$. We have $V \cap$ $S L_{2}(q)=\left\langle x^{2}\right\rangle\langle x y\rangle$.
(ii) Let $x$ and $y$ be as above, and let $a:=x^{2}$ and $b:=x y$. Let $v, w \in V$ with $v^{2}, w^{2},(v w)^{2} \in$ $Z\left(G L_{2}(q)\right)$. Then one of the following holds:
(a) $\{v, w, v w\} \cap Z\left(G L_{2}(q)\right) \neq \emptyset$.
(b) There exist $r, s \in\{v, w, v w\}$ such that $a^{r}=a, b^{r}=b^{3}$ and $a^{s}=a^{-1}$.

Proof. (i) follows from Lemma 3.16 (i), (ii).
We now prove (ii). We have $Z(V)=\left\langle x^{2^{k}}\right\rangle$ by Lemma [24, Chapter 5, Theorem 4.3]. Thus $Z\left(G L_{2}(q)\right) \cap V=\left\langle x^{2^{k}}\right\rangle$. Clearly, $\{v, w, v w\} \cap\langle x\rangle \subseteq\left\langle x^{2^{k-1}}\right\rangle$.

If $v, w \in\langle x\rangle$, then $v, w \in\left\langle x^{2^{k-1}}\right\rangle$, and it easily follows that (a) holds.
Assume now that $v \notin\langle x\rangle$ or $w \notin\langle x\rangle$. If neither $v$ nor $w$ lies in $\langle x\rangle$, then $v w \in\langle x\rangle$. Consequently, $\{v, w, v w\}$ has an element $r$ of the form $x^{\ell 2^{k-1}}$ for some $1 \leq \ell \leq 4$ and an element $s$ of the form $x^{i} y$ for some $1 \leq i \leq 2^{k+1}$. If $\ell=2$ or 4 , then (a) holds. Assume now that $\ell=1$ or 3 . It is clear that $a^{r}=a$. Furthermore, we have

$$
\begin{aligned}
b^{r} & =(x y)^{x^{\ell 2^{k-1}}} \\
& =x y^{x^{\ell 2^{k-1}}} \\
& =x x^{-\ell 2^{k-1}} y x^{\ell 2^{k-1}} y^{2} \\
& =x^{1-\ell 2^{k-1}}\left(x^{y}\right)^{\ell 2^{k-1}} y \\
& =x^{1-\ell 2^{k-1}}\left(x^{-1+2^{k}}\right)^{\ell 2^{k-1}} y \\
& =x^{1-\ell 2^{k}+\ell 2^{2 k-1}} y \\
& =x^{1-\ell 2^{k}} y \\
& \ell \stackrel{\text { odd }}{=} x^{1+2^{k}} y .
\end{aligned}
$$

On the other hand, we have

$$
b^{3}=(x y)^{3}=x^{2^{k}} x y=x^{1+2^{k}} y
$$

Consequently, $b^{r}=b^{3}$. Finally, we also have

$$
a^{s}=\left(x^{2}\right)^{x^{i} y}=\left(x^{2}\right)^{y}=\left(x^{y}\right)^{2}=\left(x^{-1+2^{k}}\right)^{2}=x^{-2}=a^{-1} .
$$

Thus (b) holds.

Proof of Lemma 3.51. If $\left.\alpha_{j}\right|_{Q, Q}=\operatorname{id}_{Q}$ for some $j \in\{1,2,3\}$, then $\alpha_{j}=\mathrm{id}_{L}$ by Lemma 3.43, which implies that

$$
\bigcap_{i=1}^{3} O\left(C_{L}\left(\alpha_{i}\right)\right) \leq O\left(C_{L}\left(\alpha_{j}\right)\right)=O(L)=1
$$

Suppose now that $\alpha_{i}$ acts nontrivially on $Q$ for all $i \in\{1,2,3\}$. Let $m \in \mathbb{N}$ with $|Q|=2^{m}$. Using Lemma 3.54 (together with Sylow's theorem) and Lemma 3.55, we see that there exist $a, b \in Q$ and $i, j \in\{1,2,3\}$ such that the following hold:
(i) $\operatorname{ord}(a)=2^{m-1}, \operatorname{ord}(b)=4, a^{b}=a^{-1}, b^{2}=a^{2^{m-2}}$;
(ii) $a^{\alpha_{i}}=a, b^{\alpha_{i}}=b^{3}, a^{\alpha_{j}}=a^{-1}$.

Clearly, $b^{\alpha_{j}}=a^{\ell} b$ for some $1 \leq \ell \leq 2^{m-1}$.
Assume that $q^{*} \equiv 1 \bmod 4$. By Lemma 3.52 , there is group isomorphism $\varphi: L \rightarrow S L_{2}\left(q^{*}\right)$ with

$$
a^{\varphi}=\left(\begin{array}{cc}
\lambda & 0 \\
0 & \lambda^{-1}
\end{array}\right)
$$

for some generator $\lambda$ of the Sylow 2-subgroup of $\left(\mathbb{F}_{q^{*}}\right)^{*}$ and

$$
b^{\varphi}=\left(\begin{array}{cc}
0 & 1 \\
-1 & 0
\end{array}\right)
$$

Set $\beta_{k}:=\varphi^{-1} \alpha_{k} \varphi$ for $k \in\{1,2,3\}$. Let $i$ and $j$ be as in (ii). Also, let

$$
m_{i}:=\left(\begin{array}{ll}
1 & \\
& -1
\end{array}\right)
$$

Then $\beta_{i}$ and $c_{m_{i}}$ normalize $Q^{\varphi}$, and we have $\left.\beta_{i}\right|_{Q^{\varphi}, Q^{\varphi}}=\left.c_{m_{i}}\right|_{Q^{\varphi}, Q^{\varphi}}$. Applying Lemma 3.43, we conclude that $\beta_{i}=c_{m_{i}}$.

Clearly,

$$
\left(\begin{array}{cc}
0 & 1 \\
-1 & 0
\end{array}\right)^{\beta_{j}}=\left(\begin{array}{cc}
0 & \mu \\
-\mu^{-1} & 0
\end{array}\right)
$$

for some 2-element $\mu$ of $\left(\mathbb{F}_{q^{*}}\right)^{*}$. Set

$$
m_{j}:=\left(\begin{array}{cc}
0 & \mu \\
-1 & 0
\end{array}\right)
$$

Then $\beta_{j}$ and $c_{m_{j}}$ normalize $Q^{\varphi}$, and we have $\left.\beta_{j}\right|_{Q^{\varphi}, Q^{\varphi}}=\left.c_{m_{j}}\right|_{Q^{\varphi}, Q^{\varphi}}$. Applying Lemma 3.43, we conclude that $\beta_{j}=c_{m_{j}}$.

It follows that $C_{S L_{2}\left(q^{*}\right)}\left(\beta_{i}\right) \cap C_{S L_{2}\left(q^{*}\right)}\left(\beta_{j}\right)=Z\left(S L_{2}\left(q^{*}\right)\right)$. So we have $C_{L}\left(\alpha_{i}\right) \cap C_{L}\left(\alpha_{j}\right)=Z(L)$, and this implies that

$$
\bigcap_{k=1}^{3} O\left(C_{L}\left(\alpha_{k}\right)\right)=1
$$

since $|Z(L)|=2$.
If $q^{*} \equiv 3 \bmod 4$, then a very similar argumentation shows that the same conclusion holds. Here, one has to use Lemma 3.53 instead of Lemma 3.52, together with the fact that $S L_{2}\left(q^{*}\right) \cong$ $S U_{2}\left(q^{*}\right)$.

We bring this section to a close with a proof of the following lemma, which will play an important role in the proof of Theorem $B$.

Lemma 3.56. Let $q$ be a nontrivial odd prime power, $\varepsilon \in\{+,-\}$ and $n \geq 2$ a natural number. Set $T:=\operatorname{Inn}\left(P S L_{n}^{\varepsilon}(q)\right)$. Let $A$ be a subgroup of $\operatorname{Aut}\left(P S L_{n}^{\varepsilon}(q)\right)$ such that $T \leq A$ and such that the index of $T$ in $A$ is odd. Let $S$ be a Sylow 2-subgroup of $T$. Then we have $\mathcal{F}_{S}(T)=\mathcal{F}_{S}(A)$.

To prove Lemma 3.56, we need to prove some other lemmas first.

Lemma 3.57. Let $q$ be a nontrivial odd prime power, $\varepsilon \in\{+,-\}$, and let $r$ be positive integer. Also, let $W$ be a Sylow 2-subgroup of $G L_{2^{r}}^{\varepsilon}(q)$. Then $\operatorname{Aut}(W)$ is a 2-group.
Proof. We proceed by induction over $r$.
Suppose that $r=1$. If $q \equiv-\varepsilon \bmod 4$, then $W$ is semidihedral by Lemmas 3.10 and 3.11, and so $\operatorname{Aut}(W)$ is a 2 -group by [19, Proposition 4.53]. If $q \equiv \varepsilon \bmod 4$, then $W \cong C_{2^{k}}$ 乙 $C_{2}$ for some positive integer $k$ by Lemmas 3.10 and 3.11 , and so $\operatorname{Aut}(W)$ is a 2 -group as a consequence of [22, Theorem 2].

Assume now that $r>1$ and that the lemma is true with $r-1$ instead of $r$. Let $W_{0}$ be a Sylow 2-subgroup of $G L_{2^{r-1}}^{\varepsilon}(q)$. Hence, $\operatorname{Aut}\left(W_{0}\right)$ is a 2-group. By Lemma 3.14, we have $W \cong W_{0}$ 乙 $C_{2}$. Applying [22, Theorem 2], we conclude that $\operatorname{Aut}(W)$ is a 2-group.
Lemma 3.58. Let $q$ be a nontrivial odd prime power, $\varepsilon \in\{+,-\}$, and let $n \geq 3$ be a natural number. Let $T:=S L_{n}^{\varepsilon}(q)$, and let $S$ be a Sylow 2-subgroup of $\operatorname{Inndiag}(T)$. Then $\operatorname{Aut}_{P \Gamma L_{n}^{\varepsilon}(q)}(S)$ is a 2-group.
Proof. Let $\alpha \in N_{P \Gamma L_{n}^{\varepsilon}(q)}(S)$. It suffices to show that $c_{\alpha} \mid S, S$ is a 2-automorphism of $S$.
Let $0 \leq r_{1}<\cdots<r_{t}$ such that $n=2^{r_{1}}+\cdots+2^{r_{t}}$. Let $W_{i} \in \operatorname{Syl}_{2}\left(G L_{2^{r_{i}}}^{\varepsilon}(q)\right)$ for all $1 \leq i \leq t$. By Lemma 3.15.

$$
\left\{\left(\begin{array}{ccc}
A_{1} & & \\
& \ddots & \\
& & A_{t}
\end{array}\right): A_{i} \in W_{i}\right\}
$$

is a Sylow 2-subgroup of $G L_{n}^{\varepsilon}(q)$.
Clearly, $\left\{\left.c_{w}\right|_{T, T} \mid w \in W\right\}$ is a Sylow 2-subgroup of $\operatorname{Inndiag}(T)$. Without loss of generality, we assume that $S=\left\{\left.c_{w}\right|_{T, T} \mid w \in W\right\}$.

Let $p$ be the odd prime number and $f$ be the positive integer with $q=p^{f}$. Since $\alpha \in P \Gamma L_{n}^{\varepsilon}(q)$, there exist some $m \in G L_{n}^{\varepsilon}(q)$ and some natural number $\ell$, where $1 \leq \ell \leq f$ if $\varepsilon=+$ and $1 \leq \ell \leq 2 f$ if $\varepsilon=-$, such that

$$
\left(a_{i j}\right)^{\alpha}=\left(\left(a_{i j}\right)^{p^{\ell}}\right)^{m}
$$

for all $\left(a_{i j}\right) \in T$.
Let

$$
\bar{\alpha}: G L_{n}^{\varepsilon}(q) \rightarrow G L_{n}^{\varepsilon}(q),\left(a_{i j}\right) \mapsto\left(\left(a_{i j}\right)^{p^{\ell}}\right)^{m} .
$$

It is easy to note that $\bar{\alpha}$ is an automorphism of $G L_{n}^{\varepsilon}(q)$. Using this, one can see that $\alpha^{-1}\left(\left.c_{w}\right|_{T, T}\right) \alpha=$ $\left.c_{w^{\bar{\alpha}}}\right|_{T, T}$ for all $w \in W$.

Let $w \in W$. Since $\alpha$ normalizes $S$, there is some $\widetilde{w} \in W$ with $\left.c_{w^{\bar{\alpha}}}\right|_{T, T}=\alpha^{-1}\left(\left.c_{w}\right|_{T, T}\right) \alpha=\left.c_{\widetilde{w}}\right|_{T, T}$. It follows that $w^{\bar{\alpha}} \in \widetilde{w} Z\left(G L_{n}^{\varepsilon}(q)\right) \subseteq W Z\left(G L_{n}^{\varepsilon}(q)\right)$. This implies $w^{\bar{\alpha}} \in W$ since $W$ is the unique Sylow 2-subgroup of $W Z\left(G L_{n}^{\varepsilon}(q)\right)$. In particular, $\bar{\alpha}$ induces an automorphism of $W$.

Let

$$
d_{i}:=\left(\begin{array}{lllll}
I_{2^{r_{1}}} & & & & \\
& \ddots & & & \\
& & -I_{2^{r_{i}}} & & \\
& & & \ddots & \\
& & & & I_{2^{r_{t}}}
\end{array}\right)
$$

for each $1 \leq i \leq t$. Then $d_{i}$ is a central involution of $W$ for each $1 \leq i \leq t$. So we have that $\left(d_{i}\right)^{\bar{\alpha}}=\left(d_{i}\right)^{m}$ is a central involution of $W$ for each $1 \leq i \leq t$. As we see from Lemma 3.17, this already implies that $\left(d_{i}\right)^{m}=d_{i}$ for each $1 \leq i \leq t$. So there is some $m_{i} \in G L_{2^{r_{i}}}^{\varepsilon}(q)$ for each $1 \leq i \leq t$ such that

$$
m=\left(\begin{array}{ccc}
m_{1} & & \\
& \ddots & \\
& & m_{t}
\end{array}\right)
$$

Now

$$
W_{r} \rightarrow W_{r},\left(a_{i j}\right) \mapsto\left(\left(a_{i j}\right)^{p^{\ell}}\right)^{m_{i}}
$$

is an automorphism of $W_{r}$ for each $1 \leq r \leq t$. Applying Lemma 3.57, we conclude that $\left.\bar{\alpha}\right|_{W, W}$ is a 2-automorphism of $W$. Since $\alpha^{-1}\left(\left.c_{w}\right|_{T, T}\right) \alpha=\left.c_{w^{\bar{\alpha}}}\right|_{T, T}$ for all $w \in W$, it follows that $\left.c_{\alpha}\right|_{S, S}$ is a 2 -automorphism of $S$, as required.

Corollary 3.59. Let $q$ be a nontrivial odd prime power, $\varepsilon \in\{+,-\}$, and let $n \geq 3$ be a natural number. Let $T:=P S L_{n}^{\varepsilon}(q)$, and let $S$ be a Sylow 2-subgroup of $\operatorname{Inndiag}(T)$. Then $\operatorname{Aut}_{P \Gamma L_{n}^{\varepsilon}(q)}(S)$ is a 2-group.

Lemma 3.60. Let $q$ be a nontrivial odd prime power, $\varepsilon \in\{+,-\}$, and $n \geq 3$ be a natural number. Let $S$ be a Sylow 2-subgroup of $S L_{n}^{\varepsilon}(q) Z\left(G L_{n}^{\varepsilon}(q)\right) / Z\left(G L_{n}^{\varepsilon}(q)\right)$, and let $S_{1}$ be a Sylow 2-subgroup of $P G L_{n}^{\varepsilon}(q)$ containing $S$. Then $N_{P G L_{n}^{\varepsilon}(q)}(S)=N_{P G L_{n}^{\varepsilon}(q)}\left(S_{1}\right)$.
Proof. Let $T_{1}$ be a Sylow 2-subgroup of $\left.G L_{n}^{\varepsilon}(q)\right)$ such that $S_{1}=T_{1} Z\left(G L_{n}^{\varepsilon}(q)\right) / Z\left(G L_{n}^{\varepsilon}(q)\right)$. Let $T:=T_{1} \cap S L_{n}^{\varepsilon}(q)$. Clearly, we have $S=T Z\left(G L_{n}^{\varepsilon}(q)\right) / Z\left(G L_{n}^{\varepsilon}(q)\right)$. It is rather easy to show $N_{P G L_{n}^{\varepsilon}(q)}(S)=N_{G L_{n}^{\varepsilon}(q)}(T) Z\left(G L_{n}^{\varepsilon}(q)\right) / Z\left(G L_{n}^{\varepsilon}(q)\right)$. By [36, Theorem 1], $N_{G L_{n}^{\varepsilon}(q)}(T)=$ $T_{1} C_{G L_{n}^{\varepsilon}(q)}\left(T_{1}\right) \leq N_{G L_{n}^{\varepsilon}(q)}\left(T_{1}\right)$. It follows that $N_{P G L_{n}^{\varepsilon}(q)}(S) \leq N_{P G L_{n}^{\varepsilon}(q)}\left(S_{1}\right)$. It is clear that we also have $N_{P G L_{n}^{\varepsilon}(q)}\left(S_{1}\right) \leq N_{P G L_{n}^{\varepsilon}(q)}(S)$.
Corollary 3.61. Let $q$ be a nontrivial odd prime power, $\varepsilon \in\{+,-\}$, and let $n \geq 3$ be a natural number. Let $T:=P S L_{n}^{\varepsilon}(q)$, let $S$ be a Sylow 2-subgroup of $\operatorname{Inn}(T)$, and let $S_{1}$ be a Sylow 2subgroup of $\operatorname{Inndiag}(T)$ containing $S$. Then $N_{\operatorname{Inndiag}(T)}(S)=N_{\operatorname{Inndiag}(T)}\left(S_{1}\right)$.

We are now ready to prove Lemma 3.56 .
Proof of Lemma 3.56. Assume that $n=2$ and $q \equiv 3$ or $5 \bmod 8$. Then $S \cong C_{2} \times C_{2}$ by Lemma 3.13. There is only one non-nilpotent fusion system on $S$. Since $T$ and $A$ are not 2nilpotent, we have that $\mathcal{F}_{S}(T)$ and $\mathcal{F}_{S}(A)$ are not nilpotent (see [39, Theorem 1.4]). It follows that $\mathcal{F}_{S}(T)=\mathcal{F}_{S}(A)$.

From now on, we assume that either $n \geq 3$, or $n=2$ and $q \equiv 1$ or $7 \bmod 8$. Let $P, Q \leq S$ and $a \in A$ such that $P^{a} \leq Q$. We are going to show that $\left.c_{a}\right|_{P, Q}$ is a morphism in $\mathcal{F}_{S}(T)$. By the Frattini argument, we have $a=w u$ for some $w \in N_{A}(S)$ and some $u \in T$. We prove that $\left.c_{w}\right|_{S, S} \in \operatorname{Inn}(S)$. This clearly implies that $\left.c_{a}\right|_{P, Q}$ is a morphism in $\mathcal{F}_{S}(T)$.

Suppose that $n=2$. Then $S$ is dihedral of order at least 8 by Lemma 3.13 , and so $\operatorname{Aut}(S)$ is a 2 -group by [19, Proposition 4.53]. This implies that $\operatorname{Aut}_{A}(S)=\operatorname{Inn}(S)$, whence $\left.c_{w}\right|_{S, S} \in \operatorname{Inn}(S)$.

Suppose now that $n \geq 3$. Let $S_{1}$ be a Sylow 2-subgroup of $\operatorname{Inndiag}\left(P S L_{n}^{\varepsilon}(q)\right)$ containing $S$. Since $T$ has odd index in $A$, we have that $A \leq P \Gamma L_{n}^{\varepsilon}(q)$. By the Frattini argument, $w=w_{1} w_{2}$ for some $w_{1} \in N_{P \Gamma L_{n}^{\varepsilon}(q)}\left(S_{1}\right)$ and some $w_{2} \in \operatorname{Inndiag}\left(P S L_{n}^{\varepsilon}(q)\right)$. Since $w_{1}$ normalizes both $S_{1}$ and $T$, we have that $w_{1}$ normalizes $S$. And since $w=w_{1} w_{2}$ normalizes $S$, we also have that $w_{2}$ normalizes $S$. So $w_{2}$ normalizes $S_{1}$ by Corollary 3.61. Consequently, $w=w_{1} w_{2} \in N_{P \Gamma L_{n}^{\varepsilon}(q)}\left(S_{1}\right)$. By Corollary 3.59, $\left.c_{w}\right|_{S_{1}, S_{1}}$ is a 2 -automorphism of $S_{1}$. So $\left.c_{w}\right|_{S, S}$ is a 2 -automorphism of $S$. Since $S \in \operatorname{Syl}_{2}(A)$ and $w \in A$, this implies that $\left.c_{w}\right|_{S, S} \in \operatorname{Inn}(S)$, as required.

## 4. The case $n \leq 5$

In this section, we verify Theorem A for the case $n \leq 5$.
Proposition 4.1. Let $q$ be a nontrivial odd prime power, and let $G$ be a finite simple group. Then the following are equivalent:
(i) the 2-fusion system of $G$ is isomorphic to the 2-fusion system of $P S L_{2}(q)$;
(ii) the Sylow 2-subgroups of $G$ are isomorphic to those of $P S L_{2}(q)$;
(iii) $G \cong P S L_{2}^{\varepsilon}\left(q^{*}\right)$ for some $\varepsilon \in\{+,-\}$ and some odd prime power $q^{*} \geq 5$ with $\varepsilon q^{*} \sim q$, or $\left|P S L_{2}(q)\right|_{2}=8$ and $G \cong A_{7}$.

In particular, Theorem A holds for $n=2$.
Proof. The implication (i) $\Rightarrow$ (ii) is clear.
(ii) $\Rightarrow$ (iii): Assume that the Sylow 2-subgroups of $G$ are isomorphic to those of $P S L_{2}(q)$. Hence, $G$ has dihedral Sylow 2-subgroups of order $\frac{1}{2}(q-1)_{2}(q+1)_{2}$. Applying a result of Gorenstein and Walter [31, Theorem 1], we may conclude that $G \cong P S L_{2}\left(q^{*}\right)$ for some odd prime power $q^{*} \geq 5$, or $G \cong A_{7}$. Suppose that the former holds. Then $\left(q^{*}-1\right)_{2}\left(q^{*}+1\right)_{2}=2|G|_{2}=(q-1)_{2}(q+1)_{2}$, whence either $q^{*} \sim q$ or $-q^{*} \sim q$. Since $\operatorname{PS} L_{2}\left(q^{*}\right) \cong P S U_{2}\left(q^{*}\right)$, this implies that the first statement in (iii) is satisfied. If $G \cong A_{7}$, then $\left|P S L_{2}(q)\right|_{2}=|G|_{2}=8$, so that the second statement in (iii) is satisfied.
(iii) $\Rightarrow$ (i): Assume that (iii) holds. Set $G_{1}:=G$ and $G_{2}:=P S L_{2}(q)$. For $i \in\{1,2\}$, let $S_{i} \in \operatorname{Syl}_{2}\left(G_{i}\right)$ and $\mathcal{F}_{i}:=\mathcal{F}_{S_{i}}\left(G_{i}\right)$. Clearly, $S_{1}$ and $S_{2}$ are dihedral groups of the same order. Let $i \in\{1,2\}$. By [24, Chapter 5, Theorem 4.3], any subgroup of $S_{i}$ is cyclic or dihedral. By [19, Proposition 4.53], a dihedral subgroup of $S_{i}$ with order greater than 4 cannot be $\mathcal{F}_{i}$-essential. Since the automorphism group of a finite cyclic 2-group is itself a 2-group, a cyclic subgroup of $S_{i}$ cannot be $\mathcal{F}_{i}$-essential either. So we have that any $\mathcal{F}_{i}$-essential subgroup of $S_{i}$ is a Klein four group. Alperin's fusion theorem [11, Part I, Theorem 3.5] implies that

$$
\left.\mathcal{F}_{i}=\left\langle\operatorname{Aut}_{\mathcal{F}_{i}}(P)\right| P \leq S_{i}, P \cong C_{2} \times C_{2} \text { or } P=S_{i}\right\rangle_{S_{i}} .
$$

If $\left|S_{i}\right|=4$, then $\operatorname{Aut}_{\mathcal{F}_{i}}\left(S_{i}\right)$ is the unique subgroup of $\operatorname{Aut}\left(S_{i}\right)$ with order 3, because otherwise Aut $_{\mathcal{F}_{i}}\left(S_{i}\right)=\operatorname{Inn}\left(S_{i}\right)$, so that [39, Theorem 1.4] would imply that $G_{i}$ is 2-nilpotent. If $\left|S_{i}\right| \geq 8$, then $\operatorname{Aut}_{\mathcal{F}_{i}}\left(S_{i}\right)=\operatorname{Inn}\left(S_{i}\right)$ since $\operatorname{Aut}\left(S_{i}\right)$ is a 2-group by [19, Proposition 4.53], and for any Klein four subgroup $P$ of $S_{i}$, we have $\operatorname{Aut}_{\mathcal{F}_{i}}(P)=\operatorname{Aut}(P)$ by [24, Chapter 7, Theorem 7.3]. As $S_{1} \cong S_{2}$ and as the preceding observations do not depend on whether $i$ is 1 or 2 , we may conclude that $\mathcal{F}_{1} \cong \mathcal{F}_{2}$, as required.

Proposition 4.2. Let $q$ be a nontrivial odd prime power, and let $G$ be a finite simple group. Then the following are equivalent:
(i) the 2-fusion system of $G$ is isomorphic to the 2-fusion system of $P S L_{3}(q)$;
(ii) the Sylow 2-subgroups of $G$ are isomorphic to those of $P S L_{3}(q)$;
(iii) $G \cong \operatorname{PSL} L_{3}^{\varepsilon}\left(q^{*}\right)$ for some $\varepsilon \in\{+,-\}$ and some nontrivial odd prime power $q^{*}$ with $\varepsilon q^{*} \sim q$, or $(q+1)_{2}=4$ and $G \cong M_{11}$.
In particular, Theorem $A$ holds for $n=3$.
Proof. The implication (i) $\Rightarrow$ (ii) is clear.
(ii) $\Rightarrow$ (iii): Assume that the Sylow 2-subgroups of $G$ are isomorphic to those of $P S L_{3}(q)$. Hence, a Sylow 2-subgroup of $G$ is wreathed (i.e. isomorphic to $C_{2^{k}}$ 久 $C_{2}$ for some positive integer $k$ ) if $q \equiv 1 \bmod 4$, and semidihedral if $q \equiv 3 \bmod 4$. Applying work of Alperin, Brauer and Gorenstein, namely [2, Third Main Theorem] and [3, First Main Theorem], we may conclude that either $G \cong P S L_{3}^{\varepsilon}\left(q^{*}\right)$ for some $\varepsilon \in\{+,-\}$ and some nontrivial odd prime power $q^{*}$ with $\varepsilon q^{*} \equiv q$ $\bmod 4$, or $q \equiv 3 \bmod 4$ and $G \cong M_{11}$. If the former holds, then $\left(\left(q^{*}-\varepsilon\right)_{2}\right)^{2}\left(q^{*}+\varepsilon\right)_{2}=|G|_{2}=$ $\left((q-1)_{2}\right)^{2}(q+1)_{2}$, and it easily follows that $\varepsilon q^{*} \sim q$. If $G \cong M_{11}$, then $16=|G|_{2}=\left((q-1)_{2}\right)^{2}(q+1)_{2}$ and hence $(q+1)_{2}=4$.
(iii) $\Rightarrow$ (i): Assume that (iii) holds. If $q \equiv 1 \bmod 4$, then Proposition 3.20 implies that the 2-fusion system of $G$ is isomorphic to the 2-fusion system of $P S L_{3}(q)$. Alternatively, this can be seen from [19, Proposition 5.87]. Now suppose that $q \equiv 3 \bmod 4$. If $(q+1)_{2} \neq 4$, then we could apply Proposition 3.20 again, but we are going to argue in a more elementary way. Let $G_{1}:=G$ and $G_{2}:=P S L_{3}(q)$. For $i \in\{1,2\}$, let $S_{i} \in \operatorname{Syl}_{2}\left(G_{i}\right)$ and $\mathcal{F}_{i}:=\mathcal{F}_{S_{i}}\left(G_{i}\right)$. Clearly, $S_{1}$ and $S_{2}$ are semidihedral groups of the same order. Let $i \in\{1,2\}$. By [24, Chapter 5, Theorem 4.3], any proper subgroup of $S_{i}$ is cyclic, dihedral or generalized quaternion. By [19, Proposition 4.53], dihedral subgroups of $S_{i}$ with order greater than 4 and generalized quaternion subgroups of $S_{i}$ with order greater than 8 cannot be $\mathcal{F}_{i}$-essential. Since the automorphism group of a finite cyclic

2-group is itself a 2-group, a cyclic subgroup of $S_{i}$ cannot be $\mathcal{F}_{i}$-essential either. Alperin's fusion theorem [11, Part I, Theorem 3.5] implies that

$$
\left.\mathcal{F}_{i}=\left\langle\operatorname{Aut}_{\mathcal{F}_{i}}(P)\right| P \cong C_{2} \times C_{2}, P \cong Q_{8}, \text { or } P=S_{i}\right\rangle_{S_{i}} .
$$

Since $\operatorname{Aut}\left(S_{i}\right)$ is a 2-group by [19, Proposition 4.53], we have $\operatorname{Aut}_{\mathcal{F}_{i}}\left(S_{i}\right)=\operatorname{Inn}\left(S_{i}\right)$. From [2, pp. 10-11, Proposition 1], one can see that $\operatorname{Aut}_{\mathcal{F}_{i}}(P)=\operatorname{Aut}(P)$ for any subgroup $P$ of $S_{i}$ isomorphic to $C_{2} \times C_{2}$ or $Q_{8}$. As $S_{1} \cong S_{2}$ and as the preceding observations do not depend on whether $i$ is 1 or 2 , we may conclude that $\mathcal{F}_{1} \cong \mathcal{F}_{2}$, as required.

The next two lemmas are required to verify Theorem $A$ for the case $n=4$.
Lemma 4.3. Let $q$ be an odd prime power with $q \equiv 3 \bmod 8$. Assume that $G=A_{10}$ or $A_{11}$. Then the 2-fusion system of $G$ is not isomorphic to the 2-fusion system of $P S L_{4}(q)$.
Proof. Set $x:=(12)(34) \in G$ and $y:=(12)(34)(56)(78) \in G$. Let $g \in G$ be an involution. Then the cycle type of $g$ is either that of $x$ or that of $y$. So, by [37, 4.3.1], $g$ is conjugate to $x$ or $y$ in the ambient symmetric group, which easily implies that $g$ is also $G$-conjugate to $x$ or $y$. The involutions $x$ and $y$ are not $G$-conjugate as they have different cycle types. It follows that $G$ has precisely two conjugacy classes of involutions with representatives $x$ and $y$.

By a direct calculation,

$$
S:=\langle(1234)(910),(12)(34),(5678)(910),(56)(78),(15)(26)(37)(48)\rangle
$$

is a Sylow 2-subgroup of $G$. Another calculation confirms that $S$ has precisely 14 involutions whose cycle type is that of $x$ and precisely 29 involutions whose cycle type is that of $y$. So there are precisely two $\mathcal{F}_{S}(G)$-conjugacy classes of involutions, one of which has 14 elements, while the other one has 29 elements. In order to prove that $\mathcal{F}_{S}(G)$ is not isomorphic to the 2 -fusion system of $P S L_{4}(q)$, we show that the 2-fusion system of $P S L_{4}(q)$ has a conjugacy class of involutions with precisely 17 elements.

Let $W_{1}$ be a Sylow 2-subgroup of $G L_{2}(q)$, and let $W_{2}$ be the Sylow 2-subgroup of $G L_{4}(q)$ obtained from $W_{1}$ by the construction given in the last statement of Lemma 3.14. Let $W:=$ $W_{2} \cap S L_{4}(q) \in \operatorname{Syl}_{2}\left(S L_{4}(q)\right)$, and let $R$ be the image of $W$ in $P S L_{4}(q)$. The involutions of $W_{2}$ are precisely the elements

$$
\left(\begin{array}{ll}
a & \\
& b
\end{array}\right) \text { and }\left(\begin{array}{ll}
c^{-1} & c
\end{array}\right)
$$

where $a, b, c \in W_{1}$ and $\max \{\operatorname{ord}(a), \operatorname{ord}(b)\}=2$. Bearing in mind that $W_{1}$ is semidihedral of order 16 , which holds because of $q \equiv 3 \bmod 8$, we may see from Lemma 3.16 that $W$ has precisely 35 involutions. As one of them is $-I_{4}$, and as the product of $-I_{4}$ with an involution of $W$ different from $-I_{4}$ is again an involution, we may conclude that $R$ has precisely 17 involutions that are images of involutions of $W$. Since any noncentral involution of $S L_{4}(q)$ is $S L_{4}(q)$-conjugate to a diagonal matrix having diagonal entries $1,1,-1,-1$, we have that all the noncentral involutions of $S L_{4}(q)$ are $S L_{4}(q)$-conjugate. Thus the 17 involutions of $R$ induced by involutions of $W$ are $P S L_{4}(q)$-conjugate. As an element of $P S L_{4}(q)$ induced by an involution cannot be conjugate to an element of $P S L_{4}(q)$ not induced by an involution, it follows that there is an $\mathcal{F}_{R}\left(P S L_{4}(q)\right)$ conjugacy class of involutions with precisely 17 elements.

Lemma 4.4. Let $q$ be an odd prime power with $q \equiv 5 \bmod 8$. Assume that $G=M_{22}, M_{23}$ or $M c L$. Then the 2 -fusion system of $G$ is not isomorphic to the 2 -fusion system of $P S L_{4}(q)$.
Proof. Let $S \in \operatorname{Syl}_{2}(G)$ and $\mathcal{F}:=\mathcal{F}_{S}(G)$. Let $x$ be an element of $S$ with order 4 such that $\langle x\rangle$ is fully $\mathcal{F}$-centralized. In other words, we have $C_{S}(x) \in \operatorname{Syl}_{2}\left(C_{G}(x)\right)$. If $G=M_{22}$ or $M_{23}$, then by [1], $C_{G}(x)$ is a 2-group, whence $C_{\mathcal{F}}(\langle x\rangle)=\mathcal{F}_{C_{S}(x)}\left(C_{G}(x)\right)=\mathcal{F}_{C_{S}(x)}\left(C_{S}(x)\right)$. If $G=M c L$, then by [1], $G$ has precisely one conjugacy class of elements of order 4 , so that all elements of $S$ with order 4 are $\mathcal{F}$-conjugate.

Consequently, we either have that $C_{\mathcal{F}}(\langle x\rangle)$ is nilpotent for all elements $x \in S$ with order 4 such that $\langle x\rangle$ is fully $\mathcal{F}$-centralized, or all elements of $S$ with order 4 are $\mathcal{F}$-conjugate. We are going to show that the 2 -fusion system of $P S L_{4}(q)$ has neither of these properties.

Let $\lambda$ be an element of $\mathbb{F}_{q}^{*}$ of order 4 and let $y$ be the image of $\operatorname{diag}\left(1,1, \lambda, \lambda^{-1}\right)$ in $P S L_{4}(q)$. Clearly, $y$ has order 4. Let $R$ be a Sylow 2-subgroup of $P S L_{4}(q)$ containing a Sylow 2-subgroup of $C:=C_{P S L_{4}(q)}(y)$. Clearly, $y \in R$. Let us denote $\mathcal{F}_{R}\left(P S L_{4}(q)\right)$ by $\mathcal{G}$. Then $\langle y\rangle$ is fully $\mathcal{G}$ centralized. The centralizer $C$ is not 2-nilpotent since it has a subgroup isomorphic to $S L_{2}(q)$. So, by [39, Theorem 1.4], $C_{\mathcal{G}}(\langle y\rangle)=\mathcal{F}_{C_{R}(y)}(C)$ is not nilpotent.

Let $m$ denote the matrix

$$
\left(\begin{array}{cccc}
0 & \lambda & & \\
1 & 0 & & \\
& & 0 & -\lambda \\
& & 1 & 0
\end{array}\right) \in S L(4, q) .
$$

A direct calculation, using $q \equiv 5 \bmod 8$, shows that $m$ has no eigenvalues, whence $m$ is in particular not diagonalizable. The image of $m$ in $P S L_{4}(q)$ has order 4, but it is not $P S L_{4}(q)$-conjugate to $y$. Therefore, $P S L_{4}(q)$ has more than one conjugacy class of elements with order 4 . Thus there is more than one $\mathcal{G}$-conjugacy class of elements with order 4 .

Proposition 4.5. Let $q$ be a nontrivial odd prime power and let $G$ be a finite simple group. Then the following are equivalent:
(i) the 2-fusion system of $G$ is isomorphic to the 2-fusion system of $P S L_{4}(q)$;
(ii) $G \cong P S L_{4}^{\varepsilon}\left(q^{*}\right)$ for some $\varepsilon \in\{+,-\}$ and some nontrivial odd prime power $q^{*}$ with $\varepsilon q^{*} \sim q$. In particular, Theorem $A$ holds for $n=4$.
Proof. The implication (ii) $\Rightarrow$ (i) is given by Proposition 3.20 .
(i) $\Rightarrow$ (ii): Assume that the 2-fusion system of $G$ is isomorphic to the 2-fusion system of $P S L_{4}(q)$. Then the Sylow 2-subgroups of $G$ are isomorphic to those of $P S L_{4}(q)$. Applying Mason's results [41, Theorem 1.1 and Corollary 1.3] and [40, Theorems 1.1 and 3.15], the latter together with [29, Theorem 4.10.5 (f)], we see that one of the following holds:
(1) $G \cong P S L_{4}^{\varepsilon}\left(q^{*}\right)$ for some nontrivial odd prime power $q^{*}$ and some $\varepsilon \in\{+,-\}$ with $\varepsilon q^{*} \equiv q$ $\bmod 4 ;$
(2) $G \cong A_{10}$ or $A_{11}$, and $q \equiv 3 \bmod 4$;
(3) $G \cong M_{22}, M_{23}$ or $M c L$, and $q \equiv 5 \bmod 8$.

Let $q_{0}$ be a nontrivial odd prime power, $\varepsilon_{0} \in\{+,-\}$, and $k_{0}, s_{0} \in \mathbb{N}$ such that $2^{k_{0}}=\left(q_{0}-\varepsilon_{0}\right)_{2}$ and $2^{s_{0}}=\left(q_{0}+\varepsilon_{0}\right)_{2}$. Then we have

$$
\left|P S L_{4}^{\varepsilon_{0}}\left(q_{0}\right)\right|_{2}=\frac{\left|G L_{4}^{\varepsilon_{0}}\left(q_{0}\right)\right|_{2}}{2^{k_{0}}\left(4,2^{k_{0}}\right)}=\frac{2\left(\left|G L_{2}^{\varepsilon_{0}}\left(q_{0}\right)\right|_{2}\right)^{2}}{2^{k_{0}}\left(4,2^{k_{0}}\right)}=\frac{2^{3 k_{0}+2 s_{0}+1}}{\left(4,2^{k_{0}}\right)} .
$$

Let $k, s \in \mathbb{N}$ such that $2^{k}=(q-1)_{2}$ and $2^{s}=(q+1)_{2}$.
Suppose that (1) holds, and let $k^{*}, s^{*} \in \mathbb{N}$ such that $2^{k^{*}}=\left(q^{*}-\varepsilon\right)_{2}$ and $2^{s^{*}}=\left(q^{*}+\varepsilon\right)_{2}$. Then we have

$$
\frac{2^{3 k^{*}+2 s^{*}+1}}{\left(4,2^{k^{*}}\right)}=|G|_{2}=\frac{2^{3 k+2 s+1}}{\left(4,2^{k}\right)}
$$

Since $\varepsilon q^{*} \equiv q \bmod 4$, this easily implies $\varepsilon q^{*} \sim q$.
Suppose that (2) holds. Then $2^{7}=|G|_{2}=2^{3+2 s}$, whence $s=2$ and thus $q \equiv 3 \bmod 8$. This is a contradiction to Lemma 4.3. So (2) does not hold.

Also, (3) cannot hold because of Lemma 4.4 .
Proposition 4.6. Let $q$ be a nontrivial odd prime power, and let $G$ be a finite simple group. Then the following are equivalent:
(i) the 2-fusion system of $G$ is isomorphic to the 2-fusion system of $P S L_{5}(q)$;
(ii) the Sylow 2-subgroups of $G$ are isomorphic to those of $P S L_{5}(q)$;
(iii) $G \cong P S L_{5}^{\varepsilon}\left(q^{*}\right)$ for some nontrivial odd prime power $q^{*}$ and some $\varepsilon \in\{+,-\}$ with $\varepsilon q^{*} \sim q$.

In particular, Theorem $A$ holds for $n=5$.
Proof. The implication (i) $\Rightarrow$ (ii) is clear, and the implication (iii) $\Rightarrow$ (i) is given by Proposition 3.20 .
(ii) $\Rightarrow$ (iii): Assume that the Sylow 2-subgroups of $G$ are isomorphic to those of $P S L_{5}(q)$. Applying work of Mason [42, Theorem 1.1], it follows that $G \cong P S L_{5}^{\varepsilon}\left(q^{*}\right)$ for some $\varepsilon \in\{+,-\}$ and some nontrivial odd prime power $q^{*}$. In view of Lemma 3.15, it is easy to see that a Sylow 2-subgroup of $G$ is isomorphic to a Sylow 2-subgroup of $G L_{4}^{\varepsilon}\left(q^{*}\right)$, while a Sylow 2-subgroup of $P S L_{5}(q)$ is isomorphic to a Sylow 2-subgroup of $G L_{4}(q)$. Now it is easy to deduce from Lemmas $3.10,3.11$ and 3.14 that a Sylow 2-subgroup of $G$ has a center of order $\left(q^{*}-\varepsilon\right)_{2}$, while a Sylow 2-subgroup of $\overline{P S L}_{5}(q)$ has a center of order $(q-1)_{2}$. It follows that $\left(q^{*}-\varepsilon\right)_{2}=(q-1)_{2}$. Let $k, s, k^{*}, s^{*} \in \mathbb{N}$ with $2^{k}=(q-1)_{2}, 2^{s}=(q+1)_{2}, 2^{k^{*}}=\left(q^{*}-\varepsilon\right)_{2}$ and $2^{s^{*}}=\left(q^{*}+\varepsilon\right)_{2}$. Then

$$
2^{4 k^{*}+2 s^{*}+1}=\left|G L_{4}^{\varepsilon}\left(q^{*}\right)\right|_{2}=|G|_{2}=\left|G L_{4}(q)\right|_{2}=2^{4 k+2 s+1}
$$

Since $2^{k^{*}}=2^{k}$, we thus have $k=k^{*}$ and $s=s^{*}$. This implies $\varepsilon q^{*} \sim q$.

## 5. The case $n \geq 6$ : Preliminary discussion and notation

Given a natural number $k \geq 6$, we say that $P(k)$ is satisfied if whenever $q_{0}$ is a nontrivial odd prime power and $H$ is a finite simple group satisfying ( $\mathcal{C K})$ and realizing the 2 -fusion system of $P S L_{k}\left(q_{0}\right)$, we have $H \cong P S L_{k}^{\varepsilon}\left(q^{*}\right)$ for some nontrivial odd prime power $q^{*}$ and some $\varepsilon \in\{+,-\}$ with $\varepsilon q^{*} \sim q_{0}$.

In order to establish Theorem for $n \geq 6$, we are going to prove by induction that $P(k)$ is satisfied for all $k \geq 6$. From now on until the end of Section 8 , we will assume the following hypothesis.

Hypothesis 5.1. Let $n \geq 6$ be a natural number such that $P(k)$ is satisfied for all natural numbers $k$ with $6 \leq k<n$, and let $q$ be a nontrivial odd prime power. Moreover, let $G$ be a finite group satisfying the following properties:
(i) $G$ realizes the 2-fusion system of $P S L_{n}(q)$;
(ii) $O(G)=1$;
(iii) $G$ satisfies (CK).

We will prove the following theorem.
Theorem 5.2. There is a normal subgroup $G_{0}$ of $G$ isomorphic to a nontrivial quotient of $S L_{n}^{\varepsilon}\left(q^{*}\right)$ for some nontrivial odd prime power $q^{*}$ and some $\varepsilon \in\{+,-\}$ with $\varepsilon q^{*} \sim q$. In particular, $P(n)$ is satisfied.

The proof of Theorem 5.2 will occupy Sections 55. In this section, we introduce some notation and prove some preliminary results needed for the proof.

For each $A \subseteq\{1, \ldots, n\}$ of even order, let $t_{A}$ be the image of the diagonal matrix $\operatorname{diag}\left(d_{1}, \ldots, d_{n}\right)$ in $P S L_{n}(q)$, where

$$
d_{i}= \begin{cases}-1 & \text { if } i \in A \\ 1 & \text { if } i \notin A\end{cases}
$$

for each $1 \leq i \leq n$. If $i$ is an even natural number with $2 \leq i<n$ and $A=\{n-i+1, \ldots, n\}$, then we write $t_{i}$ for $t_{A}$. We denote $t_{2}$ by $t$, and we write $u$ for $t_{\{1,2\}}$.

We assume $\rho$ to be an element of $\mathbb{F}_{q}^{*}$ of order $(n, q-1)$. If $\rho$ is a square in $\mathbb{F}_{q}$, then we assume $\mu$ to be a fixed element of $\mathbb{F}_{q}$ with $\rho=\mu^{2}$.

If $n$ is even, $\rho$ is a square in $\mathbb{F}_{q}$, and $i$ is an odd natural number with $1 \leq i<n$, then

$$
\left(\begin{array}{ll}
\mu I_{n-i} & \\
& -\mu I_{i}
\end{array}\right)
$$

is an element of $S L_{n}(q)$ by Proposition 3.5, and we will denote its image in $P S L_{n}(q)$ by $t_{i}$.
If $n$ is even and $\rho$ is a non-square element of $\mathbb{F}_{q}$, then we denote the matrix

$$
\left(\begin{array}{ll} 
& I_{n / 2} \\
\rho I_{n / 2} &
\end{array}\right)
$$

by $\widetilde{w}$, and if $\widetilde{w} \in S L_{n}(q)$, then we use $w$ to denote its image in $P S L_{n}(q)$.
Note that, by Proposition 3.5, any involution of $P S L_{n}(q)$ is conjugate to $t_{i}$ for some $1 \leq i<n$ such that $t_{i}$ is defined, or to $w$ (if defined).

Next, we construct a Sylow 2-subgroup of $C_{P S L_{n}(q)}(t)$ containing some "nice" elements of $P S L_{n}(q)$. Take a Sylow 2-subgroup $V$ of $G L_{2}(q)$ containing each diagonal matrix in $G L_{2}(q)$ with 2-elements of $\mathbb{F}_{q}^{*}$ along the main diagonal. Similarly, we assume $V_{2}$ to be a Sylow 2-subgroup of $G L_{n-4}(q)$ containing each diagonal matrix in $G L_{n-4}(q)$ with 2-elements of $\mathbb{F}_{q}^{*}$ along the main diagonal. Now let $W$ be a Sylow 2-subgroup of $G L_{n-2}(q)$ containing

$$
\left\{\left(\begin{array}{ll}
A & \\
& B
\end{array}\right): A \in V, B \in V_{2}\right\} .
$$

If $n=6$, then we assume that $V=V_{2}$ and that $W$ is the Sylow 2-subgroup

$$
\left\{\left(\begin{array}{ll}
A & \\
& B
\end{array}\right): A, B \in V\right\} \cdot\left\langle\left(\begin{array}{ll} 
& I_{2} \\
I_{2} &
\end{array}\right)\right\rangle
$$

of $G L_{4}(q)$.
Let $\widetilde{t}:=\operatorname{diag}(1, \ldots, 1,-1,-1) \in S L_{n}(q)$. Then we have

$$
C_{S L_{n}(q)}(\widetilde{t})=\left\{\left(\begin{array}{cc}
A & \\
& B
\end{array}\right): A \in G L_{n-2}(q), B \in G L_{2}(q), \operatorname{det}(A) \operatorname{det}(B)=1\right\}
$$

It is easy to note that

$$
\widetilde{T}:=\left\{\left(\begin{array}{ll}
A & \\
& B
\end{array}\right): A \in W, B \in V, \operatorname{det}(A) \operatorname{det}(B)=1\right\}
$$

is a Sylow 2-subgroup of $C_{S L_{n}(q)}(\widetilde{t})$. Let $T$ denote the image of $\widetilde{T}$ in $P S L_{n}(q)$. As the centralizer of $t$ in $P S L_{n}(q)$ is the image of $C_{S L_{n}(q)}(\widetilde{t})$ in $P S L_{n}(q)$, we have that $T$ is a Sylow 2-subgroup of $C_{P S L_{n}(q)}(t)$. We assume $S$ to be a Sylow 2-subgroup of $P S L_{n}(q)$ containing $T$. Since $C_{S}(t)=$ $T \in \operatorname{Syl}_{2}\left(C_{P S L_{n}(q)}(t)\right)$, we have that $\langle t\rangle$ is fully $\mathcal{F}_{S}\left(P S L_{n}(q)\right)$-centralized.

Let $K_{1}$ be the image of

$$
\left\{\left(\begin{array}{cc}
A & \\
& I_{2}
\end{array}\right): A \in S L_{n-2}(q)\right\}
$$

in $P S L_{n}(q)$, and let $K_{2}$ be the image of

$$
\left\{\left(\begin{array}{cc}
I_{n-2} & \\
& B
\end{array}\right): B \in S L_{2}(q)\right\}
$$

in $P S L_{n}(q)$. Clearly, $K_{1}$ and $K_{2}$ are normal subgroups of $C_{P S L_{n}(q)}(t)$ isomorphic to $S L_{n-2}(q)$ and $S L_{2}(q)$, respectively. Define $X_{1}$ to be the image of

$$
\left\{\left(\begin{array}{ll}
A & \\
& I_{2}
\end{array}\right): A \in W \cap S L_{n-2}(q)\right\}
$$

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in $P S L_{n}(q)$, and define $X_{2}$ to be the image of

$$
\left\{\left(\begin{array}{ll}
I_{n-2} & \\
& B
\end{array}\right): B \in V \cap S L_{2}(q)\right\}
$$

in $P S L_{n}(q)$.
Note that $X_{1}=T \cap K_{1} \in \operatorname{Syl}_{2}\left(K_{1}\right)$ and $X_{2}=T \cap K_{2} \in \operatorname{Syl}_{2}\left(K_{2}\right)$. Define

$$
\mathcal{C}_{i}:=\mathcal{F}_{X_{i}}\left(K_{i}\right)
$$

for $i \in\{1,2\}$. By [11, Part I, Proposition 6.2], $\mathcal{C}_{1}$ and $\mathcal{C}_{2}$ are normal subsystems of $\mathcal{F}_{T}\left(C_{P S L_{n}(q)}(t)\right)$.
Lemma 5.3. Let $\mathcal{F}:=\mathcal{F}_{S}\left(P S L_{n}(q)\right)$. If $q \equiv 1$ or $7 \bmod 8$, then the components of $C_{\mathcal{F}}(\langle t\rangle)$ are precisely the subsystems $\mathcal{C}_{1}$ and $\mathcal{C}_{2}$. If $q \equiv 3$ or $5 \bmod 8$, then $\mathcal{C}_{1}$ is the only component of $C_{\mathcal{F}}(\langle t\rangle)$.
Proof. Set $C:=C_{P S L_{n}(q)}(t)$. It is easy to note that the 2 -components of $C$ are precisely the quasisimple elements of $\left\{K_{1}, K_{2}\right\}$. As $n \geq 6$ and as $K_{1} \cong S L_{n-2}(q)$ and $K_{2} \cong S L_{2}(q)$, it follows that the 2-components of $C$ are $K_{1}$ and $K_{2}$ if $q \neq 3$, and that $K_{1}$ is the only 2-component of $C$ if $q=3$.

By Lemma $3.21, K_{1} / Z\left(K_{1}\right)$ is not a Goldschmidt group. If $q \neq 3$, then the lemma just cited also shows that $K_{2} / Z\left(K_{2}\right)$ is a Goldschmidt group if and only if $q \equiv 3$ or $5 \bmod 8$.

Now we apply Proposition 2.16 to conclude that $\mathcal{F}_{T \cap K_{1}}\left(K_{1}\right)$ and $\mathcal{F}_{T \cap K_{2}}\left(K_{2}\right)$ are precisely the components of $\mathcal{F}_{T}(C)$ if $q \equiv 1$ or $7 \bmod 8$, and that $\mathcal{F}_{T \cap K_{1}}\left(K_{1}\right)$ is the only component of $\mathcal{F}_{T}(C)$ if $q \equiv 3$ or $5 \bmod 8$. This completes the proof because $C_{\mathcal{F}}(\langle t\rangle)=\mathcal{F}_{T}(C), \mathcal{C}_{1}=\mathcal{F}_{T \cap K_{1}}\left(K_{1}\right)$ and $\mathcal{C}_{2}=\mathcal{F}_{T \cap K_{2}}\left(K_{2}\right)$.

Lemma 5.4. Let $\mathcal{F}:=\mathcal{F}_{S}\left(P S L_{n}(q)\right)$. Then the factor system $C_{\mathcal{F}}(\langle t\rangle) / X_{1} X_{2}$ is nilpotent.
Proof. Set $C:=C_{P S L_{n}(q)}(t)$. It is easy to note that $X_{1} X_{2}=K_{1} K_{2} \cap T$. By Lemma 2.11, $C_{\mathcal{F}}(\langle t\rangle) / X_{1} X_{2}$ is isomorphic to the 2-fusion system of $C / K_{1} K_{2}$. The factor group $C / K_{1} K_{2}$ is abelian. This easily implies that $C / K_{1} K_{2}$ has a nilpotent 2 -fusion system. Hence $C_{\mathcal{F}}(\langle t\rangle) / X_{1} X_{2}$ is nilpotent.

Lemma 5.5. Let $A \in W$ and $B \in V$ such that $\operatorname{det}(A) \operatorname{det}(B)=1$. Let

$$
m:=\left(\begin{array}{ll}
A & \\
& B
\end{array}\right) Z\left(S L_{n}(q)\right) \in T
$$

Then we have $m \in Z\left(\mathcal{C}_{1}\langle m\rangle\right)$ if and only if $A \in Z\left(G L_{n-2}(q)\right)$.
Proof. By [34, Proposition 1], we have $\mathcal{C}_{1}\langle m\rangle=\mathcal{F}_{X_{1}\langle m\rangle}\left(K_{1}\langle m\rangle\right)$.
If $A \in Z\left(G L_{n-2}(q)\right)$, then $m$ is central in $K_{1}\langle m\rangle$, which implies that $m$ lies in the center of $\mathcal{C}_{1}\langle m\rangle$.

We show now that if $A \notin Z\left(G L_{n-2}(q)\right)$, then $m \notin Z\left(\mathcal{C}_{1}\langle m\rangle\right)$. Assume to the contrary that $A \notin Z\left(G L_{n-2}(q)\right)$, but $m \in Z\left(\mathcal{C}_{1}\langle m\rangle\right)$. Clearly, $m \in Z\left(X_{1}\langle m\rangle\right)$. So $m$ centralizes $X_{1}$. It easily follows that $A$ centralizes $W \cap S L_{n-2}(q)$. Using Sylow's theorem, we may see from Lemma 3.17 that any element $A_{0}$ of $W$ which centralizes $W \cap S L_{n-2}(q)$ without being central in $G L_{n-2}(q)$ is $S L_{n-2}(q)$-conjugate to an element of $W$ different from $A_{0}$. As $A$ centralizes $W \cap S L_{n-2}(q)$, but $A \notin Z\left(G L_{n-2}(q)\right)$, it follows that $A$ is $S L_{n-2}(q)$-conjugate to an element $A^{\prime} \in W$ with $A \neq A^{\prime}$. As $\operatorname{det}(A)=\operatorname{det}\left(A^{\prime}\right)$, we have $A^{\prime}=A^{\prime \prime} A$ for some $A^{\prime \prime} \in W \cap S L_{n-2}(q)$. Now, it follows that $m$ is $K_{1}$-conjugate to

$$
\left(\begin{array}{ll}
A^{\prime} & \\
& B
\end{array}\right) Z\left(S L_{n}(q)\right)=\left(\begin{array}{ll}
A^{\prime \prime} & \\
& I_{2}
\end{array}\right)\left(\begin{array}{ll}
A & \\
& B
\end{array}\right) Z\left(S L_{n}(q)\right) \in X_{1}\langle m\rangle .
$$

So $m$ is $K_{1}$-conjugate to an element of $X_{1}\langle m\rangle$ which is different from $m$. Therefore, $m \notin Z\left(\mathcal{C}_{1}\langle m\rangle\right)$, a contradiction.

Lemma 5.6. Set $\mathcal{F}:=\mathcal{F}_{S}\left(P S L_{n}(q)\right)$ and $\mathcal{G}:=C_{\mathcal{F}}(\langle t\rangle)$. Then $\mathfrak{h n p}\left(C_{\mathcal{G}}\left(X_{1}\right)\right)=X_{2}$.

Proof. Set $C:=C_{P S L_{n}(q)}(t)$. Note that $C^{\prime}=K_{1} K_{2}$.
By [24, Chapter 7, Theorem 3.4], we have $\mathfrak{f o c}\left(C_{\mathcal{G}}\left(X_{1}\right)\right)=C_{T}\left(X_{1}\right) \cap C_{C}\left(X_{1}\right)^{\prime} \leq C_{T}\left(X_{1}\right) \cap C^{\prime}=$ $C_{T}\left(X_{1}\right) \cap X_{1} X_{2}=Z\left(X_{1}\right) X_{2}$. As $\mathfrak{h n p}\left(C_{\mathcal{G}}\left(X_{1}\right)\right) \leq \mathfrak{f o c}\left(C_{\mathcal{G}}\left(X_{1}\right)\right)$, it follows that $\mathfrak{h n p}\left(C_{\mathcal{G}}\left(X_{1}\right)\right) \leq$ $Z\left(X_{1}\right) X_{2}$.

Let $P$ be a subgroup of $C_{T}\left(X_{1}\right)$ and let $\varphi$ be a $2^{\prime}$-element of $\operatorname{Aut}_{C_{C}\left(X_{1}\right)}(P)$. By [37, 8.2.7], we have

$$
[P,\langle\varphi\rangle]=[P,\langle\varphi\rangle,\langle\varphi\rangle] \leq\left[\mathfrak{h n p}\left(C_{\mathcal{G}}\left(X_{1}\right)\right) \cap P,\langle\varphi\rangle\right] \leq\left[Z\left(X_{1}\right) X_{2} \cap P,\langle\varphi\rangle\right] .
$$

Since $\varphi \in \operatorname{Aut}_{C_{C}\left(X_{1}\right)}(P), K_{2} \unlhd C$ and $X_{2}=T \cap K_{2}$, it follows $[P,\langle\varphi\rangle] \leq X_{2}$. Consequently, $\mathfrak{h n p}\left(C_{\mathcal{G}}\left(X_{1}\right)\right) \leq X_{2}$.

On the other hand, since $K_{2} \leq O^{2}\left(C_{C}\left(X_{1}\right)\right)$, we have $X_{2} \leq \mathfrak{h n p}\left(C_{\mathcal{G}}\left(X_{1}\right)\right)$ by [19, Theorem 1.33].

Lemma 5.7. Set $C:=C_{P S L_{n}(q)}(t)$. Then $\operatorname{Aut}_{C}\left(X_{1}\right)$ is a 2-group.
Proof. Let $m \in N_{C}\left(X_{1}\right)$. We have

$$
m=\left(\begin{array}{ll}
M_{1} & \\
& M_{2}
\end{array}\right) Z\left(S L_{n}(q)\right)
$$

for some $M_{1} \in G L_{n-2}(q)$ and some $M_{2} \in G L_{2}(q)$ with $\operatorname{det}\left(M_{1}\right) \operatorname{det}\left(M_{2}\right)=1$. Let $A \in W \cap$ $S L_{n-2}(q)$ and

$$
x:=\left(\begin{array}{cc}
A & \\
& I_{2}
\end{array}\right) Z\left(S L_{n}(q)\right) \in X_{1} .
$$

As $m$ normalizes $X_{1}$, we have

$$
\left(\begin{array}{cc}
A^{M_{1}} & \\
& I_{2}
\end{array}\right) Z\left(S L_{n}(q)\right)=x^{m} \in X_{1} .
$$

This easily implies that $A^{M_{1}} \in W \cap S L_{n-2}(q)$. It follows that $M_{1}$ normalizes $W \cap S L_{n-2}(q)$. By [36, Theorem 1], we have $N_{G L_{n-2}(q)}\left(W \cap S L_{n-2}(q)\right)=W C_{G L_{n-2}(q)}(W)$. It follows that $\left.c_{m}\right|_{X_{1}, X_{1}}$ is a 2 -automorphism.

Define $T_{1}$ to be the image of

$$
\left\{\left(\begin{array}{ll}
A & \\
& I_{n-2}
\end{array}\right): A \in V \cap S L_{2}(q)\right\}
$$

in $P S L_{n}(q)$ and $T_{2}$ to be the image of

$$
\left\{\left(\begin{array}{ccc}
I_{2} & & \\
& B & \\
& & I_{2}
\end{array}\right): B \in V_{2} \cap S L_{n-4}(q)\right\}
$$

in $P S L_{n}(q)$. Clearly, $T_{1}$ and $T_{2}$ are subgroups of $X_{1}$. Recall that we use $u$ to denote $t_{\{1,2\}} \in X_{1}$. The following lemma sheds light on some properties of the centralizer fusion system $C_{\mathcal{C}_{1}}(\langle u\rangle)$.

Lemma 5.8. The following hold.
(i) We have $C_{X_{1}}(u) \in \operatorname{Syl}_{2}\left(C_{K_{1}}(u)\right)$. In particular, $\langle u\rangle$ is fully $\mathcal{C}_{1}$-centralized.
(ii) $\mathfrak{f o c}\left(C_{\mathcal{C}_{1}}(\langle u\rangle)\right)=T_{1} T_{2}$.
(iii) If $n=6$ and $q \equiv 3$ or $5 \bmod 8$, then $T_{1}$ and $T_{2}$ are the only subgroups of $\mathfrak{f o c}\left(C_{\mathcal{C}_{1}}(\langle u\rangle)\right)$ which are isomorphic to $Q_{8}$ and strongly closed in $C_{\mathcal{C}_{1}}(\langle u\rangle)$.
(iv) If $n \geq 7$ and $q \equiv 3$ or $5 \bmod 8$, then $T_{1}$ is the only subgroup of the intersection $\mathfrak{f o c}\left(C_{\mathcal{C}_{1}}(\langle u\rangle)\right) \cap C_{X_{1}}\left(T_{2}\right)$ which is isomorphic to $Q_{8}$ and strongly closed in $C_{\mathcal{C}_{1}}(\langle u\rangle)$.

A CHARACTERIZATION OF THE GROUPS $P S L_{n}(q)$ AND $P S U_{n}(q)$ BY THEIR 2-FUSION SYSTEMS, $q$ ODD45
(v) Let $C_{1}$ be the image of

$$
\left\{\left(\begin{array}{ll}
A & \\
& I_{n-2}
\end{array}\right): A \in S L_{2}(q)\right\}
$$

in $P S L_{n}(q)$ and $C_{2}$ be the image of

$$
\left\{\left(\begin{array}{ccc}
I_{2} & & \\
& B & \\
& & I_{2}
\end{array}\right): A \in S L_{n-4}(q)\right\}
$$

in $\operatorname{PS} L_{n}(q)$. Then any component of $C_{\mathcal{C}_{1}}(\langle u\rangle)$ lies in $\left\{\mathcal{F}_{T_{1}}\left(C_{1}\right), \mathcal{F}_{T_{2}}\left(C_{2}\right)\right\}$. Moreover, $\mathcal{F}_{T_{1}}\left(C_{1}\right)$ is a component if and only if $q \equiv 1$ or $7 \bmod 8$, and $\mathcal{F}_{T_{2}}\left(C_{2}\right)$ is a component if and only if $n \geq 7$ or $q \equiv 1$ or $7 \bmod 8$.

Proof. Clearly, $C_{K_{1}}(u)$ is the image of

$$
\left\{\left(\begin{array}{lll}
A & & \\
& B & \\
& & I_{2}
\end{array}\right): A \in G L_{2}(q), B \in G L_{n-4}(q), \operatorname{det}(A) \operatorname{det}(B)=1\right\}
$$

in $P S L_{n}(q)$. Let $\widetilde{W}$ be the image of

$$
\left\{\left(\begin{array}{lll}
A & & \\
& B & \\
& & I_{2}
\end{array}\right): A \in V, B \in V_{2}, \operatorname{det}(A) \operatorname{det}(B)=1\right\}
$$

in $P S L_{n}(q)$. Clearly, we have $\widetilde{W} \leq C_{X_{1}}(u)$. It is easy to note that $\widetilde{W}$ is a Sylow 2-subgroup of $C_{K_{1}}(u)$. Thus $C_{X_{1}}(u)=\widetilde{W} \in \operatorname{Syl}_{2}\left(C_{K_{1}}(u)\right)$. Hence (i) holds.

We have $C_{\mathcal{C}_{1}}(\langle u\rangle)=\mathcal{F}_{C_{X_{1}}(u)}\left(C_{K_{1}}(u)\right)=\mathcal{F}_{\widetilde{W}}\left(C_{K_{1}}(u)\right)$. The focal subgroup theorem [24, Chapter 7, Theorem 3.4] implies that $\mathfrak{f o c}\left(C_{\mathcal{C}_{1}}(\langle u\rangle)\right)=\widetilde{W} \cap\left(C_{K_{1}}(u)\right)^{\prime}$. It is easy to see that $\left(C_{K_{1}}(u)\right)^{\prime}=$ $C_{1} C_{2}$, where $C_{1}$ and $C_{2}$ are as in (v). We thus have $\mathfrak{f o c}\left(C_{\mathcal{C}_{1}}(\langle u\rangle)\right)=T_{1} T_{2}$. Hence (ii) holds.

Now we turn to the proofs of (iii) and (iv). Assume that $q \equiv 3$ or $5 \bmod 8$. Clearly, $C_{1}$ and $C_{2}$ are normal subgroups of $C_{K_{1}}(u)$ and we have $T_{1}=C_{1} \cap \widetilde{W}, T_{2}=C_{2} \cap \widetilde{W}$. This implies that $T_{1}$ and $T_{2}$ are strongly closed in $C_{\mathcal{C}_{1}}(\langle u\rangle)$. By Lemma 3.12 , we have $T_{1} \cong Q_{8}$ and, if $n=6$, we also have $T_{2} \cong Q_{8}$. Clearly, any strongly $C_{\mathcal{C}_{1}}(\langle u\rangle)$-closed subgroup of $\mathfrak{f o c}\left(C_{\mathcal{C}_{1}}(\langle u\rangle)\right)=T_{1} T_{2}$ is strongly closed in $\mathcal{F}_{T_{1} T_{2}}\left(C_{1} C_{2}\right)$. Hence, in order to prove (iii), it suffices to show that if $n=6$, then $T_{1}$ and $T_{2}$ are the only strongly $\mathcal{F}_{T_{1} T_{2}}\left(C_{1} C_{2}\right)$-closed subgroups of $T_{1} T_{2}$ which are isomorphic to $Q_{8}$. Similarly, in order to prove (iv), it suffices to show that if $n \geq 7$, then $T_{1}$ is the only subgroup of $T_{1} T_{2}$ which centralizes $T_{2}$, which is isomorphic to $Q_{8}$, and which is strongly closed in $\mathcal{F}_{T_{1} T_{2}}\left(C_{1} C_{2}\right)$.

Continue to assume that $q \equiv 3$ or $5 \bmod 8$. In order to prove the two statements just mentioned, we need some observations. As $C_{1} \cong S L_{2}(q)$, we have that $C_{1}$ is not 2-nilpotent. So $\mathcal{F}_{T_{1}}\left(C_{1}\right)$ is not nilpotent by [39, Theorem 1.4]. Again by [39, Theorem 1.4], it follows that $\operatorname{Aut}_{C_{1}}\left(T_{1}\right)$ is not a 2-group. So $\operatorname{Aut}_{C_{1}}\left(T_{1}\right)$ has an element of order 3. Similarly, if $n=6$, then $\operatorname{Aut}_{C_{2}}\left(T_{2}\right)$ has an element of order 3. It follows that there is an element $\alpha \in \operatorname{Aut}_{C_{1} C_{2}}\left(T_{1} T_{2}\right)$ such that $\left.\alpha\right|_{T_{1}, T_{1}}$ has order 3, while $\left.\alpha\right|_{T_{2}, T_{2}}=\operatorname{id}_{T_{2}}$. Moreover, if $n=6$, then there is an element $\beta \in \operatorname{Aut}_{C_{1} C_{2}}\left(T_{1} T_{2}\right)$ such that $\left.\beta\right|_{T_{1}, T_{1}}=\mathrm{id}_{T_{1}}$, while $\left.\beta\right|_{T_{2}, T_{2}}$ has order 3 .

Continue to assume that $q \equiv 3$ or $5 \bmod 8$. If $n=6$, then the observations in the preceding two paragraphs show together with Lemma 2.14 that $T_{1}$ and $T_{2}$ are the only strongly $\mathcal{F}_{T_{1} T_{2}}\left(C_{1} C_{2}\right)$ closed subgroups of $T_{1} T_{2}$ which are isomorphic to $Q_{8}$. As observed above, this is enough to conclude that (iii) holds. If $n \geq 7$, then we may apply the observations in the preceding two paragraphs together with Lemma 2.14 to conclude that if $T_{0}$ is a strongly $\mathcal{F}_{T_{1} T_{2}}\left(C_{1} C_{2}\right)$-closed subgroup of $T_{1} T_{2}$ such that $T_{0} \cong Q_{8}$ and such that $T_{0}$ centralizes $T_{2}$, then $T_{0}=T_{1}$. As observed above, this is enough to conclude that (iv) holds.

It remains to prove (v). It is easy to note that the 2 -components of $C_{K_{1}}(u)$ are precisely the quasisimple elements of $\left\{C_{1}, C_{2}\right\}$. So (v) can be obtained from Proposition 2.16 and Lemma 3.21 .

Let $G$ be as in Hypothesis 5.1. The group $G$ realizes the 2-fusion system of $P S L_{n}(q)$. So, if $R$ is a Sylow 2-subgroup of $G$, then $\mathcal{F}_{S}\left(P S L_{n}(q)\right) \cong \mathcal{F}_{R}(G)$. For the sake of simplicity, we will identify $S$ with a Sylow 2 -subgroup $R$ of $G$ and $\mathcal{F}_{S}\left(P S L_{n}(q)\right)$ with $\mathcal{F}_{R}(G)$. Hence we will work under the following hypothesis.
Hypothesis 5.9. We will treat $G$ as a group with $S \in \operatorname{Syl}_{2}(G)$ and $\mathcal{F}_{S}(G)=\mathcal{F}_{S}\left(P S L_{n}(q)\right)$.
The following lemma will play a key role in the proof of Theorem 5.2.
Lemma 5.10. Let $x$ be an involution of $S$ such that $C_{S}(x) \in \operatorname{Syl}_{2}\left(C_{G}(x)\right)$. Let $\mathcal{C}$ be a component of $\mathcal{F}_{C_{S}(x)}\left(C_{G}(x)\right)$, and let $k$ be a natural number with $3 \leq k<n$. Then the following hold.
(i) There is a unique 2-component $Y$ of $C_{G}(x)$ such that $\mathcal{C}=\mathcal{F}_{C_{S}(x) \cap Y}(Y)$.
(ii) If $\mathcal{C}$ is isomorphic to the 2 -fusion system of $S L_{k}(q)$, then we either have that $Y / O(Y) \cong$ $S L_{k}^{\varepsilon}\left(q^{*}\right) / O\left(S L_{k}^{\varepsilon}\left(q^{*}\right)\right)$ for some nontrivial odd prime power $q^{*}$ and some $\varepsilon \in\{+,-\}$ with $q \sim \varepsilon q^{*}$; or $k=3,(q+1)_{2}=4$, and $Y / Z^{*}(Y) \cong M_{11}$.
(iii) If $\mathcal{C}$ is isomorphic to the 2-fusion system of a nontrivial quotient of $S L_{k}\left(q^{2}\right)$, then $Y / O(Y)$ is isomorphic to a nontrivial quotient of $S L_{k}^{\varepsilon}\left(q^{*}\right)$ for some nontrivial odd prime power $q^{*}$ and some $\varepsilon \in\{+,-\}$ with $q^{2} \sim \varepsilon q^{*}$.

In order to prove Lemma 5.10, we need the following observation.
Lemma 5.11. Let $k \geq 6$ be a natural number satisfying $P(k)$. If $q_{0}$ is a nontrivial odd prime power and $H$ is a known finite simple group realizing the 2-fusion system of $\operatorname{PSL} L_{k}\left(q_{0}\right)$, then $H \cong P S L_{k}^{\varepsilon}\left(q^{*}\right)$ for some $\varepsilon \in\{+,-\}$ and some nontrivial odd prime power $q^{*}$ with $\varepsilon q^{*} \sim q_{0}$.

Proof. It suffices to show that any known finite simple group $H$ satisfies ( $\mathcal{C K}$ ). Without using the CFSG, this is a priori not clear. It can be deduced from [29, Proposition 5.2.9] if $H$ is alternating, from [29, Table 4.5.1] if $H$ is a finite simple group of Lie type in odd characteristic, and from [29, Table 5.3] if $H$ is sporadic. If $H$ is a finite simple group of Lie type in characteristic 2, then $H$ satisfies $(\overline{\mathcal{C K}})$ since, in this case, no involution centralizer in $H$ has a 2-component (see [5, 47.8 (3)]).

Proof of Lemma 5.10. Since $G$ satisfies (CK), we have that $Y / Z^{*}(Y)$ is a known finite simple group for each 2-component $Y$ of $C_{G}(x)$. Proposition 2.16 implies that there is a unique 2-component $Y$ of $C_{G}(x)$ with $\mathcal{C}=\mathcal{F}_{C_{S}(x) \cap Y}(Y)$. Thus (i) holds.

Suppose that $\mathcal{C}$ is isomorphic to the 2-fusion system of $S L_{k}\left(q_{0}\right) / Z$, where either $q_{0}=q$ and $Z=1$, or $q_{0}=q^{2}$ and $Z \leq Z\left(S L_{k}\left(q^{2}\right)\right.$ ). In order to prove (ii) and (iii), we need the following three claims.
(1) The 2-fusion systems of $Y / Z^{*}(Y)$ and $P S L_{k}\left(q_{0}\right)$ are isomorphic.

As $\mathcal{C}=\mathcal{F}_{C_{S}(x) \cap Y}(Y)$, we have that the 2 -fusion system of $Y$ is isomorphic to the 2-fusion system of $S L_{k}\left(q_{0}\right) / Z$. So, by Corollary 2.12, the 2-fusion system of $Y / O(Y)$ is isomorphic to the 2-fusion system of $S L_{k}\left(q_{0}\right) / Z$. Lemma 2.13 implies that the 2 -fusion systems of $Y / Z^{*}(Y)$ and $P S L_{k}\left(q_{0}\right)$ are isomorphic.
(2) We have $Y / Z^{*}(Y) \cong \operatorname{PSL} L_{k}^{\varepsilon}\left(q^{*}\right)$ for some nontrivial odd prime power $q^{*}$ and some $\varepsilon \in\{+,-\}$ with $q_{0} \sim \varepsilon q^{*}$; or $k=3,\left(q_{0}+1\right)_{2}=4$ and $Y / Z^{*}(Y) \cong M_{11}$.

If $k=3$, then it follows from (1) and Proposition 4.2. If $k \in\{4,5\}$, then it follows from (1) together with Propositions 4.5 and 4.6. Assume now that $k \geq 6$. By Hypothesis 5.1 and since $k<n$, we have that $k$ satisfies $P(k)$. Since $Y / Z^{*}(Y)$ is a known finite simple group, the claim follows from (1) and Lemma 5.11.
(3) Suppose that $Y / Z^{*}(Y) \cong P S L_{k}^{\varepsilon}\left(q^{*}\right)$, where $q^{*}$ and $\varepsilon$ are as in (2). Then we have $Y / O(Y) \cong$ $S L_{k}^{\varepsilon}\left(q^{*}\right) / U$, where $U \leq Z\left(S L_{k}^{\varepsilon}\left(q^{*}\right)\right)$ and the index of $U$ in $Z\left(S L_{k}^{\varepsilon}\left(q^{*}\right)\right)$ is equal to the 2-part of $\left|Z\left(S L_{k}\left(q_{0}\right)\right) / Z\right|$.

The group $Y / O(Y)$ is a perfect central extension of $P S L_{k}^{\varepsilon}\left(q^{*}\right)$. Since $Y / O(Y)$ is core-free, the center of $Y / O(Y)$ is a 2-group. So, by Lemmas 3.1 and 3.2, there is a central subgroup $U$ of $S L_{k}^{\varepsilon}\left(q^{*}\right)$ with $Y / O(Y) \cong S L_{k}^{\varepsilon}\left(q^{*}\right) / U$. The claim now follows from

$$
\begin{aligned}
\left|P S L_{k}\left(q_{0}\right)\right|_{2}\left|Z\left(S L_{k}\left(q_{0}\right)\right) / Z\right|_{2} & =\left|S L_{k}\left(q_{0}\right) / Z\right|_{2} \\
& =|Y|_{2} \\
& =\left|Y / Z^{*}(Y)\right|_{2}|Z(Y / O(Y))| \\
& =\left|P S L_{k}\left(q_{0}\right)\right|_{2}\left|Z\left(S L_{k}^{\varepsilon}\left(q^{*}\right)\right) / U\right|
\end{aligned}
$$

Here, the second equality follows from the fact that $Y$ realizes $\mathcal{C}$, the third one holds since $\left|Z^{*}(Y)\right|_{2}=\left|Z^{*}(Y) / O(Y)\right|_{2}=|Z(Y / O(Y))|_{2}=|Z(Y / O(Y))|$, and the fourth one follows from (1).

Assume that $q_{0}=q$ and $Z=1$. By (2) and (3), one of the following hold: either $k=3$, $(q+1)_{2}=4$ and $Y / Z^{*}(Y) \cong M_{11}$; or $Y / O(Y) \cong S L_{k}^{\varepsilon}\left(q^{*}\right) / U$, where $q^{*}$ is a nontrivial odd prime power, $\varepsilon \in\{+,-\}, q \sim \varepsilon q^{*}, U \leq Z\left(S L_{k}^{\varepsilon}\left(q^{*}\right)\right)$ and the index of $U$ in $Z\left(S L_{k}^{\varepsilon}\left(q^{*}\right)\right)$ is equal to the 2-part of $\left|Z\left(S L_{k}(q)\right)\right|$. Assume that the latter holds. As $q \sim \varepsilon q^{*}$, we have $(q-1)_{2}=\left(\varepsilon q^{*}-1\right)_{2}=$ $\left(q^{*}-\varepsilon\right)_{2}$. Since $\left|Z\left(S L_{k}(q)\right)\right|=(k, q-1)$ and $\left|Z\left(S L_{k}^{\varepsilon}\left(q^{*}\right)\right)\right|=\left(k, q^{*}-\varepsilon\right)$, it follows that the 2-part of $\left|Z\left(S L_{k}(q)\right)\right|$ is equal to the 2-part of $\left|Z\left(S L_{k}^{\varepsilon}\left(q^{*}\right)\right)\right|$. It follows that $U=O\left(Z\left(S L_{k}^{\varepsilon}\left(q^{*}\right)\right)\right)=$ $O\left(S L_{k}^{\varepsilon}\left(q^{*}\right)\right)$. This completes the proof of (ii).

Assume now that $q_{0}=q^{2}$. Then, since $q^{2} \equiv 1 \bmod 4,(2)$ und (3) imply that $Y / O(Y)$ is isomorphic to a nontrivial quotient of $S L_{k}^{\varepsilon}\left(q^{*}\right)$ for some nontrivial odd prime power $q^{*}$ and some $\varepsilon \in\{+,-\}$ with $q^{2} \sim \varepsilon q^{*}$. Thus (iii) holds.

## 6. 2-COMPONENTS OF INVOLUTION CENTRALIZERS

In this section, we continue to assume Hypotheses 5.1 and 5.9. We will use the notation introduced in the last section without further explanation.

The main goal of this section is to describe the 2 -components and the solvable 2 -components of the centralizers of involutions of $G$.
6.1. The subgroups $K$ and $L$ of $C_{G}(t)$. We start by considering $C_{G}(t)$. Let $\mathcal{F}:=\mathcal{F}_{S}(G)=$ $\mathcal{F}_{S}\left(P S L_{n}(q)\right)$. Since $\langle t\rangle$ is fully $\mathcal{F}$-centralized, we have that $T=C_{S}(t) \in \operatorname{Syl}_{2}\left(C_{G}(t)\right)$. Also, note that $\mathcal{F}_{T}\left(C_{G}(t)\right)=C_{\mathcal{F}}(\langle t\rangle)=\mathcal{F}_{T}\left(C_{P S L_{n}(q)}(t)\right)$.
Proposition 6.1. There is a unique 2-component $K$ of $C_{G}(t)$ such that $\mathcal{C}_{1}=\mathcal{F}_{T \cap K}(K)$. We have $K / O(K) \cong S L_{n-2}^{\varepsilon}\left(q^{*}\right) / O\left(S L_{n-2}^{\varepsilon}\left(q^{*}\right)\right)$ for some nontrivial odd prime power $q^{*}$ and some $\varepsilon \in\{+,-\}$ with $q \sim \varepsilon q^{*}$. Moreover, $K$ is a normal subgroup of $C_{G}(t)$.

Proof. Set $\mathcal{F}:=\mathcal{F}_{S}(G)$. By Lemma 5.3, $\mathcal{C}_{1}$ is a component of $C_{\mathcal{F}}(\langle t\rangle)$. Lemma 5.10 (i) implies that there is a unique 2-component $K$ of $C_{G}(t)$ such that $\mathcal{C}_{1}=\mathcal{F}_{T \cap K}(K)$. By definition, the component $\mathcal{C}_{1}$ is isomorphic to the 2-fusion system of $S L_{n-2}(q)$. Lemma 5.10 (ii) implies that $K / O(K) \cong S L_{n-2}^{\varepsilon}\left(q^{*}\right) / O\left(S L_{n-2}^{\varepsilon}\left(q^{*}\right)\right)$ for some nontrivial odd prime power $q^{*}$ and some $\varepsilon \in\{+,-\}$ with $q \sim \varepsilon q^{*}$.

It remains to show that $K$ is a normal subgroup of $C_{G}(t)$. Suppose that $\widetilde{K}$ is a 2-component of $C_{G}(t)$ such that $K \cong \widetilde{K}$. Set $\widetilde{\mathcal{C}}:=\mathcal{F}_{T \cap \widetilde{K}}(\widetilde{K})$. Since $\widetilde{K}$ is subnormal in $C_{G}(t)$, it easily follows from [11, Part I, Proposition 6.2] that $\widetilde{\mathcal{C}}$ is subnormal in $C_{\mathcal{F}}(\langle t\rangle)$. Moreover, $\widetilde{\mathcal{C}} \cong \mathcal{C}_{1}$ as $\widetilde{K} \cong K$. Hence $\widetilde{\mathcal{C}}$ is a component of $C_{\mathcal{F}}(\langle t\rangle)$. But as a consequence of Lemma 5.3, there is no component of $C_{\mathcal{F}}(\langle t\rangle)$ which is isomorphic to $\mathcal{C}_{1}$ and different from $\mathcal{C}_{1}$. So we have $\mathcal{C}_{1}=\widetilde{\mathcal{C}}$. The uniqueness in the first
statement of the proposition implies that $K=\widetilde{K}$. Consequently, $C_{G}(t)$ has no 2-component which is different from $K$ and isomorphic to $K$. So $K$ is characteristic and hence normal in $C_{G}(t)$.

From now on, $K, q^{*}$ and $\varepsilon$ will always have the meanings given to them by Proposition 6.1.
Our next goal is to prove the existence and uniqueness of a normal subgroup $\bar{L}$ of $\overline{C_{G}(t)}:=$ $C_{G}(t) / O\left(C_{G}(t)\right)$ such that $\bar{L} \cong S L_{2}\left(q^{*}\right)$, and to show that the image $\bar{K}$ of $K$ in $\overline{C_{G}(t)}$ and $\bar{L}$ are the only subgroups of $\overline{C_{G}(t)}$ which are components or solvable 2 -components of $\overline{C_{G}(t)}$. First, we need to prove some lemmas.

Lemma 6.2. Let $A \in W$ and $B \in V$ such that $\operatorname{det}(A) \operatorname{det}(B)=1$. Let

$$
m:=\left(\begin{array}{ll}
A & \\
& B
\end{array}\right) Z\left(S L_{n}(q)\right) \in T
$$

Set $\overline{C_{G}(t)}:=C_{G}(t) / O\left(C_{G}(t)\right)$. Then $\bar{m}$ centralizes $\bar{K}$ if and only if $A \in Z\left(G L_{n-2}(q)\right)$.
Proof. By Lemma 5.5, we have $m \in Z\left(\mathcal{C}_{1}\langle m\rangle\right)$ if and only if $A \in Z\left(G L_{n-2}(q)\right)$. Let $\overline{\mathcal{C}_{1}}$ be the subsystem of $\mathcal{F}_{\bar{T}}\left(\overline{C_{G}(t)}\right)$ corresponding to $\mathcal{C}_{1}$ under the isomorphism from $\mathcal{F}_{T}\left(C_{G}(t)\right)$ to $\mathcal{F}_{\bar{T}}\left(\overline{C_{G}(t)}\right)$ given by Corollary 2.12. Then we have $\bar{m} \in Z\left(\overline{\mathcal{C}_{1}}\langle\bar{m}\rangle\right)$ if and only if $A \in Z\left(G L_{n-2}(q)\right)$. So it is enough to show that $\bar{m} \in Z\left(\overline{\mathcal{C}_{1}}\langle\bar{m}\rangle\right)$ if and only if $\bar{m}$ centralizes $\bar{K}$. It is easy to note that $\overline{\mathcal{C}_{1}}=\mathcal{F}_{\overline{X_{1}}}(\bar{K})$. As a consequence of Proposition 6.1, we have $\bar{K} \unlhd \overline{C_{G}(t)}$. By [34, Proposition 1], we have

$$
\overline{\mathcal{C}_{1}}\langle\bar{m}\rangle=\mathcal{F}_{\overline{X_{1}}\langle\bar{m}\rangle}(\bar{K}\langle\bar{m}\rangle) .
$$

Since $\bar{m}$ is a 2-element of $\overline{C_{G}(t)}$, we have $O(\bar{K}\langle\bar{m}\rangle)=O(\bar{K})=1$. Applying [23, Corollary 1], it follows that the center of the product $\overline{\mathcal{C}_{1}}\langle\bar{m}\rangle$ is equal to the center of $\bar{K}\langle\bar{m}\rangle$. It follows that that $\bar{m} \in Z\left(\overline{\mathcal{C}_{1}}\langle\bar{m}\rangle\right)$ if and only if $\bar{m}$ centralizes $\bar{K}$, as required.
Lemma 6.3. Suppose that $q^{*}=3$. Let $C:=C_{G}(t)$ and $\bar{C}:=C / O(C)$. Then:
(i) The factor group $\bar{C} / \bar{K} C_{\bar{C}}(\bar{K})$ is a 2-group.
(ii) The centralizer $C_{\bar{C}}(\bar{u})$ is core-free.
(iii) The factor group $C_{\bar{C}}(\bar{u}) / C_{\bar{C}}(\bar{K})$ is core-free.

Proof. Clearly, $\bar{C} / \bar{K} C_{\bar{C}}(\bar{K})$ is isomorphic to a subgroup of $\operatorname{Out}(\bar{K})$. Since $q^{*}=3$, we have that $\bar{K} \cong S L_{n-2}^{\varepsilon}(3)$. By Propositions 3.40 and 3.42 . Out $(\bar{K})$ is a 2 -group. So (i) holds.

Set $\bar{C}_{0}:=\bar{K} C_{\bar{C}}(\bar{K})$. As a consequence of (i), $C_{\bar{C}}(\bar{u}) / C_{\bar{C}_{0}}(\bar{u})$ is a 2 -group. Hence, in order to prove (ii), it suffices to show that $C_{\bar{C}_{0}}(\bar{u})$ is core-free. As $\bar{u} \in \bar{K}$, we have $C_{\bar{C}_{0}}(\bar{u})=$ $C_{\bar{K}}(\bar{u}) C_{\bar{C}}(\bar{K})$. It follows that $C_{\bar{C}_{0}}(\bar{u}) / C_{\bar{C}}(\bar{K}) \cong C_{\bar{K}}(\bar{u}) /\left(C_{\bar{K}}(\bar{u}) \cap C_{\bar{C}}(\bar{K})\right)=C_{\bar{K}}(\bar{u}) / Z(\bar{K})$. By Corollary 3.8, $C_{\bar{K}}(\bar{u})$ is core-free. This easily implies that $C_{\bar{K}}(\bar{u}) / Z(\bar{K})$ is core-free. It follows that $C_{\bar{C}_{0}}(\bar{u}) / C_{\bar{C}}(\bar{K})$ is core-free. Consequently, $O\left(C_{\bar{C}_{0}}(\bar{u})\right)=O\left(C_{\bar{C}}(\bar{K})\right)=1$. So (ii) follows.

Finally, (iii) is true since $C_{\bar{C}}(\bar{u}) / C_{\bar{C}_{0}}(\bar{u})$ is a 2-group and $C_{\bar{C}_{0}}(\bar{u}) / C_{\bar{C}}(\bar{K})$ is core-free.
Lemma 6.4. Let $\overline{C_{G}(t)}:=C_{G}(t) / O\left(C_{G}(t)\right)$. Then there is a unique pair $\left(A_{1}{ }^{+}, A_{2}{ }^{+}\right)$of normal subgroups $A_{1}{ }^{+}, A_{2}{ }^{+}$of $C_{\bar{K}}(\bar{u})^{\prime}$ such that $C_{\bar{K}}(\bar{u})^{\prime}=A_{1}{ }^{+} \times A_{2}{ }^{+}, A_{1}{ }^{+} \cong S L_{2}^{\varepsilon}\left(q^{*}\right), A_{2}{ }^{+} \cong S L_{n-4}^{\varepsilon}\left(q^{*}\right)$ and $\bar{u} \in A_{1}{ }^{+}$. Moreover, the following hold.
(i) $A_{1}{ }^{+} \cap \overline{X_{1}}=\overline{T_{1}}$.
(ii) $A_{2}{ }^{+} \cap \overline{X_{1}}=\overline{T_{2}}$.
(iii) There is a group isomorphism $\varphi: \bar{K} \rightarrow S L_{n-2}^{\varepsilon}\left(q^{*}\right) / O\left(S L_{n-2}^{\varepsilon}\left(q^{*}\right)\right)$ under which $A_{1}{ }^{+}$corresponds to the image of

$$
\left\{\left(\begin{array}{cc}
A & \\
& I_{n-4}
\end{array}\right): A \in S L_{2}^{\varepsilon}\left(q^{*}\right)\right\}
$$

$$
\text { in } S L_{n-2}^{\varepsilon}\left(q^{*}\right) / O\left(S L_{n-2}^{\varepsilon}\left(q^{*}\right)\right) \text { and under which } A_{2}{ }^{+} \text {corresponds to the image of }
$$

$$
\left\{\left(\begin{array}{ll}
I_{2} & \\
& B
\end{array}\right): B \in S L_{n-4}^{\varepsilon}\left(q^{*}\right)\right\}
$$

in $S L_{n-2}^{\varepsilon}\left(q^{*}\right) / O\left(S L_{n-2}^{\varepsilon}\left(q^{*}\right)\right)$.
Proof. For each subsystem $\mathcal{G}$ of $\mathcal{F}_{T}\left(C_{G}(t)\right)$, we use $\overline{\mathcal{G}}$ to denote the subsystem of $\mathcal{F}_{\bar{T}}\left(\overline{C_{G}(t)}\right)$ corresponding to $\mathcal{G}$ under the isomorphism from $\mathcal{F}_{T}\left(C_{G}(t)\right)$ to $\mathcal{F}_{\bar{T}}\left(\overline{C_{G}(t)}\right)$ given by Corollary 2.12. Note that $\overline{\mathcal{C}_{1}}=\mathcal{F}_{\overline{X_{1}}}(\bar{K})$.

Set $H:=S L_{n-2}^{\varepsilon}\left(q^{*}\right) / O\left(S L_{n-2}^{\varepsilon}\left(q^{*}\right)\right)$. For each even natural number $i$ with $2 \leq i \leq n-2$, let $h_{i}$ be the image of $\widetilde{h_{i}}:=\operatorname{diag}(-1, \ldots,-1,1, \ldots, 1) \in S L_{n-2}^{\varepsilon}\left(q^{*}\right)$ in $H$, where -1 occurs precisely $i$ times as a diagonal entry.

We claim that there is a group isomorphism $\varphi: \bar{K} \rightarrow H$ such that $\bar{u}^{\varphi}=h_{i}$ for some even $2 \leq i<n-2$. By Proposition 6.1, we have $\bar{K} \cong K / O(K) \cong H$. As a consequence of Lemmas 3.3 and 3.4, any involution of $S L_{n-2}^{\varepsilon}\left(q^{*}\right)$ is conjugate to $\widetilde{h}_{i}$ for some even $2 \leq i \leq n-2$. Since any involution of $H$ is induced by an involution of $S L_{n-2}^{\varepsilon}\left(q^{*}\right)$, it follows that any involution of $H$ is conjugate to $h_{i}$ for some even $2 \leq i \leq n-2$. As $\bar{u}$ is an involution of $\bar{K}$, it follows that there is an isomorphism $\varphi: \bar{K} \rightarrow H$ mapping $\bar{u}$ to $h_{i}$ for some even $2 \leq i \leq n-2$. Assume that $i=n-2$. Then $\bar{u}$ is central in $\bar{K}$. Thus $\bar{u} \in Z\left(\overline{\mathcal{C}_{1}}\right)$ and hence $u \in Z\left(\mathcal{C}_{1}\right)$. This is a contradiction to Lemma 3.18 and the definition of $\mathcal{C}_{1}$. So we have $i<n-2$.

Set $h:=\bar{u}^{\varphi}=h_{i}$. Also, let $H_{1}$ be the image of

$$
\left\{\left(\begin{array}{cc}
A & \\
& I_{n-2-i}
\end{array}\right): A \in S L_{i}^{\varepsilon}\left(q^{*}\right)\right\}
$$

in $H$, and let $H_{2}$ be the image of

$$
\left\{\left(\begin{array}{ll}
I_{i} & \\
& B
\end{array}\right): B \in S L_{n-2-i}^{\varepsilon}\left(q^{*}\right)\right\}
$$

in $H$. For $j \in\{1,2\}$, let $A_{j}{ }^{+}$be the subgroup of $\bar{K}$ corresponding to $H_{j}$ under $\varphi$.
We now proceed in a number of steps in order to complete the proof.
(1) We have $C_{\bar{K}}(\bar{u})^{\prime}=A_{1}{ }^{+} A_{2}{ }^{+},\left[A_{1}{ }^{+}, A_{2}{ }^{+}\right]=1, A_{1}{ }^{+}, A_{2}{ }^{+} \unlhd C_{\bar{K}}(\bar{u}), \bar{u} \in A_{1}{ }^{+}$and $\bar{u} \notin A_{2}{ }^{+}$.

It is easy to note that $C_{H}(h)^{\prime}$ is the central product of $H_{1}$ and $H_{2}$ and that $H_{1}$ and $H_{2}$ are normal in $C_{H}(h)$. Therefore, $C_{\bar{K}}(\bar{u})^{\prime}$ is the central product of $A_{1}{ }^{+}$and $A_{2}{ }^{+}$, and $A_{1}{ }^{+}, A_{2}{ }^{+}$are normal in $C_{\bar{K}}(\bar{u})$. By definition of $H_{1}$ and $H_{2}$, we have $h \in H_{1}$ and $h \notin H_{2}$. Thus $\bar{u} \in A_{1}{ }^{+}$and $\bar{u} \notin A_{2}{ }^{+}$.
(2) We have $C_{\overline{X_{1}}}(\bar{u}) \in \operatorname{Syl}_{2}\left(C_{\bar{K}}(\bar{u})\right)$, and $\left\{\mathcal{F}_{\overline{X_{1}} \cap A_{1}+}\left(A_{1}{ }^{+}\right), \mathcal{F}_{\overline{X_{1} \cap A_{2}}}\left(A_{2}{ }^{+}\right)\right\}$contains every component of $C_{\overline{C_{1}}}(\langle\bar{u}\rangle)$.

By Lemma 5.8 (i), we have that $\langle\bar{u}\rangle$ is fully $\overline{\mathcal{C}_{1}}$-centralized. So we have $C_{\overline{X_{1}}}(\bar{u}) \in \operatorname{Syl}_{2}\left(C_{\bar{K}}(\bar{u})\right)$.
Set $P:=C_{\overline{X_{1}}}(\bar{u})^{\varphi} \in \operatorname{Syl}_{2}\left(C_{H}(h)\right)$. It is easy to note that the 2 -components of $C_{H}(h)$ are precisely the quasisimple elements of $\left\{H_{1}, H_{2}\right\}$. Proposition 2.16 implies that the components of $\mathcal{F}_{P}\left(C_{H}(h)\right)$ are precisely the quasisimple elements of $\left\{\mathcal{F}_{P \cap H_{1}}\left(H_{1}\right), \mathcal{F}_{P \cap H_{2}}\left(H_{2}\right)\right\}$.

Thus the components of $C_{\overline{\mathcal{C}_{1}}}(\langle\bar{u}\rangle)=\mathcal{F}_{C_{\overline{X_{1}}}(\bar{u})}\left(C_{\bar{K}}(\bar{u})\right)$ are precisely the quasisimple elements of $\left\{\mathcal{F}_{\overline{X_{1}} \cap A_{1}}+\left(A_{1}{ }^{+}\right), \mathcal{F}_{\overline{X_{1}} \cap A_{2}}+\left(A_{2}{ }^{+}\right)\right\}$.
(3) $\overline{X_{1}} \cap A_{1}{ }^{+}$and $\overline{X_{1}} \cap A_{2}{ }^{+}$are subgroups of $\mathfrak{f o c}\left(C_{\overline{\mathcal{C}_{1}}}(\langle\bar{u}\rangle)\right)$ and are strongly closed in $C_{\overline{\mathcal{C}_{1}}}(\langle\bar{u}\rangle)$. We have $\mathfrak{f o c}\left(C_{\overline{\mathcal{C}_{1}}}(\langle\bar{u}\rangle)\right)=C_{\overline{X_{1}}}(\bar{u}) \cap C_{\bar{K}}(\bar{u})^{\prime}$ by the focal subgroup theorem [24, Chapter 7, Theorem 3.4]. So the claim follows from (1).
(4) Suppose that $n=6$ and $\underline{q} \equiv 3$ or $5 \bmod 8$. Then we have $i=2$ and hence $A_{1}{ }^{+} \cong S L_{2}^{\varepsilon}\left(q^{*}\right) \cong$ $A_{2}{ }^{+}$. Moreover, $\overline{X_{1}} \cap A_{1}{ }^{+}=\overline{T_{1}}$ and $\overline{X_{1}} \cap A_{2}{ }^{+}=\overline{T_{2}}$.

Since $n=6$ and $2 \leq i<n-2=4$, we have $i=2$. Thus $A_{1}{ }^{+} \cong H_{1} \cong S L_{2}^{\varepsilon}\left(q^{*}\right) \cong H_{2} \cong A_{2}{ }^{+}$. By Proposition 6.1, we have $q \sim \varepsilon q^{*}$, whence $q^{*} \equiv 3$ or $5 \bmod 8$. Clearly, $\overline{X_{1}} \cap A_{1}{ }^{+} \in \operatorname{Syl}_{2}\left(A_{1}{ }^{+}\right)$ and $\overline{X_{1}} \cap A_{2}{ }^{+} \in \operatorname{Syl}_{2}\left(A_{2}{ }^{+}\right)$. Lemma 3.12 implies that $\overline{X_{1}} \cap A_{1}{ }^{+} \cong Q_{8} \cong \overline{X_{1}} \cap A_{2}{ }^{+}$. By Lemma 5.8 (iii), $\overline{T_{1}}$ and $\overline{T_{2}}$ are the only subgroups of $\mathfrak{f o c}\left(C_{\overline{\mathcal{C}_{1}}}(\langle\bar{u}\rangle)\right)$ which are isomorphic to $Q_{8}$ and strongly closed in $C_{\overline{\mathcal{C}_{1}}}(\langle\bar{u}\rangle)$. So, by (3), $\left\{\overline{X_{1}} \cap A_{1}{ }^{+}, \overline{X_{1}} \cap A_{2}{ }^{+}\right\}=\left\{\overline{T_{1}}, \overline{T_{2}}\right\}$. We have $\bar{u} \in \overline{T_{1}}$, and $\bar{u} \notin A_{2}{ }^{+}$ by (1). It follows that $\overline{X_{1}} \cap A_{1}{ }^{+}=\overline{T_{1}}$ and $\overline{X_{1}} \cap A_{2}{ }^{+}=\overline{T_{2}}$.
(5) Suppose that $n=6$ and $q \equiv 1$ or $7 \bmod 8$, or that $n \geq 7$. Then we have $i=2$, and hence $A_{1}{ }^{+} \cong S L_{2}^{\varepsilon}\left(q^{*}\right)$ and $A_{2}{ }^{+} \cong S L_{n-4}^{\varepsilon}\left(q^{*}\right)$. Moreover, $\overline{X_{1}} \cap A_{1}{ }^{+}=\overline{T_{1}}$ and $\overline{X_{1}} \cap A_{2}{ }^{+}=\overline{T_{2}}$.

We begin by proving that $\overline{X_{1}} \cap A_{2}{ }^{+}=\overline{T_{2}}$. As a consequence of Lemma $5.8(\mathrm{v}), C_{\overline{\mathcal{C}_{1}}}(\langle\bar{u}\rangle)$ has a component with Sylow group $\overline{T_{2}}$. Applying (2), we may conclude that $\overline{T_{2}}=\overline{X_{1}} \cap A_{1}{ }^{+}$or $\overline{X_{1}} \cap A_{2}{ }^{+}$. Since $\bar{u} \in A_{1}^{+}$by (1), but $\bar{u} \notin \overline{T_{2}}$, we have $\overline{X_{1}} \cap A_{2}{ }^{+}=\overline{T_{2}}$.

We show next that $i=2$. Using Proposition 3.19, or using the order formulas for $\left|S L_{n-4}\left(q^{*}\right)\right|$ and $\left|S U_{n-4}\left(q^{*}\right)\right|$ given by [33, Proposition 1.1 and Corollary 11.29], we see that

$$
\left|S L_{n-4}^{\varepsilon}\left(q^{*}\right)\right|_{2}=\left|S L_{n-4}(q)\right|_{2}=\left|T_{2}\right|=\left|A_{2}^{+}\right|_{2}=\left|H_{2}\right|_{2}=\left|S L_{n-2-i}^{\varepsilon}\left(q^{*}\right)\right|_{2}
$$

Using the order formulas cited above, we may conclude that $n-2-i=n-4$, whence $i=2$. In particular, $A_{1}{ }^{+} \cong S L_{2}^{\varepsilon}\left(q^{*}\right)$ and $A_{2}{ }^{+} \cong S L_{n-4}^{\varepsilon}\left(q^{*}\right)$.

It remains to prove $\overline{X_{1}} \cap A_{1}{ }^{+}=\overline{T_{1}}$. If $q \equiv 1$ or $7 \bmod 8$, then Lemma 5.8 (v) shows that $C_{\overline{\mathcal{C}_{1}}}(\langle\bar{u}\rangle)$ has a component with Sylow group $\overline{T_{1}}$. Since $\bar{u} \in \overline{T_{1}}$, but $\bar{u} \notin A_{2}{ }^{+}$, we have $\overline{X_{1}} \cap A_{1}{ }^{+}=\overline{T_{1}}$ by (2).

Now suppose that $q \equiv 3$ or $5 \bmod 8$. Then we have $q^{*} \equiv 3$ or $5 \bmod 8$ since $q \sim \varepsilon q^{*}$. So, by Lemma 3.12, a Sylow 2-subgroup of $A_{1}{ }^{+}$is isomorphic to $Q_{8}$. In particular, $\overline{X_{1}} \cap A_{1}{ }^{+} \cong Q_{8}$. By (3), $\overline{X_{1}} \cap A_{1}{ }^{+}$is a subgroup of $\mathfrak{f o c}\left(C_{\overline{\mathcal{C}_{1}}}(\langle\bar{u}\rangle)\right)$ and is strongly closed in $C_{\overline{\mathcal{C}_{1}}}(\langle\bar{u}\rangle)$. Moreover, by (1), $\overline{X_{1}} \cap A_{1}{ }^{+}$centralizes $\overline{X_{1}} \cap A_{2}{ }^{+}=\overline{T_{2}}$. Lemma 5.8 (iv) now implies that $\overline{T_{1}}=\overline{X_{1}} \cap A_{1}{ }^{+}$.
(6) $C_{\bar{K}}(\bar{u})^{\prime}=A_{1}{ }^{+} \times A_{2}{ }^{+}$.

We have $A_{1}{ }^{+} \cong S L_{2}^{\varepsilon}\left(q^{*}\right)$ by (4) and (5), and $\bar{u} \in Z\left(A_{1}^{+}\right)$by (1). It follows that $Z\left(A_{1}^{+}\right)=\langle\bar{u}\rangle$. By (1), $A_{1}{ }^{+} \cap A_{2}{ }^{+} \leq Z\left(A_{1}{ }^{+}\right)$and $\bar{u} \notin A_{1}{ }^{+} \cap A_{2}{ }^{+}$. It follows that $A_{1}{ }^{+} \cap A_{2}{ }^{+}=1$. So (1) implies that $C_{\bar{K}}(\bar{u})^{\prime}=A_{1}{ }^{+} \times A_{2}{ }^{+}$.
(7) Assume that $A_{1}{ }^{\circ}$ and $A_{2}{ }^{\circ}$ are normal subgroups of $C_{\bar{K}}(\bar{u})^{\prime}$ such that $C_{\bar{K}}(\bar{u})^{\prime}=A_{1}{ }^{\circ} \times A_{2}{ }^{\circ}$, $A_{1}{ }^{\circ} \cong S L_{2}^{\varepsilon}\left(q^{*}\right), A_{2}{ }^{\circ} \cong S L_{n-4}^{\varepsilon}\left(q^{*}\right)$ and $\bar{u} \in A_{1}{ }^{\circ}$. Then $A_{1}{ }^{\circ}=A_{1}{ }^{+}$and $A_{2}{ }^{\circ}=A_{2}{ }^{+}$.

Let $j \in\{1,2\}$. As a consequence of (4) and (5), $A_{j}{ }^{+}$is either quasisimple or isomorphic to $S L_{2}(3)$. In either case, it is easy to see that $A_{j}{ }^{+}$is indecomposable, i.e. $A_{j}{ }^{+}$cannot be written as an internal direct product of two proper normal subgroups. Moreover, $\left|A_{1}{ }^{+} /\left(A_{1}{ }^{+}\right)^{\prime}\right|$ and $\left|Z\left(A_{2}{ }^{+}\right)\right|$ as well as $\left|A_{2}{ }^{+} /\left(A_{2}{ }^{+}\right)^{\prime}\right|$ and $\left|Z\left(A_{1}{ }^{+}\right)\right|$are coprime. A consequence of the Krull-Remak-Schmidt theorem, namely [35, Kapitel I, Satz 12.6], implies that $\left\{A_{1}{ }^{+}, A_{2}{ }^{+}\right\}=\left\{A_{1}{ }^{\circ}, A_{2}{ }^{\circ}\right\}$. Since $\bar{u} \in A_{1}{ }^{+}$ and $\bar{u} \notin A_{2}{ }^{\circ}$, we have $A_{1}{ }^{+}=A_{1}{ }^{\circ}$ and $A_{2}{ }^{+}=A_{2}{ }^{\circ}$.
(8) The isomorphism $\varphi: \bar{K} \rightarrow H$ maps $A_{1}{ }^{+}$to the image of

$$
\left\{\left(\begin{array}{ll}
A & \\
& I_{n-4}
\end{array}\right): A \in S L_{2}^{\varepsilon}\left(q^{*}\right)\right\}
$$

in H and $\mathrm{A}_{2}{ }^{+}$to the image of

$$
\left\{\left(\begin{array}{ll}
I_{2} & \\
& B
\end{array}\right): B \in S L_{n-4}^{\varepsilon}\left(q^{*}\right)\right\}
$$

in $H$.
By (4) and (5), we have $i=2$. So the claim follows from the definitions of $A_{1}{ }^{+}$and $A_{2}{ }^{+}$.
From now on, $A_{1}{ }^{+}$and $A_{2}{ }^{+}$will always have the meanings given to them by Lemma 6.4.

Lemma 6.5. Let $C:=C_{G}(t)$ and $\bar{C}:=C / O(C)$. Then $A_{1}{ }^{+}$and $A_{2}{ }^{+}$are normal subgroups of $C_{\bar{C}}(\bar{u})$.

Proof. We have $C_{\bar{K}}(\bar{u}) \unlhd C_{\bar{C}}(\bar{u})$ as $\bar{K} \unlhd \bar{C}$. Thus $C_{\bar{K}}(\bar{u})^{\prime} \unlhd C_{\bar{C}}(\bar{u})$. Having this observed, the lemma is immediate from Lemma 6.4.

Let $C:=C_{G}(t)$ and $\bar{C}:=C / O(C)$. Next we introduce certain preimages of $A_{1}{ }^{+}$and $A_{2}{ }^{+}$in $C_{C}(u)$. By Corollary 2.2, we have $C_{\bar{C}}(\bar{u})=\overline{C_{C}(u)}$. We may see from Proposition 2.4 that there is a bijection from the set of 2-components of $C_{C}(u)$ to the set of 2-components of $\bar{C}_{\bar{C}}(\bar{u})$ sending each 2-component $A$ of $C_{C}(u)$ to $\bar{A}$.

Suppose that $q^{*} \neq 3$. Then $A_{1}{ }^{+}$is a component and hence a 2 -component of $C_{\bar{C}}(\bar{u})$. We use $A_{1}$ to denote the 2-component of $C_{C}(u)$ corresponding to $A_{1}{ }^{+}$under the bijection described above.

Suppose that $q^{*} \neq 3$ or $n \geq 7$. Then $A_{2}{ }^{+}$is a component and hence a 2 -component of $C_{\bar{C}}(\bar{u})$. We use $A_{2}$ to denote the 2-component of $C_{C}(u)$ corresponding to $A_{2}{ }^{+}$under the bijection described above.

Suppose that $q^{*}=3$. By Lemma 6.3 (ii), $O\left(C_{\bar{C}}(\bar{u})\right)=1$. So the factor group $C_{C}(u) /\left(C_{C}(u) \cap\right.$ $O(C))$ is core-free, whence $O\left(C_{C}(u)\right)=C_{C}(u) \cap O(C)$. Let $O\left(C_{C}(u)\right) \leq A_{1} \leq C_{C}(u)$ such that $A_{1} / O\left(C_{C}(u)\right)$ corresponds to $A_{1}{ }^{+}$under the natural group isomorphism $C_{C}(u) / O\left(C_{C}(u)\right) \rightarrow$ $C_{\bar{C}}(\bar{u})$. Furthermore, if $n=6$, let $O\left(C_{C}(u)\right) \leq A_{2} \leq C_{C}(u)$ such that $A_{2} / O\left(C_{C}(u)\right)$ corresponds to $A_{2}{ }^{+}$under the natural group isomorphism $C_{C}(u) / O\left(C_{C}(u)\right) \rightarrow C_{\bar{C}}(\bar{u})$.
Lemma 6.6. We have $T_{1} \leq A_{1}$ and $T_{2} \leq A_{2}$.
Proof. Let $i \in\{1,2\}$. Set $C:=C_{G}(t)$ and $\bar{C}:=C / O(C)$.
Let $C_{C}(u) \cap O(C) \leq \widetilde{A_{i}} \leq C_{C}(u)$ such that $\widetilde{A_{i}} /\left(C_{C}(u) \cap O(C)\right)$ corresponds to $A_{i}{ }^{+}$under the natural group isomorphism $C_{C}(u) /\left(C_{C}(u) \cap O(C)\right) \rightarrow C_{\bar{C}}(\bar{u})$. We have $T_{i} \leq C_{C}(u)$ and, by Lemma 6.4, $\overline{T_{i}} \leq A_{i}{ }^{+}$. Thus $T_{i} \leq \widetilde{A_{i}}$. If $A_{i}{ }^{+} \cong S L_{2}(3)$, then we have $A_{i}=\widetilde{A_{i}}$ and thus $T_{i} \leq A_{i}$. Assume now that $A_{i}{ }^{+}$is a component of $C_{\bar{C}}(\bar{u})$. Then $A_{i}$ is the 2-component of $C_{C}(u)$ associated to the 2-component $\widetilde{A}_{i} /\left(C_{C}(u) \cap O(C)\right)$ of $C_{C}(u) /\left(C_{C}(u) \cap O(C)\right)$. So, by Proposition 2.4. $A_{i}=O^{2^{\prime}}\left(\widetilde{A_{i}}\right)$, and hence $T_{i} \leq A_{i}$.

Lemma 6.7. There is an element $g \in G$ such that $T_{1}{ }^{g}=X_{2}$ and $X_{2}{ }^{g}=T_{1}$. For each such $g \in G$, we have $u^{g}=t$ and $t^{g}=u$.

Proof. The first statement easily follows from $\mathcal{F}_{S}(G)=\mathcal{F}_{S}\left(P S L_{n}(q)\right)$. By Lemma 3.12, the groups $T_{1}$ and $X_{2}$ are generalized quaternion. So $u$ is the only involution of $T_{1}$ and $t$ is the only involution of $X_{2}$. Thus $u^{g}=t$ and $t^{g}=u$ for any $g \in G$ with $T_{1}{ }^{g}=X_{2}$ and $X_{2}{ }^{g}=T_{1}$.

With the above lemmas at hand, we can now prove the following proposition.
Proposition 6.8. Take an element $g \in G$ such that $T_{1}{ }^{g}=X_{2}$ and $X_{2}{ }^{g}=T_{1}$. Set $C:=C_{G}(t)$ and $\bar{C}:=C / O(C)$. Let $L:=A_{1}{ }^{g}$. Then the following hold.
(i) $L \leq C_{C}(u)$.
(ii) $\bar{L}$ is subnormal in $\bar{C}$ and $\bar{L} \cong S L_{2}\left(q^{*}\right)$.
(iii) The subgroups $\bar{K}$ and $\bar{L}$ are the only subgroups of $\bar{C}$ which are components or solvable 2 -components of $\bar{C}$. In particular, $\bar{K}$ and $\bar{L}$ are normal subgroups of $\bar{C}$.
Proof. By Lemma 6.7, we have $t^{g}=u$ and $u^{g}=t$. Hence $C_{C}(u)^{g}=C_{C}(u)$. As $A_{1}$ is a subgroup of $C_{C}(u)$, we thus have $L=A_{1}{ }^{g} \leq C_{C}(u)$. So (i) holds.

Before proving (ii), we show that $C_{\bar{L}}(\bar{K})$ is a normal subgroup of $\bar{L}$ containing $\overline{X_{2}}$. Since $C_{\bar{C}}(\bar{K}) \unlhd \bar{C}$, we have $C_{\bar{L}}(\bar{K})=\bar{L} \cap C_{\bar{C}}(\bar{K}) \unlhd \bar{L}$. Because of Lemma 6.6, we have $X_{2}=T_{1}{ }^{g} \leq$ ${A_{1}}^{g}=L$. Thus $\overline{X_{2}} \leq \bar{L}$. By the definition of $X_{2}$ and by Lemma 6.2, we have $\overline{X_{2}} \leq C_{\bar{C}}(\bar{K})$. Thus $\overline{X_{2}} \leq C_{\bar{L}}(\bar{K})$.

Note that $\overline{X_{2}}$ is generalized quaternion by Lemma 3.12 and in particular nonabelian.
We now prove (ii) for the case $q^{*} \neq 3$. Then $A_{1}$ is a 2-component of $C_{C}(u)$. As $g$ normalizes $C_{C}(u)$ and $L=A_{1}^{g}$, it follows that $L$ is a 2-component of $C_{C}(u)$. So $\bar{L}$ is a 2-component of $C_{\bar{C}}(\bar{u})$. Moreover, we have $A_{1} / O\left(A_{1}\right) \cong S L_{2}\left(q^{*}\right)$ since $A_{1} /\left(A_{1} \cap O(C)\right) \cong \overline{A_{1}}=A_{1}{ }^{+} \cong S L_{2}\left(q^{*}\right)$. Hence $L / O(L)$ is isomorphic to $S L_{2}\left(q^{*}\right)$. The group $C_{\bar{L}}(\bar{K}) O(\bar{L}) / O(\bar{L})$ is normal in $\bar{L} / O(\bar{L})$, and it is nonabelian since $\overline{X_{2}} \leq C_{\bar{L}}(\bar{K})$. As $\bar{L} / O(\bar{L})$ is quasisimple, it follows that $C_{\bar{L}}(\bar{K}) O(\bar{L})=\bar{L}$. So $C_{\bar{L}}(\bar{K})$ has odd index in $\bar{L}$. Since $\bar{L}$ is a 2-component of $C_{\bar{C}}(\bar{u})$, we have $O^{2^{\prime}}(\bar{L})=\bar{L}$. It follows that $\bar{L}=C_{\bar{L}}(\bar{K}) \leq C_{\bar{C}}(\bar{K})$. Since $\bar{L}$ is subnormal in $C_{\bar{C}}(\bar{u})$ and $C_{\bar{C}}(\bar{K}) \leq C_{\bar{C}}(\bar{u})$, we have that $\bar{L}$ is subnormal in $C_{\bar{C}}(\bar{K})$. Hence $\bar{L}$ is subnormal in $\bar{C}$. As $\bar{C}$ is core-free, we have $O(\bar{L})=1$. It follows that $O(L)=L \cap O(C)$ and hence $\bar{L} \cong L / O(L) \cong S L_{2}\left(q^{*}\right)$. So we have proved (ii) for the case $q^{*} \neq 3$.

Assume now that $q^{*}=3$. Then $O\left(C_{C}(u)\right)=C_{C}(u) \cap O(C), O\left(C_{C}(u)\right) \leq A_{1} \leq C_{C}(u)$, and $A_{1} / O\left(C_{C}(u)\right)$ corresponds to $A_{1}{ }^{+} \cong S L_{2}(3)$ under the natural isomorphism $C_{C}(u) / O\left(C_{C}(u)\right) \rightarrow$ $C_{\bar{C}}(\bar{u})$. By Lemma 6.5. $A_{1}{ }^{+}$is normal in $C_{\bar{C}}(\bar{u})$. Hence, $A_{1} / O\left(C_{C}(u)\right)$ is a normal subgroup of $C_{C}(u) / O\left(C_{C}(u)\right)$ isomorphic to $S L_{2}(3)$. Since $g$ normalizes $C_{C}(u)$ and $L=A_{1}{ }^{g}$, it follows that $O\left(C_{C}(u)\right) \leq L$ and that $L / O\left(C_{C}(u)\right)$ is a normal subgroup of $C_{C}(u) / O\left(C_{C}(u)\right)$ isomorphic to $S L_{2}(3)$. Since $L / O\left(C_{C}(u)\right)$ corresponds to $\bar{L}$ under the natural isomorphism $C_{C}(u) / O\left(C_{C}(u)\right) \rightarrow$ $C_{\bar{C}}(\bar{u})$, it follows that $\bar{L}$ is a normal subgroup of $C_{\bar{C}}(\bar{u})$ isomorphic to $S L_{2}(3)$. Recall that $\overline{X_{2}} \leq$ $C_{\bar{L}}(\bar{K}) \unlhd \bar{L}$. As $\bar{L}$ has order 24 and $\overline{X_{2}}$ has order $8, C_{\bar{L}}(\bar{K})$ either equals $\bar{L}$ or has index 3 in $\bar{L}$. However, if the latter holds, then $\bar{L} C_{\bar{C}}(\bar{K}) / C_{\bar{C}}(\bar{K})$ is a normal subgroup of $C_{\bar{C}}(\bar{u}) / C_{\bar{C}}(\bar{K})$ of order 3, which is a contradiction to Lemma 6.3 (iii). Thus $\bar{L}=C_{\bar{L}}(\bar{K}) \leq C_{\bar{C}}(\bar{K})$. As $\bar{L} \unlhd C_{\bar{C}}(\bar{u})$ and $C_{\bar{C}}(\bar{K}) \leq C_{\bar{C}}(\bar{u})$, it follows that $\bar{L}$ is normal in $C_{\bar{C}}(\bar{K})$ and hence subnormal in $\bar{C}$. So we have proved (ii) for the case $q^{*}=3$.

We now prove (iii). Clearly, $\bar{T} \cap \bar{K}=\overline{X_{1}}$. Also $\bar{T} \cap \bar{L}=\overline{X_{2}}$ since $\left|\overline{X_{2}}\right|=\left|S L_{2}(q)\right|_{2}=$ $\left|S L_{2}\left(q^{*}\right)\right|_{2}=|\bar{L}|_{2}$ and $\overline{X_{2}} \leq \bar{L}$. As a consequence of Lemma 5.4, the fusion system $\mathcal{F}_{\bar{T}}(\bar{C}) /\left(\overline{X_{1}} \overline{X_{2}}\right)$ is nilpotent. Applying Lemma 2.17, we may conclude that $\bar{K}$ and $\bar{L}$ are the only subgroups of $\bar{C}$ which are components or solvable 2-components of $\bar{C}$. As $\bar{K}$ and $\bar{L}$ are not isomorphic, both are characteristic and hence normal in $\bar{C}$.

It is not difficult to observe that the definition of $L$ in Proposition 6.8 is independent of the choice of $g$. From now on, $L$ will always have the meaning given to it by the above proposition.
6.2. 2-components of centralizers of involutions conjugate to $t_{i}, i \neq 2$. Having described the components and the solvable 2-components of the group $C_{G}(t) / O\left(C_{G}(t)\right)$, we now turn our attention to centralizers of involutions of $G$ not conjugate to $t$.

First we recall some notation from Section 5. Let $1 \leq i<n$. If $i$ is even, then $t_{i}$ denotes the image of

$$
\left(\begin{array}{ll}
I_{n-i} & \\
& -I_{i}
\end{array}\right)
$$

in $P S L_{n}(q)$. We use $\rho$ to denote an element of $\mathbb{F}_{q}^{*}$ with order $(n, q-1)$, and if $\rho$ is a square in $\mathbb{F}_{q}$, then $\mu$ denotes an element of $\mathbb{F}_{q}^{*}$ with $\mu^{2}=\rho$. If $n$ is even, $\rho$ is a square in $\mathbb{F}_{q}$ and $i$ is odd, then $t_{i}$ is defined to be the image of

$$
\left(\begin{array}{ll}
\mu I_{n-i} & \\
& -\mu I_{i}
\end{array}\right) \in S L_{n}(q)
$$

in $P S L_{n}(q)$. It is easy to note that $t_{i}$ lies in $T$ and hence in $S$ whenever $t_{i}$ is defined.
Let $\mathcal{S}$ denote the set of all subgroups $E$ of $P S L_{n}(q)$ such that there is some elementary abelian 2-subgroup $\widetilde{E} \leq S L_{n}(q)$ with $E=\widetilde{E} Z\left(S L_{n}(q)\right) / Z\left(S L_{n}(q)\right)$. For each $3 \leq i \leq n$, we define $\mathcal{S}_{i}$ to be the set of all elements $E$ of $\mathcal{S}$ such that $E$ contains a $P S L_{n}(q)$-conjugate of $t_{j}$ for some even $2 \leq j<i$.

Lemma 6.9. Let $1 \leq i<n$ such that $t_{i}$ is defined. Assume that $i \neq 2$, and that $i \leq \frac{n}{2}$ if $n$ is even. Let $P$ be a Sylow 2-subgroup of $C_{P S L_{n}(q)}\left(t_{i}\right)$ and $\mathcal{F}:=\mathcal{F}_{P}\left(C_{P S L_{n}(q)}\left(t_{i}\right)\right)$. Then the following hold.
(i) Assume that $i \notin\{1, n-1\}$. Then $\mathcal{F}$ has precisely two components. Denoting them in a suitable way by $\mathcal{E}_{1}$ and $\mathcal{E}_{2}$, the following hold.
(a) $\mathcal{E}_{1}$ is isomorphic to the 2 -fusion system of $S L_{n-i}(q)$.
(b) $\mathcal{E}_{2}$ is isomorphic to the 2-fusion system of $S L_{i}(q)$.
(c) Let $Y_{1}$ be the Sylow group of $\mathcal{E}_{1}$ and let $Y_{2}$ be the Sylow group of $\mathcal{E}_{2}$. Then $Y_{1} Y_{2}$ is normal in $P$ and $\mathcal{F} / Y_{1} Y_{2}$ is nilpotent. The group $Y_{i}$, where $i \in\{1,2\}$, contains a PSL $L_{n}(q)$-conjugate of $t$. Moreover, any elementary abelian subgroup of $Y_{1}$ of rank at least 2 is contained in $\mathcal{S}_{n-i}$, and any elementary abelian subgroup of $Y_{2}$ of rank at least 2 is contained in $\mathcal{S}_{i}$.
(ii) Assume that $i=1$ or $i=n-1$. Then $\mathcal{F}$ has a unique component. This component is isomorphic to the 2-fusion system of $S L_{n-1}(q)$. If $Y$ is its Sylow group, then $Y \unlhd P$ and $\mathcal{F} / Y$ is nilpotent. Moreover, any elementary abelian subgroup of $Y$ of rank at least 2 is contained in $\mathcal{S}_{n-1}$.

Proof. Assume that $i \notin\{1, n-1\}$. By hypothesis, we have $i \neq 2$, and $i \leq \frac{n}{2}$ if $n$ is even. It follows that $i \geq 3$ and $n-i \geq 3$. Let $J_{1}$ be the image of

$$
\left\{\left(\begin{array}{cc}
A & \\
& I_{i}
\end{array}\right): A \in S L_{n-i}(q)\right\}
$$

in $P S L_{n}(q)$, and let $J_{2}$ be the image of

$$
\left\{\left(\begin{array}{cc}
I_{n-i} & \\
& A
\end{array}\right): A \in S L_{i}(q)\right\}
$$

in $\operatorname{PS} L_{n}(q)$. It is easy to note that $J_{1}$ and $J_{2}$ are the only 2-components of $C_{P S L_{n}(q)}\left(t_{i}\right)$. Applying Proposition 2.16 and Lemma 3.21 , we may conclude that $\mathcal{E}_{1}:=\mathcal{F}_{P \cap J_{1}}\left(J_{1}\right)$ and $\mathcal{E}_{2}:=\mathcal{F}_{P \cap J_{2}}\left(J_{2}\right)$ are the only components of $\mathcal{F}=\mathcal{F}_{P}\left(C_{P S L_{n}(q)}\left(t_{i}\right)\right)$. Clearly, $\mathcal{E}_{1}$ is isomorphic to the 2 -fusion system of $S L_{n-i}(q)$, while $\mathcal{E}_{2}$ is isomorphic to the 2-fusion system of $S L_{i}(q)$. Set $Y_{1}:=P \cap J_{1}$ and $Y_{2}:=P \cap J_{2}$. It is easy to note that $Y_{1} Y_{2}=P \cap J_{1} J_{2}$. As $J_{1} J_{2} \unlhd C_{P S L_{n}(q)}\left(t_{i}\right)$, it follows that $Y_{1} Y_{2} \unlhd P$. By Lemma 2.11, $\mathcal{F} / Y_{1} Y_{2}$ is isomorphic to the 2-fusion system of $C_{P S L_{n}(q)}\left(t_{i}\right) / J_{1} J_{2}$, and it is easy to note that $C_{P S L_{n}(q)}\left(t_{i}\right) / J_{1} J_{2}$ is 2-nilpotent. So $\mathcal{F} / Y_{1} Y_{2}$ is nilpotent by [39, Theorem 1.4]. It is clear from the definitions of $J_{1}$ and $J_{2}$ that both $J_{1}$ and $J_{2}$ contain a $P S L_{n}(q)$-conjugate of $t$. Hence $Y_{k}$ has an element which is $P S L_{n}(q)$-conjugate to $t$ for $k \in\{1,2\}$. Clearly, any elementary abelian 2-subgroup of $J_{k}, k \in\{1,2\}$, lies in $\mathcal{S}$. Moreover, any noncentral involution of $J_{1}$ is $P S L_{n}(q)$-conjugate to $t_{j}$ for some even $2 \leq j<n-i$, and any noncentral involution of $J_{2}$ is $P S L_{n}(q)$-conjugate to $t_{j}$ for some even $2 \leq j<i$. This implies that any elementary abelian subgroup of $Y_{1}$ of rank at least 2 is contained in $\mathcal{S}_{n-i}$, and that any elementary abelian subgroup of $Y_{2}$ of rank at least 2 is contained in $\mathcal{S}_{i}$. This completes the proof of (i).

We omit the proof of (ii) since it is very similar to the one of (i).
Proposition 6.10. Let $1 \leq i<n$ such that $t_{i}$ is defined. Assume that $i \notin\{1,2, n-1\}$, and that $i \leq \frac{n}{2}$ if $n$ is even. Let $x$ be an involution of $S$ which is $G$-conjugate to $t_{i}$. Then $C_{G}(x)$ has precisely two 2-components. Denoting them in a suitable way by $J_{1}$ and $J_{2}$, the following hold.
(i) $J_{1} / O\left(J_{1}\right)$ is isomorphic to $S L_{n-i}^{\varepsilon}\left(q^{*}\right) / O\left(S L_{n-i}^{\varepsilon}\left(q^{*}\right)\right)$, where $\varepsilon$ and $q^{*}$ are as in Proposition 6.1 .
(ii) $J_{2} / O\left(J_{2}\right) \cong S L_{i}^{\varepsilon}\left(q^{*}\right) / O\left(S L_{i}^{\varepsilon}\left(q^{*}\right)\right)$, where $\varepsilon$ and $q^{*}$ are as in Proposition 6.1.
(iii) Any elementary abelian 2-subgroup of $J_{1}$ of rank at least 2 is $G$-conjugate to a subgroup of $S$ lying in $\mathcal{S}_{n-i}$, and any elementary abelian 2-subgroup of $J_{2}$ of rank at least 2 is $G$-conjugate to a subgroup of $S$ lying in $\mathcal{S}_{i}$.

Proof. Let $\mathcal{F}:=\mathcal{F}_{S}(G)=\mathcal{F}_{S}\left(P S L_{n}(q)\right)$. It suffices to prove the proposition under the assumption that $\langle x\rangle$ is fully $\mathcal{F}$-centralized, and we will assume that this is the case. So we have $C_{S}(x) \in \operatorname{Syl}_{2}\left(C_{G}(x)\right)$ and $C_{S}(x) \in \operatorname{Syl}_{2}\left(C_{P S L_{n}(q)}(x)\right)$. Also, $\mathcal{F}_{C_{S}(x)}\left(C_{G}(x)\right)=C_{\mathcal{F}}(\langle x\rangle)=$ $\mathcal{F}_{C_{S}(x)}\left(C_{P S L_{n}(q)}(x)\right)$.

Clearly, $x$ is $P S L_{n}(q)$-conjugate to $t_{i}$. So Lemma 6.9 (i) shows together with Lemma 5.10 (i) that there exist two distinct 2-components $J_{1}$ and $J_{2}$ of $C_{G}(x)$ satisfying the following conditions, where $Y_{1}:=C_{S}(x) \cap J_{1}$ and $Y_{2}:=C_{S}(x) \cap J_{2}$.
(1) $\mathcal{F}_{Y_{1}}\left(J_{1}\right)$ is isomorphic to the 2 -fusion system of $S L_{n-i}(q)$.
(2) $\mathcal{F}_{Y_{2}}\left(J_{2}\right)$ is isomorphic to the 2 -fusion system of $S L_{i}(q)$.
(3) $Y_{1} Y_{2}$ is normal in $C_{S}(x)$, and $C_{\mathcal{F}}(\langle x\rangle) / Y_{1} Y_{2}$ is nilpotent.
(4) For $k \in\{1,2\}, Y_{k}$ contains a $G$-conjugate of $t$.
(5) Any elementary abelian abelian subgroup of $Y_{1}$ of rank at least 2 lies in $\mathcal{S}_{n-i}$, and any elementary abelian subgroup of $Y_{2}$ of rank at least 2 lies in $\mathcal{S}_{i}$.
By (3) and Corollary 2.18, $J_{1}$ and $J_{2}$ are the only 2-components of $C_{G}(x)$. It remains to show that $J_{1}$ and $J_{2}$ satisfy (i)-(iii). As $Y_{k} \in \operatorname{Syl}_{2}\left(J_{k}\right)$ for $k \in\{1,2\}$, (5) implies (iii).

We now prove (ii). The proof of (i) will be omitted since it is very similar to the proof of (ii).
Let $s$ be an element of $J_{1}$ which is $G$-conjugate to $t$. Set $C:=C_{G}(s), \widehat{C}:=C / O(C)$ and $\overline{C_{G}(x)}:=C_{G}(x) / O\left(C_{G}(x)\right)$.

Since $\overline{J_{1}}$ and $\overline{J_{2}}$ are distinct components of $\overline{C_{G}(x)}$, we have $\left[\overline{J_{1}}, \overline{J_{2}}\right]=1$ by [37, 6.5.3]. As $\bar{s} \in \overline{J_{1}}$, it follows that $\overline{J_{2}}$ is a component of $C_{\overline{C_{G}(x)}}(\bar{s})$. As a consequence of Corollary 2.2 and Proposition 2.4. $C_{G}(x) \cap C$ has a 2-component $H$ with $\bar{H}=\overline{J_{2}}$.

By assumption, $s$ is $G$-conjugate to $t$. So, by Proposition 6.8, $\widehat{C}$ has a unique normal subgroup $K^{+}$isomorphic to $S L_{n-2}^{\varepsilon}\left(q^{*}\right) / O\left(S L_{n-2}^{\varepsilon}\left(q^{*}\right)\right)$ and a unique normal subgroup $L^{+}$isomorphic to $S L_{2}\left(q^{*}\right)$. Moreover, $K^{+}$and $L^{+}$are the only subgroups of $\widehat{C}$ which are components or solvable 2-components of $\widehat{C}$.

Clearly, $\widehat{H}$ is a 2-component of $C_{\widehat{C}}(\widehat{x})$. Lemma 2.5 implies that $\widehat{H}$ is a 2-component of $C_{K^{+}}(\widehat{x})$ or of $C_{L^{+}}(\widehat{x})$. By Corollary 3.46 (i), we even have that $\widehat{H}$ is a component of $C_{K^{+}}(\widehat{x})$ or $C_{L^{+}}(\widehat{x})$. It is easy to note that $\widehat{H} / Z(\widehat{H}) \cong H / Z^{*}(H) \cong \overline{J_{2}} / Z\left(\overline{J_{2}}\right)$. By Corollary 3.46 (ii), we have $\widehat{H} / Z(\widehat{H}) \not \approx$ $M_{11}$, and so $\overline{J_{2}} / Z\left(\overline{J_{2}}\right) \not \models M_{11}$. Now (2) and Lemma 5.10 (ii) imply that $\overline{J_{2}} \cong S L_{i}^{\varepsilon_{0}}\left(q_{0}\right) / O\left(S L_{i}^{\varepsilon_{0}}\left(q_{0}\right)\right)$ for some nontrivial odd prime power $q_{0}$ and some $\varepsilon_{0} \in\{+,-\}$ with $q \sim \varepsilon_{0} q_{0}$. Hence $\widehat{H} / Z(\widehat{H}) \cong$ $\overline{J_{2}} / Z\left(\overline{J_{2}}\right) \cong P S L_{i}^{\varepsilon_{0}}\left(q_{0}\right)$. Note that $\varepsilon q^{*} \sim q \sim \varepsilon_{0} q_{0}$ and in particular $\left(q^{* 2}-1\right)_{2}=\left(q_{0}^{2}-1\right)_{2}$. Applying Corollary 3.46 (iii), we may conclude that $q_{0}=q^{*}$ and $\varepsilon_{0}=\varepsilon$. Consequently, we have $J_{2} / O\left(J_{2}\right) \cong S L_{i}^{\varepsilon}\left(q^{*}\right) / O\left(S L_{i}^{\varepsilon}\left(q^{*}\right)\right)$. So we have proved (ii).

The proof of the following proposition runs along the same lines as that of the previous result.
Proposition 6.11. Suppose that $n$ is odd and $i=n-1$, or that $n$ is even, $i=1$ and $t_{1}$ is defined. Let $x$ be an involution of $S$ which is $G$-conjugate to $t_{i}$. Then $C_{G}(x)$ has precisely one 2-component $J$. We have $J / O(J) \cong S L_{n-1}^{\varepsilon}\left(q^{*}\right) / O\left(S L_{n-1}^{\varepsilon}\left(q^{*}\right)\right)$, where $\varepsilon$ and $q^{*}$ are as in Proposition 6.1. Moreover, any elementary abelian 2-subgroup of $J$ of rank at least 2 is $G$-conjugate to a subgroup of $S$ lying in $\mathcal{S}_{n-1}$.

Proof. Let $\mathcal{F}:=\mathcal{F}_{S}(G)=\mathcal{F}_{S}\left(P S L_{n}(q)\right)$. It suffices to prove the proposition under the assumption that $\langle x\rangle$ is fully $\mathcal{F}$-centralized, and we will assume that this is the case. So we have $C_{S}(x) \in \operatorname{Syl}_{2}\left(C_{G}(x)\right)$ and $C_{S}(x) \in \operatorname{Syl}_{2}\left(C_{P S L_{n}(q)}(x)\right)$. Also, $\mathcal{F}_{C_{S}(x)}\left(C_{G}(x)\right)=C_{\mathcal{F}}(\langle x\rangle)=$ $\mathcal{F}_{C_{S}(x)}\left(C_{P S L_{n}(q)}(x)\right)$.

Clearly, $x$ is $P S L_{n}(q)$-conjugate to $t_{i}$. Lemma 6.9 (ii) implies that $C_{\mathcal{F}}(\langle x\rangle)$ has a unique component $\mathcal{E}$, and that $\mathcal{E}$ is isomorphic to the 2 -fusion system of $S L_{n-1}(q)$. Applying Lemma 5.10 (i), we may conclude that $C_{G}(x)$ has a unique 2-component $J$ with $\mathcal{E}=\mathcal{F}_{C_{S}(x) \cap J}(J)$. By

Lemma $5.10\left(\right.$ ii),$J / O(J) \cong S L_{n-1}^{\varepsilon_{0}}\left(q_{0}\right) / O\left(S L_{n-1}^{\varepsilon_{0}}\left(q_{0}\right)\right)$ for some nontrivial odd prime power $q_{0}$ and some $\varepsilon_{0} \in\{+,-\}$ with $\varepsilon_{0} q_{0} \sim q$.

Set $Y:=C_{S}(x) \cap J$. By Lemma 6.9 (ii), $Y \unlhd C_{S}(x)$ and $C_{\mathcal{F}}(\langle x\rangle) / Y$ is nilpotent. Applying Corollary 2.18, we may conclude that $J$ is the only 2 -component of $C_{G}(x)$. Using Lemma 6.9 (ii), we see that any elementary abelian subgroup of $Y$ of rank at least 2 lies in $\mathcal{S}_{n-1}$. As $Y \in \operatorname{Syl}_{2}(J)$, it follows that any elementary abelian 2 -subgroup of $J$ of rank at least 2 is $G$-conjugate to a subgroup of $S$ lying in $\mathcal{S}_{n-1}$.

It remains to show that $\varepsilon_{0}=\varepsilon$ and $q_{0}=q^{*}$. Define $s:=t_{i}$ if $i=1$ and $s:=t_{A}$, where $A:=\{1, \ldots, n-1\}$, if $i=n-1$. Then we have $s \in C_{G}(t)$, and $s$ is $G$-conjugate to $x$. Set $\overline{C_{G}(t)}:=C_{G}(t) / O\left(C_{G}(t)\right)$. Lemma 6.2 shows that $\bar{s}$ centralizes $\bar{K}$. Hence, $\bar{K}$ is a component of $C_{\overline{C_{G}}(t)}(\bar{s})$. As a consequence of Corollary 2.2 and Proposition 2.4, $C_{G}(t) \cap C_{G}(s)$ has a 2component $H$ with $\bar{H}=\bar{K}$. Set $C:=C_{G}(s)$ and $\widehat{C}:=C / O(C)$. Then $\widehat{H}$ is a 2-component of $C_{\widehat{C}}(\widehat{t})$. Since $s$ is $G$-conjugate to $x, \widehat{C}$ has precisely one component $J^{+}$, and $J^{+}$is isomorphic to $S L_{n-1}^{\varepsilon_{0}}\left(q_{0}\right) / O\left(S L_{n-1}^{\varepsilon_{0}}\left(q_{0}\right)\right)$. By Lemma 2.5, $\widehat{H}$ is a 2-component of $C_{J^{+}}(\widehat{t})$. We see from Corollary 3.46 (i) that $\widehat{H}$ is in fact a component of $C_{J^{+}}(\widehat{t})$. It is easy to see that $\widehat{H} / Z(\widehat{H}) \cong H / Z^{*}(H) \cong$ $\bar{K} / Z(\bar{K}) \cong P S L_{n-2}^{\varepsilon}\left(q^{*}\right)$. Note that $\varepsilon_{0} q_{0} \sim q \sim \varepsilon q^{*}$ and in particular $\left(q_{0}^{2}-1\right)_{2}=\left(q^{* 2}-1\right)_{2}$. Using this, we may deduce from Corollary 3.46 (iii) that $q_{0}=q^{*}$ and $\varepsilon_{0}=\varepsilon$.
6.3. 2-components of centralizers of involutions conjugate to $w$. Recall that we assume $\rho$ to be an element of $\mathbb{F}_{q}^{*}$ with order $(n, q-1)$. Recall moreover that if $n$ is even and $\rho$ is a non-square element of $\mathbb{F}_{q}$, then $\stackrel{\rightharpoonup}{w}$ denotes the matrix

$$
\left(\begin{array}{ll} 
& I_{n / 2} \\
\rho I_{n / 2} &
\end{array}\right.
$$

and, if $\widetilde{w} \in S L_{n}(q)$, then $w$ denotes its image in $P S L_{n}(q)$.
Lemma 6.12. Suppose that $w$ is defined. Let $P$ be a Sylow 2-subgroup of $C_{P S L_{n}(q)}(w)$, and let $\mathcal{F}$ denote the fusion system $\left.\mathcal{F}_{P}\left(C_{P S L_{n}(q)}\right)(w)\right)$. Then $\mathcal{F}$ has precisely one component. This component is isomorphic to the 2-fusion system of a nontrivial quotient of $S L_{\frac{n}{2}}\left(q^{2}\right)$. If $Y$ is its Sylow subgroup, then we have $Y \unlhd P$, and $\mathcal{F} / Y$ is nilpotent.

Proof. By Lemma 3.6 (i), $C_{P S L_{n}(q)}(w)$ has precisely one 2-component $J$, and $J$ is isomorphic to a nontrivial quotient of $S L_{\frac{n}{2}}\left(q^{2}\right)$. Applying Proposition 2.16 and Lemma 3.21, we may conclude that $\mathcal{F}_{P \cap J}(J)$ is the only component of $\mathcal{F}$. The last statement of the lemma is given by Lemma 3.6 (ii).

Proposition 6.13. Suppose that $w$ is defined. Let $x$ be an involution of $S$ which is $P S L_{n}(q)$ conjugate to $w$. Then $C_{G}(x)$ has precisely one 2-component, say $J$. The group $J / O(J)$ is isomorphic to a nontrivial quotient of $S L_{\frac{n}{2}}^{\varepsilon_{0}}\left(q_{0}\right)$ for some nontrivial odd prime power $q_{0}$ and some $\varepsilon_{0} \in\{+,-\}$ with $q^{2} \sim \varepsilon_{0} q_{0}$.

Proof. Let $\mathcal{F}:=\mathcal{F}_{S}(G)=\mathcal{F}_{S}\left(P S L_{n}(q)\right)$. It suffices to prove the proposition under the assumption that $\langle x\rangle$ is fully $\mathcal{F}$-centralized, and we will assume that this is the case. So we have $C_{S}(x) \in \operatorname{Syl}_{2}\left(C_{G}(x)\right)$ and $C_{S}(x) \in \operatorname{Syl}_{2}\left(C_{P S L_{n}(q)}(x)\right)$. Also, $\mathcal{F}_{C_{S}(x)}\left(C_{G}(x)\right)=C_{\mathcal{F}}(\langle x\rangle)=$ $\mathcal{F}_{C_{S}(x)}\left(C_{P S L_{n}(q)}(x)\right)$.

As $x$ is $P S L_{n}(q)$-conjugate to $w$, Lemma 6.12 implies that $C_{\mathcal{F}}(\langle x\rangle)$ has precisely one component, say $\mathcal{E}$, and that $\mathcal{E}$ is isomorphic to the 2 -fusion system of a nontrivial quotient of $S L_{\frac{n}{2}}\left(q^{2}\right)$. By Lemma 5.10 (i), $C_{G}(x)$ has a unique 2-component $J$ such that $\mathcal{E}=\mathcal{F}_{C_{S}(x) \cap J}(J)$. Set $Y:=$ $C_{S}(x) \cap J$. As a consequence of Lemma 6.12, we have $Y \unlhd C_{S}(x)$, and the factor system $C_{\mathcal{F}}(\langle x\rangle) / Y$ is nilpotent. So, by Corollary 2.18, $J$ is the only 2 -component of $C_{G}(x)$. Lemma 5.10 (iii) shows
that $J / O(J)$ is isomorphic to a nontrivial quotient of $S L_{\frac{n}{2}}^{\varepsilon_{0}}\left(q_{0}\right)$ for some nontrivial odd prime power $q_{0}$ and some $\varepsilon_{0} \in\{+,-\}$ with $q^{2} \sim \varepsilon_{0} q_{0}$.

## 7. The components of $C_{G}(t)$

The goal of this section is to determine the isomorphism types of $K$ and $L$. In order to do so, we will apply the signalizer functor techniques introduced by Gorenstein and Walter in [32]. In particular, we will see that $L$ is isomorphic to $S L_{2}\left(q^{*}\right)$. This will enable us in Section 8 to prove that a certain collection of conjugates of $L$ generates a subgroup $G_{0}$ of $G$ which is isomorphic to a nontrivial quotient of $S L_{n}^{\varepsilon}\left(q^{*}\right)$ and normal in $G$. This will complete the proof of Theorem 5.2.
7.1. 3-generation of involution centralizers. For each $3 \leq i \leq n$, we define $\mathcal{U}_{i}$ to be the set of all subgroups $U$ of $P S L_{n}(q)$ such that $U$ has a subgroup $E$ with $E \in \mathcal{S}_{i}$ and $m(E) \geq 3$. The following lemma will be important later in this section.

Lemma 7.1. Let $1 \leq i<n$ such that $t_{i}$ is defined. Suppose that $i \leq \frac{n}{2}$ if $n$ is even. Let $x$ be an involution of $S$ such that $x$ is $G$-conjugate to $t_{i}$ and such that $\langle x\rangle$ is fully $\mathcal{F}_{S}(G)$-centralized. Then $C_{G}(x)$ is 3-generated in the sense of Definition 3.35. Moreover, if $i \geq 4$, then we have

$$
C_{G}(x)=\left\langle N_{C_{G}(x)}(U) \mid U \leq C_{S}(x), U \in \mathcal{U}_{i}\right\rangle .
$$

If $i=2$, then we have

$$
C_{G}(x)=\left\langle N_{C_{G}(x)}(U) \mid U \leq C_{S}(x), U \in \mathcal{U}_{n-2}\right\rangle .
$$

Proof. Set $C:=C_{G}(x)$ and $\bar{C}:=C / O(C)$. Recall that $L_{2^{\prime}}(C)$ denotes the subgroup of $C$ generated by the 2-components of $C$ and that $L(\bar{C})$ denotes the product of all components of $\bar{C}$. Clearly, $\overline{L_{2^{\prime}}(C)}=L(\bar{C})$.

First we consider the case $(n, i) \neq(6,3)$. Then, by Propositions 6.1, 6.10 and 6.11, $C$ has a 2-component $J$ such that $\bar{J} \cong S L_{k}^{\varepsilon}\left(q^{*}\right) / O\left(S L_{k}^{\varepsilon}\left(q^{*}\right)\right)$ for some $k \geq 4$ and such that any elementary abelian subgroup of $Y:=C_{S}(x) \cap J$ of rank at least 2 lies in $\mathcal{S}_{k}$. If $i \geq 4$, then we may assume that $k=i$, and if $i=2$, then $k=n-2$.

Clearly, $Y \in \operatorname{Syl}_{2}(J)$. By Lemma 3.37, we have that $\bar{J}$ is 3 -generated. So we have

$$
\bar{J}=\left\langle N_{\bar{J}}(\bar{U}) \mid U \leq Y, m(U) \geq 3\right\rangle
$$

Set $X:=C_{S}(x) \cap L_{2^{\prime}}(C)$. By the Frattini argument, $L(\bar{C})=\bar{J} N_{L(\bar{C})}(\bar{Y})$ and $\bar{C}=L(\bar{C}) N_{\bar{C}}(\bar{X})$. It follows that

$$
\left.\bar{C}=\left\langle N_{\bar{C}}(\bar{U})\right| U=X, \text { or } U \leq Y \text { and } m(U) \geq 3\right\rangle .
$$

Lemma 2.1 implies that $C$ is generated by $O(C)$ together with the normalizers $N_{C}(U)$, where $U=X$, or $U \leq Y$ and $m(U) \geq 3$.

Let $E$ denote the subgroup of $S$ generated by $t, t_{\{n-2, n-1\}}, t_{\{n-3, n-2\}}$ and $t_{\{n-4, n-3\}}$. Clearly, $E \cong E_{16}$. Since $x$ is $G$-conjugate to $t_{i}$ and $E \leq C_{G}\left(t_{i}\right)$, there is a subgroup $E_{x}$ of $C_{S}(x)$ which is $G$-conjugate to $E$. By [28, Proposition 11.23], we have

$$
O(C)=\left\langle C_{O(C)}(D) \mid D \leq E_{x}, D \cong E_{8}\right\rangle .
$$

As remarked above, any elementary abelian subgroup of $Y$ of rank at least 2 lies in $\mathcal{S}_{k}$. So, if $U \leq Y$ and $m(U) \geq 3$, then $U \in \mathcal{U}_{k}$. Also $X \in \mathcal{U}_{k}$. Clearly, any $E_{8}$-subgroup of $E_{x}$ lies in $\mathcal{S}_{k}$ and hence in $\mathcal{U}_{k}$. We therefore have

$$
C=\left\langle N_{C}(U) \mid U \leq C_{S}(x), U \in \mathcal{U}_{k}\right\rangle
$$

Consequently, $C$ is 3 -generated, and the last two statements of the lemma are satisfied.

Suppose now that $(n, i)=(6,3)$. By Proposition 6.10 , $C$ has precisely two 2 -components $J_{1}$ and $J_{2}$, and we have $\overline{J_{1}} \cong P S L_{3}^{\varepsilon}\left(q^{*}\right) \cong \overline{J_{2}}$. Set $Y_{1}:=C_{S}(x) \cap J_{1}$ and $Y_{2}:=C_{S}(x) \cap J_{2}$. Since $\overline{J_{1}}$ is 2-generated by Lemma 3.36, we have

$$
\overline{J_{1}}=\left\langle N_{\overline{J_{1}}}(\bar{U}) \mid U \leq Y_{1}, m(U) \geq 2\right\rangle
$$

Let $y$ be an involution of $Y_{2}$. We have $\left[\overline{J_{1}}, \overline{J_{2}}\right]=1$ by [37, 6.5.3], and so $\bar{y}$ centralizes $\overline{J_{1}}$. As $Z\left(\overline{J_{1}}\right)=1$, we have $\bar{y} \notin \overline{J_{1}}$. Now let $U \leq Y_{1}$ with $m(U) \geq 2$. Then $\langle\bar{U}, \bar{y}\rangle$ has rank at least 3 . Moreover, it is clear that $N_{\overline{J_{1}}}(\bar{U})$ normalizes $\langle\bar{U}, \bar{y}\rangle$. Thus

$$
\overline{J_{1}}=\left\langle N_{\overline{J_{1}}}(\bar{U}) \mid U \leq Y_{1} Y_{2}, m(U) \geq 3\right\rangle
$$

Interchanging the roles of $J_{1}$ and $J_{2}$, we also see that

$$
\overline{J_{2}}=\left\langle N_{\overline{J_{2}}}(\bar{U}) \mid U \leq Y_{1} Y_{2}, m(U) \geq 3\right\rangle
$$

By the Frattini argument, $\bar{C}=\overline{J_{1}} \overline{J_{2}} N_{\bar{C}}\left(\overline{Y_{1}} \overline{Y_{2}}\right)$. Lemma 2.1 implies that $C$ is generated by $O(C)$ together with the normalizers $N_{C}(U)$, where $U \leq Y_{1} Y_{2}$ and $m(U) \geq 3$. For any $E_{16}$-subgroup $A$ of $C_{S}(x)$, we have

$$
O(C)=\left\langle C_{O(C)}(B) \mid B \leq A, B \cong E_{8}\right\rangle
$$

by [28, Proposition 11.23]. It follows that $C$ is 3 -generated. The proof is now complete.
Lemma 7.2. Suppose that $w$ is defined. Let $x$ be an involution of $S$ which is $P S L_{n}(q)$-conjugate to $w$. Then $C_{G}(x)$ is 3-generated.
Proof. Set $C:=C_{G}(x)$ and $\bar{C}:=C / O(C)$. By Proposition 6.13, $C$ has a unique 2-component $J$, and $\bar{J}$ is isomorphic to a nontrivial quotient of $S L_{\frac{n}{2}}^{\varepsilon_{0}}\left(q_{0}\right)$ for some nontrivial odd prime power $q_{0}$ and some $\varepsilon_{0} \in\{+,-\}$ with $q^{2} \sim \varepsilon_{0} q_{0}$. Note that $q_{0} \equiv \varepsilon_{0} \bmod 8$.

First we prove that $\bar{C}$ is 3 -generated. Let $R$ be a Sylow 2 -subgroup of $C$ and $Y:=R \cap J$. We consider two cases.

Case 1: $n \geq 8$.
By Lemma 3.37, $\bar{J}$ is 3-generated. Hence

$$
\bar{J}=\left\langle N_{\bar{J}}(\bar{U}) \mid U \leq Y, m(U) \geq 3\right\rangle
$$

By the Frattini argument, $\bar{C}=\bar{J} N_{\bar{C}}(\bar{Y})$. So $\bar{C}$ is 3-generated.
Case 2: $n=6$.
We have $\bar{J} \cong P S L_{3}^{\varepsilon_{0}}\left(q_{0}\right)$. By Lemma 3.36, $\bar{J}$ is 2-generated. Applying the Frattini argument, we may conclude that

$$
\bar{C}=\left\langle N_{\bar{C}}(\bar{U}) \mid U \leq Y, m(U) \geq 2\right\rangle
$$

Now let $U \leq Y$ with $m(U) \geq 2$. Since $\bar{x}$ is a central involution of $\bar{C}$ and $Z(\bar{J})$ is trivial, we have $\bar{x} \notin \bar{J}$ and hence $\bar{x} \notin \bar{U}$. It follows $\langle\bar{U}, \bar{x}\rangle$ has rank at least 3 . Moreover, as $\bar{x}$ is central in $\bar{C}$, we have $N_{\bar{C}}(\bar{U}) \leq N_{\bar{C}}(\langle\bar{U}, \bar{x}\rangle)$. Clearly, $\langle\bar{U}, \bar{x}\rangle \leq \bar{R}$. It follows that

$$
\bar{C}=\left\langle N_{\bar{C}}(\bar{U}) \mid U \leq R, m(U) \geq 3\right\rangle
$$

Hence $\bar{C}$ is 3 -generated.
Applying Lemma 2.1, we may conclude that $C$ is generated by $O(C)$ together with the normalizers $N_{C}(U)$, where $U \leq R$ and $m(U) \geq 3$. By Lemma 3.6 (iii), $R$ has an elementary abelian 2-subgroup of rank 4, say $A$. By [28, Proposition 11.23], we have

$$
O(C)=\left\langle C_{O(C)}(B) \mid B \leq A, B \cong E_{8}\right\rangle
$$

So $C$ is 3-generated.
Corollary 7.3. Let $x$ be an involution of $S$. Then $C_{G}(x)$ is 3-generated.

Proof. As a consequence of Proposition [3.5, $x$ is $G$-conjugate to $t_{i}$ for some $1 \leq i<n$ such that $t_{i}$ is defined or $P S L_{n}(q)$-conjugate to $w$ (if defined). So the statement follows from Lemmas 7.1 and 7.2 .
7.2. The case $q^{*}=3$. Recall that our goal is to determine the isomorphism types of $K$ and $L$. First we will deal with the case $q^{*}=3$. We will prove that, in this case, $O\left(C_{G}(t)\right)=1$.

Lemma 7.4. Let $x$ be an involution of $S$, and let $J$ be a 2-component of $C_{G}(x)$. Let $1 \leq i<n$ such that $t_{i}$ is defined. Suppose that $q^{*}=3$ and that $x$ is $G$-conjugate to $t_{i}$. Then $J / O(J)$ is locally balanced.
Proof. By Propositions 6.8 (iii), 6.10 and 6.11, we have $J / O(J) \cong S L_{k}^{\varepsilon}(3)$ for some $3 \leq k<n$. So $J / O(J)$ is locally balanced by Lemma 3.47.
Lemma 7.5. Let $P$ and $Q$ be subgroups of $S$.
(i) If $P \in \mathcal{S}$ and $m(P) \leq 2$, then there is a subgroup $\bar{P}$ of $S$ such that $P<\bar{P}, \bar{P} \in \mathcal{S}$ and $m(\bar{P})=3$.
(ii) If $P$ and $Q$ are elements of $\mathcal{S}$ of rank at least 3, then there exist some $m \geq 1$ and $a$ sequence

$$
P=P_{1}, \ldots, P_{m}=Q,
$$

where $P_{i}, 1 \leq i \leq m$, is a subgroup of $S$ of rank at least 2 lying in $\mathcal{S}$ and where

$$
P_{i} \subseteq P_{i+1} \text { or } P_{i+1} \subseteq P_{i}
$$

for all $1 \leq i<m$.
Proof. Suppose that $P \in \mathcal{S}$ and $m(P) \leq 2$. Let $\widetilde{S}$ be a Sylow 2-subgroup of $S L_{n}(q)$ such that $S$ is the image of $\widetilde{S}$ in $P S L_{n}(q)$. Note that $\widetilde{S}$ is unique. Since $P$ is an element of $\mathcal{S}$, there exists some elementary abelian 2-subgroup $\widetilde{P}$ of $S L_{n}(q)$ such that $P$ is the image of $\widetilde{P}$ in $P S L_{n}(q)$. Clearly, $\widetilde{P} \leq \widetilde{S}$. We have $m(\widetilde{P}) \leq 3$ as $m(P) \leq 2$. By Corollary 3.34, $\widetilde{P}$ is contained in an $E_{16}$-subgroup of $\overline{\widetilde{S}}$. This implies (i).

We now prove (ii). Suppose that $P$ and $Q$ are elements of $\mathcal{S}$ of rank at least 3. There are elementary abelian subgroups $\widetilde{P}$ and $\widetilde{Q}$ of $S L_{n}(q)$ such that $P$ is the image of $\widetilde{P}$ in $P S L_{n}(q)$ and such that $Q$ is the image of $\widetilde{Q}$ in $P S L_{n}(q)$. Clearly, $\widetilde{P}, \widetilde{Q} \leq \widetilde{S}$. Also $m(\widetilde{P}), m(\widetilde{Q}) \geq 3$. Since $\widetilde{S}$ is 3 -connected by Corollary 3.33, there exist some $m \geq 1$ and a sequence

$$
\widetilde{P}=\widetilde{P}_{1}, \ldots, \widetilde{P}_{n}=\widetilde{Q}
$$

where $\widetilde{P}_{i}(1 \leq i \leq m)$ is an elementary abelian subgroup of $\widetilde{S}$ of rank at least 3 and where

$$
\widetilde{P}_{i} \subseteq \widetilde{P}_{i+1} \text { or } \widetilde{P}_{i+1} \subseteq \widetilde{P}_{i}
$$

for all $1 \leq i<m$. Let $P_{i}, 1 \leq i \leq m$, denote the image of $\widetilde{P}_{i}$ in $S$. Then the sequence

$$
P=P_{1}, \ldots, P_{m}=Q
$$

has the desired properties.
Lemma 7.6. Suppose that $q^{*}=3$. For each elementary abelian subgroup $E$ of $S$ of rank at least 2, let

$$
W_{E}:=\left\langle O\left(C_{G}(x)\right) \mid x \in E^{\#}\right\rangle .
$$

Let $P$ and $Q$ be subgroups of $S$ with $P, Q \in \mathcal{S}$ and $m(P), m(Q) \geq 3$. Then $W_{P}=W_{Q}$.
Proof. By Lemma 7.5 (ii), there exist some $m \geq 1$ and a sequence

$$
P=P_{1}, \ldots, P_{m}=Q
$$

where $P_{i}, 1 \leq i \leq m$, is a subgroup of $S$ of rank at least 2 lying in $\mathcal{S}$ and where

$$
P_{i} \subseteq P_{i+1} \text { or } P_{i+1} \subseteq P_{i}
$$

for all $1 \leq i<m$. By Lemma 7.5 (i), there is a subgroup $\overline{P_{i}}$ of $S$ such that $\overline{P_{i}} \in \mathcal{S}, m\left(\overline{P_{i}}\right) \geq 3$ and $P_{i} \leq \overline{P_{i}}$ for each $1 \leq i \leq m$.

Let $1 \leq i \leq m$ and let $x$ be an involution of $\overline{P_{i}}$. Also let $J$ be a 2 -component of $C_{G}(x)$. As $\overline{P_{i}} \in \mathcal{S}$, we have that $x$ is $G$-conjugate to $t_{j}$ for some even $2 \leq j<n$. Therefore, by Lemma 7.4 , $J / O(J)$ is locally balanced. Applying [32, Corollary 5.6], we may conclude that $G$ is balanced with respect to $\overline{P_{i}}$.

Let $1 \leq i<m$. We have $m\left(P_{i} \cap P_{i+1}\right) \geq 2$ since $P_{i} \subseteq P_{i+1}$ or $P_{i+1} \subseteq P_{i}$ and $m\left(P_{i}\right), m\left(P_{i+1}\right) \geq 2$. Hence $m\left(\overline{P_{i}} \cap \overline{P_{i+1}}\right) \geq 2$. Proposition 2.8 (ii) implies

$$
W_{P_{i}}=W_{\overline{P_{i}}}=W_{\overline{P_{i}} \cap \overline{P_{i+1}}}=W_{\overline{P_{i+1}}}=W_{P_{i+1}} .
$$

Consequently, $W_{P}=W_{Q}$, as wanted.
Proposition 7.7. Suppose that $q^{*}=3$. Let $x$ be an involution of $S$ which is $G$-conjugate to $t_{i}$ for some even $2 \leq i<n$. Then we have $O\left(C_{G}(x)\right)=1$. In particular, $O\left(C_{G}(t)\right)=1$.

Proof. We follow the pattern of the proof of [32, Theorem 9.1]. Let $E$ be the subgroup of $S$ consisting of all $t_{A}$, where $A \subseteq\{1, \ldots, n\}$ has even order. For each elementary abelian 2-subgroup $A$ of $G$ of rank at least 2, let

$$
W_{A}:=\left\langle O\left(C_{G}(y)\right) \mid y \in A^{\#}\right\rangle .
$$

Set $W_{0}:=W_{E}$ and $M:=N_{G}\left(W_{0}\right)$. We accomplish the proof step by step.
(1) $N_{G}(S) \leq M$.

Let $g \in N_{G}(S)$. Clearly, $E \in \mathcal{S}$, and it is easy to note $E^{g}$ still lies in $\mathcal{S}$. Lemma 7.6 implies that $W_{0}=W_{E^{g}}$. On the other hand, we have $\left(W_{0}\right)^{g}=W_{E^{g}}$ by Proposition 2.8 (i). So we have $\left(W_{0}\right)^{g}=W_{0}$ and hence $g \in M$.
(2) Let $y$ be an involution of $S$ such that $y$ is $G$-conjugate to $t_{j}$ for some even $2 \leq j<n$. Then $y$ is $M$-conjugate to $t_{j}$.

We have $\langle y\rangle \in \mathcal{S}$. By Lemma 7.5 (i), there is a subgroup $A$ of $S$ with $\langle y\rangle \leq A, A \in \mathcal{S}$ and $m(A)=3$. As a consequence of Lemma 3.22, there is an element $g$ of $G$ with $A^{g} \leq E$. By Lemma 7.6 and Proposition $2.8(\mathrm{i})$, we have $\left(W_{0}\right)^{g}=\left(W_{A}\right)^{g}=W_{A^{g}}=W_{0}$. Thus $g \in M$.

We have $y^{g} \in E$, and $y^{g}$ is $G$-conjugate and hence $P S L_{n}(q)$-conjugate to $t_{j}$. It is rather easy to show that an element of $E$ is $N_{P S L_{n}(q)}(E)$-conjugate to $t_{j}$ if it is $P S L_{n}(q)$-conjugate to $t_{j}$. So $y^{g}$ is $N_{P S L_{n}(q)}(E)$-conjugate and hence $N_{G}(E)$-conjugate to $t_{j}$. As $N_{G}(E) \leq M$, it follows that $y^{g}$ is $M$-conjugate to $t_{j}$. Hence $y$ is $M$-conjugate to $t_{j}$.
(3) Let $y$ be an involution of $S$ such that $y$ is $G$-conjugate to $t_{j}$ for some even $2 \leq j<n$. Then $C_{G}(y) \leq M$.

Because of (2), we may assume that $\langle y\rangle$ is fully $\mathcal{F}_{S}(G)$-centralized. Then, by Lemma 7.1, $C_{G}(y)$ is generated by the normalizers $N_{C_{G}(y)}(U)$, where $U$ is a subgroup of $C_{S}(y)$ such that there exists $B \leq U$ with $B \in \mathcal{S}$ and $m(B) \geq 3$. It suffices to show that each such normalizer lies in $M$.

Let $U$ and $B$ be as above and let $g \in N_{C_{G}(y)}(U)$. By Lemma 7.6 and Proposition 2.8 (i), we have $\left(W_{0}\right)^{g}=\left(W_{B}\right)^{g}=W_{B^{g}}=W_{0}$. Thus $g \in M$ and hence $N_{C_{G}(y)}(U) \leq M$, as required.
(4) Let $y$ be an involution of $S$. Then $C_{G}(y) \leq M$.

We can see from Lemmas 3.14 and 3.15 that $Z(S)$ has an involution $s$ which is $G$-conjugate to $t_{j}$ for some even $2 \leq j<n$. Let $P$ be a Sylow 2-subgroup of $C_{G}(y)$ with $s \in P$. By (1), $s \in M$ and hence $s \in P \cap M$. Now let $r \in N_{P}(P \cap M)$. Then $s^{r} \in P \cap M$. As a consequence of (1) and (2), $s^{r}$ and $s$ are $M$-conjugate to $t_{j}$. Therefore, there is some $m \in M$ with $s^{r}=s^{m}$. We have $r m^{-1} \in C_{G}(s)$, and so $r m^{-1} \in M$ by (3). Hence $r \in M$. Consequently, $N_{P}(P \cap M)=P \cap M$. It follows that $P=P \cap M$.

Let $U \leq P$ with $m(U) \geq 3$ and let $g \in N_{C_{G}(y)}(U)$. By Lemma 2.3, any $E_{8}$-subgroup of $S$ has an involution which is the image of an involution of $S L_{n}(q)$. Since $m(U) \geq 3$, it follows that $U$ has
an element $u$ which is $G$-conjugate to $t_{k}$ for some even $2 \leq k<n$. By the preceding paragraph, $u, u^{g} \in U \leq P \leq M$. As a consequence of (1) and (2), $u$ and $u^{g}$ are $M$-conjugate to $t_{k}$. So there is some $m \in M$ with $u^{g}=u^{m}$. Hence $g m^{-1} \in C_{G}(u)$. From (3), we see that $C_{G}(u) \leq M$, and so $g m^{-1} \in M$. Thus $g \in M$ and hence $N_{C_{G}(y)}(U) \leq M$. Since $C_{G}(y)$ is 3-generated by Corollary 7.3, it follows that $C_{G}(y) \leq M$.
(5) $M=G$.

Assume that $M \neq G$. By [28, Proposition 17.11], we may deduce from (1) and (4) that $M$ is strongly embedded in $G$, i.e. $M \cap M^{g}$ has odd order for any $g \in G \backslash M$. Applying [49, Chapter $6,4.4]$, it follows that $G$ has only one conjugacy class of involutions. On the other hand, we see from Proposition 3.5 that $G$ has at least two conjugacy classes of involutions. This contradiction shows that $M=G$.
(6) Conclusion.

Let $y \in E^{\#}$ and let $J$ be a 2-component of $C_{G}(y)$. By Lemma 7.4, $J / O(J)$ is locally balanced. So, by [32, Corollary 5.6], $G$ is balanced with respect to $E$. Proposition 2.8 (ii) implies that $W_{0}$ has odd order. By (5), we have $M=G$ and hence $W_{0} \unlhd G$. As $O(G)=1$ by Hypothesis 5.1, it follows that $W_{0}=1$. So we have $O\left(C_{G}(y)\right)=1$ for all $y \in E^{\#}$, and the statement of the proposition follows.

Proposition 7.7 implies that if $q^{*}=3$, then $K \cong S L_{n-2}^{\varepsilon}(3)$ and $L \cong S L_{2}(3)$. Our next goal is to find the isomorphism types of $K$ and $L$ for the case $q^{*} \neq 3$.

In general, $O\left(C_{P S L_{n}(q)}(t)\right)$ is not trivial. So, if $q^{*}$ is not assumed to be 3 , we have no chance to prove that $O\left(C_{G}(t)\right)=1$. However, we will be able to show that

$$
\Delta_{G}(F)=\bigcap_{a \in F^{\#}} O\left(C_{G}(a)\right)=1
$$

for any Klein four subgroup $F$ of $G$ consisting of elements of the form $t_{A}$, where $A \subseteq\{1, \ldots, n\}$ has even order. This will later enable us to determine the isomorphism types of $K$ and $L$ for the case $q^{*} \neq 3$.
7.3. 2-balance of $G$. In this subsection, we prove that $G$ is 2 -balanced when $q^{*} \neq 3$.

Lemma 7.8. Set $C:=C_{G}(t)$ and $\bar{C}:=C / O(C)$. Let $F$ be a Klein four subgroup of $C$. Then $\left[\Delta_{\bar{C}}(\bar{F}), \bar{K}\right]=1$.

Proof. We closely follow arguments found in the proof of [32, Theorem 5.2].
First we consider the case that $F$ has a nontrivial element $y$ such that $\bar{y}$ centralizes $\bar{K}$. Then $\bar{K}$ normalizes $O\left(C_{\bar{C}}(\bar{y})\right)$ and, as $\bar{K} \unlhd \bar{C}, O\left(C_{\bar{C}}(\bar{y})\right)$ also normalizes $\bar{K}$. It follows that

$$
\left[\bar{K}, O\left(C_{\bar{C}}(\bar{y})\right)\right] \leq \bar{K} \cap O\left(C_{\bar{C}}(\bar{y})\right) .
$$

Hence, $\left[\bar{K}, O\left(C_{\bar{C}}(\bar{y})\right)\right]$ is a subgroup of $\bar{K}$ with odd order. By [37, 1.5.5], $\bar{K}$ normalizes $\left[\bar{K}, O\left(C_{\bar{C}}(\bar{y})\right)\right]$. It follows that

$$
\left[\bar{K}, O\left(C_{\bar{C}}(\bar{y})\right)\right] \leq O(\bar{K})
$$

As $O(\bar{K})=1$, this implies that $O\left(C_{\bar{C}}(\bar{y})\right)$ centralizes $\bar{K}$. By definition of $\Delta_{\bar{C}}(\bar{F})$, we have $\Delta_{\bar{C}}(\bar{F}) \leq O\left(C_{\bar{C}}(\bar{y})\right)$. Consequently, $\Delta_{\bar{C}}(\bar{F})$ centralizes $\bar{K}$.

Now we treat the case that $C_{\bar{F}}(\bar{K})=1$. For each subgroup or element $X$ of $C$, let $\widehat{X}$ denote the image of $\bar{X}$ in $\bar{C} / C_{\bar{C}}(\bar{K})$. Since $C_{\bar{F}}(\bar{K})=1$, we have $\widehat{F} \cong \bar{F}$, and so $\widehat{F}$ is a Klein four subgroup of $\widehat{C}$. As $\bar{K} \cong S L_{n-2}^{\varepsilon}\left(q^{*}\right) / O\left(S L_{n-2}^{\varepsilon}\left(q^{*}\right)\right)$, we have that $\bar{K}$ is locally 2-balanced (see Lemma 3.48. Using this together with the fact that the group $\widehat{C}=\bar{C} / C_{\bar{C}}(\bar{K})$ is isomorphic to a subgroup of $\operatorname{Aut}(\bar{K})$ containing $\operatorname{Inn}(\bar{K})$, we may conclude that $\Delta_{\widehat{C}}(\widehat{F})=1$. By [32, Proposition 3.11], if $X$ is a finite group, $B$ a 2-subgroup of $X$ and $N \unlhd X$, then the image of $O\left(C_{X}(B)\right)$ in
$X / N$ lies in $O\left(C_{X / N}(B N / N)\right)$. Thus, if $y$ is an involution of $F$, then the image of $O\left(C_{\bar{C}}(\bar{y})\right)$ in $\widehat{C}$ lies in $O\left(C_{\widehat{C}}(\widehat{y})\right)$. It follows that the image of $\Delta_{\bar{C}}(\bar{F})$ in $\widehat{C}$ is contained in $\Delta_{\widehat{C}}(\widehat{F})=1$. Hence $\Delta_{\bar{C}}(\bar{F}) \leq C_{\bar{C}}(\bar{K})$.

Lemma 7.9. Let $C:=C_{G}(t)$ and $\bar{C}:=C / O(C)$. Then $C_{\bar{C}}(\bar{K}) \cap C_{\bar{C}}(\bar{L})$ is a 2-group.
Proof. For convenience, we denote $C_{\bar{C}}(\bar{K}) \cap C_{\bar{C}}(\bar{L})$ by $C_{\bar{C}}(\bar{K}, \bar{L})$. Since $\bar{C}$ is core-free, we have that $C_{\bar{C}}(\bar{K}, \bar{L})$ is core-free. So it is enough to prove that $C_{\bar{C}}(\bar{K}, \bar{L})$ is 2-nilpotent. By [39, Theorem 1.4], it suffices to show that $C_{\bar{C}}(\bar{K}, \bar{L})$ has a nilpotent 2 -fusion system.

Let $X$ denote the subgroup of $T$ consisting of all elements of $T$ of the form

$$
\left(\begin{array}{ll}
A & \\
& B
\end{array}\right) Z\left(S L_{n}(q)\right)
$$

with $A \in W \cap Z\left(G L_{n-2}(q)\right), B \in V \cap Z\left(G L_{2}(q)\right)$ and $\operatorname{det}(A) \operatorname{det}(B)=1$.
Let $A \in W$ and $B \in V$ with $\operatorname{det}(A) \operatorname{det}(B)=1$ and

$$
m:=\left(\begin{array}{ll}
A & \\
& B
\end{array}\right) Z\left(S L_{n}(q)\right) \in T .
$$

Assume that $\bar{m}$ centralizes $\bar{K}$ and $\bar{L}$. Then we have $A \in Z\left(G L_{n-2}(q)\right)$ by Lemma 6.2. Since $\bar{m}$ centralizes $\bar{L}, \bar{m}$ also centralizes $\overline{X_{2}}$. Thus $m$ centralizes $X_{2}$, and so $B$ centralizes $V \cap S L_{2}(q)$. Lemma 3.17 implies that $B \in Z\left(G L_{2}(q)\right)$. So we have $m \in X$. Conversely, if $A \in Z\left(G L_{n-2}(q)\right)$ and $B \in Z\left(G L_{2}(q)\right)$, then $\bar{m} \in C_{\bar{C}}(\bar{K}, \bar{L})$ as a consequence of Lemmas 6.2 and 3.43 . It follows that $\bar{T} \cap C_{\bar{C}}(\bar{K}, \bar{L})=\bar{X}$.

Let $\mathcal{F}:=\mathcal{F}_{S}\left(P S L_{n}(q)\right)=\mathcal{F}_{S}(G)$. Since $X$ is central in $C_{P S L_{n}(q)}(t)$, the only subsystem of $C_{\mathcal{F}}(\langle t\rangle)$ on $X$ is the nilpotent fusion system on $X$. It follows that $\mathcal{F}_{\bar{X}}\left(C_{\bar{C}}(\bar{K}, \bar{L})\right)$ is nilpotent. So $C_{\bar{C}}(\bar{K}, \bar{L})$ has a nilpotent 2 -fusion system, as required.

In the following lemma, $A_{1}$ and $A_{2}$ have the meanings given to them after Lemma 6.5.
Lemma 7.10. Set $C:=C_{G}(t)$. Suppose that $q^{*} \neq 3$. Then $A_{1}, A_{2}$ and $L$ are the only 2components of $C_{C}(u)$. Moreover, the following hold:
(i) $A_{1}$ is the only 2-component of $C_{C}(u)$ containing $u$.
(ii) $A_{2}$ is the only 2-component of $C_{C}(u)$ containing neither $u$ nor $t$.
(iii) $L$ is the only 2 -component of $C_{C}(u)$ containing $t$.

Proof. By definition, $A_{1}$ and $A_{2}$ are 2-components of $C_{C}(u)$. Also, it is clear from the definition of $L$ (see Proposition 6.8) that $L$ is a 2 -component of $C_{C}(u)$.

Set $\bar{C}:=C / O(C)$. As a consequence of Lemma $6.4, \overline{A_{1}}$ and $\overline{A_{2}}$ are the only 2-components of $C_{\bar{K}}(\bar{u})$. Moreover, $\bar{L}$ is a component of $C_{\bar{C}}(\bar{u})$. So Lemma 2.5 shows that $\overline{A_{1}}, \overline{A_{2}}$ and $\bar{L}$ are the only 2 -components of $C_{\bar{C}}(\bar{u})$. As we have observed after Lemma 6.5 , there is a bijection from the set of 2-components of $C_{C}(u)$ to the set of 2-components of $C_{\bar{C}}(\bar{u})$ sending each 2-component $A$ of $C_{C}(u)$ to $\bar{A}$. Therefore, $A_{1}, A_{2}$ and $L$ are the only 2-components of $C_{C}(u)$.

It remains to prove (i), (ii) and (iii). We have $T_{1} \leq A_{1}$ by Lemma 6.6 and thus $u \in A_{1}$. From the definition of $L$, it is clear that $t \in L$. Moreover, $u \notin L$ since $\bar{t}$ is the only involution of $\bar{L}$. Similarly, $t \notin A_{1}$. Also, it is easy to see from Lemma 6.4 that $u$ and $t$ cannot be elements of $A_{2}$.
Lemma 7.11. Suppose that $q^{*} \neq 3$. Let $F$ be a Klein four subgroup of $T$. Then we have $\Delta_{G}(F) \cap C_{G}(t) \leq O\left(C_{G}(t)\right)$.
Proof. Set $C:=C_{G}(t), D:=\Delta_{G}(F) \cap C$ and $\bar{C}:=C / O(C)$. We are going to show that $\bar{D}$ is trivial.

A direct calculation shows that $D \leq \Delta_{C}(F)$. For each $a \in F^{\#}$, we have $\overline{O\left(C_{C}(a)\right)} \leq O\left(C_{\bar{C}}(\bar{a})\right)$ as a consequence of Corollary 2.2. Therefore, we have $\overline{\Delta_{C}(F)} \leq \Delta_{\bar{C}}(\bar{F})$, and hence $\bar{D} \leq \Delta_{\bar{C}}(\bar{F})$. Lemma 7.8 implies that $[\bar{D}, \bar{K}]=1$. In particular, $\bar{D} \leq C_{\bar{C}}(\bar{u})=\overline{C_{C}(u)}$. Fix a subgroup $D_{0}$ of $C_{C}(u)$ with $\overline{D_{0}}=\bar{D}$. Also, let $g \in G$ with $u^{g}=t$ and $t^{g}=u$ (such an element exists by Lemma 6.7). Note that $\left(D_{0}\right)^{g} \leq\left(C_{C}(u)\right)^{g}=C_{C}(u)$.

We accomplish the proof step by step.
(1) $A_{1}, A_{2}$ and $L$ are normal subgroups of $C_{C}(u)$.

This is immediate from Lemma 7.10 .
(2) There is a group isomorphism $\operatorname{Aut}\left(\overline{A_{1}}\right) \rightarrow \operatorname{Aut}(\bar{L})$ which maps $\operatorname{Inn}\left(\overline{A_{1}}\right)$ to $\operatorname{Inn}(\bar{L})$ and $\operatorname{Aut}_{\left(D_{0}\right)^{g}}\left(\overline{A_{1}}\right)$ to $\operatorname{Aut}_{\bar{D}}(\bar{L})$.

Let $\operatorname{Aut}_{D_{0}}(L / O(L))$ denote the image of $\operatorname{Aut}_{D_{0}}(L)$ under the natural group homomorphism $\operatorname{Aut}(L) \rightarrow \operatorname{Aut}(L / O(L))$. Also, let $\operatorname{Aut}_{\left(D_{0}\right)^{g}}\left(A_{1} / O\left(A_{1}\right)\right)$ denote the image of $\operatorname{Aut}_{\left(D_{0}\right)^{g}}\left(A_{1}\right)$ under the natural group homomorphism $\operatorname{Aut}\left(A_{1}\right) \rightarrow \operatorname{Aut}\left(A_{1} / O\left(A_{1}\right)\right)$.

From Lemma 7.10 , it is clear that $\left(A_{1}\right)^{g^{-1}}=L$. The group isomorphism $\left.c_{g^{-1}}\right|_{A_{1}, L}$ induces a group isomorphism $A_{1} / O\left(A_{1}\right) \rightarrow L / O(L)$, and this group isomorphism induces a group isomorphism $\operatorname{Aut}\left(A_{1} / O\left(A_{1}\right)\right) \rightarrow \operatorname{Aut}(L / O(L))$. By a direct calculation, the group isomorphism just mentioned maps $\operatorname{Aut}_{\left(D_{0}\right)^{g}}\left(A_{1} / O\left(A_{1}\right)\right)$ to $\operatorname{Aut}_{D_{0}}(L / O(L))$ and $\operatorname{Inn}\left(A_{1} / O\left(A_{1}\right)\right)$ to $\operatorname{Inn}(L / O(L))$.

We have $A_{1} /\left(A_{1} \cap O(C)\right) \cong \overline{A_{1}} \cong S L_{2}\left(q^{*}\right)$. As $S L_{2}\left(q^{*}\right)$ is core-free, it follows that $A_{1} \cap O(C)=$ $O\left(A_{1}\right)$. So the natural group homomorphism $A_{1} \rightarrow \overline{A_{1}}$ induces a group isomorphism $A_{1} / O\left(A_{1}\right) \rightarrow$ $\overline{A_{1}}$. This group isomorphism induces a group isomorphism $\operatorname{Aut}\left(A_{1} / O\left(A_{1}\right)\right) \rightarrow \operatorname{Aut}\left(\overline{A_{1}}\right)$. By a direct calculation, the group isomorphism just mentioned maps Aut ${ }_{\left(D_{0}\right)^{g}}\left(A_{1} / O\left(A_{1}\right)\right)$ to Aut $\overline{\left(D_{0}\right)^{g}}\left(\overline{A_{1}}\right)$ and $\operatorname{Inn}\left(A_{1} / O\left(A_{1}\right)\right)$ to $\operatorname{Inn}\left(\overline{A_{1}}\right)$. In a very similar way, we obtain an isomorphism $\operatorname{Aut}(L / O(L)) \rightarrow$ $\operatorname{Aut}(\bar{L})$ which maps $\operatorname{Aut}_{D_{0}}(L / O(L))$ to $\operatorname{Aut}_{\overline{D_{0}}}(\bar{L})=\operatorname{Aut}_{\bar{D}}(\bar{L})$ and $\operatorname{Inn}(L / O(L))$ to $\operatorname{Inn}(\bar{L})$.

As a consequence of the preceding observations, there is a group isomorphism $\operatorname{Aut}\left(\overline{A_{1}}\right) \rightarrow$ $\operatorname{Aut}(\bar{L})$ which maps $\operatorname{Inn}\left(\overline{A_{1}}\right)$ to $\operatorname{Inn}(\bar{L})$ and $\operatorname{Aut}_{\left(\overline{\left.D_{0}\right)^{9}}\right.}\left(\overline{A_{1}}\right)$ to $\operatorname{Aut}_{\bar{D}}(\bar{L})$, as asserted.

As observed above, $\overline{D_{0}}=\bar{D}$ centralizes $\bar{K}$. In particular, $\bar{D}$ centralizes $\overline{A_{2}}$. This implies that $\left[D_{0}, A_{2}\right] \leq O(C)$. As $D_{0}$ normalizes $A_{2}$ by (1), we also have that $\left[D_{0}, A_{2}\right] \leq A_{2}$. Consequently, $\left[D_{0}, A_{2}\right] \leq O\left(A_{2}\right)$. Because of Lemma 7.10, we have $\left(A_{2}\right)^{g}=A_{2}$. It follows that $\left[\left(D_{0}\right)^{g}, A_{2}\right] \leq$ $O\left(A_{2}\right)$. This easily implies $\left[\overline{\left(D_{0}\right)^{g}}, \overline{A_{2}}\right] \leq O\left(\overline{A_{2}}\right)$. As $\overline{A_{2}} \cong S L_{n-4}^{\varepsilon}\left(q^{*}\right)$ by Lemma 6.4, we have $O\left(\overline{A_{2}}\right) \leq Z\left(\overline{A_{2}}\right)$. It follows that $\left[\overline{A_{2}}, \overline{\left(D_{0}\right)^{g}}, \overline{A_{2}}\right]=\left[\overline{\left(D_{0}\right)^{g}}, \overline{A_{2}}, \overline{A_{2}}\right] \leq\left[Z\left(\overline{A_{2}}\right), \overline{A_{2}}\right]=1$. The Three Subgroups Lemma [37, 1.5.6] implies $\left[\overline{A_{2}}, \overline{\left(D_{0}\right)^{g}}\right]=\left[\overline{A_{2}}, \overline{A_{2}}, \overline{\left(D_{0}\right)^{g}}\right]=1$. Hence, $\overline{\left(D_{0}\right)^{g}}$ centralizes $\overline{A_{2}}$. By (1), $\overline{\left(D_{0}\right)^{g}}$ normalizes $\overline{A_{1}}$. Clearly, Aut $\overline{\left(D_{0}\right)^{g}}(\bar{K})$ has odd order. The assertion now follows from Lemmas 6.4 (iii), 3.49 and 3.50 .
(4) $\bar{D} \leq \bigcap_{y \in F^{\#}} O\left(C_{\bar{L}}(\bar{y})\right)$.

As a consequence of (2) and (3), we have $\operatorname{Aut}_{\bar{D}}(\bar{L}) \leq \operatorname{Inn}(\bar{L})$. This implies $\bar{D} \leq \bar{L}_{\bar{C}}(\bar{L})$. By [37, 6.5.3], $\bar{L} \leq C_{\bar{C}}(\bar{K})$. As observed above, $[\bar{D}, \bar{K}]=1$ and hence $\bar{D} \leq C_{\bar{C}}(\bar{K})$. It follows that $\bar{D}$ is a subgroup of $\bar{L}\left(C_{\bar{C}}(\bar{L}) \cap C_{\bar{C}}(\bar{K})\right)$. By Lemma 7.9, $C_{\bar{C}}(\bar{L}) \cap C_{\bar{C}}(\bar{K})$ is a 2-group. As $\bar{D}$ has odd
order and $\bar{L} \unlhd \bar{C}$, this implies that $\bar{D} \leq \bar{L}$. Now we see that

$$
\begin{aligned}
\bar{D} & \leq \bar{L} \cap \Delta_{\bar{C}}(\bar{F}) \\
& =\bigcap_{y \in F^{\#}}\left(\bar{L} \cap O\left(C_{\bar{C}}(\bar{y})\right)\right) \\
& =\bigcap_{y \in F^{\#}}\left(C_{\bar{L}}(\bar{y}) \cap O\left(C_{\bar{C}}(\bar{y})\right)\right) \\
& =\bigcap_{y \in F^{\#}} O\left(C_{\bar{L}}(\bar{y})\right) .
\end{aligned}
$$

(5) Conclusion.

As $F$ is a Klein four subgroup of $T$, we have $F=\left\langle y_{1}, y_{2}\right\rangle$ for two commuting involutions $y_{1}$ and $y_{2}$ of $T$. For $i \in\{1,2\}$, we have

$$
y_{i}=\left(\begin{array}{ll}
A_{i} & \\
& B_{i}
\end{array}\right) Z\left(S L_{n}(q)\right)
$$

for some $A_{i} \in W$ and $B_{i} \in V$ with $\operatorname{det}\left(A_{i}\right) \operatorname{det}\left(B_{i}\right)=1$. Let $y_{3}:=y_{1} y_{2}, A_{3}:=A_{1} A_{2}$ and $B_{3}:=B_{1} B_{2}$. As $y_{1}, y_{2}, y_{3}$ are involutions, we have $\left(B_{i}\right)^{2} \in Z\left(G L_{2}(q)\right)$ for each $i \in\{1,2,3\}$.

It is easy to note that $\overline{X_{2}} \in \operatorname{Syl}_{2}(\bar{L})$. If $B \in V \cap S L_{2}(q)$ and

$$
y:=\left(\begin{array}{ll}
I_{n-2} & \\
& B
\end{array}\right) Z\left(S L_{n}(q)\right) \in X_{2},
$$

then

$$
y^{y_{i}}=\left(\begin{array}{ll}
I_{n-2} & \\
& B^{B_{i}}
\end{array}\right) Z\left(S L_{n}(q)\right)
$$

for each $i \in\{1,2,3\}$. Applying Lemma 3.51, we deduce that

$$
\bigcap_{y \in F^{\#}} O\left(C_{\bar{L}}(\bar{y})\right)=1 .
$$

So we have $\bar{D}=1$ by (4). This completes the proof.
Lemma 7.12. Suppose that $q^{*} \neq 3$. Then $G$ is 2-balanced.
Proof. Let $F$ be a Klein four subgroup of $G$ and let $a$ be an involution of $G$ centralizing $F$. We have to show that $\Delta_{G}(F) \cap C_{G}(a) \leq O\left(C_{G}(a)\right)$.

Assume that $a$ is $G$-conjugate to $t$. Then there is some $g \in G$ with $a^{g}=t$ and $F^{g} \leq T$. By Lemma 7.11, we have $\Delta_{G}\left(F^{g}\right) \cap C_{G}(t) \leq O\left(C_{G}(t)\right)$. Clearly $\Delta_{G}(F)^{g}=\Delta_{G}\left(F^{g}\right)$. It follows that $\Delta_{G}(F) \cap C_{G}(a) \leq O\left(C_{G}(a)\right)$.

Assume now that $a$ is not $G$-conjugate to $t$. Let $J$ be a 2 -component of $C_{G}(a)$. By Propositions 6.10, 6.11 and 6.13, either $J / O(J) \cong S L_{k}^{\varepsilon}\left(q^{*}\right) / O\left(S L_{k}^{\varepsilon}\left(q^{*}\right)\right)$ for some $k \geq 3$, or $J / O(J)$ is isomorphic to a nontrivial quotient of $S L_{\frac{n}{2}}^{\varepsilon_{0}}\left(q_{0}\right)$ for some nontrivial odd prime power $q_{0}$ and some $\varepsilon_{0} \in\{+,-\}$. So $J / O(J)$ is locally 2-balanced by Lemma 3.48. Applying [32, Theorem 5.2], we may conclude that $\Delta_{C_{G}(a)}(F) \leq O\left(C_{G}(a)\right)$. A direct calculation shows that $\Delta_{G}(F) \cap C_{G}(a) \leq \Delta_{C_{G}(a)}(F)$. Hence $\Delta_{G}(F) \cap C_{G}(a) \leq O\left(C_{G}(a)\right)$.
7.4. The case $q^{*} \neq 3$ : Triviality of $\Delta_{G}(F)$.

Lemma 7.13. Suppose that $q^{*} \neq 3$. Assume moreover that $q \equiv 1 \bmod 4$ or $n \geq 7$. Then we have $\Delta_{G}(F)=1$ for each Klein four subgroup $F$ of $S$.

Proof. We follow the pattern of the proof of [32, Theorem 9.1].
For each elementary abelian 2 -subgroup $A$ of $G$ of rank at least 3 , we define

$$
W_{A}:=\left\langle\Delta_{G}(F) \mid F \leq A, m(F)=2\right\rangle .
$$

Let $P$ and $Q$ be elementary abelian subgroups of $S$ of rank at least 3 . We claim that $W_{P}=W_{Q}$. By Corollary 3.33 (iii), $S$ is 3 -connected. So there exist a natural number $m \geq 1$ and a sequence

$$
P=P_{1}, \ldots, P_{m}=Q
$$

such that $P_{i}, 1 \leq i \leq m$, is an elementary abelian subgroup of $S$ of rank at least 3 and such that

$$
P_{i} \subseteq P_{i+1} \text { or } P_{i+1} \subseteq P_{i}
$$

for all $1 \leq i<m$. By Lemma 7.12, $G$ is 2-balanced. Proposition 2.8 (ii) implies that $W_{P_{i}}=W_{P_{i+1}}$ for all $1 \leq i<m$. Therefore, $W_{P}=W_{Q}$, as asserted.

We use $W_{0}$ to denote $W_{P}$, where $P$ is an elementary abelian subgroup of $S$ of rank at least 3 . Let $M:=N_{G}\left(W_{0}\right)$. We accomplish the proof step by step.
(1) $N_{G}(S) \leq M$.

Let $g \in N_{G}(S)$. Take an elementary abelian subgroup $P$ of $S$ with $m(P) \geq 3$. By Proposition 2.8 (i), we have $\left(W_{0}\right)^{g}=\left(W_{P}\right)^{g}=W_{P^{g}}=W_{0}$. Thus $g \in M$.
(2) Let $x$ be an involution of $S$. Then $C_{G}(x) \leq M$.

By Corollary 3.34, there is an elementary abelian subgroup $P$ of $S$ with $x \in P$ and $m(P)=4$. Clearly, $P \leq C_{G}(x)$. Let $R$ be a Sylow 2-subgroup of $C_{G}(x)$ containing $P$. By Corollary 7.3 , $C_{G}(x)$ is 3 -generated. Hence, $C_{G}(x)$ is generated by the normalizers $N_{C_{G}(x)}(U)$, where $U \leq R$ and $m(U) \geq 3$. It suffices to show that each such normalizer lies in $M$.

So let $U$ be a subgroup of $R$ with $m(U) \geq 3$, and let $g \in N_{C_{G}(x)}(U)$. Let $Q$ be an elementary abelian subgroup of $U$ with $m(Q)=3$, and let $h \in G$ with $R^{h} \leq S$. Then $W_{Q^{h}}=W_{Q^{g h}}=W_{P^{h}}=$ $W_{0}$. Proposition 2.8 (i) implies that $W_{Q}=W_{Q^{g}}=W_{P}=W_{0}$. Applying Proposition 2.8 (i) again, it follows that $\left(W_{0}\right)^{g}=\left(W_{Q}\right)^{g}=W_{Q^{g}}=W_{0}$. Hence $g \in M$ and thus $N_{C_{G}(x)}(U) \leq M$.
(3) $M=G$.

Assume that $M \neq G$. By [28, Proposition 17.11], we may deduce from (1) and (2) that $M$ is strongly embedded in $G$, i.e. $M \cap M^{g}$ has odd order for any $g \in G \backslash M$. Applying [49, Chapter $6,4.4]$, it follows that $G$ has only one conjugacy class of involutions. On the other hand, we see from Proposition 3.5 that $G$ has at least two conjugacy classes of involutions. This contradiction shows that $M=G$.

## (4) Conclusion.

Let $F$ be a Klein four subgroup of $S$. By Corollary 3.34 , there is an elementary abelian subgroup $P$ of $S$ with $F \leq P$ and $m(P)=4$. Clearly, $\Delta_{G}(F) \leq W_{P}$. Since $G$ is 2-balanced, $W_{P}$ has odd order by Proposition 2.8 (ii). Since $W_{P}=W_{0}$, we have $W_{P} \unlhd G$ by (3). As $O(G)=1$ by Hypothesis 5.1, it follows that $W_{P}=1$. Hence $\Delta_{G}(F)=1$.

Next, we deal with the case that $n=6, q \equiv 3 \bmod 4$ and $q^{*} \neq 3$. We show that, in this case, $\Delta_{G}(F)=1$ for each Klein four subgroup $F$ of $S$ consisting of elements of the form $t_{A}$, where $A \subseteq\{1, \ldots, n\}$ has even order. We need the following lemma.

Lemma 7.14. Suppose that $q^{*} \neq 3$. Set $\ell:=n-4$. Let $E$ be the subgroup of $T$ consisting of all $t_{A}$, where $A \subseteq\{1, \ldots, n\}$ has even order. Let $E_{1}$ denote the subgroup of $X_{1}$ consisting of all $t_{A}$, where $A$ is a subset of $\{1, \ldots, n-2\}$ of even order. Then we may choose elements $m_{1}, \ldots, m_{\ell} \in N_{K}\left(E_{1}\right)$ and an $E_{8}$-subgroup $E_{0}$ of $E$ with

$$
K=\left\langle O(K), L_{2^{\prime}}\left(C_{K}\left(E_{0}\right)\right), L_{2^{\prime}}\left(C_{K}\left(E_{0}\right)\right)^{m_{1}}, \ldots, L_{2^{\prime}}\left(C_{K}\left(E_{0}\right)\right)^{m_{\ell}}\right\rangle .
$$

Proof. Set $C:=C_{G}(t)$ and $\bar{C}:=C / O(C)$. Let $H:=S L_{n-2}^{\varepsilon}\left(q^{*}\right) / O\left(S L_{n-2}^{\varepsilon}\left(q^{*}\right)\right)$. Let $\widetilde{D}$ be the subgroup of $S L_{n-2}^{\varepsilon}\left(q^{*}\right)$ consisting of all diagonal matrices in $S L_{n-2}^{\varepsilon}\left(q^{*}\right)$ with diagonal entries in $\{1,-1\}$, and let $D$ denote the image of $\widetilde{D}$ in $H$. Denote by $H_{1}$ the image of

$$
\left\{\left(\begin{array}{cc}
A & \\
& I_{n-4}
\end{array}\right): A \in S L_{2}^{\varepsilon}\left(q^{*}\right)\right\}
$$

in $H$.
We claim that there is a group isomorphism $\psi: \bar{K} \rightarrow H$ which maps $\overline{E_{1}}$ to $D$ and $\overline{A_{1}}$ to $H_{1}$. By Lemma 6.4(iii), there is a group isomorphism $\varphi: \bar{K} \rightarrow H$ under which $\overline{A_{1}}$ corresponds to $H_{1}$. Since $\bar{u}$ is the only involution of $\overline{A_{1}}$, we have that $\bar{u}^{\varphi}$ is the image of $\operatorname{diag}(-1,-1,1, \ldots, 1) \in S L_{n-2}^{\varepsilon}\left(q^{*}\right)$ in $H$. Clearly, $\overline{E_{1}}$ is elementary abelian of order $2^{n-3}$. Using Lemma 3.22, we conclude that $\overline{E_{1}}{ }^{\varphi}$ is $H$-conjugate to $D$. So there is some $\alpha \in \operatorname{Inn}(H)$ mapping ${\overline{E_{1}}}^{\varphi}$ to $D$. We may assume that $\alpha$ centralizes $\bar{u}^{\varphi}$. Then $H_{1}{ }^{\alpha}=H_{1}$, and the isomorphism $\psi:=\varphi \alpha$ maps $\overline{E_{1}}$ to $D$ and $\overline{A_{1}}$ to $H_{1}$, as desired.

Using Lemma 3.38, we can find elements $x_{1}, \ldots, x_{\ell} \in N_{H}(D)$ such that $H=\left\langle H_{1}, H_{1}{ }^{x_{1}}, \ldots\right.$, $\left.H_{1}{ }^{x_{\ell}}\right\rangle$. Therefore, $K$ has elements $m_{1}, \ldots, m_{\ell}$ such that

$$
\bar{K}=\left\langle\overline{A_{1}}, \overline{A_{1}} \overline{\overline{m_{1}}}, \ldots,{\overline{A_{1}}}_{\overline{m_{\ell}}}\right\rangle
$$

and $\overline{m_{1}}, \ldots, \overline{m_{\ell}} \in N_{\bar{K}}\left(\overline{E_{1}}\right)$. From Lemma 2.1, we see that $N_{\bar{K}}\left(\bar{E}_{1}\right)=\overline{N_{K}\left(E_{1}\right)}$. So we may assume $m_{i} \in N_{K}\left(E_{1}\right)$ for $i \in\{1, \ldots, \ell\}$. Let $E_{0}:=\left\langle u, t_{\{3,4\}}, t_{\{4,5\}}\right\rangle$. By Lemma 6.5, we have $\overline{A_{1}} \unlhd C_{\bar{C}}(\bar{u})$. In particular, $\overline{E_{0}}$ normalizes $\overline{A_{1}}$. Moreover, $\overline{E_{0}}$ centralizes $\overline{T_{1}}$. We have $\overline{A_{1}} \cong S L_{2}\left(q^{*}\right)$ and $\overline{T_{1}} \in \operatorname{Syl}_{2}\left(\overline{A_{1}}\right)$ (see Lemma 6.4). Applying Lemma 3.43, we conclude that $\overline{A_{1}} \leq C_{\bar{K}}\left(\overline{E_{0}}\right)$. As $\overline{A_{1}} \unlhd C_{\bar{K}}(\bar{u})$ and $\overline{A_{1}} \leq C_{\bar{K}}\left(\overline{E_{0}}\right) \leq C_{\bar{K}}(\bar{u})$, we even have that $\overline{A_{1}}$ is a component of $C_{\bar{K}}\left(\overline{E_{0}}\right)$. It follows that

$$
\bar{K}=\left\langle L_{2^{\prime}}\left(C_{\bar{K}}\left(\overline{E_{0}}\right)\right), L_{2^{\prime}}\left(C_{\bar{K}}\left(\overline{E_{0}}\right)\right)^{\overline{m_{1}}}, \ldots, L_{2^{\prime}}\left(C_{\bar{K}}\left(\overline{E_{0}}\right)\right)^{\overline{m_{\ell}}}\right\rangle
$$

Let $k \in K$ such that $\bar{k} \in C_{\bar{K}}\left(\overline{E_{0}}\right)$. As $K \unlhd C$, we have $\left[k, E_{0}\right] \leq O(C) \cap K=O(K)$. Thus $k O(K) \in C_{C / O(K)}\left(E_{0} O(K) / O(K)\right)$. By Lemma 2.1, there is an element $z \in C_{C}\left(E_{0}\right)$ such that $k O(K)=z O(K)$. Observing that $z \in C_{K}\left(E_{0}\right)$ and that $\bar{k}=\bar{z}$, we may conclude that $C_{\bar{K}}\left(\overline{E_{0}}\right)=$ $\overline{C_{K}\left(E_{0}\right)}$. If $1 \leq i \leq \ell$, then $L_{2^{\prime}}\left(C_{\bar{K}}\left(\overline{E_{0}}\right)\right)^{\overline{m_{i}}}=L_{2^{\prime}}\left(\overline{C_{K}\left(E_{0}\right)}\right)^{\overline{m_{i}}}=\overline{L_{2^{\prime}}\left(C_{K}\left(E_{0}\right)\right)^{\overline{m_{i}}}}=\overline{L_{2^{\prime}}\left(C_{K}\left(E_{0}\right)\right)^{m_{i}}}$, where the second equality follows from Proposition 2.4. It follows that

$$
K=\left\langle O(K), L_{2^{\prime}}\left(C_{K}\left(E_{0}\right)\right), L_{2^{\prime}}\left(C_{K}\left(E_{0}\right)\right)^{m_{1}}, \ldots, L_{2^{\prime}}\left(C_{K}\left(E_{0}\right)\right)^{m_{\ell}}\right\rangle .
$$

This completes the proof.
Lemma 7.15. Suppose that $n=6, q \equiv 3 \bmod 4$ and $q^{*} \neq 3$. Let $E$ denote the subgroup of $S$ consisting of all $t_{A}$, where $A$ is a subset of $\{1, \ldots, n\}$ of even order. Then $\Delta_{G}(F)=1$ for any Klein four subgroup $F$ of $E$.

Proof. We follow the pattern of the proof of [32, Theorem 9.1].
Set $W_{0}:=\left\langle\Delta_{G}(F) \mid F \leq E, m(F)=2\right\rangle$ and $M:=N_{G}\left(W_{0}\right)$. Since $T$ is the image of

$$
\left\{\left(\begin{array}{ll}
A & \\
& B
\end{array}\right): A \in W, B \in V, \operatorname{det}(A) \operatorname{det}(B)=1\right\}
$$

in $P S L_{n}(q)$, we have $T \in \operatorname{Syl}_{2}\left(P S L_{n}(q)\right)$ by Lemma 3.15. Hence $S=T$ and thus $t \in Z(S)$. By choice of $W$ (see Section 5), we have

$$
W=\left\{\left(\begin{array}{ll}
A & \\
& B
\end{array}\right): A, B \in V\right\} \cdot\left\langle\left(\begin{array}{ll} 
& I_{2} \\
I_{2} &
\end{array}\right)\right\rangle
$$

We accomplish the proof step by step.
(1) For each subgroup $E_{0}$ of $E$ with order at least 8, we have $N_{G}\left(E_{0}\right) \leq M$.

Clearly, $E \cong E_{16}$. Therefore, the statement follows from the 2-balance of $G$ (see Lemma 7.12 ) and Proposition 2.8 (ii).
(2) $N_{G}(S) \leq M$.

First we prove $S \leq M$. By (1), we have $E \leq M$. As $q \equiv 3 \bmod 4$ and $S=T$, any element of $S$ can be written as a product of an element of $E$ and an element of $S$ induced by a matrix of the form

$$
\left(\begin{array}{ll}
A & \\
& B
\end{array}\right)
$$

with $A \in W \cap S L_{4}(q)$ and $B \in V \cap S L_{2}(q)$. So, in order to prove that $S \leq M$, it suffices to show that each element of $S$ induced by a matrix of this form lies in $M$. If $B \in V \cap S L_{2}(q)$, then the image of

$$
\left(\begin{array}{ll}
I_{4} & \\
& B
\end{array}\right)
$$

in $S$ centralizes the group $\left\langle t_{\{1,2\}}, t_{\{2,3\}}, t_{\{3,4\}}\right\rangle \cong E_{8}$. So it is contained in $M$ by (1). Hence, in order to prove that $S \leq M$, it suffices to show that if $A \in W \cap S L_{4}(q)$, then the image of

$$
\left(\begin{array}{ll}
A & \\
& I_{2}
\end{array}\right)
$$

in $S$ lies in $M$. So assume that $A \in W \cap S L_{4}(q)$. By the structure of $W$, there are elements $M_{1}$, $M_{2}$ of $V$ such that $\operatorname{det}\left(M_{1}\right)=\operatorname{det}\left(M_{2}\right)$ and

$$
A=\left(\begin{array}{ll}
M_{1} & \\
& M_{2}
\end{array}\right) \text { or } A=\left(\begin{array}{ll}
M_{1} & \\
& M_{2}
\end{array}\right)\left(\begin{array}{ll} 
& I_{2} \\
I_{2} &
\end{array}\right)
$$

The image of

$$
\left(\begin{array}{ccc}
M_{1} & & \\
& M_{2} & \\
& & I_{2}
\end{array}\right)
$$

in $S$ can be written as a product of an element of $E$ and an element of $S$ induced by a matrix of the form

$$
\left(\begin{array}{ccc}
\widetilde{M_{1}} & & \\
& \widetilde{M_{2}} & \\
& & I_{2}
\end{array}\right)
$$

with $\widetilde{M}_{1}, \widetilde{M}_{2} \in V \cap S L_{2}(q)$. The images of

$$
\left(\begin{array}{cc}
\widetilde{M}_{1} & \\
& I_{4}
\end{array}\right) \text { and }\left(\begin{array}{ccc}
I_{2} & & \\
& \widetilde{M}_{2} & \\
& & I_{2}
\end{array}\right)
$$

in $S$ centralize the groups $\left\langle t_{\{3,4\}}, t_{\{4,5\}}, t_{\{5,6\}}\right\rangle$ and $\left\langle t_{\{1,2\}}, t_{\{2,5\}}, t_{\{5,6\}}\right\rangle$, respectively. So they are elements of $M$. It follows that the image of

$$
\left(\begin{array}{lll}
M_{1} & & \\
& M_{2} & \\
& & I_{2}
\end{array}\right)
$$

in $S$ lies in $M$. The image of the block matrix

$$
\left(\begin{array}{lll} 
& I_{2} & \\
I_{2} & & \\
& & I_{2}
\end{array}\right)
$$

in $S$ normalizes $E$ and is thus contained in $M$. It follows that the image of

$$
\left(\begin{array}{ll}
A & \\
& I_{2}
\end{array}\right)
$$

in $S$ lies in $M$. Consequently, $S \leq M$.
By Lemma 3.23, $\operatorname{Aut}_{P S L_{n}(q)}(S)=\operatorname{Inn}(S)$. As $\mathcal{F}_{S}(G)=\mathcal{F}_{S}\left(P S L_{n}(q)\right)$, it follows that $\operatorname{Aut}_{G}(S)=$ $\operatorname{Inn}(S)$, and so $N_{G}(S)=S C_{G}(S)$. We have seen above that $S \leq M$, and we have $C_{G}(S) \leq M$ by (1). Hence $N_{G}(S) \leq M$.
(3) $C_{G}(t) \leq M$.

Let $E_{1}$ be the subgroup of $X_{1}$ consisting of all $t_{A}$, where $A$ is a subset of $\{1, \ldots, n-2\}$ of even order. As a consequence of Lemma 7.14 , there is an $E_{8}$-subgroup $E_{0}$ of $E$ such that $K=\left\langle O(K), C_{K}\left(E_{0}\right), N_{K}\left(E_{1}\right)\right\rangle$. By (1), $C_{K}\left(E_{0}\right)$ and $N_{K}\left(E_{1}\right)$ are subgroups of $M$. By [28, Proposition 11.23], we have

$$
O(K)=\left\langle C_{O(K)}(B) \mid B \leq E, m(B)=3\right\rangle .
$$

Therefore, $O(K) \leq M$ by (1). Consequently, $K \leq M$. By the Frattini argument,

$$
C_{G}(t)=K N_{C_{G}(t)}\left(X_{1}\right) .
$$

So it suffices to show that $N_{C_{G}(t)}\left(X_{1}\right) \leq M$. Since $\mathcal{F}_{S}(G)=\mathcal{F}_{S}\left(P S L_{n}(q)\right)$, we may conclude from Lemma 5.7 that $\operatorname{Aut}_{C_{G}(t)}\left(X_{1}\right)$ is a 2-group. Hence, $N_{C_{G}(t)}\left(X_{1}\right) / C_{C_{G}(t)}\left(X_{1}\right)$ is a 2-group. As $X_{1} \unlhd T=S \in \operatorname{Syl}_{2}\left(C_{G}(t)\right)$, it follows that $N_{C_{G}(t)}\left(X_{1}\right)=S C_{C_{G}(t)}\left(X_{1}\right)$. We have $S \leq M$ by (2), and $C_{C_{G}(t)}\left(X_{1}\right) \leq C_{G}\left(E_{1}\right) \leq M$ by (1). Consequently, $N_{C_{G}(t)}\left(X_{1}\right) \leq M$, as required.
(4) Let $x$ be an involution of $S$ which is $G$-conjugate to $t$. Then $x$ is $M$-conjugate to $t$.

It is easy to see that if an element of $T$ is $\operatorname{PSL}_{n}(q)$-conjugate to $t$, then it is $C_{P S L_{n}(q)}(t)$ conjugate to an element of $E$. As $\mathcal{F}_{S}(G)=\mathcal{F}_{S}\left(P S L_{n}(q)\right)$ and $S=T$, it follows that $x$ is $C_{G}(t)$-conjugate and hence $M$-conjugate to an element $y$ of $E$. It is rather easy to show that if an element of $E$ is $P S L_{n}(q)$-conjugate to $t$, then it is $N_{P S L_{n}(q)}(E)$-conjugate to $t$. So, as $\mathcal{F}_{S}(G)=\mathcal{F}_{S}\left(P S L_{n}(q)\right)$, we have that $y$ is $N_{G}(E)$-conjugate to $t$. By $(1), N_{G}(E) \leq M$, and so $x$ is $M$-conjugate to $t$.
(5) Let $x$ be an involution of $S$. Then $C_{G}(x) \leq M$.

Let $R$ be a Sylow 2-subgroup of $C_{G}(x)$ with $C_{S}(x) \leq R$. We have $t \in Z(S) \leq C_{S}(x)$ and $t \in M$. Thus $t \in R \cap M$. Let $r \in N_{R}(R \cap M)$. Then $y:=t^{r} \in R \cap M$. As a consequence of (4), $y$ is $M$-conjugate to $t$. So there is an element $m$ of $M$ such that $t^{r}=y=t^{m}$. We have $r m^{-1} \in C_{G}(t) \leq M$ by (3), and so $r \in R \cap M$. Hence, $N_{R}(R \cap M)=R \cap M$, and thus $R=R \cap M$.

By Corollary 7.3, $C_{G}(x)$ is 3 -generated. Therefore, $C_{G}(x)$ is generated by the normalizers $N_{C_{G}(x)}(U)$, where $U \leq R$ and $m(U) \geq 3$. It suffices to show that each such normalizer lies in $M$.

So let $U \leq R$ with $m(U) \geq 3$, and let $g \in N_{C_{G}(x)}(U)$. Take an elementary abelian subgroup $Q$ of $U$ of rank 3 . Lemma 2.3 shows that any $E_{8}$-subgroup of $S$ has an involution which is the image of an involution of $S L_{n}(q)$. This implies that $Q$ has an element $s$ which is $G$-conjugate to $t$. Since $s, s^{g} \in U \leq R \leq M$, we see from (4) that $s$ and $s^{g}$ are $M$-conjugate to $t$. So there are elements $m, m^{\prime} \in M$ such that $s=t^{m}$ and $s^{g}=t^{m^{\prime}}$. We have $t^{m^{\prime}}=s^{g}=\left(t^{m}\right)^{g}=t^{m g}$. Thus $m g m^{\prime-1} \in C_{G}(t) \leq M$, and hence $g \in M$. It follows that $N_{C_{G}(x)}(U) \leq M$.
(6) $M=G$.

Assume that $M \neq G$. By [28, Proposition 17.11], we may deduce from (2) and (5) that $M$ is strongly embedded in $G$, i.e. $M \cap M^{g}$ has odd order for any $g \in G \backslash M$. Applying [49, Chapter $6,4.4$ ], it follows that $G$ has only one conjugacy class of involutions. On the other hand, we see from Proposition 3.5 that $G$ has precisely two conjugacy classes of involutions. This contradiction shows that $M=G$.
(7) Conclusion.

Let $F$ be a Klein four subgroup of $E$. Clearly, $\Delta_{G}(F) \leq W_{0}$. By (6), we have $W_{0} \unlhd G$. Since $G$ is 2-balanced, $W_{0}$ has odd order by Proposition 2.8 (ii). As $O(G)=1$ by Hypothesis 5.1, it follows that $W_{0}=1$. Hence $\Delta_{G}(F)=1$.
7.5. Quasisimplicity of the 2-components of $C_{G}(t)$. In this subsection, we determine the isomorphism types of $K$ and $L$.
Lemma 7.16. Let $x$ and $y$ be two commuting involutions of $G$. Set $C:=C_{G}(x)$ and $\bar{C}:=$ $C / O(C)$. Then any 2 -component of $C_{\bar{C}}(\bar{y})$ is a component of $C_{\bar{C}}(\bar{y})$.

Proof. By [32, Corollary 3.2], $L_{2^{\prime}}\left(C_{\bar{C}}(\bar{y})\right)=L_{2^{\prime}}\left(C_{L(\bar{C})}(\bar{y})\right)$. We know from Section 6 that $L(\bar{C})$ is a $K$-group, i.e. the composition factors of $L(\bar{C})$ are known finite simple groups. Applying [26, Theorem 3.5], we conclude that $L_{2^{\prime}}\left(C_{L(\bar{C})}(\bar{y})\right)=L\left(C_{L(\bar{C})}(\bar{y})\right)$. Therefore, any 2-component of $C_{L(\bar{C})}(\bar{y})$ is a component of $C_{L(\bar{C})}(\bar{y})$. So any 2-component of $C_{\bar{C}}(\bar{y})$ is a component of $C_{\bar{C}}(\bar{y})$.

Instead of using [26, Theorem 3.5], the lemma could be proved directly by using Corollary 3.46 (i) and the results of Section 6 .

Proposition 7.17. $K$ is isomorphic to a quotient of $S L_{n-2}^{\varepsilon}\left(q^{*}\right)$ by a central subgroup of odd order.

Proof. The proof is inspired from the proof of [32, Theorem 10.1].
For $q^{*}=3$, the proposition follows from Proposition 7.7. From now on, we assume that $q^{*} \neq 3$.
Set $C:=C_{G}(t)$. Let $E$ denote the subgroup of $T$ consisting of all $t_{A}$, where $A \subseteq\{1, \ldots, n\}$ has even order. We assume $m_{1}, \ldots, m_{\ell}$, where $\ell:=n-4$, to be elements of $K$ and $E_{0}$ to be an $E_{8}$-subgroup of $E$ with

$$
K=\left\langle O(K), L_{2^{\prime}}\left(C_{K}\left(E_{0}\right)\right), L_{2^{\prime}}\left(C_{K}\left(E_{0}\right)\right)^{m_{1}}, \ldots, L_{2^{\prime}}\left(C_{K}\left(E_{0}\right)\right)^{m_{\ell}}\right\rangle .
$$

Such elements $m_{1}, \ldots, m_{\ell}$ and such a subgroup $E_{0}$ exist by Lemma 7.14 .
The proof will be accomplished step by step.
(1) Let $f$ be an involution of $E_{0}$. Then $L_{2^{\prime}}\left(C_{K}\left(E_{0}\right)\right) \leq L_{2^{\prime}}\left(C_{C}(f)\right)$.

As $K \unlhd C$, we have $C_{K}\left(E_{0}\right) \unlhd C_{C}\left(E_{0}\right)$. This implies $L_{2^{\prime}}\left(C_{K}\left(E_{0}\right)\right) \leq L_{2^{\prime}}\left(C_{C}\left(E_{0}\right)\right)$. By [32, Theorem 3.1], we have $L_{2^{\prime}}\left(C_{C_{C}(f)}\left(E_{0}\right)\right) \leq L_{2^{\prime}}\left(C_{C}(f)\right)$. Clearly, $C_{C_{C}(f)}\left(E_{0}\right)=C_{C}\left(E_{0}\right)$. It follows that $L_{2^{\prime}}\left(C_{K}\left(E_{0}\right)\right) \leq L_{2^{\prime}}\left(C_{C}\left(E_{0}\right)\right) \leq L_{2^{\prime}}\left(C_{C}(f)\right)$.
(2) Let $F$ be a Klein four subgroup of $E_{0}$. Set $D:=\left[C_{O(K)}(F), L_{2^{\prime}}\left(C_{K}\left(E_{0}\right)\right)\right]$. Then $D=1$.

Clearly, $L_{2^{\prime}}\left(C_{K}\left(E_{0}\right)\right)$ normalizes $C_{O(K)}(F)$. Also, $O^{2^{\prime}}\left(L_{2^{\prime}}\left(C_{K}\left(E_{0}\right)\right)\right)=L_{2^{\prime}}\left(C_{K}\left(E_{0}\right)\right)$, and $C_{O(K)}(F)$ is a $2^{\prime}$-group. Applying [28, Proposition 4.3 (i)], we conclude that $D=\left[D, L_{2^{\prime}}\left(C_{K}\left(E_{0}\right)\right)\right]$.

Now let $f$ be an involution of $F$. We are going to show that $D \leq O\left(C_{G}(f)\right)$. Set $M:=$ $L_{2^{\prime}}\left(C_{C}(f)\right)$. By (1), $L_{2^{\prime}}\left(C_{K}\left(E_{0}\right)\right) \leq M$. Also, $D \leq C_{C}(F) \leq C_{C}(f)$ and $M \unlhd C_{C}(f)$. It follows that $D=\left[D, L_{2^{\prime}}\left(C_{K}\left(E_{0}\right)\right)\right] \leq\left[C_{C}(f), M\right] \leq M$.

Let $\overline{C_{G}(f)}:=C_{G}(f) / O\left(C_{G}(f)\right)$. By Corollary 2.2. $C_{\overline{C_{G}(f)}}(\bar{t})=\overline{C_{C}(f)}$. As a consequence of Proposition 2.4, $L_{2^{\prime}}\left(C_{\overline{C_{G}(f)}}(\bar{t})\right)=\bar{M}$. Lemma 7.16 implies that $\bar{M}=L\left(C_{\overline{C_{G}(f)}}(\bar{t})\right)$. It easily follows that $\bar{O}(\bar{M})$ is central in $\bar{M}$.

From the definition of $D$, it is clear that $D \leq O(K)$. So we have $D \leq M \cap O(K) \leq O(M)$. It follows that $\bar{D} \leq \overline{O(M)} \leq O(\bar{M}) \leq Z(\bar{M})$. In particular, $\overline{L_{2^{\prime}}\left(C_{K}\left(E_{0}\right)\right)}$ centralizes $\bar{D}$. Thus $D=\left[D, L_{2^{\prime}}\left(C_{K}\left(E_{0}\right)\right)\right] \leq O\left(C_{G}(f)\right)$.

Since $f$ was arbitrarily chosen, it follows that $D \leq \Delta_{G}(F)$. By Lemmas 7.13 and 7.15 , we have $\Delta_{G}(F)=1$. Consequently, $D=1$, as wanted.
(3) $O(K) \leq Z(K)$.

By [28, Proposition 11.23], we have

$$
O(K)=\left\langle C_{O(K)}(F): F \leq E_{0}, m(F)=2\right\rangle .
$$

Because of (2), it follows that $O(K)$ centralizes $L_{2^{\prime}}\left(C_{K}\left(E_{0}\right)\right)$. By choice of $E_{0}$, we have

$$
K=\left\langle O(K), L_{2^{\prime}}\left(C_{K}\left(E_{0}\right)\right), L_{2^{\prime}}\left(C_{K}\left(E_{0}\right)\right)^{m_{1}}, \ldots, L_{2^{\prime}}\left(C_{K}\left(E_{0}\right)\right)^{m_{\ell}}\right\rangle
$$

for some $m_{1}, \ldots, m_{\ell} \in K$. It follows that $K=O(K) C_{K}(O(K))$. Therefore, $C_{K}(O(K))$ has odd index in $K$. We have $O^{2^{\prime}}(K)=K$ since $K$ is a 2-component of $C$. It follows that $K=C_{K}(O(K))$. Consequently, $O(K) \leq Z(K)$.
(4) Conclusion.

Applying [28, Lemma 4.11], we deduce from (3) that $K$ is a component of $C$. Therefore, $K$ is quasisimple. We have

$$
K / Z(K) \cong(K / O(K)) / Z(K / O(K)) \cong P S L_{n-2}^{\varepsilon}\left(q^{*}\right)
$$

Applying Lemmas 3.1 and 3.2 , we conclude that $K \cong S L_{n-2}^{\varepsilon}\left(q^{*}\right) / Z$ for some central subgroup $Z$ of $S L_{n-2}^{\varepsilon}\left(q^{*}\right)$. Using Proposition 3.19, or using the order formulas for $\left|S L_{n-2}^{\varepsilon}\left(q^{*}\right)\right|$ and $\left|S L_{n-2}(q)\right|$ given by [33, Proposition 1.1 and Corollary 11.29], we see that

$$
\left|S L_{n-2}^{\varepsilon}\left(q^{*}\right)\right|_{2}=\left|S L_{n-2}(q)\right|_{2}=\left|X_{1}\right|=|K|_{2}=\left|S L_{n-2}^{\varepsilon}\left(q^{*}\right) / Z\right|_{2} .
$$

Thus $Z$ has odd order.
Proposition 7.18. We have $L \cong S L_{2}\left(q^{*}\right)$ and $L \unlhd C_{G}(t)$. Moreover, $L$ is the only normal subgroup of $C_{G}(t)$ which is isomorphic to $S L_{2}\left(q^{*}\right)$.
Proof. For $q^{*}=3$, this follows from Propositions 7.7 and 6.8 .
Assume now that $q^{*} \neq 3$. Let $\widetilde{K}:=K O\left(C_{G}(t)\right)$. By the last statement in Proposition 2.4 , $K=O^{2^{\prime}}(\widetilde{K})$. Let $i \in\{1,2\}$. Since $A_{i}$ is a 2-component of $C_{C_{G}(t)}(u)$, we have $A_{i}=O^{2^{\prime}}\left(A_{i}\right)$. Also, $A_{i} \leq \widetilde{K}$, and so $A_{i} \leq O^{2^{\prime}}(\widetilde{K})=K$. It follows that $A_{i}$ is a 2-component of $C_{K}(u)$.

By Proposition 7.17, we have $K \cong S L_{n-2}^{\varepsilon}\left(q^{*}\right) / Z$ for some central subgroup $Z$ of $S L_{n-2}^{\varepsilon}\left(q^{*}\right)$ with odd order. It is easy to see that if $m$ is a non-central involution of $S L_{n-2}^{\varepsilon}\left(q^{*}\right) / Z$ and $J$ is a 2-component of its centralizer in $S L_{n-2}^{\varepsilon}\left(q^{*}\right) / Z$, then $J \cong S L_{k}^{\varepsilon}\left(q^{*}\right)$ for some $k \geq 2$. Since $u$ is a non-central involution of $K$ and $A_{1} / O\left(A_{1}\right) \cong S L_{2}\left(q^{*}\right)$, it follows that $A_{1} \cong S L_{2}\left(q^{*}\right)$. By definition of $L$ (see Proposition 6.8), $L$ is isomorphic to $A_{1}$. So we have $L \cong S L_{2}\left(q^{*}\right)$.

Let $L_{0}$ be the 2-component of $C_{G}(t)$ associated to $L O\left(C_{G}(t)\right) / O\left(C_{G}(t)\right)$. By [37, 6.5.2], we have $\left[L_{0}, K\right]=1$. Hence $L_{0} \leq C_{C_{G}(t)}(u)$. So $L_{0}$ is a 2-component of $C_{C_{G}(t)}(u)$. Clearly $A_{1} \neq L_{0} \neq A_{2}$. Lemma 7.10 implies that $L_{0}=L$. From Proposition 6.8 (iii), we see that $L=L_{0} \unlhd C_{G}(t)$.

Proposition 6.8 (iii) also shows that $K$ and $L$ are the only 2 -components of $C_{G}(t)$. So $L$ is the only normal subgroup of $C_{G}(t)$ isomorphic to $S L_{2}\left(q^{*}\right)$.

## 8. The subgroup $G_{0}$

Let $A$ be a subset of $\{1, \ldots, n\}$ with order 2. Then $t_{A}$ is $G$-conjugate to $t$. Proposition 7.18 implies that $C_{G}\left(t_{A}\right)$ has a unique normal subgroup isomorphic to $S L_{2}\left(q^{*}\right)$. We denote this subgroup by $L_{A}$, and we define $G_{0}$ to be the subgroup of $G$ generated by the groups $L_{A}$, where $A=\{i, i+1\}$ for some $1 \leq i<n$. We are going to prove that $G_{0} \unlhd G$ and that $G_{0}$ is isomorphic to a nontrivial quotient of $S L_{n}^{\varepsilon}\left(q^{*}\right)$. This will complete the proof of Theorem 5.2.

By Proposition 7.17, $K$ is isomorphic to a quotient of $S L_{n-2}^{\varepsilon}\left(q^{*}\right)$ by a central subgroup of odd order. By the proof of Proposition 7.18, $A_{1}$ and $A_{2}$ are 2-components of $C_{K}(u)$ if $q^{*} \neq 3$.
Lemma 8.1. Let $Z \leq Z\left(S L_{n-2}^{\varepsilon}\left(q^{*}\right)\right)$ with $K \cong H:=S L_{n-2}^{\varepsilon}\left(q^{*}\right) / Z$. Let $H_{1}$ be the image of

$$
\left\{\left(\begin{array}{cc}
A & \\
& I_{n-4}
\end{array}\right): A \in S L_{2}^{\varepsilon}\left(q^{*}\right)\right\}
$$

in H and $\mathrm{H}_{2}$ the image of

$$
\left\{\left(\begin{array}{ll}
I_{2} & \\
& A
\end{array}\right): A \in S L_{n-4}^{\varepsilon}\left(q^{*}\right)\right\}
$$

in $H$. Then there is a group isomorphism $\varphi: K \rightarrow H$ which maps $A_{1}$ to $H_{1}$ and $A_{2}$ to $H_{2}$.
Proof. For $q^{*}=3$, this follows from Proposition 7.7 and Lemma 6.4 (iii).
Assume now that $q^{*} \neq 3$. Let $\varphi: K \rightarrow H$ be a group isomorphism. For each even natural number $k$ with $2 \leq k<n-2$, let $h_{k}$ be the image of

$$
\left(\begin{array}{cc}
-I_{k} & \\
& I_{n-2-k}
\end{array}\right)
$$

in $H$. It is easy to note that each non-central involution of $H$ is conjugate to $h_{k}$ for some even $2 \leq k<n-2$. As $u$ is a non-central involution of $K$, we may assume that $u^{\varphi}=t_{k}$ for some even $2 \leq k<n-2$.

Let $\widetilde{H_{1}}$ be the image of

$$
\left\{\left(\begin{array}{cc}
A & \\
& I_{n-2-k}
\end{array}\right): A \in S L_{k}^{\varepsilon}\left(q^{*}\right)\right\}
$$

in $H$ and $\widetilde{H_{2}}$ be the image of

$$
\left\{\left(\begin{array}{ll}
I_{k} & \\
& A
\end{array}\right): A \in S L_{n-2-k}^{\varepsilon}\left(q^{*}\right)\right\}
$$

in $H$. It is easy to note that the 2-components of $C_{H}\left(t_{k}\right)$ are precisely the quasisimple elements of $\left\{\widetilde{H_{1}}, \widetilde{H_{2}}\right\}$. Also, $t_{k} \in \widetilde{H_{1}}$, but $t_{k} \notin \widetilde{H_{2}}$. On the other hand, $A_{1}$ and $A_{2}$ are the 2-components of $C_{K}(u)$, and we have $u \in A_{1}$. This implies $\left(A_{1}\right)^{\varphi}=\widetilde{H_{1}}$ and $\left(A_{2}\right)^{\varphi}=\widetilde{H_{2}}$. Since $A_{1} \cong L \cong S L_{2}\left(q^{*}\right)$, we have $k=2$, and hence $\widetilde{H_{1}}=H_{1}$ and $\widetilde{H_{2}}=H_{2}$.

Lemma 8.2. Let $1 \leq i<j<n$. Set $A:=\{i, i+1\}$ and $B:=\{j, j+1\}$. Then:
(i) If $i+1<j$, then $\left[L_{A}, L_{B}\right]=1$.
(ii) Suppose that $j=i+1$. Then there is a group isomorphism from $\left\langle L_{A}, L_{B}\right\rangle$ to $S L_{3}^{\varepsilon}\left(q^{*}\right)$ under which $L_{A}$ corresponds to the subgroup

$$
\left\{\left(\begin{array}{c|c}
M & 0 \\
& 0 \\
\hline 0 & 0 \\
1
\end{array}\right): M \in S L_{2}^{\varepsilon}\left(q^{*}\right)\right\}
$$

of $S L_{3}^{\varepsilon}\left(q^{*}\right)$ and under which $L_{B}$ corresponds to the subgroup

$$
\left\{\left(\begin{array}{c|c}
1 & 0 \\
\hline 0 & 0 \\
\hline 0 & M
\end{array}\right): M \in S L_{2}^{\varepsilon}\left(q^{*}\right)\right\}
$$

of $S L_{3}^{\varepsilon}\left(q^{*}\right)$.
(iii) Suppose that $1 \leq i \leq n-3$ and that $j=i+1$. Set $k:=i+2$ and $C:=\{k, k+1\}$. Then $\left\langle L_{A}, L_{B}, L_{C}\right\rangle$ is isomorphic to $S L_{4}^{\varepsilon}\left(q^{*}\right)$.

Proof. Let $H, H_{1}, H_{2}$ and $\varphi$ be as in Lemma 8.1. For each $D \subseteq\{1, \ldots, n-2\}$ of even order, let $h_{D}$ be the image of the matrix $\operatorname{diag}\left(d_{1}, \ldots, d_{n-2}\right) \in S L_{n-2}^{\varepsilon}\left(q^{*}\right)$ in $H$, where $d_{\ell}=-1$ if $\ell \in D$ and $d_{\ell}=1$ if $\ell \in\{1, \ldots, n-2\} \backslash D$. Note that $u^{\varphi}=h_{\{1,2\}}$. Let $J$ be the subgroup of $H$ consisting of all $h_{D}$, where $D \subseteq\{1, \ldots, n-2\}$ has even order, and let $E_{1}$ denote the subgroup of $X_{1}$ consisting of all $t_{D}$, where $D \subseteq\{1, \ldots, n-2\}$ has even order. From Lemma 3.22, we see that $\left(E_{1}\right)^{\varphi}$ is $C_{H}\left(u^{\varphi}\right)$-conjugate to $J$. Upon replacing $\varphi$ by a composite of $\varphi$ and an inner automorphism of $H$, we may (and will) assume that $\left(E_{1}\right)^{\varphi}=J$.

From the definition of $L$ (Proposition 6.8), it is easy to see that $L_{\{1,2\}}=A_{1}$.
We now prove (i). Assume that $i+1<j$. Since $\mathcal{F}_{S}(G)=\mathcal{F}_{S}\left(P S L_{n}(q)\right)$, there is some $g \in G$ with $\left(t_{A}\right)^{g}=t_{\{1,2\}}=u$ and $\left(t_{B}\right)^{g}=t_{\{3,4\}}$. So it suffices to show that $\left[L_{\{1,2\}}, L_{\{3,4\}}\right]=1$. Let $h$ denote the image of $t_{\{3,4\}}$ under $\varphi$. Then $h \in H_{2}$ since $t_{\{3,4\}} \in T_{2} \leq A_{2}$. Therefore, and since $h$ is conjugate to $u^{\varphi}=h_{\{1,2\}}$, we may choose $\varphi$ such that $h=h_{\{3,4\}}$ (and for the rest of the proof of
(i), we will assume that $\varphi$ has been chosen in this way). We see from Lemma 3.38(ii) that there is an $a \in H$ with $\left(h_{\{1,2\}}\right)^{a}=h_{\{3,4\}}$ and $\left(H_{1}\right)^{a} \leq H_{2}$. In particular, $\left[H_{1},\left(H_{1}\right)^{a}\right]=1$. If $k$ is the preimage of $a$ under $\varphi$, then $u^{k}=t_{\{3,4\}}$ and $\left[A_{1},\left(A_{1}\right)^{k}\right]=1$. We also have $\left(A_{1}\right)^{k}=L_{\{3,4\}}$ and thus $\left[L_{\{1,2\}}, L_{\{3,4\}}\right]=1$.

We now prove (ii). Assume that $j=i+1$. Since $\mathcal{F}_{S}(G)=\mathcal{F}_{S}\left(P S L_{n}(q)\right)$, there is some $g \in G$ with $\left(t_{A}\right)^{g}=t_{\{1,2\}}$ and $\left(t_{B}\right)^{g}=t_{\{2,3\}}$. Therefore, it is enough to prove (ii) under the assumption that $i=1$, and we will assume that this is the case. We see from Lemmas 6.4 (ii) and 6.6 that $X_{1} \cap A_{2}=T_{2}$. Thus $t_{\{2,3\}} \notin A_{2}$. Let $h$ denote the image of $t_{\{2,3\}}$ under $\varphi$. Then $h \notin H_{2}$. Therefore, and since $h$ is conjugate to $u^{\varphi}=h_{\{1,2\}}$, we may choose $\varphi$ such that $h=h_{\{2,3\}}$ (and for the rest of the proof of (ii), we will assume that $\varphi$ has been chosen in this way). Let $\widetilde{H_{1}}$ be the image of

$$
\left\{\left(\begin{array}{ccc}
1 & & \\
& M & \\
& & I_{n-5}
\end{array}\right): M \in S L_{2}^{\varepsilon}\left(q^{*}\right)\right\}
$$

in $H$. By Lemma 3.38 (ii), there is some $a \in H$ with $\left(h_{\{1,2\}}\right)^{a}=h_{\{2,3\}}$ and $\left(H_{1}\right)^{a}=\widetilde{H_{1}}$. Let $k$ be the preimage of $a$ under $\varphi$. Then $u^{k}=t_{\{2,3\}}$ and hence $L_{\{2,3\}}=\left(L_{\{1,2\}}\right)^{k}=\left(A_{1}\right)^{k}$. We see now that $\varphi$ induces an isomorphism from $\left\langle L_{\{1,2\}}, L_{\{2,3\}}\right\rangle$ to $\left\langle H_{1}, \widetilde{H_{1}}\right\rangle$ mapping $L_{\{1,2\}}$ to $H_{1}$ and $L_{\{2,3\}}$ to $\widetilde{H_{1}}$. With this observation, it is easy to complete the proof of (ii).

We now prove (iii). Assume that $1 \leq i \leq n-3$ and that $j=i+1$. Let $k$ and $C$ be as in the statement of (iii). Since $\mathcal{F}_{S}(G)=\mathcal{F}_{S}\left(P S L_{n}(q)\right)$, there is some $g \in G$ with $\left(t_{A}\right)^{g}=t_{\{1,2\}}=u$, $\left(t_{B}\right)^{g}=t_{\{2,3\}}$ and $\left(t_{C}\right)^{g}=t_{\{3,4\}}$. Therefore, it is enough to show that $\left\langle L_{\{1,2\}}, L_{\{2,3\}}, L_{\{3,4\}}\right\rangle$ is isomorphic to $S L_{4}^{\varepsilon}\left(q^{*}\right)$. Let $h:=\left(t_{\{2,3\}}\right)^{\varphi}$ and $\widetilde{h}:=\left(t_{\{3,4\}}\right)^{\varphi}$. As in the proof of (ii), we can choose $\varphi$ such that $h=h_{\{2,3\}}$. Also, $\widetilde{h}=h_{D}$ for some $D \subseteq\{1, \ldots, n-2\}$ of order 2 . We have $t_{\{3,4\}} \in T_{2} \leq A_{2}$ and hence $h_{D}=\widetilde{h} \in H_{2}$. Therefore, $D \cap\{1,2\}=\emptyset$. We claim that $D \cap\{2,3\}=\{3\}$. Assume not. Then $D \cap\{2,3\}=\emptyset$, and it is easy to find an element $a \in N_{H}(J)$ with $h^{a}=h_{\{1,2\}}=u^{\varphi}$ and $(\widetilde{h})^{a}=h_{\{3,4\}} \in H_{2}$. So there is some $k \in N_{K}\left(E_{1}\right)$ with $\left(t_{\{2,3\}}\right)^{k}=u$ and $\left(t_{\{3,4\}}\right)^{k} \in T_{2}$. On the other hand, it is easy to see from $\mathcal{F}_{S}(G)=\mathcal{F}_{S}\left(P S L_{n}(q)\right)$ that there is no $g \in K$ with $\left(t_{\{2,3\}}\right)^{g}=u$ and $\left(t_{\{3,4\}}\right)^{g} \in T_{2}$. This contradiction shows that $D \cap\{2,3\}=\{3\}$. So we can choose $\varphi$ such that $h=h_{\{2,3\}}$ and $\widetilde{h}=h_{\{3,4\}}$. Now the proof of (iii) can be completed by using similar arguments as in the proof of (ii).

Proposition 8.3. $G_{0}$ is isomorphic to a nontrivial quotient of $S L_{n}^{\varepsilon}\left(q^{*}\right)$.
Proof. Assume that $\varepsilon=+$. By Lemma 8.2 , the groups $L_{\{1,2\}}, \ldots, L_{\{n-1, n\}}$ form a weak CurtisTits system in $G$ of type $S L_{n}\left(q^{*}\right)$ (in the sense of [30, p. 9]). Applying a version of the Curtis-Tits theorem, namely [30, Chapter 13, Theorem 1.4], we conclude that $G_{0}$ is isomorphic to a quotient of $S L_{n}\left(q^{*}\right)$.

Assume now that $\varepsilon=-$. Then Lemma 8.2 shows that $G_{0}$ has a weak Phan system of rank $n-1$ over $\mathbb{F}_{q^{* 2}}$ (in the sense of [14, p. 288]). If $q^{*} \neq 3$, then [14, Theorem 1.2] implies that $G_{0}$ is isomorphic to a quotient of $S U_{n}\left(q^{*}\right)$. If $q^{*}=3$, the same follows from [14, Theorem 1.3] and Lemma 8.2 (iii).

Lemma 8.4. Let $R$ be a Sylow 2 -subgroup of $G_{0}$. Then $R \in \operatorname{Syl}_{2}(G)$ and $\mathcal{F}_{R}\left(G_{0}\right)=\mathcal{F}_{R}(G)$.
Proof. Since $q \sim \varepsilon q^{*}$, we have that the 2-fusion system of $\operatorname{PSL} L_{n}^{\varepsilon}\left(q^{*}\right)$ is isomorphic to the 2fusion system of $P S L_{n}(q)$ (see Proposition 3.20). Clearly, $G_{0} / Z\left(G_{0}\right) \cong P S L_{n}^{\varepsilon}\left(q^{*}\right)$. So the 2fusion system of $G_{0} / Z\left(G_{0}\right)$ is isomorphic to the 2 -fusion system of $G$. It easily follows that $\left|G_{0}\right|_{2}=\left|G_{0} / Z\left(G_{0}\right)\right|_{2}=|G|_{2}$, and Lemma 2.11 shows that the 2-fusion system of $G_{0}$ is isomorphic to that of $G_{0} / Z\left(G_{0}\right)$ and hence to that of $G$. This completes the proof.

Lemma 8.5. The following hold.
(i) If $q^{*} \neq 3$, then $O^{2^{\prime}}\left(O^{2}\left(C_{G}(t)\right)\right)=K L$.
(ii) If $q^{*}=3$, then $O^{2}\left(C_{G}(t)\right)=K L$.

Proof. Set $C:=C_{G}(t)$.
Assume that $q^{*} \neq 3$. Then $K L$ is perfect. This implies that $K L=O^{2^{\prime}}\left(O^{2}(K L)\right) \leq O^{2^{\prime}}\left(O^{2}(C)\right)$. Since $T \cap K L=(T \cap K)(T \cap L)=X_{1} X_{2}$, Lemmas 5.4 and 2.11 show that $C / K L$ has a nilpotent 2 -fusion system. So $C / K L$ is 2-nilpotent by [39, Theorem 1.4]. This implies $O^{2^{\prime}}\left(O^{2}(C)\right) \leq K L$.

We assume now that $q^{*}=3$. Then $K L=O^{2}(K L)$ since $K$ is perfect and $L \cong S L_{2}(3)$. Thus $K L \leq O^{2}(C)$. In order to prove equality, it suffices to show that $C / K L$ is a 2 -group. By Proposition 7.7 and Lemma 6.3(i), $C / K C_{C}(K)$ is a 2-group. By [37, 6.5.2], we have $L \leq C_{C}(K)$. It is enough to show that $C_{C}(K) / L$ is a 2 -group.

We have $O^{2}\left(C_{C}(K)\right) \cap T \leq O^{2}\left(C_{C}\left(X_{1}\right)\right) \cap T=X_{2}$ by Lemma 5.6 and the hyperfocal subgroup theorem [19, Theorem 1.33]. On the other hand, $X_{2} \leq L=O^{2}(L) \leq O^{2}\left(C_{C}(K)\right)$. Consequently, $X_{2}=O^{2}\left(C_{C}(K)\right) \cap T \in \operatorname{Syl}_{2}\left(O^{2}\left(C_{C}(K)\right)\right)$. Set $U:=C_{O^{2}\left(C_{C}(K)\right)}\left(X_{2}\right)$. We have $X_{2} \unlhd C$ since $X_{2}$ is the unique Sylow 2-subgroup of $L \cong S L_{2}(3)$. So we have $U \unlhd C$. Hence $Z\left(X_{2}\right)=X_{2} \cap U \in$ $\operatorname{Syl}_{2}(U)$. Applying [37, 7.2.2], we conclude that $U$ is 2-nilpotent. We have $O(U)=1$ since $U \unlhd C$ and $O(C)=1$ by Proposition 7.7. It follows that $U=Z\left(X_{2}\right)$.

Clearly, $O^{2}\left(C_{C}(K)\right) / U$ is isomorphic to a subgroup of $\operatorname{Aut}\left(X_{2}\right)$. We have $\left|O^{2}\left(C_{C}(K)\right) / U\right|_{2}=4$ since $Q_{8} \cong X_{2} \in \operatorname{Syl}_{2}\left(O^{2}\left(C_{C}(K)\right)\right)$ and $U=Z\left(X_{2}\right)$. Also, $\left|O^{2}\left(C_{C}(K)\right) / U\right| \geq 12$ since $L \leq$ $O^{2}\left(C_{C}(K)\right)$. As $\operatorname{Aut}\left(X_{2}\right) \cong \operatorname{Aut}\left(Q_{8}\right) \cong S_{4}$ by [37, 5.3.3], it follows that $\left|O^{2}\left(C_{C}(K)\right) / U\right|=12$. This implies $O^{2}\left(C_{C}(K)\right)=L$. So $C_{C}(K) / L$ is a 2-group, as required.

Lemma 8.6. We have $K L \leq G_{0}$.
Proof. We have $t \in X_{2} \leq L=L_{\{n-1, n\}} \leq G_{0}$. Let $R \in \operatorname{Syl}_{2}\left(G_{0}\right)$ with $t \in R$ such that $\langle t\rangle$ is fully centralized in $\mathcal{G}:=\mathcal{F}_{R}\left(G_{0}\right)$. By Lemma $8.4, R \in \operatorname{Syl}_{2}(G)$ and $\mathcal{G}=\mathcal{F}_{R}(G)$. Therefore, $C_{R}(t) \in$ $\operatorname{Syl}_{2}\left(C_{G}(t)\right)$ and $C_{\mathcal{G}}(\langle t\rangle)=\mathcal{F}_{C_{R}(t)}\left(C_{G}(t)\right)$. Also, $T=C_{S}(t) \in \operatorname{Syl}_{2}\left(C_{G}(t)\right)$ and $C_{\mathcal{F}_{S}(G)}(\langle t\rangle)=$ $\mathcal{F}_{T}\left(C_{G}(t)\right)$. So, by Lemma 5.3, $C_{\mathcal{G}}(\langle t\rangle)$ has a component isomorphic to the 2-fusion system of $S L_{n-2}(q)$.

Let $Z \leq Z\left(S L_{n}^{\varepsilon}\left(q^{*}\right)\right)$ with $G_{0} \cong S L_{n}^{\varepsilon}\left(q^{*}\right) / Z$. By the proof of Lemma 8.4, $Z\left(G_{0}\right)$ has odd order.
Let $\widetilde{x}$ be an element of $S L_{n}^{\varepsilon}\left(q^{*}\right)$ such that $x:=\widetilde{x} Z$ is an involution of $S L_{n}^{\varepsilon}\left(q^{*}\right) / Z$. Set $C:=$ $C_{S L_{n}^{\varepsilon}\left(q^{*}\right) / Z}(x)$. It is easy to note that the 2 -components of $C$ are precisely the images of the 2-components of $C_{S L_{n}^{\varepsilon}\left(q^{*}\right)}(\widetilde{x})$ in $S L_{n}^{\varepsilon}\left(q^{*}\right) / Z$. Using this, it is not hard to see from Lemmas 3.3 and 3.4 that one of the following holds:
(1) $q^{*} \neq 3, O^{2^{\prime}}\left(O^{2}(C)\right)=K_{0} L_{0}$, where $K_{0}$ and $L_{0}$ are subnormal subgroups of $C$ such that $K_{0} \cong S L_{n-i}^{\varepsilon}\left(q^{*}\right)$ and $L_{0} \cong S L_{i}^{\varepsilon}\left(q^{*}\right)$ for some $1 \leq i<n$. Moreover, the 2 -components of $C$ are precisely the quasisimple elements of $\left\{K_{0}, L_{0}\right\}$.
(2) $q^{*}=3, O^{2}(C)=K_{0} L_{0}$, where $K_{0}$ and $L_{0}$ are subnormal subgroups of $C$ such that $K_{0} \cong S L_{n-i}^{\varepsilon}\left(q^{*}\right)$ and $L_{0} \cong S L_{i}^{\varepsilon}\left(q^{*}\right)$ for some $1 \leq i<n$. Moreover, the 2-components of $C$ are precisely the quasisimple elements of $\left\{K_{0}, L_{0}\right\}$.
(3) $C$ has precisely one 2-component, and this 2-component is isomorphic to a nontrivial quotient of $S L_{n / 2}\left(\left(q^{*}\right)^{2}\right)$.
As seen above, $C_{\mathcal{G}}(\langle t\rangle)=\mathcal{F}_{C_{R}(t)}\left(C_{G_{0}}(t)\right)$ has a component isomorphic to the 2-fusion system of $S L_{n-2}(q)$. By Proposition 2.16, this component is induced by a 2-component of $C_{G_{0}}(t)$. In view of the preceding observations, we can conclude that $C_{G_{0}}(t)$ has subgroups $K_{0}$ and $L_{0}$ with $K_{0} \cong S L_{n-2}^{\varepsilon}\left(q^{*}\right)$ and $L_{0} \cong S L_{2}\left(q^{*}\right)$ such that $O^{2^{\prime}}\left(O^{2}\left(C_{G_{0}}(t)\right)\right)=K_{0} L_{0}$ if $q^{*} \neq 3$ and $O^{2}\left(C_{G_{0}}(t)\right)=$ $K_{0} L_{0}$ if $q^{*}=3$.

Clearly, $O^{2^{\prime}}\left(O^{2}\left(C_{G_{0}}(t)\right)\right) \leq O^{2^{\prime}}\left(O^{2}\left(C_{G}(t)\right)\right)$ and $O^{2}\left(C_{G_{0}}(t)\right) \leq O^{2}\left(C_{G}(t)\right)$. Lemma 8.5 implies that $K_{0} L_{0} \leq K L$. If $n$ is odd, then it is easy to see that $\left|K_{0} L_{0}\right|=\left|K_{0}\right|\left|L_{0}\right| \geq|K||L|=|K L|$.

If $n$ is even, then one can easily see that $\left|K_{0} L_{0}\right|=\frac{1}{2}\left|K_{0}\right|\left|L_{0}\right| \geq \frac{1}{2}|K||L|=|K L|$. Consequently, $K_{0} L_{0} \leq K L$ and $\left|K_{0} L_{0}\right| \geq|K L|$. It follows that $K L=K_{0} L_{0} \leq G_{0}$.

Corollary 8.7. Let $x$ be an involution of $G_{0}$ which is $G$-conjugate to $t$. Let $L_{0}$ be the unique normal $S L_{2}\left(q^{*}\right)$-subgroup of $C_{G}(x)$, and let $K_{0}$ be the component of $C_{G}(x)$ different from $L_{0}$. Then we have $K_{0} L_{0} \leq G_{0}$.
Proof. Since $t \in G_{0}$, we see from Lemma 8.4 that there is some $g \in G_{0}$ with $x=t^{g}$. Clearly, $\left(K_{0} L_{0}\right)=(K L)^{g}$, and so $K_{0} L_{0} \leq G_{0}$ by Lemma 8.6.
Lemma 8.8. We have $N_{G}(S) \leq N_{G}\left(G_{0}\right)$.
Proof. Set $M:=N_{G}\left(G_{0}\right)$. Let $s \in N_{S}(S \cap M)$, and let $1 \leq i \leq n-1$. We have $t_{\{i, i+1\}} \in$ $S \cap L_{\{i, i+1\}} \leq S \cap G_{0} \leq S \cap M$, and hence $\left(t_{\{i, i+1\}}\right)^{s} \in S \cap M \leq M$. Since $G_{0}$ has odd index in $M$ by Lemma 8.4 , we even have $\left(t_{\{i, i+1\}}\right)^{s} \in G_{0}$. Corollary 8.7 implies that $\left(L_{\{i, i+1\}}\right)^{s} \leq G_{0}$. So we have $s \in M$ by the definition of $G_{0}$. Thus $N_{S}(S \cap M)=S \cap M$ and hence $S \leq M$. It is clear that $C_{G}(S) \leq M$. Using Lemma 3.23, we conclude that $N_{G}(S)=S C_{G}(S) \leq M$.
Lemma 8.9. If $x$ is an involution of $S$, then $C_{G}(x) \leq N_{G}\left(G_{0}\right)$.
Proof. Set $M:=N_{G}\left(G_{0}\right)$.
We begin by proving that $C_{G}(t) \leq M$. We have $K \leq G_{0} \leq M$ by Lemma 8.6 and $C_{G}(t)=$ $K N_{C_{G}(t)}\left(X_{1}\right)$ by the Frattini argument. Also, $N_{C_{G}(t)}\left(X_{1}\right)=T C_{C_{G}(t)}\left(X_{1}\right)$ as a consequence of Lemma 5.7, and $T \leq M$ by Lemma 8.8. So it suffices to show that $C_{C_{G}(t)}\left(X_{1}\right) \leq M$.

Let $z \in C_{C_{G}(t)}\left(X_{1}\right)$. In order to prove $z \in M$, it is enough to show that $\left(L_{\{i, i+1\}}\right)^{z} \leq G_{0}$ for all $1 \leq i<n$. If $1 \leq i<n$ and $i \neq n-2$, we have $z \in C_{G}\left(t_{\{i, i+1\}}\right)$ and hence $\left(L_{\{i, i+1\}}\right)^{z}=L_{\{i, i+1\}} \leq$ $G_{0}$. It remains to show that $\left(L_{\{n-2, n-1\}}\right)^{z} \leq G_{0}$. Since $\mathcal{F}_{S}(G)=\mathcal{F}_{S}\left(P S L_{n}(q)\right)$, there is some $g \in G$ with $t^{g}=u, u^{g}=t$ and $\left(t_{\{2,3\}}\right)^{g}=t_{\{n-2, n-1\}}$. From the definition of $L$ (Proposition 6.8), it is easy to see that $L_{\{1,2\}}=A_{1} \leq K$. Since $u=t_{\{1,2\}}$ and $t_{\{2,3\}}$ are $K$-conjugate, we thus have $L_{\{2,3\}} \leq K \leq L_{2^{\prime}}\left(C_{G}(t)\right)$. Hence $L_{\{n-2, n-1\}}=\left(L_{\{2,3\}}\right)^{g} \leq L_{2^{\prime}}\left(C_{G}(t)\right)^{g}=L_{2^{\prime}}\left(C_{G}(u)\right)$. Since $z$ centralizes $u$, it follows that $\left(L_{\{n-2, n-1\}}\right)^{z} \leq L_{2^{\prime}}\left(C_{G}(u)\right)$. From Corollary 8.7, we see that $L_{2^{\prime}}\left(C_{G}(u)\right) \leq G_{0}$. So we have $\left(L_{\{n-2, n-1\}}\right)^{z} \leq G_{0}$, and it follows that $C_{C_{G}(t)}\left(X_{1}\right) \leq M$. Consequently, $C_{G}(t) \leq M$.

Since $G_{0}$ has odd index in $M$ by Lemma 8.4 , we see from Lemma 8.8 that $S \leq G_{0}$. Also, $\mathcal{F}_{S}\left(G_{0}\right)=\mathcal{F}_{S}(G)$ by Lemma 8.4. As $C_{G}(t) \leq M$, it follows that $C_{G}(x) \leq M$ for any involution $x$ of $S$ which is $G$-conjugate to $t$.

Assume now that $x$ is an involution of $S$ which is $G$-conjugate to $t_{i}$ for some even natural number $i$ with $4 \leq i<n$ such that $i \leq \frac{n}{2}$ if $n$ is even. We are going to show that $C_{G}(x) \leq M$. Arguing by induction over $i$ and using the preceding observations, we may assume that for each even $2 \leq j<i$ and each involution $y$ of $S$ which is $G$-conjugate to $t_{j}$, we have $C_{G}(y) \leq M$. Furthermore, we may assume that $\langle x\rangle$ is fully $\mathcal{F}_{S}(G)$-centralized since $\mathcal{F}_{S}(G)=\mathcal{F}_{S}\left(G_{0}\right)$.

As a consequence of Lemma 7.1, $C_{G}(x)$ is generated by the normalizers $N_{C_{G}(x)}(U)$, where $U$ is a subgroup of $C_{S}(x)$ containing a $G$-conjugate of $t_{j}$ for some even $2 \leq j<i$. We show that each such normalizer is contained in $M$. Thus let $U$ be a subgroup of $C_{S}(x)$ and let $y$ be an element of $U$ which is $G$-conjugate to $t_{j}$ for some even $2 \leq j<i$. Also, let $g \in N_{C_{G}(x)}(U)$. Then $y^{g} \in U \leq C_{S}(x) \leq S$. Since $\mathcal{F}_{S}\left(G_{0}\right)=\mathcal{F}_{S}(G)$, we have that $y$ and $y^{g}$ are $G_{0}$-conjugate. Hence, there is some $m \in G_{0}$ with $y^{g}=y^{m}$. We have $m g^{-1} \in C_{G}(y) \leq M$. This implies $g \in M$ since $m \in G_{0} \leq M$. So we have $N_{C_{G}(x)}(U) \leq M$ and hence $C_{G}(x) \leq M$.

Assume now that $x$ is an arbitrary involution of $S$. We are going to prove that $C_{G}(x) \leq M$. Since $\mathcal{F}_{S}(G)=\mathcal{F}_{S}\left(G_{0}\right)$, we may assume that $\langle x\rangle$ is fully $\mathcal{F}_{S}(G)$-centralized. By Corollary 7.3 , $C_{G}(x)$ is 3 -generated. Therefore, $C_{G}(x)$ is generated by the normalizers $N_{C_{G}(x)}(U)$, where $U \leq$ $C_{S}(x)$ and $m(U) \geq 3$. Take some $U \leq C_{S}(x)$ with $m(U) \geq 3$. By Lemma 2.3, any $E_{8}$-subgroup of $S$ has an involution which is the image of an involution of $S L_{n}(q)$. It follows that $U$ has an element
$y$ which is $G$-conjugate to $t_{k}$ for some even $2 \leq k<n$. By the preceding observations, $C_{G}(y) \leq M$. Arguing as above, we can conclude that $N_{C_{G}(x)}(U) \leq M$. It follows that $C_{G}(x) \leq M$.
Proposition 8.10. We have $G_{0} \unlhd G$.
Proof. Suppose that $M:=N_{G}\left(G_{0}\right)$ is a proper subgroup of $G$. By [28, Proposition 17.11], we may deduce from Lemmas 8.8 and 8.9 that $M$ is strongly embedded in $G$. Therefore, by 49, Chapter $6,4.4], G$ has only one conjugacy class of involutions. On the other hand, we see from Proposition 3.5 that $G$ has at least two conjugacy classes of involutions. This contradiction shows that $M=G$. Hence $G_{0} \unlhd G$.

With Propositions 8.3 and 8.10 , we have completed the proof of Theorem 5.2.

## 9. Proofs of the main results

Proof of Theorem A. By Section 4, Theorem A is true for $n \leq 5$.
Suppose now that $n \geq 6$. Let $q$ be a nontrivial odd prime power, and let $G$ be a finite simple group satisfying (CK).

Recall that a natural number $k \geq 6$ is said to satisfy $P(k)$ if whenever $q_{0}$ is a nontrivial odd prime power and $H$ is a finite simple group satisfying ( $\overline{\mathcal{C K}}$ ) and realizing the 2-fusion system of $P S L_{k}\left(q_{0}\right)$, we have $H \cong P S L_{k}^{\varepsilon}\left(q^{*}\right)$ for some nontrivial odd prime power $q^{*}$ and some $\varepsilon \in\{+,-\}$ with $\varepsilon q^{*} \sim q_{0}$. Theorem 5.2 shows that $P(k)$ is satisfied for all natural numbers $k \geq 6$.

Therefore, if the 2-fusion system of $G$ is isomorphic to the 2-fusion system of $P S L_{n}(q)$, then condition (i) of Theorem A is satisfied.

Conversely, if one of the conditions (i), (ii), (iii) of Theorem A is satisfied, then this can only be condition (i), and Proposition 3.20 implies that the 2 -fusion system of $G$ is isomorphic to the 2-fusion system of $P S L_{n}(q)$.

Proof of Theorem B. Let $q$ be a nontrivial odd prime power and let $n \geq 2$ be a natural number, where $q \equiv 1$ or $7 \bmod 8$ if $n=2$. Let $G$ be a finite simple group and $S \in \operatorname{Syl}_{2}(G)$. Suppose that $\mathcal{F}_{S}(G)$ has a normal subsystem $\mathcal{E}$ on a subgroup $T$ of $S$ such that $\mathcal{E}$ is isomorphic to the 2 -fusion system of $\operatorname{PS} L_{n}(q)$ and such that $C_{S}(\mathcal{E})=1$. We have to show that $\mathcal{F}_{S}(G)$ is isomorphic to the 2-fusion system of $P S L_{n}(q)$.

By Lemma 3.21, $P S L_{n}(q)$ is not a Goldschmidt group. Applying [10, Theorem 5.6.18], we conclude that $\mathcal{E}$ is simple. We see from [16, Theorem B] that $\mathcal{E}$ is tamely realized by some finite simple group of Lie type $K$.

By Theorem A, we have $K \cong \operatorname{PS} L_{n}^{\varepsilon}\left(q^{*}\right)$ for some nontrivial odd prime power $q^{*}$ and some $\varepsilon \in\{+,-\}$ with $\varepsilon q^{*} \sim q$.

By Propositions 3.39 and 3.41, we have that $\operatorname{Out}(K)$ is 2-nilpotent. Now Proposition 2.19 implies that $\mathcal{F}_{S}(G)$ is tamely realized by a subgroup $L$ of $\operatorname{Aut}(K)$ containing $\operatorname{Inn}(K)$ such that the index of $\operatorname{Inn}(K)$ in $L$ is odd. By Lemma 3.56, the 2 -fusion system of $L$ is isomorphic to the 2-fusion system of $\operatorname{Inn}(K) \cong K$ and hence isomorphic to the 2 -fusion system of $P S L_{n}(q)$. So $\mathcal{F}_{S}(G)$ is isomorphic to the 2 -fusion system of $P S L_{n}(q)$.

Proof of Corollary $\mathbb{C}$. Let $q$ be a nontrivial odd prime power and let $n \geq 2$ be a natural number, where $q \equiv 1$ or $7 \bmod 8$ if $n=2$. Let $G$ be a finite simple group and let $S$ be a Sylow 2 -subgroup of $G$. Suppose that $F^{*}\left(\mathcal{F}_{S}(G)\right)$ is isomorphic to the 2-fusion system of $P S L_{n}(q)$.

We have $F^{*}\left(\mathcal{F}_{S}(G)\right) \unlhd \mathcal{F}_{S}(G)$ and $C_{S}\left(F^{*}\left(\mathcal{F}_{S}(G)\right)\right)=Z\left(F^{*}\left(\mathcal{F}_{S}(G)\right)\right)=1$. So Theorem B implies that $\mathcal{F}_{S}(G)$ is isomorphic to the 2-fusion system of $\operatorname{PSL} L_{n}(q)$.

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## References

1. Rachel Abbott, John Bray, Steve Linton, Simon Nickerson, Simon Norton, Richard Parker, Ibrahim Suleiman, Jonathan Tripp, Peter Walsh, and Robert Wilson, ATLAS of Finite Group Representations - Version 3, http: //brauer.maths.qmul.ac.uk/Atlas/v3/, 2021, [Online; accessed 11-May-2021].
2. J. L. Alperin, Richard Brauer, and Daniel Gorenstein, Finite groups with quasi-dihedral and wreathed Sylow 2-subgroups, Trans. Amer. Math. Soc. 151 (1970), 1-261. MR 284499
3. ___ Finite simple groups of 2-rank two, Scripta Math. 29 (1973), no. 3-4, 191-214. MR 401902
4. Kasper K. S. Andersen, Bob Oliver, and Joana Ventura, Reduced, tame and exotic fusion systems, Proc. Lond. Math. Soc. (3) $\mathbf{1 0 5}$ (2012), no. 1, 87-152. MR 2948790
5. M. Aschbacher, Finite group theory, second ed., Cambridge Studies in Advanced Mathematics, vol. 10, Cambridge University Press, Cambridge, 2000. MR 1777008
6. Michael Aschbacher, Finite groups with a proper 2-generated core, Trans. Amer. Math. Soc. 197 (1974), 87-112. MR 364427
7. __, The generalized Fitting subsystem of a fusion system, Mem. Amer. Math. Soc. 209 (2011), no. 986, vi+110. MR 2752788
8. ___ On fusion systems of component type, Mem. Amer. Math. Soc. 257 (2019), no. 1236, v+182. MR 3898993
9. __, The 2-fusion system of an almost simple group, J. Algebra 561 (2020), 5-16. MR 4135537
10. _ Quaternion fusion packets, Contemporary Mathematics, vol. 765, American Mathematical Society, [Providence], RI, [2021] ©2021. MR 4240597
11. Michael Aschbacher, Radha Kessar, and Bob Oliver, Fusion systems in algebra and topology, London Mathematical Society Lecture Note Series, vol. 391, Cambridge University Press, Cambridge, 2011. MR 2848834
12. Michael Aschbacher and Bob Oliver, Fusion systems, Bull. Amer. Math. Soc. (N.S.) 53 (2016), no. 4, 555-615. MR 3544261
13. Adolfo Ballester-Bolinches, Ramón Esteban-Romero, and Mohamed Asaad, Products of finite groups, De Gruyter Expositions in Mathematics, vol. 53, Walter de Gruyter GmbH \& Co. KG, Berlin, 2010. MR 2762634
14. Curtis D. Bennett and Sergey Shpectorov, A new proof of a theorem of Phan, J. Group Theory $\mathbf{7}$ (2004), no. 3, 287-310. MR 2062999
15. Carles Broto, Jesper M. Møller, and Bob Oliver, Equivalences between fusion systems of finite groups of Lie type, J. Amer. Math. Soc. 25 (2012), no. 1, 1-20. MR 2833477
16. , Automorphisms of fusion systems of finite simple groups of Lie type, Mem. Amer. Math. Soc. 262 (2019), no. 1267, 1-120. MR 4071770
17. Timothy C. Burness and Michael Giudici, Classical groups, derangements and primes, Australian Mathematical Society Lecture Series, vol. 25, Cambridge University Press, Cambridge, 2016. MR 3443032
18. Roger Carter and Paul Fong, The Sylow 2-subgroups of the finite classical groups, J. Algebra 1 (1964), 139-151. MR 166271
19. David A. Craven, The theory of fusion systems, Cambridge Studies in Advanced Mathematics, vol. 131, Cambridge University Press, Cambridge, 2011, An algebraic approach. MR 2808319
20. Jean Dieudonné, On the automorphisms of the classical groups. With a supplement by Loo-Keng Hua, Mem. Amer. Math. Soc. 2 (1951), vi+122. MR 45125
21. Benjamin Fine and Gerhard Rosenberger, Number theory, Birkhäuser Boston, Inc., Boston, MA, 2007, An introduction via the distribution of primes. MR 2261276
22. Francesco Fumagalli, On the group of automorphisms of finite wreath products, Rend. Sem. Mat. Univ. Padova 115 (2006), 15-28. MR 2245584
23. George Glauberman, Central elements in core-free groups, J. Algebra 4 (1966), 403-420. MR 202822
24. Daniel Gorenstein, Finite groups, second ed., Chelsea Publishing Co., New York, 1980. MR 569209
25. $\qquad$ , Finite simple groups, University Series in Mathematics, Plenum Publishing Corp., New York, 1982, An introduction to their classification. MR 698782
26. ___ The classification of finite simple groups. Vol. 1, The University Series in Mathematics, Plenum Press, New York, 1983, Groups of noncharacteristic 2 type. MR 746470
27. Daniel Gorenstein, Richard Lyons, and Ronald Solomon, The classification of the finite simple groups, Mathematical Surveys and Monographs, vol. 40, American Mathematical Society, Providence, RI, 1994. MR 1303592
28. ___, The classification of the finite simple groups. Number 2. Part I. Chapter G, Mathematical Surveys and Monographs, vol. 40, American Mathematical Society, Providence, RI, 1996, General group theory. MR 1358135
29. $\qquad$ , The classification of the finite simple groups. Number 3. Part I. Chapter A, Mathematical Surveys and Monographs, vol. 40, American Mathematical Society, Providence, RI, 1998, Almost simple $K$-groups. MR 1490581
30. $\qquad$ , The classification of the finite simple groups. Number 8. Part III. Chapters 12-17. The generic case, completed, Mathematical Surveys and Monographs, vol. 40, American Mathematical Society, Providence, RI, 2018. MR 3887657
31. Daniel Gorenstein and John H. Walter, The characterization of finite groups with dihedral Sylow 2-subgroups. I, J. Algebra 2 (1965), 85-151. MR 177032
$\qquad$ , Balance and generation in finite groups, J. Algebra 33 (1975), 224-287. MR 357583
32. Larry C. Grove, Classical groups and geometric algebra, Graduate Studies in Mathematics, vol. 39, American Mathematical Society, Providence, RI, 2002. MR 1859189
33. Ellen Henke, Products in fusion systems, J. Algebra 376 (2013), 300-319. MR 3003728
34. B. Huppert, Endliche Gruppen. I, Die Grundlehren der mathematischen Wissenschaften, Band 134, SpringerVerlag, Berlin-New York, 1967. MR 0224703
35. A. S. Kondrat'ev, Normalizers of Sylow 2-subgroups in finite simple groups, Mat. Zametki 78 (2005), no. 3, 368-376. MR 2227510
36. Hans Kurzweil and Bernd Stellmacher, The theory of finite groups, Universitext, Springer-Verlag, New York, 2004, An introduction, Translated from the 1998 German original. MR 2014408
37. Changwen Li, Xuemei Zhang, and Xiaolan Yi, On partially $\tau$-quasinormal subgroups of finite groups, Hacet. J. Math. Stat. 43 (2014), no. 6, 953-961. MR 3331152
38. Markus Linckelmann, Introduction to fusion systems, Group representation theory, EPFL Press, Lausanne, 2007, pp. 79-113. MR 2336638
39. David R. Mason, Finite simple groups with Sylow 2-subgroup dihedral wreath $Z_{2}$, J. Algebra 26 (1973), 10-68. MR 318294
40. _ Finite simple groups with Sylow 2-subgroups of type PSL(4, q), q odd, J. Algebra 26 (1973), 75-97. MR 318295
42._, Finite simple groups with Sylow 2-subgroups of type $\operatorname{PSL}(5, q), q$ odd, Math. Proc. Cambridge Philos. Soc. 79 (1976), no. 2, 251-269. MR 396737
41. Ulrich Meierfrankenfeld, Bernd Stellmacher, and Gernot Stroth, Finite groups of local characteristic p: an overview, Groups, combinatorics \& geometry (Durham, 2001), World Sci. Publ., River Edge, NJ, 2003, pp. 155192. MR 1994966
42. Bob Oliver, Reductions to simple fusion systems, Bull. Lond. Math. Soc. 48 (2016), no. 6, 923-934. MR 3608937
43. Kok-wee Phan, A theorem on special linear groups, J. Algebra 16 (1970), 509-518. MR 269732
44. $\quad$, A characterization of the finite groups $\operatorname{PSL}(n, q)$, Math. Z. 124 (1972), 169-185. MR 296173
45. Kok Wee Phan, A characterization of the finite groups $\operatorname{PSU}(n, q)$, J. Algebra 37 (1975), no. 2, 313-339. MR 390044
46. Robert Steinberg, Lectures on Chevalley groups, University Lecture Series, vol. 66, American Mathematical Society, Providence, RI, 2016, Notes prepared by John Faulkner and Robert Wilson, Revised and corrected edition of the 1968 original [ MR0466335], With a foreword by Robert R. Snapp. MR 3616493
47. Michio Suzuki, Group theory. II, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 248, Springer-Verlag, New York, 1986, Translated from the Japanese. MR 815926

Institute of Mathematics, University of Aberdeen, Fraser Noble Building, Aberdeen AB24 3UE, UK

Technische Universität Dresden, Institut für Algebra, 01069 Dresden, Germany
Email address: julian.kaspczyk@gmail.com


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